# Input-Output Stability of a Reaction Diffusion Equation with In-domain Disturbances

Suha Shreim[1], Francesco Ferrante[2] and Christophe Prieur[1]

***Abstract*— The input-output stability (IOS) of a reaction-diffusion equation by means of a finite-dimensional linear time-invariant control system is studied. The reaction-diffusion plant admits a finite number of unstable poles and is open-loop unstable. The infinite-dimensional plant is put in feedback with a dynamic controller to achieve output stability via a Dirichlet boundary measurement and regulated output. The control design problem consists of deriving sufficient conditions in the form of matrix inequalities which allows us to show that the order of the finite-dimensional controller can be selected large enough to achieve IOS even when the control design is not optimal.**

## I. Introduction

To properly model and meet the performance requirement of numerous engineering systems, control scientists have to use models given by partial differential equations (PDEs). Those types of equations are examples of distributed parameter systems (DPS) in which the state-space is infinite-dimensional. Infinite-dimensional systems emerge to be of utmost relevance when describing physical systems such as quantum systems, fluid mechanical systems, wave propagation, diffusion phenomena. Practical applications in all engineering domains are subjected to delays, reaction-diffusion dynamics, thermodynamics, etc. Therefore, the control theory of infinite-dimensional systems remains a necessary area of research and its motivation has been well established (see more in [3], [14], [5], [8], [7]).

One of the most important problems in control theory is global stabilization by means of output-feedback control. While the problem of the input-output stability (IOS) property with respect to domain or boundary disturbances has been widely investigated in the general context of finite-dimensional linear time-invariant (LTI) control systems and is considered classical, its extension to distributed parameter systems is a challenge and is still an open problem. Recently, one can find several works on the IOS property extension to PDEs ([16], [4]) and output-feedback control extensions to PDEs ([15], [2], [17], [19]).

From a practical point of view, there is an emphasized interest in designing finite-dimensional feedback control for infinite-dimensional DPS mainly because the controllers must be implemented by online digital computers with finite memory-access time and finite wordlength. In past work, finite-dimensional approximations of an infinite-dimensional system was obtained to design the controllers on those so-called reduced-order models ([1], [11], [9]). However, we cannot theoretically guarantee the stability of the closed-loop system consisting of the actual DPS and the reduced order controller. In contrast, in this paper, the controller synthesis is considered with respect to a closed-loop system consisting of the infinite-dimensional system and the finite-dimensional control system. In particular, we propose, for the first time, an output feedback control design procedure to achieve the input-output stability (IOS) of a reaction diffusion system by means of a finite-dimensional, LTI control system.

The considered reaction-diffusion plant, which may be unstable, is modeled by a Sturm-Liouville operator as classically introduced in the context of parabolic partial differential equation (see [10]). We focus on a boundary control input along with a Dirichlet boundary measurement and regulated output since this configuration is interesting from a practical engineering perspective. This renders our problem more challenging since they require to deal with unbounded control operators. Furthermore, the disturbance is in-domain. The proposed control design strategy is based on an appropriate coupling of an output-feedback with finite-dimension LTI control system and Lyapunov stability methods. It is important to note that when Dirichlet boundary conditions are considered for the control input and to-be-regulated output respectively, the solutions need to be sufficiently regular. However, our conditions do not need further regularity than the ones required by the existence results of classical solutions. The main result of this article is presented as matrix inequalities which constitute sufficient conditions for the input-output stability of the reaction-diffusion plant with respect to in-domain disturbances. We compute the control gain matrices using the method of single-objective synthesis and use numerical examples to assess the feasibility of the sufficient conditions as a function of the order of the finite-dimensional controller.

The paper is organized as follows. Section II introduces necessary notations and properties. Section III introduces the reaction-diffusion plant with Dirichlet measurement and Dirichlet regulated output. The spectral reduction of the given plant is then partitioned such that a finite-dimensional system, beyond which the system is stable, is coupled with the infinite-dimensional stable plant. Section IV presents the general framework for the output-feedback control problem by means of a finite-dimensional LTI control system. The problem of input-output stability is explicitly stated. In

This work has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003).

[1] S. Shreim and C. Prieur are with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France. suha.shreim@gipsa-lab.fr, christophe.prieur@gipsa-lab.fr

[1] F. Ferrante is with the Department of Engineering, University of Perugia, 06125 Perugia, Italy francesco.ferrante@unipg.it

Section V, the Lyapunov stability analysis is studied to derive sufficient conditions in the form of matrix inequalities. Section VI proposes the single-objective method to compute the control gain matrices and minimize the effect of the IOS gain. In Section VII, we illustrate a numerical example to verify our results and in Section VIII, we present a brief summary to the contributions of this paper and future perspective.

## II. Notations and properties

The sets $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ represent the set of nonnegative and positive real scalars, respectively. The symbol $\mathbb{S}_p^n$ denotes the set of real $n \times n$ symmetric positive definite matrices. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of $A$. For a symmetric matrix $A$, positive definiteness (negative definiteness) and positive semidefiniteness (negative semidefiniteness) are denoted, respectively, by $A > 0$ ($A < 0$) and $A \geq 0$ ($A \leq 0$). Also, $\lambda_{\min}(A)$ (respectively $\lambda_{\max}(A)$) denotes its minimal (respectively maximal) eigenvalue. In partitioned symmetric matrices, the symbol $*$ stands for symmetric blocks. For a vector $z \in \mathbb{R}^n$, $\|z\|$ denotes its Euclidean norm. For $U \subset \mathbb{R}$, $f : U \subset \mathbb{R} \to V$, we denote by $\|f\|_{\mathcal{L}^2} = (\int_U \|f(z)\|^2 dz)^{\frac{1}{2}}$, the $\mathcal{L}^2$-norm of [1] $f$ and the Fréchet derivative of $f$ at $z$ and is denoted by $Df(z)$. Given $f : U \subset \mathbb{R} \to V$, we say that $f \in \mathcal{L}^2$ if $f$ is measurable and $\|f\|_{\mathcal{L}^2}$ is finite. The symbol $\mathcal{C}^2(U, V)$ denotes the set of functions $f : U \to V$ that are continuously differentiable and their derivatives are continuously differentiable. For an integer $m \geq 1$, the Sobolev space of order $m$ is denoted by $\mathcal{H}^m(0,1)$ and the $m$-order Soblev space is denoted by $\|\cdot\|_{\mathcal{H}^m}$.

