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# An algebraic approach to vectorial programs

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## Abstract

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Vectorial programming, the combination of SIMD instructions with usual processor instructions, is known to speed-up many standard algorithms. Simple regular languages have benefited from this technology. This paper is a first step towards pushing these benefits further. We take advantage of the inner algebraic structure of regular languages and produce high level representations of efficient *vectorial programs* that recognize certain classes of regular languages.

As a technical ingredient, we establish equivalences between classes of *vectorial circuits* and logical formalisms, namely unary temporal logic and first order logic. The main result is the construction of compilation procedures that turns *syntactic semigroups* into vectorial circuits. The circuits we obtain are *small* in that they improve known upper-bounds on representations of automata within the logical formalisms. The gain is mostly due to a careful sharing of sub-formulas based on algebraic tools.

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## 1 Introduction

Finite state machines abstract the simplest class of programs. They are used everywhere: basic string manipulation functions of the C standard library like `memchr`, `strlen` or `strstr` are based on simple finite state automata, but also text-processing related tasks; checking the validity of encodings; text-mining; etc. As finite state machines are pervasive, implementing them efficiently is key in many softwares. The string related functions of the C standard library we mentioned earlier have greatly benefited from SIMD instructions built into modern CPUs. These functions can now process several characters per CPU cycle.

Single Instruction, Multiple Data (SIMD) executes an operation on several data in parallel, offering a form of low-level parallelism akin to Lamport's [12]. A function like `memchr` searches the first occurrence of a character in a string. SIMD instructions can check whether this character appears among several consecutive characters of the string in one go: each individual character is compared in parallel with the others. Other *vectorized algorithms* (see [10] for examples) benefit from these instructions. In the context of text-processing, impressive handcrafted SIMD based implementations have been proposed for string pattern matching [11], classical regular expression matching [29], Json parsing [13], checking correctness UTF-8 encoding [9], or DNA alignment in bioinformatics [5].

Though many efforts have been put into compilers to solve the problem of *auto-vectorization* [17, 28, 18, 19, 7], these optimization methods rarely succeed in accelerating text algorithms with SIMD instructions. Finding auto-vectorization methods that deal with text algorithms would have a high impact. The challenge would be to produce, among others, the clever handcrafted code in the C standard library from the description of its underlying regular expression. Text processing, however, requires some form of sequentiality simply because, in many cases, information needs to be passed sideways. In this paper we consider two of them: prefix-or and addition.

**The unreasonable power of binary addition.** For sideways information passing, addition is very interesting: carry propagation can be used to compute long distance relations words. Moreover, *Big Int* instructions of modern processors compute it efficiently over large vectors.

Several papers already explored the use of addition in relation to regular languages: Myers [16] uses it to solve approximate string matching; Bergeron and Hamel [2, 1] show that counter-free automata can also benefit from addition; Serre [24] then characterizes counter-free languages in terms of addition; on the practical side, Cameron et al. [3] use explicitly addition when compiling parts of regular expressions of the form  $B^*$  when  $B$  is a set of letters.

This work starts with the definition of a notion of *vectorial circuit* that abstracts away from the details of CPU operations. Circuits make clear which operations are independent from one another and are objects of choice when it comes to introduce parallelism in some computation. We are confident that vectorial circuits can be compiled to obtain efficient programs that use SIMD instructions. As a first step in that direction, the main focus of the paper is then to construct small vectorial circuits from various presentations of counter-free regular languages.

The results of [2, 1] heavily rely on Krohn-Rhodes' Theorem. In this paper, the construction goes through Past-LTL logic and uses the equivalence between the Yesterday-Since operation and binary addition. The relation between these two operations is actually formalized in coq [20] (for technical reasons, in this formalization, we consider Forward-LTL and Next-Until instead of Past-LTL and Yesterday-Since). A consequence is that we can obtain concise vectorial circuits from Past-LTL formulae in the sense that they have the same size

as the initial formula.

We also consider how to obtain concise vectorial circuits directly from automata. With Serre's result, it is theoretically possible to produce vectorial circuits for counter-free automata. However, only a double exponential upper bound in the size of the automaton is known on the size of Past-LTL formulas. This would make circuits far too large in practice. This upperbound comes from a construction of Wilke [31, Corollary 1] that builds formulas by induction over the structure of the *syntactic monoids* of the input automata. On other classes of automata, such as  $\text{FO}_2[<]$  [4, 27], known transformations of automata into formulas are indirect and not constructive. They are of no use to actually compile automata into formulas or circuits.

**Main contributions.** We study the notion of *vectorial circuits* as an abstraction of SIMD programs. Vectorial circuits describe the shape of actual circuits for arbitrary size of inputs. In particular they are circuits of constant depth (a uniform fragment of  $\text{AC}^0$ ). As for circuits, their expressivity depends on the set of authorized gates. Our main goal is to produce small vectorial circuits for particular classes of regular languages: the class of counter-free or  $\text{FO}[<]$ -languages and the class **DA** or  $\text{FO}_2[<]$ -languages. We measure the size of output circuits with respect to the size of the syntactic monoids. Were we to consider automata as inputs that we would need to exponentiate our bounds (e.g. the languages of the form  $A^k a A^*$  have syntactic monoids of size  $2^k$  with minimal automata of size  $O(k)$ ).

Concerning counter-free languages, we revisit Serre's result. We propose an algorithm that produces vectorial circuits (using bitwise boolean operations and addition) that are polynomial in the size of the input aperiodic monoid. Serre's result ensures here that the class of vectorial programs expressed with addition and bitwise boolean operations are equivalent to counter-free languages.

For the class **DA**, we replace addition with prefix-or and obtain a class of circuits that capture exactly that class. When transforming syntactic monoids of **DA** into vectorial circuits, the use of prefix-or makes circuit larger than they would be with addition. The transformation that we propose gives vectorial circuits which have exponential size in the  $\mathcal{J}$ -depth (see 3.1 for the definition of  $\mathcal{J}$ -depth) of the syntactic monoid.

We begin by presenting vectorial circuits in Section 2. We also show the links between vectorial circuits and fragments of logic. In Section 3, we introduce general strategies of evaluation of words. Then, in Section 4, we construct small programs that compute our strategies of evaluation. All proofs can be found in the appendices.

## 2 Compiling regular languages into vectorial circuits

### 2.1 Algebraic preliminaries

We write  $[n]$  for the set  $\{0, \dots, n-1\}$ . Given a set  $E$ , we denote by  $|E|$  its cardinality. Given some finite set  $\Sigma$ , the *alphabet*, words on  $\Sigma$  are finite sequences of elements of  $\Sigma$ . We write  $|x|$  for the length of the word  $x$ . We denote  $\Sigma^*$  the set of words on  $\Sigma$ .

**Semigroups.** A *semigroup* is a pair consisting of a set  $S$  and an associative binary operation  $\cdot_S$  on  $S$ , called the inner operation of  $S$ . We usually write that the set  $S$  is a semigroup. A *monoid* is a triple  $(M, \cdot_M, 1)$ , where  $(M, \cdot_M)$  is a semigroup and  $1 \in M$  is an identity (or a neutral element) of  $M$ . We usually write that the set  $M$  is a monoid. We only work with *finite* semigroups and monoids. We thus designate *finite semigroups* (resp. finite monoids) when we mention semigroups (resp. monoids). Given a semigroup  $S$ , any element  $e$  of  $S$  satisfying  $e \cdot_S e = e$  is called an *idempotent*. In a finite semigroup  $S$ , any element  $s$  of  $S$

admits an *idempotent power*, which is an element  $s^n$  (where  $n > 0$  is an integer) that is idempotent, where  $s^n$  denotes the iterated product of  $s$  by itself  $n$  times. We use the usual notation  $s^\omega$  to denote the idempotent power of  $s$  ( $\omega$  is the minimum integer such that, for any element  $s$ ,  $s^\omega$  is the idempotent power of  $s$ ). Given a semigroup  $S$ , we define  $S^1 = S \cup \{1\}$  as the monoid formed by the semigroup to which an identity is added if necessary. For any subsets  $X$  and  $Y$  of  $S$ , we denote by  $X \cdot_S Y$  the set  $\{x \cdot_S y \mid x \in X, y \in Y\}$ . Similarly, for any  $x \in S$  and  $Y \subseteq S$ , we write  $x \cdot_S Y$  and  $Y \cdot_S x$  respectively for  $\{x\} \cdot_S Y$  and  $Y \cdot_S \{x\}$ . Given a finite set  $\Sigma$ , we call  $\Sigma^+$  the *free semigroup over  $\Sigma$*  with the concatenation as the associative binary operation. This is the only infinite semigroup that we consider. Given a semigroup  $S$ , we will denote by  $S^+$  the free semigroup with the underlying set of  $S$  as alphabet.

**Canonical morphism.** We denote concatenation implicitly: given two words  $u, v$ , their concatenation is written  $uv$ . For instance, taking two elements  $x, y$  of  $S$ ,  $xy$  denotes a word of  $S^+$  of length 2. This notation **must not** be confused with  $x \cdot_S y$  that denotes the element of  $S$  obtained by multiplying  $x$  and  $y$  with the inner operation of  $S$ . We **never use** concatenation to mark the product within  $S$ . However, we relate words of the free semigroup  $S^+$  to their value in  $S$  by means of *the canonical morphism*:  $\pi_S: S^+ \rightarrow S$ . It is the unique associative morphism verifying both the following properties: for every  $x \in S$ ,  $\pi_S(x) = x$  and, for every  $u, v \in S^+$ ,  $\pi_S(uv) = \pi_S(u) \cdot_S \pi_S(v)$ .

**Languages and semigroups.** A link can be established between logics and semigroups by taking the *syntactic semigroup* of a language. This semigroup is defined as follows: given a language  $L$  on an alphabet  $\Sigma$ , the *syntactic congruence* of  $L$  in  $\Sigma^*$  is the relation  $\sim_L$  defined on  $\Sigma^*$  such that, for any words  $u, v \in \Sigma^*$ ,  $u \sim_L v$  if and only if, for all words  $x, y \in \Sigma^*$ ,  $xuy \in L \Leftrightarrow xvy \in L$ . The syntactic semigroup of  $L$  is the quotient  $\Sigma^+ / \sim_L$ , and the syntactic monoid of  $L$  is the quotient  $\Sigma^* / \sim_L$ .

The link between logic and semigroups has already been well studied and gave birth to very nice algebraic characterizations of some well-known classes of languages. Notably, the class of starfree languages is equivalent to the variety of aperiodic semigroups, the semigroups satisfying an equation of the form  $\pi_S(x^{\omega+1}) = \pi_S(x^\omega)$ , for any  $x \in S$ . We can also mention the class  $\text{FO}_2[<]$ , which is equivalent to the variety **DA** of semigroups, the semigroups satisfying an equation of the form  $\pi_S((xy)^\omega x (xy)^\omega) = \pi_S((xy)^\omega)$ , for any  $x, y \in S$ . We refer to [26] for a complete exposition of this class and its relationship to various logics.

## 2.2 Vectorial circuits

We call *vectors* the words on the alphabet  $\{0, 1\}$ . For a vector  $\mathbf{x}$ , we may use the term *dimension* to refer to its length  $|\mathbf{x}|$ . We refer to vectors of dimension  $n$  as  $n$ -vectors. We let  $\mathbf{1}_n$  and  $\mathbf{0}_n$  respectively denote a sequence of  $n$  1's and a sequence of  $n$  0's. When  $n$  is irrelevant or obvious for the context, we may write  $\mathbf{1}$  and  $\mathbf{0}$ .

**Vectorial circuits and their semantics.** *Vectorial circuits* are labeled directed acyclic graphs. The nodes that have no input edge are called *input nodes*. The nodes with incoming edges are called *gates* and are labeled with *commutative operations*. The in-degree of a gate should be equal to the arity of its labeling operation while the out-degree can be arbitrary. We usually write nodes in bold-face fonts:  $\mathbf{v}, \mathbf{v}_1, \dots$ . *Output nodes* are distinguished nodes of the circuit. Generally they include the nodes that have no output edge. The *size of a circuit* is the number of its nodes.

Vectorial circuits can be seen as circuit templates that, for each  $n$ , instantiate a *concrete circuit* working on vectors of dimension  $n$ . Once the dimension is fixed to  $n$ , associating  $n$ -vectors to the input nodes and flowing the values through the gates (where the right

function operating on  $n$ -vectors is used) yields output values in the output nodes. Take a circuit  $C$  with input nodes  $\mathbf{i}_1, \dots, \mathbf{i}_p$  and output nodes  $\mathbf{o}_1, \dots, \mathbf{o}_r$ , given  $p$   $n$ -vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$ , we write  $C(\mathbf{x}_1, \dots, \mathbf{x}_p)$  for the tuple of  $n$ -vectors  $\mathbf{y}_1, \dots, \mathbf{y}_r$  that are respectively yielded in the output nodes  $\mathbf{o}_1, \dots, \mathbf{o}_r$  when evaluating the  $C$  with the vector  $\mathbf{x}_1, \dots, \mathbf{x}_p$  respectively associated to the input nodes  $\mathbf{i}_1, \dots, \mathbf{i}_p$ .

**Operations for labeling gates.** We use bitwise Boolean operations: the unary negation  $\neg$  and the binary operations  $\wedge$  and  $\vee$ , respectively the bitwise conjunction and disjunction.

Given a function  $f : \{0, 1\}^+ \rightarrow \{0, 1\}$ , we define the unary operation  $\text{pref-}f$  (resp.  $\text{suf-}f$ ): given a  $n$ -vector  $\mathbf{x} = b_0 \dots b_{n-1}$ , with  $b_0, \dots, b_{n-1}$  in  $\{0, 1\}$ ,  $\text{pref-}f(\mathbf{x})$  (resp.  $\text{suf-}f(\mathbf{x})$ ) is the  $n$ -vector  $\mathbf{z} = c_0 \dots c_{n-1}$  where for each  $i \in [n]$ ,  $c_i = f(b_0 \dots b_i)$  (resp.  $c_i = f(b_i \dots b_{n-1})$ ). In this paper, we use the unary operations  $\text{pref-}\vee$ ,  $\text{suf-}\vee$ ,  $\text{pref-}\wedge$  and  $\text{suf-}\wedge$ .

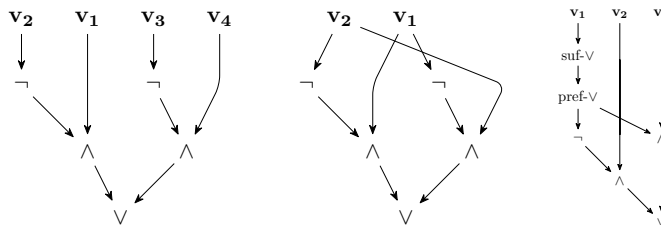
Binary vectors  $\mathbf{x}$  of dimension  $n$  naturally represent numbers in  $[2^n - 1]$ . We write  $nb(\mathbf{x})$  for the number represented by  $\mathbf{x}$  with the convention that its least significant bit is the left-most occurrence of 1. For a natural number  $k$ , we write  $\text{bin}_n(k)$  to denote the vector of dimension  $n$  that represents  $k$  modulo  $2^n$ .

