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3 **GLOBAL WELL-POSEDNESS OF THE KDV EQUATION ON A**
4 **STAR-SHAPED NETWORK AND STABILIZATION BY SATURATED**
5 **CONTROLLERS***

6 HUGO PARADA[†], EMMANUELLE CRÉPEAU[†], AND CHRISTOPHE PRIEUR[‡]

7 **Abstract.** In this work, we deal with the global well-posedness and stability of the linear and
8 nonlinear Korteweg-de Vries equations on a finite star-shaped network by acting with saturated
9 controls. We obtain the global well-posedness by using the Kato smoothing property for the linear
10 case and then using some estimates and a fixed point argument we deal with the nonlinear system.
11 Finally, we obtain the exponential stability using two different kinds of saturation by proving an
12 observability inequality via a contradiction argument.

13 **Key words.** Korteweg-de Vries equation, star-shaped network, stabilization, saturating control

14 **MSC codes.** 93C20, 93D15, 35R02, 35A01, 35Q53

15 **DOI.** 10.1137/21M1434581

16 **1. Introduction and presentation of our results.** The Korteweg-de Vries
17 (KdV) equation $u_t + u_x + u_{xxx} + uu_x = 0$ was introduced in [13] to model the prop-
18 agation of long water waves in a channel. The KdV equation has been very well
19 studied in recent years, in particular, the controllability and stabilization properties;
20 see [9, 22] for a complete introduction to these problems. With respect to the KdV
21 equation on networks, we can mention the work [8] where well-posedness of the KdV
22 equation on a star metric graph was studied. In the works [1, 10], stabilization and
23 controllability problems were studied, for the KdV equation on a star-shaped network,
24 and recently the problem of stabilization using internal delay was addressed in [16].
25 In this work, we are interested in the global well-posedness and stability properties
26 of a KdV equation posed on a star-shaped network using internal saturated feedback
27 terms. Let $K = \{k_n : 1 \leq n \leq N\}$ be the set of the N edges of a network \mathcal{T} described
28 as the intervals $[0, \ell_n]$ with $\ell_n > 0$ for $n = 1, \dots, N$, the network \mathcal{T} is defined by
29 $\mathcal{T} = \bigcup_{n=1}^N k_n$. Specifically, we are going to consider the next evolution problem for
30 the KdV equation,

(KdV-N)

$$\left\{ \begin{array}{ll} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) = 0 & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{array} \right.$$

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where $\alpha \geq \frac{N}{2}$. The central node conditions are obtained taking account of the following: If we denote by u_n and v_n the dimensionless and scaled variables standing, respectively, for the deflection from rest position and the velocity on the branch n of long water waves, then we get from [25, eq. (13.102)]

$$\begin{cases} \partial_t u_n + \partial_x u_n + \partial_x^3 u_n + u_n \partial_x u_n = 0 & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ v_n = u_n - \frac{1}{6} u_n^2 + 2 \partial_x^2 u_n & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N. \end{cases}$$

Moreover, at the central node, we can suppose that the elevation of water is the same in all branches and that the sum of the flux is null, which implies

$$\begin{cases} u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N u_n(t, 0) v_n(t, 0) = 0, & t > 0. \end{cases}$$

Then we obtain the following problem:

$$\begin{cases} u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\frac{N}{2} u_1(t, 0) + \frac{N}{6} u_1^2(t, 0), & t > 0. \end{cases}$$

We adapt the boundary condition at the central node to have a decreasing energy. The hypothesis $\alpha > \frac{N}{2}$ was introduced in [1] and then in [10] the case $\alpha = \frac{N}{2}$ was included. (KdV-N) was studied in [1] by using the following functional setting: Let $H_r^1(0, \ell_n) = \{v \in H^1(0, \ell_n), v(\ell_n) = 0\}$, where the index r is related to the null right boundary conditions, the space $\mathbb{H}_e^1(\mathcal{T})$ be the Cartesian product of $H_r^1(0, \ell_n)$ including the continuity condition on the central node ($u_n(0) = u_{n'}(0) \forall n, n' = 1, \dots, N$)

$$\mathbb{H}_e^1(\mathcal{T}) = \left\{ \underline{u} = (u_1, \dots, u_N)^T \in \prod_{n=1}^N H_r^1(0, \ell_n), u_n(0) = u_{n'}(0) \forall n, n' = 1, \dots, N \right\},$$

and

$$\|\underline{u}\|_{\mathbb{H}_e^1(\mathcal{T})}^2 = \sum_{n=1}^N \|u_n\|_{H^1(0, \ell_n)}^2,$$

where the index e is related so that each edge belongs to $H_r^1(0, \ell_n)$. Introduce also the state space

$$\mathbb{L}^2(\mathcal{T}) = \prod_{n=1}^N L^2(0, \ell_n) \quad \text{with} \quad (\underline{u}, \underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{n=1}^N \int_0^{\ell_n} u_n v_n dx \quad \forall \underline{u}, \underline{v} \in \mathbb{L}^2(\mathcal{T}).$$

We also define the space $\mathbb{B}_T = C([0, T], \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$ with $\|u\|_{\mathbb{B}_T} = \|u\|_{C([0, T], \mathbb{L}^2(\mathcal{T}))} + \|u\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}$, and \mathbb{Y}_T be the space of all functions $\underline{v} \in \mathbb{B}_T$ such that $\partial_x^\kappa v_n \in L_x^\infty(0, \ell_n; H^{\frac{1-\kappa}{3}}(0, T))$ for $\kappa = 0, 1, 2$, with the induced norm

$$\|\underline{v}\|_{\mathbb{Y}_T} = \|\underline{v}\|_{\mathbb{B}_T} + \sum_{\kappa=0}^2 \|\partial_x^\kappa \underline{v}\|_{\prod_{n=1}^N L_x^\infty(0, \ell_n; H^{\frac{1-\kappa}{3}}(0, T))}.$$

In [1, 10] the next well-posedness result was proved for small initial condition and for any time horizon.

60 THEOREM 1.1 (Theorem 2.7 of [1]). *Let $(\ell_n)_{n=1,\dots,N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$ and*
 61 *$T > 0$. Then there exist $\epsilon > 0$ and $C > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with*
 62 *$\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, there exists a unique solution of (KdV-N). Moreover, it satisfies*
 63 *$\|\underline{u}\|_{\mathbb{B}_T} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

64 The main problem to get a global well-posedness result is the action of the non-
 65 linear boundary condition on the central node. Similar boundary conditions appear
 66 for the first time to our knowledge in the work [21] where a wave maker control for
 67 a single KdV equation was studied and then in the work [5] where a well-posedness
 68 result was given. The system studied in these papers was the next one

$$(1.1) \quad \begin{cases} \partial_t u(t, x) + \partial_x u(t, x) + u(t, x) \partial_x u(t, x) + \partial_x^3 u(t, x) = 0 & \forall x \in (0, L), t > 0, \\ \partial_x^2 u(t, 0) = -u(t, 0) + \frac{1}{6} u^2(t, 0) + h(t), & t > 0, \\ u(t, L) = \partial_x u(t, L) = 0, & t > 0, \\ u(0, x) = \phi(x), & x \in (0, L), \end{cases}$$

70 and the following well-posedness result local-in-time for bounded initial data was
 71 proven in [5].

72 THEOREM 1.2 (Theorem 1.1 of [5]). *Let $T > 0$ and $\gamma > 0$ be given. There*
 73 *exists $T^* \in (0, T]$ such that for any $\phi \in L^2(0, L)$ and $h \in H^{-\frac{1}{3}}(0, T)$ satisfying,*
 74 *$\|\phi\|_{L^2(0, L)} + \|h\|_{H^{-\frac{1}{3}}(0, T)} \leq \gamma$. Then the problem (1.1) admits a unique solution $u \in$*
 75 *$C([0, T^*]; L^2(0, L)) \cap L^2(0, T^*; H^1(0, L))$. Moreover, the corresponding solution map*
 76 *is Lipschitz continuous and the solution possesses the hidden regularities (the sharp*
 77 *Kato smoothing properties) $\partial_x^\kappa u \in L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T^*))$, $\kappa = 0, 1, 2$.*

78 The first main result of our work is the following global-in-time well-posedness
 79 theorem.

80 THEOREM 1.3. *Let $(\ell_n)_{n=1,\dots,N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$, and $T > 0$. Then, for all*
 81 *$\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, there exists a unique solution $\underline{u} \in \mathbb{B}_T$ of (KdV-N). Moreover, there exist*
 82 *$0 < T^* \leq T$, $C > 0$ such that $\underline{u} \in \mathbb{Y}_{T^*}$ and $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

83 Note that our result generalized Theorem 1.1 in the sense that the smallness
 84 assumption on the initial data is not needed. Our idea is to follow [5] to obtain
 85 a similar sharp Kato smoothing regularity presented in Theorem 1.2 for a linear
 86 problem of the KdV equation on a star-shaped network. In order to deal with the
 87 nonlinear part, we use a fixed point argument to obtain global well-posedness for
 88 small time. Finally, we use an energy estimation to obtain a global well-posedness
 89 in time. Similar ideas were applied in the case of a single KdV equation in [18].
 90 From the point of view of stabilization, we can refer to the work [26] in which the
 91 boundary exponential stabilization problem in the bounded spatial domain $x \in (0, 1)$
 92 was studied. It is well known that the length L of the spatial domain plays an
 93 important role in the stabilization and controllability properties of the KdV equation.
 94 For example, when $L = 2\pi$ it is possible to find a solution of the linearization around
 95 0 of KdV ($u(t, x) = 1 - \cos(x)$) that has constant energy. More generally, if $L \in \mathcal{N}$,
 96 where \mathcal{N} is the set of critical lengths defined by

$$97 \quad \mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N}^* \right\},$$

98 we can find suitable initial data such that the solution of the linear KdV equation
 99 has constant energy. For the case of internal stabilization, it is proved in [18, 17] that

for any critical length we achieve local exponential stability for the nonlinear KdV equation by adding a localized damping. In most real-life settings we have to take into account the saturation in the input control due to some (physical, economical, etc.) constraints. With respect to saturated control in infinite-dimensional systems, we can refer to [19] where a wave equation with distributed and boundary saturated feedback law was studied, [14] where the saturated internal stabilization of a single KdV equation was studied and recently [15] where a saturated feedback control law was derived for a linear reaction-diffusion equation. Our idea closely follows works [14] and [16] to prove the stability of the KdV equation in a star-shaped network with saturated internal control. In this work, we consider a saturation map \mathbf{sat} that could be any of the following cases:

- $\mathbf{sat} = \mathbf{sat}_{1oc}$: First consider the following scalar saturation,

$$\mathbf{sat}(f) = \begin{cases} -M & \text{if } f \leq -M, \\ f & \text{if } -M \leq f \leq M, \\ M & \text{if } f \geq M, \end{cases}$$

where $M > 0$ is given and denotes the saturation level. Then we take the next extension to an infinite-dimensional setting

$$(1.2) \quad \mathbf{sat}_{1oc}(f)(x) = \mathbf{sat}(f(x)).$$

- $\mathbf{sat} = \mathbf{sat}_2$: For $f \in L^2(0, L)$ we define

$$(1.3) \quad \mathbf{sat}_2(f)(x) = \begin{cases} f(x) & \text{if } \|f\|_{L^2(0,L)} \leq M, \\ \frac{f(x)M}{\|f\|_{L^2(0,L)}} & \text{if } \|f\|_{L^2(0,L)} \geq M. \end{cases}$$

In what follows, \mathbf{sat} corresponds to either \mathbf{sat}_{1oc} or \mathbf{sat}_2 . In order to consider the saturated stabilization problem, we study the next system

(KdV-S)

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) \\ \quad + \mathbf{sat}(a_n(x)u_n(t, x)) = 0, & x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases}$$

where the damping terms $(a_n)_{n=1, \dots, N} \in \prod_{n=1}^N L^\infty(0, \ell_n)$ act locally on all branches, formally written as

$$(1.4) \quad a_n \geq c_n > 0 \text{ in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n = 1, \dots, N.$$

In this work, we are going to consider the following energy $E(t)$ of $\underline{u} = (u_1, \dots, u_N)^T \in \mathbb{L}^2(\mathcal{T})$ by

$$(1.5) \quad E(t) = \frac{1}{2} \|\underline{u}\|_{\mathbb{L}^2(\mathcal{T})}^2.$$

The second main result of this paper states the semiglobal exponential stability of (KdV-S).

