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Aiman Mbarek, Sana Ben Hafsia

► **To cite this version:**

Aiman Mbarek, Sana Ben Hafsia. LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATORS WITH A PERTURBED WIGNER-VON NEUMANN POTENTIAL. 2022. hal-03826160

HAL Id: hal-03826160

<https://hal.science/hal-03826160>

Preprint submitted on 23 Oct 2022

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LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATORS WITH A PERTURBED WIGNER-VON NEUMANN POTENTIAL

MBAREK AIMAN, BEN HAFSIA SANA

October 23, 2022

Abstract.

Making use of the localised Putnam theory developed in [GJ1], applied in [JM],..., we show the limiting absorption principle for Dirac operators with a perturbed Wigner-Von Neumann potential on appropriate energy intervals. Furthermore we can allow longrange and short-range components.

1 Introduction

We will start this paper by discussing some commutator methods. The first stone was set by C.R. Putnam, see [P] and for instance [RSim][Theorem XIII.28]. Let H be a self-adjoint operator acting in a Hilbert space \mathcal{H} . One supposes there is a bounded operator A so that

$$C := [H, iA]^\circ > 0 \tag{1.0.1}$$

where " $>$ " means non-negative and injective. The commutator has to be understood in the form sense. When it extends into a bounded operator between some spaces, we denote this extension with the symbol \circ in subscript. The operator A is said to be conjugate to H . One deduces some estimation on the imaginary part of the resolvent, i.e., one finds some weight B , a closed injective operator with dense domain, so that

$$\sup_{\mathcal{R}z \in \mathbb{R}, \mathcal{I}z > 0} \langle f, (H - z)^{-1} f \rangle \leq \|Bf\|^2. \tag{1.0.2}$$

This estimation is equivalent to the global propagation estimate, c.f. [K] and [RSim][Theorem XIII.25]:

$$\int_{\mathbb{R}} \|B^{-1} e^{itH} f\|^2 dt \leq 2\|f\|^2. \tag{1.0.3}$$

One infers that the spectrum of H is purely and absolutely continuous with respect to the Lebesgue measure. E. Mourre has the idea to localise the estimates in energy and to allow a compact perturbation, see [M]. With further hypotheses, one shows an estimate of the resolvent (and not only on the imaginary part). The applications of this theory are numerous. The theory was finally improved in many directions and optimized in many ways, see [ABG] for a more thorough discussion of these matters, many papers have shown the power of Mourre's commutator theory to study the point and continuous spectra of a quite wide class of self-adjoint operators. Among others, we refer to [BG], [BFS], [BCHM],[CGH], [DJ], [FH], [GGéM], [GGo], [HuSi], [JMP], [Sa] and to the book [ABG]. One can also find parameter dependent versions of the theory (a semi-classical one for instance) in [RoT], [W], [WZ]. Recently it has been extended to (non self-adjoint) dissipative operators (cf., [BG], [Roy]).

We mention also [GGéM, Gé] for recent developments. In [MM] one generalizes the result of Putnam's approach. Under some conditions, one allows A to be unbounded. They obtain a global estimate of the resolvent. Note this implies the absence of eigenvalue. In [FS], in the non-relativistic context, by asking some positivity on the Virial of the potential, see below, one is able to conciliate

the estimation of the resolvent above the threshold energy and the accumulation of eigenvalues under it. In [R], one presents an abstract version of the method of [FS]. In [GJ], they introduced a new approach of Mourre's commutator theory, which is strongly inspired by results in semi-classical analysis (cf., [Bu], [CJ], [J1], [J2]). In [Gé], Gérard had a very close approach to that of [GJ]. These approaches furnished an alternative way to develop the original Mourre Theory and do not use differential inequalities. Our example is a perturbation of a Dirac operator with a Wigner-Von Neumann potential. Furthermore we can allow a long range perturbation and short-range. I want to quote that the T.A.L. of the Dirac operator was the objective of Golénia-Boussaid paper [BG] in another context.

Let I, J be two open intervals of \mathbb{R} . Given $k \in \mathbb{N}$, we say that $H \in \mathcal{C}_J^k(A)$ if for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with support in J , for all $f \in \mathcal{H}$, the map $\mathbb{R} \ni t \rightarrow e^{itA}\chi(H)e^{-itA}f \in \mathcal{H}$ has the usual \mathcal{C}^k regularity. Denote by $E_I(H)$ the spectral measure of H above I . We say that the Mourre estimate holds true for H on I if there exist $c > 0$ and a compact operator K such that

$$E_I(H)[H, iA]E_I(H) \geq E_I(H)(c + K)E_I(H), \quad (1.0.4)$$

in the form sense on $(\mathcal{D}(A) \cap \mathcal{D}(H)) \times (\mathcal{D}(A) \cap \mathcal{D}(H))$. In general, the l.h.s. of (1.0.4) does not extend, as a form, on $\mathcal{H} \times \mathcal{H}$ but it is the case if $H \in \mathcal{C}_J^1(A)$ and $I \subset J$ (cf., [Sa, GJ]). We say that the strict Mourre estimate holds true if the Mourre estimate (1.1) holds true with $K = 0$. In the first case (resp. the second case), it turns out that the point spectrum of H is finite (resp. empty) in compact sub-intervals I' of I if $H \in \mathcal{C}_J^k(A)$ and $I \subset J$. The main aim of Mourre's commutator theory is to show, when the strict Mourre estimate holds true for H on I , the following limiting absorption principle (LAP) on compact sub-intervals I' of I . Given such a I' and $s > \frac{1}{2}$, we say that the LAP, respectively to the triplet (I', s, A) , holds true for H if

$$\sup_{\mathcal{R}z \in I', \mathcal{I}z \neq 0} \|\langle A \rangle^{-s}(H - z)^{-1}\langle A \rangle^{-s}\| < +\infty, \quad (1.0.5)$$

where $\langle t \rangle = (1 + |t|^2)^{\frac{1}{2}}$, $\mathcal{I}z$ and $\mathcal{R}z$ are respectively the imaginary and the real of z . In that case, it turns out that the spectrum of H is purely absolutely continuous in I' (cf., Theorem XIII.20 in [RS4]). Notice that (1.2) holds true for $s = 0$ if and only if $I' \cap \sigma(H) = \emptyset$.

In [[ABG], [Sa]], such LAPs are derived under a slightly stronger regularity assumption than $H \in \mathcal{C}_J^1(A)$ with $I \subset J$. Actually, stronger results are proved. In particular, in the norm topology of bounded operators, one can defined the boundary values of the resolvent:

$$\mathcal{I}' \ni \lambda \mapsto \lim_{\epsilon \rightarrow 0^\pm} \langle A \rangle^{-s}(H - \lambda - i\epsilon)^{-1}\langle A \rangle^{-s} \quad (1.0.6)$$

and show some Hölder continuity for them.

Implicitly in [GJ] and explicitly in [Gé], one can derive, using $H \in \mathcal{C}_J^2(A)$ with $I \subset J$, the LAP (1.0.5) on compact sub-intervals I' of I from the Mourre estimate (1.0.4) with $K = 0$ via a strict, weighted Mourre estimate:

$$E_I(H)[H, i\varphi(A)]E_I(H) \geq c_1 E_I(H)\langle A \rangle^{-1-\epsilon} E_I(H), \quad (1.0.7)$$

where $\epsilon = 2s - 1 > 0$, c_1 is a strictly non-negative real, and φ is some appropriate non-negative, bounded, smooth function on \mathbb{R} . Note that the l.h.s. of (1.0.7) is a well defined form on $\mathcal{H} \times \mathcal{H}$. It seems that the use of such inequality to derive resolvent estimates appears in [J1] for the first time.