The unary operations LSB (Least Significant Bit) and MSB (Most Significant Bit) replace by 0 respectively the left-most 1 or right-most 1 of their argument vector. For these two operations, when the argument vector is  $\mathbf{0}_n$ , the resulting vector is also  $\mathbf{0}_n$ . The binary operation  $+$  is defined by the family  $(\text{plus}_n)_{n \in \mathbb{N}}$  so that for any two vectors of dimension  $n$ , denoted by  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\text{plus}_n(\mathbf{x}, \mathbf{y}) = \text{bin}_n(nb(\mathbf{x}) + nb(\mathbf{y}))$ .

We study two families of vectorial circuits:

- Sweeping-vectorial circuits, circuits built only with the operations,  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\text{pref-}\vee$ ,  $\text{pref-}\wedge$ ,  $\text{suf-}\vee$ ,  $\text{suf-}\wedge$ , LSB, MSB,
- and ADD-vectorial circuits, circuits built only with the operations  $+$ ,  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\text{pref-}\vee$ ,  $\text{pref-}\wedge$ ,  $\text{suf-}\vee$ ,  $\text{suf-}\wedge$ , LSB, MSB.

**Term notation for vectorial circuits.** Trees are a particular kind of directed acyclic graphs. Circuits that are tree shaped are those where nodes have at most one out-going edge and exactly one output node. These particular circuits can be advantageously denoted by terms built with operations (respecting their arity) and input nodes. The term  $(\mathbf{v}_1 \wedge \neg \mathbf{v}_2) \vee (\neg \mathbf{v}_3 \wedge \mathbf{v}_4)$  represents the circuit of Figure 1. Allowing input nodes to have several occurrences in terms gives access to some limited kind of sharing. This is exemplified with Figure 1. So as to fully capture such sharing capabilities with the term notation, we use equations: a term  $t$  that is to be shared is associated to a node  $\mathbf{v}$  with the equation  $\mathbf{v} = t$  and, when  $\mathbf{v}$  is used in another term, this refers to the *shared* circuit  $t$ . For example, we write  $\mathbf{v} = \text{pref-}\vee(\text{suf-}\vee(\mathbf{v}_1))$ ,  $(\neg \mathbf{v} \wedge \mathbf{v}_2) \vee (\mathbf{v} \wedge \mathbf{v}_3)$  to denote the third circuit in Figure 1.



■ **Figure 1** Graph representation of terms:  $(\mathbf{v}_1 \wedge \neg \mathbf{v}_2) \vee (\neg \mathbf{v}_3 \wedge \mathbf{v}_4)$ ;  $(\mathbf{v}_1 \wedge \neg \mathbf{v}_2) \vee (\neg \mathbf{v}_1 \wedge \mathbf{v}_2)$ ; and  $\mathbf{v} = \text{pref-}\vee(\text{suf-}\vee(\mathbf{v}_1))$ ,  $(\neg \mathbf{v} \wedge \mathbf{v}_2) \vee (\mathbf{v} \wedge \mathbf{v}_3)$

Terms also offer a convenient way for reusing certain circuits in several places: it suffices to change the input nodes at their leaves. We adopt a notation that denotes *parametrized*

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*circuits:*  $c(\mathbf{v}_1, \dots, \mathbf{v}_n) := t$  where  $t$  is a term built with the nodes  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . For circuits  $t_1, \dots, t_n$  we write  $c(t_1, \dots, t_n)$  the circuit described by the term obtained by replacing  $\mathbf{v}_1, \dots, \mathbf{v}_n$  respectively with  $t_1, \dots, t_n$  in  $t$ . For example, we define the bitwise exclusive-or as  $\mathbf{v}_1 \oplus \mathbf{v}_2 := (\mathbf{v}_1 \wedge \neg \mathbf{v}_2) \vee (\neg \mathbf{v}_1 \wedge \mathbf{v}_2)$ . We can compose parametrized circuits to define others et construct complex circuits.

**The addition Lemma.** We say that a vector  $\mathbf{x}$  is contained in a vector  $\mathbf{y}$  if for any position  $i$ ,  $\mathbf{x}_i = 1$  implies  $\mathbf{y}_i = 1$ . Let  $\mathbf{x}, \mathbf{y}$  be two disjoint vectors of dimension  $n$  and  $\mathbf{z}$  a vector of dimension  $n$  that contains both  $\mathbf{x}$  and  $\mathbf{y}$ . We denote by  $\mathbf{v} = \text{Successor}(\mathbf{x}, \mathbf{y}, \mathbf{z})$  the vector such that for all  $i < n$ ,  $\mathbf{v}_i = 1$  if and only if  $\mathbf{x}_i = 1$ , there exists  $j < i$  such that  $\mathbf{y}_j = 1$  and, for all  $k \in \mathbb{N}$  such that  $j < k < i$ ,  $\mathbf{z}_k = 0$ . In other words,  $\text{Successor}(\mathbf{x}, \mathbf{y}, \mathbf{z})$  indicates the positions marked in  $\mathbf{x}$  that follow a marked position of  $\mathbf{y}$ , with no other position marked by  $\mathbf{z}$  in-between.

► **Lemma 1 (Addition lemma).** *Let  $\mathbf{x}, \mathbf{y}$  be two disjoint vectors of dimension  $n$  and  $\mathbf{z}$  a vector of dimension  $n$  that contains both  $\mathbf{x}$  and  $\mathbf{y}$ . Then, we have  $\text{Successor}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := (\mathbf{y} \vee (\mathbf{y} \vee \neg \mathbf{z})) \wedge \mathbf{x}$*

This lemma has a rather tedious proof. So as to relieve the reader from checking its details, we provide a formalization and a proof this Addition Lemma in Coq [20].

### Vectorial circuits as language recognizers and functions from words to finite sets.

Given a fixed alphabet  $\Sigma$ , we say that a vectorial circuit  $C$  *recognizes* a set of words in  $\Sigma^+$  when it has a unique output node and there is a bijection between the letters  $a_1, \dots, a_p$  of  $\Sigma$  and its input nodes  $\mathbf{a}_1, \dots, \mathbf{a}_p$ . We consider a particular bijection: given a word  $u$  of length  $n$ , we write  $\mathbb{1}_a(u)$  for the  $n$ -vector  $\mathbf{x}$  so that  $\mathbf{x}_i = 1$  if and only if  $u_i = a$  for every  $i$  in  $[n-1]$ . We say that  $u$  is *accepted* or *recognized* by the circuit when  $C(\mathbb{1}_{a_1}(u), \dots, \mathbb{1}_{a_p}(u)) \neq \mathbf{0}$ . As a shorthand, we write  $\text{enc}(u)$  for the tuple  $(\mathbb{1}_{a_1}(u), \dots, \mathbb{1}_{a_p}(u))$  and thus  $C(\text{enc}(u))$  for  $C(\mathbb{1}_{a_1}(u), \dots, \mathbb{1}_{a_p}(u))$ .

Vectorial circuits can also represent functions  $f$  from  $\Sigma^+$  to a finite domain  $E$ . It suffices to consider circuits  $C$  which have a bijection between their input nodes and the letters of  $\Sigma$ , but also a bijection between the elements  $e_1, \dots, e_r$  of  $E$  and their output nodes  $\mathbf{e}_1, \dots, \mathbf{e}_r$ . We say that  $C$  *represents*  $f$  when for every  $u$ , the output  $(\mathbf{z}_1, \dots, \mathbf{z}_r) = C(\text{enc}(u))$  is such that, for every  $i$  in  $\{1, \dots, r\}$ ,  $\mathbf{z}_i \neq \mathbf{0}$  if and only if  $f(u) = e_i$ .

**Example.** Consider the alphabet  $\Sigma = \{a, b, c\}$ . The language  $c^*a\Sigma^*$  is recognized by the Sweeping-vectorial circuit  $C = \neg(\text{suf-}\vee(\text{pref-}\vee(\text{suf-}\vee(F_a) \wedge \mathbf{b})))$ , where the circuit  $F_a = \text{LSB}(\mathbf{a}) \oplus \mathbf{a}$  gives the position of the first occurrence of  $a$ .

The language  $\Sigma^*ac^*a\Sigma^*$  is recognized by the ADD-vectorial circuit  $C = (\mathbf{a} + \neg \mathbf{b}) \wedge \mathbf{a}$ , which is equal to  $(\mathbf{a} + (\mathbf{a} \vee \neg(\mathbf{a} \vee \mathbf{b}))) \wedge \mathbf{a}$  (the circuit that corresponds to the definition of  $\text{Successor}(\mathbf{a}, \mathbf{a}, \mathbf{a} \vee \mathbf{b})$ ). An example is showed in Table 1; note that every 1 in the last vector indicates, as we want, each letter  $a$  such that there is no letter  $b$  between this position and the previous letter  $a$ , and no other letter is indicated.

$\mathbb{1}_b$	0	0	1	0	0	0	0
$\mathbb{1}_a$	0	1	0	1	0	1	1
$\mathbb{1}_a + \neg \mathbb{1}_b$	1	0	1	0	0	1	1
$(\mathbb{1}_a + \neg \mathbb{1}_b) \wedge \mathbb{1}_a$	0	0	0	0	0	1	1

■ **Table 1** Example of application of the operation  $(\mathbb{1}_a + \neg \mathbb{1}_b) \wedge \mathbb{1}_a$



## 2.3 Going through first-order logic

The computational model we propose is actually very close to some classical fragments of logic over words. For a full exposition of this subject, we refer to [4]. Informally, we can describe some regular languages with the help of first-order formulas. In those formulas, quantifications range over positions of the word and atomic predicates can check either numeric constraints between positions (typically their order) or the label of the position. For instance, the formula  $\exists x, y \forall z, x < z \wedge z < y \wedge a(x) \wedge a(y) \Rightarrow c(z)$  describes the language over the alphabet  $\Sigma = \{a, b, c\}$  which words belong to  $D = \Sigma^*ac^*a\Sigma^*$ .

In this context, linear temporal logic (LTL for short) has an interesting role that we will rely on. LTL, even restricted to future or past operators, allows to define the same languages as full first-order logic [8]. For instance, the language  $D$  can be described with the following LTL formula:  $F(a \wedge X(cUa))$ .

This class of languages has many equivalent characterizations. In [23], Schützenberger proved for regular languages the equivalence between First-Order/LTL-describable regular languages and *star-free* languages. Later, McNaughton and Papert proved that star-free languages are equivalent with LTL and first-order logic [14, 4]. In a similar fashion to McNaughton and Papert, we can prove that languages computed by ADD-vectorial circuits are exactly the starfree languages. Both directions of the equivalence have been proved by Serre [24, Theorem 1.] by relying on a syntactic logical rewriting of LTL formulas.

► **Proposition 2.** *Let  $L$  be a regular language. The following propositions are equivalent.*

1.  $L$  is computed by an ADD-vectorial circuit.
2.  $L$  is starfree.
3.  $L$  is definable in Forward-LTL.

Another fragment of interest is the class of languages definable by two-variable first-order logic, or equivalently by LTL without the Until operation but with neXt and Yesterday operations. This fragment has been studied a lot and is well understood [8, 4, 30, 32].

In the proof of the next result, we rely on the known equivalence between  $\text{FO}_2[<]$  and a logic called  $\text{TL}[X_a, Y_a]$ . More formally, the languages describable in  $\text{FO}_2[<]$  are exactly the languages describable by the logic  $\text{TL}[X_a, Y_a]$  presented in [4]. This logic uses only two kinds of operators:  $X_a$ , for  $a$  any letter, verifies that there exists a position greater than the current one which holds an  $a$ , and if so, it sets the new current position at the position of the closest  $a$ . If no occurrence of  $a$  is found, the word is not recognized. The operator  $Y_a$ , for any letter  $a$ , is the symmetrical operator: it operates the same way except for the fact that it searches for the closest occurrence of  $a$  before the current position.

► **Proposition 3.** *A language is recognized by a Sweeping-vectorial circuit if and only if it is definable in  $\text{TL}[X_a, Y_a]$ .*

► **Remark 4.** We can always build a vectorial circuit of size linear in the size of the equivalent formula. Conversely, if we consider formulas with sharing of sub-formulas, we can also build an equivalent circuit of size linear in the size of the formula.

## 2.4 Direct compilation scheme

In the previous section, we have seen the relationship between vectorial circuit classes and LTL logic formulas. However, the proofs of those results do not give satisfying algorithms for building a vectorial circuit recognizing a given formula. Indeed, the proofs going from the automata to the formulas are either indirect and not constructive, and they do not give



any upperbound on the sizes of the formulas, or they are constructive but give formulas (and thus circuits) of too large sizes.

Our goal is to reduce the size of the formulas we obtain through semigroups, so that we can obtain tractable algorithms for the languages that have a small syntactic monoid. Because of this point of view, all complexity measures of our circuits are provided in terms of the semigroup size.

► **Theorem 5.** *Let  $S$  be a semigroup in **DA** of  $\mathcal{J}$ -depth  $d$ . We can construct a Sweeping-vectorial circuit of size  $O(2^d|S|^3)$  that computes the operation  $\pi_S$ .*

Moving from Sweeping-vectorial circuits to Unary-LTL could cost an exponential blowup, which in total gives a doubly-exponential blowup for constructing a Unary-LTL formula from a semigroup in **DA**. Remark that the classical constructions do not provide upperbounds on the minimum size of a Unary-LTL formula equivalent to a given semigroup in **DA**. However, these constructions already have considerable sizes of formulas.

We conjecture that no Sweeping-vectorial circuit of polynomial size exists and believe that it is an interesting open question to provide lowerbounds for those circuit models.

► **Theorem 6.** *Let  $S$  be an aperiodic semigroup of  $\mathcal{J}$ -depth  $d$ . We can construct an ADD-vectorial circuit of size  $O(d|S|^3)$  that computes the operation  $\pi_S$ .*

### Proofs' organisation.

We chose to separate the proofs into two main parts. In the first one, we introduce the notion of evaluation program, which is a very generic tool allowing to replace a sub-word by a single letter equal to its product, and we define two generic evaluation strategies which are evaluation programs. These strategies were chosen accordingly to the properties of the classes of semigroup that will be considered later but, supposing that we can provide an efficient encoding of those strategies, they are valid on any semigroup. Then, in the second part, we provide vectorial circuits encoding the two evaluation strategies on the classes of semigroups that we are interested in.

## 3 Semigroups evaluation strategies

### 3.1 Green's relations

We refer the reader to [22, 21] for a complete exposition of algebraic automata theory. We remind here some more basic notations and definitions about semigroups.