129 **THEOREM 1.4.** *Assume that the damping terms $(a_n)_{n=1,\dots,N}$ satisfy (1.4). Let*
 130 *$(\ell_n)_{n=1}^N \subset (0, \infty)$ and $R > 0$, then there exist $C(R) > 0$ and $\mu(R) > 0$ such that for*
 131 *all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, the energy of any solution of (KdV-S) defined by*
 132 *(1.5) satisfies $E(t) \leq C(R)E(0)e^{-\mu(R)t}$ for all $t > 0$.*

133 Then, in order to add damped terms only on the critical lengths as in [1], we ne-
 134 glect the term $u_n \partial_x u_n$ in the KdV equation (KdV-N). Let $I_c = \{n \in \{1, \dots, N\}; \ell_n \in$
 135 $\mathcal{N}\}$ be the set of critical lengths and I_c^* be the subset of I_c where we remove one index.
 136 We consider now the following problem,

$$137 \text{ (LKdV-S)} \quad \begin{cases} (\partial_t u_n + \partial_x u_n + \partial_x^3 u_n)(t, x) \\ \quad + \text{sat}(a_n(x)u_n(t, x)) = 0, & x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0), & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x), & x \in (0, \ell_n), \end{cases}$$

138 where the damping $(a_n)_{n=1,\dots,N} \in \prod_{n=1}^N L^\infty(0, \ell_n)$ satisfy

$$139 \text{ (1.6)} \quad \begin{cases} a_n = 0 \text{ for } n \in \{1, \dots, N\} \setminus I_c^*, \\ a_n \geq c_n \text{ in an open nonempty set } \omega_n \text{ of } (0, \ell_n), \text{ for all } n \in I_c^*, \\ \text{and } c_n > 0 \text{ is a constant.} \end{cases}$$

140 Then we are able to prove the following global stabilization result, which is the last
 141 main result.

142 **THEOREM 1.5.** *Assume that the damping terms $(a_n)_{n=1,\dots,N}$ satisfy (1.6) and let*
 143 *$(\ell_n)_{n=1}^N \subset (0, \infty)$. Then, there exist $C > 0$ and $\mu > 0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$,*
 144 *the energy of any solution of (LKdV-S) defined by (1.5) satisfies $E(t) \leq CE(0)e^{-\mu t}$*
 145 *for all $t > 0$.*

146 *Remark 1.* Note that for the system (LKdV-S) the stabilization result is global,
 147 instead of the one for (KdV-S) which is semiglobal. This difference comes from the
 148 action of the term $u_n \partial_x u_n$: The condition $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ is necessary to handle this
 149 term. ◦

150 *Remark 2.* A global stabilization result for (KdV-S) is, to our knowledge, an open
 151 problem. ◦

152 **2. Well-posedness.** This section is devoted to prove well-posedness results for
 153 (KdV-N)-(KdV-S) and (LKdV-S); in particular, we focus on Theorem 1.3. Our
 154 scheme will be to consider appropriate linear systems to derive regularity proper-
 155 ties. Then, using a fixed point result, we obtain the well-posedness for the nonlinear
 156 systems.

157 **2.1. Linear problems.** We start by considering the following linear system for
 158 the KdV equation on a star-shaped network \mathcal{T} :

$$159 \text{ (LKdV-N)} \quad \begin{cases} \partial_t u_n + \partial_x^3 u_n = f_n & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = h(t), & t > 0, \\ u_n(t, \ell_n) = 0, \quad \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0(x) & \forall x \in (0, \ell_n), j = 1, \dots, N. \end{cases}$$

160 The terms f_n and h are internal and boundary functions that are useful for the fixed
 161 point approach. First, we deal with the linear system (LKdV-N) with homogeneous
 162 initial condition and homogeneous internal source terms ($f_n = 0$):

$$163 \quad (2.1) \quad \begin{cases} \partial_t u_n + \partial_x^3 u_n = 0 & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0), & t > 0, \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = h(t), & t > 0, \\ u_n(t, \ell_n) = 0, \quad \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = 0, & \forall x \in (0, \ell_n), n = 1, \dots, N, \end{cases}$$

164 The fact that we work with the linear system $\partial_t u_n + \partial_x^3 u_n = 0$ instead of $\partial_t u_n +$
 165 $\partial_x u_n + \partial_x^3 u_n = 0$ is motivated by [3, 5]. It is well known, that the term $\partial_x u_n$ yields
 166 problematic behaviors with respect to regularity and controllability properties, as well
 167 noted Rosier in [20] and then in several works [7, 27, 4]. Now, formally we apply the
 168 usual Laplace transform with respect to time to the system (2.1) and obtain

$$169 \quad (2.2) \quad \begin{cases} s \hat{u}_n + \partial_x^3 \hat{u}_n = 0 & \forall x \in (0, \ell_n), n = 1, \dots, N, \\ \hat{u}_n(s, 0) = \hat{u}_{n'}(s, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 \hat{u}_n(s, 0) = \hat{h}(s), \\ \hat{u}_n(s, \ell_n) = 0, \quad \partial_x \hat{u}_n(s, \ell_n) = 0, & n = 1, \dots, N, \\ \hat{u}_n(0, x) = 0 & \forall x \in (0, \ell_n), n = 1, \dots, N, \end{cases}$$

170 where

$$171 \quad \hat{u}_n(s, x) = \int_0^\infty e^{-st} u_n(t, x) dt, \quad \hat{h}(s) = \int_0^\infty e^{-st} h(t) dt \quad \forall x \in (0, \ell_n).$$

172 Following [3], we can see that the N component solutions to (2.2) can be written as

$$173 \quad (2.3) \quad \hat{u}_n(s, x) = \sum_{j=1}^3 c_{3(n-1)+j}^N e^{\lambda_j(s)x},$$

174 where $\lambda_j(s)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $s + \lambda^3 = 0$ and
 175 $c^N = (c_k)_{k=1, \dots, 3N}^N$ solves the following linear system

$$176 \quad (2.4) \quad \left\{ \begin{array}{l} \sum_{n=1}^N \sum_{j=1}^3 c_{3(n-1)+j}^N \lambda_j^2 = \hat{h}, \\ \sum_{j=1}^3 c_j^N e^{\lambda_j \ell_1} = 0, \\ \sum_{j=1}^3 c_j^N \lambda_j e^{\lambda_j \ell_1} = 0, \\ \sum_{j=1}^3 c_j^N = \sum_{j=1}^3 c_{3(n-1)+j}^N, \\ \sum_{j=1}^3 c_{3(n-1)+j}^N e^{\lambda_j \ell_n} = 0, \\ \sum_{j=1}^3 c_{3(n-1)+j}^N \lambda_j e^{\lambda_j \ell_n} = 0 \end{array} \right\} \quad \forall n = 2, \dots, N.$$

177 We write this previous system in its matrix form $A_N c^N = \hat{h} e_1$, where e_1 is the first
 178 vector of the canonical basis in \mathbb{R}^{3N} . We can see easily that $A_N \in M_{3N}$ can be
 179 decomposed by induction in blocks as

$$(2.5) \quad A_1 = \begin{bmatrix} (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ e^{\lambda_1 \ell_1} & e^{\lambda_2 \ell_1} & e^{\lambda_3 \ell_1} \\ \lambda_1 e^{\lambda_1 \ell_1} & \lambda_2 e^{\lambda_2 \ell_1} & \lambda_3 e^{\lambda_3 \ell_1} \end{bmatrix},$$

$$(2.6) \quad A_N = \left[\begin{array}{ccc|ccc} & & & (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ & A_{N-1} & & & & \\ & & & \mathbf{0}_{3(N-1)-1 \times 3} & & \\ \hline 1 & 1 & 1 & & & \\ 0 & 0 & 0 & \mathbf{0}_{3 \times 3(N-2)} & & \\ 0 & 0 & 0 & & D_N & \end{array} \right] = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix}$$

183 for an appropriate choice of B_N , C_N , and

$$(2.7) \quad D_N = \begin{bmatrix} -1 & -1 & -1 \\ e^{\lambda_1 \ell_N} & e^{\lambda_2 \ell_N} & e^{\lambda_3 \ell_N} \\ \lambda_1 e^{\lambda_1 \ell_N} & \lambda_2 e^{\lambda_2 \ell_N} & \lambda_3 e^{\lambda_3 \ell_N} \end{bmatrix}.$$

185 Formally, taking the inverse of the Laplace transform of \hat{u}_n in (2.3), we get for $t \geq 0$
 186 and $x \in (0, \ell_n)$

$$187 \quad u_n(t, x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \hat{u}_n(s, x) ds = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds.$$

188 If we denote, for $t \geq 0$ and $x \in (0, \ell_n)$,

$$189 \quad I_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi i} \int_0^{i\infty} e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds,$$

$$190 \quad J_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{-i\infty}^0 e^{st} c_{3(n-1)+j}^N \hat{h}(s) e^{\lambda_j(s)x} ds,$$

191

192 we have

$$193 \quad (2.8) \quad u_n(t, x) = I_n(t, x) + J_n(t, x).$$

194 Now we introduce the notation, super index, $+\setminus^-$ which corresponds to taking $s =$
 195 $\pm i\rho^3$, $\rho > 0$, in the characteristic equation. Then the roots of the characteristic
 196 equation are given by

$$197 \quad \begin{cases} \lambda_1^+(\rho) = i\rho, & \lambda_2^+(\rho) = \frac{1}{2}\rho(\sqrt{3} - i), & \lambda_3^+(\rho) = \frac{1}{2}\rho(-\sqrt{3} - i), \\ \lambda_j^-(\rho) = \overline{\lambda_j^+(\rho)}, & j = 1, 2, 3. \end{cases}$$

198 Let $\Delta^{N,+}(\rho)$ be the determinant of $A_N(i\rho^3)$ and $\Delta_{3(n-1)+j}^{N,+}(s)$ be the determinant of
 199 the matrix that is obtained by replacing the column $3(n-1)+j$ of the matrix $A_N(i\rho^3)$
 200 by $[1 \ 0 \ \dots \ 0]^T$ and $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$. Assuming that $\Delta^{N,+}(\rho) \neq 0$ (this property will be

justified in Proposition 2.1), Cramer's rule implies that $c_{3(n-1)+j}^{N,+}(\rho) = c_{3(n-1)+j}^N(i\rho^3)$ is given by

$$(2.9) \quad c_{3(n-1)+j}^{N,+}(\rho) = \frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)} \hat{h}^+(\rho).$$

Thus, I_n and J_n can be seen as

$$(2.10) \quad I_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)} \hat{h}^+(\rho) 3\rho^2 d\rho,$$

206

$$(2.11) \quad J_n(t, x) = \sum_{j=1}^3 \frac{1}{2\pi} \int_0^\infty e^{-i\rho^3 t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{3(n-1)+j}^{N,-}(\rho)}{\Delta^{N,-}(\rho)} \hat{h}^-(\rho) 3\rho^2 d\rho,$$

where we use the notation $\Delta_k^{N,-}(\rho) = \overline{\Delta_k^{N,+}(\rho)}$, $\Delta^{N,-}(\rho) = \overline{\Delta^{N,+}(\rho)}$, and $\hat{h}^-(\rho) = \overline{\hat{h}^+(\rho)}$. Our idea now is to obtain estimates for u_n ; for that we are going to prove some asymptotic properties for $\frac{\Delta_{3(n-1)+j}^{N,+}(\rho)}{\Delta^{N,+}(\rho)}$, the following proposition collects these properties.

