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# Chordal directed graphs are not directed $\chi$ -bounded

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## Abstract

We show that digraphs with no transitive tournament on 3 vertices and in which every induced directed cycle has length 3 can have arbitrarily large dichromatic number. This answers in the negative a question of Carbonero, Hompe, Moore, and Spirkl (and strengthens one of their results).

**Mathematics Subject Classifications:** 05C15, 05C20

## 1 Introduction

Throughout this paper, we only consider simple graphs (resp. directed graphs)  $G$ , that is, for every two distinct vertices  $u$  and  $v$ , the graph  $G$  does not have multiple edges (resp. both arcs  $uv$  and  $vu$ ).

Relations between the chromatic number  $\chi(G)$  and the clique number  $\omega(G)$  of a graph  $G$  have been studied for decades in structural graph theory. In particular, it is well known that there exist triangle-free graphs  $G$  with arbitrarily large chromatic number (see e.g. [4, 9]). A hereditary class of graphs is  $\chi$ -bounded if there exists a function  $f$  such that for every  $G \in \mathcal{G}$ ,  $\chi(G) \leq f(\omega(G))$  (see e.g. a recent survey [7] on the topic). The following question received considerable attention in the last few years: Consider a hereditary class of graphs  $\mathcal{G}$  in which every triangle-free graph has bounded chromatic number. Is it true that  $\mathcal{G}$  is  $\chi$ -bounded? Carbonero, Hompe, Moore and Spirkl [2] answered this in the negative in a recent breakthrough paper.

Their initial motivation was actually to prove a result on digraphs. Let  $D$  be a digraph. A  $k$ -dicolouring of  $D$  is a  $k$ -partition  $(V_1, \dots, V_k)$  of  $V(D)$  such that  $D[V_i]$  is acyclic for every  $1 \leq i \leq k$ . Such a partition is also called an *acyclic colouring* of  $D$ . The *dichromatic number* of  $D$ , denoted by  $\vec{\chi}(D)$  and introduced by Neumann-Lara in [6], is the smallest integer  $k$  such that  $D$  admits a  $k$ -dicolouring. We denote by  $\omega(D)$  the size of a largest clique in the underlying graph of  $D$ . A *directed triangle* is a directed cycle of length 3. As for unoriented graphs, we say that a hereditary class of digraphs  $\mathcal{G}$  is  $\vec{\chi}$ -bounded if for every  $G \in \mathcal{G}$ ,  $\vec{\chi}(G) \leq f(\omega(G))$ .

Carbonero, Hompe, Moore and Spirkl [2] proved that the class of digraphs with no induced directed cycle of odd length at least 5 is not  $\vec{\chi}$ -bounded by giving a collection of digraphs with no induced directed cycle of odd length at least 5, no  $K_4$  and with arbitrarily large dichromatic number. They ask (Question 3.2) if the class of digraphs in which every induced directed cycle has length 3 is  $\vec{\chi}$ -bounded. These digraphs can be seen as directed analogues of chordal graphs, where a *chordal directed graph* is a directed graph with no induced directed cycle of length at least 4.

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We give a negative answer to this question (and thus strengthen the construction of [2]). Let us denote by  $TT_3$  the *transitive tournament* on 3 vertices (i.e. the triangle which is oriented acyclically). Let  $\mathcal{C}_3$  be the class of digraphs with no  $TT_3$  nor induced directed cycle of length at least 4. We prove the following.

**Theorem 1.** *For every  $k$ , there exists  $G \in \mathcal{C}_3$  such that  $\bar{\chi}(G) \geq k$ .*

Since any orientation of a  $K_4$  contains a  $TT_3$ , Theorem 1 answers Question 3.2 of [2].