Consider a function  $F: S \rightarrow \mathcal{P}(S)$ , where  $\mathcal{P}(S)$  denotes the power set of  $S$ . We write  $x \mathcal{F} y$  when  $F(x) = F(y)$ ;  $x \leq_{\mathcal{F}} y$  when  $F(x) \subseteq F(y)$ ; and  $x <_{\mathcal{F}} y$  when  $x \leq_{\mathcal{F}} y$  and  $F(x) \neq F(y)$ . The relation  $\mathcal{F}$  is an equivalence relation and  $\leq_{\mathcal{F}}$  is a partial pre-order. We also write  $\mathcal{F}(x) = \{y \mid y \mathcal{F} x\}$ , the  $\mathcal{F}$ -class of  $x$ . We say that a semigroup is  $\mathcal{F}$ -trivial when  $\mathcal{F}(x)$  is a singleton for any element  $x \in S$ . Green's relations are defined with the following functions:  $R: x \mapsto x \cdot_S S^1$ ,  $L: x \mapsto S^1 \cdot_S x$ ,  $J: x \mapsto S^1 \cdot_S x \cdot_S S^1$ ,  $H: x \mapsto R(x) \cap L(x)$ . Note that the respective relations obtained from  $R$ ,  $L$ ,  $J$  and  $H$  are denoted by  $\mathcal{R}$ ,  $\leq_{\mathcal{R}}$ ,  $<_{\mathcal{R}}$ ,  $\mathcal{L}$ ,  $\leq_{\mathcal{L}}$ ,  $<_{\mathcal{L}}$ ,  $\mathcal{J}$ ,  $\leq_{\mathcal{J}}$ ,  $<_{\mathcal{J}}$ ,  $\mathcal{H}$ ,  $\leq_{\mathcal{H}}$  and  $<_{\mathcal{H}}$ . For a complete introduction to Green's relations, we refer to [22, 21].

**The  $\mathcal{J}$ -depth of a semigroup.** Let  $S$  be a semigroup. The  $\mathcal{J}$ -depth of a  $\mathcal{J}$ -class is the length of a maximal strictly decreasing sequence of  $\mathcal{J}$ -classes to it. Formally, given a semigroup  $S$  and a  $\mathcal{J}$ -class  $J$ , we say that  $J$  is of  $\mathcal{J}$ -depth  $i$  if there exist  $i$   $\mathcal{J}$ -classes  $J_1 >_{\mathcal{J}} J_2 \dots >_{\mathcal{J}} J_i$  such that  $J_i = J$  and  $J_1$  is maximal, but there exists no decreasing

sequence  $J'_1 >_{\mathcal{J}} J'_2 \dots >_{\mathcal{J}} J'_{i+1}$  such that  $J'_{i+1} = J$  and  $J'_1$  is maximal. The  $\mathcal{J}$ -depth of a semigroup is the maximum  $\mathcal{J}$ -depth of its  $\mathcal{J}$ -classes. By definition of a  $\mathcal{J}$ -class, for any semigroup, there exists a unique  $\mathcal{J}$ -class of maximum  $\mathcal{J}$ -depth. Given  $d$  the  $\mathcal{J}$ -depth of  $S$ , for each integer  $i$  such that  $1 \leq i \leq d$ , we denote by  $D_i(S)$  the union of all the  $\mathcal{J}$ -classes of depth  $i$  and we denote by  $S_i$  the sub-semigroup composed exactly of all the elements of  $S$  of  $\mathcal{J}$ -depth at least  $i$ .

**$\mathcal{J}$ -constant words.** Let  $S$  be a semigroup. A word  $s_0 \cdots s_k$  in  $S^+$  is *left  $\mathcal{J}$ -constant* if, for any index  $i$  such that  $0 \leq i \leq k$ , we have  $\pi_S(s_0 \cdots s_i) \mathcal{J} s_0$ . Symmetrically,  $s_0 \cdots s_k$  is *right  $\mathcal{J}$ -constant* if, for any index  $i$  such that  $0 \leq i \leq k$ ,  $\pi_S(s_i \cdots s_k) \mathcal{J} s_k$ . Finally, a word in  $S^+$  is  $\mathcal{J}$ -constant if it is both left and right  $\mathcal{J}$ -constant. The latter property is equivalent to having a left  $\mathcal{J}$ -constant word such that  $s_0 \mathcal{J} s_k$ .

### 3.2 Evaluation programs

In this paper, we consider words over some semigroup  $S$ . Our goal is to compute the product in  $S$  of the letters composing these words. For any word  $u \in S^+$ , this amounts to compute  $\pi_S(u)$ . This computation can be performed by vectorial circuits. Instead of directly building these circuits, we first pay attention to evaluation strategies that we call *evaluation programs*. These strategies form an overlay of abstraction over the intricacy of circuits. They are meant to modularize the construction of vectorial circuits.

Given a semigroup  $S$ , an evaluation program over  $S$  transforms words in  $S^+$  by replacing some of the factors by their values (through the canonical morphism) in  $S$ . In this section, we consider a fixed semigroup  $S$ .

► **Definition 7 (Partial evaluation).** A partial evaluation step over  $S$  is a relation over  $S^+$  denoted by  $\rightarrow_S$  and defined as  $uvw \rightarrow_S u\pi_S(v)w$  for any  $v$  in  $S^+$  and  $u, w \in S^*$ . We denote by  $\rightarrow_S^+$  the transitive closure of  $\rightarrow_S$ . We say that  $v$  is a partial evaluation of  $u$  when  $u \rightarrow_S^+ v$ .

Remark that if  $v$  is a partial evaluation of  $u$  over  $S$ , then  $\pi_S(u) = \pi_S(v)$ . Usually, the context makes it clear which semigroup is considered. Thus, we generally leave the semigroup implicit and only say that  $v$  is a partial evaluation of  $u$ . Remark that each word  $u$  is a partial evaluation of itself.

► **Definition 8 (Partial evaluation programs).** A partial evaluation program over  $S$  is a partial function  $P : S^+ \rightarrow S^+$  such that, for any word  $u \in S^+$  that is in the domain of  $P$ ,  $P(u)$  is a partial evaluation of  $u$ . If the domain of  $P$  is  $S^+$ , then we say that  $P$  is total.

The function  $\pi_S$  is an example of a partial evaluation (which is, in this case, total). Another example is the function  $\text{LProd}_S : S^+ \rightarrow S^+$  that performs the product of the first two elements of the input, if there are at least two elements, and otherwise returns the input word. In symbols, it is defined by  $\text{LProd}_S(s_0 \cdots s_n) = \pi_S(s_0 s_1) s_2 \cdots s_n$  for any word  $u = s_0 \cdots s_n$  of length at least two, and  $\text{LProd}_S(s_0) = s_0$  for any element  $s_0 \in S$ . Similarly, we define  $\text{RProd}_S$  as the partial evaluation program which performs the product of the two last elements, if there are at least two elements, and otherwise returns the input word. Evaluation programs are closed under composition.

### 3.3 Waterfall evaluation programs

In this section, we are designing a first set of specific evaluation programs. These programs work by evaluating the semigroup in a top-down fashion (with respect to the  $\mathcal{J}$ -order). We

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first detect each maximal subword which product is maximal for the  $\mathcal{J}$ -order, evaluate those subwords and multiply the results with the letters that immediately follow.

► **Definition 9** ( $\mathcal{J}$ -maximal falling words). *A word  $u$  in  $S^+$  is  $\mathcal{J}$ -maximal falling whenever for every  $p, s \in S^*$  and  $v, w \in S^+$  so that  $u = pvws$ , we have  $\pi_S(vw) \in S_2$ .*

Remark that when  $u$  is  $\mathcal{J}$ -maximal falling and  $|u| > 1$ ,  $\pi_S(u) \in S_2$ .

► **Definition 10** (Collapsing evaluation). *A total evaluation program is collapsing when, for any input word  $u$  in  $S^+$ , it produces a  $\mathcal{J}$ -maximal falling word.*

► **Definition 11** ( $\mathcal{J}$ -maximal decomposition). *Consider a word  $u \in S^+$ . If  $u \notin S_2^+$ , let  $t \in \mathbb{N}$  and some words  $w_0, \dots, w_{t+1} \in S^*$ ,  $v_0, \dots, v_t \in S^+$ , that define a decomposition  $u = w_0v_0w_1 \cdots v_t w_{t+1}$ . This decomposition is called  $\mathcal{J}$ -maximal if we have*

- *for any integer  $i$  such that  $0 \leq i \leq t+1$ ,  $w_i$  is a word in  $S_2^*$*
- *for any integer  $i$  such that  $1 \leq i \leq t$ ,  $v_i$  is a maximal subword of  $u$  verifying  $\pi_S(v_i) \in D_1(S)$ . More formally, if we consider the decomposition  $u = pav_i bs$ , with  $p, s \in S^*$ , and  $a, b \in S \cup \epsilon$  ( $a$  is the letter preceding  $v_i$ , if it exists, and  $b$  is the letter following  $v_i$ , if it exists) then, if  $a \neq \epsilon$ ,  $\pi_S(av_i) <_{\mathcal{J}} \pi_S(v_i)$  and, if  $b \neq \epsilon$ ,  $\pi_S(v_i b) <_{\mathcal{J}} \pi_S(v_i)$*

We call  $t+1$  the size of the decomposition.

If  $u \in S_2^+$ , the  $\mathcal{J}$ -maximal decomposition of  $u$  is composed of only one element equal to  $u$  itself. In this case, the decomposition is unique and by convention of size 0.

► **Remark 12.** The property of the  $(v_i)$ 's implies that each  $v_i$  is a word of  $D_1(S)^+$ . Indeed, the product of a word is at most  $\mathcal{J}$ -equivalent to the letter of greatest  $\mathcal{J}$ -depth, so all the letters must be of  $\mathcal{J}$ -depth 1 for the product to be in  $D_1(S)$ .

We are now proving the existence and uniqueness of  $\mathcal{J}$ -maximal decomposition. The proof works by considering a variant of  $\mathcal{J}$ -maximal decomposition by enforcing the maximality constraint only at the right and showing that the two variants are actually equivalent.

► **Lemma 13.** *Let  $S$  be a semigroup. For any finite word  $u \in S^+$ , there exists a unique  $\mathcal{J}$ -maximal decomposition of  $u$ .*

Now, we need to refine the  $\mathcal{J}$ -maximal decomposition. Keeping the same notation for  $u$ , and its  $\mathcal{J}$ -maximal decomposition  $u = w_0v_0 \cdots v_t w_{t+1}$ , we set, for any  $0 \leq i \leq t$ ,  $v_i = x_i v'_i$  ( $x_i \in S$  is the first letter of  $v_i$ ),  $w'_i = w_i x_i$  and  $w''_i = y_i w'_i$  ( $y_i \in S$  is the first letter of  $w'_i$ ). The evaluation program  $\text{Collapse}_S$  takes a word  $u \in S^+$  and performs at once all the products  $\pi_S(v'_i y_{i+1})$ , the operation  $\pi_S(v'_i)$  if  $w_{t+1} = \epsilon$ , or  $\pi_S(v'_i y_{t+1})$  otherwise, with  $w'_{t+1} = y_{t+1} w_{t+1}$  ( $y_{t+1} \in S$  is the first letter of  $w_{t+1}$ ). In symbols:

$\text{Collapse}_S(u) = w'_0 \pi_S(v'_0 y_1) w''_1 \cdots \pi_S(v'_i x'_{i+1}) w''_{i+1} \cdots w''_t z$ , where  $z$  denotes either  $\pi_S(v'_t)$  if  $w_{t+1}$  is the empty word, or  $\pi_S(v'_t y_{t+1}) w'_{t+1}$ . The image of any word  $u \in S^+$  by  $\text{Collapse}_S$  is a  $\mathcal{J}$ -maximal falling word.

Given an element  $s \in D_1(S)$ , we also define the partial evaluation program  $\text{Falling}_S(s)$  which takes a word  $u \in S^+$  which is a  $\mathcal{J}$ -maximal falling word such that the last letter of  $u$  is in  $S_2$ , and returns a partial evaluation of  $u$  in which there is no occurrence of  $s$ . Given a word  $u$ , we call  $s$ -decomposition of  $u$  the decomposition of the form  $u = w_0 s^{k_0} x_0 w_1 \cdots s^{k_t} x_t w_{t+1}$  where the  $k_i$ 's are positive integers, the  $x_i$ 's are non- $s$  elements of  $S$ , except for  $x_t$  which can also be equal to  $\epsilon$  if  $w_{t+1} = \epsilon$ , and the  $w_i$ 's are words without any occurrence of  $s$ . The  $s$ -decomposition of a word always exists and is unique. Given the  $s$ -decomposition of some word  $u \in S^+$ , keeping the same notation, we define

$$\text{Falling}_S(s)(u) = w_0 \pi_S(s^{k_1} x_1) \cdots \pi_S(s^{k_t} x_t) w_t$$

► **Lemma 14.** *Let  $S$  be a semigroup of  $\mathcal{J}$ -depth  $d$ . The evaluation program  $\pi_S$  is equal to the composition of  $O(|S|)$  evaluation programs among  $\text{RProd}_S$ ,  $\text{Collapse}_{S_i}$  and  $\text{Falling}_{S_i}(s)$  for all  $1 \leq i \leq d$  and  $s \in D_1(S_i)$ .*

### 3.4 Sweeping evaluation programs

We introduce an evaluation program which processes words in a more lateral fashion – from left to right or right to left. In the following section,  $S$  is a semigroup of  $\mathcal{J}$ -depth  $d$ .

In the proofs of Section 4, we will perform evaluations that produce left (resp. right)  $\mathcal{J}$ -constant prefixes (resp. suffixes). We define those programs depending on the  $\mathcal{J}$ -depth of the semigroup that is considered. First, we introduce a *left splitting* higher order operation that applies an evaluation program over a prefix of the input word defined by some constraints over Green's relations. Formally, for any integer  $i \leq d$ , we define the operation  $\text{LSplit}_{S,i}$  as follows. Consider a word  $u = s_0 \cdots s_k \in S^+$  and an evaluation program  $P$  defined at least on all left  $\mathcal{J}$ -constant words of depth  $i$ . If  $s_0$  is not of  $\mathcal{J}$ -depth  $i$ , we set  $\text{LSplit}_{S,i}(P)(u) = u$ . Otherwise, there exist two uniquely defined words  $p \in S^+$ ,  $s \in S^*$  such that  $u = ps$ , where  $p$  is left  $\mathcal{J}$ -constant and either  $s$  is nonempty or, if we denote by  $x \in S$  its first letter,  $\pi_S(px) <_{\mathcal{J}} \pi_S(p)$ . In this case,  $\text{LSplit}_{S,i}(P)(u) = P(p)s$ . We define similarly the symmetric operation  $\text{RSplit}_{S,i}$ . Finally, we define the partial function  $\text{JProd}_S$ , that is the restriction of  $\pi_S$  over words that are  $\mathcal{J}$ -constant. Formally,  $\text{JProd}_S$  is defined only on  $\mathcal{J}$ -constant words and, given  $u \in S^+$  such a word,  $\text{JProd}_S(u) = \pi_S(u)$ .

