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The initial boundary value problem for second order traffic flow models with vacuum: existence of entropy weak solutions

Paola Goatin¹ Alexandra Würth¹

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Abstract

We study the IBVP for a non-strictly hyperbolic 2×2 system of conservation laws on an interval with characteristic boundaries, modeling traffic dynamics including vacuum states on a road stretch. After giving a detailed characterization of the admissible states at the boundary in terms of Riemann solver and entropy conditions, we prove existence of entropy weak solutions for data of bounded variation in the Riemann invariant coordinates by convergence of wave-front tracking approximations.

2020 Mathematics Subject Classification: 35L65; 90B20

Key words: Macroscopic traffic flow models; hyperbolic systems of conservation laws; weak boundary conditions; wave-front tracking.

1 Introduction

The Generic Second Order Model (GSOM in short) was introduced in [32] to provide a general framework for macroscopic traffic flow modeling. It consists in the 2×2 hyperbolic system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0, \end{cases} \quad x \in \mathbb{R}, t > 0, \quad (1.1)$$

defined on a domain $\Omega \subset \{(\rho, w) \in \mathbb{R}^2, \rho \geq 0, w \geq 0\}$, where the average speed of vehicles is a function of the density $\rho = \rho(t, x)$ and a Lagrangian vehicle property $w = w(t, x)$, namely $v = \mathcal{V}(\rho, w)$ for some speed function $\mathcal{V} : \Omega \rightarrow \mathbb{R}_{\geq 0}$. Notice that, setting $\mathcal{V}(\rho, w) = w - p(\rho)$ for a suitable “pressure” function p , system (1.1) corresponds to the celebrated Aw-Rascle-Zhang (ARZ) model [5, 40]. We also remark that, taking $w = \bar{w}$ constant, we recover the classical Lighthill-Whitham-Richards (LWR) model [33, 35].

Under standard hypotheses (see (2.1)), system (1.1) is strictly hyperbolic for $\rho > 0$, but the two eigenvalues coalesce at vacuum states, inducing instabilities [5, Section 4]. Besides, one characteristic field is genuinely non-linear with coinciding shock and rarefaction curves and the other one is linearly degenerate (but not straight). Thus, the GSOM system can be related to Temple class [38].

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In traffic flow applications (see e.g. [39]), it is natural to consider the Initial Boundary Value Problem (IBVP) for (1.1) on a bounded interval $]x_{in}, x_{out}[\subset \mathbb{R}$, namely

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0, \end{cases} \quad x \in]x_{in}, x_{out}[, t > 0, \quad (1.2a)$$

$$(\rho, w)(0, x) = (\rho_0, w_0)(x), \quad x \in]x_{in}, x_{out}[, \quad (1.2b)$$

$$(\rho, w)(t, x_{in}) = (\rho_{in}, w_{in})(t), \quad t > 0, \quad (1.2c)$$

$$(\rho, w)(t, x_{out}) = (\rho_{out}, w_{out})(t), \quad t > 0. \quad (1.2d)$$

As usual with hyperbolic equations, solutions to (1.2) have to be intended in the weak sense. In particular, boundary conditions (1.2c) and (1.2d) may not be satisfied in the classical sense, i.e. the traces of the solutions at the boundaries may not be equal the corresponding boundary values. In the literature, two definitions of boundary conditions for systems of conservation laws are commonly considered: a *boundary entropy inequality* derived by viscosity approximation [8, 9] and a *Riemann boundary condition* based on the Riemann solver associated to (1.1). Dubois and LeFloch [21] showed that the two formulations are equivalent for scalar conservation laws, linear systems and a 2×2 system whose fields are both linearly degenerate. For non-linear hyperbolic systems, Benabdallah and Serre [10] proved that the Riemann boundary condition implies the entropy one, and equivalence holds in the case of 2×2 systems with straight-line characteristic fields and never vanishing eigenvalues. This result was further extended to $n \times n$ strictly hyperbolic Temple class systems with non-characteristic boundary in [3, Section 8]. We remark that these results do not apply in the present setting, since (1.2a) is non-strictly hyperbolic and the boundaries (1.2c) and (1.2d) can be characteristic [1]. We just note that [34, Section 4] gives a characterization of the boundary entropy set for the ARZ model with Chaplygin pressure (fully linearly degenerate).

Well-posedness results for the IBVP with both characteristic and non-characteristic boundary for general, strictly hyperbolic systems of conservation laws were provided in [1, 2] for data with small total variation. For strictly hyperbolic Temple class systems with BV data, no monotonicity assumption on the eigenvalues along the Lax curves and possibly characteristic boundary we refer to [17] and with \mathbf{L}^∞ data, genuinely non-linear characteristic fields and non-characteristic boundary see [3].

Concerning second order traffic flow models, and in particular ARZ model with vacuum (i.e. non-strictly hyperbolic), existence results for the Cauchy problem were provided in [4, 27], while \mathbf{L}^1 stability is provided in [6, 28] for the system in Lagrangian coordinates. We notice that, to avoid problems at vacuum, several “phase-transition” or “collapsed” models were introduced in the literature, see e.g.. [11, 16, 25, 26]. Vacuum issues can also be avoided by suitably modifying the speed function near the vacuum, as proposed in [31].

In this article, we prove the existence of entropy weak solutions of (1.2) with possibly characteristic boundaries for BV initial and boundary data on domains including vacuum states. Our approach is the following. After introducing the Riemann solver for (1.1) based on [4] in Section 2, in Section 3 we describe the sets of admissible traces at the boundaries given both by the Riemann solver and the boundary entropy inequality. Based on the available entropy families, we can prove the equivalence of these conditions at the right boundary, while at the left boundary the two sets differ for a subset of the vacuum states. Possibly, the selection of further entropies could allow to remove these unphysical states. This analysis ensures that the wave-front tracking approximations constructed in Section 4.1 are approximate entropy

weak solutions in the sense of [13, 14]. Uniform BV bounds allow to pass to the limit in the sequence of approximate solutions in Section 4.2, thus guaranteeing the desired existence result. This strategy allows to avoid the cumbersome technicalities related to a limiting procedure involving the solution boundary traces, see e.g. [1]. Some complementary material is deferred to the Appendix.

2 The Riemann solver for the GSOM model

In this work, we consider speed functions \mathcal{V} satisfying the following hypotheses on the domain of definition [24]:

$$\mathcal{V}(\rho, w) \geq 0, \quad \mathcal{V}(0, w) = w, \quad (2.1a)$$

$$2\mathcal{V}_\rho(\rho, w) + \rho\mathcal{V}_{\rho\rho}(\rho, w) < 0 \text{ for } w > 0, \quad (2.1b)$$

$$\mathcal{V}_w(\rho, w) > 0, \quad (2.1c)$$

$$\forall w > 0 \quad \exists R(w) > 0 : \quad \mathcal{V}(R(w), w) = 0. \quad (2.1d)$$

As in [15, 24], we observe that (2.1b) implies that $Q(\rho, w) := \rho\mathcal{V}(\rho, w)$ is strictly concave and $\mathcal{V}_\rho(\rho, w) < 0$ for $w > 0$, if \mathcal{V} is a \mathbf{C}^2 function in ρ . We also remark that in (2.1d) we can have $R(w) = \bar{R}$ for all $w > 0$.

Under the above hypotheses, system (1.1) is strictly hyperbolic for $\rho > 0$, with eigenvalues

$$\lambda_1(\rho, w) = \mathcal{V}(\rho, w) + \rho\mathcal{V}_\rho(\rho, w), \quad \lambda_2(\rho, w) = \mathcal{V}(\rho, w), \quad (2.2)$$

and corresponding eigenvectors

$$r_1(\rho, w) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad r_2(\rho, w) = \begin{pmatrix} \mathcal{V}_w(\rho, w) \\ -\mathcal{V}_\rho(\rho, w) \end{pmatrix}, \quad (2.3)$$

with the first characteristic field being genuinely non-linear and the second linearly degenerate. The associated Riemann invariants [20, Chapter 7.3] are

$$z_1(\rho, w) = \mathcal{V}(\rho, w), \quad z_2(\rho, w) = w.$$

In the present setting, we are interested in the IBVP (1.2) with values in an invariant domain of the form

$$\Omega := \left\{ U = (\rho, w) \in \mathbb{R}^2 : \rho \in [0, R(w_{max})], w \in [w_{min}, w_{max}] \right\}, \quad (2.4)$$

for some $0 < w_{min} \leq w_{max} < +\infty$. Since $\mathcal{V}_\rho(\rho, w) < 0$ and $\mathcal{V}(0, w) = w$, the range of $v = \mathcal{V}(\rho, w)$ is given by $v \in [0, w]$ for any $w \in [w_{min}, w_{max}]$. Therefore, the inverse function $\rho = \mathcal{R}(v, w)$ is uniquely defined in the invariant domain

$$\mathcal{W} := \left\{ W = (v, w) \in \mathbb{R}^2 : 0 \leq v \leq w, w \in [w_{min}, w_{max}] \right\}. \quad (2.5)$$

Following [4], the vacuum set, i.e. $\rho = 0$, corresponds to $\mathcal{W}_0 := \{(v, w) \in \mathcal{W} : v = w\}$ and the non-vacuum set is denoted $\mathcal{W}_0^c = \mathcal{W} \setminus \mathcal{W}_0$.

For later use in the definition of boundary conditions and in the construction of approximate solutions, we recall in this section the definition of the Riemann solver for the GSOM model (1.1) with initial conditions of the form

$$(\rho, w)(0, x) = \begin{cases} U_L = (\rho_L, w_L) & \text{if } x < 0, \\ U_R = (\rho_R, w_R) & \text{if } x > 0, \end{cases} \quad (2.6)$$

and their corresponding velocities denoted by $v_L = \mathcal{V}(\rho_L, w_L)$, $v_R = \mathcal{V}(\rho_R, w_R)$.

It is well known that the solution of a Riemann problem is based on the theory of elementary waves, such as rarefaction waves, shock waves and contact discontinuities. To define the solution, we introduce the notion of intermediate state $U_M = (\rho_M, w_M)$: in general, the left state U_L is connected to U_M by a first family wave (rarefaction or shock), i.e. $z_2(\rho_L, w_L) = z_2(\rho_M, w_M)$, while U_M is connected to the right state U_R by a contact-discontinuity with $z_1(\rho_M, w_M) = v_M = \mathcal{V}(\rho_M, w_M) = v_R = z_1(\rho_R, w_R)$. Thus, the intermediate state U_M is identified by the system of equations

$$\begin{cases} w_M = w_L, \\ v_M = v_R, \\ \rho_M = \mathcal{R}(v_R, w_L). \end{cases}$$

If $w_L \leq v_R$, we set $\rho_M = 0$, meaning that U_M corresponds to the vacuum. This case is treated separately in Definition 1 (see case 6).

Remark 1. The propagation speed σ of a shock wave between two states U_- and U_+ is given by the Rankine-Hugoniot condition

$$\sigma(U_-, U_+) = \frac{\rho_+ v_+ - \rho_- v_-}{\rho_+ - \rho_-}. \quad (2.7)$$

In this work, we will rely on the following solutions of (1.1), (2.6).

Definition 1 ([4]). *For any $U_L, U_R \in \Omega$, the Riemann solver*

$$\mathcal{RS} : \Omega \times \Omega \rightarrow \mathbf{C}^0 \left(]0, +\infty[; \mathbf{L}_{loc}^1(\mathbb{R}; \Omega) \right), \quad (U_L, U_R) \mapsto \mathcal{RS}(U_L, U_R)$$

is defined as follows:

1. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$, $w_L = w_R$ and $v_L > v_R$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \sigma(U_L, U_R)t, \\ U_R & \text{if } x > \sigma(U_L, U_R)t, \end{cases}$$

with σ defined in (2.7).

2. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$, $w_L \neq w_R$ and $v_L > v_R$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \sigma(U_L, U_M)t, \\ U_M & \text{if } \sigma(U_L, U_M)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with σ defined in (2.7).

3. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$, $w_L = w_R$ and $v_L < v_R$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(\rho_R, w_R)t, \\ U_R & \text{if } \lambda_1(\rho_R, w_R)t < x, \end{cases}$$

with $\hat{U} = (\rho, w_L)$ solving $\lambda_1(\rho, w_L) = \frac{x}{t}$.

4. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$, $w_L \neq w_R$ and $v_L < v_R < w_L$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(\rho_M, w_M)t, \\ U_M & \text{if } \lambda_1(\rho_M, w_M)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with $\hat{U} = (\rho, w_L)$ solving $\lambda_1(\rho, w_L) = \frac{x}{t}$.

5. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$ and $v := v_L = v_R$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < vt, \\ U_R & \text{if } x > vt. \end{cases}$$

6. If $(v_L, w_L), (v_R, w_R) \in \mathcal{W}_0^c$ and $w_L \leq v_R$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(\rho_M, w_M)t, \\ U_M & \text{if } \lambda_1(\rho_M, w_M)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with $\hat{U} = (\rho, w_L)$ solving $\lambda_1(\rho, w_L) = \frac{x}{t}$ and $U_M = (0, w_L)$.

7. If $(v_L, w_L) \in \mathcal{W}_0$ and $(v_R, w_R) \in \mathcal{W}_0^c$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < v_R t, \\ U_R & \text{if } x > v_R t. \end{cases} \quad (2.8)$$

8. If $(v_L, w_L) \in \mathcal{W}_0^c$ and $(v_R, w_R) \in \mathcal{W}_0$, then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(0, w_L)t, \\ \tilde{U}_R & \text{if } \lambda_1(0, w_L)t < x, \end{cases} \quad (2.9)$$

with $\hat{U} = (\rho, w_L)$ solving $\lambda_1(\rho, w_L) = \frac{x}{t}$ and $\tilde{U}_R = (0, w_L)$.

9. If $(v_L, w_L) \in \mathcal{W}_0$ and $(v_R, w_R) \in \mathcal{W}_0$, then

$$\mathcal{RS}(U_L, U_R)(t, x) \equiv U_L. \quad (2.10)$$

Remark 2. We emphasize that in case 7 of Definition 1, if $\rho_M \neq 0$ (i.e. $w_L > v_R$) the solution is either a juxtaposition of a shock wave and a contact discontinuity (if $w_L \neq w_R$) or a shock wave with $U_R = U_M$ (if $w_L = w_R$). The speed of the wave connecting U_L to U_M is given by $\sigma(U_L, U_M) = \frac{\rho_M v_M - \rho_L v_L}{\rho_M - \rho_L} \stackrel{\rho_L=0}{=} v_M = v_R$. Thus, the solution can be also seen as a contact discontinuity. This justifies the definition of the Riemann solver in (2.8).

Remark 3. Case 8 of Definition 1 is based on [4], see also [5, Section 3, Case 4]. The solution in the right vacuum case is obtained by a rarefaction wave independent of $v_R = w_R$. The right state \tilde{U}_R of the solution is not the original state U_R anymore since the speed v and the Lagrangian vehicle property w are set equal to w_L (see Figure 1). This choice is the one better matching real observations: if the road is empty downstream, for example when a traffic light turns green, the solution is expected to be a rarefaction wave, and not a juxtaposition of a rarefaction wave and contact discontinuity or vacuum wave as it is proposed in [23] (see Figure 2a and 2b). Nevertheless, this choice is not compatible with Case 7: the solution to piece-wise constant initial data consisting of three states U_L, U_M and U_R , with $U_M = (0, w_M)$, can be constructed by gluing together the Riemann solutions defined in Cases 7 and 8 if and only if $w_M = w_L$. This is why, for the construction of approximate solutions in Section 4.1, we will need well-prepared initial data, as in [4, Section 2.1]. Moreover, the choice made in Case 8 do not provide a \mathbf{L}^1 -stable Riemann solver close to the vacuum.

The solution for the case $v_L = v_R = w_R$ (resp. $v_R = w_R < v_L$) could also consist of a contact discontinuity (resp. shock wave and contact discontinuity) to U_R (see resp. Figure 2c and Figure 2d) instead of a rarefaction wave to the state \tilde{U}_R . This would be consistent with the structure of the solutions corresponding to U_R close to the vacuum (with $\rho_R > 0$), guaranteeing the \mathbf{L}^1 continuity of the Riemann solver. We emphasize that, as remarked in [4], the set of entropies considered later does not allow to select a unique solution when vacuum is involved. However, the above mentioned alternative choices look unrealistic for traffic applications.

Remark 4. Case 9 of Definition 1 is motivated by coherence with case 8. In general, from a practical point of view, the interpretation of the speed v and the Lagrangian vehicle property w is lost in the vacuum.

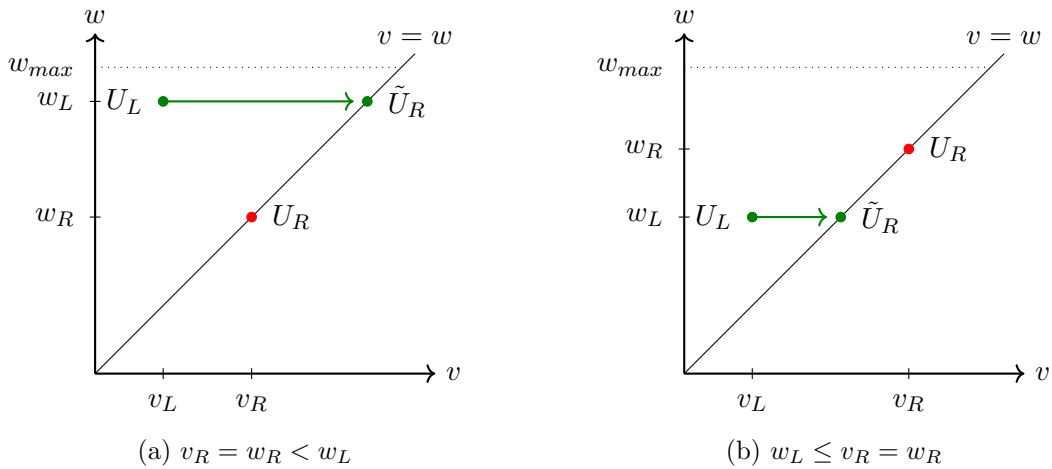


Figure 1: Definition 1, case 8: the solution consists of a rarefaction wave from U_L to \tilde{U}_R .

