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► **To cite this version:**

Pierre Aboulker, Nicolas Bousquet, Rémi de Verclos. Chordal Directed Graphs Are Not χ -Bounded. The Electronic Journal of Combinatorics, 2022, 29 (2), 10.37236/11050 . hal-03829690

HAL Id: hal-03829690

<https://hal.science/hal-03829690v1>

Submitted on 21 Oct 2024

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Chordal directed graphs are not directed χ -bounded

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Submitted: Feb 12, 2022; Accepted: Apr 2, 2022; Published: May 6, 2022

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Abstract

We show that digraphs with no transitive tournament on 3 vertices and in which every induced directed cycle has length 3 can have arbitrarily large dichromatic number. This answers in the negative a question of Carbonero, Hompe, Moore, and Spirkl (and strengthens one of their results).

Mathematics Subject Classifications: 05C15, 05C20

1 Introduction

Throughout this paper, we only consider simple graphs (resp. directed graphs) G , that is, for every two distinct vertices u and v , the graph G does not have multiple edges (resp. both arcs uv and vu).

Relations between the chromatic number $\chi(G)$ and the clique number $\omega(G)$ of a graph G have been studied for decades in structural graph theory. In particular, it is well known that there exist triangle-free graphs G with arbitrarily large chromatic number (see e.g. [4, 9]). A hereditary class of graphs is χ -bounded if there exists a function f such that for every $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$ (see e.g. a recent survey [7] on the topic). The following question received considerable attention in the last few years: Consider a hereditary class of graphs \mathcal{G} in which every triangle-free graph has bounded chromatic number. Is it true that \mathcal{G} is χ -bounded? Carbonero, Hompe, Moore and Spirkl [2] answered this in the negative in a recent breakthrough paper.

Their initial motivation was actually to prove a result on digraphs. Let D be a digraph. A k -dicolouring of D is a k -partition (V_1, \dots, V_k) of $V(D)$ such that $D[V_i]$ is acyclic for every $1 \leq i \leq k$. Such a partition is also called an *acyclic colouring* of D . The *dichromatic number* of D , denoted by $\vec{\chi}(D)$ and introduced by Neumann-Lara in [6], is the smallest integer k such that D admits a k -dicolouring. We denote by $\omega(D)$ the size of a largest clique in the underlying graph of D . A *directed triangle* is a directed cycle of length 3. As for unoriented graphs, we say that a hereditary class of digraphs \mathcal{G} is $\vec{\chi}$ -bounded if for every $G \in \mathcal{G}$, $\vec{\chi}(G) \leq f(\omega(G))$.

Carbonero, Hompe, Moore and Spirkl [2] proved that the class of digraphs with no induced directed cycle of odd length at least 5 is not $\vec{\chi}$ -bounded by giving a collection of digraphs with no induced directed cycle of odd length at least 5, no K_4 and with arbitrarily large dichromatic number. They ask (Question 3.2) if the class of digraphs in which every induced directed cycle has length 3 is $\vec{\chi}$ -bounded. These digraphs can be seen as directed analogues of chordal graphs, where a *chordal directed graph* is a directed graph with no induced directed cycle of length at least 4.

*Supported by ANR project DAGDigDec (JCJC) ANR-21-CE48-0012

†Supported by by ANR project DAGDigDec (JCJC) ANR-21-CE48-0012

We give a negative answer to this question (and thus strengthen the construction of [2]). Let us denote by TT_3 the *transitive tournament* on 3 vertices (i.e. the triangle which is oriented acyclically). Let \mathcal{C}_3 be the class of digraphs with no TT_3 nor induced directed cycle of length at least 4. We prove the following.

Theorem 1. *For every k , there exists $G \in \mathcal{C}_3$ such that $\bar{\chi}(G) \geq k$.*

Since any orientation of a K_4 contains a TT_3 , Theorem 1 answers Question 3.2 of [2].