The new idea is to take the strict weighted Mourre estimate (1.0.7) as starting point, instead of the strict Mourre estimate. This costs actually less regularity of H w.r.t. A . Precisely, they show

Theorem 1.1 *Let I be a bounded, open interval of \mathbb{R} and assume that $H \in \mathcal{C}_I^1(A)$. Assume that, for some $\epsilon_0 > 0$, for any $\epsilon \in (0; \epsilon_0]$, there exist some real borelian bounded function φ such that the strict, weighted Mourre estimate, i.e. (1.0.7), holds true. Then, for any $s > \frac{1}{2}$ and for any closed sub-interval I' of I , the LAP (1.0.5) for H respectively to (I', s, A) holds true.*

Remark 1.2 *Notice that the LAP (1.0.5) for H respectively to (I', s, A) implies the LAP (1.0.5) for H respectively to (I', s', A) , for any $s' \geq s$. Therefore, it is enough to prove 1.1 for s close to $\frac{1}{2}$.*

Remark 1.3 *By using Gérard's energy method in [Gé], they can upper bound the size of the l.h.s. of (1.0.5) in terms of the constant c_1 appeared in (1.0.7).*

The weighted Mourre's theory is essentially a part of the variant Mourre's theory in [Gé, GJ]. It is simpler than the usual Mourre's theory (it does not use differential inequalities).

Acknowledgement: The authors thank S. Golénia, who drew our attention to the fact that a formula of the type (2.1.10) should exist and would be very interesting and useful. We also thank T. Jecko for guiding, abetting and counseling me throughout the preparation of this article.

2 SOME RESULTS ON THE HELFFER-SJÖSTRAND FORMULA .

The first objective of this section is to give a formula for unitary operators that corresponds to the Helffer-Sjöstrand formula for self-adjoint operators. Secondly, we generalize the standard Helffer-Sjöstrand formula for a vector of self-adjoint operators that commute two by two and we give some applications of this new formula in the expansion of commutator.

2.1 Regularity and Usual HELFFER-SJÖSTRAND formula.

We introduce some notation and recall known results. For details, we refer to [ABG], [DG], [GJ], [Sa] on regularity. For an interval I of \mathbb{R} , we denote by \bar{I} (resp. $\circ I$) its closure (resp. its interior). The scalar product $\langle \cdot, \cdot \rangle$ in \mathcal{H} is right linear and $\| \cdot \|$ denotes the corresponding norm and also the norm in $\mathcal{B}(\mathcal{H})$, the space of bounded operators on \mathcal{H} . Let A be a self-adjoint operator. Let T be a closed operator. The form $[T, A]$ is defined on $(\mathcal{D}(A) \cap \mathcal{D}(T)) \times (\mathcal{D}(A) \cap \mathcal{D}(T))$ by

$$\langle f, [T, A]g \rangle = \langle T^* f, Ag \rangle - \langle Af, Tg \rangle. \quad (2.1.1)$$

If T is a bounded operator on \mathcal{H} and $k \in \mathbb{N}$, we say that $T \in \mathcal{C}^k(A)$ if, for all $f \in \mathcal{H}$, the map $\mathbb{R} \ni t \rightarrow e^{itA} T e^{-itA} f \in \mathcal{H}$ has the usual \mathcal{C}^k regularity. The following characterizations are available.

Proposition 2.1 ([ABG], p. 250). *Let $T \in \mathcal{B}(\mathcal{H})$. These three assertions are equivalent:*

- (1). $T \in \mathcal{C}^1(A)$.
- (2). *The form $[T, A]$ defined on $\mathcal{D}(A) \times \mathcal{D}(A)$ extends to a bounded form on $\mathcal{H} \times \mathcal{H}$ associated to a bounded operator denoted by $ad_A^1(T) := [T, A]^\circ$.*
- (3). *T preserves $\mathcal{D}(A)$ and the operator $TA - AT$, defined on $\mathcal{D}(A)$, extends to a bounded operator on \mathcal{H} .*

It follows that $T \in \mathcal{C}^k(A)$ if and only if the iterated commutators $ad_A^p(T) := [ad_A^{p-1}(T), A]^\circ$ are bounded for $p \leq k$. In particular, for $T \in \mathcal{C}^1(A)$, $T \in \mathcal{C}^2(A)$ if and only if $[T, A]^\circ \in \mathcal{C}^1(A)$.

Let H be a self-adjoint operator and I be an open interval. As it is mentioned in our introduction, we say that H is locally of class $\mathcal{C}^k(A)$ on I , we write $H \in \mathcal{C}_I^k(A)$, if, for all $\varphi \in \mathcal{C}_c^\infty(I)$, $\varphi(H) \in \mathcal{C}^k(A)$.

It turns out that $T \in \mathcal{C}^k(A)$ if and only if, for a z outside $\sigma(T)$, the spectrum of T , $(T-z)^{-1} \in \mathcal{C}^k(A)$. It is natural to say that $H \in \mathcal{C}^k(A)$ if $(H-z)^{-1} \in \mathcal{C}^k(A)$ for some $z \notin \sigma(H)$. In that case, $(H-z)^{-1} \in \mathcal{C}^k(A)$, for all $z \notin \mathbb{R}$. This regularity is stronger than the local one that it will be asserted in the following proposition:

Proposition 2.2 ([ABG], p. 244) *If $H \in \mathcal{C}^k(A)$ then $H \in \mathcal{C}_I^k(A)$ for all open interval I of \mathbb{R} .*

For $\rho \in \mathbb{R}$, let \mathcal{S}^ρ be the class of functions $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\forall k \in \mathbb{N}, \mathcal{C}^k(\varphi) := \sup_{\{t \in \mathbb{R}\}} \langle t \rangle^{-\rho+k} |\varphi^{(k)}(t)| < \infty. \quad (2.1.2)$$

Here $\varphi^{(k)}$ denotes the k -th derivative of φ . Equipped with the semi-norms defined by (2.1.2), \mathcal{S}^ρ is a Fréchet space. We recall the following result from [DG] in almost analytic extension.

Proposition 2.3 ([DG]) *Let φ be a smooth function with compact support in \mathbb{R} . Then there exist a function $\varphi^{\mathbb{C}} \in \mathcal{C}_0^\infty(\mathbb{C})$, called an almost analytic extension of f , such that there exist $0 < C < 1$,*

$$\varphi^{\mathbb{C}}|_{\mathbb{R}} = \varphi, \text{ supp } \varphi^{\mathbb{C}} \subset \{x + iy \in \mathbb{C} ; x \in \text{supp } \varphi ; |y| \leq C\} \quad (2.1.3)$$

$$\varphi^{\mathbb{C}}(x + iy) = 0, \text{ if } x \in \text{supp } \varphi. \quad (2.1.4)$$

We have the following Helffer-Sjöstrand's functional calculus (cf., [HeS, DG]) : If A a self-adjoint operator on a Hilbert space \mathcal{H} , we have

$$\varphi(A) = \frac{1}{2i\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - A)^{-1} dz \wedge d\bar{z}, \quad (2.1.5)$$

and for all $k \in \mathbb{N}$,

$$\varphi^{(k)}(A) = \frac{k!}{2i\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - A)^{-1-k} dz \wedge d\bar{z}, \quad (2.1.6)$$

where $dz \wedge d\bar{z}$ is the Lebesgue measure on the complex plane and the integral exists in the norm topology, by (2.1.3). This formula also holds true for a larger class of functions f having some prescribed behaviour at infinity (See. {[DG],[GJ],...}). For instance (See [DG]), one can require, for $\varphi \in \mathcal{S}^\rho$ with $\rho \geq 0$. We rely on the following approximation:

Proposition 2.4 ([DG]) *Let $\rho \geq 0$ and $\varphi \in \mathcal{S}^\rho$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi = 1$ near 0, $0 \leq \chi \leq 1$, and for $R > 0$, $\chi_R(t) = \chi(t/R)$. For $f \in \mathcal{D}(\langle A \rangle^\rho)$, there exist*

$$\varphi^{(k)}(A)f = \lim_{R \rightarrow +\infty} \frac{k!}{2i\pi} \int_{\mathbb{C}} \partial_{\bar{z}} (\varphi \chi_R)^{\mathbb{C}}(z) (z - A)^{-1-k} f dz \wedge d\bar{z}, \quad (2.1.7)$$

The r.h.s. converges for the norm in \mathcal{H} . It is independent of the choice of χ .