PROPOSITION 2.1. *For all $\rho > 0$, $\Delta^{N,+}(\rho) \neq 0$. Moreover, the following asymptotic properties hold, for $\rho \rightarrow \infty$,*

$$(2.12) \quad \begin{aligned} \frac{\Delta_{3(n-1)+1}^{N,+}}{\Delta^{N,+}} &\sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, & \sum_{j=1}^3 \frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \quad n = 1, \dots, N, \end{aligned}$$

where $\delta_N > 0$ only depends on N and satisfies $\delta_N = \frac{\delta_{N-1}}{\delta_{N-1} + 1}$.

Proof. The main problem in this proof is to deal with the determinant of the matrix without making explicit computations. Recall that, in the case of N branches, the matrix A_N has size $3N \times 3N$. Our proof is based on an induction argument over the number N of branches of the network.

- $N = 1$: in this case, system (2.4) is exactly the system studied in [5] for $\ell_1 = 1$. By Appendix B, it holds that $\Delta^{1,+}(\rho) \neq 0$ for all $\rho > 0$. Moreover, following the explicit calculations given in [5] we can deduce

$$\begin{aligned} \frac{\Delta_1^{1,+}}{\Delta^{1,+}} &\sim -\rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_1 - i\frac{3}{2}\rho\ell_1}, & \frac{\Delta_2^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-\rho\sqrt{3}\ell_1 + i\frac{\pi}{3}}, & \frac{\Delta_3^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-i\frac{\pi}{3}}, \\ & & \sum_{j=1}^3 \frac{\Delta_j^{1,+}}{\Delta^{1,+}} &\sim \rho^{-2} e^{-i\frac{\pi}{3}}. \end{aligned}$$

That gives (2.12) in the case $N = 1$.

- Suppose now that $\Delta^{N-1,+}(\rho) \neq 0$ for all $\rho > 0$ and that the asymptotic property (2.12) is true for any network of $N - 1$ branches. Let us prove that

227 $\Delta^{N,+}(\rho) \neq 0$ for all $\rho > 0$ and that the asymptotic property (2.12) holds for
 228 a network of N branches. As

$$229 \quad A_N = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix},$$

230 and we have $\det(A_{N-1}) = \Delta^{N-1,+} \neq 0$ by hypothesis, we can write

$$231 \quad A_N = \begin{bmatrix} I_{3(N-1)} & \mathbf{0}_{3(N-1)} \\ C_N A_{N-1}^{-1} & I_{3(N-1)} \end{bmatrix} \begin{bmatrix} A_{N-1} & \mathbf{0}_{3(N-1)} \\ \mathbf{0}_{3(N-1)} & D_N - C_N A_{N-1}^{-1} B_N \end{bmatrix} \\
 232 \quad \times \begin{bmatrix} I_{3(N-1)} & A_{N-1}^{-1} B_N \\ \mathbf{0}_{3(N-1)} & I_{3(N-1)} \end{bmatrix}, \\
 233$$

234 which implies directly that

$$235 \quad (2.13) \quad \Delta^{N,+} = \det(A_N) = \det(A_{N-1}) \det(D_N - C_N A_{N-1}^{-1} B_N).$$

236 The difficulty of the last expression is the role of the matrix A_{N-1}^{-1} . In fact,
 237 to calculate this inverse explicitly is quite complicated. Note now that if

$$238 \quad A_{N-1}^{-1} = \begin{bmatrix} x_1 & \vdots & \dots & \vdots \\ x_2 & \vdots & \dots & \vdots \\ x_3 & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

239 then, we have

$$240 \quad (2.14) \quad C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2(x_1 + x_2 + x_3) & (\lambda_2^+)^2(x_1 + x_2 + x_3) & (\lambda_3^+)^2(x_1 + x_2 + x_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

241 from here we can see that it is not necessary to calculate all the entries of
 242 the matrix A_{N-1}^{-1} . Indeed, we only need the 3 first entries of the first column.
 243 Straightforward calculations show that

$$244 \quad (2.15) \quad x_1 = \frac{\Delta_1^{N-1,+}}{\Delta^{N-1,+}}, \quad x_2 = \frac{\Delta_2^{N-1,+}}{\Delta^{N-1,+}}, \quad x_3 = \frac{\Delta_3^{N-1,+}}{\Delta^{N-1,+}}.$$

245 Using (2.14) and (2.15) we get

$$246 \quad (2.16) \quad C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_2^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_3^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \\
 247 \\
 248$$

Then with (2.7)

(2.17)

$$D_N - C_N A_{N-1}^{-1} B_N = \begin{bmatrix} -1 - (\lambda_1^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1 - (\lambda_2^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1 - (\lambda_3^+)^2 \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix}$$

and using the multilinearity of the determinant

$$\det(D_N - C_N A_{N-1}^{-1} B_N) = - \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N),$$

where

$$F_N = \begin{bmatrix} (\lambda_1^+)^2 & (\lambda_2^+)^2 & (\lambda_3^+)^2 \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix}.$$

Then, it holds that

$$\Delta^{N,+} = \Delta^{N-1,+} \left[- \sum_{j=1}^3 \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N) \right].$$

Using (2.7) and (2.18) we can derive

$$\det(D_N) = \rho\sqrt{3}e^{-i\rho\ell_N} + \left(-\frac{\rho\sqrt{3}}{2} - \frac{3}{2}i\rho \right) e^{(-\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_N} + \left(-\frac{\rho\sqrt{3}}{2} + \frac{3}{2}i\rho \right) e^{(\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_N},$$

$$\det(F_N) = \sqrt{3}\rho^3 e^{-i\rho\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_N}.$$

Now, to compute $\Delta_{3(n-1)+j}^{N,+}$, let $A_{N,j}^n$ the matrix obtained by replacing the column $3(n-1)+j$ of A_N by $[10 \cdots 0]^T$, for $j=1, 2, 3$ and $n=1, \dots, N-1$, that is

$$A_{N,j}^n = \left[\begin{array}{c|c} \begin{matrix} (j+3(n-1)-th) \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} B_N \\ \hline D_N \end{matrix} \end{array} \right] = \begin{cases} \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n & B_N \\ \hline 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} D_N \\ \hline D_N \end{matrix} \end{array} \right] & \text{if } j=1, n=1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n & B_N \\ \hline 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} D_N \\ \hline D_N \end{matrix} \end{array} \right] & \text{if } j=2, n=1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n & B_N \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} D_N \\ \hline D_N \end{matrix} \end{array} \right] & \text{if } j=3, n=1 \\ \left[\begin{array}{c|c} \begin{matrix} A_{N-1,j}^n & B_N \\ \hline C_N & D_N \end{matrix} & \begin{matrix} D_N \\ \hline D_N \end{matrix} \end{array} \right] & \text{if } j=1, 2, 3, n=2, \dots, N-1. \end{cases}$$

We claim the following property of $\Delta_{3(n-1)+j}^{N,+}$.

LEMMA 2.2.

$$(2.23) \quad \Delta_{3(n-1)+j}^{N,+} = \Delta_{3(n-1)+j}^{N-1,+} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3.$$

Proof. Using the decomposition given by (2.22), we get

$$A_{N,j}^n = \begin{bmatrix} A_{N-1,j}^n & B_N \\ C_{N,j}^n & D_N \end{bmatrix}$$

for an appropriate choice of $C_{N,j}^n$. Thus, with the same idea as (2.13) it holds that

$$(2.24) \quad \Delta_{3(n-1)+j}^{N,+} = \det(A_{N,j}^n) = \det(A_{N-1,j}^n) \det(D_N - C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N).$$

Similarly, as before, we need to study the product $C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N$, in particular, the first column of the matrix $(A_{N-1,k}^n)^{-1}$. To do that, note that

$$A_{N-1,j}^n v = \left[\begin{array}{c|c} \begin{matrix} \overbrace{1}^{(j+3(n-1)-th)} \\ 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} B_{N-1} \\ \\ \\ \end{matrix} \\ \hline \begin{matrix} \vdots \\ \vdots \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \\ \\ D_N \\ \end{matrix} \end{array} \right] v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

by simple inspection; the solution of this problem is $v = [0 \cdots \overbrace{1}^{j+3(n-1)} \cdots 0]^T$ which we know coincides with the first column of $(A_{N-1,j}^n)^{-1}$ and, therefore, $C_{N,j}^n (A_{N-1,j}^n)^{-1} B_N = \mathbf{0}_{3 \times 3}$; therefore, with (2.24)

$$\Delta_{3(n-1)+j}^{N,+} = \Delta_{3(n-1)+j}^{N-1,+} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3,$$

which finishes the proof of Lemma 2.2. \square

In order to show that $\Delta^{N,+} \neq 0$, note that by (2.19) we get

$$\Delta^{N,+} = - \sum_{j=1}^3 \Delta_j^{N-1,+} \det(F_N) + \Delta^{N-1,+} \det(D_N), \quad j = 1, 2, 3.$$

Using (2.23) recursively, we get

$$\Delta_j^{N-1,+} = \Delta_j^{1,+} \prod_{\ell=2}^{N-1} \det(D_\ell).$$

Noticing that $\Delta^{1,+} = \det(F_1)$, $-\sum_{j=1}^3 \Delta_j^{1,+} = \det(D_1)$ and invoking inductively (2.19), we deduce

$$\Delta^{N,+} = \sum_{j=1}^N \det(F_j) \prod_{\ell=1, \ell \neq j}^N \det(D_\ell).$$

Then, from Appendix C, it holds, for all $j = 1, \dots, N$, $\det(D_j) \neq 0$, thus

$$\Delta^{N,+} = \left(\prod_{\ell=1}^N \det(D_\ell) \right) \sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)},$$

and from Appendix D, $\sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} \neq 0$, thus $\Delta^{N,+} \neq 0$. Now as $\Delta^{N,+} \neq 0$, we can obtain using (2.19) and (2.23) that

$$(2.25) \quad \frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} = \frac{\Delta_{3(n-1)+j}^{N-1,+}}{\Delta^{N-1,+}} \frac{\det(D_N)}{-\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)}$$

for $j = 1, 2, 3$, $n = 1, \dots, N-1$. Then, using (2.21) we get $\det(F_N) \sim \sqrt{3}\rho^3 e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N}$ and by the induction assumption $\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \sim \delta_{N-1}\rho^{-2}e^{-i\frac{\pi}{3}}$. Thus $\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) \sim \delta_{N-1}\sqrt{3}\rho e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N - i\frac{\pi}{3}}$ and then for $\rho \rightarrow \infty$

$$(2.26) \quad \frac{\det(D_N)}{-\sum_{l=1}^3 \frac{\Delta_l^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)} \sim \frac{1}{\delta_{N-1} + 1}.$$