Let $p \in \mathcal{C}^1([0,1])$ and $q \in \mathcal{C}^0([0,1])$ with $p, q > 0$. Let the Strum-Liouville operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathcal{L}^2(0,1) \to \mathcal{L}^2(0,1)$ be defined by

$$\mathbb{A}f = -(pf')' + qf$$

on the domain $\mathcal{D}(\mathbb{A}) \subset \mathcal{L}^2(0,1)$ given by $\mathcal{D}(\mathbb{A}) = \{f \in \mathcal{H}^2(0,1) : f'(0) = f(1) = 0\}$. The eigenvalues $\lambda_n, n \geq 1$ of $\mathbb{A}$ are simple, non-negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. Moreover the associated unit eigenvectors $\phi_n \in \mathcal{L}^2(0,1)$ form an orthonormal basis and we also have $\mathcal{D}(\mathbb{A}) = \{f \in \mathcal{H}^2(0,1) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\}$ and $\mathbb{A}f = \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle \phi_n$. Let $p_*, p^*, q^*, q_* \in \mathbb{R}$ be such that $0 < p_* \leq p(x) \leq p^*$ and $0 < q_* \leq q(x) \leq q^*$ for all $x \in [0,1]$, then it holds (see e.g [10]):

$$0 \leq \pi^2(n-1)^2 p_* + q_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^* \tag{1}$$

for all $n \geq 1$.

Finally, one can check that for all $f \in \mathcal{D}(\mathbb{A})$

$$\langle \mathbb{A}f, f\rangle = \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle^2 = \int_0^1 p(x) f'(x)^2 + q(x) f(x)^2 dx. \tag{2}$$

Moreover, for any $f \in \mathcal{D}(\mathbb{A})$, we have $f(x) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n(x)$ and $f'(x) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n'(x)$.

[1] In this paper, we only consider Lebesgue measurable functions.

## III. Preliminaries

We consider the one-dimensional reaction-diffusion system with Dirichlet boundary measurement and regulated output described for $t > 0$ and $x \in (0,1)$ by

$$\begin{aligned} &z_t(t,x) = (p(x) z_x(t,x))_x + (q_c - q) z(t,x) + m(x) d(t) \\ &z_x(t,0) = 0, \qquad z(t,1) = u(t) \\ &z(0,x) = z_0(x) \\ &y(t) = z(t,0) \end{aligned} \tag{3}$$

where $p \in \mathcal{C}^1([0,1])$. Here $q_c \in \mathbb{R}$ is a constant, $u(t) \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is a boundary measurement and the to-be-regulated output, $z_0 \in \mathcal{H}^2(0,1)$ is the initial condition, $z(t,\cdot) \in \mathcal{H}^2(0,1)$ is the state and $d \in \mathbb{R}$ is the disturbance with $m \in \mathcal{C}^0([0,1])$. The objective is to design a finite-dimensional controller to achieve input-output stability (IOS) with respect to the in-domain disturbance.

### A. Spectral Reduction

Following [6], and as classically done in the context of boundary control systems (see [3, Sec 3.3]), we transform problem (3) into an in-domain controlled equivalent system. To this end, we introduce the change of variable

$$w(t,x) \coloneqq z(t,x) - x^2 u(t). \tag{4}$$

For which we have

$$\begin{aligned} &w_t(t,x) = (p(x) w_x(t,z))_x + (q_c - q) w(t,x)) \\ &+ a(x) u(t) + b(x) \dot{u}(t) + m(x) d(t) \\ &w_x(t,0) = 0, \qquad w(t,1) = 0 \\ &y(t) = w(t,0) \end{aligned} \tag{5}$$

with $a, b \in \mathcal{L}^2(0,1)$ defined by $a(x) = 2p(x) + 2xp'(x) + (q_c - q)x^2$, and $b(x) = -x^2$. In addition, $w_0(x) = z_0(x) - x^2 u(0)$. Considering that $v(t) = \dot{u}(t)$ is an auxiliary command input, we obtain

$$\begin{aligned} &\dot{w}(t) = -\mathbb{A}w(t) + q_c w(t) + a u(t) + b v(t) + m d(t) \\ &\dot{u}(t) = v(t) \end{aligned} \tag{6}$$

with $\mathcal{D}(\mathbb{A}) \coloneqq \{f \in \mathcal{H}^2(0,1) : f'(0) = f(1) = 0\}$. We introduce the coefficients of projection $w_n = \langle w(\cdot), \phi_n\rangle, a_n = \langle a, \phi_n\rangle$, $b_n = \langle b, \phi_n\rangle$ and $m_n = \langle m, \phi_n\rangle$. We consider the classical solutions associated with any $z_0 \in \mathcal{H}^2(0,1)$ and any $u(0) \in \mathbb{R}$ such that $z'(0) = 0$ and $z_0(1) = u(0)$ (their existence for the upcoming closed-loop dynamics is a direct consequence of [3, Chapter 10, Thm 10.1.4] with $m$ continuously differentiable and $d \in \mathcal{C}^2(0,\tau)$ for all $\tau > 0$). We have $w(t) \in \mathcal{D}(\mathbb{A})$ for all $t \geq 0$ and for $n \geq 1$

$$\begin{aligned} &\dot{w}_n(t) = (-\lambda_n + q_c) w_n(t) + a_n u(t) + b_n v(t) + m_n d(t), \\ &\dot{u}(t) = v(t), \\ &y(t) = \sum_{i\geq 1} \phi_i(0) w_i(t). \end{aligned} \tag{7}$$

Let $N_0 \geq 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. We now introduce an arbitrary integer

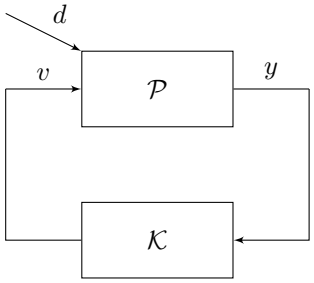



Fig. 1. Closed-loop system.

