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# A Lower Bound on the Relative Entropy with Respect to a Symmetric Probability

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## Abstract

Let  $\rho$  and  $\mu$  be two probability measures on  $\mathbb{R}$  which are not the Dirac mass at 0. We denote by  $H(\mu|\rho)$  the relative entropy of  $\mu$  with respect to  $\rho$ . We prove that, if  $\rho$  is symmetric and  $\mu$  has a finite first moment, then

$$H(\mu|\rho) \geq \frac{\left(\int_{\mathbb{R}} z d\mu(z)\right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)},$$

with equality if and only if  $\mu = \rho$ . We give an application to the Curie-Weiss model of self-organized criticality.

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# 1 Introduction

Given two probability measures  $\mu$  and  $\rho$  on  $\mathbb{R}$ , the relative entropy of  $\mu$  with respect to  $\rho$  (or the Kullback-Leibler divergence of  $\rho$  from  $\mu$ ) is

$$H(\mu|\rho) = \begin{cases} \int_{\mathbb{R}} f(z) \ln f(z) d\rho(z) & \text{if } \mu \ll \rho \text{ and } f = \frac{d\mu}{d\rho} \\ +\infty & \text{otherwise,} \end{cases}$$

where  $d\mu/d\rho$  denotes the Radon-Nikodym derivative of  $\mu$  with respect to  $\rho$  when it exists. In this paper, we prove the following theorem:

**Theorem 1.** *Let  $\rho$  and  $\mu$  be two probability measures on  $\mathbb{R}$  which are not the Dirac mass at 0. If  $\rho$  is symmetric and if  $\mu$  has a finite first moment, then*

$$H(\mu|\rho) \geq \frac{\left(\int_{\mathbb{R}} z d\mu(z)\right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)},$$

with equality if and only if  $\mu = \rho$ .

A remarkable feature of this inequality is that the lower bound does not depend on the symmetric probability measure  $\rho$ . We found the following related inequality in the literature (see lemma 3.10 of [1]): if  $\rho$  is a probability measure on  $\mathbb{R}$  whose first moment  $m$  exists and such that

$$\exists v > 0 \quad \forall \lambda \in \mathbb{R} \quad \int_{\mathbb{R}} \exp(\lambda(z - m)) d\rho(z) \leq \exp\left(\frac{v\lambda^2}{2}\right),$$

then, for any probability measure  $\mu$  on  $\mathbb{R}$  having a first moment, we have

$$H(\mu|\rho) \geq \frac{1}{2v} \left(\int_{\mathbb{R}} z d\mu(z) - m\right)^2.$$

Our inequality does not require an integrability condition. Instead we assume that  $\rho$  is symmetric.

The proof of the theorem is given in the following section. It consists in relating the relative entropy  $H(\cdot|\rho)$  and the Cramér transform  $I$  of  $(Z, Z^2)$  when  $Z$  is a random variable with distribution  $\rho$ . We then use an inequality on  $I$  which we proved initially in [2]. We give here a simplified proof of this inequality.

In section 3, we apply the inequality of theorem 1 to the Curie-Weiss model of self-organized criticality we designed in [2]. We prove that, if  $(X_n^1, \dots, X_n^n)$  has the distribution

$$d\tilde{\mu}_{n,\rho}(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

for any  $n \geq 1$ , and if  $\rho$  is symmetric with compact support and such that  $\rho(\{0\}) < 1/\sqrt{e}$ , then, for any continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \tilde{\mu}_{n,\rho} \left( \left| \frac{1}{n} \sum_{k=1}^n f(X_n^k) - \int_{\mathbb{R}} f(z) d\rho(z) \right| \geq \varepsilon \right) = 0.$$

## 2 Proof of the theorem

Let  $\rho$  and  $\mu$  be two probability measures on  $\mathbb{R}$  which are not the Dirac mass at 0. We first recall that  $H(\mu|\rho) \geq 0$ , with equality if and only if  $\mu = \rho$ .