Now by the induction assumption

$$\begin{aligned} \frac{\Delta_{3(n-1)+1}^{N-1,+}}{\Delta^{N-1,+}} &\sim -\delta_{N-1}\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N-1,+}}{\Delta^{N-1,+}} &\sim \delta_{N-1}\rho^{-2}e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N-1,+}}{\Delta^{N-1,+}} &\sim \delta_{N-1}\rho^{-2}e^{-i\frac{\pi}{3}}, \end{aligned}$$

and (2.25)–(2.26) we have

$$(2.27) \quad \begin{aligned} \frac{\Delta_{3(n-1)+1}^{N,+}}{\Delta^{N,+}} &\sim -\delta_N\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, & \frac{\Delta_{3(n-1)+2}^{N,+}}{\Delta^{N,+}} &\sim \delta_N\rho^{-2}e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \\ \frac{\Delta_{3(n-1)+3}^{N,+}}{\Delta^{N,+}} &\sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}}, & \sum_{j=1}^3 \frac{\Delta_{3(n-1)+j}^{N,+}}{\Delta^{N,+}} &\sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}}, \quad n = 1, \dots, N-1, \end{aligned}$$

where $\delta_N = \frac{\delta_{N-1}}{\delta_{N-1}+1}$. It just remains to study the case $n = N$. Note that using the block decomposition of A_N we get

$$C_N \begin{bmatrix} \frac{\Delta_1^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_2^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_3^{N,+}}{\Delta^{N,+}} \\ \vdots \\ \frac{\Delta_{3N-5}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N-4}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N-3}^{N,+}}{\Delta^{N,+}} \end{bmatrix} + D_N \begin{bmatrix} \frac{\Delta_{3N-2}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N-1}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N}^{N,+}}{\Delta^{N,+}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and recalling (2.6) and (2.7) explicit calculations show that

$$(2.28) \quad \begin{bmatrix} \frac{\Delta_{3N-2}^{N,+}}{\Delta_{N,+}^{N,+}} \\ \frac{\Delta_{3N-1}^{N,+}}{\Delta_{N,+}^{N,+}} \\ \frac{\Delta_{3N}^{N,+}}{\Delta_{N,+}^{N,+}} \end{bmatrix} = \frac{\left(-\sum_{j=1}^3 \frac{\Delta_j^{N,+}}{\Delta_{N,+}^{N,+}}\right)}{\det(D_N)} \begin{bmatrix} -\rho\sqrt{3}e^{-i\rho\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2} + \frac{3}{2}i\rho\right)e^{\left(-\frac{\rho\sqrt{3}}{2} + \frac{i\rho}{2}\right)\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2} - \frac{3}{2}i\rho\right)e^{\left(\frac{\rho\sqrt{3}}{2} + \frac{i\rho}{2}\right)\ell_N} \end{bmatrix},$$

and using (2.27) we can conclude from (2.28)

$$\begin{aligned} \frac{\Delta_{3N-2}^{N,+}}{\Delta_{N,+}^{N,+}} &\sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_N - i\frac{3}{2}\rho\ell_N}, & \frac{\Delta_{3N-1}^{N,+}}{\Delta_{N,+}^{N,+}} &\sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_N + i\frac{\pi}{3}}, \\ \frac{\Delta_{3N}^{N,+}}{\Delta_{N,+}^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, & \sum_{j=1}^3 \frac{\Delta_{3(N-1)+j}^{N,+}}{\Delta_{N,+}^{N,+}} &\sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \end{aligned}$$

which gives the induction and concludes the proof of Proposition 2.1. \square

Remark 3. Recently, in [11], the problem of small-time local controllability of the nonlinear single KdV equation was addressed. To reach the obstruction to small-time controllability in [11] new regularity results in the spirit of [2] were established. Those results have some connections with the analysis developed in this work. Here, the analysis of the linear problem (2.4) is based on the estimate of the terms I_n and J_n ((2.10) and (2.11)). These involve two integrals of ρ from 0 to infinity, and Proposition 2.1 shows the integrands are well-defined ($\Delta^{N,+} \neq 0$) and deal with their behavior at infinity. However, in [11] the behavior of the integrands might be infinite for finite ρ . This is the case where $L \in \mathcal{N}$, with $2k + l \notin 3\mathbb{N}^*$ [11, Lemma B1]. The main difference between these two different behaviors is because in [11] they worked with the linear system including the term, u_x which is necessary to study controllability issues. \circ

Now we are going to state the next regularity result for the solution (2.1) using the Laplace representation obtained in (2.8) and Proposition 2.1.

PROPOSITION 2.3. *Let $T > 0$ and $h \in H^{-\frac{1}{3}}(0, T)$, then we have a unique solution $\underline{u} \in \mathbb{Y}_T$ of (2.1). Moreover, there exists $C > 0$ such that for all $h \in H^{-\frac{1}{3}}(0, T)$, $\|\underline{u}\|_{\mathbb{Y}_T} \leq C \|h\|_{H^{-\frac{1}{3}}(0, T)}$.*

Proof. This proof uses Proposition 2.1 and follows closely [5, Proposition 2.2] and [3], thus it is omitted here. \square

Note that Proposition 2.3 justifies the formal computations given in (2.8). Let \underline{W} the operator that corresponds to the integral representation obtained in Proposition 2.3, i.e., given $T > 0$ and $h \in H^{-\frac{1}{3}}(0, T)$, the unique solution \underline{u} of (2.1) is given by

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \underline{W}h \in \mathbb{B}_T.$$

Our next step is to consider the linear problem including nonhomogeneous initial data and source terms, as follows:

$$(2.29) \quad \begin{cases} \partial_t v_n(t, x) + \partial_x^3 v_n(t, x) = f_n(t, x) & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ v_n(t, 0) = v_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n(t, 0) = h(t), & t > 0, \\ v_n(t, \ell_n) = 0, \quad \partial_x v_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ v_n(0, x) = v_n^0, & x \in (0, \ell_n). \end{cases}$$

342 We know from [1] that in the case, $h = 0$ the solution of (2.29) can be written as

$$343 \quad \underline{v}(t, x) = \underline{W}_0(t)\underline{v}^0 + \int_0^t \underline{W}_0(t - \tau)\underline{f}(\tau)d\tau,$$

344 for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$ and $\underline{f} \in L^1(0, T; \mathbb{L}^2(\mathcal{T}))$, where $\{\underline{W}_0(t)\}_{t \geq 0}$ is the C_0 -semigroup
345 in the space $\mathbb{L}^2(\mathcal{T})$ generated by the operator $\mathcal{A}\underline{v} = -\partial_x^3 \underline{v}$, with domain

$$346 \quad D(\mathcal{A}) = \left\{ \underline{v} \in \left(\prod_{n=1}^N H^3(0, \ell_n) \right) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{n=1}^N \frac{d^2 v_n}{dx^2}(0) = 0, \right\}$$

347 where $H_r^2(0, \ell_n) = \{v \in H^2(0, \ell_n), \left(\frac{d}{dx}\right)^{i-1} v(\ell_n) = 0, 1 \leq i \leq 2\}$ and the space
348 $\mathbb{H}_e^2(\mathcal{T})$ is the Cartesian product of $H_r^2(0, \ell_n)$ including the continuity condition on the
349 central node ($u_n(0) = u_{n'}(0) \forall n, n' = 1, \dots, N$). Using semigroup theory it is possible
350 to show that $\underline{v} \in C([0, T]; \mathbb{L}^2(\mathcal{T}))$ and also using multipliers we can obtain the classical
351 Kato smoothing result $\underline{v} \in L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$, but it is difficult (if not impossible) to
352 derive the sharp Kato smoothing property established in Proposition 2.3 using energy
353 methods. Now we use the following result obtained in [3] for a single KdV equation
354 posed on a bounded domain.

$$355 \quad (2.30) \quad \begin{cases} \partial_t \psi + \partial_x^3 \psi = f, & x \in (0, L), \quad t \geq 0, \\ \psi(t, 0) = \psi(t, L) = \partial_x \psi(t, L) = 0, & t \geq 0, \\ \psi(0, x) = \psi^0(x), & x \in (0, L), \end{cases}$$

356

357 PROPOSITION 2.4 (Lemma 3.3 of [3]). *Let $T > 0$ and $L > 0$ be given. For any*
358 *$\psi^0 \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, the problem (2.30) admits a unique solution*
359 *$\psi \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$, with $\partial_x^\kappa \psi \in L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T))$, $\kappa =$*
360 *$0, 1, 2$. Moreover, there exists $C > 0$ depending only on T and L such that*

$$361 \quad \|\psi\|_{C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))} + \sum_{\kappa=0}^2 \|\partial_x^\kappa \psi\|_{L_x^\infty(0, L; H^{\frac{1-\kappa}{3}}(0, T))}$$

$$362 \quad \leq C (\|\psi^0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; H^1(0, L))}).$$

363

364 Now for any $v_n^0 \in L^2(0, \ell_n)$ and $f_n \in L^1(0, T; L^2(0, \ell_n))$, consider

$$365 \quad \psi_n = \psi_n(t, \cdot) = W_1^n(t)v_n^0(\cdot) + \int_0^t W_1^n(t - \tau)f_n(\tau, \cdot)d\tau,$$

366 where $W_1^n(t)$ is the C_0 -semigroup associated with the boundary-value problem (2.30)
367 on $(0, \ell_n)$. Let $\mathfrak{h}(t) = \sum_{n=1}^N \partial_x^2 \psi_n(t, 0) \in H^{-\frac{1}{3}}(0, T)$ by Proposition 2.4. Now take
368 $h \in H^{-\frac{1}{3}}(0, T)$, then by Proposition 2.3 the function $\underline{w} = \underline{W}(t)(h - \mathfrak{h})$ is well-defined
369 and is the solution of (2.1) with boundary data $h - \mathfrak{h}$. Finally, the solution \underline{v} of (2.29)
370 can be expressed as

$$371 \quad \underline{v}(t, \cdot) = \underline{W}_1(t)\underline{v}^0(\cdot) + \int_0^t \underline{W}_1(t - \tau)\underline{f}(\tau, \cdot)d\tau + \underline{W}(t)(h - \mathfrak{h})(t).$$

372 The next result encapsulates these ideas.

373 PROPOSITION 2.5. *Let $T > 0$ be given, then, for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$, $h \in H^{-\frac{1}{3}}(0, T)$,*
374 *and $\underline{f} \in L^1(0, T; \mathbb{L}^2(\mathcal{T}))$, the problem (2.29) admits a unique solution $\underline{v} \in \mathbb{Y}_T$. More-*
375 *over, there exists $C > 0$ depending only on T and ℓ_1, \dots, ℓ_n such that*

$$376 \quad \|\underline{v}\|_{\mathbb{Y}_T} \leq C \left(\|h\|_{H^{-\frac{1}{3}}(0, T)} + \|\underline{f}\|_{L^1(0, T; \mathbb{L}^2(\mathcal{T}))} + \|\underline{v}^0\|_{\mathbb{L}^2(\mathcal{T})} \right).$$

377 **2.2. Nonlinear problem.** With all the tools developed in the last sections we
 378 are ready to prove the global well-posedness result established on Theorem 1.3; the
 379 main ingredients of this proof are the regularity obtained in the linear cases, energy
 380 and multiplier estimates, and a fixed point argument. Let $T > 0$ and define $\mathbb{X}_T =$
 381 $\mathbb{L}^2(\mathcal{T}) \times H^{-\frac{1}{3}}(0, T)$.