Notice that, for some $c > 0$ and $s \in [0, 1]$, there exist some $C > 0$ such that, for all $z = x + iy \in \{a + ib | 0 < |b| \leq c\langle a \rangle\}$

$$\|\langle A \rangle^s (A - z)^{-1}\| \leq C \langle x \rangle^s \cdot |y|^{-1} \quad (2.1.8)$$

Observing that the self-adjointness assumption on B is useless, we pick from [GJ] the following result in two parts

Proposition 2.5 ([GJ]) *Let $k \in \mathbb{N}^*$, $\rho < k$, $\varphi \in \mathcal{S}^\rho$, and B be a bounded operator in $\mathcal{C}^k(A)$. As forms on $\mathcal{D}(\langle A \rangle^{k-1}) \times \mathcal{D}(\langle A \rangle^{k-1})$,*

$$\begin{aligned} [\varphi(A), B] &= \sum_{j=1}^{k-1} \frac{1}{j!} \varphi^{(j)}(A) ad_A^j(B) \\ &+ \frac{1}{2i\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - A)^{-k} ad_A^k(B) (z - A)^{-1} dz \wedge d\bar{z}, \end{aligned} \quad (2.1.9)$$

In particular, if $\rho \leq 1$, then $B \in \mathcal{C}^1(\varphi(A))$. The rest of the previous expansion is estimated in

Proposition 2.6 ([GJ]) *Let $B \in \mathcal{C}^k(A)$ bounded. Let $\varphi \in \mathcal{S}^\rho$, with $\rho < k$. Let $I_k(\varphi)$ be the rest of the development of order k (2.1.9) of $[\varphi(A), B]$. Let $s, s' \geq 0$ such that $s' < 1, s < k$, and $\rho + s + s' < k$. Then, for φ staying in a bounded subset of \mathcal{S}^ρ , $\langle A \rangle^s I_k(\varphi) \langle A \rangle^{s'}$ is bounded and there exist a A and φ independent constant $C > 0$ such that $\|\langle A \rangle^s I_k(\varphi) \langle A \rangle^{s'}\| \leq C \|ad_A^k(B)\|$.*

We refer to [BG] for some generalizations of Propositions 2.5 and 2.6 to the case where B is unbounded and $[A, B]^\circ$ is bounded.

A second application is provided in [GN]. Given a self-adjoint operator A on \mathcal{H} and a smooth function φ with compact support in \mathbb{R} , one gets a control on the norm $\|\varphi(A)\|_{\mathcal{B}(\mathcal{H})}$ in terms of the norm of the resolvent of A , $\|(A - z_0)^{-1}\|_{\mathcal{B}(\mathcal{H})}$, for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$. There are also results of this kind where $\mathcal{B}(\mathcal{H})$ is replaced by a Schatten class norm.

Let us mention one last application in the context of linear PDE. For $\zeta > 0$, let $V(Q)$ be the multiplication operator by a function, with $V : \mathbb{R}^d \rightarrow \mathbb{R}, x \mapsto V(x)$, such that $x \mapsto |\langle x \rangle^\zeta V|$ is bounded, where $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$. Let φ be a smooth function with compact support in \mathbb{R} and let

$H_0 := -\Delta$ be the positive Laplacian on \mathbb{R}^d . Let $H_1 = H_0 + V = -\Delta + V$ be the self-adjoint realization on $\mathbf{L}^2(\mathbb{R}^d)$. By the Helffer-Sjöstrand formula one can show that

$$((\varphi(H_1) - \varphi(H_0))\langle Q \rangle^\epsilon \langle H_0 \rangle^\alpha)_{0 \leq \epsilon < \zeta, 0 \leq \alpha < 1}$$

is a family of compact operators.

Our objective is to find a kind of Helffer-Sjöstrand formula for unitary operators U and for smooth functions defined in the 1-dimensional sphere \mathbb{S}^1 . Since U is a unitary operator, its spectrum is contained in \mathbb{S}^1 and the functional calculus is well defined. Here we shall take a smooth function $f : \mathbb{S}^1 \rightarrow \mathbb{C}$ with compact support in the interior of $\mathbb{S}^1 \setminus \{1\}$. The latter condition corresponds to the condition of compact support for the Helffer-Sjöstrand formula. For a unitary operator on some Hilbert space \mathcal{H} , we shall derive the formula

$$f(U) = (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} f_{\mathbb{S}^1}^{\mathbb{C}}(z) (z - U)^{-1} dz \wedge d\bar{z}, \quad (2.1.10)$$

where $f_{\mathbb{S}^1}^{\mathbb{C}}$ is some almost analytic extension of f , $dz \wedge d\bar{z}$ is the Lebesgue measure on the complex plane and $f(U)$ is given by the functional calculus for the unitary operator U . We will show the formula (2.1.10) in Theorem 2.19. We expect that our formula (2.1.10) holds true under a weaker assumption on the behaviour of f near 1.

The proof is organized as follows. First, we formulate the theorem of Helffer-Sjöstrand and we give a slightly different proof from that of [HeS], afterwards, we recall some properties on the Cayley transform and we prove some complex analysis results, and finally we prove our theorem.

Proof : "Proposition 2.3" we follow the argument of [DG], checking that we can ensure the formula (2.1.3) with $C < 1$.

Let $M_n := \sup_{0 \leq k \leq n} \sup_{x \in \mathbb{R}} |\varphi^{(k)}(x)|$, for all $n \in \mathbb{N}$. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi = 1$ on $[-1/2, 1/2]$ and $\chi = 0$ on $\mathbb{R} \setminus [-1, 1]$. We choose

$$\varphi^{\mathbb{C}}(z) = \varphi^{\mathbb{C}}(x + iy) = \sum_{n=0}^{+\infty} \frac{(iy)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{y}{T_n}\right), \quad (2.1.11)$$

for a decreasing and positive sequence $(T_n)_n$ satisfying, $T_0 < 1$ and

$$\forall n \in \mathbb{N}, \quad T_n M_{2n} \leq 2^{-n}. \quad (2.1.12)$$

For example we can take $(T_n)_n$ defined by $0 < T_0 < 1$ and $T_n = \min(T_{n-1}, \frac{1}{2^n M_{2n}})$, for $n \geq 1$. With this choice we will see that our sum (2.1.11) is uniformly convergent. Then, for x real,

$$\varphi^{\mathbb{C}}(x) = \sum_{n=0}^{+\infty} \frac{(i0)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{0}{T_n}\right) = \frac{(i0)^0}{0!} \varphi(x) \chi\left(\frac{0}{T_0}\right) = \varphi(x).$$

We deduce that $\varphi^{\mathbb{C}}$ is an extension of φ to \mathbb{C} . Let $C := T_0$. Let $H_{\varphi, C} := \{x + iy; x \in \text{supp}\varphi, |y| \leq C\}$. If $x_0 + iy_0 \in \mathbb{C}$ such that the distance $d(x_0 + iy_0, H_{\varphi, C}) > 0$, then $x_0 \notin \text{supp}\varphi$ or $|y_0| > C$. If $x_0 \notin \text{supp}\varphi$, therefore $\varphi^{(n)}(x_0) = 0, n \geq 0$ for all $n \in \mathbb{N}$, and $\varphi^{\mathbb{C}}(x_0 + iy_0) = 0$. If $|y_0| > C$, then for all $n \in \mathbb{N}, |y_0| > C \geq T_n$, since $(T_n)_n$ decreases. Thus, for all $n, \chi(\frac{y_0}{T_n}) = 0$ and $\varphi^{\mathbb{C}}(x_0 + iy_0) = 0$, we get (2.1.4).

To show that our sum (2.1.11) exists and belongs to $C^\infty(\mathbb{C})$, we simply have to show that, for $(p, q) \in \mathbb{N}^2$,

$$\sum_{n \in \mathbb{N}} \partial_x^p \partial_y^q \left(\frac{(iy)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{y}{T_n}\right) \right) \quad (2.1.13)$$

is uniformly convergent.