## 2 Proof of Theorem 1

Our proof technique can be seen as a generalization of the construction of triangle-free graphs with arbitrarily large chromatic number due to Zykov [9]. Assume that we are given a triangle-free graph  $G_k$  with chromatic number at least  $k$ , and let us define  $G_{k+1}$  as follows (note that we can set  $G_1$  as a single vertex graph). Let  $G$  be the graph made of  $k$  disjoint induced copies of  $G_k$ . Set  $\mathcal{I}$  to be the set of all  $k$ -subsets of vertices of  $G$  containing exactly one vertex in each copy of  $G_k$ . Now, build the graph  $G_{k+1}$  from  $G$  as follows: for every set  $I \in \mathcal{I}$ , create a new vertex  $x_I$  adjacent to every vertex in  $I$ . The key observation is that, for any colouring of  $G_{k+1}$ , for each  $I \in \mathcal{I}$ , the vertex  $x_I$  forces  $I$  to miss at least one colour, namely the one received by  $x_I$ . This easily implies that  $G_{k+1}$  is not  $k$ -colourable. Indeed, if one tries to  $k$ -colour  $G_{k+1}$ , since  $G_k$  has chromatic number  $k$ , there must be a vertex  $x_i$  coloured  $i$  in the  $i^{\text{th}}$  copy of  $G_k$  for every  $i \leq k$ . A contradiction with the key observation above. Moreover, since each set of  $\mathcal{I}$  is an independent set,  $G_{k+1}$  is triangle-free.

For digraphs, such a naive construction fails since adjacent vertices are allowed to receive the same colour. A way to force a given independent set  $I$  of a digraph  $D$  to avoid a colour (without creating induced directed cycle of length at least 4 nor  $TT_3$ ) is to connect each vertex of  $I$  to an arc  $uv$  (instead of a single vertex as in the directed case) in such a way that each vertex of  $I$  forms a directed triangle with  $uv$  and then hope that the two vertices  $u$  and  $v$  receive the same colour. Unfortunately we cannot force an arc to have both endpoints of the same colour. But we have the following slightly weaker property.

*Remark 2.* Let  $G \in \mathcal{C}_3$  be a directed graph with at least one arc. Any  $\bar{\chi}(G)$ -dicolouring of  $G$  contains at least one monochromatic arc.

*Proof.* The result trivially holds if  $\bar{\chi}(G) = 1$ , so we may assume that  $\bar{\chi}(G) \geq 2$ . Let  $V_1, \dots, V_{\bar{\chi}(G)}$  be a  $\bar{\chi}(G)$ -dicolouring of  $G$ . The set  $V_1 \cup V_2$  must contain an induced directed cycle  $C$  since otherwise  $G$  would be  $(\bar{\chi}(G) - 1)$ -colourable. (Indeed, a colouring of the vertices of a digraph is acyclic if and only if none of its induced directed cycles is monochromatic). Hence, by definition of  $\mathcal{C}_3$ ,  $V_1 \cup V_2$  contains a directed triangle, and an arc of this directed triangle must have both endpoints in  $V_1$  or both endpoints in  $V_2$ .  $\square$

Let  $G$  be a  $k$ -chromatic digraph and  $I$  be an independent set of  $G$ . Using Remark 2, we prove that we can create a graph  $G'$  containing many induced copies of  $G$  such that, for every  $k$ -coloring of  $G'$ , there is one copy of  $G$  in  $G'$  where the vertices of  $I$  (in that copy) miss at least one color (Lemma 3). We then extend this result for arbitrarily many independent sets (Lemma 4). We then prove Theorem 1 using Lemma 4 as in Zykov's construction.

**Lemma 3.** *Let  $k$  be an integer. Let  $G \in \mathcal{C}_3$  with  $n$  vertices and  $m$  arcs, and such that  $\bar{\chi}(G) = k$ . Let  $I$  be an independent set of  $G$ . Then there exists a digraph  $H \in \mathcal{C}_3$  such that  $H$  contains  $m$  pairwise disjoint induced copies  $G_1, \dots, G_m$  of  $G$  and satisfy the following:*

- For every  $1 \leq i \neq j \leq m$ , there is no arc between  $G_i$  and  $G_j$ ;
- For every  $k$ -dicolouring of  $H$ , there exists an index  $i \leq m$  and a colour  $\alpha \in \{1, \dots, k\}$  such that no vertex of the copy of  $I$  in  $G_i$  is coloured with  $\alpha$ .

