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ON A PROJECTION LEAST SQUARES ESTIMATOR FOR JUMP DIFFUSION PROCESSES

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ABSTRACT. This paper deals with a projection least squares estimator of the drift function of a jump diffusion process X computed from multiple independent copies of X observed on $[0, T]$. Risk bounds are established on this estimator and on an associated adaptive estimator. Finally, some numerical experiments are provided.

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1. INTRODUCTION

Let $Z = (Z_t)_{t \in [0, T]}$ be the compound Poisson process defined by

$$Z_t := \sum_{n=1}^{\nu_t} \zeta_n = \int_0^t \int_{-\infty}^{\infty} z \mu(ds, dz)$$

for every $t \in [0, T]$, where $\nu = (\nu_t)_{t \in [0, T]}$ is a (usual) Poisson process of intensity $\lambda > 0$, independent of the ζ_n 's which are i.i.d. random variables of probability distribution π , and μ is the Poisson random measure of intensity $m(ds, dz) := \lambda \pi(dz) ds$ defined by

$$\mu([0, t] \times dz) := |\{s \in [0, t] : Z_s - Z_{s-} \in dz\}| ; \forall t \in [0, T].$$

In the sequel, Z is *replaced* by the centered martingale $\mathfrak{Z} = (\mathfrak{Z}_t)_{t \in [0, T]}$ defined by

$$\mathfrak{Z}_t := Z_t - \int_0^t \int_{-\infty}^{\infty} z m(ds, dz) = Z_t - \mathfrak{c}_\zeta \lambda t$$

for every $t \in [0, T]$, where \mathfrak{c}_ζ is the (common) expectation of the ζ_n 's. Now, let us consider the stochastic differential equation

$$(1) \quad X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s + \int_0^t \gamma(X_s) d\mathfrak{Z}_s ; t \in [0, T],$$

Key words and phrases. Projection least squares estimator ; Model selection ; Jump diffusion processes.

where $x_0 \in \mathbb{R}$, $B = (B_t)_{t \in [0, T]}$ is a Brownian motion independent of Z , $b \in C^1(\mathbb{R})$ and its derivative is bounded, and $\sigma, \gamma : \mathbb{R} \rightarrow \mathbb{R}$ are bounded Lipschitz continuous functions such that $\inf_{x \in \mathbb{R}} \sigma(x)^2 \wedge \gamma(x)^2 > 0$. Under these conditions on b , σ and γ , Equation (1) has a unique (strong) solution $X = (X_t)_{t \in [0, T]}$.

As for continuous diffusion processes, the major part of the estimators of the drift function in stochastic differential equations driven by jump processes are computed from one path of the solution to Equation (1) and converges when $T \rightarrow \infty$ (see Schmisser [18], Gloter et al. [15], Amorino et al. [2], etc.). The existence and the uniqueness of the ergodic stationary solution to Equation (1) is then required, and obtained thanks to a restrictive dissipativity condition on b . For stochastic differential equations driven by a pure-jump Lévy process, some authors have also studied estimation methods based on high frequency observations, on a fixed time interval, of one path of the solution (see Clément and Gloter [5, 6]).

Now, consider $X^i := \mathcal{I}(x_0, B^i, \mathfrak{Z}^i)$ for every $i \in \{1, \dots, N\}$, where $\mathcal{I}(\cdot)$ is the Itô map associated to Equation (1) and $(B^1, \mathfrak{Z}^1), \dots, (B^N, \mathfrak{Z}^N)$ are $N \in \mathbb{N}^*$ independent copies of (B, \mathfrak{Z}) . The estimation of the drift function b from a continuous-time or a discrete-time observation of (X^1, \dots, X^N) is a functional data analysis problem already investigated in the parametric and in the nonparametric frameworks for continuous diffusion processes (see Ditlevsen and De Gaetano [12], Picchini and Ditlevsen [17], Delattre et al. [10], Comte and Genon-Catalot [9], Denis et al. [11], Marie and Rosier [16], etc.). Up to our knowledge, no such estimator of the drift function has been already proposed for jump diffusion processes. So, our paper deals with a projection least squares estimator \hat{b}_m of b computed from X^1, \dots, X^N , which means that \hat{b}_m is minimizing the objective function

$$\tau \longmapsto \gamma_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \left(\int_0^T \tau(X_s^i)^2 ds - 2 \int_0^T \tau(X_s^i) dX_s^i \right)$$

on a m -dimensional function space \mathcal{S}_m . Precisely, risk bounds are established on \hat{b}_m and on the adaptive estimator $\hat{b}_{\hat{m}}$, where

$$\hat{m} = \arg \min_{m \in \hat{\mathcal{M}}_N} \{-\|\hat{b}_m\|_N^2 + \text{pen}(m)\}$$

with $\hat{\mathcal{M}}_N \subset \{1, \dots, N\}$,

$$\text{pen}(m) := \mathfrak{c}_{\text{cal}} \frac{m}{N}; \forall m \in \mathbb{N}$$

and $\mathfrak{c}_{\text{cal}} > 0$ is a constant to calibrate in practice.

In Section 2, a detailed definition of the projection least squares estimator of b is provided. Section 3 deals with a risk bound on \hat{b}_m and Section 4 with a risk bound on the adaptive estimator $\hat{b}_{\hat{m}}$. Finally, some numerical experiments are provided in Section 5. The proofs (resp. tables and figures) are postponed to Appendix A (resp. Appendix B).

2. A PROJECTION LEAST SQUARES ESTIMATOR OF THE DRIFT FUNCTION

2.1. The objective function. Assume that the probability distribution of X_s has a density $p_s(x_0, \cdot)$ with respect to Lebesgue's measure for every $s \in (0, T]$, and let f_T be the density function defined by

$$f_T(x) := \frac{1}{T} \int_0^T p_s(x_0, x) ds; \forall x \in \mathbb{R}.$$

Remark 2.1. Assume that b and π satisfy the following additional conditions:

(1) The function b belongs to the Kato class

$$\mathbb{K}_2 := \left\{ \varphi : \mathbb{R} \rightarrow \mathbb{R} : \limsup_{\delta \rightarrow 0} \int_0^\delta \int_{-\infty}^\infty |f(x+y) + f(x-y)| s^{(\lambda-1)/2} (|y| + s^{1/2})^{-(1+\lambda)} dy ds = 0 \right\}.$$

(2) The Lévy measure $\pi_\lambda(\cdot) := \lambda\pi(\cdot)$ has a density θ with respect to Lebesgue's measure. Moreover, there exists $\alpha \in (0, 2)$ such that $z \in \mathbb{R} \mapsto \theta(z)|z|^{1+\alpha}$ is bounded, and if $\alpha = 1$, then

$$\int_{r < |z| < R} z\theta(z)dz = 0 ; \forall R > r > 0.$$

By Chen et al. [4], Theorem 1.1 and the remark p. 126, l. 5-7, in Amorino and Gloter [1], for every $s \in (0, T]$, the probability distribution of X_s has a density $p_s(x_0, \cdot)$ with respect to Lebesgue's measure, and there exists a constant $\mathfrak{c}_p > 0$, not depending on s , such that

$$p_s(x_0, x) \leq \mathfrak{c}_p \left[s^{-1/2} e^{-\lambda \frac{(x-x_0)^2}{s}} + \frac{s}{(s^{1/2} + |x-x_0|)^{1+\alpha}} \right] ; \forall x \in \mathbb{R}.$$

So, f_T is bounded, which is crucial in Section 4. Indeed, since $-1/2 < 1 - (1 + \alpha)/2 < 1/2$, for every $x \in \mathbb{R}$,

$$\sup_{x \in \mathbb{R}} f_T(x) \leq \frac{\mathfrak{c}_p}{T} \left(\int_0^T s^{-1/2} ds + \int_0^T s^{1-(1+\alpha)/2} ds \right) = \frac{\mathfrak{c}_p}{T} \left[2T^{1/2} + \frac{T^{2-(1+\alpha)/2}}{2 - (1+\alpha)/2} \right] < \infty.$$

Now, let us consider the objective function $\gamma_N(\cdot)$ defined by

$$\gamma_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \left(\int_0^T \tau(X_s^i)^2 ds - 2 \int_0^T \tau(X_s^i) dX_s^i \right)$$

for every $\tau \in \mathcal{S}_m$, where $m \in \{1, \dots, N_T\}$, $N_T := [NT] + 1$, $\mathcal{S}_m := \text{span}\{\varphi_1, \dots, \varphi_m\}$, $\varphi_1, \dots, \varphi_{N_T}$ are continuous functions from I into \mathbb{R} such that $(\varphi_1, \dots, \varphi_{N_T})$ is an orthonormal family in $\mathbb{L}^2(I, dx)$, and $I \subset \mathbb{R}$ is a non-empty interval. For any $\tau \in \mathcal{S}_m$,

$$\begin{aligned} \mathbb{E}(\gamma_N(\tau)) &= \frac{1}{T} \int_0^T \mathbb{E}(|\tau(X_s) - b(X_s)|^2) ds - \frac{1}{T} \int_0^T \mathbb{E}(b(X_s)^2) ds \\ &= \int_{-\infty}^{\infty} (\tau(x) - b(x))^2 f_T(x) dx - \int_{-\infty}^{\infty} b(x)^2 f_T(x) dx. \end{aligned}$$

Then, the closer τ is to b , the smaller $\mathbb{E}(\gamma_N(\tau))$. For this reason, the estimator of b minimizing $\gamma_N(\cdot)$ is studied in this paper.