382 *Proof of Theorem 1.3.* Let $(\underline{u}^0, 0) \in \mathbb{X}_T$ and $R, \theta > 0$ that will be chosen after.
 383 Consider the closed ball $B_{\mathbb{Y}_\theta}(0, R) := \{\underline{v} \in \mathbb{Y}_\theta, \|\underline{v}\|_{\mathbb{Y}_\theta} \leq R\}$. Then $B_{\mathbb{Y}_\theta}(0, R)$ is a
 384 complete metric space. Consider the map $\Phi : \mathbb{Y}_\theta \rightarrow \mathbb{Y}_\theta$ defined by $\Phi(\underline{v}) = \underline{u}$, where \underline{u}
 385 is the solution of

$$(2.31) \quad \begin{cases} (\partial_t u_n + \partial_x^3 u_n)(t, x) = -(\partial_x v_n + v_n \partial_x v_n)(t, x) & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_n^0(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha v_1(t, 0) - \frac{N}{3}(v_1(t, 0))^2, & t > 0, \\ u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0, & x \in (0, \ell_n). \end{cases}$$

387 Clearly, $\underline{u} \in \mathbb{Y}_\theta$ is solution of (KdV-N) if \underline{u} is a fixed point of Φ . Now we write two
 388 lemmas to deal with the source term and boundary conditions. \square

389 **LEMMA 2.6** (Lemma 3.1 of [3]). *There exists a constant $C > 0$ such that for any*
 390 *$T > 0$ and $u, v \in Y_T$*

$$391 \quad \int_0^T \|u(t, \cdot) \partial_x v(t, \cdot)\|_{L^2(0, L)} dt \leq C(T^{1/2} + T^{1/3}) \|u\|_{Y_T} \|v\|_{Y_T},$$

392 where Y_T is \mathbb{Y}_T for $N = 1$.

393 **LEMMA 2.7** (Lemma 3.2 of [12]). *There exist of constants $C, \beta > 0$ such that*
 394 *for any $T > 0$ and $g_1, g_2 \in H^{\frac{1}{3}}(0, T)$, it holds that, $g_1 g_2 \in H^{-\frac{1}{3}}(0, T)$ and*

$$395 \quad \|g_1 g_2\|_{H^{-\frac{1}{3}}(0, T)} \leq C T^\beta \|g_1\|_{H^{\frac{1}{3}}(0, T)} \|g_2\|_{H^{\frac{1}{3}}(0, T)}.$$

396 From Proposition 2.5 and Lemmas 2.6 and 2.7 we get for all $\underline{v} \in \mathbb{Y}_\theta$

$$\begin{aligned} \|\Phi(\underline{v})\|_{\mathbb{Y}_\theta} &\leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \left\| -\alpha v_1(t, 0) - \frac{N}{3}(v_1(t, 0))^2 \right\|_{H^{-\frac{1}{3}}(0, \theta)} \right. \\ &\quad \left. + \int_0^\theta \|\partial_x \underline{v}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt + \int_0^\theta \|\underline{v}(t, \cdot) \partial_x \underline{v}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt \right) \\ &\leq C \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \theta^\beta (\|\underline{v}\|_{\mathbb{Y}_\theta} + \|\underline{v}\|_{\mathbb{Y}_\theta}^2) + (\theta^{1/2} + \theta^{1/3}) \|\underline{v}\|_{\mathbb{Y}_\theta}^2 + \theta^{1/2} \|\underline{v}\|_{\mathbb{Y}_\theta} \right). \end{aligned}$$

398 We consider Φ restricted to the closed ball $B_{\mathbb{Y}_\theta}(0, R)$ and choose $\theta, R > 0$ such that

$$(2.32) \quad \begin{cases} R = 3C \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}, \\ C(\theta^\beta + \theta^{1/2}) \leq \frac{1}{3}, \\ C(\theta^\beta + \theta^{1/2} + \theta^{1/3})R \leq \frac{1}{6}. \end{cases}$$

399 Thus, for $\underline{u} \in B_{\mathbb{Y}_\theta}(0, R)$, Φ maps $B_{\mathbb{Y}_\theta}(0, R)$ into itself. Take now \underline{v} and $\tilde{v} \in B_{\mathbb{Y}_\theta}(0, R)$,
 400 then $\underline{w} = \Phi(\underline{v}) - \Phi(\tilde{v})$ solves the equation

$$401 \quad \begin{cases} \partial_t w_n + \partial_x^3 w_n = -(\partial_x v_n - \partial_x \tilde{v}_n) \\ \quad - \frac{1}{2} \partial_x ((v_n - \tilde{v}_n)(v_n + \tilde{v}_n)) & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ w_n(t, 0) = w_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 w_n(t, 0) = -\alpha(v_1(t, 0) - \tilde{v}_1(t, 0)), \\ \quad - \frac{1}{3} ((v_1(t, 0) - \tilde{v}_1(t, 0))(v_1(t, 0) + \tilde{v}_1(t, 0))), & t > 0, \\ w_n(t, \ell_n) = \partial_x w_n(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ w_n(0, x) = 0, & x \in (0, \ell_n). \end{cases}$$

402 Now from Proposition 2.5 we obtain

$$403 \quad \begin{aligned} \|\Phi(\underline{v}) - \Phi(\tilde{v})\|_{\mathbb{Y}_\theta} &\leq C \left(\theta^{1/2} \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} + \frac{1}{2} (\theta^{1/2} + \theta^{1/3}) \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} \|\underline{v} + \tilde{v}\|_{\mathbb{Y}_\theta} \right. \\ &\quad \left. + \theta^\beta \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} + \theta^\beta \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} \|\underline{v} + \tilde{v}\|_{\mathbb{Y}_\theta} \right) \\ &\leq C \left((\theta^{1/2} + \theta^\beta) \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} + \frac{1}{2} (\theta^{1/2} + \theta^{1/3} + 2\theta^\beta) \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} 2R \right); \end{aligned}$$

404 then with (2.32)

$$405 \quad \|\Phi(\underline{v}) - \Phi(\tilde{v})\|_{\mathbb{Y}_\theta} \leq \left(\frac{1}{3} + \frac{1}{3} \right) \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta} = \frac{2}{3} \|\underline{v} - \tilde{v}\|_{\mathbb{Y}_\theta}.$$

406 That means that the map Φ is a contraction on $B_{\mathbb{Y}_\theta}$ and by the Banach fixed point
 407 theorem has a unique fixed point $\underline{u} \in \mathbb{Y}_\theta$. It gives the local-in-time well-posedness for
 408 bounded initial data. Now taking $T > 0$, we can check using integration by parts and
 409 the boundary conditions that every solution of (KdV-N) satisfies

$$410 \quad (2.33) \quad \frac{d}{dt} E(t) = - \left(\alpha - \frac{N}{2} \right) |u_1(t, 0)|^2 - \frac{1}{2} \sum_{n=1}^N |\partial_x u_n(t, 0)|^2 \leq 0$$

411 since $N \leq 2\alpha$. This dissipation law tells us that the energy is a nonincreasing function
 412 of the time variable, that means

$$413 \quad (2.34) \quad E(t) \leq E(\theta) \leq E(0) = \frac{1}{2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \quad \forall t > \theta > 0.$$

414 From here, taking the maximum for $t \in [0, T]$ we can see that

$$415 \quad (2.35) \quad \|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))} \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}.$$

416 Finally, following [16, 10] we multiply (KdV-N) by $q_n u_n$, integrate over $(0, T) \times (0, \ell_n)$,
 417 and sum over $n = 1, \dots, N$ to obtain the following equality:

$$418 \quad (2.36) \quad \begin{aligned} &\sum_{n=1}^N \int_0^{\ell_n} q_n(t, x) |u_n(t, x)|^2 dx \Big|_0^T - \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (\partial_t q_n + \partial_x q_n + \partial_x^3 q_n) |u_n|^2 dx dt \\ &\quad + 3 \sum_{n=1}^N \int_0^T \int_0^{\ell_n} |\partial_x u_n|^2 \partial_x q_n dx dt - \frac{2}{3} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} |u_n|^3 \partial_x q_n dx dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^N \int_0^T \left[(q_n + \partial_x^2 q_n) |u_n|^2 + 2q_n u_n \partial_x^2 u_n \right. \\
&\quad \left. - 2\partial_x q_n u_n \partial_x u_n - q_n |\partial_x u_n|^2 + \frac{2}{3} q_n |u_n|^3 \right] (t, 0) dt.
\end{aligned}$$

- Taking $q_n = 1$ in (2.36) we can derive

$$(2.37) \quad \sum_{n=1}^N \int_0^T |\partial_x u_n(t, 0)|^2 dt \leq \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2.$$

- If we take $q_n = \frac{x(2\ell_n - x)}{\ell_n^2}$ in (2.36), defining $L = \max_{n=1, \dots, N} \ell_n$ and $\ell = \min_{n=1, \dots, N} \ell_n$, we can obtain

$$\begin{aligned}
\frac{2N}{L^2} \|u_1(\cdot, 0)\|_{L^2(0, T)}^2 &\leq \frac{2T}{\ell^2} \|\underline{u}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))}^2 - 2 \int_0^T u_1(t, 0) \sum_{n=1}^N \partial_x u_n(t, 0) \frac{2}{\ell_n} dt \\
&\quad + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \frac{4}{3\ell} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} u_n^3(t, x) dx dt.
\end{aligned}$$

Using (2.35)–(2.37) and Young’s inequality we derive

$$(2.38) \quad \|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + C \sum_{n=1}^N \int_0^T \int_0^{\ell_n} u_n^3(t, x) dx dt.$$

As $H^1(0, \ell_n)$ embeds compactly into $C([0, \ell_n])$ we get

$$\sum_{n=1}^N \int_0^T \int_0^{\ell_n} |u_n|^3 dx dt \leq CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}$$

and then with (2.38)

$$(2.39) \quad \|u_1(t, 0)\|_{L^2(0, T)}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}.$$

- Finally, considering $q_j = x$ and using (2.35)–(2.37)–(2.39)

$$\|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq C(T+1) \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + CT^{1/2} \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \|\underline{u}\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))}.$$

Using Young’s inequality, we can find $C > 0$ which does not depend on $T > 0$ such that

$$(2.40) \quad \|\partial_x \underline{u}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 \leq C(T+1) \left(\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^4 \right),$$

which concludes the proof of Theorem 1.3. \square

To obtain a well-posedness result for the systems (KdV-S) and (LKdV-S) we can use the same idea presented in Theorem 1.3 and Lemma A.3 to take into account the saturation. It is very important that in Lemma A.3, time appears on the right-hand side; this estimate gives us the possibility of using small time in the fixed point approach. Then to derive the global-in-time well-posedness similar estimates to (2.35)–(2.40) can be obtained.