Let $(p, q) \in \mathbb{N}^2$ and we take $n \geq \max(p, q)$. Using the Leibnitz formula we have

$$\begin{aligned} & \left| \partial_x^p \partial_y^q \left(\frac{(iy)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{y}{T_n}\right) \right) \right| \\ &= \left| i^n (\partial_x^{p+n}) \varphi(x) \sum_{r=0}^q \frac{C_q^r}{(n-q+r)! T_n^r} y^{n-q+r} \chi^{(r)}\left(\frac{y}{T_n}\right) \right|, \end{aligned}$$

where $C_q^r = \frac{q!}{r!(q-r)!}$, $0 \leq r \leq q$. For all $0 \leq r \leq q \leq n$, $\chi^{(r)}\left(\frac{\cdot}{T_n}\right)$ is supported in $\{0 \leq |\cdot| \leq T_n\}$. Let $x, y \in \mathbb{R}$, we have

$$\begin{aligned} \left| \partial_x^p \partial_y^q \left(\frac{(iy)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{y}{T_n}\right) \right) \right| &\leq M_{p+n} \sum_{r=0}^q \frac{C_q^r}{(n-q+r)! T_n^r} |y|^{n-q+r} |\chi^{(r)}\left(\frac{y}{T_n}\right)| \\ &\leq M_{p+n} T_n^{n-q} \sum_{r=0}^q \frac{C_q^r}{(n-q+r)!} \|\chi^{(r)}\|_\infty \\ &\leq D_q M_{p+n} T_n^{n-q}. \end{aligned}$$

As $T_n < 1$, we have $T_n^{n-q} M_{p+n} \leq T_n M_{2n} \leq 2^{-n}$, by (2.1.12). Thus

$$\sup_{x,y} \left| \partial_x^p \partial_y^q \left(\frac{(iy)^n}{n!} \varphi^{(n)}(x) \chi\left(\frac{y}{T_n}\right) \right) \right| \leq D_q 2^{-n}.$$

This yields the normal convergence of the series (2.1.13). In particular, $\varphi^{\mathbb{C}} \in \mathcal{C}^\infty(\mathbb{C})$.

It remains to show (2.1.3). For $x, y \in \mathbb{R}$

$$\begin{aligned} 2\partial_{\bar{z}}\varphi^{\mathbb{C}}(z) &= (\partial_x + i\partial_y)\varphi^{\mathbb{C}}(x + iy) \\ &= \sum_{n=0}^{+\infty} \left(\frac{(iy)^n}{n!} \varphi^{(n+1)}(x) \chi\left(\frac{y}{T_n}\right) + i \frac{i^n}{(n-1)!} \varphi^{(n)}(x) y^{n-1} \chi\left(\frac{y}{T_n}\right) \right. \\ &\quad \left. + i \frac{(iy)^n}{n!} \varphi^{(n)}(x) \frac{1}{T_n} \chi'\left(\frac{y}{T_n}\right) \right) \\ &= \sum_{n=0}^{+\infty} \frac{(iy)^n}{n!} \varphi^{(n+1)}(x) \chi\left(\frac{y}{T_n}\right) + i \sum_{n=0}^{+\infty} \frac{\varphi^{(n)}(x)}{(n-1)!} \left(i^n n y^{n-1} \chi\left(\frac{y}{T_n}\right) \right. \\ &\quad \left. + (iy)^n \frac{1}{T_n} \chi'\left(\frac{y}{T_n}\right) \right) \\ &= \sum_{n=0}^{+\infty} \frac{(iy)^n}{n!} \varphi^{(n+1)}(x) \left(\chi\left(\frac{y}{T_n}\right) - \chi\left(\frac{y}{T_{n+1}}\right) \right) \\ &\quad + \sum_{n=0}^{+\infty} \frac{\varphi^{(n)}(x)}{n! T_n} i^{n+1} y^n \chi'\left(\frac{y}{T_n}\right). \end{aligned}$$

Let $l \in \mathbb{N}$. Denoting by $\mathbf{1}_I$ the characteristic function of I , we have, for $y \neq 0$,

$$\begin{aligned} 2\| |y|^{-l} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) \| &\leq \sum_{n=0}^{+\infty} M_{n+1} |y|^{n-l} 2\|\chi\|_\infty \mathbf{1}_{\{T_{n+1}/2 \leq |\cdot| \leq T_n\}}(y) \\ &\quad + \sum_{n=0}^{+\infty} \frac{M_n}{T_n} |y|^{n-l} \|\chi'\|_\infty \mathbf{1}_{\{T_n/2 \leq |\cdot| \leq T_n\}}(y) \\ &\leq C \sum_{n=1}^l M_{n+1} \left(\frac{T_{n+1}}{2}\right)^{n-l} + C' \sum_{n=l+1}^{+\infty} M_{n+1} (T_n)^{n-l} \\ &\quad + C'' \sum_{n=0}^l \frac{M_n}{T_n} \left(\frac{T_n}{2}\right)^{n-l} + C''' \sum_{n=l+1}^{+\infty} \frac{M_n}{T_n} (T_n)^{n-l} \\ &\leq C'_l + D \sum_{n=l+2}^{+\infty} (M_n + M_{n+1}) T_n. \end{aligned}$$

By (2.1.12) and the definition of $(M_n)_n$, we have $(M_n + M_{n+1}) T_n \leq 2T_n M_{2n} \leq 2 \cdot 2^{-n}$. This yields (2.1.3). \blacksquare

Proposition 2.7 [H] Let ω be an open set in the complex plane \mathbb{C} and $f \in \mathcal{C}^1(\omega)$. For all $\xi \in \omega$, the second integral in (2.1.14) is convergent. Moreover, for all $\xi \in \omega$,

$$f(\xi) = (2i\pi)^{-1} \left\{ \int_{\partial\omega} \frac{f(z)}{z - \xi} dz + \int_{\omega} \frac{\partial_{\bar{z}} f(z)}{z - \xi} dz \wedge d\bar{z} \right\}. \quad (2.1.14)$$

Corollary 2.8 If $f \in \mathcal{C}^\infty(\mathbb{C})$ with compact support in ω then, for all $\xi \in \omega$,

$$f(\xi) = (2i\pi)^{-1} \int_{\omega} \frac{\partial_{\bar{z}} f(z)}{z - \xi} dz \wedge d\bar{z}. \quad (2.1.15)$$

Proof : Just apply the proposition 2.7 and use the fact that f is zero on $\partial\omega$. ■

We will recall some results of the theory of normal operators. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a compact supported function in K and let N be a normal operator on a Hilbert space \mathcal{H} .

Proposition 2.9 [Co] For z outside $\sigma(N)$, the spectrum of the operator N , we have

$$\|(z - N)^{-1}\|_{\mathcal{B}(\mathcal{H})} = \frac{1}{d(z, \sigma(N))}.$$

Proposition 2.10 [Co] Let K be a compact of \mathbb{C} . Let, for $n \in \mathbb{N}$, $f_n \in \mathcal{C}^\infty(\mathbb{C})$ with $\text{supp } f_n \subset K$ and $f \in \mathcal{C}^\infty(\mathbb{C})$ with $\text{supp } f \subset K$ such that $(f_n)_n$ converges to f , uniformly on K . Then $(f_n(N))_n$ converges to $f(N)$ for the operator norm.