2 Proof of Theorem 1

Our proof technique can be seen as a generalization of the construction of triangle-free graphs with arbitrarily large chromatic number due to Zykov [9]. Assume that we are given a triangle-free graph G_k with chromatic number at least k , and let us define G_{k+1} as follows (note that we can set G_1 as a single vertex graph). Let G be the graph made of k disjoint induced copies of G_k . Set \mathcal{I} to be the set of all k -subsets of vertices of G containing exactly one vertex in each copy of G_k . Now, build the graph G_{k+1} from G as follows: for every set $I \in \mathcal{I}$, create a new vertex x_I adjacent to every vertex in I . The key observation is that, for any colouring of G_{k+1} , for each $I \in \mathcal{I}$, the vertex x_I forces I to miss at least one colour, namely the one received by x_I . This easily implies that G_{k+1} is not k -colourable. Indeed, if one tries to k -colour G_{k+1} , since G_k has chromatic number k , there must be a vertex x_i coloured i in the i^{th} copy of G_k for every $i \leq k$. A contradiction with the key observation above. Moreover, since each set of \mathcal{I} is an independent set, G_{k+1} is triangle-free.

For digraphs, such a naive construction fails since adjacent vertices are allowed to receive the same colour. A way to force a given independent set I of a digraph D to avoid a colour (without creating induced directed cycle of length at least 4 nor TT_3) is to connect each vertex of I to an arc uv (instead of a single vertex as in the directed case) in such a way that each vertex of I forms a directed triangle with uv and then hope that the two vertices u and v receive the same colour. Unfortunately we cannot force an arc to have both endpoints of the same colour. But we have the following slightly weaker property.

Remark 2. Let $G \in \mathcal{C}_3$ be a directed graph with at least one arc. Any $\bar{\chi}(G)$ -dicolouring of G contains at least one monochromatic arc.

Proof. The result trivially holds if $\bar{\chi}(G) = 1$, so we may assume that $\bar{\chi}(G) \geq 2$. Let $V_1, \dots, V_{\bar{\chi}(G)}$ be a $\bar{\chi}(G)$ -dicolouring of G . The set $V_1 \cup V_2$ must contain an induced directed cycle C since otherwise G would be $(\bar{\chi}(G) - 1)$ -colourable. (Indeed, a colouring of the vertices of a digraph is acyclic if and only if none of its induced directed cycles is monochromatic). Hence, by definition of \mathcal{C}_3 , $V_1 \cup V_2$ contains a directed triangle, and an arc of this directed triangle must have both endpoints in V_1 or both endpoints in V_2 . \square

Let G be a k -chromatic digraph and I be an independent set of G . Using Remark 2, we prove that we can create a graph G' containing many induced copies of G such that, for every k -coloring of G' , there is one copy of G in G' where the vertices of I (in that copy) miss at least one color (Lemma 3). We then extend this result for arbitrarily many independent sets (Lemma 4). We then prove Theorem 1 using Lemma 4 as in Zykov's construction.

Lemma 3. *Let k be an integer. Let $G \in \mathcal{C}_3$ with n vertices and m arcs, and such that $\bar{\chi}(G) = k$. Let I be an independent set of G . Then there exists a digraph $H \in \mathcal{C}_3$ such that H contains m pairwise disjoint induced copies G_1, \dots, G_m of G and satisfy the following:*

- For every $1 \leq i \neq j \leq m$, there is no arc between G_i and G_j ;
- For every k -dicolouring of H , there exists an index $i \leq m$ and a colour $\alpha \in \{1, \dots, k\}$ such that no vertex of the copy of I in G_i is coloured with α .

Moreover H has $n \cdot (m + 1) \leq n^4$ vertices and at most $m(m + 1) + mn^2 \leq n^4$ arcs.

Proof. Let us first describe the construction of H . We first create $m + 1$ pairwise disjoint induced copies of G denoted by G_1, \dots, G_m, G_{m+1} . For every $i \leq m$, let I_i be the copy of I in G_i . Let us denote $u_1^{m+1}v_1^{m+1}, \dots, u_m^{m+1}v_m^{m+1}$ the arcs of G_{m+1} . We add in H some arcs between the G_i ($i \leq m$) and G_{m+1} as follows. For every $i \leq m$ and for every vertex $x \in I_i$, add the arcs $v_i^{m+1}x$ and xu_i^{m+1} in H .

Observe that H has $n \cdot (m + 1)$ vertices and $m \cdot (m + 1) + 2m \cdot |I| \leq n^4$ arcs as announced.

By construction, for every $1 \leq i \neq j \leq m$, there is no arc between G_i and G_j , so the first bullet holds.