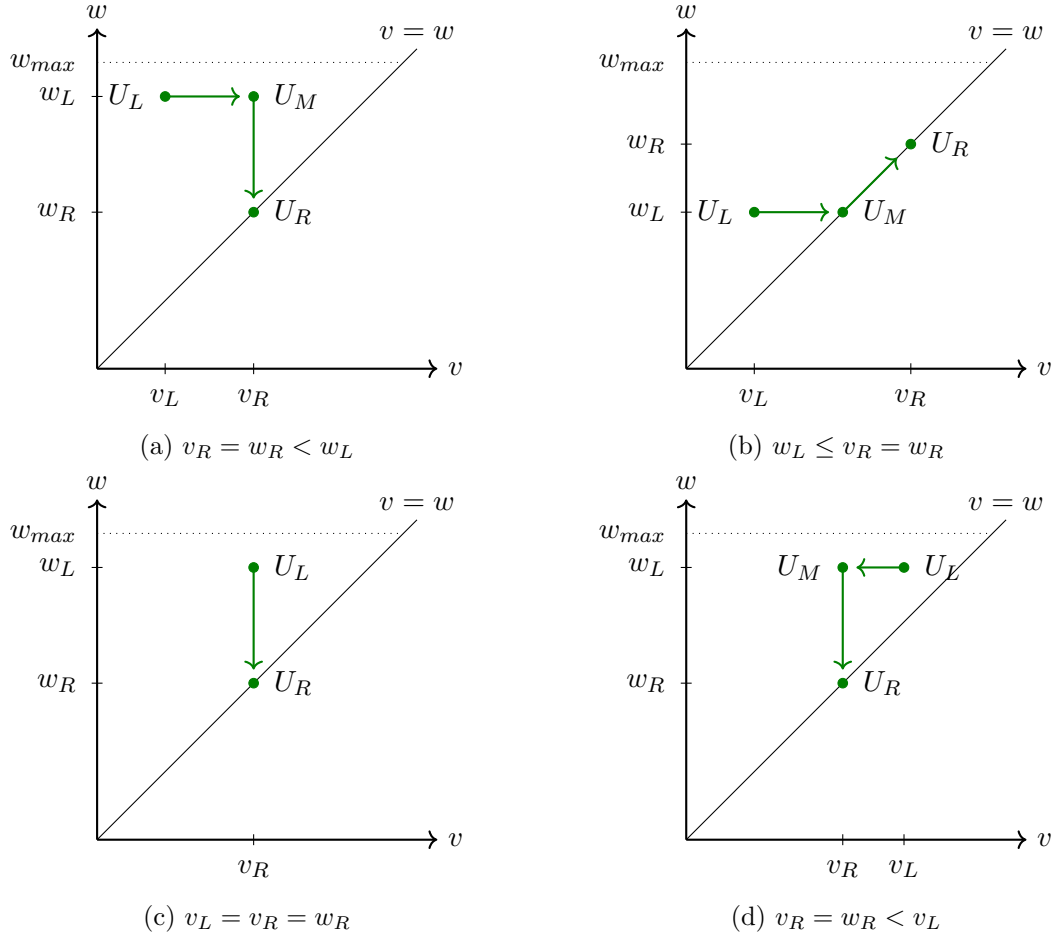


Figure 2: Alternative solutions for the right vacuum case. (a) A rarefaction wave from U_L to U_M and a contact discontinuity from U_M to U_R . (b) A rarefaction wave from U_L to U_M and a vacuum wave from U_M to U_R . (c) A contact discontinuity from state U_L to U_R . (d) A shock wave from U_L to U_M and a contact discontinuity from U_M to U_R .

3 Admissible boundary sets

Since we are dealing with an initial boundary value problem, we describe in this section the sets of admissible values for both the left and right boundaries of (1.2).

3.1 Riemann boundary sets

On the left boundary, only the states (ρ_0, w_0) reachable from a constant boundary datum (ρ_B, w_B) with non-positive waves in the Riemann problem (1.1), (2.6) with data

$$(\rho, w)(0, x) = \begin{cases} (\rho_B, w_B) & \text{if } x < x_{in}, \\ (\rho_0, w_0) & \text{if } x > x_{in}, \end{cases}$$

are admissible. Since second family wave speeds are positive, except those with zero speed, the remaining admissible waves are shock or rarefactions of the first family with non-positive

speed. In this case, the admissible states at the left boundary belong to the curve

$$w = w_B. \quad (3.1)$$

From (2.1b) we know that the curve (3.1) is strictly concave in the $(\rho, \rho v)$ -plane. In particular, there exists a *critical density* $\rho_{cr}(w)$ which maximizes the flow ρv on the curve (3.1), i.e.

$$\rho_{cr}(w) = \underset{\rho}{\operatorname{argmax}} Q(\rho, w) = \underset{\rho}{\operatorname{argmax}} (\rho \mathcal{V}(\rho, w)) \quad \text{for any } w \in [w_{min}, w_{max}]. \quad (3.2)$$

Additionally, there exists a unique density $\tau(\rho) \neq \rho$ such that $Q(\tau(\rho), w) = Q(\rho, w)$ for each $\rho \neq \rho_{cr}(w)$ and any $w \in [w_{min}, w_{max}]$.

Remark 5. From a geometrical point of view, it is worth noticing that the slope of the tangent to the curve (3.1) in the $(\rho, \rho v)$ -plane coincides with the first eigenvalue, indeed

$$Q_\rho(\rho, w) = \mathcal{V}(\rho, w) + \rho \mathcal{V}_\rho(\rho, w) = \lambda_1(\rho, w) \quad \text{for any } w \in [w_{min}, w_{max}].$$

Moreover, the slope of the secant between any two points, U_- and U_+ , in the $(\rho, \rho v)$ -plane is given by the shock wave speed (2.7).

Proposition 1 describes the admissible states on the left boundary (see also [29, 30]):

Proposition 1. *Let $U_B := (\rho_B, w_B) \in \Omega$ with $\rho_B > 0$ be the left boundary datum at $x = x_{in}$. The Riemann admissible boundary set $\mathcal{B}_L^{Rie}(\rho_B, w_B)$ is composed of the following states $U_0 = (\rho_0, w_0)$:*

- $w_0 = w_B$ and
 1. if $\rho_B < \rho_{cr}(w_B)$: $U_0 = U_B$ or $\rho_0 \geq \tau(\rho_B)$ (see Figure 3a);
 2. if $\rho_B \geq \rho_{cr}(w_B)$: $\rho_0 \geq \rho_{cr}(w_B)$ (see Figure 3b);
- the set of points $\{U_0 = (R(w_0), w_0) : w_0 \in [w_{min}, w_{max}]\}$, which can be reached from U_B with a negative 1-shock to $(R(w_B), w_B)$, followed by a contact discontinuity with zero speed.

In the vacuum case $\rho_B = 0$ and $v_B = \mathcal{V}(0, w_B) = w_B$, then the admissible states are $U_0 = U_B$ and $\{U_0 = (R(w_0), w_0) : w_0 \in [w_{min}, w_{max}]\}$ (contact discontinuity with zero speed) (see Figure 3c).

Concerning the right boundary, first and second family curves with non-negative wave speeds are admissible when solving the Riemann problem (1.1), (2.6) with data

$$(\rho, w)(0, x) = \begin{cases} (\rho_0, w_0) & \text{if } x < x_{out}, \\ (\rho_B, w_B) & \text{if } x > x_{out}. \end{cases}$$

The admissible set will thus be two-dimensional.

Proposition 2. *Let $U_B := (\rho_B, w_B) \in \Omega$ with $\rho_B > 0$ be the right boundary datum at $x = x_{out}$. Then, the Riemann admissible boundary set $\mathcal{B}_R^{Rie}(\rho_B, w_B)$ is composed of the states $U_0 = (\rho_0, w_0)$ such that (see Figure 4a):*

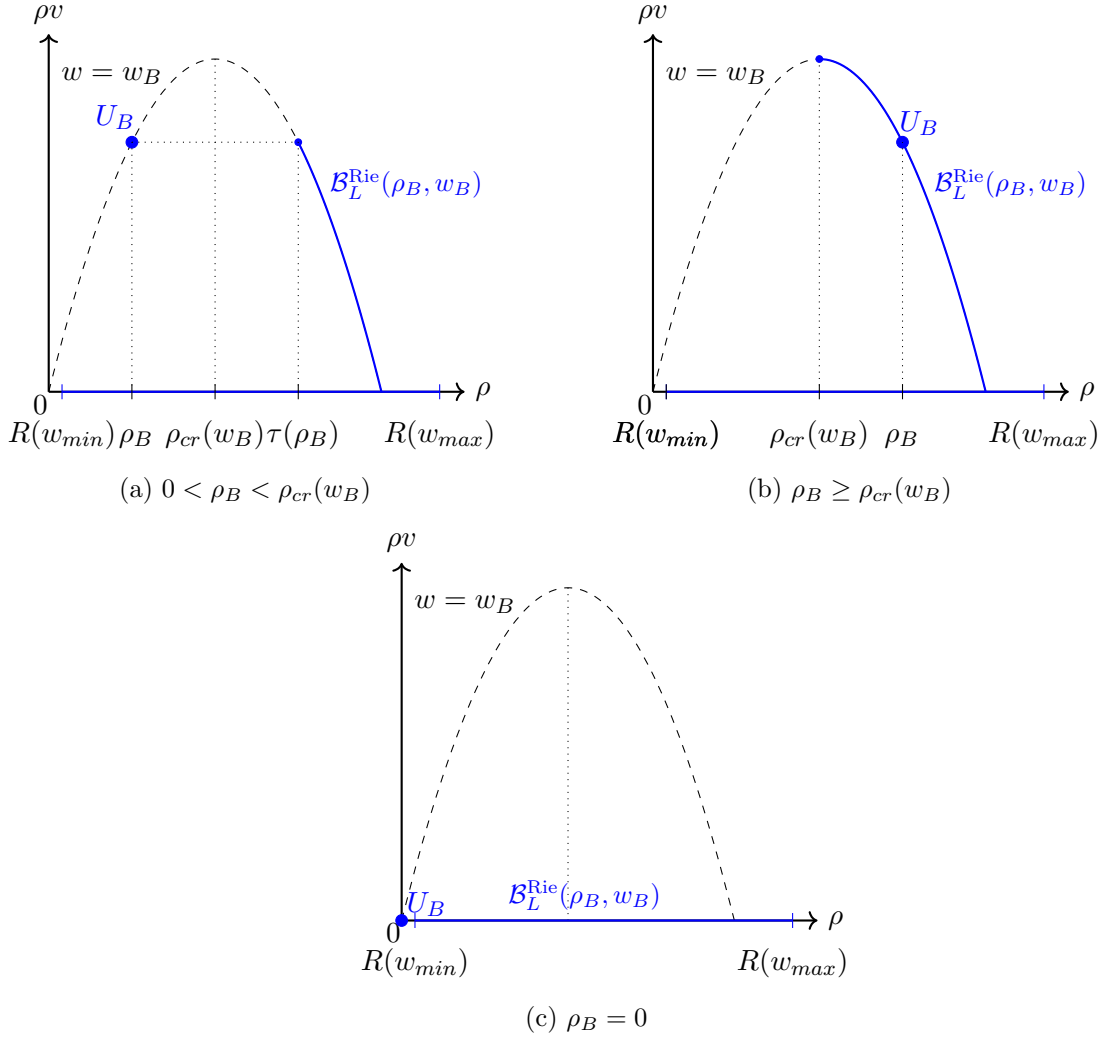


Figure 3: Riemann admissible boundary sets $\mathcal{B}_L^{\text{Rie}}(\rho_B, w_B)$ at the left boundary.

- $z_1(\rho_0, w_0) = v_B$;
- $\mathcal{V}(\rho_{cr}(w_0), w_0) \leq v_B$ and $\rho_0 \leq \rho_{cr}(w_0)$;
- $\mathcal{V}(\rho_{cr}(w_0), w_0) > v_B$ and $\rho_0 \leq \tau(\mathcal{R}(v_B, w_0))$.

In the vacuum case $\rho_B = 0$ and $v_B = \mathcal{V}(0, w_B) = w_B$, then U_0 is admissible if and only if $\rho_0 \leq \rho_{cr}(w_0)$ (rarefaction wave with non-negative speed, see Figure 4b).

Remark 6. We emphasize that in Proposition 2 we do not need to distinguish between two different cases for the vacuum case $\rho_B = 0$ (in contrast to $\rho_B > 0$), due to the definition of the Riemann solver (case 8 in Definition 1): the solution is always a rarefaction wave.

Remark 7. In the case $\rho_B > 0$, any state on the curve $\{z_1(\rho, w) = v_B\}$ is admissible since we can connect it to ρ_B by a contact discontinuity (see case 5 in Definition 1).

Remark 8. We note that the right Riemann boundary set $\mathcal{B}_R^{Rie}(\rho_B, w_B)$ is independent of the variable w_B , i.e. $\mathcal{B}_R^{Rie}(\rho_B, w_B) = \mathcal{B}_R^{Rie}(\mathcal{R}(v_B, w), w)$ for all $w \in [w_{min}, w_{max}]$, with $v_B = \mathcal{V}(\rho_B, w_B)$. This holds for all $(\rho_B, w_B) \in \Omega$.

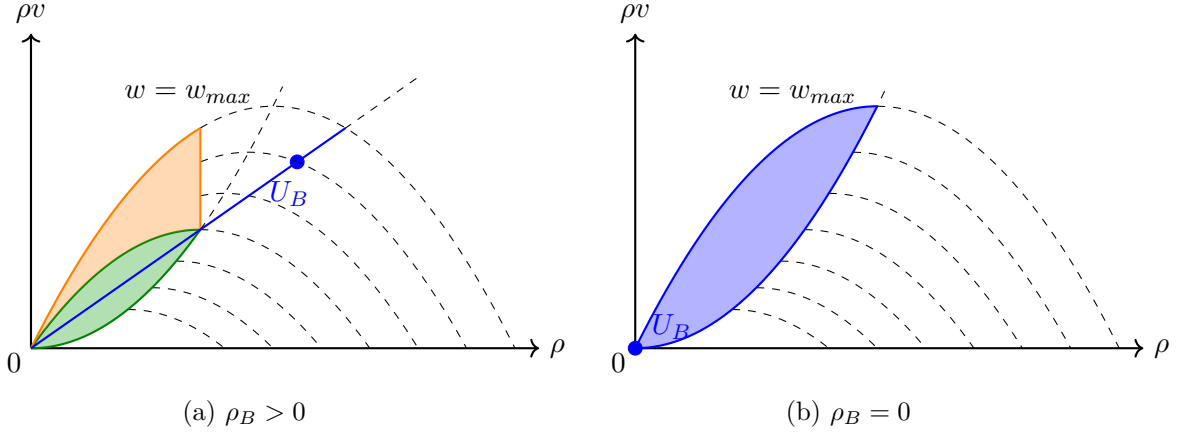


Figure 4: Riemann admissible boundary sets $\mathcal{B}_R^{Rie}(\rho_B, w_B)$ at the right boundary for $Q(\rho, w) = \rho(w - \rho)$.

(a) The green (resp. orange) region refers to case $\mathcal{V}(\rho_{cr}(w_0), w_0) \leq v_B$ (resp. $\mathcal{V}(\rho_{cr}(w_0), w_0) > v_B$) in Proposition 2. The blue line represents the admissible points on the curve $\{z_1(\rho, w) = v_B\}$.

(b) The admissible region for the vacuum boundary datum is indicated in blue.

3.2 Entropy boundary sets

Defining $u = (\rho, \rho w)^\top \in \tilde{\Omega}$ with $\tilde{\Omega} = \{(\rho, \rho w) \in \mathbb{R}^2 : \rho \in [0, R(w_{max})], w \in [w_{min}, w_{max}]\}$ and $f(u) = (\rho v, \rho w v)^\top$, system (1.1) can be written more compactly as

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0.$$

The definition of admissible values at the boundary is based on the notion of boundary entropy inequality [8, 21]:

Definition 2. For each boundary state $u_B = (\rho_B, \rho_B w_B) \in \tilde{\Omega}$, the set of entropy admissible values at the left (resp. right) boundary, denoted by $\mathcal{B}_L^{Ent}(\rho_B, w_B)$ (resp. $\mathcal{B}_R^{Ent}(\rho_B, w_B)$), is defined as all the states $u = (\rho, \rho w) \in \tilde{\Omega}$ satisfying

$$\beta(u, u_B) = \mathcal{Q}(u) - \mathcal{Q}(u_B) - \nabla \mathcal{E}(u_B) \cdot \{f(u) - f(u_B)\} \leq (\geq) 0 \quad (3.3)$$

for each entropy-flux pair $(\mathcal{E}, \mathcal{Q})$.

Following [20, Section 7.4], we seek for entropy-flux pairs $(\mathcal{E}^j, \mathcal{Q}^j)$, $j \in \{1, 2\}$, which are functions of the Riemann invariants $W = (v, w)$, then setting $u = u(W)$. In particular, we consider the family of entropy-flux pairs derived in [4, Equation 2.13]:

$$\mathcal{E}^1(u(W)) = \begin{cases} 0 & \text{if } v \leq \bar{v}, \\ 1 - \frac{\mathcal{R}(v, w)}{\mathcal{R}(\bar{v}, w)} & \text{if } v > \bar{v}, \end{cases} \quad (3.4a)$$

$$\mathcal{Q}^1(u(W)) = \begin{cases} 0 & \text{if } v \leq \bar{v}, \\ \bar{v} - \frac{v\mathcal{R}(v,w)}{\mathcal{R}(\bar{v},w)} & \text{if } v > \bar{v}, \end{cases} \quad (3.4b)$$

for any $\bar{v} \in [0, w_{max}]$.

Additionally, we consider the pairs identified by the left eigenvector $l_2(u(W)) = \begin{pmatrix} w & -1 \end{pmatrix}$ (see [37, Chapter 13]):

$$\mathcal{E}^2(u(W)) = |l_2(u(\bar{W})) \cdot (u(W) - u(\bar{W}))| = \mathcal{R}(v, w)|\bar{w} - w| \quad (3.5a)$$

$$\begin{aligned} \mathcal{Q}^2(u(W)) &= l_2(u(\bar{W})) \cdot (f(u(W)) - f(u(\bar{W}))) \operatorname{sgn} \left(l_2(u(\bar{W})) \cdot (u(W) - u(\bar{W})) \right) \\ &= v\mathcal{R}(v, w)|\bar{w} - w|, \end{aligned} \quad (3.5b)$$

for any $\bar{w} \in [w_{min}, w_{max}]$.