Proof : By combining the lemma 1.9 page 257 and Theorem 4.7 page 321 in the book [Co], we found the following inequality

$$\|f_n(N) - f(N)\|_{\mathcal{B}(\mathcal{H})} \leq \|f_n - f\|_{\infty, K} := \sup_{x \in K} |f_n(x) - f(x)|.$$

Therefore by passing to the limit we conclude

$$\lim_{n \rightarrow +\infty} \|f_n(N) - f(N)\|_{\mathcal{B}(\mathcal{H})} = 0.$$

If we replace u by an almost analytic extension satisfying $\omega = \mathbb{C}$ and the corollary 2.8 and (2.1.14), the generalized integral in (2.1.15) converges uniformly, w.r.t. ξ , as proved in the following proposition 2.10. ■

For a map $k : \mathbb{C} \rightarrow \mathbb{C}$ and $D \subset \mathbb{C}$, we define

$$\|k\|_{\infty, D} := \sup_{x \in D} |k(x)|. \quad (2.1.16)$$

Proposition 2.11 Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a compact supported function in K and $\varphi^{\mathbb{C}}$ be an almost analytic extension of φ given by the proposition 2.3. We have the following uniform convergence on K

$$\lim_{n \rightarrow +\infty} \left\| \int_{|\text{Im}(z)| > 1/n} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - \cdot)^{-1} dz \wedge d\bar{z} - \varphi(\cdot) \right\|_{\infty, K} = 0. \quad (2.1.17)$$

Proof : Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a compact supported function in K . By the proposition 2.3, $\varphi^{\mathbb{C}}$ is an almost analytic extension of φ such that $\text{supp } \varphi^{\mathbb{C}} \subset K^{\mathbb{C}}$, where $K^{\mathbb{C}} := \{(x, y) \in \mathbb{R}^2; x \in K, |y| \leq C < 1\}$. Let ω be an open in the complex plane \mathbb{C} that contains $K^{\mathbb{C}}$. From the corollary 2.8 with $u = \varphi^{\mathbb{C}}$, we have, for all $\xi \in K$,

$$\varphi(\xi) = (2i\pi)^{-1} \int_{\mathbb{C}} \frac{\partial_{\bar{z}} \varphi^{\mathbb{C}}(z)}{z - \xi} dz \wedge d\bar{z}. \quad (2.1.18)$$

Using the (2.1.15) with $l = 1$, we have, for $\xi \in K$,

$$\int_{K^{\mathbb{C}}} |\partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - \xi)^{-1}| dz \wedge d\bar{z} \leq C_1 \int_{K^{\mathbb{C}}} |\mathcal{I}z| |\mathcal{I}z|^{-1} dz \wedge d\bar{z} < +\infty. \quad (*)$$

Thus this integral (2.1.18) is absolutely convergent. Now we will show the uniform convergence on K , let $\xi \in K$

$$\begin{aligned} & \left| \int_{|\mathcal{I}z| > 1/n} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z)(z - \xi)^{-1} dz \wedge d\bar{z} - \varphi(\xi) \right| \\ &= \left| \int_{|\mathcal{I}z| < 1/n} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z)(z - \xi)^{-1} dz \wedge d\bar{z} \right| \\ &\leq C_1 \int_{|\mathcal{I}z| < 1/n} |\mathcal{I}z| |\mathcal{I}z|^{-1} dz \wedge d\bar{z} \\ &\leq \frac{C}{n}. \end{aligned}$$

where C is independent of ξ . Therefore

$$\sup_{\xi \in K} \left| \int_{|\mathcal{I}z| > 1/n} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z)(z - \xi)^{-1} dz \wedge d\bar{z} - \varphi(\xi) \right| < \frac{C}{n},$$

proves (2.1.17). ■

We are able to reprove Helffer-Sjöstrand theorem.

Theorem 2.12 [HeS] Let $\varphi \in C^\infty(\mathbb{R})$ be a function with compact support K and $\varphi^{\mathbb{C}}$ be an almost analytic extension of f given by the proposition 2.3. Let A be a self-adjoint operator on a Hilbert space \mathcal{H} . The integral

$$\int_{\mathbb{C}} \|\partial_{\bar{z}} \varphi^{\mathbb{C}}(z)(z - A)^{-1}\|_{\mathcal{B}(\mathcal{H})} dz \wedge d\bar{z} \quad (**)$$

converges, the integral in the following formula converges in operator norm in $\mathcal{B}(\mathcal{H})$, and we have

$$\varphi(A) = (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z)(z - A)^{-1} dz \wedge d\bar{z}. \quad (2.1.19)$$

Proof : By the proposition 2.9, (*) holds true with ξ replaced by A . This proves the convergence of (**). In particular, the integral in (2.1.19) converges in the operator norm of $\mathcal{B}(\mathcal{H})$. By (2.1.17) and the proposition 2.10 we get (2.1.19). ■

2.2 Some complex analysis results

We give now some properties on the Cayley transform and we prove a known result in complex analysis that will be used in the proof of our main theorem.

We will need some estimates on the Cayley transform on specific regions.

Definition 2.13 The Cayley transform is the map

$$\begin{aligned} \psi : \mathbb{C} \setminus \{i\} &\longrightarrow \mathbb{C} \setminus \{1\} \\ z &\longmapsto \psi(z) = \frac{z+i}{z-i}. \end{aligned}$$

We denote by \mathbb{S}^1 the unit sphere in \mathbb{C} , $D(0, 1)$ the open disk with center the origin and of radius 1, $\overline{D(0, 1)}$ the closure of $D(0, 1)$, $\mathbb{C}^+ = \{z \in \mathbb{C}; \mathcal{I}z > 0\}$ and $\mathbb{C}^- = \{z \in \mathbb{C}; \mathcal{I}z < 0\}$.

Proposition 2.14 [H] ψ is an analytic, bijective function and ψ^{-1} is given by $\psi^{-1}(\xi) = i \frac{\xi+1}{\xi-1}$, for all $\xi \in \mathbb{C} \setminus \{1\}$. Furthermore,
 $\psi(\mathbb{R}) = \mathbb{S}^1 \setminus \{1\}$, $\psi(\mathbb{C}^+ \setminus \{i\}) = \mathbb{C} \setminus \{\overline{D(0, 1)}\}$, $\psi(\mathbb{C}^-) = D(0, 1)$,
 $\psi(0) = -1$, $\psi(-1) = i$, $\lim_{z \rightarrow \infty} \psi(z) = 1$, $\lim_{z \rightarrow i} |\psi(z)| = \infty$.

Let $a, b, c \in \mathbb{R}$ be such that $a < b$ and $0 \leq c < 1$. We define

$$\Omega := \psi([a, b] \times [-c, c]).$$

Lemma 2.15 *There exist $C > 0$, such that*

$$\|\partial_x(\psi^{-1})\|_{\infty, \Omega} \leq C, \|\partial_y(\psi^{-1})\|_{\infty, \Omega} \leq C, \quad (2.2.1)$$

and, for all $\xi \in \Omega$,

$$|\mathcal{I}\psi^{-1}(\xi)| \leq Cd(\xi, \mathbb{S}^1).$$

where $\|\cdot\|_{\infty, \Omega}$ defined in (2.1.16) and $d(\xi, \mathbb{S}^1)$ is the distance between ξ and \mathbb{S}^1 .

Proof : Let $\xi \in \Omega$. Then $|\mathcal{I}\psi^{-1}(\xi)| = \left| \frac{|\xi|^2 - 1}{|\xi - 1|^2} \right|$,

$$\frac{|\mathcal{I}\psi^{-1}(\xi)|}{d(\xi, \mathbb{S}^1)} = \frac{|\xi| + 1}{|\xi - 1|^2}.$$

Call $C_\Omega = d(1, \Omega)$, $d_\Omega = \sup_{\xi \in \Omega} |\xi|$. We have, for all $\xi \in \Omega$,

$$C_1 := \frac{1}{(2d_\Omega)^2} \leq \frac{|\mathcal{I}(\psi^{-1}(\xi))|}{d(\xi, \mathbb{S}^1)} = \frac{|\xi| + 1}{|\xi - 1|^2} \leq \frac{d_\Omega + 1}{C_\Omega^2} =: C_2.$$

$\partial_x(\psi^{-1}) = \frac{-2i}{(\xi-1)^2}$ and $\partial_y(\psi^{-1}) = \frac{2}{(\xi-1)^2}$. As there exist $r > 0$, such that, for all $\xi \in \Omega$, $|\xi - 1| \geq r$, we get (2.2.1). \blacksquare

Lemma 2.16 *Let h be an analytic function on \mathbb{C} and g be a smooth function on $\mathbb{C} \simeq \mathbb{R}^2$. We have*

$$\forall z \in \mathbb{C}, \quad \partial_{\bar{z}}(g \circ h)(z) = (\partial_{\bar{z}}g)(h(z))\partial_{\bar{z}}\overline{h(z)}.$$