$N \geq N_0$ which will be further constrained later. We design an output feedback controller that will act on and modify the first $N$ modes of the plant. First, we introduce the following vectors:

$$\begin{aligned} W^N(t) &\coloneqq [w_1(t) \dots w_N(t)]^\top \\ A_u &\coloneqq [a_1 \dots a_N]^\top \\ B_1 &\coloneqq [b_1 \dots b_N]^\top \\ B_2 &\coloneqq [m_1 \dots m_N]^\top. \end{aligned}$$

$$A_0 \coloneqq \begin{pmatrix} -\lambda_1 + q_c & 0 & \cdots & 0 \\ 0 & -\lambda_2 + q_c & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\lambda_N + q_c \end{pmatrix}$$

and, we focus on the following finite-dimension truncation of (7) which will be later used in the stability analysis:

$$\begin{aligned} \dot{W}^N(t) &= A_0 W^N(t) + A_u u(t) + B_1 v(t) + B_2 d(t) \\ \dot{u}(t) &= v(t). \end{aligned} \tag{8}$$

Our objective is to design a finite-dimensional linear time invariant control system that renders the closed-loop system (7) exponentially stable in the $\mathcal{H}_1$-norm while achieving a quadratic performance specification on the controlled system. This controller serves to achieve IOS for the infinite-dimensional system (3) with respect to the disturbance $d$. The general framework of such a controller is given in the next section.

## IV. General Set-Up for the Output-Feedback Control Problem

In this section, we present the general framework of the control problem using output measurement feedback. This is presented in Fig. 1 which conveys the problem of a feedback interconnection of a plant $\mathcal{P}$ given by (6) and controller $\mathcal{K}$ with inputs $d$, $u$ and regulated output $y$.

We suppose that plant (7) is controlled by the following dynamic output feedback controller $\mathcal{K}$ given below:

$$\mathcal{K} : \begin{pmatrix} \dot{X}_c \\ v \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} X_c \\ y_m \end{pmatrix} \tag{9}$$

where $A_c \in \mathbb{R}^{(N+1)\times(N+1)}, B_c \in \mathbb{R}^{(N+1)}, C_c \in \mathbb{R}^{1\times(N+1)}$ and $D_c \in \mathbb{R}^2$ are to be designed, with $N$ being the dimension of $W^N$. Let $C \coloneqq [\phi_1(0) \dots \phi_N(0)]^\top$, and $\tilde{y} \coloneqq \sum_{i \geq N+1} \phi_i(0) w_i$. Then, the closed-loop system dynamics $(\mathcal{P}, \mathcal{K})$ can be formally written as follows:

$$\begin{cases} \begin{pmatrix} \dot{W}^N(t) \\ \dot{u}(t) \\ \dot{X}_c(t) \\ \dot{w}_n(t) \end{pmatrix} = \begin{pmatrix} A_0 + B_1 D_c C & A_u & B_1 C_c & 0 \\ D_c C & 0 & C_c & 0 \\ B_c C & 0 & A_c & 0 \\ b_n D_c C & a_n & b_n C_c & -\lambda_n + q_c \end{pmatrix} \\ \begin{pmatrix} W^N(t) \\ u(t) \\ X_c(t) \\ w_n(t) \end{pmatrix} + \begin{pmatrix} B_1 D_c \\ D_c \\ B_c \\ b_n D_c \end{pmatrix} \tilde{y}(t) + \begin{pmatrix} B_2 \\ 0 \\ 0 \\ m_n \end{pmatrix} d(t) \qquad n \geq N+1 \\ y(t) = C W^N(t) + \tilde{y} \end{cases} \tag{10}$$

From now on, we denote $\zeta \coloneqq \big(w, u, X_c\big)$ and for all $\zeta \in \mathcal{D}(\mathcal{A}) \times \mathbb{R} \times \mathbb{R}^{N+1}$:

$$\|\zeta\|_s \coloneqq \sqrt{\|w\|^2_{\mathcal{H}^1} + u^2 + X_c^\top X_c}. \tag{11}$$

We will refer to this norm as the $\mathcal{H}^1_s$-norm.

We are now able to formally state the problem we solve in this paper.

**Problem 1.** *Given $p \in \mathcal{C}^1([0,1])$, $q \in \mathcal{C}^0([0,1])$ with $p, q > 0$ and $q_c \in \mathbb{R}$. Design the control parameters $\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$ such that the following properties hold for the closed-loop system:* (10)*:*

- *the origin of the closed-loop system is zero-input globally exponentially stable;*
- *for some (solution independent) $\rho > 0$ for each classical solution pair $(\zeta, d)$ with $\zeta(0) = 0$ to the closed-loop system* (10) *the bound:*

$$|y(t)| \leq \rho \sqrt{\int_0^t d(\theta)^2 d\theta} \qquad \forall t \geq 0 \tag{12}$$

*holds for all $t \in \mathtt{dom} z$.*

Inequality (12) corresponds to an input-output stability (IOS) bound for the closed-loop system (3) with (9). The main contribution of this paper is to design an optimal controller $\mathcal{K}$ in order to minimize the effect of the gain $\rho$. In Section V, we provide an explicit estimate of the IOS gain $\rho$.