A *sweeping evaluation program* is an evaluation program built with the following operations:  $\text{LProd}_S$ ,  $\text{RProd}_S$ , the partial function  $\text{JProd}_S$ , and the higher order operations  $\text{LSplit}_{S,i}$  and  $\text{RSplit}_{S,i}$  for any integer  $i$  such that  $1 \leq i \leq d$ .

► **Lemma 15** (Sweeping evaluation programs). *Let  $S$  be a semigroup. There exists a sweeping evaluation program computing  $\pi_S$ . Moreover, there exists such a program that is equal to the composition of  $O(2^d)$  operations.*

## 4 Proof of the main results

We recall the proof strategy. Given a regular language which is aperiodic (resp. in **DA**), we prove that we can compute  $\pi_S$  with a **ADD**- (resp. **Sweeping**-) vectorial circuit. To build such circuits, we rely on respectively waterfall and sweeping evaluations programs. As the considered vectorial circuit classes are closed under functional composition, it is sufficient to prove that every basic operation in our evaluation programs is computable with the desired vectorial circuit class. We provide those arguments in this section.

**Vectorial encoding of partial evaluation.** Our sweeping and waterfall evaluation programs perform operations on partial evaluations of a word, so we need to provide an explanation on how we encode these partial evaluations. Informally, a vectorial representation of a partial evaluation is a set of vectors, each vector corresponding to a semigroup element. Each bit set to one denotes the presence of the element in the partial evaluation, and the order of the bits set to one determines the order of the letters in the word. As in the case of characteristic functions of letters, in the vector encoding of a word, vectors in an encoding do not overlap, however their union may not cover all the positions. More formally, a *vectorial encoding of a partial evaluation* is a mapping  $\mathbf{c}: S \rightarrow \{0,1\}^n$ , for some integer  $n \geq 1$ , such that we have the following constraint: for any  $s, t \in S$  such that  $s \neq t$ , we have  $\mathbf{c}(s) \wedge \mathbf{c}(t) = \mathbf{0}$ . Note that, by definition, a word is a partial encoding of itself, so

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$\text{enc}(u)$  is a vectorial encoding of a partial evaluation, with the additionnal property that  $\bigvee_{a \in S} \mathbf{1}_a(u) = \mathbf{1}$ .

Given such a function  $\mathbf{c}$  outputting vectors of dimension  $n$ , we can interpret it as a word of length at most  $n$  by respecting the order of apparition of the bits. Formally, a word  $s_0 \cdots s_k \in S^{\leq n}$  is represented by a vectorial encoding  $\mathbf{c}$  of dimension  $n$  if, for every index  $j \leq k$ , there exists some integer  $i < n$  such that  $\mathbf{c}(s_j)_i = 1$  and  $|\{t \in \mathbb{N} \mid t < i \text{ and } \bigvee_{s \in S} \mathbf{c}(s)_t = 1\}| = j - 1$ .

In this section, we use only circuits of the following form: the input is a vectorial encoding of a partial evaluation  $\mathbf{c}: S \rightarrow \{0, 1\}^n$  (for some  $n \in \mathbb{N}^+$ ) representing some word  $u \in S^{\leq n}$ , and the output is a vectorial encoding  $\mathbf{out}: S \rightarrow \{0, 1\}^n$  representing the partial evaluation of  $u$  obtained by applying the operation corresponding to the circuit. To prove our theorems, we construct vectorial circuits using this encoding that implement the partial evaluation functions we presented earlier.

**Sketch of proof of Theorem 6.** To prove Theorem 6, we consider a fixed aperiodic semigroup  $S$ , and we denote by  $d$  its  $\mathcal{J}$ -depth. Thanks to Lemma 14, it is sufficient to provide ADD-vectorial circuits computing the operations  $\text{Collapse}_S$ ,  $\text{RProd}_S$  and  $\text{Falling}_S(s)$  (for any aperiodic semigroup  $S$  and  $s \in D_1(S)$ ) over some vectorial encoding of any partial evaluation of a word. Once we have those, we proceed as in Lemma 14, in which we prove that, for any integer  $i$  such that  $1 \leq i \leq d$ ,  $\pi_{S_i}$  is equal to the composition of  $\pi_{S_{i+1}}$  and  $O(|S|)$  operations among  $\text{Collapse}_{S_i}$ ,  $\text{RProd}_S$  and  $\text{Falling}_S(s)$  (for any  $s \in D_i(S)$ ). More precisely, the evaluation program for  $\pi_S$  uses  $|S|$  operations of the form  $\text{Falling}_{S'}(s)$  (where  $s \in S$ ),  $d$  operations of the form  $\text{RProd}_S$  and  $d$  operations of the form  $\text{Collapse}_{S_i}$  (recall that  $d$  is the  $\mathcal{J}$ -depth of  $S$ ). Now, we can obtain the result by using the following Lemmas:

- **Lemma 16.** *For any aperiodic semigroup  $S$ , we can compute  $\text{Collapse}_S$  over any vectorial encoding of a partial evaluation with an ADD-vectorial circuit of size  $O(|S|^3)$ .*
- **Lemma 17.** *For any aperiodic semigroup  $S$ , we can compute  $\text{RProd}_S$  on any vectorial encoding of a partial evaluation of a word over  $S$  with an ADD-vectorial circuit of size  $O(|S|^2)$ .*
- **Lemma 18.** *Let  $S$  be an aperiodic semigroup of  $\mathcal{J}$ -depth  $d$ . For any element  $s \in D_1(S)$ , we can compute  $\text{Falling}_S(s)$  over any vectorial encoding of a partial evaluation in its domain with an ADD-vectorial circuit of size  $O(d|S|)$ .*

With this, we can conclude that  $\pi_S$  can be computed with an ADD-vectorial circuit of size  $O(d|S|^3)$ .

**Sketch of proof of Theorem 5.** To prove Theorem 5, we consider a fixed semigroup  $S \in \mathbf{DA}$ , and we denote by  $d$  its  $\mathcal{J}$ -depth. Thanks to Lemma 15, it is sufficient to provide Sweeping-vectorial circuits computing the base operations over some vectorial encoding of any partial evaluation of a word. Those operations are  $\text{LProd}_S$ ,  $\text{RProd}_S$ ,  $\text{JProd}_S$  and, for each integer  $i$  such that  $1 \leq i \leq d$ , the operations  $\text{LSplit}_{S,i}$  and  $\text{RSplit}_{S,i}$ . Once we have those, we proceed as in Lemma 15, in which we prove that  $\pi_S$  is equal to the composition of  $O(2^d)$  operations among  $\text{JProd}_S$ ,  $\text{LProd}_S$ ,  $\text{RProd}_S$  and  $\text{LSplit}_{S,i}(E)$  (for any integer  $i$  such that  $1 < i \leq d$  and some sweeping program  $E$  composed of the same operations, that are taken into account in the  $O(2^d)$ ). To prove this, we have  $\pi_S = P_{d,l} = P_{d,r}$  where, for each integer  $i$  such that  $1 \leq i \leq d$ , the program  $P_{i,l}$  computes  $\pi_S$  on the maximal prefix of  $\mathcal{J}$ -depth at most  $i$  (included), and the symmetric program  $P_{i,r}$  which acts on suffixes instead of prefixes. Our proof constructs Sweeping-vectorial circuits for those programs, by induction on the depth  $i$ . Now, we can obtain the result by using the following Lemmas:

► **Lemma 19.** *For any semigroup  $S \in \mathbf{DA}$ , we can compute  $\text{JProd}_S$  over any vectorial encoding of a partial evaluation in its domain with a Sweeping-vectorial circuit of size  $O(|S|^2)$ .*

► **Lemma 20.** *For any semigroup  $S \in \mathbf{DA}$ , we can compute  $\text{LProd}_S$  and  $\text{RProd}_S$  over any vectorial encoding of a partial evaluation in their domains with Sweeping-vectorial circuits of size  $O(|S|^2)$ .*

► **Lemma 21.** *Let  $S$  be a semigroup in  $\mathbf{DA}$  of  $\mathcal{J}$ -depth  $d$ ,  $i$  be an integer such that  $1 \leq i \leq d$ , let  $P$  be a sweeping evaluation program defined at least on all left  $\mathcal{J}$ -constant words of depth  $i$ , and suppose that we have a Sweeping-vectorial circuit of size  $s_P$  that computes  $P$  over any vectorial encoding of a partial evaluation. Then we can compute  $\text{LSplit}_{S,i}(P)$  and  $\text{RSplit}_{S,i}(P)$  over any vectorial encoding of a partial evaluation in their respective domains with Sweeping-vectorial circuits of size  $O(|S|^3 + s_P)$ .*

Thus, following the construction in the proof of Lemma 15, for each integer  $i$  such that  $1 \leq i < d$ , we have a Sweeping-vectorial circuit computing  $P_{i+1,l}$  (or  $P_{i+1,r}$ , these are symmetrical operations) given some circuits computing  $P_{i,l}$  and  $P_{i,r}$ , and its size is equal to  $O(|S|^3 + 2c_i)$ , where  $c_i$  is the size of the circuits computing  $P_{i,l}$  and  $P_{i,r}$ . Since, by the construction of Lemma 15, we also have  $c_1 = O(|S|^3)$ , we then have  $c_i = (2^i - 1)|S|^3$  for each integer  $i$  such that  $1 < i \leq d$ . Thus, the circuit computing  $\pi_S = P_{d,l}$  is of size  $O(2^d|S|^3)$ .

## 5 Conclusion

We introduce vectorial circuits as abstractions of low level hardware architectures. As circuits, they put forward dependencies between steps of computation and thus opportunities of parallelism. Taking stock on previous work [15, Theorem 1], the next step is then to adapt these circuits to a streaming context. This streaming setting is closer to text processing problems as only small chunks of input data can hold at the level of CPUs. We shall then study how to take advantage of SIMD instructions to implement these machines. The vast number of SIMD instructions give many possibilities to compile vectorial circuits for streaming text. The challenge is then to find the right set of instructions, combine them sufficiently well together and obtain efficient programs. Here again, we hope that algebra can back our efforts up.

Concerning the circuits themselves, we can consider extensions with operations that have natural correspondence in CPU instructions such as *shifts*, *prefix-xor*, etc. The question is then to understand what classes of regular languages we can describe. Hopefully, the combinations of particular sets of operations could correspond to well-studied algebraic operations [25]. Instead of operations, we can also consider the use of arbitrary constants, (i.e. particular vectors that can be used as gates inputs). We think that using constants in circuits can be related to that of arbitrary monadic advices [6].

In the paper, we limit ourselves to consider vectorial circuits as recognizers. As these circuits produce outputs, they should be more adequately viewed as transducers. On the theoretical side, there is a need to explore their expressivity. This requires an adaptation of the algebraic tools we use, i.e. an understanding of classes of logical transducers. On the practical side, in the context of streaming, this point of view calls for composing stream processing programs. This probably requires to explore ideas from synchrone programming in combination with vectorial circuits.

This paper also calls for more foundational work concerning circuit complexity. In particular we conjecture that the bounds provided in Theorem 5 are tight. However, finding an adequate lower bound is a challenging open problem.



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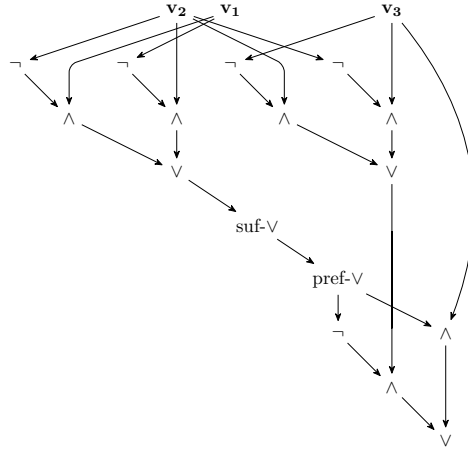
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**A Proofs for Section 2 (Compiling regular languages into vectorial circuits)**

**Parameterized circuits.** We can compose parametrized circuits to define others:

- $\text{NotZero}(\mathbf{v}) := \text{pref-}\vee(\text{suf-}\vee(\mathbf{v}))$
- $\text{IsZero}(\mathbf{v}) := \neg\text{NotZero}(\mathbf{v})$
- $\text{IfThenElse}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \mathbf{v} = \text{NotZero}(\mathbf{v}), (\neg\mathbf{v} \wedge \mathbf{v}_1) \vee (\mathbf{v} \wedge \mathbf{v}_2)$
- $\text{Thr2}(\mathbf{v}) := \text{NotZero}(\text{LSB}(\mathbf{v}))$

The parametrized circuits  $\text{NotZero}(\mathbf{v})$ ,  $\text{IsZero}(\mathbf{v})$ ,  $\text{Thr2}(\mathbf{v})$  perform tests on their input: on an  $n$ -vector, they return  $\mathbf{1}_n$  when the test succeeds and  $\mathbf{0}_n$  when it fails.  $\text{NotZero}(\mathbf{v})$  tests whether its input vector contains an occurrence of 1,  $\text{IsZero}(\mathbf{v})$  tests whether its input vector is  $\mathbf{0}$  and  $\text{Thr2}(\mathbf{v})$  tests whether its input vector contains at least two occurrences of 1. The parametrized circuit  $\text{IfThenElse}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ , tests whether  $\mathbf{v}_1$  is  $\mathbf{0}$  and in this case outputs  $\mathbf{v}_2$ . Otherwise, it outputs  $\mathbf{v}_3$ . With parameterized circuits, we can construct complex circuits. An example is the circuit  $\text{IfThenElse}(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{v}_2 \oplus \mathbf{v}_3, \mathbf{v}_1)$  which is depicted Figure 2.



■ **Figure 2** Circuit represented by  $\text{IfThenElse}(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{v}_2 \oplus \mathbf{v}_3, \mathbf{v}_1)$  .

► **Proposition 2.** *Let  $L$  be a regular language. The following propositions are equivalent.*

1.  $L$  is computed by an ADD-vectorial circuit.
2.  $L$  is starfree.
3.  $L$  is definable in Forward-LTL.