For notational simplicity, throughout this section we will drop the u -variable dependency and we write $\mathcal{E}^j(W) = \mathcal{E}^j(v, w)$ (resp. $\mathcal{Q}^j(W) = \mathcal{Q}^j(v, w)$) instead of $\mathcal{E}^j(u(W))$ (resp. $\mathcal{Q}^j(u(W))$) and $f(W) = f(v, w) = \begin{pmatrix} \mathcal{R}(v, w)v \\ \mathcal{R}(v, w)wv \end{pmatrix}$ instead of $f(u(W))$ when possible. Thus, the entropy boundary condition (3.3) expressed in Riemann invariants reads

$$\beta^j(W, W_B) := \mathcal{Q}^j(W) - \mathcal{Q}^j(W_B) - \nabla_u \mathcal{E}^j(W_B) \cdot (f(W) - f(W_B)) \leq (\geq) 0 \quad (3.6)$$

for $j \in \{1, 2\}$, where

$$\nabla_u \mathcal{E}^j(W) = \nabla_u \mathcal{E}^j(v, w) = \begin{pmatrix} \frac{\partial v}{\partial \rho} \mathcal{E}_v^j(v, w) + \frac{\partial w}{\partial \rho} \mathcal{E}_w^j(v, w) \\ \frac{\partial v}{\partial y} \mathcal{E}_v^j(v, w) + \frac{\partial w}{\partial y} \mathcal{E}_w^j(v, w) \end{pmatrix},$$

with $y = \rho w$ and

$$\begin{aligned} \frac{\partial v}{\partial \rho}(v, w) &= \frac{\partial \mathcal{V}(\rho, \frac{y}{\rho})}{\partial \rho} = \mathcal{V}_\rho(\rho, \frac{y}{\rho}) - \frac{y}{\rho^2} \mathcal{V}_w(\rho, \frac{y}{\rho}) = \mathcal{V}_\rho(\mathcal{R}(v, w), w) - \frac{w}{\mathcal{R}(v, w)} \mathcal{V}_w(\mathcal{R}(v, w), w), \\ \frac{\partial v}{\partial y}(v, w) &= \frac{\partial \mathcal{V}(\rho, \frac{y}{\rho})}{\partial y} = \frac{1}{\rho} \mathcal{V}_w(\rho, \frac{y}{\rho}) = \frac{1}{\mathcal{R}(v, w)} \mathcal{V}_w(\mathcal{R}(v, w), w), \\ \frac{\partial w}{\partial \rho}(v, w) &= -\frac{y}{\rho^2} = -\frac{w}{\mathcal{R}(v, w)}, \\ \frac{\partial w}{\partial y}(v, w) &= \frac{1}{\rho} = \frac{1}{\mathcal{R}(v, w)}. \end{aligned}$$

In the case $v \neq \bar{v}$, the partial derivatives of \mathcal{E}^1 are given by

$$\begin{aligned} \mathcal{E}_v^1(v, w) &= \begin{cases} 0 & \text{if } v < \bar{v}, \\ -\frac{\mathcal{R}_v(v, w)}{\mathcal{R}(\bar{v}, w)} & \text{if } v > \bar{v}, \end{cases} \\ \mathcal{E}_w^1(v, w) &= \begin{cases} 0 & \text{if } v < \bar{v}, \\ -\frac{\mathcal{R}_w(v, w)\mathcal{R}(\bar{v}, w) - \mathcal{R}(v, w)\mathcal{R}_w(\bar{v}, w)}{\mathcal{R}(\bar{v}, w)^2} & \text{if } v > \bar{v}. \end{cases} \end{aligned}$$

If $v = \bar{v}$, the sub-differential of \mathcal{E}^1 reads as

$$\mathcal{E}_v^1(v, w) = \left\{ \alpha: \alpha \in \left[0, -\frac{\mathcal{R}_v(v, w)}{\mathcal{R}(v, w)} \right] \right\},$$

$$\mathcal{E}_w^1(v, w) = 0.$$

By (3.5a), the gradient of \mathcal{E}^2 can be computed directly by

$$\nabla \mathcal{E}_u^2(W) = \begin{cases} l_2(u(\bar{W})) \operatorname{sgn}(l_2(u(\bar{W}))) \cdot (u(W_B) - u(\bar{W})) & \text{if } u(W_B) \neq u(\bar{W}), \\ \{\gamma l_2(u(\bar{W})); \gamma \in [-1, 1]\} & \text{if } u(W_B) = u(\bar{W}). \end{cases}$$

Remark 9. Deriving $\mathcal{V}(\mathcal{R}(v, w), w) = v$ with respect to v , we get

$$\mathcal{V}_\rho(\mathcal{R}(v, w), w) \mathcal{R}_v(v, w) = 1 \text{ and thus } \mathcal{R}_v(v, w) = \frac{1}{\mathcal{V}_\rho(\mathcal{R}(v, w), w)}.$$

Moreover, deriving $\mathcal{V}(\mathcal{R}(v, w), w) = v$ with respect to w , we obtain

$$\mathcal{V}_\rho(\mathcal{R}(v, w), w) \mathcal{R}_w(v, w) + \mathcal{V}_w(\mathcal{R}(v, w), w) = 0 \text{ and thus } \mathcal{R}_w(v, w) = -\frac{\mathcal{V}_w(\mathcal{R}(v, w), w)}{\mathcal{V}_\rho(\mathcal{R}(v, w), w)}.$$

Using the above identities, we can prove that (3.4) and (3.5) satisfy

$$\mathcal{Q}_v^j = \lambda_1(v, w) \mathcal{E}_v^j(v, w), \quad \mathcal{Q}_w^j = \lambda_2(v, w) \mathcal{E}_w^j(v, w)$$

with $\lambda_1(v, w) = v + \mathcal{R}(v, w) \mathcal{V}_\rho(\mathcal{R}(v, w), w)$ and $\lambda_2(v, w) = v$, and are therefore entropy-flux pairs, see [20, Equation (7.4.12)].

Finally, for $j = 1$ we obtain

$$\beta^1(W, W_B) = \begin{cases} \begin{cases} \rho v \left(\frac{1}{\mathcal{R}(\bar{v}, w_B)} - \frac{1}{\mathcal{R}(\bar{v}, w)} + \frac{1}{\mathcal{R}(\bar{v}, w_B)^2} \frac{\mathcal{V}_w(\mathcal{R}(\bar{v}, w_B), w_B)}{\mathcal{V}_\rho(\mathcal{R}(\bar{v}, w_B), w_B)} (w - w_B) \right) & \text{if } v > \bar{v}, \\ -\bar{v} + \rho v \left(\frac{1}{\mathcal{R}(\bar{v}, w_B)} + \frac{1}{\mathcal{R}(\bar{v}, w_B)^2} \frac{\mathcal{V}_w(\mathcal{R}(\bar{v}, w_B), w_B)}{\mathcal{V}_\rho(\mathcal{R}(\bar{v}, w_B), w_B)} (w - w_B) \right) & \text{if } v \leq \bar{v}, \end{cases} & \text{if } v_B > \bar{v}, \\ \begin{cases} \bar{v} - \rho v \frac{1}{\mathcal{R}(\bar{v}, w)} & \text{if } v > \bar{v}, \\ 0 & \text{if } v \leq \bar{v}, \end{cases} & \text{if } v_B < \bar{v}, \end{cases} \quad (3.9a)$$

if $\bar{v} \neq v_B$ and

$$\beta^1(W, W_B) = \begin{cases} \begin{cases} \rho v \left(\frac{1}{\rho_B} - \frac{1}{\mathcal{R}(v_B, w)} + \frac{1}{\rho_B^2} \frac{\mathcal{V}_w(\rho_B, w_B)}{\mathcal{V}_\rho(\rho_B, w_B)} (w - w_B) \right) & \text{if } v > v_B, \\ -v_B + \rho v \left(\frac{1}{\rho_B} + \frac{1}{\rho_B^2} \frac{\mathcal{V}_w(\rho_B, w_B)}{\mathcal{V}_\rho(\rho_B, w_B)} (w - w_B) \right) & \text{if } v \leq v_B, \end{cases} & \text{if } \alpha = -\frac{\mathcal{R}_v(v_B, w_B)}{\mathcal{R}(v_B, w_B)}, \\ \begin{cases} v_B - \rho v \frac{1}{\mathcal{R}(v_B, w)} & \text{if } v > v_B, \\ 0 & \text{if } v \leq v_B, \end{cases} & \text{if } \alpha = 0, \end{cases} \quad (3.9b)$$

if $\bar{v} = v_B$.

For $j = 2$, it holds

$$\beta^2(W, W_B) = \rho v (w - \bar{w}) \left(\operatorname{sgn}(\mathcal{R}(v_B, w_B)(\bar{w} - w_B)) - \operatorname{sgn}(\rho(\bar{w} - w)) \right), \quad (3.10a)$$

if $u(\bar{W}) \neq u(W_B)$ and

$$\beta^2(W, W_B) = \begin{cases} \begin{cases} 0 & \text{if } w > w_B, \\ 2\rho v(w_B - w) & \text{if } w \leq w_B, \end{cases} & \text{if } \gamma = -1, \\ \begin{cases} 2\rho v(w - w_B) & \text{if } w > w_B, \\ 0 & \text{if } w \leq w_B, \end{cases} & \text{if } \gamma = 1, \end{cases} \quad (3.10b)$$

if $u(\bar{W}) = u(W_B)$.

We now state that the Riemann boundary condition implies the entropy one, as already proven in [10, Theorem 1].

Proposition 3. *The admissible states defined by the Riemann solver satisfy the entropy boundary condition, i.e. $\mathcal{B}_i^{Rie}(\rho_B, w_B) \subseteq \mathcal{B}_i^{Ent}(\rho_B, w_B)$ for $i \in \{L, R\}$.*

For a detailed proof of Proposition 3, we refer to Appendix A. Finally, we end this section by verifying the following Proposition.

Proposition 4. *The following equalities hold for the boundary sets:*

$$\begin{aligned} \mathcal{B}_L^{Ent}(\rho_B, w_B) &= \mathcal{B}_L^{Rie}(\rho_B, w_B) \cup \mathcal{B}_L^*(\rho_B, w_B), \\ \mathcal{B}_R^{Ent}(\rho_B, w_B) &= \mathcal{B}_R^{Rie}(\rho_B, w_B), \end{aligned} \quad (3.11)$$

with $\mathcal{B}_L^*(\rho_B, w_B) = \{(\rho, w) \mid \rho = 0, w < w_B, \mathcal{V}(\rho, w) \leq \mathcal{V}(\rho_B, w_B)\}$.

Proof. Due to Proposition 3, it suffices to prove

1. $\mathcal{B}_L^*(\rho_B, w_B) \subset \mathcal{B}_L^{Ent}(\rho_B, w_B)$;
2. $\mathcal{B}_L^{Ent}(\rho_B, w_B) \subseteq \mathcal{B}_L^{Rie}(\rho_B, w_B) \cup \mathcal{B}_L^*(\rho_B, w_B)$;
3. $\mathcal{B}_R^{Ent}(\rho_B, w_B) \subseteq \mathcal{B}_R^{Rie}(\rho_B, w_B)$.

We treat separately the above points.

1. We observe that $\beta^2(W, W_B) = 0$ in (3.10a) for any $\bar{w} \in [w_{min}, w_{max}]$ since $\rho = 0$. Moreover, for any $\bar{v} \in [0, w_{max}]$, it holds in (3.9a) and (3.9b) that $\beta^1(W, W_B) \leq 0$ due to $v \leq w_B$ and $\rho = 0$.

To prove inclusion 2 and 3, we will show that for any $(\rho, w) \notin \mathcal{B}_i^{Rie}(\rho_B, w_B)$ (and $(\rho, w) \notin \mathcal{B}_L^*(\rho_B, w_B)$ if $i = L$), there exists a $\bar{v} \in [0, w_{max}]$ or $\bar{w} \in [w_{min}, w_{max}]$ for which $(\rho, w) \notin \mathcal{B}_i^{Ent}(\rho_B, w_B)$, $i \in \{L, R\}$.

2. From Proposition 1, we observe that $\Omega \setminus (\mathcal{B}_L^{Rie}(\rho_B, w_B) \cup \mathcal{B}_L^*(\rho_B, w_B)) = \bigcup_{i=1}^5 \mathcal{K}_L^i(\rho_B, w_B)$,

where

$$\begin{aligned} \mathcal{K}_L^1(\rho_B, w_B) &= \{(\rho, w) \mid \rho \in [0, \rho_B[, w = w_B, 0 < \rho_B < \rho_{cr}(w_B)\}, \\ \mathcal{K}_L^2(\rho_B, w_B) &= \{(\rho, w) \mid \rho \in]\rho_B, \tau(\rho_B)[, w = w_B, 0 < \rho_B < \rho_{cr}(w_B)\}, \\ \mathcal{K}_L^3(\rho_B, w_B) &= \{(\rho, w) \mid \rho < \rho_{cr}(w_B), w = w_B, \rho_B \geq \rho_{cr}(w_B)\}, \end{aligned}$$

$$\begin{aligned}\mathcal{K}_L^4(\rho_B, w_B) &= \{(\rho, w) \mid \mathcal{V}(\rho, w) > 0, w \neq w_B, \rho_B \geq 0\} \setminus \mathcal{B}_L^*(\rho_B, w_B), \\ \mathcal{K}_L^5(\rho_B, w_B) &= \{(\rho, w) \mid \mathcal{V}(\rho_B, w_B) \neq \mathcal{V}(\rho, w) > 0, w = w_B, \rho_B = 0\}.\end{aligned}$$

Thus, $(\rho, w) \notin \left(\mathcal{B}_L^{Rie}(\rho_B, w_B) \cup \mathcal{B}_L^*(\rho_B, w_B) \right)$ if and only if $(\rho, w) \in \bigcup_{i=1}^5 \mathcal{K}_L^i(\rho_B, w_B)$.

Let $(\rho, w) \in \mathcal{K}_L^1(\rho_B, w_B)$. Since $v > v_B$, it holds $\rho v < \rho_B v_B$. Choosing $v_B < \bar{v} < v$, we have $\mathcal{R}(\bar{v}, w) > \rho v$ and thus, by (3.9a), $\beta^1(W, W_B) > 0$.

If $(\rho, w) \in \mathcal{K}_L^2(\rho_B, w_B)$, it holds $v < v_B$. Setting $\bar{v} = v_B$, we obtain $\rho v > \rho_B v_B = \mathcal{R}(\bar{v}, w_B)\bar{v}$ and, by (3.9b), again that $\beta^1(W, W_B) > 0$.

If $(\rho, w) \in \mathcal{K}_L^3(\rho_B, w_B)$, we have $v > v_B$. Choosing $\bar{v} = V(\rho_{cr}(w_B), w_B)$, it holds $v_B \leq \bar{v} < v$ leading to $\rho v < \mathcal{R}(\bar{v}, w)\bar{v}$ and therefore $\beta^1(W, W_B) > 0$.

If $(\rho, w) \in \mathcal{K}_L^4(\rho_B, w_B)$, we distinguish between the following two cases:

- $w > w_B$:
 - * If $v > v_B$, we either have $\mathcal{R}(v_B, w) \leq \rho_{cr}(w)$, resulting in $\mathcal{R}(v_B, w)v_B > \rho v$. Choosing $\bar{v} = v_B < v$, we obtain $\beta^1(W, W_B) > 0$. Or it holds $\mathcal{R}(v_B, w) > \rho_{cr}(w)$, leading to $\mathcal{R}(\bar{v}, w)\bar{v} > \rho v$ by considering $\bar{v} = \mathcal{V}(\rho_{cr}(w), w)$. Thus, it holds again $\beta^1(W, W_B) > 0$.
 - * If $0 < v \leq v_B$, we know that $\rho \neq 0$. Choosing $\bar{w} \in]w_B, w[$ implies, by (3.10a), $\beta^2(W, W_B) > 0$.
- $w < w_B$.
 - * If $v > v_B$, we obtain with the same argumentation as in the case $w > w_B$ that $\beta^1(W, W_B) > 0$.
 - * If $0 < v \leq v_B$, $\rho \neq 0$ implies $\beta^2(W, W_B) > 0$ by choosing $\bar{w} \in]w, w_B[$.

Finally, let $(\rho, w) \in \mathcal{K}_L^5(\rho_B, w_B)$. It holds $w = w_B = v_B > v$ (otherwise $(\rho, w) = (\rho_B, w_B) \in \mathcal{B}^{Rie}(\rho_B, w_B)$). Since $0 = \rho_B v_B < \rho v$ and considering $\bar{v} = v_B$, we conclude $\beta^1(W, W_B) > 0$.

This shows that $\mathcal{B}_L^{Ent}(\rho_B, w_B) \subseteq \mathcal{B}_L^{Rie}(\rho_B, w_B) \cup \mathcal{B}_L^*(\rho_B, w_B)$.

3. By Proposition 2, we observe that $\Omega \setminus \mathcal{B}_R^{Rie}(\rho_B, w_B) = \bigcup_{i=1}^3 \mathcal{K}_R^i(\rho_B, w_B)$, where

$$\begin{aligned}\mathcal{K}_R^1(\rho_B, w_B) &= \{(\rho, w) \mid \rho > \rho_{cr}(w), V(\rho_{cr}(w), w) \leq v_B\}, \\ \mathcal{K}_R^2(\rho_B, w_B) &= \{(\rho, w) \mid \rho > \mathcal{R}(v_B, w), V(\rho_{cr}(w), w) > v_B\}, \\ \mathcal{K}_R^3(\rho_B, w_B) &= \{(\rho, w) \mid \rho \in]\tau(\mathcal{R}(v_B, w), \mathcal{R}(v_B, w)[, V(\rho_{cr}(w), w) > v_B\}.\end{aligned}$$

Thus, $(\rho, w) \notin \mathcal{B}_R^{Rie}(\rho_B, w_B)$ if and only if $(\rho, w) \in \bigcup_{i=1}^3 \mathcal{K}_R^i(\rho_B, w_B)$. Moreover, as in

Remark 8, it holds $\mathcal{K}_R^i(\rho_B, w_B) = \mathcal{K}_R^i(\mathcal{R}(v_B, w), w)$ for $i \in \{1, 2, 3\}$, since the sets are defined by v_B . Thus it suffices to consider $w = w_B$.

Let $(\rho, w) \in \mathcal{K}_R^1(\rho_B, w_B)$. Since $v < v_B$, we choose $v < \bar{v} = \mathcal{V}(\rho_{cr}(w), w) \leq v_B$ implying $\bar{v}\mathcal{R}(\bar{v}, w_B) > \rho v$ and leading to $\beta^1(W, W_B) < 0$.

If $(\rho, w) \in \mathcal{K}_R^2(\rho_B, w_B)$ we consider $\bar{v} = v_B > v$ which results in $\rho_B v_B > \rho v$ and thus $\beta^1(W, W_B) < 0$.

Finally, let $(\rho, w) \in \mathcal{K}_R^3(\rho_B, w_B)$. Thus, taking $\bar{v} = v_B < v$, it follows $v_B \mathcal{R}(v_B, w) < \rho v$

and again $\beta^1(W, W_B) < 0$.

This shows that $\mathcal{B}_R^{Ent}(\rho_B, w_B) \subseteq \mathcal{B}_R^{Rie}(\rho_B, w_B)$.

□

Remark 10. The family of entropy-flux pairs $(\mathcal{E}^2, \mathcal{Q}^2)$ defined by (3.5) are essential to obtain the first equality in (3.11). If we considered only the family $(\mathcal{E}^1, \mathcal{Q}^1)$ constructed by (3.5), there would be points (away from the vacuum) which are admissible for the left entropy but not for the left Riemann boundary set: as a demonstration, we choose $Q(\rho, w) = \rho(w - \rho)$ (see Figure 4), $W_B = (v_B, w_B) = (1, 1.6)$,

$W = (v, w) = (0.4417, 1.8)$. In this specific case, we observe for any $\bar{v} \in [0, w_{max}]$ that $\beta^1(W, W_B) \leq 0$ (see Figure 5). However, $(\rho, w) \notin \mathcal{B}_L^{Rie}(\rho_B, w_B)$ since $w \neq w_B$.