Moreover  $H$  has  $n \cdot (m + 1) \leq n^4$  vertices and at most  $m(m + 1) + mn^2 \leq n^4$  arcs.

*Proof.* Let us first describe the construction of  $H$ . We first create  $m + 1$  pairwise disjoint induced copies of  $G$  denoted by  $G_1, \dots, G_m, G_{m+1}$ . For every  $i \leq m$ , let  $I_i$  be the copy of  $I$  in  $G_i$ . Let us denote  $u_1^{m+1}v_1^{m+1}, \dots, u_m^{m+1}v_m^{m+1}$  the arcs of  $G_{m+1}$ . We add in  $H$  some arcs between the  $G_i$  ( $i \leq m$ ) and  $G_{m+1}$  as follows. For every  $i \leq m$  and for every vertex  $x \in I_i$ , add the arcs  $v_i^{m+1}x$  and  $xu_i^{m+1}$  in  $H$ .

Observe that  $H$  has  $n \cdot (m + 1)$  vertices and  $m \cdot (m + 1) + 2m \cdot |I| \leq n^4$  arcs as announced.

By construction, for every  $1 \leq i \neq j \leq m$ , there is no arc between  $G_i$  and  $G_j$ , so the first bullet holds.

Let  $c$  be a  $k$ -dicolouring of  $H$ . By Remark 2,  $G_{m+1}$  has a monochromatic arc, say  $u_i^{m+1}v_i^{m+1}$ . Let  $\alpha$  be the colour of  $u_i^{m+1}$  and  $v_i^{m+1}$  in  $c$ . Then, for every vertex  $x \in I_i$ ,  $x$  is not coloured with  $\alpha$  since  $H[\{u_i, v_i, x\}]$  is a directed triangle. This proves the second bullet.

To conclude, we simply have to prove that  $H \in \mathcal{C}_3$ . First assume for contradiction that  $H$  contains a copy  $X$  of a  $TT_3$  as a subgraph. Since there is no arc between  $G_i$  and  $G_j$  for  $1 \leq i \neq j \leq m$ ,  $X$  intersects at most one of the graphs  $G_i$  for  $i \leq m$ . Moreover, since  $G$  is in  $\mathcal{C}_3$ ,  $X$  is not included in  $G_i$  for  $i \leq m + 1$ . So  $X$  must intersect  $G_{m+1}$  and  $G_i$  for some  $i \leq m$ . Assume first that  $X$  contains two vertices of  $G_{m+1}$ . By construction, the only vertices of  $G_{m+1}$  connected to  $G_i$  are  $u_i^{m+1}$  and  $v_i^{m+1}$ . So both vertices are in  $X$ . Moreover, the only vertices of  $G_i$  connected to  $G_{m+1}$  are the vertices of  $I_i$  so the third vertex must be a vertex  $x$  of  $I_i$ . But by construction,  $G[\{x, u_i^{m+1}, v_i^{m+1}\}]$  is a directed triangle, a contradiction. So we can assume that  $X$  contains two vertices of  $G_i$ . Since  $X$  is a  $TT_3$ , they must be adjacent and both be adjacent to a vertex of  $G_{m+1}$ . But, by construction, the only vertices of  $G_i$  connected to  $G_{m+1}$  are the vertices of  $I_i$  which is an independent set, a contradiction. So  $H$  contains no  $TT_3$ .

Finally, assume for contradiction that  $H$  contains a directed cycle  $C$  of length at least 4 as an induced subgraph. Since  $G \in \mathcal{C}_3$ ,  $C$  is not contained in  $G_i$  for  $i = 1, \dots, m + 1$ . Since there is no arc between  $G_i$  and  $G_j$  for  $1 \leq i \neq j \leq m$ , the cycle  $C$  intersects  $G_{m+1}$  and we may assume without loss of generality that  $C$  also intersects  $G_1$ . So  $C$  contains  $u_1^{m+1}$  or  $v_1^{m+1}$ . Since, by construction,  $u_1^{m+1}$  has no out-neighbour in  $G_1$  and  $v_1^{m+1}$  has no in-neighbour in  $G_1$ ,  $C$  must contain both  $u_1^{m+1}$  and  $v_1^{m+1}$  (since the deletion of  $u_1^{m+1}$  and  $v_1^{m+1}$  disconnects  $G_1$  from the rest of the graph). But now all the vertices of  $G_1$  incident to  $u_1$  or  $v_1$  are the vertices  $x$  of  $I$ . And by construction, for every  $x \in I$ ,  $H[\{u_1^{m+1}, v_1^{m+1}, x\}]$  is a directed triangle, a contradiction.  $\square$