**Proof.** The equivalence between 1 and 2 is from [24], Corollary 1. In this paper, the vectorial computational model consists of straight-line expressions over a basis which includes addition, Boolean operations and right shift. This basis is equivalent to ADD-vectorial circuits. The equivalence between 2 and 3 is standard. It can be proved by combining theorems proved by Kamp [8] and McNaughton and Papert [14]. The former proves the equivalence between being definable in Forward-LTL and First-Order logic for regular languages and the later the equivalence between First-Order logic and star-free languages. ◀

► **Proposition 3.** *A language is recognized by a Sweeping-vectorial circuit if and only if it is definable in  $T\mathcal{L}[X_a, Y_a]$ .*

**Proof.** We rely on the equivalence between  $\text{FO}_2[<]$  and  $\text{TL}[X_a, Y_a]$ , to prove the result. First, we show that  $\text{TL}[X_a, Y_a]$  is captured by Sweeping-vectorial circuits, i.e. that the formulas of  $\text{FO}_2[<]$  define the very same functions from words to Boolean vectors as these circuits; here we interpret a formula of  $\text{FO}_2[<]$  as a function from words to Boolean vectors, by considering the truth vector of a formula on every position of the input word. Clearly, Boolean operations are equivalent in both models. Now, suppose that we have a Sweeping-vectorial circuit  $C_\alpha$  that is equivalent to a  $\text{TL}[X_a, Y_a]$  formula  $\alpha$ , i.e. given a word  $u \in \Sigma^+$  and a vector **pos** (of the same dimension as  $u$ ) that has a unique 1 in some position  $x$ , we have  $C_\alpha(\mathbf{pos}, \text{enc}(u)) \neq \mathbf{0}$  if and only if  $u, x \models \alpha$ . Then, for any letter  $a \in \Sigma$ , the formula  $\beta = X_a\alpha$  can be modeled by the circuit  $C_\beta$  defined as follows: for any word  $u \in \Sigma^*$  and any vector **pos** having a unique 1, we begin by finding the first letter  $a$  strictly after the position marked by **pos**, by sequentially computing the following vectors **authorizedPos** =  $\text{LSB}(\text{pref-}\vee(\mathbf{pos}))$ , **next** = **authorizedPos**  $\wedge$   $\mathbb{1}_a(u)$ , **nextPos** = **next**  $\oplus$   $\text{LSB}(\mathbf{next})$ . This gives us the new position vector to give as parameter to the new circuit. Then we can define  $C_\beta(\mathbf{pos}, \text{enc}(u)) := C_\alpha(\mathbf{nextPos}, \text{enc}(u))$ , which is equivalent to  $\beta$ . Finally, we can model the formula  $\gamma = Y_a\alpha$  using the same formulas, except for the vector **authorizedPos**, which is computed as **authorizedPos** =  $\text{MSB}(\text{suf-}\vee(\mathbf{pos}))$ , in order to get the positions situated strictly before the position  $x$  marked by **pos**.

Now, we prove that Sweeping-vectorial circuits are captured by formulas in  $\text{FO}_2[<]$ . Since Sweeping-vectorial circuits output a truth vector indicating if some property is true for each position in the input word, we must consider  $\text{FO}_2[<]$  formulas which can take into account a starting position. Thus, we consider only  $\text{FO}_2[<]$  formulas with one free variable, which represents the position in which we begin to evaluate the formula in the input word.

Clearly, Boolean operations are equivalent in both models. We can also interpret  $\text{pref-}\vee$  as an  $\text{FO}_2[<]$  formula: consider an  $\text{FO}_2[<]$  formula  $\varphi$  with one free variable and suppose that it is computed by a circuit  $C_\varphi$ . Then, the formula  $\varphi_{\text{pref-}\vee}$  with one free variable defined by  $\varphi_{\text{pref-}\vee}(y) := \exists x \leq y, \varphi(x)$  has its truth vector equal to the output of  $\text{pref-}\vee(C_\varphi)$ . The other prefix and suffix operations can be dealt with in a similar way. Finally, we can interpret  $\text{LSB}$  and  $\text{MSB}$  as  $\text{FO}_2[<]$  formulas. Indeed, consider an  $\text{FO}_2[<]$  formula  $\varphi$  with one free variable and suppose that it is computed by a circuit  $C_\varphi$ . Then, the formula  $\varphi_{\text{LSB}}$  with one free variable defined by  $\varphi_{\text{LSB}}(y) := \varphi(y) \wedge \exists x < y, \varphi(x)$  has its truth vector equal to the output of  $\text{LSB}(C_\varphi)$ .  $\text{MSB}$  is equivalent to the same formula where  $x < y$  is replaced by  $x > y$ .  $\blacktriangleleft$

## B Proofs for Section 3 (Semigroups evaluation strategies)

Every word in  $S^+$  admits a unique  $\mathcal{J}$ -maximal decomposition of a word. This property hinges on the Localisation Theorem of Clifford and Miller [22, Proposition 1.6, page 48].

► **Lemma 22** (Localisation Theorem). *Let  $S$  be a semigroup and  $x, y$  be in  $S$ . We have  $xy\mathcal{J}x$  if and only if there exists an idempotent  $e$  in  $R(y) \cap L(x)$ .*

We will use the following useful technical lemma.

► **Lemma 23.** *Let  $S$  be a semigroup. Let  $x, y$  be  $\mathcal{J}$ -equivalent elements of  $S$  and  $z$  another element of  $S$ .*

- *If  $\pi_S(xy)\mathcal{J}x$  and  $\pi_S(zxy) <_{\mathcal{J}} x$ , then  $\pi_S(zx) <_{\mathcal{J}} x$ .*
- *If  $\pi_S(xy)\mathcal{J}x$  and  $\pi_S(xyz) <_{\mathcal{J}} x$ , then  $\pi_S(yz) <_{\mathcal{J}} x$ .*

**Proof.** Both cases are symmetric, so we only prove the first one. Suppose that  $\pi_S(zx)\mathcal{J}x$ ,  $\pi_S(xy)\mathcal{J}x$  and  $\pi_S(zxy) <_{\mathcal{J}} x$ . Lemma 22 implies that there is no idempotent in  $R(y) \cap L(zx)$ .

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But, as  $\pi_S(zx)\mathcal{J}x$ , by definition of  $\mathcal{L}$ , we have  $\mathcal{L}(zx) = \mathcal{L}(x)$ . Therefore  $R(y) \cap L(x)$  does not contain an idempotent. Finally Lemma 22 entails  $xy <_{\mathcal{J}} x$ , a contradiction. ◀

► **Lemma 13.** *Let  $S$  be a semigroup. For any finite word  $u \in S^+$ , there exists a unique  $\mathcal{J}$ -maximal decomposition of  $u$ .*

**Proof.** We focus on proving the existence of such a decomposition. Unicity follows from the proof of existence. First, remark that  $S = D_1(S) \uplus S_2$ . Thus, for each any word  $u \in S^+$ , there necessarily exists a unique decomposition of  $u$  of the form  $u = w_0v_0w_1 \cdots w_tv_tv_{t+1}$  such that

- $w_0, w_{t+1} \in S_2^*$
- for each integer  $i$  such that  $1 \leq i \leq t$ ,  $w_i \in S_2^+$
- for each  $i \in [t+1]$ ,  $v_i \in D_1(S)^+$

Informally, this is a decomposition of  $u$  based only on the  $\mathcal{J}$ -depth of each letter. Thanks to Remark 12, we know that the words  $w_i$  that are not empty in any  $\mathcal{J}$ -maximal decomposition of  $u$  correspond exactly to the words  $w_i$  of this decomposition. Thus, we only need to prove that any word over  $D_1(S)$  can be decomposed into a uniq sequence of maximal subwords which product is in  $D_1(S)$ .

Consider a fixed word  $u \in D_1(S)^+$ . We prove that there exists a decomposition of  $u$  of the form  $u = v_0 \cdots v_s$ , where each word  $v_i$  is maximal in  $u$  such that  $\pi_S(v_i) \in D_1(S)$ .

If  $\pi_S(u) \in D_1(S)$ , then both existence and unicity follow from the definition.

Suppose now that  $\pi_S(u) \in S_2$ . Then we define a decomposition of  $u$  with weaker properties than the  $\mathcal{J}$ -maximal decompositions, and we prove that it is a  $\mathcal{J}$ -maximal decomposition of  $u$  (with empty words as the  $(w_i)$ 's). This decomposition is of the form  $u = v_0v_1 \cdots v_t$  for some integer  $t \in \mathbb{N}$  and is defined from left to right such that, for each integer  $i \in [t+1]$ ,  $\pi_S(v_i) \in D_1(S)$  and, for each integer  $i \in [t]$ ,  $\pi_S(v_ix_{i+1}) \in S_2$ , where  $x_{i+1}$  is the first letter of  $v_{i+1}$ . By construction, this decomposition exists and is unique. We prove that this decomposition satisfies the properties of the  $(v_i)$ 's given in Definition 11, that is, we prove that each word  $v_i$  is maximal such that  $\pi_S(v_i) \in D_1(S)$ . To do that, we only need to prove that, for each integer  $i$  such that  $1 \leq i \leq t$ ,  $\pi_S(y_{i-1}v_i) \in S_2$ , where  $y_{i-1}$  is the last letter of the word  $v_{i-1}$ .

Thus, consider some integer  $i \in [t]$ . We focus on the subword  $v_iv_{i+1}$  in order to prove that  $\pi_S(y_iv_{i+1}) \in S_2$ , where  $y_i$  is the last letter of  $v_i$ . This fact is a consequence of Lemma 23. Since we know that  $\pi_S(v_ix_{i+1}) \in S_2$ , where  $x_{i+1}$  is the first letter of  $v_{i+1}$ , and that  $\pi_S(v_i) \in D_1(S)$ , Lemma 23 implies that  $\pi_S(y_ix_{i+1}) \in S_2$ . Thus,  $\pi_S(y_iv_{i+1}) \in S_2$ . Thus, this decomposition is the unique  $\mathcal{J}$ -maximal decomposition of  $u$ . ◀

► **Lemma 14.** *Let  $S$  be a semigroup of  $\mathcal{J}$ -depth  $d$ . The evaluation program  $\pi_S$  is equal to the composition of  $O(|S|)$  evaluation programs among  $\text{RProd}_S$ ,  $\text{Collapse}_{S_i}$  and  $\text{Falling}_{S_i}(s)$  for all  $1 \leq i \leq d$  and  $s \in D_1(S_i)$ .*

**Proof.** For any integer  $i$  such that  $1 \leq i \leq d$ , we define  $O_i = (s_1, \dots, s_k)$  to be any enumeration of  $D_1(S_i)$ . Given such an enumeration, we define as follows the operation  $\text{Falling}_{S_i}[O_i]$ :

$$\text{Falling}_{S_i}[O_i] = \text{Falling}_{S_i}(s_k) \circ \cdots \circ \text{Falling}_S(s_1)$$

We define an intermediate partial evaluation program  $f_i$  which is a restriction of  $\pi_S$  over the domain  $S_i^+ \cup S$ . In symbols, for any word  $u \in S_i^+ \cup S$ ,  $f_i(u) = \pi_S(u)$ . Remark that  $f_1$  is

equal to  $\pi_S$  on any word of  $S^+$ . We prove by induction over  $j = d - i$  with  $i$  ranging from 0 to  $d - 1$  that

$$f_j = f_{j+1} \circ \text{Falling}_{S_j}[O_j] \circ \text{RProd}_S \circ \text{Collapse}_{S_j}$$

with the convention that  $f_{d+1}$  is the identity. The base case ( $f_{d+1}$ ) being fixed, we only need to prove the result by induction on  $i$  for  $0 \leq i \leq d-1$ . We remark that the image of  $\text{Collapse}_{S_j}$  produces a  $\mathcal{J}$ -maximal falling word, and then the application of  $\text{RProd}_S$  guarantees that the last element is not in  $D_1(S)$ . Finally, the application of  $\text{Falling}_{S_i}(s_i)$  on the resulting elements produces new elements that are only in  $S_{i+1}$  and removes all occurrences of the element  $s_i$ . Hence by applying  $\text{Falling}_{S_i}[O_i]$  we obtain a word in  $S_{i+1}$ , which concludes the proof.  $\blacktriangleleft$

To prove Lemma 15, we introduce an intermediate operation. Let  $i$  be an integer such that  $1 \leq i \leq d$ . We denote by  $P_{i,l}$  the *left sweeping evaluation program* of  $\mathcal{J}$ -depth  $i$ , which computes  $\pi_S$  on the maximal prefix of  $\mathcal{J}$ -depth at most  $i$  (included). Formally, given a word  $u = s_0 \cdots s_k \in S^+$ ,  $P_{i,l}(u)$  is equal to  $u$  if  $s_0$  is of  $\mathcal{J}$ -depth strictly greater than  $i$ . Otherwise, there exist  $p \in S^+$ ,  $q \in S^*$  such that  $u = pq$ , where either  $q$  is empty, or  $x \in S$  is its first letter and then  $\pi_S(p)$  is of  $\mathcal{J}$ -depth at most  $i$  and  $\pi_S(px)$  is of  $\mathcal{J}$ -depth strictly greater than  $i$ . In this case,  $P_{i,l}(u) = \pi_S(p)q$ . We define symmetrically  $P_{i,r}$ , the *right sweeping evaluation program* of  $\mathcal{J}$ -depth  $i$ .

The next lemma allows to conclude the proof of Lemma 15 since  $\pi_S = P_{d,l} = P_{d,r}$ .