2.2. The projection least squares estimator and some related matrices. Consider

$$J := \sum_{j=1}^m \theta_j \varphi_j \quad \text{with} \quad \theta_1, \dots, \theta_m \in \mathbb{R}.$$

Then,

$$\begin{aligned} \nabla \gamma_N(J) &= \left(\frac{1}{NT} \sum_{i=1}^N \left(2 \sum_{\ell=1}^m \hat{\theta}_\ell \int_0^T \varphi_j(X_s^i) \varphi_\ell(X_s^i) ds - 2 \int_0^T \varphi_j(X_s^i) dX_s^i \right) \right)_{j \in \{1, \dots, m\}} \\ &= 2(\hat{\Psi}_m(\theta_1, \dots, \theta_m)^* - \hat{\mathbf{X}}_m) \end{aligned}$$

where

$$\hat{\Psi}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_\ell(X_s^i) ds \right)_{j, \ell \in \{1, \dots, m\}}$$

and

$$\hat{\mathbf{X}}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) dX_s^i \right)_{j \in \{1, \dots, m\}}.$$

The symmetric matrix $\hat{\Psi}_m$ is positive semidefinite because

$$\mathbf{u}^* \hat{\Psi}_m \mathbf{u} = \frac{1}{NT} \sum_{i=1}^N \int_0^T \left(\sum_{j=1}^m \mathbf{u}_j \varphi_j(X_s^i) \right)^2 ds \geq 0$$

for every $\mathbf{u} \in \mathbb{R}^m$. If in addition $\widehat{\Psi}_m$ is invertible, it is positive definite, and then

$$\widehat{b}_m = \sum_{j=1}^m \widehat{\theta}_j \varphi_j \quad \text{with} \quad \widehat{\Theta}_m := (\widehat{\theta}_1, \dots, \widehat{\theta}_m)^* = \widehat{\Psi}_m^{-1} \widehat{\mathbf{X}}_m$$

is the only minimizer of $\gamma_N(\cdot)$ on \mathcal{S}_m , called the projection least squares estimator of b .

Remarks:

(1) $\widehat{\Psi}_m = (\langle \varphi_j, \varphi_\ell \rangle_N)_{j,\ell}$, where

$$\langle \varphi, \psi \rangle_N := \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi(X_s^i) \psi(X_s^i) ds$$

for every continuous functions $\varphi, \psi : \mathbb{R} \rightarrow \mathbb{R}$.

(2) $\widehat{\mathbf{X}}_m = (\langle b, \varphi_j \rangle_N)_j^* + \widehat{\mathbf{E}}_m$, where

$$\widehat{\mathbf{E}}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i) \right)_{j \in \{1, \dots, m\}}^*$$

Let us introduce the two following deterministic matrices related to the previous random ones:

$$\Psi_{m,\sigma} := NT \mathbb{E}(\widehat{\mathbf{E}}_m \widehat{\mathbf{E}}_m^*) \quad \text{and} \quad \Psi_m := \mathbb{E}(\widehat{\Psi}_m) = (\langle \varphi_j, \varphi_\ell \rangle_{f_T})_{j,\ell},$$

where $\langle \cdot, \cdot \rangle_{f_T}$ is the usual scalar product in $\mathbb{L}^2(I, f_T(x) dx)$. Finally, the following lemma will be crucial in Section 3 to evaluate the order of the variance term in the risk bound on our projection least squares estimator of b .

Lemma 2.2. $\text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}) \leq (\|\sigma\|_\infty^2 + \lambda \mathfrak{c}_{\zeta^2} \|\gamma\|_\infty^2) m$.

3. RISK BOUND ON THE PROJECTION LEAST SQUARES ESTIMATOR

This section deals with a risk bound on the truncated estimator

$$\widetilde{b}_m := \widehat{b}_m \mathbf{1}_{\Lambda_m},$$

where

$$\Lambda_m := \left\{ L(m) (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1) \leq \mathfrak{c}_T \frac{NT}{\log(NT)} \right\}$$

with

$$L(m) = 1 \vee \left(\sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 \right) \quad \text{and} \quad \mathfrak{c}_T = \frac{3 \log(3/2) - 1}{8T}.$$

On the event Λ_m , $\widehat{\Psi}_m$ is invertible because

$$\inf\{\text{sp}(\widehat{\Psi}_m)\} \geq \frac{L(m)}{\mathfrak{c}_T} \cdot \frac{\log(NT)}{NT},$$

and then \widetilde{b}_m is well-defined. In the sequel, m fulfills the following assumption.

Assumption 3.1. $L(m) (\|\Psi_m^{-1}\|_{\text{op}} \vee 1) \leq \frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)}$.

The above condition is a generalization of the so-called *stability condition* introduced in the nonparametric regression framework in Cohen et al. [7], and already extended to the independent copies of continuous diffusion processes framework in Comte and Genon-Catalot [9].

Theorem 3.2. *Under Assumption 3.1, there exists a constant $\mathfrak{c}_{3.2} > 0$, not depending on m and N , such that*

$$\mathbb{E}(\|\widetilde{b}_m - b_I\|_N^2) \leq \min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 + \mathfrak{c}_{3.2} \left(\frac{m}{NT} + \frac{1}{N} \right).$$

Remarks:

- Note that Theorem 3.2 says first that the bound on the variance of \tilde{b}_m is of order m/N as in the nonparametric regression framework.
- The order of the bias term

$$\min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2,$$

as well as $L(m)$ and $\|\Psi_m^{-1}\|_{\text{op}}$, depend on the φ_j 's. For instance, assume that $I = [\ell, \mathbf{r}]$ with $\ell, \mathbf{r} \in \mathbb{R}$ satisfying $\ell < \mathbf{r}$, and that

$$\begin{aligned} \varphi_1(x) &:= \sqrt{\frac{1}{\mathbf{r} - \ell}} \mathbf{1}_{[\ell, \mathbf{r}]}(x), \\ \varphi_{2j+1}(x) &:= \sqrt{\frac{2}{\mathbf{r} - \ell}} \sin\left(2\pi j \frac{x - \ell}{\mathbf{r} - \ell}\right) \mathbf{1}_{[\ell, \mathbf{r}]}(x) \quad \text{and} \\ \varphi_{2j}(x) &:= \sqrt{\frac{2}{\mathbf{r} - \ell}} \cos\left(2\pi j \frac{x - \ell}{\mathbf{r} - \ell}\right) \mathbf{1}_{[\ell, \mathbf{r}]}(x) \end{aligned}$$

for every $x \in [\ell, \mathbf{r}]$ and $j \in \mathbb{N}^*$. On the one hand, since $\cos(\cdot)^2 + \sin(\cdot)^2 = 1$, there exists a constant $\mathbf{c}_\varphi > 0$, not depending on m , N and T , such that

$$L(m) = 1 \vee \left[\sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 \right] \leq \mathbf{c}_\varphi m.$$

