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Nonlinear Helmholtz equations with sign-changing diffusion coefficient

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Abstract

We study nonlinear Helmholtz equations with sign-changing diffusion coefficients on bounded domains. The existence of an orthonormal basis of eigenfunctions is established making use of weak T-coercivity theory. All eigenvalues are proved to be bifurcation points and the bifurcating branches are investigated both theoretically and numerically. In a one-dimensional model example we obtain the existence of infinitely many bifurcating branches that are mutually disjoint, unbounded, and consist of solutions with a fixed nodal pattern. We also extend the numerics to a Drude model.

Keywords: Helmholtz equation; Bifurcation Theory; Sign-changing; T-coercivity.

1 Problem setting

We are interested in the nonlinear Helmholtz equations in dimension $N \in \{1, 2, 3\}$ of the form:

$$-\operatorname{div}(\sigma(x)\nabla u) - \lambda c(x)u = \kappa(x)u^3, \quad \text{in } \Omega, \quad (1)$$

for $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ where Ω is a bounded open domain of \mathbb{R}^N , the diffusion coefficient $\sigma \in L^\infty(\Omega)$ is sign-changing, $0 < c \in L^\infty(\Omega)$, and $\kappa \in L^\infty(\Omega)$. Being sign-changing means that the domain Ω is partitioned in two open subdomains Ω_\pm such that $\overline{\Omega_-} \cup \overline{\Omega_+} = \overline{\Omega}$, $\Omega_- \cap \Omega_+ = \emptyset$, and the function σ is negative on Ω_- and positive on Ω_+ . The main goal is to detect nontrivial solutions of Eq. (1) that bifurcate from the trivial solution family $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$.

Equation (1) occurs in the study of time-harmonic wave propagation across an interface between a dielectric and a metamaterial with negative permeability and nonlinear Kerr-type permittivity [3]. Those two effects have been studied separately. The term κu^3 is a classic manifestation of the nonlinear Kerr-type permittivity. In the case of positive diffusion coefficients σ on the whole domain, it is well known

that all the eigenvalues give rise to bifurcation branches. The metamaterial property manifests by having σ negative on the subdomain Ω_- . The linear theory dealing with the well-posedness of such problems for right-hand sides $f(x)$ instead of $\kappa(x)u^3$ has been studied both analytically and numerically [1]. The main difficulty with the indefinite operator $u \mapsto -\operatorname{div}(\sigma(x)\nabla u)$ is that the standard theory for elliptic boundary value problems based on the Lax-Milgram Lemma does not apply. The (weak) T-coercivity has been introduced to recover a linear theory in the Fredholm sense. We assume that the operator $u \mapsto -\operatorname{div}(\sigma(x)\nabla u)$ is weakly T-coercive, which means that there exists an isomorphism $\mathbf{T} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ and a compact operator $\mathbf{K} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$(u, v) \mapsto \int_{\Omega} \sigma \nabla u \cdot \nabla(\mathbf{T}v) + \int_{\Omega} \nabla(\mathbf{K}u) \cdot \nabla v$$

is coercive. Being weakly T-coercive depends on the precise shape of the interface $\partial\Omega_- \cap \partial\Omega_+$ and jumps of σ across the interface but, in dimension 1 and 2, it is known to be the case in many settings.

2 Main results

The weak T-coercivity ensured the existence of an orthonormal basis $(\phi_j)_{j \in \mathbb{Z}}$ consisting of eigenfunctions associated to the linear differential operator $u \mapsto -c^{-1} \operatorname{div}(\sigma \nabla u)$. Due to the sign-change of σ the corresponding sequence of eigenvalues $(\lambda_j)_{j \in \mathbb{Z}}$ satisfy $\lambda_j \rightarrow \pm\infty$ as $j \rightarrow \pm\infty$. Using some recent bifurcation result for strongly indefinite operators needed because of the sign-change of σ , we can show [3]:

Theorem 1 *Each eigenpair (λ_j, ϕ_j) is a bifurcation point of Eq. (1). If λ_j has an odd geometric multiplicity then the connected component $\mathcal{C}_j \subset \mathbb{R} \times H_0^1(\Omega)$ containing $(\lambda_j, 0)$ satisfies Rabinowitz' alternative:*

- (1) \mathcal{C}_j is unbounded in $\mathbb{R} \times H_0^1(\Omega)$;
- (2) \mathcal{C}_j contains another trivial solution $(\lambda_i, 0)$.