**Lemma 4.** *Let  $k, r$  be two integers. Let  $G \in \mathcal{C}_3$  such that  $\vec{\chi}(G) = k$  and let  $I_1, \dots, I_r$  be  $r$  independent sets of  $G$ . There exist an integer  $\ell_r$  and a digraph  $H \in \mathcal{C}_3$  such that  $H$  contains  $\ell_r$  pairwise disjoint induced copies  $G_1, \dots, G_{\ell_r}$  of  $G$  such that:*

- For every  $1 \leq i \neq j \leq \ell_r$ , there is no arc between  $G_i$  and  $G_j$ ;
- For every  $k$ -dicolouring of  $H$ , there exists an index  $j \leq \ell_r$  such that, for every  $s \leq r$ , there exists a colour  $\alpha_s \in \{1, \dots, k\}$  such that no vertex of the copy of  $I_s$  in  $G_j$  is coloured with  $\alpha_s$ .

Moreover  $H$  contains at most  $n^{4r}$  vertices and arcs.

*Proof.* We now have all the ingredients to prove Lemma 4 by induction on  $r$ . By Lemma 3, the case  $r = 1$  holds.

Assume that the conclusion holds for  $r \geq 1$  and let us prove the result for  $r + 1$ . Let  $G \in \mathcal{C}_3$  with  $\vec{\chi}(G) = k$  and let  $I_1, \dots, I_{r+1}$  be  $r + 1$  independent sets of  $G$ . By induction applied to  $G$  and independent sets  $I_1, \dots, I_r$ , there exists an integer  $\ell_r$  and a digraph  $H_r \in \mathcal{C}_3$  such that  $H_r$  contains  $\ell_r$  pairwise disjoint induced copies  $G_1, \dots, G_{\ell_r}$  of  $G$  such that:

- For every  $1 \leq i \neq j \leq \ell_r$ , there is no arc between  $G_i$  and  $G_j$ ;
- For every  $k$ -dicolouring of  $H_r$ , there exists an index  $j \leq \ell_r$  such that, for  $s = 1, \dots, r$ , there exists a colour  $\alpha_s$  such that no vertex of the copy of  $I_s$  in  $G_j$  is coloured with  $\alpha_s$ .

Note that by induction,  $H_r$  has at most  $n^{4r}$  vertices and edges. Let us denote by  $J$  the union of the vertices of the copies of  $I_{r+1}$  in the subgraphs  $G_1, \dots, G_{\ell_r}$  and observe that  $J$  is an independent set. By Lemma 3 applied to  $H_r$  and  $J$ , there exists a digraph  $H_{r+1} \in \mathcal{C}_3$  that contains  $m = |E(H_r)|$  pairwise disjoint induced copies  $H_r^1, \dots, H_r^m$  of  $H_r$  such that:

- For  $1 \leq i \neq j \leq m$ , there is no arc between  $H_r^i$  and  $H_r^j$ ;
- For every  $k$ -dicolouring of  $H_{r+1}$ , there exists an index  $j \leq m$  and a colour  $\alpha_{r+1}$  such that no vertex of the copy of  $J$  in  $H_r^j$  is coloured with  $\alpha_{r+1}$ .

Moreover,  $H$  has at most  $|V(H_r)|^4 = n^{4(r+1)}$  vertices and arcs.