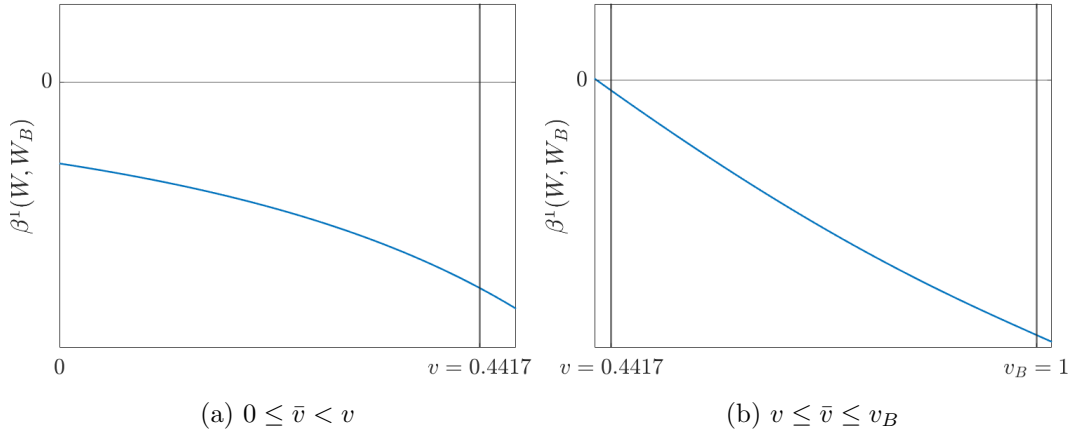


Figure 5: Illustration of the entropy boundary condition $\beta^1(W, W_B)$ for a point (ρ, w) which is not admissible for the left Riemann boundary set. The case $\bar{v} > v_B$ is not depicted in the Figure since it holds $\beta^1(W, W_B) = 0$ due to (3.9a).

Remark 11. The chosen entropy families (3.4) and (3.5) do not provide the equality between the left Riemann and entropy boundary sets. In fact, the two sets differ for points in the vacuum positioned as in Figures 2c and 2d with respect to the boundary datum $(v_L = v_B \geq v = v_R$ and $w_L = w_B > w = w_R)$, which are described by the set $\mathcal{B}_L^*(\rho_B, w_B)$. We emphasize that this is the only case where the two sets do not coincide.

One could avoid this problem by setting $w = w_{max}$ whenever $\rho = 0$. However, even if this choice would allow to prove the equivalence between the two boundary sets, it cannot be guaranteed when passing to the limit in approximate solutions (as done in the proof of Proposition 7), since we can end up with vacuum states $U = (0, w)$, $w \notin w_{max}$, which do not belong to the Riemann solver.

Nevertheless, if we consider an invariant domain not including the vacuum, i.e.

$$\mathcal{W} := \left\{ W = (v, w) \in \mathbb{R}^2 : 0 \leq v < w, w \in [w_{min}, w_{max}], v \in [v_{min}, v_{max}], v_{max} < w_{min} \right\},$$

for some $0 \leq v_{min} < v_{max}$, it holds $\mathcal{B}_i^{Rie}(\rho_B, w_B) = \mathcal{B}_i^{Ent}(\rho_B, w_B)$ for $i \in \{L, R\}$ (and not only $i = R$). This is a new result compared to [3, 10, 21], since it applies to a Temple class system whose characteristic lines of the second family are not straight (see Remark 12) and the boundary is possibly characteristic (the first eigenvalue can change sign).

Remark 12. Unlike in [3], the family of functions

$$\eta(u(W)) = |l_1(u(\bar{W})) \cdot (u(W) - u(\bar{W}))|, \quad \bar{W} \in \mathcal{W},$$

corresponding to the first left eigenvector $l_1(u) = \begin{pmatrix} \rho \mathcal{V}_\rho(\rho, w) - w \mathcal{V}_w(\rho, w) & \mathcal{V}_w(\rho, w) \end{pmatrix}$, are not of use in this setting, since the level sets of the first Riemann invariant $z_1(\rho, w)$ are not straight lines in the conservative variables $u = (\rho, y)$, see for example [5] in the case of the ARZ model.

4 Existence of entropy weak solutions

Since we are dealing with entropy-flux pairs $(\mathcal{E}, \mathcal{Q})$ expressed in Riemann invariants, it is convenient to rewrite (1.2) into the same variables, and on a limited time interval, i.e.

$$\partial_t u(W) + \partial_x f(u(W)) = 0, \quad x \in]x_{in}, x_{out}[, t \in]0, T[, \quad (4.1a)$$

$$W(0, x) = W_0(x), \quad x \in]x_{in}, x_{out}[, \quad (4.1b)$$

$$W(t, x_{in}) = W_{in}(t) = (v_{in}, w_{in})(t), \quad t \in]0, T[, \quad (4.1c)$$

$$W(t, x_{out}) = W_{out}(t) = (v_{out}, w_{out})(t), \quad t \in]0, T[, \quad (4.1d)$$

where $u(W) = (\mathcal{R}(v, w), \mathcal{R}(v, w)w)^\top$, $f(u(W)) = v \cdot u(W)$. Observe that, since we deal with solutions in weak (distributional) sense, problem (4.1), set on a bounded time interval, is equivalent to (1.2).

It is also convenient, for the existence proof, to set $W_0(x+) = (w_0(x-), w_0(x-))$ whenever $W_0(x+) \in \mathcal{W}_0$, assuming $W_0 \in \text{BV}(]x_{in}, x_{out}[; \mathcal{W})$ (see Remark 13) and therefore traces are defined at each point $x \in]x_{in}, x_{out}[$. Indeed, this does not change the initial condition in conservative variables, since $u(W_1) = u(W_2) = (0, 0)^\top$ if $W_1, W_2 \in \mathcal{W}_0$. Moreover, we set $W_{out}(t) = (w_{max}, w_{max})$ whenever $W_{out}(t) \in \mathcal{W}_0$. This does not impact the solution, which is independent of w_{out} (as $\mathcal{B}_R^{Ent}(\rho_B, w_B) = \mathcal{B}_R^{Rie}(\rho_B, w_B)$ is independent of w_B).

Remark 13. We remark that if $W_0 = W_0^- \in \text{BV}(]x_{in}, x_{out}[; \mathcal{W})$ before the above mentioned replacement of vacuum states, then also the new initial datum $W_0 = W_0^+ \in \text{BV}(]x_{in}, x_{out}[; \mathcal{W})$. Indeed, for each state $W_M^- \in \mathcal{W}_0$ of $W_0^-(\cdot)$, let W_L, W_R the left and right values involved in the computation of the total variation (so that W_M^- is replaced by $W_M^+ = (w_L, w_L)$). Then we have, applying twice the triangle inequality,

$$\begin{aligned} & \left\{ |W_L - W_M^+| + |W_M^+ - W_R| \right\} - \left\{ |W_L - W_M^-| + |W_M^- - W_R| \right\} \\ &= \left\{ |v_L - w_L| + |w_L - v_R| + |w_L - w_R| \right\} - \left\{ |v_L - w_M^-| + |w_L - w_M^-| + |w_M^- - v_R| + |w_M^- - w_R| \right\} \\ &\leq \left\{ |w_L - v_R| + |w_L - w_R| \right\} - \left\{ |w_M^- - v_R| + |w_M^- - w_R| \right\} \\ &\leq |w_L - v_R| + |w_L - w_R| \\ &\leq 2|w_L - w_M^-| + |w_M^- - v_R| + |w_M^- - w_R| \\ &\leq 2 \left\{ |W_L - W_M^-| + |W_M^- - W_R| \right\}, \end{aligned}$$

leading to the bound $\text{TV}(W_0^+) \leq 3\text{TV}(W_0^-)$.

In the following subsections, we construct a sequence of approximate solutions and we show that it converges to an entropy weak solution of (4.1), which is defined below. First, we need to recall the notion of boundary entropy pairs, see [13, Definition 4.1], where we drop the convexity assumption.

Definition 3 (Boundary entropy pair). *An entropy pair*

$$(\alpha(u(W_1), u(W_2)), \beta(u(W_1), u(W_2))), \quad W_1, W_2 \in \mathcal{W},$$

is called a boundary entropy pair if for every fixed $W_2 \in \mathcal{W}$ it satisfies

$$\alpha(u(W_2), u(W_2)) = \beta(u(W_2), u(W_2)) = \nabla_1 \alpha(u(W_2), u(W_2)) = (0, 0)^\top.$$

Definition 4 (Entropy weak solution). *A function $W \in \mathbf{L}^\infty\left(\left]0, T[\times]x_{in}, x_{out}[\right]; \mathcal{W}\right)$ is an entropy weak solution of IBVP (4.1) if*

- *for any entropy-flux pair $(\mathcal{E}, \mathcal{Q})$ and any test function $\phi \in \mathbf{C}_c^\infty\left(\left] - \infty, T[\times]x_{in}, x_{out}[\right]; \mathbb{R}_{\geq 0}\right)$, it holds*

$$\int_0^T \int_{x_{in}}^{x_{out}} \{\mathcal{E}(u(W)) \partial_t \phi + \mathcal{Q}(u(W)) \partial_x \phi\} dx dt + \int_{x_{in}}^{x_{out}} \mathcal{E}(u(W_0(x))) \phi(0, x) dx \geq 0; \quad (4.2)$$

- *for any boundary entropy pair (α, β) and any $\gamma(t) \in \mathbf{L}^1\left(\left]0, T[; \mathbb{R}_{\geq 0}\right)$, it holds*

$$\begin{aligned} \text{ess lim}_{x \rightarrow x_{in}^+} \int_0^T \beta(u(W(t, x)), u(W_{in}(t))) \gamma(t) dt &\leq 0, \\ \text{ess lim}_{x \rightarrow x_{out}^-} \int_0^T \beta(u(W(t, x)), u(W_{out}(t))) \gamma(t) dt &\geq 0. \end{aligned} \quad (4.3)$$

For future reference, we also recall the corresponding definition of a weak solution [12].

Definition 5 (Weak solution). *We call $W \in \mathbf{L}^\infty\left(\left]0, T[\times]x_{in}, x_{out}[\right]; \mathcal{W}\right)$ a weak solution to the IBVP (4.1), if for any test function $\phi \in \mathbf{C}_c^\infty\left(\left] - \infty, T[\times]x_{in}, x_{out}[\right]; \mathbb{R}\right)$ it satisfies*

$$\int_0^T \int_{x_{in}}^{x_{out}} \{u(W) \phi_t + f(u(W)) \phi_x\} dx dt + \int_{x_{in}}^{x_{out}} u(W_0(x)) \phi(0, x) dx = 0. \quad (4.4)$$

We can now state the main result of this paper:

Theorem 1. *Let us assume $W_0 \in \text{BV}\left(\left]x_{in}, x_{out}[; \mathcal{W}\right)$, $W_{in}, W_{out} \in \text{BV}\left(\left]0, T[; \mathcal{W}\right)$. Then, for any $T > 0$, the IBVP (4.1) admits an entropy weak solution $W \in \mathbf{L}^\infty\left(\left]0, T[\times]x_{in}, x_{out}[; \mathcal{W}\right)$ in the sense of Definition 4. Additionally, W satisfies the following bounds:*

$$\text{TV}(W(t, \cdot)) \leq \gamma_0 \text{ and } \|W(t)\|_\infty \leq \|W_0\|_\infty \quad \forall t \in \left]0, T[\text{ and } x \in \left]x_{in}, x_{out}[,$$

where $\gamma_0 = \text{TV}(W_0) + |W_{in}(0) - W_0(x_{in}^+)| + |v_{out}(0) - v(0, x_{out}^-)| + 3\text{TV}(W_{in}(s); s \in \left]0, T[) + \text{TV}(v_{out}(s); s \in \left]0, T[)$.

The proof is postponed to Section 4.2 (see Propositions 6 and 7).

4.1 Wave-front tracking (WFT) algorithm

The WFT algorithm [19, 36] allows to construct piece-wise constant approximate entropy weak solutions W^h of the IBVP problem (4.1) by means of an approximate Riemann solver obtained by approximating the rarefaction waves by piece-wise constant functions with values in a fixed grid of mesh size ε^h , see [3, 4] for an implementation in the case of Temple and ARZ systems. The procedure is briefly summarized below.

1. Fix $h \in \mathbb{N}$ sufficiently large, $\varepsilon^h = 2^{-h}\|W_0\|_\infty$, $\mathcal{W}^h = \mathcal{W} \cap [\varepsilon^h\mathbb{N}^2]$ (see Figure 6).

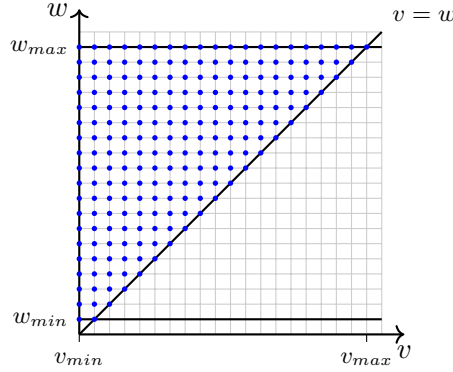


Figure 6: Illustration of the discretized domain \mathcal{W}^h . The grid points are illustrated in blue.

2. Approximate the initial and boundary data with piece-wise constant functions, i.e. $W_0^h \in PC([x_{in}, x_{out}]; \mathcal{W}^h)$, $W_{in}^h, W_{out}^h \in PC([0, T]; \mathcal{W}^h)$ such that [1]:
 - $\text{TV}(W_0^h) \leq \text{TV}(W_0)$, $\text{TV}(W_{in}^h) \leq \text{TV}(W_{in})$, $\text{TV}(W_{out}^h) \leq \text{TV}(W_{out})$,
 - $\lim_{h \rightarrow \infty} \|W_0 - W_0^h\|_1 = 0$, $\lim_{h \rightarrow \infty} \|W_{in} - W_{in}^h\|_1 = 0$, $\lim_{h \rightarrow \infty} \|W_{out} - W_{out}^h\|_1 = 0$,
 - $W_0^h(x+) = (w_0^h(x-), w_0^h(x-))$ whenever $W_0^h(x+) \in \mathcal{W}_0$,
 $|W_{in}^h(0+) - W_0^h(x_{in}+)| \leq |W_{in}(0+) - W_0(x_{in}+)|$,
 $|W_{out}^h(0+) - W_0^h(x_{out}-)| \leq |W_{out}(0+) - W_0(x_{out}-)|$,
 - $\|W_0^h\|_\infty \leq \|W_0\|_\infty$, $\|W_{in}^h\|_\infty \leq \|W_{in}\|_\infty$, $\|W_{out}^h\|_\infty \leq \|W_{out}\|_\infty$.
3. Approximately solve the Riemann problem at $x = x_{in}, x_{out}$ and at every jump discontinuity in the approximate initial data. Notice that the total variation of these approximations is bounded by $\text{TV}(W_0^h) + |W_{in}^h(0) - W_0^h(x_{in}+)| + |v_{out}^h(0) - v^h(0, x_{out}-)|$ even in the presence of vacuum states.
4. Glue together these solutions to obtain a piece-wise constant approximate solution $W^h = (v^h, w^h)$ defined up to the first time \hat{t} at which an interaction between two or more wave-fronts takes place, or a wave hits the boundary, or a jump discontinuity occurs in the boundary data (see [2, page 240] or [18, page 690]).
5. Solve the new Riemann problem arisen at $t = \hat{t}$ and prolong the solution until the next interaction.

This process can be extended to any time $t > 0$, as proven by the following result.

Proposition 5. For any h fixed, the number of waves in the approximate solution W^h is finite for all $t \in]0, T[$ and the functional $\gamma^h : [0, T] \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$\begin{aligned} \gamma^h(t) = & \text{TV}(W^h(t, \cdot)) + \left| W_{in}^h(t) - W^h(t, x_{in+}) \right| + \left| v_{out}^h(t) - v^h(t, x_{out-}) \right| \\ & + 3 \text{TV} \left(W_{in}^h(s); s \in]t, T[\right) + \text{TV} \left(v_{out}^h(s); s \in]t, T[\right) \end{aligned} \quad (4.5)$$

is non-increasing.

Remark 14. Note that (4.5) does not depend on the total variation of w_{out}^h , which is in line with the fact that the set $\mathcal{B}_R^{Rie}(U_B)$ is independent of the w -variable (see Remark 8).

Proof. By construction, W^h is a piece-wise constant function, i.e.

$W^h(t, \cdot) \in PC \left(]x_{in}, x_{out}[; \mathcal{W}^h \right)$ for all $t \geq 0$ for which it is defined. By slightly changing the wave positions, it is not restrictive to assume that at any interaction time \hat{t} , either two waves interact in the interior of the domain, or a single wave hits the boundary, or a change in the boundary state occurs.

Regarding interactions not involving vacuum states occurring in $]x_{in}, x_{out}[$, the number of waves can increase only if one of the outgoing waves is a rarefaction. However, for Temple class systems, a rarefaction wave can only occur if one of the incoming waves already was a rarefaction. Thus, the number of waves does not increase. Additionally, we know from the standard theory of Temple class systems that the space total variation in the Riemann invariants is non-increasing [7, 27] as long as the waves have finite speeds.

Therefore it suffices to focus on the following three cases that may occur at $t = \hat{t}$:

- (A) an interaction between waves in $]x_{in}, x_{out}[$ involving at least a vacuum state;
- (B) a wave hitting the boundary at $x = x_{in}$ or $x = x_{out}$;
- (C) a jump in the approximate boundary data W_{in}^h or W_{out}^h .

For notational simplicity, we will drop the h , t and x dependencies in the rest of the proof, thus writing W instead of $W^h(t, x)$, ε instead of ε^h and γ instead of γ^h . Additionally, we still denote the critical density, defined in (3.2), as $\rho_{cr}(w)$ (instead of expressing it in Riemann invariant coordinates). We also set

$$\Delta\gamma = \gamma(\hat{t}+) - \gamma(\hat{t}-)$$

for the variation of the functional (4.5) at $t = \hat{t}$. Finally, we recall that the absolute difference between a left (non-vacuum) state $W_L = (v_L, w_L) \in \mathcal{W}_0^c$ and a right (vacuum) state $W_R \in \mathcal{W}_0$ is computed by

$$|W_L - W_R| = |v_L - w_L| + |w_L - w_L| = |v_L - w_L|.$$

Remark that, since the vacuum states in the interior of the domain are the results of Riemann problem solutions, we must have $w_R = w_L$ by case 8 in Definition 1.

Let us first consider case (A). Following [4], we will look at the solution of the Riemann problem after the interaction of a wave connecting the state W_L to W_M and a wave connecting W_M to W_R (see Figure 7).

(A.1) If $W_L = (w_L, w_L) \in \mathcal{W}_0$, we know that $W_M \in \mathcal{W}_0^c$ (otherwise case 9 in Definition 1 implies $W_M = W_L$). By case 7 in Definition 1, it holds that the first wave travels with propagation speed v_M . If the second wave was a contact discontinuity, it would travel with the same speed of propagation leading to no interaction between the waves. Thus, the second wave must be either a shock or a rarefaction, i.e. $w_M = w_R$. Moreover it holds that $W_R \in \mathcal{W}_0^c$ otherwise the solution of the Riemann problem between the states W_M and $W_R \in \mathcal{W}_0$ would be an ε -step size rarefaction with the same speed as the first wave (\rightarrow no interaction). Finally, the solution of the Riemann problem associated to the interaction is a discontinuity travelling with speed v_R . Thus, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |w_L - v_R| + |w_L - w_R| \right\} - \left\{ |w_L - v_M| + |w_L - \underbrace{w_M}_{=w_R}| + |v_M - v_R| \right\} \leq 0.$$

(A.2) If $W_M \in \mathcal{W}_0$, we know with the same argument as in (A.1) that $W_L \in \mathcal{W}_0^c$ and $W_R \in \mathcal{W}_0^c$. The first wave, connecting W_L to $W_M = (w_L, w_L)$, is an ε -step size rarefaction with propagation speed $v_L = w_L - \varepsilon$. The propagation speed of the second wave (discontinuity) is v_R . Moreover it must hold that $v_R < w_L - \varepsilon$ (otherwise the waves do not interact). Finally, the solution of the Riemann problem associated to the interaction is a shock-wave travelling from W_L to an intermediate state $W_{M'}$, followed by a contact discontinuity from $W_{M'}$ to W_R with propagation speed $v_{M'} = v_R$. Thus, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |v_L - \underbrace{v_{M'}}_{=v_R}| + |w_{M'} - w_R| \right\} - \left\{ |v_L - w_L| + |w_L - v_R| + |w_L - w_R| \right\} \leq 0.$$

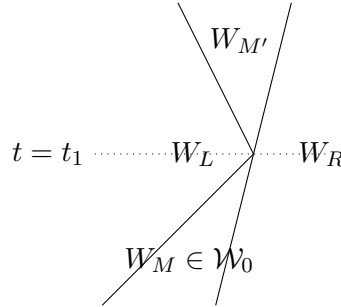


Figure 7: Sample illustration of the interacting waves in the vacuum case (A.2).