Proof : Writing $z = x + iy$ and $h(z) = h_1(x, y) + ih_2(x, y)$, we have

$$\begin{aligned} 2\partial_{\bar{z}}(g \circ h)(z) &= \partial_x(g \circ h)(z) + i\partial_y(g \circ h)(z) \\ &= (\partial_xg)(h(z))\partial_xh_1(x, y) + (\partial_yg)(h(z))\partial_xh_2(x, y) \\ &\quad + i((\partial_xg)(h(z))\partial_yh_1(x, y) + (\partial_yg)(h(z))\partial_yh_2(x, y)) \\ &= ((\partial_z + \partial_{\bar{z}})g)(h(z))\partial_xh_1(x, y) + \left(\frac{\partial_{\bar{z}} - \partial_z}{i}\right)g(h(z))\partial_xh_2(x, y) \\ &\quad + i(((\partial_z + \partial_{\bar{z}})g)(h(z))\partial_yh_1(x, y) + \left(\frac{\partial_{\bar{z}} - \partial_z}{i}\right)g(h(z))\partial_yh_2(x, y)) \\ &= (\partial_zg)(h(z))(\partial_xh_1(x, y) + i\partial_xh_2(x, y) + i\partial_yh_1(x, y) - \partial_yh_2(x, y)) \\ &\quad + (\partial_{\bar{z}}g)(h(z))(\partial_xh_1(x, y) - i\partial_xh_2(x, y) + i\partial_yh_1(x, y) + \partial_yh_2(x, y)). \end{aligned}$$

As h is analytic, h satisfies the Cauchy Riemann relations $\partial_xh_1(x, y) = \partial_yh_2(x, y)$ and $\partial_yh_1(x, y) = -\partial_xh_2(x, y)$, therefore

$$\begin{aligned} 2\partial_{\bar{z}}(g \circ h)(z) &= 2(\partial_zg)(h(z))(\partial_xh_1(x, y) + i\partial_yh_1(x, y)) \\ &= 2(\partial_zg)(h(z))\left(\frac{\partial_x + i\partial_y}{2}h_1(x, y) + \frac{\partial_x + i\partial_y}{2}h_1(x, y)\right) \\ &= 2(\partial_zg)(h(z))\left(\frac{\partial_x + i\partial_y}{2}h_1(x, y) + \frac{\partial_y - i\partial_x}{2}h_2(x, y)\right) \\ &= 2(\partial_zg)(h(z))\left(\frac{\partial_x + i\partial_y}{2}h_1(x, y) - i\frac{\partial_x + i\partial_y}{2}h_2(x, y)\right) \\ &= 2(\partial_zg)(h(z))\partial_{\bar{z}}\overline{h(z)}. \end{aligned}$$

\blacksquare

2.3 Helffer-Sjöstrand formula for Unitary Operators

We want a formula similar to (2.1.19), replacing the self-adjoint operator A by a unitary operator U . We start with some reminders on unitary operators.

Definition 2.17 U is called a unitary operator on a Hilbert space \mathcal{H} , if

$$UU^* = U^*U = I.$$

Proposition 2.18 *The spectrum of a unitary operator U on a Hilbert space \mathcal{H} is included in \mathbb{S}^1 .*

Proof : Let $z \in \mathbb{C}$ such that $|z| > 1$. We have $\|z^{-1}U\| < 1$, then $(I - z^{-1}U)$ is invertible, its inverse is bounded and it is given by the following series $\sum_{n=0}^{+\infty} (z^{-1}U)^n$. Therefore

$$z^{-1}(I - z^{-1}U)^{-1} = (z - U)^{-1}.$$

Let now $z \in \mathbb{C}$ be such that $|z| < 1$, we have $|zU^*| < 1$, then

$$U^*(zU^* - I)^{-1} = (zU^*U - U)^{-1} = (z - U)^{-1}.$$

This shows that the spectrum of U is included in \mathbb{S}^1 . ■

Theorem 2.19 Let $f \in C^\infty(\mathbb{S}^1)$ be supported away from 1 and U a unitary operator on a Hilbert space \mathcal{H} . Let $(f \circ \psi)^{\mathbb{C}}$ be an almost analytic extension of $f \circ \psi$ given by the proposition 2.3 such that $(f \circ \psi)^{\mathbb{C}}$ is supported in $\{x + iy; x \in \text{supp}(f \circ \psi), |y| \leq C\}$, $0 < C < 1$. Let $f_{\mathbb{S}^1}^{\mathbb{C}} = (f \circ \psi)^{\mathbb{C}} \circ \psi^{-1}$. The integral

$$\int_{\mathbb{C}} \|\partial_{\bar{z}} f_{\mathbb{S}^1}^{\mathbb{C}}(z)(z - U)^{-1}\|_{\mathcal{B}(\mathcal{H})} dz \wedge d\bar{z} \quad (2.3.1)$$

converges, the integral in the following formula converges in operator norm in $\mathcal{B}(\mathcal{H})$, and we have

$$f(U) = (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} f_{\mathbb{S}^1}^{\mathbb{C}}(z)(z - U)^{-1} dz \wedge d\bar{z}. \quad (2.3.2)$$

Proof : Let $f \in C^\infty(\mathbb{S}^1)$ be supported away from 1 and U a unitary operator on a Hilbert space \mathcal{H} . $f \circ \psi$ is a smooth function with compact support in \mathbb{R} . By the proposition 2.3, there exist an almost analytic extension $(f \circ \psi)^{\mathbb{C}}$ of $f \circ \psi$, such that $(f \circ \psi)^{\mathbb{C}}$ is supported in $\{x + iy; x \in \text{supp} f \circ \psi, |y| \leq C < 1\}$.

Let $f_{\mathbb{S}^1}^{\mathbb{C}} := (f \circ \psi)^{\mathbb{C}} \circ \psi^{-1}$. $f_{\mathbb{S}^1}^{\mathbb{C}}$ is defined on \mathbb{C} . It is a smooth function, supported in $\Omega_0 := \psi(\text{supp}(f \circ \psi) \times [-C, C])$.

For $\xi \in \mathbb{S}^1$, $f^{\mathbb{C}}(\xi) = (f \circ \psi)^{\mathbb{C}} \circ \psi^{-1}(\xi) = f \circ \psi \circ \psi^{-1}(\xi) = f(\xi)$, since $\psi^{-1}(\xi) \in \mathbb{R}$.

As in the proof of theorem 2.12 we show that (2.3.3) converges.

Making a call to lemma 2.16 with $h = \psi^{-1}$ and $g = (f \circ \psi)^{\mathbb{C}}$ and to proposition 2.3 with $l = 1$, and to lemma 2.15, we get the following uniform convergence on \mathbb{S}^1 ,

$$\lim_{n \rightarrow +\infty} \left\| \int_{d(z, \mathbb{S}^1) > 1/n} \partial_{\bar{z}} f_{\mathbb{S}^1}^{\mathbb{C}}(z)(z - \cdot)^{-1} dz \wedge d\bar{z} - f(\cdot) \right\|_{\infty, \mathbb{S}^1} = 0.$$

Therefore, by Proposition 2.10,

$$\int_{\mathbb{C}} \partial_{\bar{z}} f_{\mathbb{S}^1}^{\mathbb{C}}(z)(z - U)^{-1} dz \wedge d\bar{z}$$

converges in operator norm in $\mathcal{B}(\mathcal{H})$ and we have (2.3.4). ■

Theorem 2.20 Let $f \in C^\infty(\mathbb{S}^1)$ and U be a unitary operator on a Hilbert space \mathcal{H} . there exist two almost analytic functions $g^{\mathbb{C}}$ and $h^{\mathbb{C}}$ such that, the integral

$$\int_{\mathbb{C}} \|(|\partial_{\bar{z}} g^{\mathbb{C}}(z)| + |\partial_{\bar{z}} h^{\mathbb{C}}(z)|)(z - U)^{-1}\|_{\mathcal{B}(\mathcal{H})} dz \wedge d\bar{z} \quad (2.3.3)$$

converges, the integral in the following formula converges in operator norm in $\mathcal{B}(\mathcal{H})$, and we have

$$f(U) = (2i\pi)^{-1} \int_{\mathbb{C}} (\partial_{\bar{z}} g^{\mathbb{C}}(z) + \partial_{\bar{z}} h^{\mathbb{C}}(z))(z - U)^{-1} dz \wedge d\bar{z}. \quad (2.3.4)$$