We assume that  $\rho$  is symmetric and that  $\mu$  has a finite first moment. We denote

$$\mathcal{F}(\mu) = \frac{\left( \int_{\mathbb{R}} z d\mu(z) \right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)}.$$

If  $\mu = \rho$  then  $\mathcal{F}(\mu) = 0 = H(\mu|\rho)$ . From now onwards we suppose that  $\mu \neq \rho$ . If the first moment of  $\mu$  vanishes or if its second moment is infinite, then we have  $\mathcal{F}(\mu) = 0 < H(\mu|\rho)$ . Finally, if  $\mu$  is such that  $H(\mu|\rho) = +\infty$ , then Jensen's inequality implies that

$$\mathcal{F}(\mu) \leq 1/2 < H(\mu|\rho).$$

In the following, we suppose that

$$\int_{\mathbb{R}} z d\mu(z) \neq 0, \quad \int_{\mathbb{R}} z^2 d\mu(z) < +\infty,$$

and that  $H(\mu|\rho) < +\infty$ . This implies that  $\mu \ll \rho$  and we set  $f = d\mu/d\rho$ . It follows from Jensen's inequality that, for any  $\mu$ -integrable function  $\Phi$ ,

$$\int_{\mathbb{R}} \Phi d\mu - H(\mu|\rho) = \int_{\mathbb{R}} \ln \left( \frac{e^\Phi}{f} \right) d\mu \leq \ln \int_{\mathbb{R}} \frac{e^\Phi}{f} d\mu = \ln \int_{\mathbb{R}} e^\Phi d\rho.$$

As a consequence

$$\sup_{\Phi \in L^1(\mu)} \left\{ \int_{\mathbb{R}} \Phi d\mu - \ln \int_{\mathbb{R}} e^\Phi d\rho \right\} \leq H(\mu|\rho).$$

In order to make appear the first and second moments of  $\rho$ , we consider functions  $\Phi$  of the form  $z \mapsto uz + vz^2$ ,  $(u, v) \in \mathbb{R}^2$ . This way we obtain

$$I \left( \int_{\mathbb{R}} z d\mu(z), \int_{\mathbb{R}} z^2 d\mu(z) \right) \leq H(\mu|\rho),$$

where

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) = \sup_{(u, v) \in \mathbb{R}^2} \left\{ ux + vy - \ln \int_{\mathbb{R}} e^{uz + vz^2} d\rho(z) \right\}.$$

The function  $I$  is the Cramér transform of  $(Z, Z^2)$  when  $Z$  is a random variable with distribution  $\rho$ . In our paper dealing with a Curie-Weiss model of self-organized criticality [2], we proved with the help of the following inequality that, under some integrability condition, the function  $(x, y) \mapsto I(x, y) - x^2/(2y)$  has a unique global minimum on  $\mathbb{R} \times ]0, +\infty[$  at  $(0, \int x^2 d\rho(x))$ .

**Proposition 2.** *If  $\rho$  is a symmetric probability measure which is not the Dirac mass at 0, then*

$$\forall x \neq 0 \quad \forall y \neq 0 \quad I(x, y) > \frac{x^2}{2y}.$$

We present here a proof of this proposition which is simpler than in [2].

**Proof.** Let  $x \neq 0$  and  $y \neq 0$ . By definition of  $I(x, y)$ , we have

$$\begin{aligned} I(x, y) &\geq x \times \frac{x}{y} + y \times \left( -\frac{x^2}{2y^2} \right) - \ln \int_{\mathbb{R}} \exp \left( \frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) \\ &= \frac{x^2}{2y} - \ln \int_{\mathbb{R}} \exp \left( \frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z). \end{aligned}$$