Moreover, since under the same conditions than in Remark 2.1, there exists $\underline{\mathbf{m}} > 0$ such that $f_T(\cdot) \geq \underline{\mathbf{m}}$ on I by Chen et al. [4], Theorem 1.3,

$$\begin{aligned} \|\Psi_m^{-1}\|_{\text{op}} &= \frac{1}{\lambda_{\min}(\Psi_m)} = \left(\inf_{\theta: \|\theta\|_{2,m}=1} \sum_{j,\ell=1}^m \theta_j \theta_\ell [\Psi_m]_{j,\ell} \right)^{-1} \\ &= \left[\inf_{\theta: \|\theta\|_{2,m}=1} \int_\ell^{\mathbf{r}} \left(\sum_{j=1}^m \theta_j \varphi_j(x) \right)^2 f_T(x) dx \right]^{-1} \leq \frac{1}{\underline{\mathbf{m}}}. \end{aligned}$$

On the other hand, consider the Fourier-Sobolev space

$$\mathbb{W}_2^\beta([\ell, \mathbf{r}]) := \left\{ \varphi \in C^\beta([\ell, \mathbf{r}]; \mathbb{R}) : \int_\ell^{\mathbf{r}} \varphi^{(\beta)}(x)^2 dx < \infty \right\}$$

with $\beta \in \mathbb{N}^*$, and assume that $b_I \in \mathbb{W}_0^\beta([\ell, \mathbf{r}])$. By DeVore and Lorentz [14], Corollary 2.4 p. 205, there exists a constant $\mathbf{c}_{\beta,\ell,\mathbf{r}} > 0$, not depending on m and N , such that

$$\|\Pi_m(b_I) - b_I\|^2 \leq \mathbf{c}_{\beta,\ell,\mathbf{r}} m^{-2\beta},$$

where Π_m is the orthogonal projection from $\mathbb{L}^2(I, dx)$ onto \mathcal{S}_m . Since under appropriate conditions on b and π , as explained in Remark 2.1, f_T is upper bounded on I by a constant $\bar{\mathbf{m}} > 0$,

$$\begin{aligned} \min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 &\leq \bar{\mathbf{m}} \|\Pi_m(b_I) - b_I\|^2 \\ &\leq \bar{\mathbf{c}}_{\beta,\ell,\mathbf{r}} m^{-2\beta} \quad \text{with} \quad \bar{\mathbf{c}}_{\beta,\ell,\mathbf{r}} = \mathbf{c}_{\beta,\ell,\mathbf{r}} \bar{\mathbf{m}}. \end{aligned}$$

So, by Theorem 3.2, there exists a constant $\bar{\mathbf{c}}_{3.2,2} > 0$, not depending m and N , such that

$$\mathbb{E}(\|\tilde{b}_m - b_I\|_N^2) \leq \bar{\mathbf{c}}_{3.2,2} \left(m^{-2\beta} + \frac{m}{N} \right),$$

and then the bias-variance tradeoff is reached by (the risk bound on) \tilde{b}_m for m of order $N^{1/(1+2\beta)}$.

4. MODEL SELECTION

First of all, let us state some additional assumptions on the φ_j 's, on the Lévy measure π_λ and on the density function f_T .

Assumption 4.1. *The spaces \mathcal{S}_m , $m \in \{1, \dots, N_T\}$, satisfy the following conditions:*

- (1) *There exists a constant $\mathfrak{c}_\varphi \geq 1$, not depending on N , such that for every $m \in \{1, \dots, N_T\}$, the basis $(\varphi_1, \dots, \varphi_m)$ of \mathcal{S}_m satisfies*

$$L(m) = 1 \vee \left[\sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 \right] \leq \mathfrak{c}_\varphi^2 m.$$

- (2) *(Nested spaces) For every $m, m' \in \{1, \dots, N_T\}$, if $m > m'$, then $\mathcal{S}_{m'} \subset \mathcal{S}_m$.*

The proof of the oracle inequality satisfied by the adaptive estimator relies on a Bernstein type inequality (see Lemma A.4). In order to control the *big jumps* of X in the proof of Lemma A.4, the Lévy measure π_λ needs to fulfill the following assumption.

Assumption 4.2. *The Lévy measure π_λ is sub-exponential: there exist positive constants $\mathfrak{a}, \alpha > 0$ such that, for every $x > 1$,*

$$\pi_\lambda((-x, x)^c) \leq \mathfrak{a} e^{-\alpha|x|}.$$

Assumption 4.3. *The density function f_T is bounded.*

Note that Remark 2.1 provides a sufficient condition for Assumption 4.3 to be satisfied by f_T . Now, let us introduce the set

$$\widehat{\mathcal{M}}_N := \left\{ m \in \{1, \dots, N_T\} : \mathfrak{c}_\varphi^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \vee 1) \leq \mathfrak{d}_T \frac{NT}{\log(NT)} \right\}$$

with

$$\mathfrak{d}_T = \frac{1}{60T(\|f_T\|_\infty + 1/3)} < 2\mathfrak{c}_T,$$

as well as its theoretical counterpart

$$\mathcal{M}_N := \left\{ m \in \{1, \dots, N_T\} : \mathfrak{c}_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}}^2 \vee 1) \leq \frac{\mathfrak{d}_T}{4} \cdot \frac{NT}{\log(NT)} \right\}.$$

Let us also consider

$$\widehat{m} := \arg \min_{m \in \widehat{\mathcal{M}}_N} \{-\|\widehat{b}_m\|_N^2 + \text{pen}(m)\},$$

where

$$\text{pen}(m) := \mathfrak{c}_{\text{cal}} \frac{m}{NT}; \quad \forall m \in \{1, \dots, N_T\}$$

and $\mathfrak{c}_{\text{cal}} > 0$ is a deterministic constant to calibrate.

Theorem 4.4. *Under Assumptions 3.1, 4.1, 4.2 and 4.3, there exists a constant $\mathfrak{c}_{4.4} > 0$, not depending on N , such that*

$$\mathbb{E}(\|\widehat{b}_{\widehat{m}} - b_I\|_N^2) \leq \mathfrak{c}_{4.4} \left[\min_{m \in \widehat{\mathcal{M}}_N} \left\{ \min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 + \frac{m}{NT} \right\} + \frac{1}{N} \right].$$

Note that Theorem 4.4 provides a risk bound on the adaptive estimator $\widehat{b}_{\widehat{m}}$ of same order than the minimal risk bound on \widetilde{b}_m (see Theorem 3.2) for m taken in \mathcal{M}_N .

5. NUMERICAL EXPERIMENTS

First of all, let us recall that all figures and tables related to this section are postponed to Appendix B.

Some numerical experiments on our estimation method of b in Equation (1) are presented in this subsection when the common distribution π of the ζ_n 's is standard normal, which implies that $\mathfrak{Z} = Z$, and the intensity of the (usual) Poisson process ν is $\lambda = 0.5$. The estimation method investigated on the theoretical side in Sections 3 and 4 is implemented here for the three following models:

- (1) $X_t = 0.5 - \int_0^t X_s ds + 0.5B_t + Z_t$ (linear/additive noise).
- (2) $X_t = 0.5 + 0.5 \int_0^t \sqrt{1 + X_s^2} ds + 0.5B_t + Z_t$ (nonlinear/additive noise).
- (3) $X_t = 0.5 + 0.5 \int_0^t \sqrt{1 + X_s^2} ds + 0.5 \int_0^t (1 + \cos(X_s)^2) dB_s + Z_t$ (nonlinear/multiplicative noise).

For each of the three previous models, our adaptive estimator of b is computed on $I = [-3, 3]$ from $N = 400$ paths of the process X observed along the dissection $\{lT/n ; l = 1, \dots, n\}$ of $[0, T]$ with $n = 200$ and $T = 5$, when $(\varphi_1, \dots, \varphi_m)$ is the m -dimensional trigonometric basis for every $m \in \{1, \dots, 6\}$. This experiment is repeated 100 times, and the means and the standard deviations of the MISE of $\widehat{b}_{\widehat{m}}$ are stored in Table 1. Moreover, for each model, 10 estimations (dashed black curves) of b (red curve) are plotted on Figures 1, 2 and 3.