Proof : Let $\chi_1, \chi_2 \in C^\infty(\mathbb{S}^1)$; $\text{supp } \chi_1 \subset \mathbb{S}^1 \setminus \{1\}$ and $\text{supp } \chi_2 \subset \mathbb{S}^1 \setminus \{-1\}$, such that $\chi_1 + \chi_2 = 1$. We have $f(x) = (\chi_1 f)(x) + (\chi_2 f)(x)$. As $\chi_1 f \in C^\infty(\mathbb{S}^1)$; $\chi_1 f = 0$ near to 1, by the theorem 2.19 we obtained

$$(\chi_1 f)(U) = (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}} g^{\mathbb{C}}(z)(z - U)^{-1} dz \wedge d\bar{z}.$$

We will study $(\chi_2 f)(U)$. By the same computation of the paragraph of Cayley transform with the following map $\psi : \mathbb{C} \setminus \{i\} \rightarrow \mathbb{C} \setminus \{-1\}; z \mapsto \tilde{\psi}(z) = \frac{z+i}{i-z}$, we can show that $\tilde{\psi}$ is an analytic bijective function and $\tilde{\psi}^{-1}$ is given by $\tilde{\psi}^{-1}(\xi) = i \frac{\xi-1}{\xi+1}$, for all $\xi \in \mathbb{C} \setminus \{-1\}$. Furthermore,

$$\tilde{\psi}(\mathbb{R}) = \mathbb{S}^1 \setminus \{-1\}, \tilde{\psi}(\mathbb{C}^+ \setminus \{i\}) = \mathbb{C} \setminus \{\overline{D(0,1)}\}, \tilde{\psi}(\mathbb{C}^-) = D(0,1), \tilde{\psi}(0) = 1,$$

$$\lim_{z \rightarrow \infty} \tilde{\psi}(z) = -1, \lim_{z \rightarrow i} |\tilde{\psi}(z)| = \infty.$$

For $a, b, c \in \mathbb{R}$ with $a < b$ and $0 \leq c < 1$, and $\tilde{\Omega} := \tilde{\psi}([a, b] \times [-c, c])$. there exist $C > 0$, such that $\|\partial_x(\tilde{\psi}^{-1})\|_{\infty, \tilde{\Omega}} \leq C, \|\partial_y(\tilde{\psi}^{-1})\|_{\infty, \tilde{\Omega}} \leq C$, and, for all $\xi \in \tilde{\Omega}, |\mathcal{I}\tilde{\psi}^{-1}(\xi)| \leq Cd(\xi, \mathbb{S}^1)$. As $\text{supp } \chi_2 f$ is away from -1 , using the same reasoning of the theorem 2.19, we can obtain the results of this theorem for $\chi_2 f$. ■

2.4 The Helffer-Sjöstrand formula for a vector of self-adjoint operators that commute two by two.

The objective in this part is to give a Helffer-Sjöstrand formula for a vector of self-adjoint operators that commute two by two. First, we will start with constructing a spectral measure associated with $f(H_1, \dots, H_n)$ and by the construction of almost analytic extensions of f which is a smooth function on \mathbb{R}^n with compact support.

Definition 2.21 Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a smooth function with compact support and H_1, H_2, \dots, H_n be a self-adjoint operators that commute two by two. We define

$$\begin{aligned} f(H_1, \dots, H_n) &:= \int_{\sigma(H_1) \times \dots \times \sigma(H_n)} f(\lambda_1, \dots, \lambda_n) d(E_{\lambda_1}(H_1) \otimes \dots \otimes E_{\lambda_n}(H_n)) \\ &:= \int_{\sigma(H_1) \times \dots \times \sigma(H_n)} f(\lambda_1, \dots, \lambda_n) dE_{\lambda_1}(H_1) \dots dE_{\lambda_n}(H_n). \end{aligned} \quad (2.4.1)$$

Proposition 2.22 Let f be a smooth function on \mathbb{R}^n with compact support. Then, there exist a smooth function $f^{\mathbb{C}} : \mathbb{C}^n \rightarrow \mathbb{C}$, called an almost analytic extension of f , $0 < C < 1$ such that for all $l = (l_1, \dots, l_n) \in \mathbb{N}^n$ there exist $C_l > 0$ that the following assertions hold true:

$$*f|_{\mathbb{R}} = f,$$

$$* \text{supp } f^{\mathbb{C}} \subset \{(x_1 + iy_1, \dots, x_n + iy_n); (x_1, \dots, x_n) \in \text{supp } f; \forall 1 \leq j \leq n; |y_j| \leq C\},$$

$$* |\partial_{\bar{z}} f^{\mathbb{C}}(z_1, \dots, z_n)| \leq C_l |\mathcal{I}z_1|^{l_1} \dots |\mathcal{I}z_n|^{l_n}.$$

Proof : Let f be a smooth function on \mathbb{R}^n with compact support. We can take $f^{\mathbb{C}}$, given by the following expression on \mathbb{C}^n :

$$f^{\mathbb{C}}(z_1, \dots, z_n) = \sum_{\alpha \in \mathbb{N}^n, |\alpha|=0}^{+\infty} \frac{(iy)^\alpha}{\alpha!} f^\alpha(x) \prod_{j=1}^n \chi\left(\frac{y_j}{T_{\alpha_j}}\right)$$

where $z_j = x_j + iy_j$, for all $1 \leq j \leq n$, $(iy)^\alpha = \prod_{j=1}^n (iy_j)^{\alpha_j}$, $\alpha! = \prod_{j=1}^n \alpha_j!$, and $f^\alpha(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f(x)$.

With this choice of extension, and by the same reasoning of the proof of the proposition 2.3, one can show our result. ■

Theorem 2.23 Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a smooth function with compact support and $f^{\mathbb{C}}$ be an almost analytic extension of f given by the proposition 2.22 Let H_1, H_2, \dots, H_n be a self-adjoint operators that commute two by two. The integral

$$\int_{\mathbb{C}^n} \|\partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - H_1)^{-1} \dots (z_n - H_n)^{-1}\| dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n$$

converges, the integral in the following formula converges in operator norm, and we have

$$f(H_1, \dots, H_n) = (2i\pi)^{-n} \int_{\mathbb{C}^n} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - H_1)^{-1} \dots (z_n - H_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n$$

Lemma 2.24 If $u \in C^\infty(\mathbb{C}^n)$ with compact support in ω then, for all $(\xi_1, \dots, \xi_n) \in \omega$;

$$u(\xi_1, \dots, \xi_n) = (2i\pi)^{-n} \int_{\mathbb{C}^n} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} u^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \xi_1)^{-1} \dots (z_n - \xi_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \quad (2.4.2)$$

Proof : Let $u_n; \mathbb{C} \rightarrow \mathbb{C}; \xi_n \mapsto u_n(\xi_n)$ be defined by $u_n(\xi_n) := u(\xi_1, \dots, \xi_n)$. By the corollary 2.8, we have

$$u(\xi_1, \dots, \xi_n) = u_n(\xi_n) = (2i\pi)^{-1} \int_{\mathbb{C}} \partial_{\bar{z}_n} u^{\mathbb{C}}(\xi_1, \dots, \xi_n) (z_n - \xi_n)^{-1} dz_n \wedge d\bar{z}_n$$

By repeated use of the corollary 2.8, we obtain (2.4.2). \blacksquare

Proposition 2.25 Let K be a compact of \mathbb{R}^n . Let, for $k \in \mathbb{N}$, $f_k \in C^\infty(\mathbb{R}^n)$ with $\text{supp} f_k \subset K$ and $f \in C^\infty(\mathbb{R}^n)$ such that $(f_k)_k$ converges uniformly to f on K . Then $(f_k(H_1, \dots, H_n))_k$ converges to $f(H_1, \dots, H_n)$ for the operator norm.