THEOREM 2.8. *Let $(\ell_n)_{n=1, \dots, N} \in (0, \infty)^N$, $\alpha \geq \frac{N}{2}$, and $T > 0$. Then, for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, there exists a unique solution $\underline{u} \in \mathbb{B}_T$ of (KdV-S) or (LKdV-S). Moreover, there exist $0 < T^* \leq T$ and $C > 0$ such that $\underline{u} \in \mathbb{Y}_{T^*}$ and $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.*

449 **3. Stabilization.** In this section, we are going to prove our stabilization results
 450 inspired by [14]. The proofs are based on observability inequalities for (KdV-S) and
 451 (LKdV-S), respectively. These inequalities imply the exponential stability. First, note
 452 that, given $T > 0$, we can check that every solution of (KdV-S) and (LKdV-S) has a
 453 nonincreasing energy,

(3.1)

$$454 \quad \frac{d}{dt} E(t) = - \left(\alpha - \frac{N}{2} \right) |u_1(t, 0)|^2 - \frac{1}{2} \sum_{n=1}^N |\partial_x u_n(t, 0)|^2 - \sum_{n=1}^N \int_0^{\ell_n} u_n \mathbf{sat}(a_n u_n) dx \leq 0.$$

455 **3.1. Stability of (KdV-S).** We start by studying (KdV-S). First, note that
 456 multiplying (KdV-S) by u_n and integrating on $(0, s) \times (0, \ell_n)$ we get

$$457 \quad \sum_{n=1}^N \int_0^{\ell_n} |u_n(s, x)|^2 dx + \sum_{n=1}^N \int_0^s \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt + (2\alpha - N) \int_0^s |u_1(t, 0)|^2 dt \\ + \sum_{n=1}^N \int_0^s |\partial_x u_n(t, 0)|^2 dt = \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2.$$

458 Integrating again this expression with respect to time on $(0, T)$ we obtain

$$459 \quad (3.2) \quad T \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq \int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + (2\alpha - N) T \int_0^T |u_1(t, 0)|^2 dt \\ + T \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt.$$

460 Our goal here is to prove the following observability inequality:

$$461 \quad (\text{Obs}) \quad \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq C \left((2\alpha - N) \int_0^T |u_1(t, 0)|^2 dt + \sum_{n=1}^N \int_0^T |\partial_x u_n(t, 0)|^2 dt \right. \\ \left. + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right).$$

462 Note that (Obs) is quite similar to (3.2). From (3.2) we can observe that to get (Obs)
 463 it is enough to prove the following inequality:

$$464 \quad \int_0^T \|\underline{u}(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt \leq C \left((2\alpha - N) \int_0^T |u_1(t, 0)|^2 dt + \sum_{n=1}^N \int_0^T |\partial_x u_n(t, 0)|^2 dt \right. \\ \left. + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right).$$

465 Suppose that it is false and take $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, then we can find $(\underline{u}^{0,j})_{j \in \mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$
 466 such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and

$$467 \quad \lim_{j \rightarrow \infty} \frac{\|\underline{u}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}^2}{(2\alpha - N) \|u_1^j(\cdot, 0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{u}^j(\cdot, 0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n^j) u_n^j dx dt} = \infty,$$

468 where \underline{u}^j is the corresponding solution of (KdV-S) with initial data $\underline{u}^{0,j}$. Note now
469 that using (2.33), we deduce

$$470 \quad (3.3) \quad \|\underline{u}^j(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq \|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R.$$

471 Take $\lambda^j = \|\underline{u}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$, then $\lambda^j \leq T^{1/2}\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq T^{1/2}R$. Thus $(\lambda^j)_{j \in \mathbb{N}} \subset \mathbb{R}$
472 is bounded. Taking $v_n^j = \frac{\underline{u}_n^j}{\lambda^j}$, then \underline{v}^j fulfills

$$473 \quad (3.4) \quad \begin{cases} \left(\partial_t v_n^j + \partial_x v_n^j + \partial_x^3 v_n^j + \lambda^j v_n^j \partial_x v_n^j + \frac{\mathfrak{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right) (t, x) = 0 & \forall x \in (0, \ell_n), t > 0, \\ & n = 1, \dots, N, \\ v_n^j(t, 0) = v_{n'}^j(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n^j(t, 0) = -\alpha v_1^j(t, 0) - \lambda^j \frac{N}{3} (v_1^j(t, 0))^2, & t > 0, \\ v_n^j(t, \ell_n) = \partial_x v_n^j(t, \ell_n) = 0, & t > 0, n = 1, \dots, N, \\ \|\underline{v}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1, \end{cases}$$

474 and satisfies

$$475 \quad (3.5) \quad (2\alpha - N)\|v_1^j(t, 0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}(a_n \lambda^j v_n^j) v_n^j dx dt \rightarrow 0.$$

476 First, note that multiplying (3.4) by v_n^j and integrating on $(0, s) \times (0, \ell_n)$ we get

$$477 \quad (3.6) \quad \begin{aligned} & \sum_{n=1}^N \int_0^{\ell_n} |v_n^j(s, x)|^2 dx + \sum_{n=1}^N \int_0^s \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}(a_n \lambda^j v_n^j) v_n^j dx dt + (2\alpha - N) \int_0^s |v_1^j(t, 0)|^2 dt \\ & + \sum_{n=1}^N \int_0^s |\partial_x v_n^j(t, 0)|^2 dt = \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2, \end{aligned}$$

478 which gives us, using that \mathfrak{sat} is odd,

$$479 \quad (3.7) \quad \|\underline{v}^j\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))}^2 \leq \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2, \quad \|\partial_x \underline{v}^j(t, 0)\|_{L^2(0,T)}^2 \leq \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2.$$

480 Now integrating (3.6) again with respect to time on $(0, T)$ we obtain

$$481 \quad (3.8) \quad \begin{aligned} T\|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 & \leq \int_0^T \|\underline{v}^j(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 dt + (2\alpha - N)T \int_0^T |v_1^j(t, 0)|^2 dt \\ & + T \sum_{n=1}^N \int_0^T |\partial_x v_n^j(t, 0)|^2 dt + 2T \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}(a_n \lambda^j v_n^j) v_n^j dx dt. \end{aligned}$$

482 This last inequality implies that $(\underline{v}^j(0, \cdot))_{j \in \mathbb{N}}$ is bounded in $\mathbb{L}^2(\mathcal{T})$. Again using that
483 \mathfrak{sat} is odd and similar estimates in (2.37)–(2.39)–(2.40) we conclude

$$484 \quad (3.9) \quad \|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}^2 \leq C \left(\|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^2 + \|\underline{v}^j(0, \cdot)\|_{\mathbb{L}^2(\mathcal{T})}^4 \right).$$

485 Thus $(\underline{v}^j)_{j \in \mathbb{N}} \subset L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))$ is bounded and it holds that

$$486 \quad \|\underline{v}_n^j \partial_x v_n^j\|_{L^2(0,T;L^1(0,\ell_n))} \leq \|\underline{v}^j\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} \|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))},$$

487 which implies that $(v_n^j \partial_x v_n^j)_{j \in \mathbb{N}}$ is a subset of $L^2(0, T; L^1(0, \ell_n))$. Using Lemma A.1
488 we have

$$489 \quad \left\| \frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right\|_{L^2(0, T; L^2(0, \ell_n))} \leq 3 \|a_n\|_{L^\infty(0, \ell_n)} \ell_n^{1/2} \|\underline{v}^j\|_{L^2(0, T; \mathbb{H}_e^1(\mathcal{T}))},$$

490 and then $(\frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j})_{j \in \mathbb{N}}$ is a subset of $L^2(0, T; L^2(0, \ell_n))$. From this, we can see that
491 $\partial_t v_n^j = -(\partial_x^3 v_n^j + \partial_x v_n^j + \lambda^j v_n^j \partial_x v_n^j + \frac{\mathbf{sat}(a_n \lambda^j v_n^j)}{\lambda^j})$ is bounded in $L^2(0, T; H^{-2}(0, \ell_n))$.
492 Hence, by the Aubin–Lions lemma ([24, Chapter III, Proposition 1.3]) we can deduce
493 that $(\underline{v}^j)_{j \in \mathbb{N}}$ is relatively compact in $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ and we can assume that \underline{v}^j
494 converges strongly at \underline{v} in $L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ with $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$. Now we are
495 going to study the case $\mathbf{sat} = \mathbf{sat}_2$ and $\mathbf{sat} = \mathbf{sat}_{1oc}$ separately.

496 **3.1.1. Case $\mathbf{sat} = \mathbf{sat}_2$.** First, we consider the case $\mathbf{sat} = \mathbf{sat}_2$. We know that
497 by (3.3), $\|\underline{v}^j(t, \cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and then by Lemma A.2 we have that

$$498 \quad 0 \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathbf{sat}_2(a_n \lambda^j v_n^j) v_n^j,$$

499 which gives us using (3.5), as $j \rightarrow \infty$,
(3.10)

$$500 \quad (2\alpha - N) \|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x v_1^j(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt \rightarrow 0.$$

501 Furthermore, passing to the limit in (3.10) we get

$$\begin{aligned} & (2\alpha - N) \|v_1(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x v_1(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) |v_n|^2 dx dt \\ 502 \quad & \leq \liminf \left((2\alpha - N) \|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x v_1^j(t, 0)\|_{L^2(0, T)}^2 \right. \\ & \quad \left. + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt \right) = 0. \end{aligned}$$

503 Thus, $v_n(t, x) = 0$ in $(0, T) \times \omega_n$ and $(2\alpha - N)v_1(t, 0) = \partial_x v_1(t, 0) = 0$ in $(0, T)$
504 for all $n = 1, \dots, N$. Additionally, as $(\lambda^j)_{j \in \mathbb{N}}$ is bounded and nonnegative, we can
505 extract a convergent subsequence such that $\lambda^j \rightarrow \lambda \geq 0$, consequently \underline{v} satisfies
506 $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$ and solves the following system:

$$507 \quad (3.11) \quad \begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n + \lambda v_n \partial_x v_n = 0 & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = \partial_x v_n(t, 0) = 0, & t \in (0, T) \forall n = 1, \dots, N, \\ (2\alpha - N)v_n(t, 0) = 0, & t \in (0, T), \\ v_n(t, x) = 0, & (t, x) \in (0, T) \times \omega_n. \end{cases}$$

508 1. If $\lambda = 0$ the system satisfied by \underline{v} is linear, then we can use Holmgren's
509 theorem as in [18] to conclude that $\underline{v} = 0$, which contradicts the fact that
510 $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.

511 2. If $\lambda > 0$, we have to prove that $v_n \in L^2(0, T; H^3(0, \ell_n))$ in order to apply [23,
512 Theorem 4.2]. Consider $w_n = \partial_t v_n$ then

$$513 \left\{ \begin{array}{l} \partial_t w_n + \partial_x w_n + \partial_x^3 w_n + \lambda w_n \partial_x v_n + \lambda v_n \partial_x w_n = 0 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ w_n(t, \ell_n) = \partial_x w_n(t, \ell_n) = \partial_x w_n(t, 0) = 0 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \forall n = 1, \dots, N, \\ (2\alpha - N)w_n(t, 0) = 0, \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad t \in (0, T) \forall j = 1, \dots, N, \\ w_n(t, x) = 0, \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad (t, x) \in (0, T) \times \omega_n, \\ w_n(0, x) = -v'_n(0, x) - v_n'''(0, x) - \lambda v_n(0, x)v'_n(0, x) \in H^{-3}(0, \ell_n), \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad x \in (0, \ell_n), j = 1, \dots, N. \end{array} \right.$$

514 With [9, Lemma A.2] we can get that $w_n(0, x) \in L^2(0, \ell_n)$ and $w_n \in C([0, T],$
515 $L^2(0, \ell_n)) \cap L^2(0, T; H^1(0, \ell_n))$. Thus, $\partial_x^3 v_n = -(\partial_t v_n - \partial_x v_n - \lambda v_n \partial_x v_n) \in$
516 $L^2(0, T; L^2(0, \ell_n))$ which implies $v_n \in L^2(0, T; H^3(0, \ell_n))$. Applying [23, The-
517 orem 4.2] we obtain that $v_n = 0$ for all $j = 1, \dots, N$ that contradicts the fact
518 that $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$.

519 **3.1.2. Case $\mathfrak{sat} = \mathfrak{sat}_{\text{loc}}$.** Let us consider the case where $\mathfrak{sat} = \mathfrak{sat}_{\text{loc}}$, by the
520 injection of $H^1(0, \ell_n)$ into $C([0, \ell_n])$, we can derive using similar estimate as in (3.9),

$$521 (3.12) \quad \int_0^T |u_n^j(t, x)|^2 dt \leq \ell_n \|\underline{u}^j\|_{L^2(0, T; \mathbb{H}_x^1(\mathcal{T}))}^2 \leq \ell_n \beta$$

522 for $\beta = (R^2 + R^4)$. Now, inspired by [14], take $\Omega_{n,i} \subset [0, T]$ defined as follows:

$$523 (3.13) \quad \Omega_{n,i} = \left\{ t \in [0, T], \sup_{x \in [0, \ell_n]} |u_n(t, x)| > i \right\}.$$

524 Then denote $\Omega_{n,i}^c$ as the complement of $\Omega_{n,i}$ and observe that

$$525 \int_0^T \sup_{x \in [0, \ell_n]} |u_n^j(t, x)|^2 dt \geq \int_{\Omega_{n,i}} \sup_{x \in [0, \ell_n]} |u_n^j(t, x)|^2 dt \geq i^2 \nu(\Omega_{n,i})$$

526 for $\nu(\Omega_{n,i})$ the Lebesgue measure of $\Omega_{n,i}$. Thus, using (3.12) we obtain $\nu(\Omega_{n,i}) \leq \frac{\ell_n \beta}{i^2}$.