On the one hand, on average, the MISE of our adaptive estimator is slightly increasing with the *complexity* of the model: 0.1251 (Model 1) < 0.1469 (Model 2) < 0.1825 (Model 3). The same remark holds for its standard deviation: 0.0950 (Model 1) < 0.1688 (Model 2) < 0.1928 (Model 3). This means that the more the model is *complex*, the more the quality of the estimation degrades, and it's visible on Figures 1, 2 and 3. However, for all the three previous models, the MISE of $\widehat{b}_{\widehat{m}}$ remains good and of same order than for models without jump component (see Comte and Genon-Catalot [9], Section 4). On the other hand, the model selection procedure works well because it doesn't systematically select the lowest or the highest possible value of m (see \widehat{m} on Figures 1, 2 and 3). For Model 1, the procedure is stable because \widehat{m} has a low estimated standard deviation (0.4830). The procedure remains satisfactory for Models 2 and 3, with $\text{StD}(\widehat{m}) = 1.1353$ and $\text{StD}(\widehat{m}) = 1.2867$ respectively, but not as much as for Model 1.

REFERENCES

- [1] C. Amorino and A. Gloter. *Invariant Density Adaptive Estimation for Ergodic Jump Diffusion Processes over Anisotropic Classes*. Journal of Statistical Planning and Inference 213, 106-129, 2021.
- [2] C. Amorino, C. Dion-Blanc, A. Gloter and S. Lemler. *On the Nonparametric Inference of Coefficients of Self-Exciting Jump-Diffusion*. Electronic Journal of Statistics 16, 3212-3277, 2022.
- [3] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, 2009.
- [4] Z.Q. Chen, E. Hu, L. Xie and X. Zhang. *Heat Kernels for Nonsymmetric Diffusion Operators with Jumps*. Journal of Differential Equations 263, 10, 6576-6634, 2017.
- [5] E. Clément and A. Gloter. *Estimating Functions for SDE Driven by Stable Lévy Processes*. Annales de l'Institut Henri Poincaré (B) 55, 3, 1316-1348, 2019.
- [6] E. Clément and A. Gloter. *Joint Estimation for SDE Driven by Locally Stable Lévy Processes*. Electronic Journal of Statistics 14, 2, 2922-2956, 2020.
- [7] A. Cohen, M.A. Davenport and D. Leviatan. *On the Stability and Accuracy of Least Squares Approximations*. Foundations of Computational Mathematics 13, 819-834, 2013.
- [8] F. Comte. *Adaptive Estimation of the Spectrum of a Stationary Gaussian Sequence*. Bernoulli 7, 2, 267-198, 2001.
- [9] F. Comte and V. Genon-Catalot. *Nonparametric Drift Estimation for i.i.d. Paths of Stochastic Differential Equations*. The Annals of Statistics 48, 6, 3336-3365, 2020.
- [10] M. Delattre, V. Genon-Catalot and A. Samson. *Maximum Likelihood Estimation for Stochastic Differential Equations with Random Effects*. Scandinavian Journal of Statistics 40, 322-343, 2013.
- [11] C. Denis, C. Dion-Blanc and M. Martinez. *A Ridge Estimator of the Drift from Discrete Repeated Observations of the Solution of a Stochastic Differential Equation*. Bernoulli 27, 2675-2713, 2021.

- [12] S. Ditlevsen and A. De Gaetano. *Mixed Effects in Stochastic Differential Equation Models*. REVSTAT 3, 137-153, 2005.
- [13] C.A. Dellacherie and P.A. Meyer. *Probabilités et potentiel : théorie des martingales*. Hermann, 1980.
- [14] R.A. DeVore and G.G. Lorentz. *Constructive Approximation*. Springer, 1993.
- [15] A. Gloter, D. Loukianova and H. Mai. *Jump Filtering and Efficient Drift Estimation for Lévy-Driven SDEs*. The Annals of Statistics 46, 4, 1445-1480, 2018.
- [16] N. Marie and A. Rosier. *Nadaraya-Watson Estimator for I.I.D. Paths of Diffusion Processes*. Scandinavian Journal of Statistics (to appear), 2022.
- [17] U. Picchini and S. Ditlevsen. *Practical Estimation of High Dimensional Stochastic Differential Mixed-Effects Models*. Computational Statistics and Data Analysis 55, 1426-1444, 2011.
- [18] E. Schmisser. *Non-Parametric Adaptive Estimation of the Drift for a Jump Diffusion Process*. Stochastic Processes and their Applications 124, 883-914, 2014.

APPENDIX A. PROOFS

A.1. Proof of Lemma 2.2. First, let us show that the symmetric matrix $\Psi_{m,\sigma}$ is positive semidefinite. Indeed, for any $y \in \mathbb{R}^m$,

$$\begin{aligned} y^* \Psi_{m,\sigma} y &= \frac{1}{NT} \sum_{j,\ell=1}^m y_j y_\ell \sum_{i,k=1}^N \mathbb{E} \left(\left(\int_0^T \varphi_j(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i) \right) \right. \\ &\quad \left. \times \left(\int_0^T \varphi_\ell(X_s^k) (\sigma(X_s^k) dB_s^k + \gamma(X_s^k) d\mathfrak{Z}_s^k) \right) \right) \\ &= \frac{1}{NT} \mathbb{E} \left[\left(\sum_{i=1}^N \int_0^T \tau_y(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i) \right)^2 \right] \geq 0 \quad \text{with} \quad \tau_y(\cdot) := \sum_{j=1}^m y_j \varphi_j(\cdot). \end{aligned}$$

Moreover, by the isometry property of Itô's integral (with respect to B), by the isometry type property of the stochastic integral with respect to \mathfrak{Z} , and since σ and γ are bounded, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned} y^* \Psi_{m,\sigma} y &= \frac{1}{T} \mathbb{E} \left[\left(\int_0^T \tau_y(X_s) \sigma(X_s) dB_s \right)^2 \right] + \frac{1}{T} \mathbb{E} \left[\left(\int_0^T \tau_y(X_s) \gamma(X_s) d\mathfrak{Z}_s \right)^2 \right] \\ &= \frac{1}{T} \int_0^T \mathbb{E}(\tau_y(X_s)^2 \sigma(X_s)^2) ds + \frac{\lambda \mathfrak{c}_{\zeta^2}}{T} \int_0^T \mathbb{E}(\tau_y(X_s)^2 \gamma(X_s)^2) ds \\ (2) \quad &\leq (\|\sigma\|_\infty^2 + \lambda \mathfrak{c}_{\zeta^2} \|\gamma\|_\infty^2) \int_{-\infty}^{\infty} \left(\sum_{j=1}^m y_j \varphi_j(x) \right)^2 f_T(x) dx = (\|\sigma\|_\infty^2 + \lambda \mathfrak{c}_{\zeta^2} \|\gamma\|_\infty^2) \|\Psi_m^{1/2} y\|_{2,m}^2. \end{aligned}$$

Therefore, since $\Psi_{m,\sigma}$ is positive semidefinite, and by Inequality (2),

$$\begin{aligned} \text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}) &\leq m \|\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}\|_{\text{op}} \\ &= m \cdot \sup\{y^* \Psi_{m,\sigma} y ; y \in \mathbb{R}^m \text{ and } \|\Psi_m^{1/2} y\|_{2,m} = 1\} \\ &\leq (\|\sigma\|_\infty^2 + \lambda \mathfrak{c}_{\zeta^2} \|\gamma\|_\infty^2) m. \end{aligned}$$

A.2. Proof of Theorem 3.2. The proof of Theorem 3.2 relies on the two following lemmas.

Lemma A.1. *There exists a constant $\mathfrak{c}_{A.1} > 0$, not depending on m and N , such that*

$$\mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2) \leq \mathfrak{c}_{A.1} \frac{mL(m)^2}{N^2}.$$

Lemma A.2. *Consider the event*

$$\Omega_m := \left\{ \sup_{\tau \in \mathcal{S}_m} \left| \frac{\|\tau\|_N^2}{\|\tau\|_{f_T}^2} - 1 \right| \leq \frac{1}{2} \right\}.$$

Under Assumptions 3.1, there exists a constant $\mathfrak{c}_{A.2} > 0$, not depending on m and N , such that

$$\mathbb{P}(\Omega_m^c) \leq \frac{\mathfrak{c}_{A.2}}{N^7} \quad \text{and} \quad \mathbb{P}(\Lambda_m^c) \leq \frac{\mathfrak{c}_{A.2}}{N^7}.$$

The proof of Lemma A.1 is postponed to Subsubsection A.2.2, and the proof of Lemma A.2 remains the same than the proof of Comte and Genon-Catalot [9], Lemma 6.1, because $(B^1, Z^1), \dots, (B^N, Z^N)$ are independent.