- (A.3) If $W_R \in \mathcal{W}_0$, the second wave connecting W_M to $W_R = (w_M, w_M)$ is an ε -rarefaction with speed $v_M = w_M - \varepsilon$. Thus, the first wave cannot be a contact discontinuity since it would have the same speed as the second wave (\rightarrow no interaction), this implies $w_L = w_M$. However, an interaction between the two waves can only occur if the first wave travels faster than $w_M - \varepsilon$ which is not possible.
- (A.4) If $W_{M'} \in \mathcal{W}_0$ (and $W_L \in \mathcal{W}_0^c$, $W_M \in \mathcal{W}_0^c$, $W_R \in \mathcal{W}_0^c$), we know by the Riemann solver that $W_{M'} = (w_L, w_L)$ with $v_{M'} = w_L > v_L$. Moreover, we have that $v_L < v_{M'} \leq v_R$, otherwise $W_{M'} \in \mathcal{W}_0^c$. Since the speed of the first wave before the interaction is higher

than the second wave speed, it must be a contact discontinuity (from W_L to W_M) followed by a ε -rarefaction from $W_M = (v_L, w_R)$ to W_R with $v_R = v_L + \varepsilon$. Assuming that $v_{M'} < v_R$, we obtain a contradiction due to our ε -discretization: $v_L < v_{M'} < v_R = v_L + \varepsilon$. Thus, it must hold $w_L = v_{M'} = v_R = v_L + \varepsilon$ and $W_{M'} = (v_R, w_L)$. Finally, the number of waves does not increase (since we only have an ε -rarefaction wave) and it holds

$$\Delta\gamma = \left\{ |v_L - v_R| + |w_L - w_R| \right\} - \left\{ |w_L - \underbrace{w_M}_{=v_R}| + |\underbrace{v_M}_{=v_L} - v_R| \right\} = 0.$$

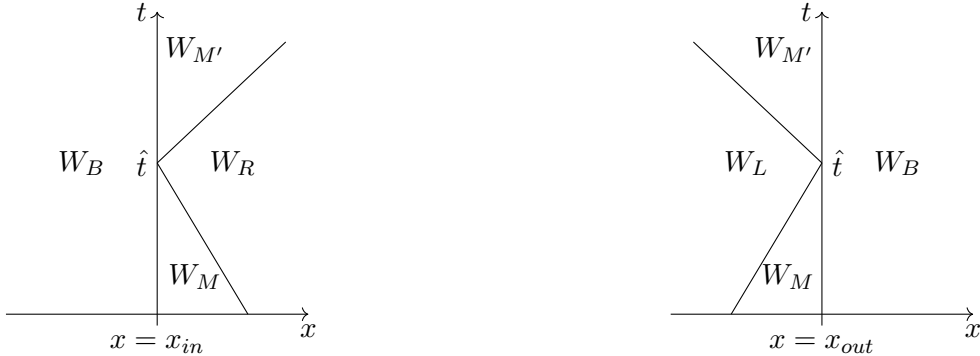
Next, we consider case (B). For the left (resp. right) boundary case, the states before the interaction will be denoted by W_B, W_M and W_R (resp. W_L) (see Figure 8a (resp. 8b)). To prove that the functional γ is non-increasing, it suffices to show that

$$\Delta\gamma = \left\{ |W_B - W_{M'}| + |W_{M'} - W_R| \right\} - \left\{ |W_B - W_M| + |W_M - W_R| \right\} \leq 0$$

(resp.

$$\Delta\gamma = \left\{ |W_L - W_{M'}| + |v_{M'} - v_B| \right\} - \left\{ |W_L - W_M| + |v_M - v_B| \right\} \leq 0.)$$

Moreover, we assume that $U_M = (\mathcal{R}(v_M, w_M), w_M) \in \mathcal{B}_i^{Rie}(U_B)$ with $i \in \{L, R\}$ and $U_B = (\mathcal{R}(v_B, w_B), w_B)$.



(a) Illustration of the states and the corresponding interacting waves at the left boundary (B.L).

(b) Illustration of the states and the corresponding interacting waves at the right boundary (B.R).

Figure 8: Case (B).

In the left boundary case (B.L), i.e. $i = L$, we define the subset of the admissible states with zero speed (see Proposition 1) as

$$\mathcal{W}_L = \{(v, w) \in \mathcal{W}_0^c \mid v = 0\}. \quad (4.6)$$

(B.L.1) If $W_B \in \mathcal{W}_0^c$, $W_M \notin \mathcal{W}_L$ and $\mathcal{R}(v_B, w_B) < \rho_{cr}(w_B)$, we know that $w_B = w_M = w_R = w_{M'}$. By case 1 in Proposition 1, it follows that W_B is connected to W_M by a negative shock. Thus, the only possible wave with negative speed (joining W_M to W_R) leading to a visible wave (with positive speed) after the interaction has to be a negative rarefaction, i.e. $v_R = v_M + \varepsilon$ and $\mathcal{R}(v_R, w_R)v_R > \mathcal{R}(v_B, w_B)v_B$. Finally, the solution of

the Riemann problem associated to the interaction is a positive shock-wave travelling from $W_B = W_{M'}$ to W_R . Thus, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |v_B - v_R| \right\} - \left\{ |v_B - v_M| + |v_M - v_R| \right\} \leq 0.$$

(B.L.2) If $W_B \in \mathcal{W}_0^c$, $W_M \notin \mathcal{W}_L$ and $\mathcal{R}(v_B, w_B) \geq \rho_{cr}(w_B)$, we know that $w_B = w_M = w_R$. It holds that both the wave connecting W_M to W_R and the wave travelling from W_B via $W_{M'}$ to W_R are negative. Therefore, no wave emerges from the interaction and $\Delta\gamma \leq 0$. Additionally, as shown later in the cases (C.L.1) and (C.L.3), it can appear the situation where $U_M \notin \mathcal{B}_L^{Rie}(U_B)$. However, applying the same argumentation as before, the result does not change.

(B.L.3) If $W_B \in \mathcal{W}_0^c$, $W_M \in \mathcal{W}_L$ (see Figure 9), it holds that $v_M = 0$ and $w_M = w_R$. Thus, the boundary wave travelling from W_M to W_R is a negative ε -rarefaction, implying $v_R = v_M + \varepsilon = \varepsilon$. The solution of the Riemann problem associated to the interaction is a first family wave travelling from W_B to $W_{M'} = (v_{M'}, w_B)$ followed by a contact discontinuity from $W_{M'}$ to W_R , i.e. $v_{M'} = v_R$. If $W_{M'} \neq W_B$ is admissible (see Figure 9b), we have only one outgoing wave, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |v_B - \underbrace{v_{M'}}_{=v_R}| + \underbrace{|w_{M'} - w_R|}_{=w_B} \right\} - \left\{ |v_B - v_M| + \underbrace{|w_B - w_M|}_{=w_R} + |v_M - v_R| \right\} \leq 0.$$

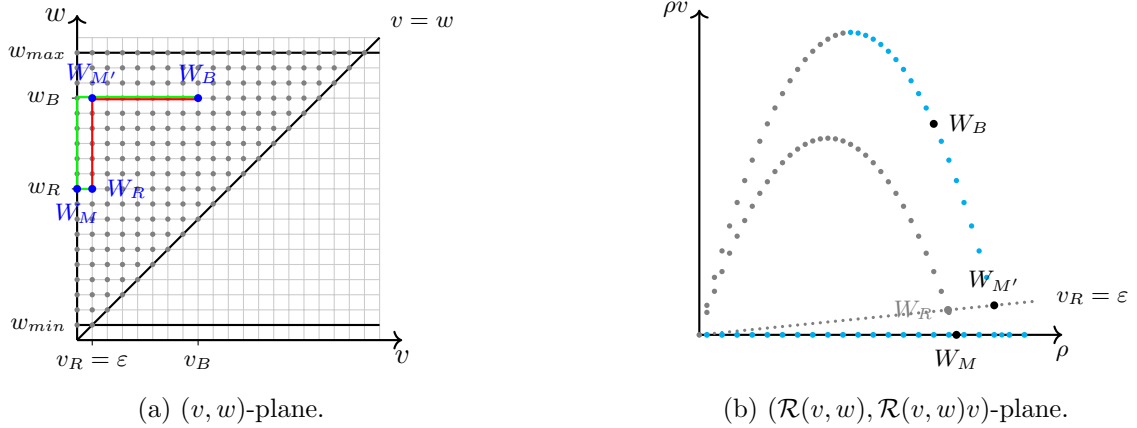


Figure 9: Sample illustration of the states in the case (B.L.3) in two different planes.
(a) waves before (resp. after) the interaction are drawn in green (resp. red).
(b) the admissible states for U_B are indicated in light blue.

Otherwise, we have two outgoing waves (see Figure 10). We thus consider an additional intermediate state $W_{M''}$ between $W_{M'} (= W_B)$ and W_R . The first wave is a positive shock (from W_B to $W_{M''}$), followed by a contact discontinuity (from $W_{M''}$ to W_R). It holds $v_B > v_R \neq 0$. Thus, the number of waves increases by one but it holds that

$$\Delta\gamma = \left\{ v_B - v_R + \underbrace{|w_{M'} - w_R|}_{=w_B} \right\} - \left\{ v_B + v_R + \underbrace{|w_B - w_M|}_{=w_R} \right\} = -2v_R = -2\varepsilon.$$

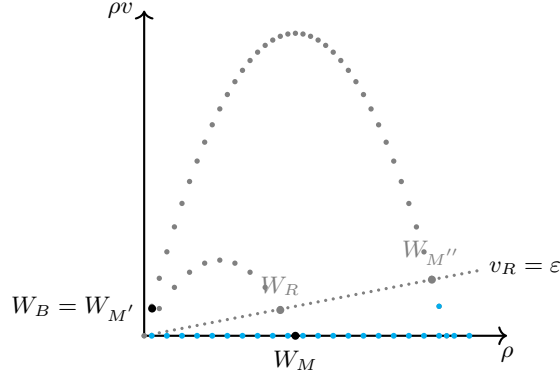


Figure 10: Illustration of case (B.L.3) with $U_{M''} = (\mathcal{R}(v_R, w_B), w_B) \notin \mathcal{B}_L^{Rie}(U_B)$, i.e. $\mathcal{R}(v_B, w_B)v_B < \mathcal{R}(v_R, w_B)v_R$. The admissible states for U_B are indicated in light blue.

- (B.L.4) If $W_B = (w_B, w_B) \in \mathcal{W}_0$, it holds that $v_M = 0$ and $w_M = w_R$. Thus again, the boundary wave travelling from W_M to W_R is a negative ε -rarefaction, i.e. $v_R = v_M + \varepsilon = \varepsilon$. The solution of the Riemann problem associated to the interaction is a discontinuity from $W_B = W_{M'}$ to W_R travelling with speed v_R . Hence, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |w_B - v_R| + |w_B - w_R| \right\} - \left\{ |w_B - v_M| + |w_B - \underbrace{w_M}_{=w_R}| + |v_M - v_R| \right\} \leq 0.$$

We now focus on the right boundary case (B.R), i.e. $i = R$.

- (B.R.1) If $W_B \in \mathcal{W}_0^c$ and the boundary wave from W_L to W_M is a contact discontinuity travelling with speed $v_M = v_L$ (see Figure 11), the solution to the boundary Riemann problem associated to the interaction displays at most a first family-curve travelling with negative speed from W_L to $W_{M'} = (v_{M'}, w_L)$. It holds either that $v_L < v_{M'} \leq v_B$ (rarefaction wave from W_L to $W_{M'}$) or $v_{M'} = v_B < v_L$ (shock wave from W_L to $W_{M'} = \tilde{W}$). In the first case, the number of waves can increase; in the second case, the number of waves does not change. By means of the triangle inequality it follows that

$$\Delta\gamma = \left\{ \underbrace{|v_L - v_{M'}| + |v_{M'} - v_B|}_{=|v_L - v_B|} \right\} - \left\{ |w_L - w_M| + \underbrace{|v_M - v_B|}_{=v_L} \right\} \leq -\varepsilon < 0,$$

since $W_L \neq W_M$.

Remark: In the case of a negative rarefaction wave from W_L to $W_{M'}$ (only possible if $\mathcal{R}(v_L, w_L) > \rho_{cr}(w_L)$) and $v_{M'} \neq v_B$, then $W_{M'}$ will be the point the propagation speed of the rarefaction wave changes from negative into positive speed. If the state $U_{cr} = (\rho_{cr}(w_L), w_L) \in \mathcal{B}_R^{Rie}(U_B)$ is lying on the grid, we know that $U_{cr} = U_{M'} = (\mathcal{R}(v_{M'}, w_L), w_L)$. On the contrary, if the state U_{cr} is not lying on the grid (see Figure 11b), then it can appear the situation that $U_{M'} \notin \mathcal{B}_R^{Rie}(U_B)$. However, since we move with an ε -step size along the v -variable on the grid, it holds that $|v_{M'} - v_{cr}| < \varepsilon$ and $v_{cr} = \mathcal{V}(\rho_{cr}(w_L), w_L)$.

Due to the previous remark (and also case (C.R.2)), we can also have $U_M \notin \mathcal{B}_R^{Rie}(U_B)$. However, applying the same argumentation as before, the result does not change.

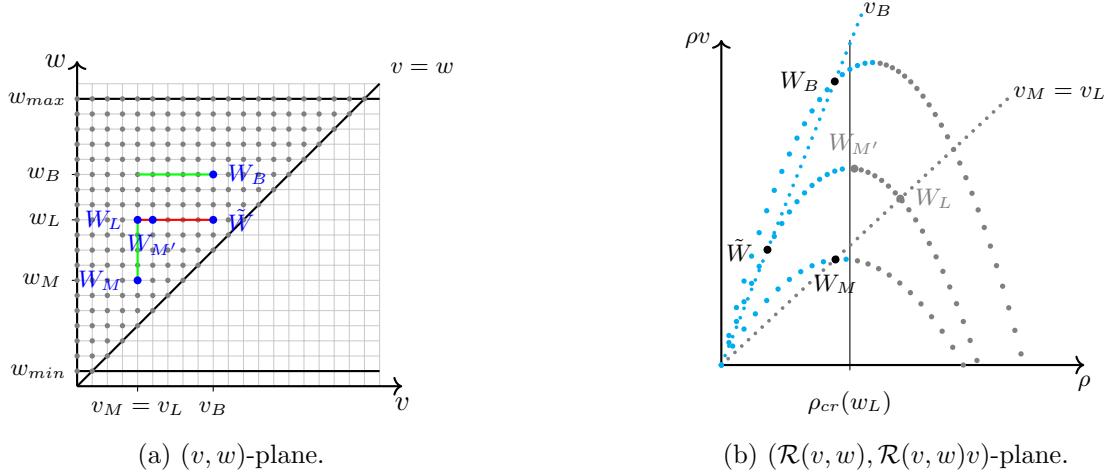


Figure 11: Sample illustration of the states in the case (B.R.1) in two different planes.
(a) waves before (resp. after) the interaction are drawn in green (resp. red).
(b) the admissible states for U_B are indicated in light blue. In this case, $W_{M'} \notin \mathcal{B}_R^{Rie}(U_B)$.

(B.R.2) If $W_B \in \mathcal{W}_0^c$ and the boundary wave from W_L to W_M is a first family curve, i.e. $w_M = w_L$, travelling with positive speed, the boundary solution after the interaction displays at most a negative shock-wave travelling from W_L to $W_{M'}$. Thus, the number of waves does not increase and it holds by means of the triangle inequality

$$\Delta\gamma = \left\{ |v_L - \underbrace{v_{M'}}_{=v_B} | \right\} - \left\{ |v_L - w_M| + |w_M - v_B| \right\} \leq 0.$$

As we have seen in the case (B.R.1) (see also (C.R.2)), it can happen that $U_M \notin \mathcal{B}_R^{Rie}(U_B)$. However, due to $\mathcal{R}(v_B, w_M) < \rho_{cr}(w_M)$, there will be no visible solution.

(B.R.3) If $W_B = (w_{max}, w_{max}) \in \mathcal{W}_0$, we know that the wave travelling from W_L to W_M is either a positive first family curve or a contact discontinuity. In the first case, the solution to the Riemann problem after the interaction will be a positive rarefaction wave which is not visible in the domain. However, in the case of a contact discontinuity, the solution to the boundary Riemann problem between W_L and W_B may consist of a negative rarefaction fan, travelling from W_L to $W_{M'}$ and it holds $v_L = v_M < v_{M'} < w_B = w_{max}$. Thus, the number of waves can increase, but

$$\Delta\gamma = \left\{ \underbrace{|v_L - v_{M'}| + |v_{M'} - w_{max}|}_{=|v_L - w_{max}|} \right\} - \left\{ |w_L - w_M| + \underbrace{|w_M - w_{max}|}_{=v_L} \right\} = -|w_L - w_M| \leq -\varepsilon.$$

We remark that in this case, we use the assumption of $w_B = w_{max}$ (if $W_B \in \mathcal{W}_0$) in order to obtain a negative value for $\Delta\gamma$.

As shown later in the case (C.R.4), we may have $U_M \notin \mathcal{B}_R^{Rie}(U_B)$. However, this does not change the above argumentation.

Finally, we consider case (C): we analyse the solution of the boundary Riemann problem after a jump discontinuity in the boundary state from W_B^- to W_B^+ . From Figure 12, we see

that in this scenario the number of waves can increase. However, since there is a finite number of jumps in the approximate boundary states, the total number of new waves remains finite. Therefore, it is sufficient to prove that the functional γ is non-increasing, i.e.

$$\Delta\gamma = \left\{ \left| W_B^+ - W_{M'} \right| + \left| W_{M'} - W_R \right| \right\} - \left\{ \left| W_B^- - W_R \right| + 3 \left| W_B^- - W_B^+ \right| \right\} \leq 0$$

(resp.

$$\Delta\gamma = \left\{ \left| W_L - W_{M'} \right| + \left| v_{M'} - v_B^+ \right| \right\} - \left\{ \left| v_L - v_B^- \right| + \left| v_B^- - v_B^+ \right| \right\} \leq 0).$$

Moreover, we assume that $U_R = (\mathcal{R}(v_R, w_R), w_R) \in \mathcal{B}_L^{Rie}(U_B^-)$ and $U_L = (\mathcal{R}(v_L, w_L), w_L) \in \mathcal{B}_R^{Rie}(U_B^-)$ with $U_B^- = (\mathcal{R}(v_B^-, w_B^-), w_B^-)$.