Let c be a k -dicolouring of H . By Remark 2, G_{m+1} has a monochromatic arc, say $u_i^{m+1}v_i^{m+1}$. Let α be the colour of u_i^{m+1} and v_i^{m+1} in c . Then, for every vertex $x \in I_i$, x is not coloured with α since $H[\{u_i, v_i, x\}]$ is a directed triangle. This proves the second bullet.

To conclude, we simply have to prove that $H \in \mathcal{C}_3$. First assume for contradiction that H contains a copy X of a TT_3 as a subgraph. Since there is no arc between G_i and G_j for $1 \leq i \neq j \leq m$, X intersects at most one of the graphs G_i for $i \leq m$. Moreover, since G is in \mathcal{C}_3 , X is not included in G_i for $i \leq m + 1$. So X must intersect G_{m+1} and G_i for some $i \leq m$. Assume first that X contains two vertices of G_{m+1} . By construction, the only vertices of G_{m+1} connected to G_i are u_i^{m+1} and v_i^{m+1} . So both vertices are in X . Moreover, the only vertices of G_i connected to G_{m+1} are the vertices of I_i so the third vertex must be a vertex x of I_i . But by construction, $G[\{x, u_i^{m+1}, v_i^{m+1}\}]$ is a directed triangle, a contradiction. So we can assume that X contains two vertices of G_i . Since X is a TT_3 , they must be adjacent and both be adjacent to a vertex of G_{m+1} . But, by construction, the only vertices of G_i connected to G_{m+1} are the vertices of I_i which is an independent set, a contradiction. So H contains no TT_3 .

Finally, assume for contradiction that H contains a directed cycle C of length at least 4 as an induced subgraph. Since $G \in \mathcal{C}_3$, C is not contained in G_i for $i = 1, \dots, m + 1$. Since there is no arc between G_i and G_j for $1 \leq i \neq j \leq m$, the cycle C intersects G_{m+1} and we may assume without loss of generality that C also intersects G_1 . So C contains u_1^{m+1} or v_1^{m+1} . Since, by construction, u_1^{m+1} has no out-neighbour in G_1 and v_1^{m+1} has no in-neighbour in G_1 , C must contain both u_1^{m+1} and v_1^{m+1} (since the deletion of u_1^{m+1} and v_1^{m+1} disconnects G_1 from the rest of the graph). But now all the vertices of G_1 incident to u_1 or v_1 are the vertices x of I . And by construction, for every $x \in I$, $H[\{u_1^{m+1}, v_1^{m+1}, x\}]$ is a directed triangle, a contradiction. \square

Lemma 4. *Let k, r be two integers. Let $G \in \mathcal{C}_3$ such that $\vec{\chi}(G) = k$ and let I_1, \dots, I_r be r independent sets of G . There exist an integer ℓ_r and a digraph $H \in \mathcal{C}_3$ such that H contains ℓ_r pairwise disjoint induced copies G_1, \dots, G_{ℓ_r} of G such that:*

- For every $1 \leq i \neq j \leq \ell_r$, there is no arc between G_i and G_j ;
- For every k -dicolouring of H , there exists an index $j \leq \ell_r$ such that, for every $s \leq r$, there exists a colour $\alpha_s \in \{1, \dots, k\}$ such that no vertex of the copy of I_s in G_j is coloured with α_s .

Moreover H contains at most n^{4r} vertices and arcs.

Proof. We now have all the ingredients to prove Lemma 4 by induction on r . By Lemma 3, the case $r = 1$ holds.

Assume that the conclusion holds for $r \geq 1$ and let us prove the result for $r + 1$. Let $G \in \mathcal{C}_3$ with $\vec{\chi}(G) = k$ and let I_1, \dots, I_{r+1} be $r + 1$ independent sets of G . By induction applied to G and independent sets I_1, \dots, I_r , there exists an integer ℓ_r and a digraph $H_r \in \mathcal{C}_3$ such that H_r contains ℓ_r pairwise disjoint induced copies G_1, \dots, G_{ℓ_r} of G such that:

- For every $1 \leq i \neq j \leq \ell_r$, there is no arc between G_i and G_j ;
- For every k -dicolouring of H_r , there exists an index $j \leq \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour α_s such that no vertex of the copy of I_s in G_j is coloured with α_s .