A.2.1. *Steps of the proof.* First of all,

$$(3) \quad \begin{aligned} \|\tilde{b}_m - b_I\|_N^2 &= \|b_I\|_N^2 \mathbf{1}_{\Lambda_m^c} + \|\hat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m} \\ &= \|b_I\|_N^2 \mathbf{1}_{\Lambda_m^c} + \|\hat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m} + \|\hat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c} =: U_1 + U_2 + U_3. \end{aligned}$$

Let us find suitable bounds on $\mathbb{E}(U_1)$, $\mathbb{E}(U_2)$ and $\mathbb{E}(U_3)$.

- **Bound on $\mathbb{E}(U_1)$.** By Cauchy-Schwarz's inequality,

$$\begin{aligned} \mathbb{E}(U_1) &\leq \mathbb{E}(\|b_I\|_N^4)^{1/2} \mathbb{P}(\Lambda_m^c)^{1/2} \leq \mathbb{E}\left(\frac{1}{T} \int_0^T b_I(X_t)^4 dt\right)^{1/2} \mathbb{P}(\Lambda_m^c)^{1/2} \\ &\leq \mathbf{c}_1 \mathbb{P}(\Lambda_m^c)^{1/2} < \infty \quad \text{with} \quad \mathbf{c}_1 = \left(\int_{-\infty}^{\infty} b_I(x)^4 f_T(x) dx\right)^{1/2} < \infty. \end{aligned}$$

- **Bound on $\mathbb{E}(U_2)$.** Let $\Pi_{N,m}(\cdot)$ be the orthogonal projection from $\mathbb{L}^2(I, f_T(x) dx)$ onto \mathcal{S}_m with respect to the empirical scalar product $\langle \cdot, \cdot \rangle_N$. Then,

$$(4) \quad \|\hat{b}_m - b_I\|_N^2 = \|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 + \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_N^2.$$

As in the proof of Comte and Genon-Catalot [9], Proposition 2.1, on Ω_m ,

$$\|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 = \hat{\mathbf{E}}_m^* \hat{\Psi}_m^{-1} \hat{\mathbf{E}}_m \leq 2 \hat{\mathbf{E}}_m^* \Psi_m^{-1} \hat{\mathbf{E}}_m.$$

So,

$$\begin{aligned} \mathbb{E}(\|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m}) &\leq 2 \mathbb{E}\left(\sum_{j,\ell=1}^m [\hat{\mathbf{E}}_m]_j [\hat{\mathbf{E}}_m]_\ell \Psi_m^{-1}(j, \ell)\right) \\ &= \frac{2}{NT} \sum_{j,\ell=1}^m \Psi_{m,\sigma}(j, \ell) \Psi_m^{-1}(j, \ell) = \frac{2}{NT} \text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}). \end{aligned}$$

Then, by Equality (4) and Lemma 2.2,

$$\begin{aligned} \mathbb{E}(U_2) &\leq \mathbb{E}\left(\min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_N^2\right) + \frac{2}{NT} \text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}) \\ &\leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 + \frac{2m}{NT} (\|\sigma\|_\infty^2 + \lambda \mathbf{c}_2 \|\gamma\|_\infty^2). \end{aligned}$$

- **Bound on $\mathbb{E}(U_3)$.** Since

$$\|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 = \hat{\mathbf{E}}_m^* \hat{\Psi}_m^{-1} \hat{\mathbf{E}}_m,$$

by the definition of the event Λ_m , and by Lemma A.1,

$$\begin{aligned} \mathbb{E}(\|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) &\leq \mathbb{E}(\|\hat{\Psi}_m^{-1}\|_{\text{op}} |\hat{\mathbf{E}}_m^* \hat{\mathbf{E}}_m| \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) \\ &\leq \frac{\mathbf{c}_T}{L(m)} \cdot \frac{NT}{\log(NT)} \mathbb{E}(|\hat{\mathbf{E}}_m^* \hat{\mathbf{E}}_m|^2)^{1/2} \mathbb{P}(\Omega_m^c)^{1/2} \leq \frac{\mathbf{c}_2 m^{1/2}}{\log(NT)} \mathbb{P}(\Omega_m^c)^{1/2}, \end{aligned}$$

where the constant $\mathbf{c}_2 > 0$ doesn't depend on m and N . Then,

$$\begin{aligned} \mathbb{E}(U_3) &\leq \mathbb{E}(\|\hat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) + \mathbb{E}(\|b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) \\ &\leq \frac{\mathbf{c}_2 m^{1/2}}{\log(NT)} \mathbb{P}(\Omega_m^c)^{1/2} + \mathbf{c}_1 \mathbb{P}(\Omega_m^c)^{1/2}. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{E}(\|\tilde{b}_m - b_I\|_N^2) &\leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 \\ &\quad + \frac{2m}{NT} (\|\sigma\|_\infty^2 + \lambda \mathbf{c}_{\zeta^2} \|\gamma\|_\infty^2) + \mathbf{c}_2 \frac{\sqrt{m\mathbb{P}(\Omega_m^c)}}{\log(NT)} + \mathbf{c}_1 (\mathbb{P}(\Lambda_m^c)^{1/2} + \mathbb{P}(\Omega_m^c)^{1/2}). \end{aligned}$$

Therefore, by Lemma A.2, there exists a constant $\mathbf{c}_3 > 0$, not depending on m and N , such that

$$\mathbb{E}(\|\tilde{b}_m - b_I\|_N^2) \leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 + \mathbf{c}_3 \left(\frac{m}{NT} + \frac{1}{N} \right).$$

A.2.2. *Proof of Lemma A.1.* In the sequel, the quadratic variation of any piecewise continuous stochastic process $(\Gamma_t)_{t \in [0, T]}$ is denoted by $([\Gamma]_t)_{t \in [0, T]}$. First of all, note that since B and Z are independent, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned} \left[\int_0^\cdot \varphi_j(X_s) (\sigma(X_s) dB_s + \gamma(X_s) d\mathfrak{Z}_s) \right]_T &= \int_0^T \varphi_j(X_s)^2 \sigma(X_s)^2 ds + \int_0^T \varphi_j(X_s)^2 \gamma(X_s)^2 dZ_s^{(2)} \\ &= \int_0^T \varphi_j(X_s)^2 (\sigma(X_s)^2 + \mathbf{c}_{\zeta^2} \lambda \gamma(X_s)^2) ds \\ &\quad + \int_0^T \varphi_j(X_s)^2 \gamma(X_s)^2 d\mathfrak{Z}_s^{(2)} \end{aligned}$$

where, for every $t \in [0, T]$,

$$Z_t^{(2)} := [\mathfrak{Z}]_t = \sum_{n=1}^{\nu_t} \zeta_n^2 \quad \text{and} \quad \mathfrak{Z}_t^{(2)} := Z_t^{(2)} - \mathbf{c}_{\zeta^2} \lambda t.$$

By Jensen's inequality and Burkholder-Davis-Gundy's inequality (see Dellacherie et Meyer [13], p. 303), there exists a constant $\mathbf{c}_1 > 0$, not depending on m and N , such that

$$\begin{aligned} \mathbb{E}(|\hat{\mathbf{E}}_m^* \hat{\mathbf{E}}_m|^2) &\leq m \sum_{j=1}^m \mathbb{E}(\hat{\mathbf{E}}_m(j)^4) \leq \frac{\mathbf{c}_1 m}{N^4 T^4} \sum_{j=1}^m \mathbb{E} \left(\left[\sum_{i=1}^N \int_0^T \varphi_j(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i) \right]_T^2 \right) \\ &\leq \frac{2\mathbf{c}_1 m}{N^2 T^4} \sum_{j=1}^m \mathbb{E} \left(\left[\int_0^T \varphi_j(X_s) (\sigma(X_s) dB_s + \gamma(X_s) d\mathfrak{Z}_s) \right]_T^2 \right) \\ &\leq \frac{4\mathbf{c}_1 m}{N^2 T^4} \sum_{j=1}^m \left[\mathbb{E} \left[\left(\int_0^T \varphi_j(X_s)^2 (\sigma(X_s)^2 + \mathbf{c}_{\zeta^2} \lambda \gamma(X_s)^2) ds \right)^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_0^T \varphi_j(X_s)^2 \gamma(X_s)^2 d\mathfrak{Z}_s^{(2)} \right)^2 \right] \right]. \end{aligned}$$