(a) Illustration of the states and the corresponding waves at the left boundary (C.L).

(b) Illustration of the states and the corresponding waves at the right boundary (C.R).

Figure 12: Case (C).

First, we consider the left boundary case (C.L), i.e. $i = L$.

(C.L.1) If $W_B^- \in \mathcal{W}_0^c$, $W_B^+ \in \mathcal{W}_0^c$ and $W_R \notin \mathcal{W}_L$ (defined in (4.6)), it holds that W_B^- and W_R are connected by a first family wave (possibly null if $W_B^- = W_R$), i.e. $w_B^- = w_R$. The solution of the Riemann problem, associated to the change of the boundary state, is a first family curve from W_B^+ to $W_{M'}$, i.e. $w_B^+ = w_{M'}$, followed by either a contact discontinuity, travelling with propagation speed $v_{M'} = v_R$, or a first family wave, i.e. $w_B^- = w_R = w_{M'} = w_B^+$, from the state $W_{M'}$ to W_R . The boundary Riemann problem consists than in (a first family wave followed by) a contact discontinuity, it holds by means of the triangle inequality

$$\begin{aligned} \Delta\gamma &= \left\{ \left| v_B^+ - \underbrace{v_{M'}}_{=v_R} \right| + \left| \underbrace{w_{M'}}_{=w_B^+} - w_R \right| \right\} - \left\{ \left| v_B^- - v_R \right| + 3 \left| v_B^- - v_B^+ \right| + 3 \left| \underbrace{w_B^- - w_B^+}_{=w_R} \right| \right\} \\ &\leq -2 \left\{ \left| v_B^- - v_B^+ \right| + \left| w_B^- - w_B^+ \right| \right\} \leq -2\varepsilon < 0. \end{aligned}$$

Remark: We can have two outgoing waves, this means having an additional state $W_{M''} = (v_R, w_B^+)$ between $W_{M'}$ and W_R . This can happen in two situations: either if $\mathcal{R}(v_B^+, w_B^+) < \rho_{cr}(w_B^+)$ and $W_{M'} = W_B^+$ or if $\mathcal{R}(v_B^+, w_B^+) \geq \rho_{cr}(w_B^+)$ and $\mathcal{R}(v_R, w_B^+) < \rho_{cr}(w_B^+)$. In the first case, we observe a positive shock wave (from W_B^+ to $W_{M''}$) and a

contact discontinuity (from $W_{M''}$ to W_R). In the second case, the wave connecting $W_{M'}$ to $W_{M''}$ is a positive rarefaction followed again by a contact discontinuity (from $W_{M''}$ to W_R). However, both scenarios do not change the computation of the total variation, hence it still holds $\Delta\gamma \leq 0$.

Assuming the solution is a first family shock wave which implies $W_B^+ = W_{M'}$, it holds

$$\Delta\gamma = \left\{ |v_B^+ - v_R| \right\} - \left\{ |v_B^- - v_R| + 3|v_B^- - v_B^+| + 3\underbrace{|w_B^- - w_B^+|}_{=0} \right\} \leq 0.$$

Finally, assuming the outgoing wave is a first family rarefaction, the only possible solution, which is visible in the domain, leads to $W_R = W_B^-$. Thus,

$$\Delta\gamma = \left\{ \underbrace{|v_B^+ - v_{M'}| + |v_{M'} - v_R|}_{|v_B^+ - v_R|} \right\} - \left\{ \underbrace{|v_B^- - v_R|}_{=0} + 3\underbrace{|v_B^- - v_B^+|}_{=v_R} + 3\underbrace{|w_B^- - w_B^+|}_{=0} \right\} \leq 0.$$

Remark: For the first (resp. third) case above, if we are in the case of a positive rarefaction wave from $W_{M'}$ to $W_{M''}$ (resp. W_R) (only possible if $\mathcal{R}(v_B^+, w_B^+) > \rho_{cr}(w_B^+)$), then $W_{M'}$ will be the point the propagation speed of the rarefaction wave changes from negative into positive speed. If the state $U_{cr} = (\rho_{cr}(w_B^+), w_B^+) \in \mathcal{B}_L^{Rie}(U_B^+)$ with $U_B^+ = (\mathcal{R}(v_B^+, w_B^+), w_B^+)$ is lying on the grid, we know that $U_{cr} = U_{M'} = (\mathcal{R}(v_{M'}, w_B^+))$. On the contrary, if the state U_{cr} is not lying on the grid, then it can appear the situation that $U_{M'} \notin \mathcal{B}_L^{Rie}(U_B^+)$. However, since we move with an ε -step size along the v -variable on the grid, it holds that $|v_{M'} - v_{cr}| < \varepsilon$ and $v_{cr} = \mathcal{V}(\rho_{cr}(w_B^+), w_B^+)$.

By the remark above (and also case (C.L.3)), it may happen that $U_R \notin \mathcal{B}_L^{Rie}(U_B^-)$. However, the computations for the contact discontinuity and first family shock do not change. The rarefaction case cannot appear anymore since $W_R \neq W_B^-$.

(C.L.2) If $W_R \in \mathcal{W}_L$, we have that $W_R \in \mathcal{B}_L^{Rie}(U_B^+)$ and no new wave is created, thus $\Delta\gamma \leq 0$.

(C.L.3) If $W_B^- = (w_B^-, w_B^-) \in \mathcal{W}_0$ and $W_B^+ \in \mathcal{W}_0^c$ (see Figure 13), it follows that $W_R \in \mathcal{W}_L \cup \{W_B^-\}$. If $W_R \in \mathcal{W}_L$, then case (C.L.2) applies. If $W_R = W_B^-$, the visible solution of the Riemann problem after the change of the boundary state will be the positive part of the rarefaction fan wave travelling from W_B^- to $W_R^+ = (w_B^+, w_B^+)$. **In particular, we have an infinite speed wave jump from $W_R = W_B^- = (w_B^-, w_B^-)$ to $W_R^+ = (w_B^+, w_B^+)$** (see case 8 in Definition 1 and Figure 14).

If $\mathcal{R}(v_B^+, w_B^+) > \rho_{cr}(w_B^+)$, it can exist an intermediate state $W_{M'} = (v_{M'}, w_M^+)$ with $v_{M'} > v_B^+$. Otherwise, it holds $v_{M'} = v_B^+$, implying $W_{M'} = W_B^+$. Thus, it holds by means of the triangle inequality

$$\begin{aligned} \Delta\gamma &= \left\{ \underbrace{|v_B^+ - v_{M'}| + |v_{M'} - w_B^+|}_{=|v_B^+ - w_B^+|} \right\} - 3 \left\{ |w_B^- - v_B^+| + |w_B^- - w_B^+| \right\} \\ &\leq -2 \left\{ |w_B^- - v_B^+| + |w_B^- - w_B^+| \right\} \leq -2\varepsilon < 0. \end{aligned}$$

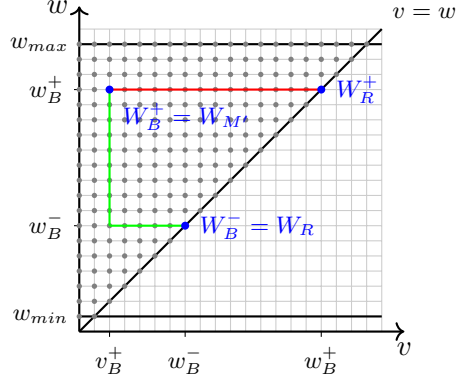


Figure 13: Sample illustration of the states in the case (C.L.3) with $\mathcal{R}(v_B^+, w_B^+) \leq \rho_{cr}(w_B^+)$, i.e. $W_{M'} = W_B^+$. The waves before (resp. after) the interaction are drawn in green (resp. red).

Due to the presence of the infinite speed wave between W_R and W_R^+ , we additionally have to check the change of the total variation when this wave interacts with others in the interior of the domain. Thus, we are interested in the solution of the Riemann problem from the state W_R^+ to a state $\hat{W} = (\hat{v}, \hat{w})$ after the interaction of the wave between W_R and \hat{W} , which travels with propagation speed \hat{v} (by case 7 in Definition 1). The solution of the new Riemann problem is again a discontinuity with speed \hat{v} . This scenario is illustrated in Figure 14.

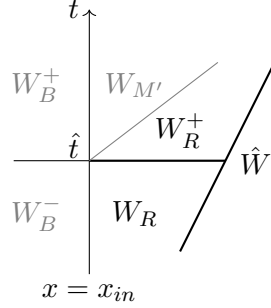


Figure 14: Interaction of an infinite speed wave with a wave in the interior of the domain.

Finally, thanks to the triangle inequality, we have:

$$\begin{aligned}
\Delta\gamma &= |W_B^+ - W_{M'}| + |W_{M'} - W_R^+| + |W_R^+ - \hat{W}| - \\
&\quad \left\{ \underbrace{|W_B^- - W_R|}_{=0} + |W_R - \hat{W}| + 3|W_B^- - W_B^+| \right\} \\
&= \underbrace{|v_B^+ - v_{M'}| + |v_{M'} - w_B^+|}_{=|v_B^+ - w_B^+|} + |w_B^+ - \hat{v}| + |w_B^+ - \hat{w}| - \\
&\quad \left\{ |w_B^- - \hat{v}| + |w_B^- - \hat{w}| + 3|w_B^- - v_B^+| + 3|w_B^- - w_B^+| \right\} \leq -2|w_B^- - v_B^+| \leq 0.
\end{aligned}$$

Remark: $W_{M'}$ is the point the propagation speed of the rarefaction wave changes from negative to positive (only possible if $\mathcal{R}(v_B^+, w_B^+) > \rho_{cr}(w_B^+)$). If the state $U_{cr} =$

$(\rho_{cr}(w_B^+), w_B^+) \in \mathcal{B}_L^{Rie}(U_B^+)$ is lying on the grid, it holds $U_{cr} = U_{M'} = (\mathcal{R}(v_{M'}, w_B^+), w_B^+)$. Otherwise, we may have $U_{M'} \notin \mathcal{B}_L^{Rie}(U_B^+)$. However, since we move with an ε -step size along the v -variable on the grid, it holds that $|v_{M'} - v_{cr}| < \varepsilon$ and $v_{cr} = \mathcal{V}(\rho_{cr}(w_B^+), w_B^+)$.

(C.L.4) If $W_B^- \in \mathcal{W}_0^c$, $W_B^+ = (w_B^+, w_B^+) \in \mathcal{W}_0$ and $W_R \notin \mathcal{W}_L$, we know that $w_B^- = w_R$. Thus, the solution of the Riemann problem associated to the change of the boundary state is a discontinuity from $W_B^+ = W_{M'}$ to W_R travelling with speed v_R . Thus, a new wave is produced and it holds

$$\begin{aligned} \Delta\gamma &= \left\{ |w_B^+ - v_R| + |w_B^+ - \underbrace{w_R}_{=w_B^-}| \right\} - \left\{ |v_B^- - v_R| + 3|v_B^- - w_B^+| + 3|w_B^- - w_B^+| \right\} \\ &\leq -2 \left\{ |v_B^- - w_B^+| + |w_B^- - w_B^+| \right\} \leq -2\varepsilon < 0. \end{aligned}$$

As seen in cases (C.L.1) and (C.L.3), it can happen that $U_R \notin \mathcal{B}_L^{Rie}(U_B^-)$. However, the computation above remains unchanged.

(C.L.5) If $W_B^- \in \mathcal{W}_0$ and $W_B^+ \in \mathcal{W}_0$, it follows that $W_R \in \mathcal{W}_L \cup \{W_B^-\}$ and no new wave is produced and $\Delta\gamma \leq 0$.

We now turn to the study of the right boundary case (C.R), i.e. $i = R$.

(C.R.1) If $W_B^- \in \mathcal{W}_0^c$, $W_B^+ \in \mathcal{W}_0^c$ and $R(v_L, w_L) \leq \rho_{cr}(w_L)$, then the solution of the boundary Riemann problem associated to the change of the boundary state displays at most a negative shock-wave travelling from W_L to an intermediate state $W_{M'}$ with $v_{M'} = v_B^+$. Thus, the number of waves may increase and it holds by means of the triangle inequality

$$\Delta\gamma = |v_L - \underbrace{v_{M'}}_{=v_B^+}| - \left\{ |v_L - v_B^-| + |v_B^- - v_B^+| \right\} \leq 0.$$

(C.R.2) If $W_B^- \in \mathcal{W}_0^c$, $W_B^+ \in \mathcal{W}_0^c$ and $R(v_L, w_L) > \rho_{cr}(w_L)$, we know that $v_L = v_B^-$. Thus, the solution to the Riemann problem is a first family-curve travelling with negative propagation speed from W_L to $W_{M'}$, possibly continued by another (first family) curve travelling with positive speed from $W_{M'}$ to the point $\tilde{W} = (v_B^+, w_{M'})$ and finally followed by a contact discontinuity from \tilde{W} to W_B^+ . It holds either that $v_L < v_{M'} \leq v_B^+$ (rarefaction wave from W_L to $W_{M'}$) or $v_{M'} = v_B^+ < v_L$ (shock wave from W_L to $W_{M'} = \tilde{W}$). By means of the triangle inequality it follows that

$$\Delta\gamma = \left\{ \underbrace{|v_L - v_{M'}| + |v_{M'} - v_B^+|}_{=|v_L - v_B^+|} \right\} - \underbrace{|v_B^- - v_B^+|}_{=v_L} = 0.$$

Remark: If the solution is a negative rarefaction wave from W_L to $W_{M'}$ and $v_{M'} \neq v_B^+$, then $W_{M'}$ will be the point the propagation speed of the rarefaction wave changes from negative into positive speed. If the state $U_{cr} = (\rho_{cr}(w_L), w_L) \in \mathcal{B}_R^{Rie}(U_B^+)$ is lying on the grid, we know that $U_{cr} = U_{M'} = (\mathcal{R}(v_{M'}, w_L))$. On the contrary, if the state U_{cr} is not lying on the grid, then we may have $U_{M'} \notin \mathcal{B}_R^{Rie}(U_B^+)$. However, since we move with an ε -step size along the v -variable on the grid, it holds that $|v_{M'} - v_{cr}| < \varepsilon$ and $v_{cr} = \mathcal{V}(\rho_{cr}(w_L), w_L)$.

Due to the previous remark (and also case (B.R.1)), we can have $U_L \notin \mathcal{B}_R^{Rie}(U_B^-)$. In contrast to above, it holds $v_B^- > v_L$ which still leads to the desired inequality, i.e.

$$\Delta\gamma = |v_L - v_B^+| - \{|v_L - v_B^-| + |v_B^- - v_B^+|\} = -2(v_B^- - v_L) \leq -2\varepsilon < 0.$$

(C.R.3) If $W_B^- = (w_{max}, w_{max}) \in \mathcal{W}_0$ and $W_B^+ \in \mathcal{W}_0^c$, then the solution of the boundary Riemann problem from W_L to W_B^+ is at most a negative shock wave from W_L to $W_{M'} = (v_{M'}, w_L)$ with $v_{M'} = v_B^+$. Thus, again, the number of waves can increase and

$$\Delta\gamma = |v_L - \underbrace{v_{M'}}_{=v_B^+}| - \{|v_L - w_{max}| + |w_{max} - v_B^+|\} = -2(w_{max} - v_L) \leq -2\varepsilon.$$

As we will see in case (C.R.4), we can have $U_L \notin \mathcal{B}_R^{Rie}(U_B^-)$. However, this does not change anything in the above argumentation.

(C.R.4) If $W_B^- \in \mathcal{W}_0^c$ and $W_B^+ = (w_{max}, w_{max}) \in \mathcal{W}_0$, new waves are produced only if $v_L = v_B^-$. In this case, we may have a negative rarefaction wave travelling from W_L to $W_{M'}$. Finally, it holds

$$\Delta\gamma = \left\{ \underbrace{|v_L - v_{M'}| + |v_{M'} - w_{max}|}_{=|v_L - w_{max}|} \right\} - \underbrace{|v_B^- - w_{max}|}_{=v_L} = 0.$$

Remark: Again, if $W_{M'}$ is the point the propagation speed of the rarefaction wave changes from negative into positive speed and if the state $U_{cr} = (\rho_{cr}(w_L), w_L)$ is not lying on the grid, it may happen that $U_{M'} \notin \mathcal{B}_R^{Rie}(U_B^+)$ with $U_{M'} = (\mathcal{R}(v_{M'}, w_L), w_L)$ and $U_B^+ = (0, w_{max})$. However, since we move with an ε -step size along the v -variable on the grid, it holds that $|v_{M'} - v_{cr}| < \varepsilon$ and $v_{cr} = \mathcal{V}(\rho_{cr}(w_L), w_L)$.

As we have seen in the cases (B.R.1) and (C.R.2), we may have $U_L \notin \mathcal{B}_R^{Rie}(U_B^-)$. Since it also holds that $v_B^- > v_L$, no new wave emerges from the interaction and $\Delta\gamma \leq 0$.

The proof of Proposition 5 is now complete. In particular, the number of waves can increase only a finite number of times and we have the following uniform bound for γ :

$$\begin{aligned} \gamma(t) \leq \gamma(0) &= \text{TV}(W_0^h) + \left| W_{in}^h(0) - W_0^h(x_{in+}) \right| + \left| v_{out}^h(0) - v^h(0, x_{out-}) \right| \\ &+ 3\text{TV}\left(W_{in}^h(s); s \in]0, T[\right) + \text{TV}\left(v_{out}^h(s); s \in]0, T[\right). \end{aligned} \quad (4.7)$$

□

4.2 Convergence to an entropy weak solution

We first prove the convergence of the sequence of approximate solutions constructed in Section 4.1.

Proposition 6. *The sequence $\{W^h\}_h$ converges up to a subsequence to a function W in \mathbf{L}^1_{loc} .*

Proof. In our case, we cannot apply Helly's Theorem in the form of [12, Theorem 2.4] to prove the convergence to W . This is due to the possible occurrence of infinite speed waves in the case (C.L.3) of Proposition 5, which prevents us from obtaining the \mathbf{L}^1 Lipschitz continuity in time of approximate solutions. Therefore, we have to prove explicitly the space-time BV bounds, which will lead to convergence (see e.g. [22, Lemma 5.6]). To this end, we need to show that for every $x \in]x_{in}, x_{out}[$, $t \in]0, T[$, it holds

$$\|W^h\|_{\mathbf{L}^\infty(]0, T[\times]x_{in}, x_{out}[)} \leq M, \quad (4.8a)$$

$$\mathrm{TV}_{(t,x)}(W^h) \leq C, \quad (4.8b)$$

where the constants $C, M > 0$ are independent of h and $\mathrm{TV}_{(t,x)}$ denotes the total variation in time and space, defined as

$$\mathrm{TV}_{(t,x)}(W^h) := \sup \left\{ \int_0^T \int_{x_{in}}^{x_{out}} W^h \cdot (\partial_t \phi + \partial_x \phi) \, dx \, dt : \phi \in \mathbf{C}_c^1(]0, T[\times]x_{in}, x_{out}[; \mathbb{R}^2), \|\phi\|_\infty \leq 1 \right\}.$$

To prove that W^h has uniformly bounded total variation, it is therefore sufficient to show that there exists C such that

$$\left| \int_0^T \int_{x_{in}}^{x_{out}} W^h \cdot (\partial_t \phi + \partial_x \phi) \, dx \, dt \right| \leq C \|\phi\|_\infty \quad (4.9)$$

for all $\phi \in \mathbf{C}_c^1(]0, T[\times]x_{in}, x_{out}[; \mathbb{R}^2)$ (see [12, Equation 2.29]).