527 Hence,

$$528 (3.14) \quad \max \left(T - \frac{\ell_n \beta}{i^2}, 0 \right) \leq \nu(\Omega_{n,i}^c) \leq T.$$

529 Now using Lemma A.2

$$\begin{aligned} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j dx dt &= \sum_{n=1}^N \int_{\Omega_{n,i}} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j dx dt \\ &\quad + \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j \\ 530 &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\text{loc}}(a_n \lambda^j v_n^j) v_n^j \\ &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt, \end{aligned}$$

531 which gives us, using (3.5),
 (3.15)

$$532 \quad (2\alpha - N)\|v_1^j(t, 0)\|_{L^2(0, T)}^2 + \|\partial_x v^j(t, 0)\|_{L^2(0, T)}^2 + \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt \rightarrow 0.$$

533 Thus, the limit function v satisfies $(2\alpha - N)v_1(t, 0) = \partial_x v_n(t, 0) = 0$ in $(0, T)$ for
 534 all $n = 1, \dots, N$ and $v_n(t, x) = 0$ in $\cup_{i \in \mathbb{N}} \Omega_{n,i}^c \times \omega_n$. Using (3.14), we know that
 535 $\nu(\cup_{i \in \mathbb{N}} \Omega_{n,i}^c) = T$, thus we get that, for almost every $t \in [0, T]$, $v_n(t, x) = 0$ for $x \in \omega_n$.
 536 Last v is a solution to (3.11) and we conclude as we do in the case $\mathbf{sat} = \mathbf{sat}_2$.

537 Finally, we obtain that (Obs) is valid for a solution (KdV-S) with $\|u_n^n\|_{\mathbb{L}^2(\mathcal{T})} \leq R$.

538 **Proof of Theorem 1.4.** The proof closely follows [16] (see also [14]). Note that
 539 for $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ using that the energy is nonincreasing using
 540 (3.1) and (Obs) we argue the existence of $C = C(R) > 0$ such that.

$$541 \quad (3.16) \quad E(T) \leq \gamma E(0) \quad \text{with } \gamma = \frac{C}{1+C} < 1.$$

542 Now as the system is invariant by translation in time, we can repeat this argument
 543 on $[(m-1)T, mT]$ for $m = 1, 2, \dots$ to obtain

$$544 \quad E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

545 Hence we have $E(mT) \leq e^{-\mu m T} E(0)$, where $\mu = \frac{1}{T} \ln(\frac{1}{\gamma}) > 0$. Let $t > 0$ then there
 546 exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and then again using the nonincreasing
 547 property of the energy we get

$$548 \quad E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

549 This concludes the proof of Theorem 1.4. \square

550 **3.2. Stability (LKdV-S).** Now we study the stabilization of (LKdV-S). For
 551 doing that, we follow the approach of section 3.1, and we prove the following observ-
 552 ability inequality

$$553 \quad (\text{Obs2}) \quad \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq C \left((2\alpha - N) \int_0^T |u_1(t, 0)|^2 dt + \sum_{j=1}^N \int_0^T |\partial_x u_n(t, 0)|^2 dt \right. \\ \left. + \sum_{j \in I_c^*} \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n) u_n dx dt \right)$$

554 for any solution \underline{u} of (LKdV-S). Suppose that it is false, then there exists a se-
 555 quence $(\underline{u}^{0,j})_{j \in \mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and the corresponding solution
 556 of (LKdV-S) satisfies

$$557 \quad (2\alpha - N)\|u_1^j(\cdot, 0)\|_{L^2(0, T)}^2 + \|\partial_x u^j(\cdot, 0)\|_{L^2(0, T)}^2 + \sum_{n \in I_c^*} \int_0^T \int_0^{\ell_n} \mathbf{sat}(a_n u_n^j) u_n^j dx dt \rightarrow 0,$$

558 as $j \rightarrow \infty$. Using the same arguments as in Theorem 1.4 we can find a nontrivial
 559 solution $\underline{v} \in \mathbb{B}_T$ of (LKdV-S) such that

$$560 \quad \begin{cases} (2\alpha - N) \|v_1(\cdot, 0)\|_{L^2(0, T)} = 0, \\ \|\partial_x v(\cdot, 0)\|_{L^2(0, T)} = 0, \\ v_n = 0 \quad \text{in } (0, T) \times \omega_n, \quad n \in I_c^*, \\ \|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1. \end{cases}$$

561 We distinguish three cases:

- 562 • For $n \in I_c^*$, $v_n = 0$ in $(0, T) \times \omega_n$. Then, $\partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0$ and thanks
563 to Holmgren's theorem, $v_n = 0$ for all $n \in I_c^*$. Note that this implies that
564 $v_n(t, 0) = 0$ for all $n \in I_c^*$ and by the continuity condition $v_n(t, 0) = 0$ for all
565 $n = 1, \dots, N$.
- 566 • For $n \in \{1, \dots, N\} \setminus I_c$, v_n is the solution to

$$567 \quad \begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0, & x \in (0, \ell_n), t \in (0, T), n = 1, \dots, N, \\ v_n(t, 0) = 0, & t \in (0, T) \forall j = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 v_n(t, 0) = 0, & t \in (0, T), \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = 0, & t \in (0, T), n = 1, \dots, N, \\ v_n(0, x) = v_n^0, & x \in (0, \ell_n). \end{cases}$$

568 Then thanks to [1, Lemma 3.2], $v_n = 0$.

- 569 • For $n \in I_c \setminus I_c^*$, v_n then satisfies

$$570 \quad \begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0, & t \in (0, T) \forall x \in (0, \ell_n), \\ v_n(t, 0) = \partial_x v_n(t, 0) = \partial_x^2 v_n(t, 0) = 0, & t \in (0, T), \\ v_n(t, \ell_n) = \partial_x v_n(t, \ell_n) = 0, & t \in (0, T), \\ v_n(0, x) = v_n^0, & x \in (0, \ell_n). \end{cases}$$

571 Due to the three null conditions at the central node, we obtain that $v_n = 0$.

572 Thus $\underline{v} = 0$ and we get a contradiction, with $\|\underline{v}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))} = 1$ which ends the proof
573 of (Obs2). As we have the observability inequality (Obs2), to derive the exponential
574 decay of the energy of (LKdV-S) given in Theorem 1.5, it is enough to follow the
575 proof of Theorem 1.4.

576 **4. Conclusions and remarks.** In this paper, the global well-posedness was
577 studied and the exponential stability of a KdV equation on a star-shaped network
578 with internal saturated feedback terms has been established. The well-posedness
579 was addressed using the Laplace transform of the linearization and obtaining Kato
580 smoothing properties which gave the local-in-time well-posedness, then using multi-
581 pplier estimates the global-in-time result was deal with.

582 **4.1. Generalization of the well-posedness result.** In the work [7] a complete
583 result for general linear boundary conditions for the KdV equation on a bounded
584 domain was derived. In this work, homogeneous Dirichlet and Neumann right condi-
585 tions ($u_n(t, \ell_n) = \partial_x u_n(t, \ell_n) = 0$) were considered. These conditions come from the
586 problems studied in [1, 16], but in a more general framework the following problem
587 could be studied:

$$588 \quad (4.1) \quad \begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t, x) = f_n(t, x), & \forall x \in (0, \ell_n), t > 0, n = 1, \dots, N, \\ u_n(t, 0) = u_{n'}(t, 0) & \forall n, n' = 1, \dots, N, \\ \sum_{n=1}^N \partial_x^2 u_n(t, 0) = -\alpha u_1(t, 0) - \frac{N}{3} u_1^2(t, 0) + h(t), & t > 0, \\ u_n(t, \ell_n) = g_n(t), \quad \partial_x u_n(t, \ell_n) = p_n(t), & t > 0, n = 1, \dots, N, \\ u_n(0, x) = u_n^0, & x \in (0, \ell_n). \end{cases}$$

590 We expected that adapting the ideas introduced in this paper and in [3], it could be
591 possible to obtain the following result.

592 CONJECTURE 4.1. Let $(\ell_n)_{n=1,\dots,N} \in (0, \infty)^N$, $0 \leq s \leq 3$, and $T > 0$. There
 593 exists $0 < T^* \leq T$ such that for all

$$\begin{aligned} \underline{u}^0 \in \prod_{n=1}^N H^s(0, \ell_n), \quad (h, \underline{g}, \underline{p}) \in H^{\frac{s-1}{3}}(0, T) \times \prod_{n=1}^N H^{\frac{s+1}{3}}(0, T) \times \prod_{n=1}^N H^{\frac{s}{3}}(0, T), \\ \underline{f} \in \prod_{n=1}^N W^{\frac{s}{3}, 1}(0, T; L^2(0, \ell_n)), \end{aligned}$$

595 satisfying the compatibility condition,

$$\begin{cases} u_n^0(\ell_n) = g_n(0) & n = 1, \dots, N \text{ if } \frac{1}{2} < s \leq 3, \\ \partial_x u_n^0(\ell_n) = p_n(0) & n = 1, \dots, N \text{ if } \frac{3}{2} < s \leq 3, \\ \sum_{n=1}^N \partial_x^2 u_n^0(0) = h(0) & \text{if } \frac{5}{2} < s \leq 3, \end{cases}$$

597 there exists a unique solution $\underline{u} \in \prod_{n=1}^N C([0, T]; H^s(0, \ell_n)) \cap L^2(0, T^*; H^{s+1}(0, \ell_n))$
 598 of (4.1). Moreover $\partial_x^\kappa u_n \in L_x^\infty(0, \ell_n; H^{\frac{s+1-\kappa}{3}}(0, T^*))$ for $\kappa = 0, 1, 2$.

599 The complications would come from the study of the matrix, which is obtained
 600 by replacing the column $j + 3(n - 1)$ of A_N by $[0 \ 1 \ 0 \ \dots \ 0]^T$ for the g_n case and
 601 $[0 \ 0 \ 1 \ \dots \ 0]^T$ for the p_n case. It is not clear how to derive a result similar to (2.23).

602 **4.2. Exact controllability in the network.** In the paper [1] the exact con-
 603 trollability of linearization around 0 of (KdV-N) was achieved by acting with $N + 1$
 604 boundary controls (N controls in the external nodes and one in the central node) if
 605 $\#\{\ell_n \in \mathcal{N}\} \leq 1$. Recently, in [10] the authors could reduce the numbers of controls
 606 (N controls acting on the external nodes), but the controllability holds for a large
 607 time and small lengths. This raises the question of what happens for the boundary
 608 control and how many components corresponding to the critical lengths one needs
 609 to control in the network case. In particular, we can mention the following open
 610 problems:

- 611 • Is the linearization around 0 of (KdV-N) exactly controllable with N controls
 612 acting in the external nodes for $T > 0$ and $\ell_n \notin \mathcal{N}$ for all $n \in \{1, \dots, N\}$?
- 613 • Is (KdV-N) exactly controllable from the boundary in the case where for some
 614 lengths we have $\ell_n \in \mathcal{N}$? A starting point could be, to consider the smallest
 615 critical lengths ($k = l = 1$ or $k = l = 2$).