## V. Main results

In this section we provide sufficient for the solution to Problem 1. First, we propose sufficient conditions for $\mathcal{H}^1_s$ stability and then construct a Lyapunov functional to derive the sufficient conditions in the form of quadratic inequalities.

### A. Sufficient conditions

The following section presents sufficient conditions for the solution to Problem 1. The result relies on a dissipation inequality. This is done by proving the following proposition:

**Proposition 2.** *Assume there exist a Fréchet differentiable functional $V \colon \mathcal{H}^1(0,1) \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}_{\geq 0}$ and*

$c_1, c_2, c_3, \chi \in \mathbb{R}_{>0}$ *such that for each* $d \in \mathbb{R}$ *and* $\zeta \in \mathcal{D}(\mathcal{A}) \times \mathbb{R} \times \mathbb{R}^{N+1}$

$$c_1 \|\zeta\|_s^2 \leq V(\zeta) \leq c_2 \|\zeta\|_s^2, \tag{13}$$

$$DV(\zeta)\dot{\zeta} \leq -c_3 V(\zeta) + \chi^2 d^2. \tag{14}$$

*Then, the origin of the closed-loop system (10) is zero-input globally exponentially stable and* (12) *holds with*

$$\rho = \frac{\chi}{\sqrt{c_1}} \tag{15}$$

*Proof.* First we consider a classical solution pair $(\zeta(t), d(t))$, i.e, $\zeta \in \mathcal{H}^1(0,1) \times \mathbb{R} \times \mathbb{R}^{N+1}$ and $d \in \mathbb{R}$ for all $t \in \texttt{dom}\zeta$ where $\texttt{dom}\zeta$ is an interval of $\mathbb{R}_{\geq 0}$ including zero. Now, consider the following function:

$$\begin{aligned} \mathcal{W} : \texttt{dom}\zeta &\to \mathbb{R} \\ t &\mapsto (V \circ \zeta)(t) \end{aligned} \tag{16}$$

Then, since $V : \mathcal{H}^1(0,1) \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}_{\geq 0}$ is Fréchet differentiable everywhere and $w : \texttt{dom}\zeta \to \mathcal{H}^1(0,1) \times \mathbb{R} \times \mathbb{R}^{N+1}$ is differentiable everywhere, it follows that for all $t \geq 0$:

$$\dot{\mathcal{W}}(t) = DV(\zeta)\dot{\zeta}(t).$$

Thus we have for all $t \in \texttt{dom}\zeta$

$$\dot{\mathcal{W}}(t) = DV(\zeta) \begin{pmatrix} -\mathbb{A}w(t,\cdot) + q_c w(t,\cdot) + au(t) \\ +bv(t) + md(t) \\ v(t) \\ A_c X_c(t) + B_c y(t) \end{pmatrix}$$

Using (14), one gets, for all $t \in \texttt{dom}\zeta$,

$$\dot{\mathcal{W}}(t) \leq -c_3 \mathcal{W}(t) + \chi^2 d(t)^2.$$

Therefore, since $\mathcal{W}$ is continuous on $\texttt{dom}\zeta$, from comparison lemma, we have:

$$\mathcal{W}(t) \leq e^{-c_3 t}\mathcal{W}(0) + \chi^2 \int_0^t e^{-c_3(t-\theta)} d(\theta)^2 d\theta, \quad \forall t \in \texttt{dom}\,\zeta.$$

This bound, thanks to (13), ensures that the origin of the closed-loop system is globally exponentially stable with respect to the norm defined in (11) when $d = 0$. At this stage, notice that for all $t \in \texttt{dom}\zeta$, one has:

$$\int_0^t e^{-c_3(t-\theta)} d(\theta)^2 d\theta \leq \int_0^t d(\theta)^2 d\theta$$

which allows one to conclude that for all $t \in \texttt{dom}\zeta$

$$\mathcal{W}(t) \leq e^{-c_3 t}\mathcal{W}(0) + \chi^2 \int_0^t d(\theta)^2 d\theta.$$

Finally by using (13), it follows that for all $t \in \texttt{dom}\zeta$

$$\|\zeta(t)\|_s \leq e^{-\frac{c_3}{2}t} \sqrt{\frac{c_2}{c_1}} \|\zeta(0)\|_s + \frac{\chi}{\sqrt{c_1}} \sqrt{\int_0^t d(\theta)^2 d\theta}. \tag{17}$$

Assume that $\zeta(0) = 0$, then we have

$$\|\zeta(t)\|_s^2 \leq \frac{\chi^2}{c_1} \int_0^t d(t)^2 dt. \tag{18}$$

Since $y(t) = w(t,0)$, and we know that since $w(t,1) = 0$, we have

$$y(t)^2 = \left( \int_0^1 w_x(t,s) ds \right)^2 \leq \int_0^1 w_x^2(t,s) ds \leq \|w(t)\|_{\mathcal{H}^1}^2 \tag{19}$$

Then, since

$$\|w(t)\|_{\mathcal{H}^1}^2 \leq \|\zeta\|_s^2$$

using (19) and (18), we get

$$y(t)^2 \leq \frac{\chi^2}{c_1} \int_0^t d(t)^2 dt. \tag{20}$$

This concludes the proof. □

**Remark 3.** *Inequality (17) in the proof corresponds to a classical input-to-state stability (ISS) bound for the closed-loop system (10) with respect to the disturbance d. The ISS asymptotic gain is equal to the IOS gain* $\rho = \frac{\chi}{\sqrt{c_1}}$ *given in (15) to-be-minimized [13].*

### *B. Construction of the functional V*

Proposition 2 provides sufficient conditions for input-output stability (IOS) for the closed-loop system in the form of functional inequality. In this section we provide a specific structure for the functional $V$ in Proposition 2, which allows one to cast the solution of Problem 1 in the solution to some matrix inequalities.