The \mathbf{L}^∞ -bound (4.8a) follows easily from the invariance of the domain $\mathcal{W}^h \subset \mathcal{W}$, which is bounded by $w_{min}, w_{max}, v_{min} \geq 0$ and $v_{max} = w_{max}$ (see Figure 6).

To prove (4.8b), it suffices to prove the boundedness of the total variation in time and space separately.

Proposition 5 guarantees that the approximate solutions are uniformly BV in space for all $t \in [0, T[$:

$$\begin{aligned} \mathrm{TV}(W^h(t, \cdot)) &\leq \mathrm{TV}(W_0) + |W_{in}(0) - W_0(x_{in}+)| + |v_{out}(0) - v(0, x_{out}-)| \\ &\quad + 3\mathrm{TV}(W_{in}(s); s \in]0, T[) + \mathrm{TV}(v_{out}(s); s \in]0, T[) := \gamma_0. \end{aligned}$$

Therefore, we have in (4.9)

$$\begin{aligned} \left| \int_0^T \int_{x_{in}}^{x_{out}} W^h \cdot \partial_x \phi \, dx \, dt \right| &= \left| \lim_{h \rightarrow 0} \int_0^T \int_{x_{in}}^{x_{out}} W^h \cdot \frac{\phi(t, x+h) - \phi(t, x)}{h} \, dx \, dt \right| \\ &= \left| \lim_{h \rightarrow 0} \int_0^T \int_{x_{in}}^{x_{out}} \frac{W^h(t, x) - W^h(t, x-h)}{h} \cdot \phi(t, x) \, dx \, dt \right| \\ &\leq \int_0^T \left\{ \limsup_{h \rightarrow 0} \frac{1}{h} \int_{x_{in}}^{x_{out}} |W^h(t, x) - W^h(t, x-h)| \, dx \right\} \|\phi\|_\infty \, dt \\ &\leq \gamma_0 T \|\phi\|_\infty. \end{aligned} \quad (4.10)$$

Concerning the time component, let us assume first that in the interval $[s, t] \subset]0, T[$ there are no infinite speed waves (see case (C.L.3) of Proposition 5). In this case the \mathbf{L}^1 -continuity

in time holds, i.e.

$$\begin{aligned}
\|W^h(t) - W^h(s)\|_1 &= \int_{x_{in}}^{x_{out}} |W^h(t, x) - W^h(s, x)| dx \\
&\leq \gamma^h(0) \underbrace{\max_{W^h \in \mathcal{W}^h} \left\{ |\lambda_1(W^h)|, |\lambda_2(W^h)| \right\}}_{=: \Lambda} |t - s| \\
&\leq L|t - s|
\end{aligned} \tag{4.11}$$

with $L = \gamma_0 \Lambda$.

Let us assume now that a wave with infinite speed occurs at time t_i , $i = 1, \dots, Z^h$, which can only happen through a change in the left boundary state. Since the number of changes is bounded by construction, Z^h is finite. Referring to case (C.L.3), Figure 14, and fixing $x \in]x_{in}, x_{out}[$, we compute

$$\begin{aligned}
\text{TV}(W^h(\cdot, x)) &= \text{TV}(W^h(s, x); s \in]0, t_1]) + \text{TV}(W^h(s, x); s \in]t_Z, T]) + \\
&\quad \sum_{i=1}^{Z^h} \text{TV}(W^h(s, x); s \in]t_i, t_{i+1}]) + \sum_{i=1}^{Z^h} \underbrace{\left| W^h(t_i^-, x) - W^h(t_i^+, x) \right|}_{=2|w_{in}^h(t_{i+}) - w_{in}^h(t_{i-})|} \\
&\leq LT + 2\text{TV}(W_{in}).
\end{aligned}$$

Acting as in (4.10), we recover the estimate for the time-component of (4.9), thus showing that the sequence $\{W^h\}_h$ has uniformly bounded total variation.

Hence, by Helly's Theorem [22, Lemma 5.6], there exists $W \in \mathbf{L}^\infty([0, T[\times]x_{in}, x_{out}[; \mathcal{W})$ and a subsequence, still denoted by $\{W^h\}_h$, which converges to W in $\mathbf{L}_{\text{loc}}^1([0, T[\times]x_{in}, x_{out}[; \mathcal{W})$ as $h \rightarrow \infty$. Additionally, W satisfies the following inequalities:

$$\text{TV}(W(t, \cdot)) \leq \gamma_0 \text{ and } \|W(t, \cdot)\|_\infty \leq M \quad \forall t \in [0, T[\text{ and } x \in]x_{in}, x_{out}[.$$

At this point, we emphasize that, in contrast to [12, Theorem 2.4], we loose the \mathbf{L}^1 -continuity in time for the limit function W . \square

We are now left to show that the limit function W is indeed an entropy weak solution of the IBVP (4.1).

As in Section 3.2, we drop the u -variable dependency for notational simplicity, i.e. we write W instead of $u(W)$. Following [13, Theorem 4.1], we consider the following boundary entropy pairs:

$$\alpha^j(W_1, W_2) = \mathcal{E}^j(W_1) - \mathcal{E}^j(W_2) - \nabla_u \mathcal{E}^j(W_2) (W_1 - W_2), \tag{4.12}$$

$$\beta^j(W_1, W_2) = \mathcal{Q}^j(W_1) - \mathcal{Q}^j(W_2) - \nabla_u \mathcal{E}^j(W_2) (f(W_1) - f(W_2)) \tag{4.13}$$

where $\mathcal{E}^j, \mathcal{Q}^j$ are defined as in (3.4) for $j = 1$ and (3.5) for $j = 2$. We remark that, by setting $W_2 = W_B = (v_B, w_B)$ in (4.13), we obtain the entropy boundary condition defined in (3.6).

Proposition 7. *The limit function W defined in Proposition 6 is an entropy weak solution of the IBVP (4.1) in the sense of Definition 4.*

Proof. We follow [13, Theorem 4.1]. We start by verifying that W^h satisfies Definition 4 up to an error which decreases to 0 for h going to infinity.

We know that $W^h \in \mathbf{L}^\infty$. Let us now consider $\phi \in \mathbf{C}_c^\infty \left((]-\infty, T[\times]x_{in}, x_{out}[) ; \mathbb{R}_{\geq 0} \right)$. Since $\phi(0, x) \geq 0$ and, for our choice of the entropies, $\mathcal{E}^j(W) \geq 0$ for all $W \in \mathcal{W}$ and $j \in \{1, 2\}$ (see Equation (3.4a) and (3.5a)), we directly obtain $\int_{x_{in}}^{x_{out}} \mathcal{E}^j(W_0^h(x)) \phi(0, x) dx \geq 0$. Regarding the term

$$\int_0^T \int_{x_{in}}^{x_{out}} \left\{ \mathcal{E}^j(W^h) \partial_t \phi + \mathcal{Q}^j(W^h) \partial_x \phi \right\} dx dt, \quad (4.14)$$

following the proof of [4, Proposition 5.2], we need to consider the three different types of discontinuities (i.e. shocks, contact discontinuities and rarefaction shocks) that may arise at some point $x_i \in]x_{in}, x_{out}[$ with left and right values W_i and W_{i+1} respectively. By the Green Gauss-Formula, (4.14) is equivalent to

$$\int_0^T \left\{ \sum_i \dot{x}_i(t) \Delta \mathcal{E}_i^j(t) - \Delta \mathcal{Q}_i^j(t) \right\} \phi(t, x_i(t)) dt$$

where $\Delta \mathcal{E}_i^j = \mathcal{E}^j(W_{i+1}) - \mathcal{E}^j(W_i)$, $\Delta \mathcal{Q}_i^j = \mathcal{Q}^j(W_{i+1}) - \mathcal{Q}^j(W_i)$ and $\dot{x}_i = \sigma(W_i, W_{i+1})$ is the speed of the discontinuity given by the Rankine-Hugoniot condition (2.7). Neglecting the time dependence, we set

$$\hat{s}_i^j := \dot{x}_i \Delta \mathcal{E}_i^j - \Delta \mathcal{Q}_i^j$$

and we consider separately the different types of waves in the following.

1. If the discontinuity is a shock, it holds $w_i = w_{i+1}$ and $v_i > v_{i+1}$.

For $j = 1$:

- If $\bar{v} < v_{i+1}$, it follows that also $\bar{v} < v_i$. Thus,

$$\begin{aligned} \hat{s}_i^1 &= \frac{\mathcal{R}(v_{i+1}, w_i) v_{i+1} - \mathcal{R}(v_i, w_i) v_i}{\mathcal{R}(v_{i+1}, w_i) - \mathcal{R}(v_i, w_i)} \left\{ 1 - \frac{\mathcal{R}(v_{i+1}, w_i)}{\mathcal{R}(\bar{v}, w_i)} - \left(1 - \frac{\mathcal{R}(v_i, w_i)}{\mathcal{R}(\bar{v}, w_i)} \right) \right\} \\ &\quad - \left\{ \bar{v} - \frac{\mathcal{R}(v_{i+1}, w_i) v_{i+1}}{\mathcal{R}(\bar{v}, w_i)} - \left(\bar{v} - \frac{\mathcal{R}(v_i, w_i) v_i}{\mathcal{R}(\bar{v}, w_i)} \right) \right\} = 0. \end{aligned}$$

- If $\bar{v} \geq v_i$, it follows that $\bar{v} > v_{i+1}$ and we directly obtain $\hat{s}_i^1 = 0$ since $\mathcal{E}^1(W_i) = \mathcal{E}^1(W_{i+1}) = \mathcal{Q}^1(W_i) = \mathcal{Q}^1(W_{i+1}) = 0$.
- If $\bar{v} \in [v_{i+1}, v_i[$ and thus $\mathcal{E}^1(W_{i+1}) = \mathcal{Q}^1(W_{i+1}) = 0$, it holds

$$\hat{s}_i^1 = \mathcal{Q}^1(W_i) - \dot{x}_i \mathcal{E}^1(W_i) = \bar{v} - \frac{\mathcal{R}(v_i, w_i) v_i}{\mathcal{R}(\bar{v}, w_i)} - \dot{x}_i \left(1 - \frac{\mathcal{R}(v_i, w_i)}{\mathcal{R}(\bar{v}, w_i)} \right) \geq 0 \quad (4.15)$$

since $\mathcal{R}(\bar{v}, w_i) \bar{v} \geq \mathcal{R}(v_i, w_i) v_i + \dot{x}_i (\mathcal{R}(\bar{v}, w_i) - \mathcal{R}(v_i, w_i)) \geq 0$ by concavity of $\rho \mapsto \mathcal{Q}(\rho, w_i) = \rho \mathcal{V}(\rho, w_i)$, which is illustrated by Figure 15.

For $j = 2$:

$$\begin{aligned} \hat{s}_i^2 &= \frac{\mathcal{R}(v_{i+1}, w_i) v_{i+1} - \mathcal{R}(v_i, w_i) v_i}{\mathcal{R}(v_{i+1}, w_i) - \mathcal{R}(v_i, w_i)} \left\{ \mathcal{R}(v_{i+1}, w_i) |\bar{w} - w_i| - \mathcal{R}(v_i, w_i) |\bar{w} - w_i| \right\} \\ &\quad - \left\{ \mathcal{R}(v_{i+1}, w_i) v_{i+1} |\bar{w} - w_i| - \mathcal{R}(v_i, w_i) v_i |\bar{w} - w_i| \right\} = 0. \end{aligned}$$

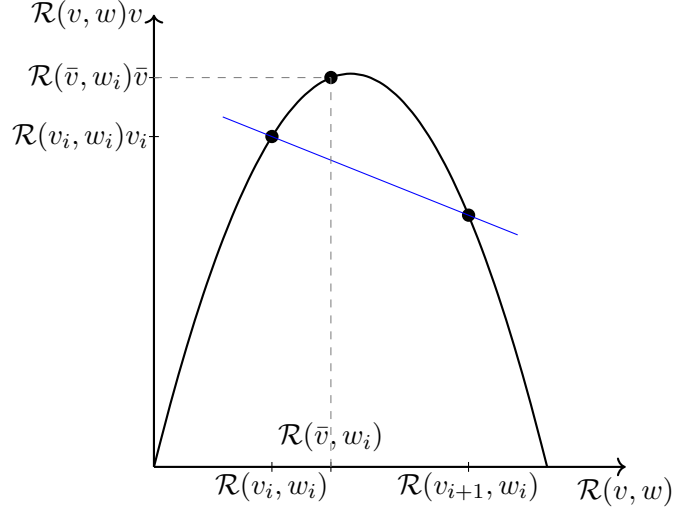


Figure 15: Graphical proof of inequality (4.15). The slope of the blue line is given by \dot{x}_i .

2. If the jump is a contact discontinuity, it holds $\dot{x}_i = v_i = v_{i+1}$.

For $j = 1$:

- If $v_i = v_{i+1} \leq \bar{v}$, we directly obtain $\hat{s}_i^1 = 0$
since $\mathcal{E}^1(W_i) = \mathcal{E}^1(W_{i+1}) = \mathcal{Q}^1(W_i) = \mathcal{Q}^1(W_{i+1}) = 0$.
- If $v_i = v_{i+1} > \bar{v}$, then

$$\begin{aligned} \hat{s}_i^1 = \dot{x}_i & \left\{ 1 - \frac{\mathcal{R}(v_{i+1}, w_{i+1})}{\mathcal{R}(\bar{v}, w_{i+1})} - \left(1 - \frac{\mathcal{R}(v_i, w_i)}{\mathcal{R}(\bar{v}, w_i)} \right) \right\} \\ & - \left\{ \bar{v} - \frac{\mathcal{R}(v_{i+1}, w_{i+1})\dot{x}_i}{\mathcal{R}(\bar{v}, w_{i+1})} - \left(\bar{v} - \frac{\mathcal{R}(v_i, w_i)\dot{x}_i}{\mathcal{R}(\bar{v}, w_i)} \right) \right\} = 0. \end{aligned}$$

For $j = 2$:

$$\begin{aligned} \hat{s}_i^2 = \dot{x}_i & \left\{ \mathcal{R}(v_{i+1}, w_{i+1})|\bar{w} - w_{i+1}| - \mathcal{R}(v_i, w_i)|\bar{w} - w_i| \right\} \\ & - \left\{ \mathcal{R}(v_{i+1}, w_{i+1})\dot{x}_i|\bar{w} - w_{i+1}| - \mathcal{R}(v_i, w_i)\dot{x}_i|\bar{w} - w_i| \right\} = 0. \end{aligned}$$

3. Finally, if the discontinuity is a ε -rarefaction, it holds $w_i = w_{i+1}$ and $v_i < v_{i+1}$ with $v_{i+1} = v_i + \varepsilon$. By similar calculations as for the shock case, we obtain that $\hat{s}_i^2 = 0$, and $\hat{s}_i^1 = 0$ if $\bar{v} < v_i$ or $\bar{v} \geq v_{i+1}$. However, if $\bar{v} \in [v_i, v_{i+1}[$, it follows that $\mathcal{E}^1(W_i) = \mathcal{Q}^1(W_i) = 0$ and, by the same concavity argument as above, we compute that $\hat{s}_i^1 \leq 0$. Moreover,

$$\begin{aligned} \hat{s}_i^1 &= \dot{x}_i \mathcal{E}^1(W_{i+1}) - \mathcal{Q}^1(W_{i+1}) \\ &= \frac{\mathcal{R}(v_{i+1}, w_{i+1})v_{i+1} - \mathcal{R}(v_i, w_i)v_i}{\mathcal{R}(v_{i+1}, w_{i+1}) - \mathcal{R}(v_i, w_i)} \left(1 - \frac{\mathcal{R}(v_{i+1}, w_{i+1})}{\mathcal{R}(\bar{v}, w_{i+1})} \right) - \bar{v} + \frac{\mathcal{R}(v_{i+1}, w_{i+1})v_{i+1}}{\mathcal{R}(\bar{v}, w_{i+1})}. \end{aligned}$$

Let us set $\varphi(\rho) := \mathcal{R}(\mathcal{V}(\rho, w_i), w_i)\mathcal{V}(\rho, w_i) = \rho\mathcal{V}(\rho, w_i)$, which is a strictly concave function by assumption (2.1b), and rewrite the above quantity as

$$\hat{s}_i^1 = \frac{\varphi(\rho_{i+1}) - \varphi(\rho_i)}{\rho_{i+1} - \rho_i} \left(1 - \frac{\rho_{i+1}}{\bar{\rho}} \right) - \mathcal{V}(\bar{\rho}, w_i) + \frac{\varphi(\rho_{i+1})}{\bar{\rho}}$$

$$\begin{aligned}
&= \dot{\varphi}(\rho_{i+1/2}) \frac{\bar{\rho} - \rho_{i+1}}{\bar{\rho}} - \frac{\varphi(\bar{\rho}) - \varphi(\rho_{i+1})}{\bar{\rho}} \\
&= \dot{\varphi}(\rho_{i+1/2}) \frac{\bar{\rho} - \rho_{i+1}}{\bar{\rho}} - \dot{\varphi}(\tilde{\rho}) \frac{\bar{\rho} - \rho_{i+1}}{\bar{\rho}} \\
&= \ddot{\varphi}(\hat{\rho})(\rho_{i+1/2} - \tilde{\rho}) \frac{\bar{\rho} - \rho_{i+1}}{\bar{\rho}} \\
&\geq \min_{\rho \in [0, R(w_i)]} \ddot{\varphi}(\rho) (\rho_i - \rho_{i+1}) \\
&\geq -C(v_{i+1} - v_i)
\end{aligned}$$

for some constant $C > 0$, with $0 \leq \rho_{i+1} < \tilde{\rho} < \hat{\rho} < \rho_{i+1/2} < \rho_i$ and $\rho_{i+1} < \tilde{\rho} < \bar{\rho} \leq \rho_i$. Above, we observed that $\frac{\bar{\rho} - \rho_{i+1}}{\bar{\rho}} < 1$.