Note that by induction, H_r has at most n^{4r} vertices and edges. Let us denote by J the union of the vertices of the copies of I_{r+1} in the subgraphs G_1, \dots, G_{ℓ_r} and observe that J is an independent set. By Lemma 3 applied to H_r and J , there exists a digraph $H_{r+1} \in \mathcal{C}_3$ that contains $m = |E(H_r)|$ pairwise disjoint induced copies H_r^1, \dots, H_r^m of H_r such that:

- For $1 \leq i \neq j \leq m$, there is no arc between H_r^i and H_r^j ;
- For every k -dicolouring of H_{r+1} , there exists an index $j \leq m$ and a colour α_{r+1} such that no vertex of the copy of J in H_r^j is coloured with α_{r+1} .

Moreover, H has at most $|V(H_r)|^4 = n^{4(r+1)}$ vertices and arcs.

Let us prove that H_{r+1} satisfies the conclusion of Lemma 4. For every $i \leq m$, H_r^i being an induced copy of H_r , it contains ℓ_r pairwise disjoint induced copies of G , denoted by $G_1^i, \dots, G_{\ell_r}^i$. Thus, by construction of H_{r+1} , the graph H_{r+1} contains $\ell_{r+1} := m \cdot \ell_r$ induced copies of G and by construction there is no arc linking any of these copies.

Fix a k -dicolouring of H_{r+1} . There exists an index $j \leq m$ and a colour α_{r+1} such that no vertex of the induced copy of J in H_r^j is coloured α_{r+1} . Since H_r^j is an induced copy of H_r there exists an index $k \leq \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour α_s such that no vertex of the induced copy of I_s in G_k^j is coloured with α_s . Hence, the second bullet holds, which completes the proof. \square

Proof of Theorem 1. Let us construct a sequence $(G_k)_{k \in \mathbb{N}}$ such that for every k , $G_k \in \mathcal{C}_3$ and $\bar{\chi}(G_k) \geq k$. Let G_1 be the graph consisting of a single vertex and let G_2 be the directed triangle. Let $k \geq 2$ and assume that we have obtained a k -dichromatic digraph G_k which is in \mathcal{C}_3 , let us define G_{k+1} as follows. Let G be the digraph consisting of k pairwise disjoint induced copies of G_k , denoted by G_k^1, \dots, G_k^k . Let \mathcal{I} be the set of independent sets that intersect each G_k^i in a single vertex. Since $\bar{\chi}(G_k) \geq k$, in any k -dicolouring of G , there exists a vertex x_i coloured i in G_k^i for every $i = 1, \dots, k$. By definition of \mathcal{I} , $\{x_1, \dots, x_k\} \in \mathcal{I}$. Hence, for every k -dicolouring of G , a set of \mathcal{I} receives all the colours.

By Lemma 4 applied to G and \mathcal{I} , there exists a digraph $G_{k+1} \in \mathcal{C}_3$ such that, for every k -dicolouring of G_{k+1} (if such a colouring exists), there exists an induced copy of G in G_{k+1} such that each set \mathcal{I} in that copy of G avoids a colour, a contradiction. So $\bar{\chi}(G_{k+1}) \geq k + 1$. \square

3 Further works

Our $(k+1)$ -dichromatic graph G_{k+1} has size $|G_k|^{2^{\text{poly}(|G_k|)}}$, which is larger than the graphs obtained using Zykov's construction which have size of order $2^{\text{poly}(|G_k|)}$. It would be interesting to know if the size of our example can be reduced.

One can wonder if directed triangles play a particular role in Theorem 1. More formally, one can wonder (as also asked in [2], Question 3.3) for which integer k , the class of digraphs which only contain induced directed cycles of length exactly k are $\bar{\chi}$ -bounded? Our main result is that it is not the case for $k = 3$. Based on the idea of our construction, Carborero et al. [3] very recently answered in the negative this question for any possible value of k .

On the same flavour, we recall here the following conjecture of Aboulker, Charbit and Naserasr which can be seen as a directed analogue of the well-known Gyárfás-Sumner conjecture [5, 8]. An *oriented tree* is an orientation of a tree.

Conjecture 5. [1] For every oriented tree T , the class of digraphs with no induced T is $\bar{\chi}$ -bounded.

Acknowledgments

This research was supported by ANR project DAGDigDec (JCJC) ANR-21-CE48-0012.

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