► **Lemma 24.** *For any integer  $i$  such that  $1 \leq i \leq d$ , there exist sweeping evaluation programs computing  $P_{i,l}$  and  $P_{i,r}$ .*

**Proof of Lemma 24.** We will prove by induction on the  $\mathcal{J}$ -depth  $i$  that we can implement a left (resp. right) sweeping evaluation program  $P_{i,l}$  (resp.  $P_{i,r}$ ). In this proof, we consider a word  $u = s_0 \cdots s_{k-1}$  over  $S$ . For the base case, we first suppose that  $i = 1$ , i.e. we consider maximal  $\mathcal{J}$ -classes. Thus, if  $s_0$  is of  $\mathcal{J}$ -depth 1, we will compute the product of the unique prefix  $s_0 \cdots s_p$  of  $u$  such that  $\pi_S(s_0 \cdots s_p) \mathcal{J} s_0$  and  $\pi_S(s_0 \cdots s_{p+1}) <_{\mathcal{J}} s_0$ . If  $s_{p+1}$  does not exist, we want to compute  $\pi_S(u)$ . Remark that  $s_0 \cdots s_p$  is  $\mathcal{J}$ -constant, hence we can apply  $\text{JProd}_S$  to it. Thus, we can compute the base case  $P_{1,l}$  using the program  $\text{LSplit}_{S,1}(\text{JProd}_S)$ . Note that this program is well defined since  $\text{JProd}_S$  is in particular defined on all  $\mathcal{J}$ -constant words of depth 1, which are exactly the left  $\mathcal{J}$ -constant words of depth 1. Symmetrically,  $P_{1,r} = \text{RSplit}_{S,1}(\text{JProd}_S)$ . To prove the induction case, we will rely on the following fact:

► **Fact.** *For any left  $\mathcal{J}$ -constant word  $v \in S^+$  of  $\mathcal{J}$ -depth  $i$ , the word  $\text{RProd}_S \circ P_{i-1,r}(v)$  is  $\mathcal{J}$ -constant.*

**Proof.** The result is obtained from the fact that the last element of  $w = \text{RProd}_S \circ P_{i-1,r}(v)$  is necessarily of  $\mathcal{J}$ -depth  $i$ . Indeed, by definition, the word  $x = P_{i-1,r}(v)$  is such that the product of its last two elements (if there are at least two elements) is at least of  $\mathcal{J}$ -depth  $i$ . Since we supposed that  $v$  is left  $\mathcal{J}$ -constant of  $\mathcal{J}$ -depth  $i$ , it is guaranteed that this product is defined and is exactly of  $\mathcal{J}$ -depth  $i$ . Thus, both the first and last elements of  $w$  are of  $\mathcal{J}$ -depth  $i$ , as well as  $\pi_S(w)$ . Thus the product of any prefix or suffix of  $w$  will be of  $\mathcal{J}$ -depth  $i$ , and in the same  $\mathcal{J}$ -class as the first and last elements of  $w$ , which corresponds to the definition of  $\mathcal{J}$ -constant.  $\blacktriangleleft$

For the induction step, we assume to have sweeping evaluation programs  $P_{i,r}$  and  $P_{i,l}$  for any integer  $i < d$ . We prove the result for  $P_{i+1,r}$  and  $P_{i+1,l}$ . These two cases being symmetrical, we only show the result for  $P_{i+1,l}$ . Let  $v = \text{LProd}_S \circ P_{i,l}(u)$ . If  $|P_{i,l}(u)| \neq 1$ ,

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we have necessary that the first letter of  $v$  is of  $\mathcal{J}$ -depth strictly greater than  $i$ . Otherwise  $v = P_{i,l}(u) = \pi_S(u)$ . We are going to split  $v$  with respect to the  $\mathcal{J}$ -depth  $i$  and apply the program  $E = \text{JProd}_S \circ \text{RProd}_S \circ P_{i,r}$  to the prefix. Indeed, after the split, and thanks to the previous Fact, we can apply  $\text{JProd}_S$  over the factor  $\text{RProd}_S \circ P_{i,r}(p)$ , where  $p$  is the prefix obtained after the split. Indeed, this factor is  $\mathcal{J}$ -constant. To conclude,  $P_{i+1,l} = \text{LSplit}_{S,i+1}(E) \circ \text{LProd}_S \circ P_{i,l}$ .

Thus, each  $P_i$  is defined using  $O(2^i)$  operations.  $\blacktriangleleft$

### C Proofs for Section 4 (Proof of the main results)

A *vectorial encoding of a partial evaluation* is a mapping  $\mathbf{c}: S \rightarrow \{0,1\}^n$ , for some integer  $n \geq 1$ , such that we have the following constraint: for any  $s, t \in S$  such that  $s \neq t$ , we have  $\mathbf{c}(s) \wedge \mathbf{c}(t) = \mathbf{0}$ . Note that, by definition, a word is a partial encoding of itself, so  $\text{enc}(u)$  is a vectorial encoding of a partial evaluation, with the additionnal property that  $\bigvee_{a \in S} \mathbf{1}_a(u) = \mathbf{1}$ . Given such a function  $\mathbf{c}$  outputting vectors of dimension  $n$ , we can interpret it as a word of length at most  $n$  by respecting the order of apparition of the bits. Formally, a word  $s_0 \cdots s_k \in S^{\leq n}$  is represented by a vectorial encoding  $\mathbf{c}$  of dimension  $n$  if, for every index  $j \leq k$ , there exists some integer  $i < n$  such that  $\mathbf{c}(s_j)_i = 1$  and  $|\{t \in \mathbb{N} \mid t < i \text{ and } \bigvee_{s \in S} \mathbf{c}(s)_t = 1\}| = j - 1$ .

We introduce two vectors that we will use extensively in our circuits: given  $S$  a semigroup and  $\mathbf{c}$  a vectorial encoding of a partial evaluation, we define the universe vector  $\mathbf{U} = \bigvee_{s \in S} \mathbf{c}(s)$ . We also define the vector marking the end of the vectors:  $\mathbf{End} = \neg \text{MSB}(\mathbf{1})$ .

We use only circuits of the following form: the input is a vectorial encoding of a partial evaluation  $\mathbf{c}: S \rightarrow \{0,1\}^n$  (for some  $n \in \mathbb{N}^+$ ) representing some word  $u \in S^{\leq n}$ , and the output is a vectorial encoding  $\mathbf{out}: S \rightarrow \{0,1\}^n$  representing the partial evaluation of  $u$  obtained by applying the operation corresponding to the circuit. To prove our theorems, we construct vectorial circuits using this encoding that implement the partial evaluation functions we need.

**Proofs for Theorem 6.** In our proofs involving aperiodic semigroups, we will rely on some classical equivalent characterizations of this variety of semigroups.

► **Proposition 25** ([23]). *Let  $S$  be a semigroup. The following conditions are equivalent:*

- $S$  is aperiodic
- there exists an integer  $\omega$  such that for all  $s \in S$ ,  $\pi_S(s^\omega) = \pi_S(s^{\omega+1})$
- All  $\mathcal{H}$ -classes of  $S$  are trivial

Thanks to this property, when considering an aperiodic semigroup, we only need to know the  $\mathcal{H}$ -class of an element to precisely characterize it. This fact will be extensively used in the proofs of our main results.

In the remaining of this section,  $S$  is an aperiodic semigroup of  $\mathcal{J}$ -depth  $d$ . Now, here is a technical property of the aperiodic semigroups that will be useful in the proofs.

► **Lemma 26.** *Let  $S$  be an aperiodic semigroup. Suppose that  $u = s_0 \cdots s_k \in S^+$  is a  $\mathcal{J}$ -constant word. Then,  $\pi_S(u)$  is the unique element of  $\mathcal{R}(s_0) \cap \mathcal{L}(s_k)$ . If  $k > 0$ , this also implies that  $\pi_S(u) = \pi_S(s_0 \cdot s_k)$ .*

**Proof.** Since  $S$  is aperiodic, and thanks to Proposition 25, we know that  $\pi_S(u)$  is the unique element of  $\mathcal{R}(u) \cap \mathcal{L}(u)$ , so we only have to find  $\mathcal{R}(\pi_S(u))$  and  $\mathcal{L}(\pi_S(u))$ . By hypothesis,  $u$  is  $\mathcal{J}$ -constant, so  $\mathcal{J}(\pi_S(u)) = \mathcal{J}(s_0) = \mathcal{J}(s_k)$ . By definition of  $\mathcal{R}$  and  $\mathcal{L}$ , this implies that  $\mathcal{R}(\pi_S(u)) = \mathcal{R}(s_0)$  and  $\mathcal{L}(\pi_S(u)) = \mathcal{L}(s_k)$ . Thus we have that  $\pi_S(u)$  is the unique element

of  $\mathcal{R}(s_0) \cap \mathcal{L}(s_k)$ . If  $k > 1$ , then  $s_0$  and  $s_k$  are separate occurrences of elements of  $S$ , so  $\pi_S(u) = \pi_S(s_0 \cdot s_k)$ , which is the unique element of  $\mathcal{R}(s_0) \cap \mathcal{L}(s_k)$ . ◀

► **Lemma 16.** *For any aperiodic semigroup  $S$ , we can compute  $\text{Collapse}_S$  over any vectorial encoding of a partial evaluation with an ADD-vectorial circuit of size  $O(|S|^3)$ .*

**Proof.** Consider a word  $u \in S^+$  and its  $\mathcal{J}$ -maximal decomposition  $u = w_0 v_1 x_1 \cdots v_t x_t w_t v_{t+1}$ . Let  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of  $u$ . Our goal is to compute a vectorial encoding  $(\mathbf{out}(s))_{s \in S}$  of the partial evaluation  $u' = w_0 \pi_S(v_1 x_1) w_1 \cdots w_{i-1} \pi_S(v_i x_i) w_i \cdots w_t \pi_S(v_{t+1})$  from the encoding  $(\mathbf{c}(s))_{s \in S}$ . To do that, we begin by finding all the letters that belong to the subwords  $v_i$ . By definition, for each index  $i \leq t + 1$ ,  $\pi_S(v_i) \in D_1(S)$  and, if the prefix  $w_0 \cdots w_{i-1}$  is not empty, then  $\pi_S(v_i) \mathcal{J} \pi_S(y_{i-1} v_i)$ , where  $y_{i-1}$  denotes the last letter of the prefix. Similarly, for each index  $i \leq t$ ,  $\pi_S(v_i x_i) <_{\mathcal{J}} \pi_S(v_i)$ . We can deduce that each  $v_i$  is exactly a maximal block which product is  $\mathcal{J}$ -maximal minus its first letter (since  $\pi_S(v_i) \mathcal{J} \pi_S(y_{i-1} v_i)$ ). Thus, we will now mark the elements of those blocks.

For each pair  $(s, t)$  of  $\mathcal{J}$ -maximal elements such that  $\pi_S(st)$  is  $\mathcal{J}$ -maximal, we compute the vector marking each occurrence of  $t$  that is preceded by an  $s$ :

$$\mathbf{Stays}(s, t) = \text{Successor}(\mathbf{c}(t), \mathbf{c}(s), \mathbf{U})$$

We then aggregate those vectors to obtain, for each  $\mathcal{J}$ -maximal element  $t$ , the vector marking the occurrences of  $t$  which product with the previous letter in the word  $u$  is  $\mathcal{J}$ -maximal: we define the set  $E_t = \{s \in D_1(S) \mid \pi_S(st) \in D_1(S)\}$ , with which we can compute the desired vector:

$$\mathbf{Stays}(t) = \bigvee_{s \in E_t} \mathbf{Stays}(s, t)$$

We also compute the union of all those vectors: we define the set  $R = \{t \in S \mid \exists s \in D_1(S), \pi_S(st) \in D_1(S)\}$ , with which we can compute the desired vector:

$$\mathbf{Stays} = \bigvee_{s \in R} \mathbf{Stays}(s)$$

With this, we can find the limits of the maximal subwords which product is  $\mathcal{J}$ -maximal, without counting the first element, which does not belong to the same  $v_i$  by definition of the  $\mathcal{J}$ -maximal decomposition. This is due to the fact that this first letter could be used to make the product of the previous subword fall in  $S_2$ . Thus, the vector **Stays** marks exactly the elements that belong to any of the subwords  $v_i$ . Then, we mark the first letter of each  $v_i$ . These letters are exactly the elements of  $S$  that are marked by **Stays** and such that the previous element marked by **U** is not marked by **Stays**. Thus, for each element  $s \in D_1(S)$ , we compute the vector which labels the first letter of each  $v_i$  with its value if it is an  $s$ :

$$\mathbf{SecondEl}(s) = \text{Successor}(\mathbf{Stays}(s), \mathbf{U} \wedge \neg \mathbf{Stays}, \mathbf{U})$$

We aggregate those vectors:

$$\mathbf{SecondEl} = \bigvee_{s \in D_1(S)} \mathbf{SecondEl}(s)$$

Then we mark the letters  $x_i$ , which are the elements just after the end of a maximal subword that is  $\mathcal{J}$ -maximal, and we label them by the last element of the corresponding  $v_i$ . For each



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element  $s \in D_1(S)$ , we compute the vector marking each  $x_i$  if and only if the last letter of  $v_i$  is an  $s$ :

$$\mathbf{BlockFall}(s) = \text{Successor}(\mathbf{U} \wedge \neg \mathbf{Stays}, \mathbf{Stays}(s), \mathbf{U})$$

We aggregate those results:

$$\mathbf{BlockFall} = \bigvee_{s \in D_1(S)} \mathbf{BlockFall}(s)$$

With with, we can compute the vector which marks the limits of the subwords  $v_i x_i$ :

$$\mathbf{Limits} = \mathbf{SecondEl} \vee \mathbf{BlockFall}$$

Note that this vector also marks the first letter of  $v_{t+1}$  (if it exists), but not its last letter.

With these vectors, we can compute all the products  $\pi_S(v_i x_i)$ . To this end, for each element  $p \in S_2$ , we want to define the set of triplets  $(s, t, x) \in (D_1(S))^2 \times S$  such that, for any word  $v \in S^+$  such that  $v$  begins by  $s$ , ends by  $t$  and is  $\mathcal{J}$ -constant, the product of the word  $vx$  is  $p$ . Thus, for any element  $p \in S_2$ , we define the set

$$F_p = \{(s, t, x) \in (D_1(S))^2 \times S \mid \pi_S(\alpha \cdot x) = p, \text{ where } \alpha = \mathcal{R}(s) \cap \mathcal{L}(t)\}$$

This set is exactly what we need since, thanks to Lemma 23, we know that the product of the subword  $v$  is only determined by  $s$  and  $t$ . Moreover, that product is not always equal to  $\pi_S(st)$ . Indeed, the subword  $v$  can be of size 1, in which case  $s = t$  and there is only one occurrence to consider. In the case where  $\pi_S(ss) \neq s$ , the product of  $v$  is equal to  $s$  and not to  $\pi_S(ss)$ . However, since  $S$  is aperiodic, we know thanks to Proposition 25 that the element  $\pi_S(st)$  is uniquely characterized by its  $\mathcal{H}$ -class, which is equal to  $\mathcal{R}(s) \cap \mathcal{L}(t)$ . Thus, the set  $F_p$  defines exactly the triplets that we need. Note that each triplet  $(s, t, x) \in (D_1(S))^2 \times S$  can only appear in at most one set  $F_p$ , so the size of the union of all these sets is only  $O(|D_1(S)|^2 * |S|) \leq O(|S|^3)$ .