We define the following matrices which will be necessary for the proof of the next theorem.

$$\begin{aligned} A_1 &\coloneqq \begin{pmatrix} A_0 + B_1 D_c C & A_u & B_1 C_c \\ D_c C & 0 & C_c \\ B_c C & 0 & A_c \end{pmatrix}, \\ B_{11} &\coloneqq \begin{pmatrix} B_1 D_c \\ D_c \\ B_c \end{pmatrix}, \qquad B_{12} \coloneqq \begin{pmatrix} B_2 \\ 0 \\ 0 \end{pmatrix}, \\ A_2 &\coloneqq \begin{pmatrix} 4D_c^2 C^\top C & 0 & 0 \\ * & \alpha \|a\|_{\mathcal{L}^2}^2 & 0 \\ * & * & 2\alpha \|b\|_{\mathcal{L}^2}^2 C_c^\top C_c \end{pmatrix}. \end{aligned} \tag{21}$$

We also define the constant $M_\phi = \sum_{n \geq N+1} \frac{\phi_n(0)^2}{\lambda_n}$ which is finite when $p \in \mathcal{C}^2([0,1])$ since $\phi_n(0) = O(1)$ as $n \to +\infty$ (see [10]) and (1) holds.

**Theorem 4.** *Assume that* $p \in \mathcal{C}^2([0,1])$. *Suppose there exist* $P \in \mathbb{S}_p^{2N+2}$, $A_c \in \mathbb{R}^{N \times N}, B_c \in \mathbb{R}^N, C_c \in \mathbb{R}^{1 \times N}, D_c \in \mathbb{R}$ *and* $\alpha, \beta, \chi, \gamma \in \mathbb{R}_{>0}$ *such that:*

$$\Theta = \begin{pmatrix} A_3 & PB_{12} & PB_{11} \\ * & \alpha \|m\|_{\mathcal{L}^2}^2 - \chi^2 & 0 \\ * & * & 2\alpha \|b\|_{\mathcal{L}^2}^2 D_c^2 - \beta \end{pmatrix} \leq 0 \tag{22}$$

*where* $A_3 \coloneqq A_1^\top P + P A_1 + A_2 + 2\eta P$.

*Select* $N$ *sufficiently large such that for all* $n \geq N+1$

$$\Gamma_n \coloneqq \lambda_n \left( -\lambda_n + q_c + \eta + \frac{3}{\alpha} + \frac{\beta}{2\gamma} M_\phi \right) \leq 0. \tag{23}$$

*Then, the parameters $A_c, B_c, C_c, D_c$ solve Problem 1. In particular (12) holds with:*

$$\rho = \frac{\chi}{\sqrt{\min\{\lambda_{\min}(P), \gamma p_*, \gamma q_*\}}} \quad . \tag{24}$$

*Proof.* Let $P \in \mathbb{S}_p^{2N+2}, \gamma > 0$, and $X_1 \coloneqq (W^N, u, X_c)$. Consider the following Fréchet differentiable Lyapunov functional:

$$\begin{aligned} V \colon & \mathcal{D}(\mathbb{A}) \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R} \\ & \begin{pmatrix} w \\ u \\ X_c \end{pmatrix} \mapsto X_1^\top P X_1 + \gamma \sum_{n \geq N+1} \lambda_n \langle w, \phi_n \rangle^2. \end{aligned} \tag{25}$$

The first term corresponds to the dynamics of the truncated model (8) and the control model (9) while the second term, which is related to the $\mathcal{H}^1$-norm of the PDE trajectories as depicted in 24, is used to handle the modes $w_n$ for $n \geq N+1$. We have $c_1 \coloneqq \min\{\lambda_{\min}(P), \gamma p_*, \gamma q_*\}$ and $c_2 \coloneqq \max\{\lambda_{\max}(P), \gamma p^*, \gamma q^*\}$ are strictly positive. Consider the first term of the functional $V$:

$$V_1(X_1) \coloneqq {X_1}^\top P X_1 \tag{26}$$

and the closed-loop system (10). Then,

$$\begin{aligned} DV_1(X_1)\dot{X}_1 = & X_1^\top (A_1^\top P + P A_1) X_1 \\ & + \tilde{y}^\top B_{11}^\top P X_1 + d^\top B_{21}^\top P X_1 \\ & + X_1^\top P B_{11} \tilde{y} + X_1^\top P B_{12} d. \end{aligned} \tag{27}$$

We obtain the computation of the time derivative of $V_1$ along the solution pair to (7) and (8):

$$DV_1(X_1)\dot{X}_1 = \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix}^\top \Omega_1 \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix} \tag{28}$$

where the matrix $\Omega_1$ is given by:

$$\Omega_1 = \begin{pmatrix} A_1^\top P + P A_1 & P B_{12} & P B_{11} \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}.$$

Now, consider $V_2(w) \coloneqq \gamma \sum_{n \geq N+1} \lambda_n \langle w, \phi_n \rangle$ with $\gamma > 0$. The time derivative of $V_2$ along the solution pair to (7), (8) yields:

$$\begin{aligned} DV_2(w)\dot{w} = 2\gamma \sum_{n \geq N+1} & \lambda_n ((-\lambda_n + q_c) w_n^2 + a_n u w_n \\ & + b_n v w_n + m_n d w_n). \end{aligned} \tag{29}$$