By Jensen's inequality,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E} \left[\left(\int_0^T \varphi_j(X_s)^2 (\sigma(X_s)^2 + \mathbf{c}_{\zeta^2} \lambda \gamma(X_s)^2) ds \right)^2 \right] &\leq 2T \sum_{j=1}^m \int_0^T \mathbb{E}(\varphi_j(X_s)^4 (\sigma(X_s)^4 + \mathbf{c}_{\zeta^2}^2 \lambda^2 \gamma(X_s)^4)) ds \\ &\leq 2T^2 L(m)^2 \int_{-\infty}^{\infty} (\sigma(x)^4 + \mathbf{c}_{\zeta^2}^2 \lambda^2 \gamma(x)^4) f_T(x) dx, \end{aligned}$$

and by the isometry type property of the stochastic integral with respect to \mathfrak{Z} ,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E} \left[\left(\int_0^T \varphi_j(X_s)^2 \gamma(X_s)^2 d\mathfrak{Z}_s^{(2)} \right)^2 \right] &= \lambda \mathfrak{c}_{\zeta^4} \sum_{j=1}^m \int_0^T \mathbb{E}(\varphi_j(X_s)^4 \gamma(X_s)^4) ds \\ &\leq \lambda \mathfrak{c}_{\zeta^4} TL(m)^2 \int_{-\infty}^{\infty} \gamma(x)^4 f_T(x) dx. \end{aligned}$$

Therefore,

$$\mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2) \leq \frac{\mathfrak{c}_2}{N^2 T^2} mL(m)^2$$

with

$$\mathfrak{c}_2 = 8\mathfrak{c}_1 \left(\int_{-\infty}^{\infty} (\sigma(x)^4 + \mathfrak{c}_{\zeta^2}^2 \lambda^2 \gamma(x)^4) f_T(x) dx + \lambda \mathfrak{c}_{\zeta^4} \int_{-\infty}^{\infty} \gamma(x)^4 f_T(x) dx \right).$$

A.3. Proof of Theorem 4.4. Let us consider the events

$$\Omega_N := \bigcap_{m \in \mathcal{M}_N^+} \Omega_m \quad \text{and} \quad \Xi_N := \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathcal{M}_N^+\},$$

where

$$\mathcal{M}_N^+ := \left\{ m \in \{1, \dots, N_T\} : \mathfrak{c}_{\varphi}^2 m (\|\Psi_m^{-1}\|_{\text{op}}^2 \vee 1) \leq 4\mathfrak{d}_T \frac{NT}{\log(NT)} \right\},$$

and let us recall that

$$\mathfrak{d}_T = \frac{1}{60T(\|f_T\|_{\infty} + 1/3)}.$$

As a reminder, the sets $\widehat{\mathcal{M}}_N$ and \mathcal{M}_N introduced in Section 4 are respectively defined by

$$\widehat{\mathcal{M}}_N = \left\{ m \in \{1, \dots, N_T\} : \mathfrak{c}_{\varphi}^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \vee 1) \leq \mathfrak{d}_T \frac{NT}{\log(NT)} \right\}$$

and

$$\mathcal{M}_N = \left\{ m \in \{1, \dots, N_T\} : \mathfrak{c}_{\varphi}^2 m (\|\Psi_m^{-1}\|_{\text{op}}^2 \vee 1) \leq \frac{\mathfrak{d}_T}{4} \cdot \frac{NT}{\log(NT)} \right\}.$$

The proof of Theorem 4.4 relies on the three following lemmas.

Lemma A.3. *Under Assumptions 3.1, 4.1 and 4.2, there exists a constant $\mathfrak{c}_{A.3} > 0$, not depending on N , such that*

$$\mathbb{P}(\Xi_N^c) \leq \frac{\mathfrak{c}_{A.3}}{N^6}.$$

Lemma A.4. *[Bernstein type inequality] Consider the empirical process*

$$\nu_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \int_0^T \tau(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i); \quad \tau \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}.$$

Under Assumption 4.2, for every $\xi, v > 0$,

$$\mathbb{P}(\nu_N(\tau) \geq \xi, \|\tau\|_N^2 \leq v^2) \leq \exp \left[- \frac{NT\xi^2}{4[\mathfrak{c}_{A.4}(\|\sigma\|_{\infty}^2 + \|\gamma\|_{\infty}^2)v^2 + \xi\|\tau\|_{\infty, I}\|\gamma\|_{\infty}]} \right]$$

with

$$\mathfrak{c}_{A.4} = \frac{1}{2} \left[1 \vee \int_{-\infty}^{\infty} e^{|z|} \pi_{\lambda}(dz) \right].$$

Lemma A.5. *Under Assumptions 3.1 and 4.2, there exists a constant $\mathfrak{c}_{A.5} > 0$, not depending on N , such that for every $m \in \mathcal{M}_N$,*

$$\mathbb{E} \left[\left(\left[\sup_{\tau \in \mathcal{B}_{m, m'}} \nu_n(\tau) \right]^2 - p(m, \widehat{m}) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right] \leq \frac{\mathfrak{c}_{A.5}}{NT}$$

where, for every $m' \in \mathcal{M}_N$,

$$\mathcal{B}_{m,m'} := \{\tau \in \mathcal{S}_{m \wedge m'} : \|\tau\|_{f_T} = 1\} \quad \text{and} \quad p(m, m') := \frac{\mathbf{c}_{\text{cal}}}{8} \cdot \frac{m \vee m'}{NT}.$$

The proof of Lemma A.4 is postponed to Subsubsection A.3.2. Lemma A.5 is a consequence of Lemma A.4 thanks to the $\mathbb{L}_{f_T}^2$ - \mathbb{L}^∞ chaining technique (see Comte [8], Proposition 4). Finally, the proof of Lemma A.3 remains the same than the proof of Comte and Genon-Catalot [9], Eq. (6.17), because $(B^1, \mathfrak{Z}^1), \dots, (B^N, \mathfrak{Z}^N)$ are independent.

A.3.1. *Steps of the proof.* First of all,

$$(5) \quad \begin{aligned} \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 &= \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N^c} + \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N} \\ &=: U_1 + U_2. \end{aligned}$$

Let us find suitable bounds on $\mathbb{E}(U_1)$ and $\mathbb{E}(U_2)$.

- **Bound on $\mathbb{E}(U_1)$.** Since

$$\|\widehat{b}_{\widehat{m}} - \Pi_{N, \widehat{m}}(b_I)\|_N^2 = \widehat{\mathbf{E}}_{\widehat{m}}^* \widehat{\Psi}_{\widehat{m}}^{-1} \widehat{\mathbf{E}}_{\widehat{m}},$$

by the definition of $\widehat{\mathcal{M}}_N$, and by Lemma A.1,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_{\widehat{m}} - \Pi_{N, \widehat{m}}(b_I)\|_N^2 \mathbf{1}_{\Xi_N^c}) &\leq \mathbb{E}(\|\widehat{\Psi}_{\widehat{m}}^{-1}\|_{\text{op}} |\widehat{\mathbf{E}}_{NT}^* \widehat{\mathbf{E}}_{NT}| \mathbf{1}_{\Xi_N^c}) \\ &\leq \left[\mathfrak{d}_T \frac{NT}{\log(NT)} \right]^{1/2} \mathbb{E}(|\widehat{\mathbf{E}}_{NT}^* \widehat{\mathbf{E}}_{NT}|^2)^{1/2} \mathbb{P}(\Xi_N^c)^{1/2} \leq \frac{\mathbf{c}_1 N}{\log(NT)} \mathbb{P}(\Xi_N^c)^{1/2}, \end{aligned}$$

where the constant $\mathbf{c}_1 > 0$ doesn't depend on N . Then,

$$\begin{aligned} \mathbb{E}(U_1) &\leq \mathbb{E}(\|\widehat{b}_{\widehat{m}} - \Pi_{N, \widehat{m}}(b_I)\|_N^2 \mathbf{1}_{\Xi_N^c}) + \mathbb{E}(\|b_I\|_N^2 \mathbf{1}_{\Xi_N^c}) \\ &\leq \frac{\mathbf{c}_1 N}{\log(NT)} \mathbb{P}(\Xi_N^c)^{1/2} + \mathbf{c}_2 \mathbb{P}(\Xi_N^c)^{1/2} \end{aligned}$$

with

$$\mathbf{c}_2 = \left(\int_{-\infty}^{\infty} b_I(x)^4 f_T(x) dx \right)^{1/2}.$$