Then we use this to compute the products of the subwords. For each element  $p \in S_2$ , we label each  $x_i$  by  $p$  if and only if there exists a triplet  $(s, t, x_i) \in F_p$  such that the first letter of  $v_i$  is an  $s$  (which can be determined using  $\mathbf{Stays}(s)$ ) and the last letter of  $v_i$  is a  $t$  (and thus  $x_i$  is marked by the vector  $\mathbf{BlockFall}(t)$ ):

$$\mathbf{Prod}(p) = \bigvee_{(s,t,x) \in F_p} \text{Successor}(\mathbf{BlockFall}(t), \mathbf{SecondEl}(s), \mathbf{Limits}) \wedge \mathbf{c}(x)$$

Note that the union of all the vectors  $\mathbf{Prod}(p)$  contains exactly all the elements obtained with the products  $\pi_S(v_i x_i)$ . With these vectors, we will be able to compute a vectorial encoding of the word  $w_0 \pi_S(v_1 x_1) \cdots w_i \pi_S(v_i x_i) \cdots w_t v_{t+1}$ . Before computing the output vectors, we consider the potential suffix  $v_{t+1}$  which product is  $\mathcal{J}$ -maximal. This case has not been treated yet since there is no letter after  $v_{t+1}$  which makes the product fall in  $S_2$ , however the first letter of  $v_{t+1}$  has been marked in  $\mathbf{SecondEl}$  (if  $v_{t+1}$  is not empty). Thus, we only need to mark the last letter of the word  $v_{t+1}$ . For each element  $s \in D_1(S)$ , we label by  $s$  the last element of  $u$  if and only if  $v_{t+1}$  is not empty and begins with an  $s$ :

$$\mathbf{FirstLeftover}(s) = \text{Successor}(\mathbf{End}, \mathbf{SecondEl}(s), \mathbf{Limits} \vee \mathbf{End})$$

Then we compute the product of that suffix ( $\pi_S(v_{t+1})$ ). For each element  $p \in D_1(S)$ , we define the set  $G_p = \{(s, t) \in (D_1(S))^2 \mid \pi_S(\alpha) = p, \text{ where } \alpha = \mathcal{R}(s) \cap \mathcal{L}(t)\}$  and we compute

the vector  $\mathbf{Prod}(p)$  which is equal to  $\mathbf{End}$  if and only if  $v_{t+1}$  is not empty, starts with some  $s$  and ends with some  $t$  such that  $(s, t)$  is in  $G_p$ :

$$\mathbf{Prod}(p) = \bigvee_{(s,t) \in G_p} \mathbf{FirstLeftover}(s) \wedge \mathbf{c}(t)$$

Now we have everything we need to compute the output vectors. To do this, we remove every element marked by  $\mathbf{Stays}$  (the letters of every  $v_i$ ) and  $\mathbf{BlockFall}$  (every  $x_i$ ), and we add the elements computed in the vectors  $\mathbf{Prod}(s)$  (the elements  $\pi_S(v_i x_i)$  and  $\pi_S(v_{t+1})$ ). For each element  $s \in S$ :

$$\mathbf{out}(s) = \mathbf{Prod}(s) \vee (\mathbf{c}(s) \wedge \neg(\mathbf{Stays} \vee \mathbf{BlockFall}))$$

◀

► **Lemma 17.** *For any aperiodic semigroup  $S$ , we can compute  $\mathbf{RProd}_S$  on any vectorial encoding of a partial evaluation of a word over  $S$  with an ADD-vectorial circuit of size  $O(|S|^2)$ .*

**Proof.** Let  $u \in S^+$  be a partial evaluation such that  $u$  is a  $\mathcal{J}$ -maximal falling word. Let  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of  $u$ . We want to multiply the last two elements of  $u$  (if  $|u| = 1$ , we return  $u$ ) and to obtain a vectorial encoding  $(\mathbf{out}(s))_{s \in S}$  of the new word. We begin by computing a mask which unmarks the last two elements of the word:

$$\mathbf{RemoveLastTwo} = \mathbf{MSB}(\mathbf{MSB}(\mathbf{U}))$$

This will allow us to remove these elements after the multiplication. However, we need to verify that there are at least two elements to multiply. For this, we will use the vector  $\mathbf{Thr2}(\mathbf{U})$  defined in Section 2.2, which is equal to  $\mathbf{1}$  if and only if  $u \in S^{\geq 2}$  (else it is equal to  $\mathbf{0}$ ). Now we perform the multiplication itself: for every  $t \in S$ , we label the last element by  $t$  if and only if the last letter of  $u$  is a  $t$ :

$$\mathbf{LastEl}(t) = \mathbf{c}(t) \wedge \mathbf{End}$$

Now we can update the vectors accordingly. We begin by removing the last two elements: for each element  $s \in S$ , we set

$$\mathbf{Others}(s) = \mathbf{IfThenElse}(\mathbf{Thr2}(\mathbf{U}), \mathbf{c}(s) \wedge \mathbf{RemoveLastTwo}, \mathbf{c}(s))$$

Note that, if  $|u| = 1$ , we have  $\mathbf{Others}(s) = \mathbf{c}(s)$  since we do not want to change  $u$  in this case. Now we can add the product of the last two elements. For every  $p \in S_2$ , we define the set  $M_p = \{(s, t) \in S^2 \mid \pi_S(st) = p\}$ , and with we define the following vector:

$$\mathbf{out}(p) = \mathbf{Others}(p) \vee \bigvee_{(s,t) \in M_p} \mathbf{Successor}(\mathbf{LastEl}(t), \mathbf{c}(s), \mathbf{U})$$

If  $|u| = 1$ , all the applications of  $\mathbf{Successor}$  will yield the vector  $\mathbf{0}$ , so in this case we will have  $\mathbf{out}(p) = \mathbf{c}(p)$ . Note that, during the computation of the vectors  $\mathbf{out}(p)$ , each pair  $(s, t) \in S^2$  can only be used once, so the complexity of the whole operation is only  $O(|S|^2)$ .

Finally, we consider all the  $\mathcal{J}$ -maximal elements and we keep every letter marked by some vector  $\mathbf{Others}$ : for each element  $p \in D_1(S)$ , we set

$$\mathbf{out}(p) = \mathbf{c}(p) \wedge \bigvee_{s \in S} \mathbf{Others}(s)$$

And thus we computed a vectorial encoding of the word obtained by multiplying the last two elements. ◀

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Before proving Lemma 18, we prove the following technical lemma.

► **Lemma 27.** *Let  $S$  be an aperiodic semigroup. Let  $u \in S^+$  be a word over  $S$ . Suppose that the last letter of  $u$  is not an element of  $D_1(S)$  and that  $u$  is a  $\mathcal{J}$ -maximal falling word. Consider a fixed element  $s \in D_1(S)$  and write  $u$  as its  $s$ -decomposition  $w_0 s^{k_1} x_1 w_1 \cdots w_{k-1} s^{i_k} x_k w_k$ . Then, we can use an ADD-vectorial circuit of size  $O(|S|)$  which take the vectorial encoding as input and produces a vectorial encoding of the word  $w_0 s^{k_1-1} \pi_S(sx_1) \cdots s^{k_t-1} \pi_S(sx_t) w_t$ .*

**Proof.** Let  $(\mathbf{c}(t))_{t \in S}$  be a vectorial encoding of  $u$ . We begin by marking the elements that are preceded by an occurrence of  $s$  and are not occurrences of  $s$ .

$$\mathbf{Last} = \text{Successor}(\mathbf{U} \wedge \neg \mathbf{c}(s), \mathbf{c}(s), \mathbf{U})$$

We separate these elements depending on their value: for each element  $t \in S \setminus \{s\}$ , we set

$$\mathbf{Last}(t) = \mathbf{Last} \wedge \mathbf{c}(t)$$

In order to remove the occurrences of  $s$  that will be used, we also mark the occurrences of  $s$  that are not preceded by another occurrence of  $s$ .

$$\mathbf{First} = \text{Successor}(\mathbf{c}(s), \mathbf{U} \wedge \neg \mathbf{c}(s), \mathbf{U})$$

These are not exactly the occurrences that should be removed, but each one belongs to the same block of occurrences of  $s$  as an occurrence that is used in one of the products we will perform, so removing them is sufficient to obtain the result we want. Moreover, by definition, we will still obtain a valid vectorial encoding of the partial evaluation of  $u$  that we want to obtain.

Now we can remove the occurrences of  $s$  that have been used:

$$\mathbf{rm}(s) = \mathbf{c}(s) \wedge \neg \mathbf{First}$$

Then we can remove the elements that are preceded by an occurrence of  $s$ : for each element  $t \in S \setminus \{s\}$ , we set

$$\mathbf{rm}(t) = \mathbf{c}(t) \wedge \neg \mathbf{Last}(t)$$

Finally, we perform the products: for each element  $p \in S_2$ , we define the set  $P_p = \{t \in S \setminus \{s\} \mid \pi_S(st) = p\}$  of elements  $t$  such that  $p$  is equal to  $\pi_S(st)$ . With this, we can produce the vectors  $\mathbf{out}(p)$ :

$$\mathbf{out}(p) = \mathbf{rm}(p) \vee \bigvee_{t \in P_p} \mathbf{Last}(t)$$

Note that each element  $t \in S \setminus \{s\}$  appears in at most one set  $P_p$ , and that consequently the computation of all the vectors  $\mathbf{out}(p)$  is only of complexity  $O(|S|)$ .

This gives us a vectorial encoding of the partial evaluation we wanted. ◀

► **Lemma 18.** *Let  $S$  be an aperiodic semigroup of  $\mathcal{J}$ -depth  $d$ . For any element  $s \in D_1(S)$ , we can compute  $\text{Falling}_S(s)$  over any vectorial encoding of a partial evaluation in its domain with an ADD-vectorial circuit of size  $O(d|S|)$ .*

**Proof.** Let  $u$  be a word of  $S^+$  and the set  $(\mathbf{c}(t))_{t \in S}$  be a vectorial encoding of  $u$ . Using Lemma 27, we can prove the result quite easily by repeating the Lemma a fixed number of times.

Since we are sure that the last element of  $u$  is not in  $D_1(S)$ , we know that each block of occurrences of  $s$  is followed by at least one element. By applying Lemma 27, we can reduce by 1 the size of each of these blocks. Moreover, the semigroup  $S$  is aperiodic, so by Proposition 25 there necessarily exists an integer  $\omega_s$  such that  $\pi_S(s^{\omega_s+1}) = \pi_S(s^{\omega_s})$  ( $s^{\omega_s}$  is the idempotent power of  $s$ ). Thus, if we apply Lemma 27  $\omega_s$  times, the only occurrences of  $s$  that will be left will be any letter  $s$  that was originally followed by at least  $\omega_s$  other occurrences of  $s$ . Since  $\pi_S(s^{\omega_s+1}) = \pi_S(s^{\omega_s})$ , we can just forget those occurrences without changing anything else. We denote by  $(\mathbf{omega}(t))_{t \in S}$  the vectorial encoding of the decomposition of  $u$  obtained after applying Lemma 27 sequentially  $\omega_s$  times. Since the products that we could compute from now on will only remove occurrences of  $s$  without changing the other elements, we can now remove all the occurrences of  $s$  that are left in the new partial evaluation of  $u$ . Thus, we can set

$$\mathbf{out}(s) = \mathbf{0}$$

and, for each element  $t \neq s$ ,

$$\mathbf{out}(t) = \mathbf{omega}(t)$$

then we obtain a vectorial encoding of the partial evaluation that we wanted.

Note that, for any  $t \in S$ , we have  $\omega_t \leq d$ , where  $d$  is the  $\mathcal{J}$ -depth of  $S$ , so we apply Lemma 27 at most  $d$  times.  $\blacktriangleleft$

**Proofs for Theorem 5.** In our proofs involving semigroups in  $\mathbf{DA}$ , we will rely on some classical equivalent characterizations of the variety  $\mathbf{DA}$ .

► **Proposition 28.** [26, Theorem 2.] *Let  $M$  be a monoid. The following conditions are equivalent:*

- $M$  is in the variety  $\mathbf{DA}$
- if  $J$  is a regular  $\mathcal{J}$ -class of  $M$ , then  $J$  is an aperiodic semigroup
- $\forall x, y, z \in M, (xyz)^\omega y (xyz)^\omega = (xyz)^\omega$  (we will refer to this as the algebraic characterization of  $\mathbf{DA}$ )

By definition, any semigroup in  $\mathbf{DA}$  is aperiodic, so we will also be able to use the characterizations presented in Proposition 25. Moreover, the class  $\mathbf{DA}$  admits the following nice property that we will rely on in the proofs.

► **Lemma 29.** *Let  $S$  be a semigroup in  $\mathbf{DA}$  and  $R$  an  $\mathcal{R}$ -class of  $S$ . Then there exist two sets  $T, K \subseteq S$  such that  $S = T \uplus K$  and, for all  $x \in R$ , we have*

- $\forall s \in T, xs\mathcal{R}x$
- $\forall s \in K, xs <_{\mathcal{J}} x$

*Moreover, both  $T$  and  $K$  are sub-semigroups of  $S$  such that, if we denote by  $J$  is the  $\mathcal{J}$ -class containing  $R$ ,  $J \subseteq T$  if  $R$  is regular and  $J \subseteq K$  otherwise. It follows that if  $S$  is a monoid, then  $T$  is also a monoid.*

**Proof.** First, we prove that the sets  $T$  and  $K$  exist and are well defined. Consider  $s \in S$  and  $x, y \in R$ . We need to prove that  $xs\mathcal{R}x \Leftrightarrow ys\mathcal{R}y$ . Note that, since  $x\mathcal{R}y$ , there exist  $p, q \in S$  such that  $xp = y$  and  $yq = x$ . Suppose that  $xs\mathcal{R}x$ . Then, we want to prove that  $ys\mathcal{R}y$ . By

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definition,  $ys \leq_{\mathcal{R}} y$ , so we only need to find some element  $z \in S$  such that  $ysz = y$ . Since  $xs\mathcal{R}x$ ,  $\exists r \in S, xsr = x$ . Thus,  $yqsrp = y$  and  $y(qsrp)^{\omega} = y$ . Since  $S$  is in **DA**, we then have  $y(qsrp)^{\omega}s(qsrp)^{\omega} = y$ , so  $ys(qsrp)^{\omega} = y$ , which proves that  $ys\mathcal{R}y$ . Since  $x$  and  $y$  have symmetric roles, the converse follows directly. Thus, the sets  $T$  and  $K$  exist and are well defined.

Now, we prove that  $T$  is a sub-semigroup of  $S$ : let  $a$  and  $b$  be two elements of  $T$ . By definition of  $T$ , for any  $x \in R$ ,  $xa\mathcal{R}x$  and  $xb\mathcal{R}x$ ; thus there exist  $p, q \in S$  such that  $xap = x$  and  $xbq = x$ . So,  $xapbq = x$  and  $x(apbq)^{\omega} = x$ . Since  $S$  is in **DA**, it is aperiodic, so  $x(apbq)^{\omega} = x(apbq)^{\omega+1}$ . Then, we can get one letter  $a$  out of the parentheses as follows:  $x(apbq)^{\omega+1} = xa(pbqa)^{\omega}pbq$ . Now, we can isolate a letter  $b$ : since  $S$  is in **DA**, we have  $xa(pbqa)^{\omega}pbq = xa(pbqa)^{\omega}b(pbqa)^{\omega}pbq$ . To summarize our results so far, we have  $x = xa(pbqa)^{\omega}b(pbqa)^{\omega}pbq$ . Now, remember that  $xapbq = x$ . Thus,  $xapbqa = xa$ , and  $xa(pbqa)^{\omega} = xa$ . This way, we can get rid of the leftmost factor  $(pbqa)^{\omega}$  in our result and obtain the following equality:  $x = xab(pbqa)^{\omega}pbq$ . This proves that  $xab\mathcal{R}x$ . Consequently,  $ab$  is also in  $T$  and  $T$  is a semigroup. Moreover, the set  $K$  is an ideal of  $S$  since, for any elements  $x \in R$ ,  $s \in K$  and  $t \in S$ , we have  $\pi_S(xst) \leq_{\mathcal{J}} \pi_S(xs) <_{\mathcal{J}} x$ . Thus,  $K$  is a sub-semigroup of  $S$ .