Thus, knowing that $V = V_1 + V_2$, we conclude the following expression:

$$\begin{aligned} & DV(X_1, w) \begin{pmatrix} \dot{X}_1 \\ \dot{w} \end{pmatrix} + 2\eta V = \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix}^\top \Omega_2 \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix} \\ & + 2\gamma \sum_{n \geq N+1} \lambda_n [(-\lambda_n + q_c + \eta) w_n^2 + a_n u w_n \\ & + b_n v w_n + m_n d w_n] \end{aligned} \tag{30}$$

and $\Omega_2 = \Omega_1 + \begin{pmatrix} 2\eta P & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}$. Using Young's inequality, we have:

$$2 \sum_{n \geq N+1} \lambda_n a_n w_n u \leq \frac{1}{\alpha} \sum_{n \geq N+1} \lambda_n^2 w_n^2 + \alpha ||a||_{\mathcal{L}^2}^2 u^2$$

$$2 \sum_{n \geq N+1} \lambda_n b_n w_n v \leq \frac{1}{\alpha} \sum_{n \geq N+1} \lambda_n^2 w_n^2 + \alpha ||b||_{\mathcal{L}^2}^2 v^2$$

$$2 \sum_{n \geq N+1} \lambda_n m_n w_n d \leq \frac{1}{\alpha} \sum_{n \geq N+1} \lambda_n^2 w_n^2 + \alpha ||m||_{\mathcal{L}^2}^2 d^2$$

for any $\alpha > 0$.Recall that $v = C_c X_c + D_c(C W^N + \tilde{y})$. We have the following inequality:

$$\begin{aligned} v^2 & \leq 2(X_c C_c^\top C_c X_c) + 2 D_c^2 (C W^N + \tilde{y})^2 \\ & \leq 2(X_c C_c^\top C_c X_c) + 4 D_c^2 ({W^N}^\top C^\top C W^N) \\ & + 4 D_c^2 \tilde{y}^2 \end{aligned} \tag{31}$$

Hence, using (31), (30) is bounded by:

$$\begin{aligned} & DV(X_1, w) \begin{pmatrix} \dot{X}_1 \\ \dot{w} \end{pmatrix} + 2\eta V \leq \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix}^\top \Omega_3 \begin{pmatrix} X_1 \\ d \\ \tilde{y} \end{pmatrix} \\ & + 2\gamma \sum_{n \geq N+1} \lambda_n (-\lambda_n + q_c + \eta + \frac{3}{\alpha}) w_n^2 \end{aligned} \tag{32}$$

and $\Omega_3$ is given by:

$$\Omega_3 = \begin{pmatrix} A_1^\top P + P A_1 + A_2 & P B_{12} & P B_{11} \\ * & \alpha \|m\|_{\mathcal{L}^2}^2 & 0 \\ * & * & 4\alpha \|b\|_{\mathcal{L}^2}^2 D_c^2 \end{pmatrix}$$

where

$$A_2 = \begin{pmatrix} 4 D_c^2 C^\top C & 0 & 0 \\ * & \alpha \|a\|_{\mathcal{L}^2}^2 & 0 \\ * & * & 2\alpha \|b\|_{\mathcal{L}^2}^2 C_c^\top C_c \end{pmatrix}.$$

Knowing that

$$\tilde{y}^2 = \left( \sum_{i \geq N+1} \phi_i(0) w_i \right)^2$$

and $M_\phi = \sum_{i \geq N+1} \frac{\phi_i(0)^2}{\lambda_i}$ is finite and $\sum_{i \geq N+1} \lambda_i w_i^2$ is finite due to (2), we use Cauchy-Schwarz inequality to obtain the following

$$\tilde{y}^2 \leq \sum_{i \geq N+1} \frac{\phi_i(0)^2}{\lambda_i} \sum_{i \geq N+1} \lambda_i w_i^2$$

Hence, for any $\beta > 0$,

$$\beta M_\phi \sum_{i \geq N+1} \lambda_i w_i^2 - \beta \tilde{y}^2 \geq 0 \tag{33}$$

Combining (32) and (33), we obtain

$$DV(X_1,w)\begin{pmatrix}\dot{X}_1\\ \dot{w}\end{pmatrix}+2\eta V\leq\begin{pmatrix}X_1\\ d\\ \tilde{y}\end{pmatrix}^{\top}\Omega_4\begin{pmatrix}X_1\\ d\\ \tilde{y}\end{pmatrix}+2\gamma\sum_{n\geq N+1}\lambda_n\left(-\lambda_n+q_c+\eta+\frac{3}{\alpha}+\frac{\beta}{2\gamma}M_\phi\right)w_n^2 \tag{34}$$

where $\Omega_4$ is given by:

$$\Omega_4=\begin{pmatrix}A_1^{\top}P+PA_1+A_2 & PB_{12} & PB_{11}\\ * & \alpha\|m\|_{\mathcal{L}^2}^2 & 0\\ * & * & 2\alpha\|b\|_{\mathcal{L}^2}^2D_c^2-\beta\end{pmatrix}$$

and $A_2$ is given in (21).

The latter implies that, for all $d\in\mathbb{R}$, $(w,u,X_c)\in\mathcal{D}(\mathcal{A})\times\mathbb{R}\times\mathbb{R}^{N+1}$, we have

$$DV(X_1,w)\begin{pmatrix}\dot{X}_1\\ \dot{w}\end{pmatrix}-\chi^2d^{\top}d\leq-2\eta V(X_1,w)+\begin{pmatrix}X_1\\ d\\ \tilde{y}\end{pmatrix}^{\top}\Theta\begin{pmatrix}X_1\\ d\\ \tilde{y}\end{pmatrix}+2\gamma\sum_{n\geq N+1}\Gamma_nw_n^2 \tag{35}$$

where $\Theta=\Omega_4-\begin{pmatrix}0&0&0\\ *&\chi^2I&0\\ *&*&0\end{pmatrix}$. At this stage, notice that (22), (23) state that $\Theta\leq0$ and $\Gamma_n\leq0$ for sufficiently large $N$; therefore, (35) implies that

$$DV(X_1,w)\begin{pmatrix}\dot{X}_1\\ \dot{w}\end{pmatrix}\leq-2\eta V(X_1,w)+\chi^2d^{\top}d$$

The previous equation reads as (14) and the proof is complete. □

Notice that the sufficient condition (22) is a nonlinear matrix inequality in the decision variables $P, A_c, B_c, C_c$, and $D_c$. Therefore (22) is hard to exploit from a numerical standpoint for the design of the controller. To overcome this drawback, next we provide sufficient conditions in the form of linear matrix inequalities. To achieve this, we adapt the approach introduced in [12] for the design of dynamical output feedback controllers. This is the objective of the upcoming section.