So, by Lemma A.3, there exists a constant $\mathbf{c}_3 > 0$, not depending on N , such that

$$\mathbb{E}(U_1) \leq \frac{\mathbf{c}_3}{N}.$$

- **Bound on $\mathbb{E}(U_2)$.** Note that

$$\begin{aligned} U_2 &= \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N \cap \Omega_N^c} + \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N \cap \Omega_N} \\ &=: U_{2,1} + U_{2,2}. \end{aligned}$$

On the one hand, by Lemma A.2, there exists a constant $\mathbf{c}_4 > 0$, not depending on N , such that

$$\mathbb{P}(\Xi_N \cap \Omega_N^c) \leq \sum_{m \in \mathcal{M}_N^+} \mathbb{P}(\Omega_m^c) \leq \frac{\mathbf{c}_4}{N^6}.$$

Then, as for $\mathbb{E}(U_1)$, there exists a constant $\mathbf{c}_5 > 0$, not depending on N , such that

$$\mathbb{E}(U_{2,1}) \leq \frac{\mathbf{c}_5}{N}.$$

On the other hand,

$$\gamma_N(\tau') - \gamma_N(\tau) = \|\tau' - b\|_N^2 - \|\tau - b\|_N^2 - 2\nu_N(\tau' - \tau)$$

for every $\tau, \tau' \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}$. Moreover, since

$$\widehat{m} = \arg \min_{m \in \widehat{\mathcal{M}}_N} \{-\|\widehat{b}_m\|_N^2 + \text{pen}(m)\} = \arg \min_{m \in \widehat{\mathcal{M}}_N} \{\gamma_N(\widehat{b}_m) + \text{pen}(m)\},$$

for every $m \in \widehat{\mathcal{M}}_N$,

$$(6) \quad \gamma_N(\widehat{b}_{\widehat{m}}) + \text{pen}(\widehat{m}) \leq \gamma_N(\widehat{b}_m) + \text{pen}(m).$$

On the event $\Xi_N = \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathcal{M}_N^+\}$, Inequality (6) remains true for every $m \in \mathcal{M}_N$. Then, on Ξ_N , for any $m \in \mathcal{M}_N$, since $\mathcal{S}_m + \widehat{\mathcal{S}}_{\widehat{m}} \subset \mathcal{S}_{m \vee \widehat{m}}$ under Assumption 4.1,

$$\begin{aligned} \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 &\leq \|\widehat{b}_m - b_I\|_N^2 + 2\|\widehat{b}_{\widehat{m}} - \widehat{b}_m\|_{f_T} \nu_N \left(\frac{\widehat{b}_{\widehat{m}} - \widehat{b}_m}{\|\widehat{b}_{\widehat{m}} - \widehat{b}_m\|_{f_T}} \right) + \text{pen}(m) - \text{pen}(\widehat{m}) \\ &\leq \|\widehat{b}_m - b_I\|_N^2 + \frac{1}{8}\|\widehat{b}_{\widehat{m}} - \widehat{b}_m\|_{f_T}^2 \\ &\quad + 8 \left(\left[\sup_{\tau \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\tau)| \right]^2 - p(m, \widehat{m}) \right) + \text{pen}(m) + 8p(m, \widehat{m}) - \text{pen}(\widehat{m}). \end{aligned}$$

Since $\|\cdot\|_{f_T}^2 \mathbf{1}_{\Omega_N} \leq 2\|\cdot\|_N^2 \mathbf{1}_{\Omega_N}$ on $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_{\max(\mathcal{M}_N^+)}$, and since $8p(m, \widehat{m}) \leq \text{pen}(m) + \text{pen}(\widehat{m})$, on $\Xi_N \cap \Omega_N$,

$$\|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \leq 3\|\widehat{b}_m - b_I\|_N^2 + 4\text{pen}(m) + 16 \left(\left[\sup_{\tau \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\tau)| \right]^2 - p(m, \widehat{m}) \right).$$

So, by Lemma A.5,

$$\begin{aligned} \mathbb{E}(U_{2,2}) &\leq \min_{m \in \mathcal{M}_N} \{ \mathbb{E}(3\|\widehat{b}_m - b_I\|_N^2 \mathbf{1}_{\Xi_N}) + 4\text{pen}(m) \} + \frac{16\mathbf{c}_{A.5}}{NT} \\ &\leq \mathbf{c}_6 \min_{m \in \mathcal{M}_N} \left\{ \inf_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 + \frac{m}{NT} \right\} + \frac{\mathbf{c}_6}{N} \end{aligned}$$

where $\mathbf{c}_6 > 0$ is a deterministic constant not depending on N .

A.3.2. Proof of Lemma A.4. Consider $\tau \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}$ and, for any $i \in \{1, \dots, N\}$, let $M^i(\tau) = (M^i(\tau))_{t \in [0, T]}$ be the martingale defined by

$$M^i(\tau) := \int_0^t \tau(X_s^i) (\sigma(X_s^i) dB_s^i + \gamma(X_s^i) d\mathfrak{Z}_s^i); \quad \forall t \in [0, T].$$

Moreover, for every $\varepsilon > 0$, consider

$$Y_\varepsilon^i(\tau) := \varepsilon M^i(\tau) - A_\varepsilon^i(\tau) - B_\varepsilon^i(\tau),$$

where $A_\varepsilon^i(\tau) = (A_\varepsilon^i(\tau)_t)_{t \in [0, T]}$ and $B_\varepsilon^i(\tau) = (B_\varepsilon^i(\tau)_t)_{t \in [0, T]}$ are the stochastic processes defined by

$$\begin{aligned} A_\varepsilon^i(\tau)_t &:= \frac{\varepsilon^2}{2} \int_0^t \tau(X_s^i)^2 \sigma(X_s^i)^2 ds \\ \text{and } B_\varepsilon^i(\tau)_t &:= \int_0^t \left[\int_{-\infty}^{\infty} (e^{\varepsilon z \tau(X_s^i) \gamma(X_s^i)} - \varepsilon z \tau(X_s^i) \gamma(X_s^i) - 1) \pi_\lambda(dz) \right] ds \end{aligned}$$

for every $t \in [0, T]$. The proof is dissected in three steps.

Step 1. Note that for any $i \in \{1, \dots, N\}$ and $t \in [0, T]$,

$$|\tau(X_t^i) \gamma(X_t^i)| \leq \|\tau\|_{\infty, I} \|\gamma\|_{\infty}$$

and then, by Assumption 4.2,

$$\mathbb{E} \left(\int_0^t \int_{|z| > 1} |e^{\varepsilon z \tau(X_s^i) \gamma(X_s^i)} - 1| \pi_\lambda(dz) ds \right) < \infty$$

for any $\varepsilon \in (0, \varepsilon^*)$ with $\varepsilon^* = (\|\tau\|_{\infty, I} \|\gamma\|_{\infty})^{-1}$. So, $(\exp(Y_\varepsilon^i(\tau)_t))_{t \in [0, T]}$ is a local martingale by Applebaum [3], Corollary 5.2.2. In other words, there exists an increasing sequence of stopping times $(T_n^i)_{n \in \mathbb{N}}$ such

that $\lim_{n \rightarrow \infty} T_n^i = \infty$ a.s. and $(\exp(Y_\varepsilon^i(\tau)_{t \wedge T_n^i}))_{t \in [0, T]}$ is a martingale. Therefore, by Lebesgue's theorem and Markov's inequality, for every $\rho > 0$, the stochastic process $Y_{N, \varepsilon}(\tau) := Y_\varepsilon^1(\tau) + \dots + Y_\varepsilon^N(\tau)$ satisfies

$$\begin{aligned} \mathbb{P}(e^{Y_{N, \varepsilon}(\tau)T} > \rho) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\exp\left[\sum_{i=1}^N Y_\varepsilon^i(\tau)_{T \wedge T_n^i}\right] > \rho\right) \\ &\leq \frac{1}{\rho} \lim_{n \rightarrow \infty} \mathbb{E}(\exp(Y_\varepsilon^1(\tau)_{T \wedge T_n^1}))^N = \frac{1}{\rho} \mathbb{E}(\exp(Y_\varepsilon^1(\tau)_0))^N = \frac{1}{\rho}. \end{aligned}$$