Applying the same argument as in the proof of [4, Proposition 5.2], we conclude that for any fixed $\delta > 0$, there exists $\hat{h} > 0$ such that for all $h \geq \hat{h}$ it holds

$$\int_0^T \int_{x_{in}}^{x_{out}} \left\{ \mathcal{E}^j(W^h) \partial_t \phi + \mathcal{Q}^j(W^h) \partial_x \phi \right\} dx dt + \int_{x_{in}}^{x_{out}} \mathcal{E}^j(W_0^h(x)) \phi(0, x) dx \geq -\delta. \quad (4.16)$$

Concerning the entropy boundary condition (4.3), we observe that it is guaranteed by Proposition 3 for those cases in Proposition 5 where it holds $W^h(t, x_{in}+) \in \mathcal{B}_L^{Rie}(\rho_{in}^h(t), w_{in}^h(t)) \subset \mathcal{B}_L^{Ent}(\rho_{in}^h(t), w_{in}^h(t))$ and $W^h(t, x_{out}-) \in \mathcal{B}_R^{Rie}(\rho_{out}^h(t), w_{out}^h(t)) = \mathcal{B}_R^{Ent}(\rho_{out}^h(t), w_{out}^h(t))$. However, due to the discretization of the domain \mathcal{W}^h , in some cases the approximate solution's traces (v^h, w^h) are states that do not belong to the admissible Riemann set, but $|v^h - v_{cr}^h| < \varepsilon$. In these cases, it holds

$$\beta^j(W^h(t, x_{in}+), W_{in}^h(t)) \leq C\varepsilon \quad \text{or} \quad \beta^j(W^h(t, x_{out}-), W_{out}^h(t)) \geq -C\varepsilon,$$

for some constant $C > 0$. We refer to Appendix B for a detailed analysis of these cases. Therefore, for any fixed $\delta > 0$, there exists $\hat{h} > 0$ such that for all $h \geq \hat{h}$ it holds

$$\begin{aligned}
&\text{ess lim}_{x \rightarrow x_{in}+} \int_0^T \beta^j(W^h(t, x), W_{in}^h(t)) \gamma(t) dt \leq \delta, \\
&\text{ess lim}_{x \rightarrow x_{out}-} \int_0^T \beta^j(W^h(t, x), W_{out}^h(t)) \gamma(t) dt \geq -\delta.
\end{aligned}$$

Thus, following the proof of [13, Theorem 4.1], the approximate WFT-solution W^h satisfies for any test function $\phi \in \mathbf{C}_c^\infty([-\infty, T] \times \mathbb{R}; \mathbb{R}^{0+})$ and any $W_2 \in \mathcal{W}$ the inequality

$$\begin{aligned}
&\int_0^T \int_{x_{in}}^{x_{out}} \left\{ \alpha^j(W^h(t, x), W_2) \partial_t \phi + \beta^j(W^h(t, x), W_2) \partial_x \phi \right\} dx dt + \int_{x_{in}}^{x_{out}} \alpha^j(W_0^h(x), W_2) \phi(0, x) dx \\
&\quad + K \left\{ \int_0^t |W_{in}^h(t) - W_2| \phi(t, x_{in}) dt + \int_0^t |W_{out}^h(t) - W_2| \phi(t, x_{out}) dt \right\} \geq -3\delta
\end{aligned} \quad (4.17)$$

for some $K > 0$ and h sufficiently large.

Moreover, since the construction of W^h is based on the Riemann solver (see Definition 1) and the Rankine-Hugoniot conditions hold at rarefaction fronts, the approximate solution is

a weak solution in the sense of Definition 5. Therefore, letting $h \rightarrow \infty$, we show that W is a weak solution.

Letting now $h \rightarrow \infty$ in (4.17), due to the \mathbf{L}^1 convergence of $\{W^h\}_h$ to W , the equation yields to

$$\int_0^T \int_{x_{in}}^{x_{out}} \left\{ \alpha^j(W(t, x), W_2) \partial_t \phi + \beta^j(W(t, x), W_2) \partial_x \phi \right\} dx dt + \int_{x_{in}}^{x_{out}} \alpha^j(W_0(x), W_2) \phi(0, x) dx + K \left\{ \int_0^t |W_{in} - W_2| \phi(t, x_{in}) dt + \int_0^t |W_{out} - W_2| \phi(t, x_{out}) dt \right\} \geq 0$$

for $j \in \{1, 2\}$. Using again [13, Theorem 4.1], we conclude that the limit function W is indeed a entropy weak solution in the sense of Definition 4. \square

We emphasize again that, with our choice of the entropy-flux pairs, the entropy weak solution W can include vacuum states which do not belong to the Riemann boundary set.

Remark 15. We remark that our definition of the entropy weak solution is a weaker formulation than the one in [13, Definition 4.2] due to the loss of the \mathbf{L}^1 continuity in time. Moreover, our entropies \mathcal{E}^1 in (3.4a) are not strictly convex. However, [13, Theorem 4.1] still applies, since convexity there is only needed to deal with the strong initial condition [13, equation (4.8)], which we consider in weak form.

Remark 16. The choice of the Riemann solver of Definition 1, Case 8, although inspired by reality, induces some extra difficulties in treating the problem, as the presence of infinite speed waves and the need of “well-prepared” initial data, see Remark 3. We conjecture that a similar existence result could be proved, maybe with less adjustments, using the Riemann solver proposed in Figure 2.

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A Proof of Proposition 3

We start considering $\mathcal{B}_L^{Rie}(\rho_B, w_B)$, which is described by Proposition 1.

Let us assume first that $\rho_B > 0$ and $w_B = w$, which implies $\beta^2(W, W_B) = 0$. For $j = 1$, it suffices to consider the case $\bar{v} \in [\min\{v, v_B\}, \max\{v, v_B\}]$, since otherwise $\beta^1(W, W_B) = 0$.

- If $\rho_B < \rho_{cr}(w_B)$, it holds $(\rho, w) = (\rho_B, w_B)$ or $\rho \geq \tau(\rho_B)$. Therefore, we have $v \leq \bar{v} \leq v_B$, $\rho_B \leq \mathcal{R}(\bar{v}, w_B) \leq \rho$ and $\mathcal{R}(\bar{v}, w_B) \bar{v} \geq \rho v$ (see Figure 3a). Thus,

$$\beta^1(W, W_B) = -\bar{v} + \frac{\rho v}{\mathcal{R}(\bar{v}, w_B)} \leq 0.$$

- If $\rho_B \geq \rho_{cr}(w_B)$, it holds $\rho \geq \rho_{cr}(w_B)$. Thus, we either have $v \leq \bar{v} \leq v_B$ with $\mathcal{R}(\bar{v}, w_B)\bar{v} \geq \rho v$ and again $\beta^1(W, W_B) \leq 0$ or $v_B \leq \bar{v} < v$, $\rho < \mathcal{R}(\bar{v}, w_B) \leq \rho_B$ and $\mathcal{R}(\bar{v}, w_B)\bar{v} < \rho v$ (see Figure 3b), leading to

$$\beta^1(W, W_B) = \bar{v} - \frac{\rho v}{\mathcal{R}(\bar{v}, w_B)} < 0.$$

Finally, we consider the set of points

$$\{(R(w), w) : w \in [w_{min}, w_{max}]\}.$$

Since $v = 0$, it holds $\beta^2(W, W_B) = 0$. Concerning $\beta^1(W, W_B)$, the only possible cases are $0 = v \leq \bar{v} \leq v_B$ and $0 = v \leq v_B < \bar{v}$. In any case, we obtain $\beta^1(W, W_B) \leq 0$.

Let us now consider the vacuum case, i.e. $\rho_B = 0$ (see Figure 3c). Since $v = \mathcal{V}(R(w), w) = 0$, we get $\beta^2(W, W_B) = 0$.

If $\bar{v} \neq v_B$, it holds that $\mathcal{Q}^1(W) = 0$,

$$\mathcal{Q}^1(W_B) = \begin{cases} 0 & \text{if } v_B \leq \bar{v} \\ \bar{v} & \text{if } v_B > \bar{v} \end{cases}$$

and $f(W) = f(W_B) = 0$, implying $\beta^1(W, W_B) = -\mathcal{Q}^1(W_B) \leq 0$.

If $\bar{v} = v_B$, it holds $\mathcal{Q}^1(W) = \mathcal{Q}^1(W_B) = 0$ and

$$\begin{aligned} \beta^1(W, W_B) &= -\alpha \begin{pmatrix} \frac{\partial v}{\partial \rho}(W_B) \\ \frac{\partial v}{\partial y}(W_B) \end{pmatrix} \cdot \underbrace{\{f(W) - f(W_B)\}}_{(0,0)^\top} \\ &= \alpha \begin{pmatrix} \frac{\partial v}{\partial \rho}(W_B) \\ \frac{\partial v}{\partial y}(W_B) \end{pmatrix} \cdot f(W_B) \\ &= -\frac{1}{\mathcal{V}_\rho(\rho_B, w_B)\rho_B} \mathcal{V}_\rho(\rho_B, w_B)\rho_B v_B = -v_B \leq 0. \end{aligned}$$

If instead $\rho = 0$, implying again $\beta^2(W, W_B) = 0$, we know from the Riemann solver (Definition 1, case 9) that $(\rho, w) = (0, w) = (0, w_B) = (\rho_B, w_B)$ which directly leads to $\beta^1(W, W_B) = 0$.

This shows that $\mathcal{B}_L^{Rie}(\rho_B, w_B) \subseteq \mathcal{B}_L^{Ent}(\rho_B, w_B)$.

We now consider the right boundary case, i.e. $i = R$ (see Proposition 2). We recall that $\mathcal{B}_R^{Rie}(\rho_B, w_B) = \mathcal{B}_R^{Rie}(\mathcal{R}(v_B, w), w)$ for all $w \in [w_{min}, w_{max}]$, with $v_B = \mathcal{V}(\rho_B, w_B)$ (see Remark 8). Therefore we can assume $w = w_B$, leading to $\beta^2(W, W_B) = 0$ due to $w = w_B$ or $\rho = 0$.

We first look at the case $\rho_B > 0$. As for the left boundary, it suffices to treat the cases $\bar{v} \in [\min\{v, v_B\}, \max\{v, v_B\}]$, otherwise we directly obtain $\beta^1(W, W_B) = 0$.

- If $\mathcal{V}(\rho_{cr}(w), w) \leq v_B$ and $\rho \leq \rho_{cr}(w)$, we either have $v_B \leq \bar{v} < v$ with $\mathcal{R}(\bar{v}, w_B)\bar{v} > \rho v$ and

$$\beta^1(W, W_B) = \bar{v} - \frac{\rho v}{\mathcal{R}(\bar{v}, w_B)} \geq 0,$$

or it holds $v \leq \bar{v} \leq v_B$, $\rho_B \leq \mathcal{R}(\bar{v}, w_B) \leq \rho$ together with $\mathcal{R}(\bar{v}, w_B)\bar{v} \leq \rho v$, which implies

$$\beta^1(W, W_B) = -\bar{v} + \frac{\rho v}{\mathcal{R}(\bar{v}, w_B)} \geq 0.$$

- If $\mathcal{V}(\rho_{cr}(w), w) > v_B$ and $\rho \leq \tau(\mathcal{R}(v_B, w))$, we have $v_B \leq \bar{v} \leq v$, $\rho \leq \mathcal{R}(\bar{v}, w_B) \leq \rho_B$ and $\mathcal{R}(\bar{v}, w_B)\bar{v} \geq \rho v$. Thus, we have again

$$\beta^1(W, W_B) = \bar{v} - \frac{\rho v}{\mathcal{R}(\bar{v}, w_B)} \geq 0.$$

Concerning the vacuum case, i.e. $\rho_B = 0$, which implies $\text{sgn}(\rho_B(\bar{w} - w_B)) \in [-1, 1]$, we know that the admissible points satisfy $\rho \leq \rho_{cr}(w)$. Moreover, since $\mathcal{B}_R^{Rie}(\rho_B, w_B) = B_R^{Rie}(\mathcal{R}(v_B, w), w)$, we can again consider $w = w_B$. For $j = 2$, we have to distinguish between $\bar{w} > w, \bar{w} < w$ and $\bar{w} = w = w_B$. In any case, we obtain $\beta^2(W, W_B) \geq 0$ in (3.10).

Next, assuming $\rho = \mathcal{R}(v, w) > 0$, we know that $v < v_B = w_B$.

It suffices now to consider the case $v \leq \bar{v} \leq v_B$ (and $v < v_B$) and thus $\rho v \geq \mathcal{R}(\bar{v}, w_B)\bar{v}$. It holds

$$\beta^1(W, W_B) = -\bar{v} + \frac{\rho v}{\mathcal{R}(\bar{v}, w)} \geq 0.$$

On the other hand, if $\rho = 0$, we can set $(\rho_B, w_B) = (0, w_B) = (0, w) = (\rho, w)$, which directly leads to $\beta^1(W, W_B) = 0$.

This shows that $\mathcal{B}_R^{Rie}(\rho_B, w_B) \subseteq \mathcal{B}_R^{Ent}(\rho_B, w_B)$.

B Approximate entropy boundary condition

In cases (B.R.1), (C.L.1), (C.L.3), (C.R.2) and (C.R.4) of Proposition 5, we may observe boundary states $W_{M'}^h$ of the discretized solution not belonging to the boundary Riemann set, i.e. $W_{M'}^h \notin B_i^{Rie}, i \in \{R, L\}$. For these particular cases, we prove below that the entropy boundary condition (3.6) is satisfied up to an error which vanishes with the discretization grid mesh when passing to the limit in Proposition 7. Again, for notational simplicity, we drop the h -index in the following.

Starting with the left boundary cases ((C.L.1) and (C.L.3)), we define $W_{in} = (v_{in}, w_{in}) = W_B^+$. It holds $\rho_{in} = \mathcal{R}(v_{in}, w_{in}) > \rho_{cr}(w_{in})$, but we may have $\rho_{M'} = \mathcal{R}(v_{M'}, w_{in}) < \rho_{cr}(w_{in})$, which implies $v_{M'} > v_{in}$. Therefore, it suffices to consider in Equation (3.9) the case $\bar{v} \in [v_{in}, v_{M'}[$, otherwise we directly obtain that $\beta^1(W_{M'}, W_{in}) = 0$. Since we reach $W_{M'}$ from W_{in} by a negative rarefaction wave, it must hold that $\rho_{M'} v_{M'} \geq \rho_{in} v_{in}$. If $\bar{v} \in [v_{in}, \mathcal{V}(\tau(\rho_{M'}), w_{in})]$, we also have $\rho_{M'} v_{M'} \geq \mathcal{R}(\bar{v}, w_{in})\bar{v} = \bar{\rho}\bar{v}$ which leads to $\beta^1(W_{M'}, W_{in}) \leq 0$ in (3.9). However, if $\bar{v} \in]\mathcal{V}(\tau(\rho_{M'}), w_{in}), v_{M'}[$, it holds $\rho_{M'} v_{M'} < \bar{\rho}\bar{v}$ and $\beta^1(W_{M'}, W_{in}) > 0$.

By defining $\varphi_{in}(\rho) := \mathcal{R}(\mathcal{V}(\rho, w_{in}), w_{in})\mathcal{V}(\rho, w_{in})$, we compute

$$\begin{aligned} \beta^1(W_{M'}, W_{in}) &= \frac{1}{\bar{\rho}}(\varphi_{in}(\bar{\rho}) - \varphi_{in}(\rho_{M'})) \\ &= \frac{1}{\bar{\rho}}\dot{\varphi}_{in}(\hat{\rho})(\bar{\rho} - \rho_{M'}) \\ &= \frac{1}{\bar{\rho}}\dot{\varphi}_{in}(\hat{\rho})(\mathcal{R}(\bar{v}, w_{in}) - \mathcal{R}(v_{M'}, w_{in})) \\ &= \frac{1}{\bar{\rho}}\dot{\varphi}_{in}(\hat{\rho})\mathcal{R}_v(\bar{v}, w_{in})(\bar{v} - v_{M'}) \\ &\leq \frac{1}{\bar{\rho}} \max_{\rho \in [0, R(w_{max})]} |\dot{\varphi}_{in}(\rho)| \max_{v \in [0, w_{max}]} |\mathcal{R}_v(v, w_{in})| (v_{M'} - \bar{v}) \\ &\leq C\varepsilon \end{aligned}$$

for some $C > 0$, $\rho_{M'} < \hat{\rho} < \bar{\rho}$ and $\bar{v} < \tilde{v} < v_{M'}$.

Moreover, since $w_{M'} = w_{in}$ and therefore $\text{sgn}(\mathcal{R}(v_{in}, w_{in})(\bar{w} - w_{in})) = \text{sgn}(\mathcal{R}(v_{M'}, w_{M'})(\bar{w} - w_{M'}))$ in (3.10a), we compute $\beta^2(W_{M'}, W_{in}) = 0$.

Considering now the right boundary cases (B.R.1), (C.R.2) and (C.R.4), we define

$$W_{out} = (v_{out}, w_{out}) = \begin{cases} W_B & \text{if (B.R.1),} \\ W_B^+ & \text{if (C.R.2), (C.R.4).} \end{cases}$$

Moreover, we use again the fact that $B_R^{Rie}(\mathcal{R}(v_{out}, w_{out}), w_{out}) = B_R^{Rie}(\mathcal{R}(v_{out}, w), w)$ (see Remark 8).

It holds $\mathcal{R}(v_{out}, w_{M'}) < \rho_{cr}(w_{M'})$, but we may have $\rho_{M'} = \mathcal{R}(v_{M'}, w_{M'}) > \rho_{cr}(w_{M'})$, which implies $v_{M'} < v_{out}$. As before, it suffices to consider the case $\bar{v} \in]v_{M'}, v_{out}]$ in (3.9), otherwise we directly obtain that $\beta^1(W_{M'}, W_{out}) = 0$. Since we reach W_{out} from $W_{M'}$ by a positive rarefaction wave, it must hold that $\rho_{M'} v_{M'} \geq \mathcal{R}(v_{out}, w_{M'}) v_{out}$. If $\bar{v} \in [\mathcal{V}(\tau(\rho_{M'}), w_{M'}), v_{out}]$, we also have $\rho_{M'} v_{M'} \geq \mathcal{R}(\bar{v}, w_{M'}) \bar{v} = \bar{\rho} \bar{v}$, which leads to $\beta^1(W_{M'}, W_{out}) \geq 0$. However, if $\bar{v} \in]v_{M'}, \mathcal{V}(\tau(\rho_{M'}), w_{M'})[$, we obtain $\rho_{M'} v_{M'} < \bar{\rho} \bar{v}$ and $\beta^1(W_{M'}, W_{out}) < 0$. By defining $\varphi_{out}(\rho) := \mathcal{R}(\mathcal{V}(\rho, w_{M'}), w_{M'}) \mathcal{V}(\rho, w_{M'})$, we compute, as in the left boundary case,

$$\begin{aligned} \beta^1(W_{M'}, W_{out}) &= -\frac{1}{\bar{\rho}}(\varphi_{out}(\bar{\rho}) - \varphi_{out}(\rho_{M'})) \\ &= -\frac{1}{\bar{\rho}} \dot{\varphi}_{out}(\hat{\rho}) \mathcal{R}_v(\tilde{v}, w_{M'}) (\bar{v} - v_{M'}) \\ &\geq -\frac{1}{\bar{\rho}} \max_{\rho \in [0, R(w_{max})]} |\dot{\varphi}_{out}(\rho)| \max_{v \in [0, w_{max}]} |\mathcal{R}_v(v, w_{M'})| (\bar{v} - v_{M'}) \\ &\geq -C\varepsilon \end{aligned}$$

for some $C > 0$, $\bar{\rho} < \hat{\rho} < \rho_{M'}$ and $v_{M'} < \tilde{v} < \bar{v}$.

Finally, we have $\beta^2(W_{M'}, W_{out}) \geq 0$, since it holds $w_{M'} = w_{out}$ in (B.R.1) and (C.R.2), and $\rho_{out} = 0$ in (C.R.4).

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