## VI. Control Synthesis

To simply our approach, we enforce $D_c=0$. This is not restrictive and leads to strictly proper controllers that can be more appealing in practice.

Let $X,Y\in\mathbb{S}_p^{N+1}$ and $U,V\in\mathbb{R}^{(N+1)\times(N+1)}$ be nonsingular matrices such that $YX+VU^{\top}=I$. Let $\mathcal{Y}=\begin{pmatrix}Y&I\\ V^{\top}&0\end{pmatrix}$ and $P=\begin{pmatrix}X&U\\ U^{\top}&\bullet\end{pmatrix}$ where "$\bullet$" denotes "don't care" symmetric positive definite matrix. Under the considered assumptions $\mathcal{Y}$ is nonsingular and a simple congruence transformation shows that

$$\Theta\leq0\iff\Sigma\leq0$$

with

$$\Sigma\coloneqq\begin{pmatrix}\mathcal{Y}^{\top}A_3\mathcal{Y} & \mathcal{Y}^{\top}PB_{12} & \mathcal{Y}^{\top}PB_{11}\\ & \alpha\|m\|_{\mathcal{L}^2}^2-\chi^2 & 0\\ * & * & -\beta\end{pmatrix}$$

We define the following matrices:

$$\hat{A}\coloneqq\begin{pmatrix}A_0&A_u\\0&0\end{pmatrix},\hat{B}_1\coloneqq\begin{pmatrix}B_1\\1\end{pmatrix},\quad\hat{B}_2\coloneqq\begin{pmatrix}B_2\\0\end{pmatrix},\qquad\hat{C}\coloneqq(C,0).$$

At this stage, as in [12], we consider the following invertible change of variables:

$$\begin{pmatrix}K&L\\M&0\end{pmatrix}=\begin{pmatrix}X\hat{A}Y&0\\0&0\end{pmatrix}+\begin{pmatrix}U&X\hat{B}_1\\0&I\end{pmatrix}\begin{pmatrix}A_c&B_c\\C_c&0\end{pmatrix}\begin{pmatrix}V^{\top}&0\\\hat{C}Y&I\end{pmatrix} \tag{36}$$

This transforms the old variables $(P,A_c,B_c,C_c)$ into the new variables $\nu=(X,Y,K,L,M)$. Using the change of variable (36), applying the Schur compliment lemma [18] and by following [12], one can show that $\Sigma\leq0$ is equivalent to the following linear matrix inequalities:

$$X(\nu)>0 \tag{37}$$

and

$$\begin{pmatrix}A(\nu)^{\top}+A(\nu) & B_1(\nu) & B_2(\nu)^{\top} & C(\nu)\\ * & \alpha\|m\|_{\mathcal{L}^2}^2-\chi^2 & 0 & 0\\ * & * & -\beta & 0\\ * & * & * & -S^{-1}\end{pmatrix}\leq0 \tag{38}$$

where

$$X(\nu)=\begin{pmatrix}Y&I\\I&X\end{pmatrix},$$

$$A(\nu)=\begin{pmatrix}\hat{A}Y+\hat{B}_1M & \hat{A}\\ K & X\hat{A}+L\hat{C}\end{pmatrix},$$

$$B_1(\nu)=\begin{pmatrix}0\\L\end{pmatrix},\qquad B_2(\nu)=\begin{pmatrix}\hat{B}_2\\X\hat{B}_2\end{pmatrix},$$

$$C(\nu)=\begin{pmatrix}\mathcal{Y}\begin{pmatrix}0\\I\end{pmatrix} & VC_c^{\top}\\ \begin{pmatrix}0\\I\end{pmatrix} & 0\end{pmatrix},\quad S=\begin{pmatrix}\alpha\|a\|_{\mathcal{L}^2}^2 & 0\\ * & 2\alpha\|b\|_{\mathcal{L}^2}^2\end{pmatrix}.$$

If conditions (37), (38) are feasible for certain $K,L,M$, one can use equation (36) to deduce the values of $A_c, B_c, C_c$ that satisfy condition (22).

In the formulation of Problem 1, no specific requirements on the scalar $\rho$ are considered. On the other hand, it is obvious that to minimize the effect of the disturbance $d$ on the closed-loop system, the control parameters should be designed so that (12) holds with a minimal $\rho$. This goal can be achieved by choosing $\gamma=\beta$, $\chi=3$ and considering the following optimization problem

$$\begin{aligned}&\sup_{P,c}\quad c\\ &\text{s.t: (37), (38) hold } P\in\mathbb{S}_p^{2N+2}, P-cI>0.\end{aligned} \tag{39}$$

The optimization problem (39) is equivalent to finding a maximal $c$ such that

$$\mathcal{Y}^\top (P - cI)\mathcal{Y} > 0$$

We apply the Schur compliment lemma [18] on the latter nonlinear inequality to obtain the equivalent condition:

$$\begin{pmatrix} \frac{1}{c}I & Y & V \\ * & Y & I \\ * & * & X \end{pmatrix} > 0 \tag{40}$$

which is linear in $\mu = \frac{1}{c}$. Problem (39) can now be seen as designing minimal $\mu$ such that (40) holds.