Let  $(s, t) \in \mathbb{R}^2$ . By using the symmetry of  $\rho$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}} \exp(sz - tz^2) d\rho(z) &= \int_{\mathbb{R}} \exp(-sz - tz^2) d\rho(z) \\ &= \frac{1}{2} \left( \int_{\mathbb{R}} \exp(sz - tz^2) d\rho(z) + \int_{\mathbb{R}} \exp(-sz - tz^2) d\rho(z) \right) \\ &= \int_{\mathbb{R}} \cosh(sz) \exp(-tz^2) d\rho(z). \end{aligned}$$

We choose now  $t = s^2/2$ . We have the inequality

$$\forall u \in \mathbb{R} \setminus \{0\} \quad \cosh(u) \exp(-u^2/2) < 1.$$

Since  $\rho$  is not the Dirac mass at 0, the above inequality implies that

$$\forall s \neq 0 \quad \int_{\mathbb{R}} \cosh(sz) \exp\left(-\frac{s^2 z^2}{2}\right) d\rho(z) < 1.$$

We finally choose  $s = x/y$  and we get

$$\int_{\mathbb{R}} \exp \left( \frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) < 1.$$

As a consequence

$$I(x, y) \geq \frac{x^2}{2y} - \ln \int_{\mathbb{R}} \exp \left( \frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) > \frac{x^2}{2y},$$

which is the desired inequality. □

By applying the above proposition with

$$x = \int_{\mathbb{R}} z d\mu(z) \neq 0, \quad y = \int_{\mathbb{R}} z^2 d\mu(z) \in ]0, +\infty[,$$

we obtain

$$H(\mu|\rho) \geq I \left( \int_{\mathbb{R}} z d\mu(z), \int_{\mathbb{R}} z^2 d\mu(z) \right) > \mathcal{F}(\mu).$$

This ends the proof of theorem 1.

### 3 Application to the Curie-Weiss model of SOC

In [2], we designed the following model: Let  $\rho$  be a probability measure on  $\mathbb{R}$ , which is not the Dirac mass at 0. We consider an infinite triangular array of real-valued random variables  $(X_n^k)_{1 \leq k \leq n}$  such that for all  $n \geq 1$ ,  $(X_n^1, \dots, X_n^n)$  has the distribution  $\tilde{\mu}_{n,\rho}$ , where

$$d\tilde{\mu}_{n,\rho}(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

and  $Z_n$  is the renormalization constant. In [2] and [4], we proved that this model exhibits self-organized criticality: for a large class of symmetric distributions, we proved the fluctuations of  $S_n = X_n^1 + \dots + X_n^n$  are of order  $n^{3/4}$  and the limiting law is  $C \exp(-\lambda x^4) dx$  for some  $C, \lambda > 0$ .

For any  $n \geq 1$ , let us introduce the empirical measure

$$M_n = \frac{1}{n} (\delta_{X_n^1} + \dots + \delta_{X_n^n}).$$

The inequality of theorem 1 is the key ingredient to prove the following theorem:

**Theorem 3.** *Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$  with compact support and such that  $\rho(\{0\}) < 1/\sqrt{e}$ . Then, under  $\tilde{\mu}_{n,\rho}$ , the sequence  $(M_n)_{n \geq 1}$  converges weakly in probability to  $\rho$ , i.e., for any continuous function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$ , we have*

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} \tilde{\mu}_{n,\rho} \left( \left| M_n(f) - \int_{\mathbb{R}} f d\rho \right| \geq \varepsilon \right) = 0.$$

Let us prove this theorem. We suppose that there exists  $L > 0$  such that the support of  $\rho$  is  $[-L, L]$  or  $] -L, L[$ . We denote by  $\mathcal{M}_1^L$  the space of all probability measures on  $[-L, L]$  endowed with the topology of weak convergence. Let  $\varepsilon > 0$  and let  $f$  be a continuous function from  $\mathbb{R}$  to  $\mathbb{R}$ . The set

$$\mathcal{U}_\varepsilon = \left\{ \mu \in \mathcal{M}_1^L : \left| \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\rho \right| < \varepsilon \right\},$$