In a 1D setting with the subdomain $\Omega_- = (a_-, 0)$ and $\Omega_+ = (0, a_+)$, and piecewise constant functions σ, c , we can strengthen our result. Using the almost explicit expression of the eigenvalues, we can use the distribution of zeros of the eigenfunction to show:

Corollary 2 *The connected component $\mathcal{C}_j \subset \mathbb{R} \times H_0^1(\Omega)$ is unbounded and $\mathcal{C}_j \cap \mathcal{C}_i = \emptyset$ for $i \neq j$.*

Using additional assumptions, see [3, Theorem 6.3], which has been verified in the 1D settings, we can show a variational result for our problem. Meaning, for a fix $\lambda \in \mathbb{R}$, there exists infinitely many solutions of Eq. (1).

3 Bifurcation visualization

We consider $\Omega_- = (-5, 0)$, $\Omega_+ = (0, 5)$, the diffusion coefficient is chosen piecewise constant $\sigma|_{\Omega_-} \equiv -1.005$ and $\sigma|_{\Omega_+} \equiv 1$, $c \equiv 1$, and $\kappa \equiv 1$. The numerics have been done using a finite element discretization and the Matlab package `pde2path` [4]. The finite element discretization use a T-conform mesh which is refined close to the interface $\{0\}$ to faithfully represent the interface behavior. Figure 1 is the 2D bifurcation diagram (abscissa λ and ordinate $\|u\|_{L^2(\Omega)}$) where λ is initialized in $[-10, 15]$. All branches are unbounded and seemingly do not contain points of secondary bifurcation.

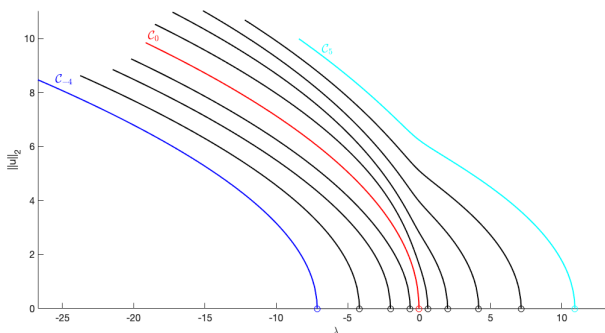


Figure 1: Bifurcation diagram.

4 Extension: The Drude model

A simple known model to have negative diffusion parameter is the Drude model, however in that model the coefficients σ and c depend on the spectral parameter λ . In [2], the diffusion coefficient σ and coefficient c are now given by

$$\sigma_\lambda(x) = \frac{1}{1 - \mathbf{1}_{\Omega_-}(x) \frac{\Lambda_c}{\lambda}}, \quad c_\lambda(x) = 1 - \mathbf{1}_{\Omega_-}(x) \frac{\Lambda_c}{\lambda}$$

where $\mathbf{1}_{\Omega_-}(x)$ is the indicator function of Ω_- and with $0 < \Lambda_c < \Lambda_\sigma$ constants.

For the numerical study, we use the same 1D subdomain $\Omega_- = (-5, 0)$ and $\Omega_+ = (0, 5)$, and $\Lambda_c = 4$, $\Lambda_\sigma = 7$, and $\kappa \equiv 1$. Using the same numerical method as before, Figure 2 we show the 2D bifurcation diagram (abscissa λ and ordinate $\|u\|_{L^2(\Omega)}$) where λ is initialized in $[0.1, 9]$. We observe that the branches seems to accumulate at $\lambda = 0$ and, around $\lambda = \Lambda_\sigma$, we see a branch that seems to have a vertical asymptote and another that non-smoothly cross the values Λ_σ . We expect more intricate branch behavior around the values $\lambda = 0$ or Λ_σ as either c_λ or Λ_σ blow-up around those points but a more rigorous study is needed.

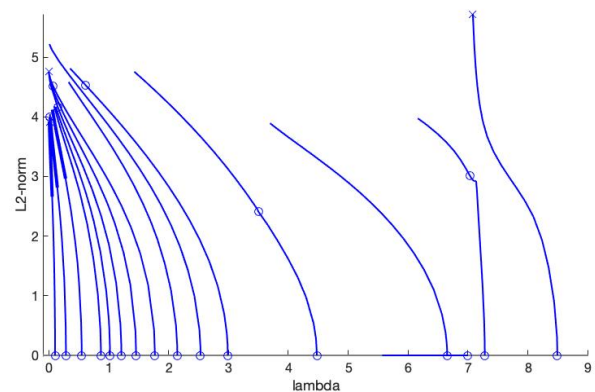


Figure 2: Drude model: bifurcation diagram.

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