616 **4.3. Generalization of stabilization results.** The stabilization results were
 617 obtained, proving appropriate observability inequalities working directly on the non-
 618 linear systems. In the work [19] more general feedback laws were considered as cone
 619 bounded control laws. Note that Theorems 1.4 and 1.5 hold, replacing **sat** by any odd
 620 nonlinearity that satisfies the properties given in Lemmas A.1, A.2, and A.3.

621 **Appendix A. Useful lemmas.** In this section, we present some technical
 622 lemmas about the regularity and sector condition of the saturation maps **sat**. Let
 623 $a : [0, L] \rightarrow \mathbb{R}$ such that

$$(A.1) \quad a^* \geq a \geq a_* > 0 \text{ in an open nonempty set } \omega \text{ of } (0, L).$$

624 LEMMA A.1 (Lemma 3.2 of [14]). For all $(f, \tilde{f}) \in L^2(0, L)$, we have

$$625 \quad (\text{A.2}) \quad \|\mathbf{sat}(f) - \mathbf{sat}(\tilde{f})\|_{L^2(0,L)} \leq 3\|f - \tilde{f}\|_{L^2(0,L)}.$$

626 LEMMA A.2 (Lemma 4.3 of [14]). Let r be a positive value and $a : [0, L] \rightarrow \mathbb{R}$ be
627 a function satisfying (A.1) and $k(r)$ defined by

$$628 \quad (\text{A.3}) \quad k(r) = \min \left\{ \frac{M}{a_* r}, 1 \right\} :$$

629 1. Given $\mathbf{sat} = \mathbf{sat}_2$ and $f \in L^2(0, L)$ such that $\|f\|_{L^2(0,L)} \leq r$, we have

$$630 \quad (\text{A.4}) \quad (\mathbf{sat}_2(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0 \quad \forall x \in [0, L].$$

631 2. Given $\mathbf{sat} = \mathbf{sat}_{loc}$ and $f \in L^\infty(0, L)$ such that $\forall x \in [0, L]$, $|f(x)| \leq r$, we
632 have

$$633 \quad (\text{A.5}) \quad (\mathbf{sat}_{loc}(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0 \quad \forall x \in [0, L].$$

634 LEMMA A.3 (Proposition 3.4 of [14]). Let $a : [0, L] \rightarrow \mathbb{R}$ satisfy (A.1). If
635 $y \in L^2(0, T; H^1(0, L))$, then $\mathbf{sat}(ay) \in L^1(0, T; L^2(0, L))$ is continuous and $\forall y, z \in$
636 $L^2(0, T; H^1(0, L))$ we have

$$637 \quad \|\mathbf{sat}(ay) - \mathbf{sat}(az)\|_{L^1(0,T;L^2(0,L))} \leq 3L^{1/2}T^{1/2}a^*\|y - z\|_{L^2(0,T;H^1(0,L))}.$$

638 **Appendix B. For all $s \neq 0$ with $\text{Re}(s) \geq 0$, it holds that $\Delta^1(s) \neq 0$.**

639 This property was stated in [7, Remark 2.5] without proof; here, for the sake of
640 completeness, we give a proof based on [6]. Suppose that $\Delta^1(s) = 0$ for some s with
641 $\text{Re}(s) \geq 0$. Then, there exists $f \in H^3(0, \ell_1)$, a nontrivial solution of

$$642 \quad (\text{B.1}) \quad \begin{cases} sf(x) + f'''(x) = 0, & x \in (0, \ell_1), \\ f''(0) = f'(\ell_1) = f(\ell_1) = 0. \end{cases}$$

643 Now, consider the conjugate of (B.1):

$$644 \quad (\text{B.2}) \quad \begin{cases} \overline{sf(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_1), \\ \overline{f''(0)} = \overline{f'(\ell_1)} = \overline{f(\ell_1)} = 0. \end{cases}$$

645 Multiplying (B.1) by \bar{f} , integrating over $(0, \ell_1)$, and performing integration by parts,
646 we get

$$647 \quad (\text{B.3}) \quad s \int_0^{\ell_1} |f|^2 dx - \int_0^{\ell_1} f \bar{f}''' dx + |f'(0)|^2 = 0.$$

648 Similarly, multiplying (B.2) by f and integrating over $(0, \ell_1)$, we get

$$649 \quad (\text{B.4}) \quad \bar{s} \int_0^{\ell_1} |f|^2 dx + \int_0^{\ell_1} \bar{f}''' f dx = 0.$$

650 Then adding (B.3) and (B.4) yields

$$651 \quad (\text{B.5}) \quad 2\text{Re}(s) \int_0^{\ell_1} |f|^2 dx = -|f'(0)|^2.$$

652 As f is nontrivial and $\operatorname{Re}(s) \geq 0$, we get $f'(0) = 0$. Then, by (B.5) $\operatorname{Re}(s) = 0$. Thus,
 653 we can make the change of variable $s = i\rho^3$ for $\rho \in \mathbb{R}$. Multiplying (B.1) by $x\bar{f}$,
 654 integrating over $(0, \ell_1)$, and performing integration by parts, we get

$$655 \quad (B.6) \quad i\rho^3 \int_0^{\ell_j} x|f|^2 dx + 3 \int_0^{\ell_j} |f'|^2 dx - \int_0^{\ell_j} x f \overline{f'''} dx = 0.$$

656 Similarly, multiplying (B.2) by xf and integrating over $(0, \ell_j)$, we get

$$657 \quad (B.7) \quad -i\rho^3 \int_0^{\ell_j} x|f|^2 dx + \int_0^{\ell_j} x \overline{f'''} f dx = 0.$$

658 Then, adding (B.6) and (B.7), we obtain $f' \equiv 0$. Using the boundary conditions of
 659 (B.1) we deduce $f \equiv 0$ which is a contradiction. Finally $f \equiv 0$ and $\Delta^1(s) \neq 0$ for all
 660 $s \neq 0$ with $\operatorname{Re}(s) \geq 0$.

661 **Appendix C. For all $\rho > 0$ and $j \in \{1, \dots, N\}$, it holds that $\det(D_j) \neq 0$.**

662 Let $j \in \{1, \dots, N\}$. Following [6] and Appendix B, suppose that $\det(D_j) = 0$ for
 663 some $\rho > 0$. Then, there exists $f \in H^3(0, \ell_j)$, a nontrivial solution of

$$664 \quad (C.1) \quad \begin{cases} i\rho^3 f(x) + f'''(x) = 0, & x \in (0, \ell_j), \\ f(0) = f(\ell_j) = f'(\ell_j) = 0. \end{cases}$$

665 Now, consider the conjugate of (C.1),

$$666 \quad (C.2) \quad \begin{cases} -i\rho^3 \overline{f(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_j), \\ \overline{f(0)} = \overline{f(\ell_j)} = \overline{f'(\ell_j)} = 0. \end{cases}$$

667 Multiplying (C.1) by \bar{f} , integrating over $(0, \ell_j)$, and performing integration by parts,
 668 we get

$$669 \quad (C.3) \quad i\rho^3 \int_0^{\ell_j} |f|^2 dx - \int_0^{\ell_j} f \overline{f'''} dx + |f'(0)|^2 = 0.$$

670 Similarly, multiplying (C.2) by f and integrating over $(0, \ell_j)$, we get

$$671 \quad (C.4) \quad -i\rho^3 \int_0^{\ell_j} |f|^2 dx + \int_0^{\ell_j} \overline{f'''} f dx = 0.$$

672 Then, adding (C.3) and (C.4) yields $f'(0) = 0$. Multiplying (C.1) by $x\bar{f}$, integrating
 673 over $(0, \ell_j)$, and performing integration by parts, we get

$$674 \quad (C.5) \quad i\rho^3 \int_0^{\ell_j} x|f|^2 dx + 3 \int_0^{\ell_j} |f'|^2 dx - \int_0^{\ell_j} x f \overline{f'''} dx = 0.$$

675 Similarly, multiplying (C.2) by xf and integrating over $(0, \ell_j)$, we get

$$676 \quad (C.6) \quad -i\rho^3 \int_0^{\ell_j} x|f|^2 dx + \int_0^{\ell_j} x \overline{f'''} f dx = 0.$$

677 Then, adding (C.5) and (C.6), we obtain $f' \equiv 0$. Using the boundary conditions of
 678 (C.1) we deduce $f \equiv 0$ which is a contradiction. Hence, $\det(D_j) \neq 0$ for all $\rho > 0$.

679 **Appendix D.** For all $\rho > 0$, it holds that $\sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} \neq 0$. Letting

680 $j \in \{1, \dots, N\}$, we are going to show that $\operatorname{Re} \left(\frac{\det(F_j)}{\det(D_j)} \right) < 0$. Using (2.20) and
681 (2.21) we get

$$682 \frac{\det(F_j)}{\det(D_j)} = \frac{\sqrt{3}\rho^3 \left(e^{-i\rho\ell_j} + e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_j} + e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_j} \right)}{\sqrt{3}\rho \left(e^{-i\rho\ell_j} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) e^{(-\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_j} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) e^{(\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2})\ell_j} \right)}$$

$$683 = \frac{\rho^2 \left(e^{-i\ell_j\rho} + 2e^{\frac{i\ell_j\rho}{2}} \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) \right)}{e^{-i\ell_j\rho} - e^{\frac{i\ell_j\rho}{2}} \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) + \sqrt{3}ie^{\frac{i\ell_j\rho}{2}} \sinh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right)}.$$

685 After some algebraic manipulations and writing the complex numbers in their binomial
686 form ($\operatorname{Re} + i\operatorname{Im}$), we obtain

$$687 \frac{\det(F_j)}{\det(D_j)} = \frac{\rho^2 \left(\cos \left(\frac{3\ell_j\rho}{2} \right) + 2 \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) - i \sin \left(\frac{3\ell_j\rho}{2} \right) \right)}{\cos \left(\frac{3\ell_j\rho}{2} \right) - \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) + i \left(\sqrt{3} \sinh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) - \sin \left(\frac{3\ell_j\rho}{2} \right) \right)}$$

688 Letting $\zeta = \cos \left(\frac{3\ell_j\rho}{2} \right) - \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) + i \left(\sqrt{3} \sinh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) - \sin \left(\frac{3\ell_j\rho}{2} \right) \right)$, and mul-
689 tipling the previous equation by $\frac{\bar{\zeta}}{\zeta}$ we get

$$690 \operatorname{Re} \left(\frac{\det(F_j)}{\det(D_j)} \right) = \frac{\rho^2}{|\zeta|^2} \left(1 + \cos \left(\frac{3\ell_j\rho}{2} \right) \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) - 2 \cosh^2 \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) \right.$$

$$\left. - \sqrt{3} \sin \left(\frac{3\ell_j\rho}{2} \right) \sinh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) \right).$$

691 By analyzing the function

$$692 F(\rho, \ell_j) = 1 + \cos \left(\frac{3\ell_j\rho}{2} \right) \cosh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right) - 2 \cosh^2 \left(\frac{\sqrt{3}\ell_j\rho}{2} \right)$$

$$693 - \sqrt{3} \sin \left(\frac{3\ell_j\rho}{2} \right) \sinh \left(\frac{\sqrt{3}\ell_j\rho}{2} \right),$$

695 it can be shown that for all $\rho > 0$, $\ell_j > 0$ it holds that $F(\rho, \ell_j) < 0$. Thus,
696 $\operatorname{Re} \left(\sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} \right) < 0$, and thus $\sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} \neq 0$.

697 *Remark 4.* In the case $\ell_1 = \dots = \ell_N$, the proof become easier. In fact,

$$698 \sum_{j=1}^N \frac{\det(F_j)}{\det(D_j)} = N \frac{\det(F_1)}{\det(D_1)} \neq 0$$

699 because, $\det(F_1) = \Delta^{1,+} \neq 0$ thanks to Appendix B. ◻

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