### A. Control Design Algorithm

We present the following algorithm to design the dynamic output feedback controller given in (9).

**Input:** Specify the values of the system parameters $p(x)$, $q(x)$ and $q_c$.

**Step 1:** Calculate the value of $N_0$, the minimum value of $n$ for which $-\lambda_n + q_c < 0$.

**Step 2:** Calculate the value of $N \geq N_0$, the minimum value of $n$ for which (23) holds. This $N$ exists due to the fact that $\lambda_n$ is positive definite and increasing. This will be the dimension of the dynamic controller.

**Step3:** Use a numerical solver to find a feasible solution $X, Y, K, L, M, N, c$ under which (37), (38), (39) hold.

**Step4:** Derive the control parameters $A_c, B_c, C_c, D_c$ using (36).

**Output:** The designed control parameters are given by $A_c, B_c, C_c, D_c$ and $\rho$ given by (24) is computed.

In the next section, we use a numerical example to showcase our method.

## VII. Numerical Analysis

In this section, we use the YALMIP package in MATLAB to solve linear matrix inequalities. We illustrate the result of this paper using a modal approximation that captures the 50 dominant modes of the reaction-diffusion plant with an in-domain disturbance given by:

$$d(t) = 0.1\sin(2t)$$

We set $p = 1$, $q = 1$, $m = 1$, and $q_c = 4$ for which the open-loop plant is unstable. We select $\delta = 0$, and we obtain that $N_0 = 1$ and $N = 2$, which is the minimum $n$ for which (23) holds. We choose the dimension of $W^N$ to be $N = 3$. The following control matrices renders (37), (38) and (39) feasible.

$$\begin{aligned} A_c &= \begin{pmatrix} -15.24 & 11.9 & -6.47 & 14.62 \\ -4.84 & -24.87 & -25.51 & -2.25 \\ 145.35 & -73.53 & -126.97 & -49.13 \\ -36.28 & -3.48 & -5.86 & -14.43 \end{pmatrix} \\ B_c &= \begin{pmatrix} 5.46 \\ 0.32 \\ -29.98 \\ 0.66 \end{pmatrix} \\ C_c &= \begin{pmatrix} 6.91 & -0.45 & -0.57 & 8.8 \end{pmatrix} \qquad D_c = 0 \end{aligned} \tag{41}$$

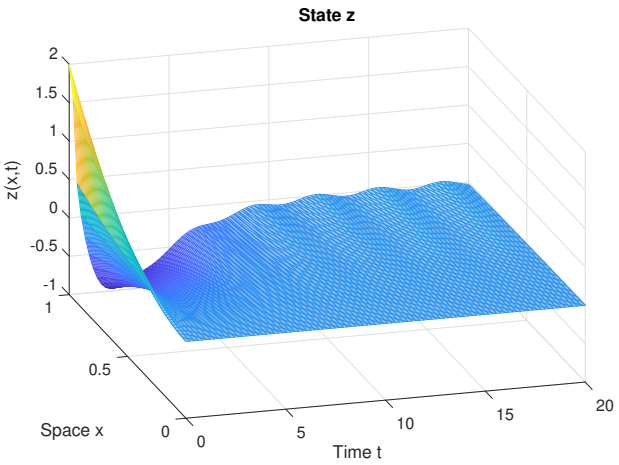



Fig. 2. Time and space-evolution of the state $z(t, x)$

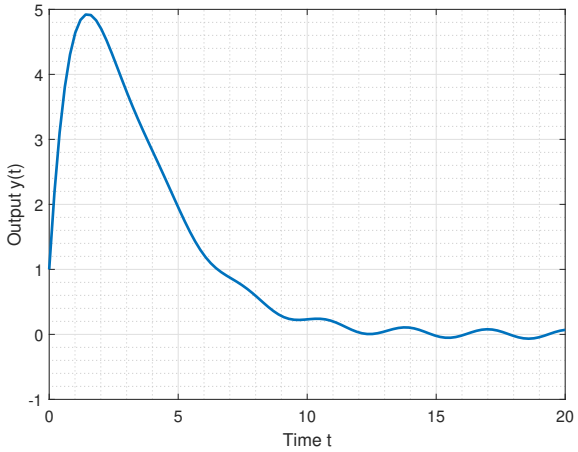



Fig. 3. Time and-evolution of the output $y(t)$

We solve inequality (22) for the designed control parameters $\{A_c, B_c, C_c, D_c\}$ and notice that we do in fact have a feasible solution. The initial condition of the reaction-diffusion system described by (3) is $z_0(x) = 1 + x^2$. We simulate the closed-loop system (10) for the first 50 modes. Thus, we can deduce the time-evolution of the state $z(x, t)$ in closed-loop system and it is depicted in Fig. 2 which showcases a convergence to a neighborhood of the origin as predicted in Theorem 4 and one can observe the effect of the disturbance at steady state. Finally, Fig. 3 demonstrates the decay in the output $y(t)$ verifying the dissipation inequality in Proposition 2 with $\rho = 1.8$.

## VIII. Conclusion

The design of a finite-dimensional linear time-invariant (LTI) control system is proposed in order to achieve the input-output stability (IOS) of a reaction-diffusion equation with a Dirichlet regulated output and in-domain disturbance. Sufficient conditions in the form of matrix inequalities are derived to solve the control design problem. Control Synthesis method proposed by [12] is applied to the nonlinear

matrix inequality to derive a suitable LMI. A control design algorithm is proposed to provide a solution to the sufficient conditions. Finally, a numerical example is presented to showcase the effectiveness and validity of our method. For future work, we propose to extend the study to robust stability analysis using the sub-optimal small-gain theorem.

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