Consequently, if we consider any element  $u \in S$  such that, for some element  $r$  of  $R$ ,  $\pi_S(ru) <_{\mathcal{J}} r$ , this property applies to all elements of  $R$ , which means that for any element  $r \in R$ ,  $\pi_S(ru) <_{\mathcal{J}} r$ . Finally, thanks to Proposition 28, we know that if  $R$  is regular, then all the elements of its  $\mathcal{J}$ -class are idempotent, so  $J \subseteq T$ . On the contrary, if  $R$  is not regular, then no element of  $J$  can be idempotent, so  $J \subseteq K$ . ◀

Before proving the Lemmas necessary for Theorem 5, we define some intermediary operations that will allow us to decompose the functions that we want to compute. These intermediary operations will all be of the same form: for any integer  $i \in \mathbb{N}$ , we define the operation  $\text{Value}_{i,S}$  that identifies the  $i^{\text{th}}$  semigroup element that occurs in the word represented by the input vectors. Formally, we define this operation as follows:

► **Definition 30.** Let  $S$  be a semigroup in **DA** and let  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of some word  $u \in S^+$ . For each element  $s \in S$  and integer  $i \in \mathbb{N}$ , we define the vecteur  $\text{Value}_{i,S}(s)$  as follows:

- $\text{Value}_{i,S}(s) = \mathbf{1}$  if and only if there exists an integer  $j$  such that the  $j^{\text{th}}$  element of  $\mathbf{c}(s)$  is a 1 and the position  $j$  is the  $i^{\text{th}}$  position of the vector  $\mathbf{U}$  to hold a 1.
- Otherwise,  $\text{Value}_{i,S}(s) = \mathbf{0}$ .

► **Lemma 31.** For any integer  $i \geq 1$ , we can compute the function  $\text{Value}_{i,S}$  over any vectorial encoding of a partial evaluation with a Sweeping-vectorial circuit of size  $O(i + |S|)$ .

**Proof.** Given a set of input vectors  $I = (\mathbf{c}(s))_{s \in S}$ ,  $\text{Value}_{i,S}(I)$  is a set of vectors  $(\mathbf{out}(s))_{s \in S}$  such that, for each element  $s \in S$ ,  $\mathbf{out}(s)$  is computed as follows. We begin by removing the first  $i - 1$  bits set to 1 in the union of the inputs by defining the following vectors:

$$\mathbf{U}_0 = \mathbf{U}$$

$$\forall j < i - 1, \mathbf{U}_{j+1} = \text{LSB}(\mathbf{U}_j)$$

Then, for each  $x \in S$ , we set to 0, in  $\mathbf{c}(x)$ , the  $i - 1$  first bits set to 1 in the union of the inputs:

$$\mathbf{rm}(x) = \mathbf{c}(x) \wedge \mathbf{U}_{i-1}$$

Now, to detect the element associated to the  $i^{\text{th}}$  bit set to 1 in  $\mathbf{U}$ , we only need to detect the value associated to the first bit set to 1 in  $\bigvee_{x \in S} \mathbf{rm}_x$ , which is done as follows: for any  $s \in S$ , we compute the vector  $\mathbf{out}(s)$  that is full of ones if and only if the position of the first bit set to 1 in  $\mathbf{U}_{i-1}$  (that is the union of the vectors  $\mathbf{rm}(x)$ ) is set to 1 in the vector  $\mathbf{rm}(s)$ .

$$\mathbf{out}(s) = \text{Eq}(\text{pref-}\bigvee(\mathbf{rm}(s)), \text{pref-}\bigvee(\mathbf{U}_{i-1}))$$

◀

► **Lemma 19.** *For any semigroup  $S \in \mathbf{DA}$ , we can compute  $\text{JProd}_S$  over any vectorial encoding of a partial evaluation in its domain with a Sweeping-vectorial circuit of size  $O(|S|^2)$ .*

**Proof.** Let  $u = u_0 \cdots u_k$  be a word of  $S^+$  and the set  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of  $u$ . To compute  $\text{JProd}_S(u)$ , we want to detect the first and last bits set to 1 in  $\mathbf{U}$  in order to compute the vectors corresponding to the word composed only of the element  $u_0 \cdot_S u_k$ . The first element is directly indicated by the vectors  $\text{Value}_{1,S}(s)$  for each element  $s \in S$ . Now we detect the last element by computing the similar vectors  $\mathbf{Last}(s)$  for each  $s \in S$ :

$$\mathbf{Last}(s) = \text{Eq}(\text{suf-}\bigvee(\mathbf{c}(s)), \text{suf-}\bigvee(\mathbf{U}))$$

To complete the product we need to set to 1 the bit at the end of the word in the right vector. To do this, we detect the position of the last bit set to 1 in the union of the inputs and store it in the vector  $\mathbf{PosLast}$ , which then has a unique bit set to 1.

$$\mathbf{PosLast} = \mathbf{U} \wedge \neg \text{MSB}(\mathbf{U})$$

Now we can perform the multiplication. To do this, we define, for each element  $s \in S$ , the set  $M_s = \{(t, p) \in S^2 \mid \mathcal{R}(t) \cap \mathcal{L}(p) = \{s\}\}$ . Now, we compute the vectors  $\mathbf{out}(s)$  for each element  $s \in S$ . These vectors contain only the product of the first and last elements, or the first element if  $u$  contains only one element. With our definition of  $M_s$ , for the same reasons as in the proof of Lemma 16, we do not need to verify that there were two elements to multiply, since if there is only one element  $s$ , it is equal to the unique element of  $\mathcal{R}(s) \cap \mathcal{L}(s)$  and thus will stay unchanged.

$$\mathbf{out}(s) = \mathbf{PosLast} \wedge \bigvee_{(t,p) \in M_s} \text{Value}_{1,S}(t) \wedge \mathbf{Last}(p)$$

Note that, since a given pair of elements  $(t, p)$  can only belong to one set  $M_s$ , this computation has a complexity of  $O(|S|^2)$ . ◀

► **Lemma 20.** *For any semigroup  $S \in \mathbf{DA}$ , we can compute  $\text{LProd}_S$  and  $\text{RProd}_S$  over any vectorial encoding of a partial evaluation in their domains with Sweeping-vectorial circuits of size  $O(|S|^2)$ .*

**Proof.** The two operations are symmetrical, so we will just present the circuit for  $\text{LProd}_S$ . Let  $u = u_0 \cdots u_k$  be a word of  $S^+$  and  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of  $u$ . We want to detect the first and last bits set to 1 in  $\mathbf{U}$  in order to compute the vectors corresponding to the word composed only of the element  $u_0 \cdot_S u_1$ . We can use  $\text{Value}_{1,S}$  and  $\text{Value}_{2,S}$  to compute vectors that give the values of the first and second elements.

If the word  $u$  is composed of only one element, we cannot detect a second element, so the vector  $\text{Thr2}((\mathbf{c}(s))_{s \in S})$  can be used as a way to know if the vectors obtained after using  $\text{Value}_{1,S}$  and  $\text{Value}_{2,S}$  have a meaning. If that vector is not null, we want to compute the

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product of the elements  $u_0$  and  $u_1$ . To complete the product, we need to set to 1 the bit at the position of the second element in the vector corresponding to the value of the product. To do this, we detect the position of the second element and store it in the vector **PosSec**, which has a unique 1 in the position of the second element.

$$\mathbf{AfterSec} = \text{pref-}\vee(\text{LSB}(\mathbf{U}))$$

$$\mathbf{PosSec} = \mathbf{AfterSec} \wedge \neg \text{LSB}(\mathbf{AfterSec})$$

Now we start to prepare the multiplication. We compute the vectors  $\mathbf{rm}(x)$  (for each element  $x \in S$ ), in which we set to 0 the bits of the first two elements of the word.

$$\mathbf{Other} = \text{LSB}(\text{LSB}(\mathbf{U}))$$

$$\mathbf{rm}(x) = \mathbf{c}(x) \wedge \mathbf{Other}$$

Now we only have to add the product of the two first elements in the right vector. So, for each element  $s \in S$ , we compute the vector  $\mathbf{Prod}(s)$ , in which we add the position of the second element if that second element is an  $s$ . To do that, for each element  $s \in S$ , we define  $M_s$  to be the set of pairs  $(t, p)$  such that  $s = \pi_S(tp)$ . In symbols,  $M_s = \{(t, p) \in S^2 \mid s = \pi_S(tp)\}$ .

$$\mathbf{Prod}(s) = \mathbf{rm}(s) \vee \bigvee_{(t,p) \in M_s} \text{Value}_{1,S}(t) \wedge \text{Value}_{2,S}(p) \wedge \mathbf{PosSec}$$

Note that, since a given pair of elements  $(t, p)$  can only belong to one set  $M_s$ , this computation has a complexity of  $O(|S|^2)$ . Finally, the vectors  $\mathbf{Prod}(s)$  only have meaning if there was two elements to multiply in the first place, so the vectors  $\mathbf{out}(s)$  are computed as follows:

$$\mathbf{out}(s) = \text{IfThenElse}(\text{Thr2}((\mathbf{c}(s))_{s \in S}), \mathbf{Prod}(s), \mathbf{c}(s))$$

◀

► **Lemma 21.** *Let  $S$  be a semigroup in **DA** of  $\mathcal{J}$ -depth  $d$ ,  $i$  be an integer such that  $1 \leq i \leq d$ , let  $P$  be a sweeping evaluation program defined at least on all left  $\mathcal{J}$ -constant words of depth  $i$ , and suppose that we have a Sweeping-vectorial circuit of size  $s_P$  that computes  $P$  over any vectorial encoding of a partial evaluation. Then we can compute  $\text{LSplit}_{S,i}(P)$  and  $\text{RSplit}_{S,i}(P)$  over any vectorial encoding of a partial evaluation in their respective domains with Sweeping-vectorial circuits of size  $O(|S|^3 + s_P)$ .*

**Proof.** The two operations are symmetrical, so we will only present the circuit for  $\text{LSplit}_{S,i}(P)$ . Let  $u = u_0 \cdots u_k$  be a word of  $S^+$  and the set  $(\mathbf{c}(s))_{s \in S}$  be a vectorial encoding of  $u$ . If  $u_0$  is of  $\mathcal{J}$ -depth  $i$ , we want to detect the first element  $u_i$  such that  $\pi_S(u_0 \cdots u_i)$  is of  $\mathcal{J}$ -depth at least  $i + 1$  in order to replace the prefix of  $u_0 \cdots u_{i-1}$  by its image through  $P$ . To do that, we begin by checking if the first element of the word is of  $\mathcal{J}$ -depth  $i$ . We want to have an output different from the input if and only if  $u_0$  is of  $\mathcal{J}$ -depth  $i$ , i.e. if  $u_0 \in D_i(S)$ :

$$\mathbf{DoSmth} = \bigvee_{s \in D_i(S)} \text{Value}_{1,S}(s)$$

For now, suppose that  $\mathbf{DoSmth} \neq \mathbf{0}$ . We can now determine the  $\mathcal{R}$ -class of the first element. For each  $\mathcal{R}$ -class  $R$  of  $\mathcal{J}$ -depth  $i$ , we compute the vector  $\mathbf{RClass}(R)$  that is full of ones if and only if the first element is in  $R$ :

$$\mathbf{RClass}(R) = \bigvee_{s \in R} \text{Value}_{1,S}(s)$$



Now, we want to find the first position such that the product of the prefix is of  $\mathcal{J}$ -depth at least  $i + 1$ . To find that position, we use Lemma 29, which tells us that the set of elements that make that product fall in a  $\mathcal{J}$ -class of greater  $\mathcal{J}$ -depth depends only on the  $\mathcal{R}$ -class of the prefix, which is uniquely determined by the first element, since that element is necessarily of  $\mathcal{J}$ -depth  $i$ . We already have the vectors indicating what  $\mathcal{R}$ -class should be considered. Now, for each  $\mathcal{R}$ -class, let  $F_R$  be the set of elements of  $S$  such that for any pair  $(r, s) \in R \times F_R$ ,  $\pi_S(rs) \notin R$  (the set  $K$  defined in Lemma 29). In symbols,  $F_R = \{s \in S \mid \forall r \in R, \pi_S(rs) \notin R\}$ . We want to consider only the set  $F_R$  corresponding to the  $\mathcal{R}$ -class of the first element and to search for the first element of that set that is not the first element of the factorization. To do this, we begin by computing, for any element  $a \in S$ , the vector  $\mathbf{Next}(a)$  which allows us to know where the first element  $a$  after the first element is:

$$\mathbf{Next}(a) = \text{pref-}\vee(\mathbf{c}(a) \wedge \text{LSB}(\mathbf{U}))$$

Then we assemble those results depending on the set  $F_R$  each element  $a$  belongs to. For each  $\mathcal{R}$ -class  $R$  of  $\mathcal{J}$ -depth  $i$ , we compute the following vector:

$$\mathbf{BeforeFall}(R) = \neg \bigvee_{a \in F_R} \mathbf{Next}(a)$$

Each vector  $\mathbf{BeforeFall}(R)$  indicates all the positions strictly before the first element  $s \in F_R$  that is at least the second element of the word. With this, we know that the prefix indicated by  $\mathbf{BeforeFall}(R)$ , where  $R$  is the  $\mathcal{R}$ -class of the first element, is the unique maximal prefix which product belongs to  $R$ . Thus, we want to apply the circuit corresponding to the program  $P$  to the prefix that is indicated by  $\mathbf{BeforeFall}(R)$ , where  $R$  is the  $\mathcal{R}$ -class of the first element. Now, we need to introduce a mask which will indicate the prefix considered in the rest of the split operation. This mask is constructed from the vectors  $\mathbf{BeforeFall}(R)$  for each  $\mathcal{R}$ -class  $R$  that belongs to the set  $\mathcal{R}_i$  of  $\mathcal{R}$ -classes of  $\mathcal{J}$ -depth  $i$ , and it is equal to the vector  $\mathbf{BeforeFall}(R)$  corresponding to the unique vector  $\mathbf{RClass}(R)$  that is not null:

$$\mathbf{mask} = \bigvee_{R \in \mathcal{R}_i} \mathbf{RClass}(R) \wedge \mathbf{BeforeFall}(R)$$

Now we can use the circuit  $C_P$  on the set of vectors  $(\mathbf{c}(s) \wedge \neg \mathbf{mask})_{s \in S}$ . That call returns a set of vectors  $(\mathbf{r}(s))_{s \in S}$  corresponding to the application of  $P$  to the prefix. Now we only have to add back the rest of the word. For each element  $s \in S$ , the vector  $\mathbf{out}(s)$  is computed as follows:

$$\mathbf{out}(s) = \text{IfThenElse}(\mathbf{DoSmth}, \mathbf{r}(s) \vee (\mathbf{c}(s) \wedge \neg \mathbf{mask}), \mathbf{c}(s))$$