is open in  $\mathcal{M}_1^L$ . Let  $n \geq 1$ . We denote by  $\tilde{\theta}_{n,\rho}$  the law of  $(\delta_{Y_1} + \dots + \delta_{Y_n})/n$  when  $Y_1, \dots, Y_n$  are  $n$  independent random variables with distribution  $\rho$ . We have

$$\tilde{\mu}_{n,\rho}(M_n \in \mathcal{U}_\varepsilon^c) = \frac{1}{Z_n} \int_{\mathcal{U}_\varepsilon^c} \exp(n\mathcal{F}(\mu)) \mathbb{1}_{\mu \neq \delta_0} d\tilde{\theta}_{n,\rho}(\mu).$$

The function  $\mathcal{F}$  is continuous on  $\mathcal{M}_1^L \setminus \{\delta_0\}$ . Next, since  $\mathcal{F}(\delta_{1/k}) = 1/2$  for any  $k \geq 1$ , we notice that putting  $\mathcal{F}(\delta_0) \geq 1/2$  is necessary to ensure that  $\mathcal{F}$  is upper semi-continuous. As a consequence we extend the definition of  $\mathcal{F}$  on  $\mathcal{M}_1^L$  by putting  $\mathcal{F}(\delta_0) = 1/2$ . We suppose that  $\rho(\{0\}) < 1/\sqrt{e}$  so that

$$\mathcal{F}(\delta_0) = 1/2 < -\ln \rho(\{0\}) = H(\delta_0|\rho).$$

If  $\mu \in \mathcal{M}_1^L \setminus \{\delta_0\}$  then theorem 1 implies that  $\mathcal{F}(\mu) \leq H(\mu|\rho)$  with equality if and only if  $\mu = \rho$ . Hence the function  $\mathcal{F} - H(\cdot|\rho)$  has a unique maximum on  $\mathcal{M}_1^L$  at  $\rho$ .

Sanov's theorem (theorem 6.2.10 of [3]) states that  $(\tilde{\theta}_{n,\rho})_{n \geq 1}$  satisfies the large deviation principle in  $\mathcal{M}_1^L$  with speed  $n$  and governed by the good rate function  $H(\cdot|\rho)$ . As a consequence

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\theta}_{n,\rho}(\{\delta_0\}^c) \geq - \inf_{\mu \neq \delta_0} H(\mu|\rho) = 0.$$

Since  $\mathcal{F}$  is bounded (by  $1/2$ ) and is upper semi-continuous on  $\mathcal{M}_1^L$ , Varadhan's lemma (see section 4.3 of [3]) implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\mu}_{n,\rho}(M_n \in \mathcal{U}_\varepsilon^c) &\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\mathcal{U}_\varepsilon^c} e^{n\mathcal{F}(\mu)} d\tilde{\theta}_{n,\rho}(\mu) - \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln Z_n \\ &\leq \sup \{ \mathcal{F}(\mu) - H(\mu|\rho) : \mu \in \mathcal{U}_\varepsilon^c \}. \end{aligned}$$

Since  $H(\cdot|\rho)$  is a good rate function,  $\mathcal{F}$  is upper semi-continuous and  $\mathcal{U}_\varepsilon^c$  is a closed subset of  $\mathcal{M}_1^L$  which does not contain  $\rho$ , the unique maximum of the function  $\mathcal{F} - H(\cdot|\rho)$ , we get

$$\sup \{ \mathcal{F}(\mu) - H(\mu|\rho) : \mu \in \mathcal{U}_\varepsilon^c \} < 0.$$

As a consequence, there exists  $c_\varepsilon > 0$  and  $n_\varepsilon \geq 1$  such that

$$\forall n \geq n_\varepsilon \quad \tilde{\mu}_{n,\rho}(M_n \in \mathcal{U}_\varepsilon^c) \leq \exp(-nc_\varepsilon).$$

This implies the convergence in theorem 3.

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