Step 2. For any $\varepsilon \in (0, \varepsilon^*)$ and $t \in [0, T]$, let us find suitable bounds on

$$A_{N, \varepsilon}(\tau)_t := \sum_{i=1}^N A_\varepsilon^i(\tau)_t \quad \text{and} \quad B_{N, \varepsilon}(\tau)_t := \sum_{i=1}^N B_\varepsilon^i(\tau)_t.$$

On the one hand,

$$(7) \quad A_{N, \varepsilon}(\tau)_t \leq \frac{\varepsilon^2 \|\sigma\|_\infty^2}{2} \sum_{i=1}^N \int_0^t \tau(X_s^i)^2 ds \leq \frac{\varepsilon^2 \|\sigma\|_\infty^2 \|\tau\|_N^2 NT}{2}.$$

On the other hand, for every $\beta < 1$, by Taylor's formula and Assumption 4.2,

$$\begin{aligned} \int_{-\infty}^{\infty} (e^{\beta z} - \beta z - 1) \pi_\lambda(dz) &= \beta^2 \int_{-\infty}^{\infty} \left(\int_0^1 (1 - \theta) e^{\theta \beta z} d\theta \right) \pi_\lambda(dz) \\ &\leq \frac{\mathbf{c}_1}{2} \beta^2 \quad \text{with} \quad \mathbf{c}_1 = \int_{-\infty}^{\infty} e^{|z|} \pi_\lambda(dz) < \infty. \end{aligned}$$

Since $\varepsilon \in (0, \varepsilon^*)$, one can take $\beta = \varepsilon \tau(X_s^i) \gamma(X_s^i)$ for any $s \in [0, t]$ and $i \in \{1, \dots, N\}$, and then

$$(8) \quad B_{N, \varepsilon}(\tau)_t \leq \frac{\mathbf{c}_1 \varepsilon^2}{2} \sum_{i=1}^N \int_0^t \tau(X_s^i)^2 \gamma(X_s^i)^2 ds \leq \frac{\mathbf{c}_1 \varepsilon^2 \|\gamma\|_\infty^2 \|\tau\|_N^2 NT}{2}.$$

Therefore, Inequalities (7) and (8) lead to

$$A_{N, \varepsilon}(\tau)_t + B_{N, \varepsilon}(\tau)_t \leq \mathbf{c}_2 \varepsilon^2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) \|\tau\|_N^2 NT \quad \text{with} \quad \mathbf{c}_2 = \frac{1}{2} (1 \vee \mathbf{c}_1).$$

Step 3 (conclusion). Consider $M_N(\tau) := M^1(\tau) + \dots + M^N(\tau)$. For any $\varepsilon \in (0, \varepsilon^*)$ and $\xi, v > 0$, thanks to Step 2,

$$\begin{aligned} \mathbb{P}(\nu_N(\tau) \geq \xi, \|\tau\|_N^2 \leq v^2) &\leq \mathbb{P}(e^{\varepsilon M_N(\tau)T} \geq e^{NT\varepsilon\xi}, A_{N, \varepsilon}(\tau)_T + B_{N, \varepsilon}(\tau)_T \leq \mathbf{c}_2 \varepsilon^2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) NT v^2) \\ &\leq \mathbb{P}(e^{Y_{N, \varepsilon}(\tau)T} \geq \exp(NT\varepsilon\xi - \mathbf{c}_2 \varepsilon^2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) NT v^2)). \end{aligned}$$

Moreover, to take

$$\varepsilon = \frac{\xi}{2\mathbf{c}_2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) v^2 + \xi/\varepsilon^*} < \varepsilon^*$$

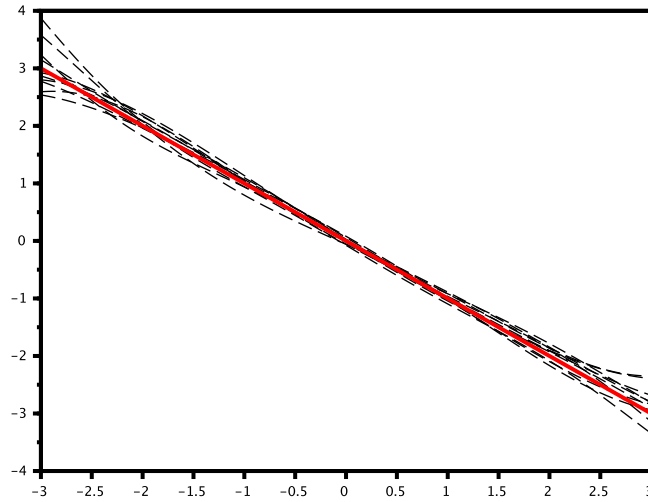
leads to

$$\begin{aligned} NT\varepsilon\xi - \mathbf{c}_2 \varepsilon^2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) NT v^2 &= \frac{NT\xi^2 [\mathbf{c}_2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) v^2 + \xi/\varepsilon^*]}{[2\mathbf{c}_2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) v^2 + \xi/\varepsilon^*]^2} \\ &\geq \frac{NT\xi^2}{4[\mathbf{c}_2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) v^2 + \xi/\varepsilon^*]}. \end{aligned}$$

Therefore, by Step 1,

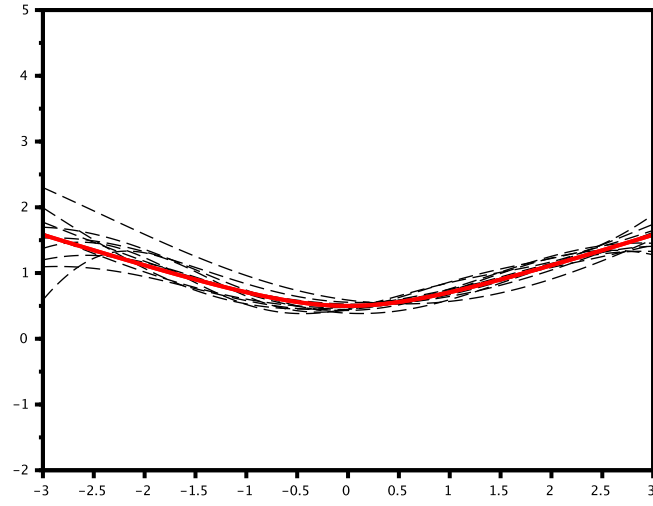
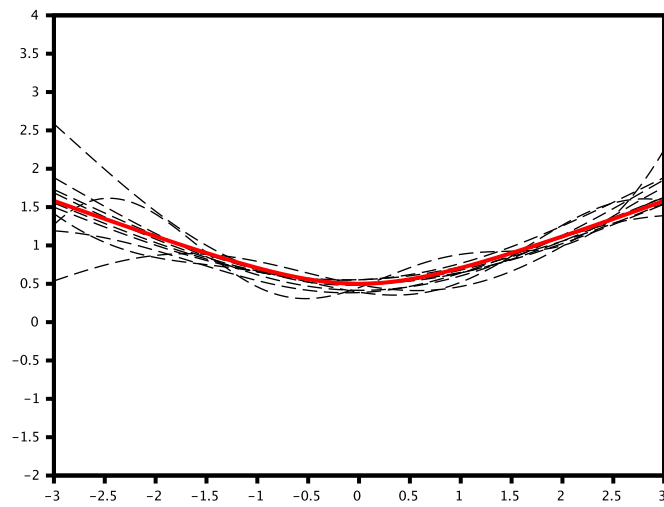
$$\mathbb{P}(\nu_N(\tau) \geq \xi, \|\tau\|_N^2 \leq v^2) \leq \exp\left(-\frac{NT\xi^2}{4[\mathbf{c}_2 (\|\sigma\|_\infty^2 + \|\gamma\|_\infty^2) v^2 + \xi\|\tau\|_{\infty, I}\|\gamma\|_\infty]}\right).$$

	Model 1	Model 2	Model 3
Mean MISE	0.1251	0.1469	0.1825
StD MISE	0.0950	0.1688	0.1928

TABLE 1. Means and StD of the MISE of $\widehat{b}_{\widehat{m}}$ (100 repetitions).FIGURE 1. Plots of b and of 10 adaptive estimations for Model 1 ($\widehat{m} = 5.3$).

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FIGURE 2. Plots of b and of 10 adaptive estimations for Model 2 ($\widehat{m} = 4.2$).FIGURE 3. Plots of b and of 10 adaptive estimations for Model 3 ($\widehat{m} = 4.1$).