Let us prove that  $H_{r+1}$  satisfies the conclusion of Lemma 4. For every  $i \leq m$ ,  $H_r^i$  being an induced copy of  $H_r$ , it contains  $\ell_r$  pairwise disjoint induced copies of  $G$ , denoted by  $G_1^i, \dots, G_{\ell_r}^i$ . Thus, by construction of  $H_{r+1}$ , the graph  $H_{r+1}$  contains  $\ell_{r+1} := m \cdot \ell_r$  induced copies of  $G$  and by construction there is no arc linking any of these copies.

Fix a  $k$ -dicolouring of  $H_{r+1}$ . There exists an index  $j \leq m$  and a colour  $\alpha_{r+1}$  such that no vertex of the induced copy of  $J$  in  $H_r^j$  is coloured  $\alpha_{r+1}$ . Since  $H_r^j$  is an induced copy of  $H_r$  there exists an index  $k \leq \ell_r$  such that, for  $s = 1, \dots, r$ , there exists a colour  $\alpha_s$  such that no vertex of the induced copy of  $I_s$  in  $G_k^j$  is coloured with  $\alpha_s$ . Hence, the second bullet holds, which completes the proof.  $\square$

*Proof of Theorem 1.* Let us construct a sequence  $(G_k)_{k \in \mathbb{N}}$  such that for every  $k$ ,  $G_k \in \mathcal{C}_3$  and  $\bar{\chi}(G_k) \geq k$ . Let  $G_1$  be the graph consisting of a single vertex and let  $G_2$  be the directed triangle. Let  $k \geq 2$  and assume that we have obtained a  $k$ -dichromatic digraph  $G_k$  which is in  $\mathcal{C}_3$ , let us define  $G_{k+1}$  as follows. Let  $G$  be the digraph consisting of  $k$  pairwise disjoint induced copies of  $G_k$ , denoted by  $G_k^1, \dots, G_k^k$ . Let  $\mathcal{I}$  be the set of independent sets that intersect each  $G_k^i$  in a single vertex. Since  $\bar{\chi}(G_k) \geq k$ , in any  $k$ -dicolouring of  $G$ , there exists a vertex  $x_i$  coloured  $i$  in  $G_k^i$  for every  $i = 1, \dots, k$ . By definition of  $\mathcal{I}$ ,  $\{x_1, \dots, x_k\} \in \mathcal{I}$ . Hence, for every  $k$ -dicolouring of  $G$ , a set of  $\mathcal{I}$  receives all the colours.

By Lemma 4 applied to  $G$  and  $\mathcal{I}$ , there exists a digraph  $G_{k+1} \in \mathcal{C}_3$  such that, for every  $k$ -dicolouring of  $G_{k+1}$  (if such a colouring exists), there exists an induced copy of  $G$  in  $G_{k+1}$  such that each set  $\mathcal{I}$  in that copy of  $G$  avoids a colour, a contradiction. So  $\bar{\chi}(G_{k+1}) \geq k + 1$ .  $\square$

### 3 Further works

Our  $(k+1)$ -dichromatic graph  $G_{k+1}$  has size  $|G_k|^{2^{\text{poly}(|G_k|)}}$ , which is larger than the graphs obtained using Zykov's construction which have size of order  $2^{\text{poly}(|G_k|)}$ . It would be interesting to know if the size of our example can be reduced.

One can wonder if directed triangles play a particular role in Theorem 1. More formally, one can wonder (as also asked in [2], Question 3.3) for which integer  $k$ , the class of digraphs which only contain induced directed cycles of length exactly  $k$  are  $\bar{\chi}$ -bounded? Our main result is that it is not the case for  $k = 3$ . Based on the idea of our construction, Carborero et al. [3] very recently answered in the negative this question for any possible value of  $k$ .

On the same flavour, we recall here the following conjecture of Aboulker, Charbit and Naserasr which can be seen as a directed analogue of the well-known Gyárfás-Sumner conjecture [5, 8]. An *oriented tree* is an orientation of a tree.

**Conjecture 5.** [1] For every oriented tree  $T$ , the class of digraphs with no induced  $T$  is  $\bar{\chi}$ -bounded.

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