Proof : By the same reason of the proof of the proposition 2.10, we find the following inequality,

$$\|f_k(H_1, \dots, H_n) - f(H_1, \dots, H_n)\|_{\mathcal{B}(H)} \leq \|f_k - f\|_{\infty, K} := \sup_{(x_1, \dots, x_n) \in K} |f_k(x) - f(x)|$$

If we replace u by an almost analytic extension $f^{\mathbb{C}}$ given by the proposition 2.22 and we take $\omega = \mathbb{C}^n$, the generalized integral in (2.4.1) converges uniformly, w.r.t. (ξ_1, \dots, ξ_n) , as proved in the following proposition. \blacksquare

Proposition 2.26 Let $f \in C^\infty(\mathbb{R}^n)$ be a function with compact support K and $f^{\mathbb{C}}$ be an almost analytic extension of f given by the proposition 2.22. We have the following uniform convergence on K

$$\lim_{k \rightarrow +\infty} (2\pi)^{-n} \left\| \int_{\prod_{j=1}^n \{z_j; |\mathcal{I}z_j| > \frac{1}{k}\}} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \cdot)^{-1} \dots (z_n - \cdot)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n - f(\cdot) \right\|_{\infty, \omega} = 0 \quad (2.4.3)$$

Proof : Let $f \in C^\infty(\mathbb{R}^n)$ be a function with compact support K . By the proposition 2.22, $f^{\mathbb{C}}$ is an almost analytic extension of f such that $\text{supp} f^{\mathbb{C}} \subset K^{\mathbb{C}}$, where

$K^{\mathbb{C}} := \{(x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}^n; (x_1, \dots, x_n) \in \omega; \forall 1 \leq j \leq n; 0 \leq |y_j| \leq C < 1\}$. Let ω be an open in \mathbb{C}^n that contains $K^{\mathbb{C}}$. From the lemma 2.24 with taking $u = f^{\mathbb{C}}$, we have, for all $(l_1, \dots, l_n) \in K$,

$$f(\xi_1, \dots, \xi_n) = (2i\pi)^{-n} \int_{\omega} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \xi_1)^{-1} \dots (z_n - \xi_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \quad (2.4.4)$$

Using the proposition 2.22 with $l_1 = \dots = l_n = 1$, we have, for $(l_1, \dots, l_n) \in K$,

$$\begin{aligned} & (2\pi)^{-n} \int_{K^{\mathbb{C}}} |\partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \xi_1)^{-1} \dots (z_n - \xi_n)^{-1}| dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \\ & \leq C_1 \int_{K^{\mathbb{C}}} \prod_{j=1}^n |\mathcal{I}z_j| \cdot \prod_{j=1}^n |\mathcal{I}z_j|^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \\ & < +\infty. \end{aligned} \quad (2.4.5)$$

Thus this integral (5.0.4) is absolutely convergent. Now we will show the uniform convergence on K , let $(l_1, \dots, l_n) \in K$,

$$\begin{aligned} & |(2i\pi)^{-n} \int_{\prod_{j=1}^n \{z_j; |\mathcal{I}z_j| > \frac{1}{k}\}} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \xi_1)^{-1} \dots (z_n - \xi_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n - f(\xi_1, \dots, \xi_n)| \\ & = |(2i\pi)^{-n} \int_{\prod_{j=1}^n \{z_j; |\mathcal{I}z_j| < \frac{1}{k}\}} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - \xi_1)^{-1} \dots (z_n - \xi_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n| \\ & \leq C_1 \int_{\prod_{j=1}^n \{z_j; |\mathcal{I}z_j| < \frac{1}{k}\}} \prod_{j=1}^n |\mathcal{I}z_j| \cdot \prod_{j=1}^n |\mathcal{I}z_j|^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \\ & < \frac{C}{k^n}, \text{ where } C \text{ is independent of } (\xi_1, \dots, \xi_n). \end{aligned} \quad (2.4.6)$$

Therefore (2.4.3) yields. ■

We are able to reprove Helffer-Sjöstrand theorem.

Theorem 2.27 Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with compact support K and $f^{\mathbb{C}}$ be an almost analytic extension of f given by the proposition 2.22. Let H_1, \dots, H_n be a self-adjoint operators that commute two by two. The integral

$$\int_{\mathbb{C}^n} \|\partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - H_1)^{-1} \dots (z_n - H_n)^{-1}\| dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \quad (2.4.7)$$

converges, the integral in the following formula converges in the operator norm, and we have

$$f(H_1, \dots, H_n) = (2i\pi)^{-n} \int_{\mathbb{C}^n} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) (z_1 - H_1)^{-1} \dots (z_n - H_n)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n \quad (2.4.8)$$

Proof : By the proposition 2.26, (2.4.5) holds true with (ξ_1, \dots, ξ_n) replaced by (H_1, \dots, H_n) . This proves the convergence of (2.4.7). In particular, the integral in (2.4.8) converges in the operator norm. By the proposition 2.26 and proposition 2.25 we get (2.4.8). ■

2.5 Application

Proposition 2.28 Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth function with compact support K . Let H_1, \dots, H_n be a self-adjoint operators that commute two by two, and let B be a self-adjoint operator such that for all $j \in \{1, 2, \dots, n\}$, $H_j \in C^m(B)$. Then

$$[f(H_1, \dots, H_n), B] = (2i\pi)^{-n} \int_{\mathbb{C}^n} \partial_{\bar{z}_1} \dots \partial_{\bar{z}_n} f^{\mathbb{C}}(z_1, \dots, z_n) \sum_{\substack{1 \leq k_1 \leq \dots \leq k_m \leq \dots \leq n \\ j=1}}^{k_m} \prod_{j=1}^{k_m} (z_j - H_j)^{-1 - \#\{p: k_p=j\}} ad_{H_{k_1}, \dots, H_{k_m}}^n \prod_{j=k_m}^n (z_j - H_j)^{-1} dz_1 \wedge d\bar{z}_1 \dots dz_n \wedge d\bar{z}_n$$

where $ad_{H_{k_1}, \dots, H_{k_m}}^{k_m} = [H_{k_m}, \dots, [H_{k_1}, B], \dots]$.

3 LIMITING ABSORPTION PRINCIPLE FOR DIRAC OPERATORS WITH A PERTURBED WIGNER-VON NEUMANN POTENTIAL

In this section, we are interested in the behaviour on the real axis of the resolvent of a class of continuous Dirac operators. We shall prove a so called “limiting absorption principle”, a very useful result to develop the scattering theory associated with those Dirac operators. It also gives information on the nature of their essential spectrum. The main interest of our study relies on the fact that we include some oscillating contribution in the potential of our Dirac operator.

To set up our framework and precisely formulate our results, we need to introduce some notations. We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the right linear scalar product and the norm in $\mathbf{L}^2(\mathbb{R}^3)$, the space of squared integrable, complex functions on \mathbb{R}^3 . We also denote by $\|\cdot\|$ the norm of bounded operators on $\mathbf{L}^2(\mathbb{R}^3)$. By writing $x = (x_1; x_2; x_3)$ the variable in \mathbb{R}^3 , we set $x^2 := \sum_{j=1}^3 x_j^2$ and where $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$. Let Q_j be the multiplication operator in $\mathbf{L}^2(\mathbb{R}^3)$ by x_j and P_j the self-adjoint realization of $-i\partial_{x_j}$ in $\mathbf{L}^2(\mathbb{R}^3)$. We set $Q = (Q_1; Q_2; Q_3)^T$ and $P = (P_1; P_2; P_3)^T$, where T denotes the transposition. Let $-\Delta = |P|^2 := \sum_{j=1}^3 P_j^2 = P^T \cdot P$ be the self-adjoint realization of the nonnegative Laplace operator $-\Delta$ in $\mathbf{L}^2(\mathbb{R}^3)$.

Let us introduce now the Dirac operators we shall consider. Let

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

be the Dirac 4×4 matrices.

The free Dirac operator D_0 in \mathbb{R}^3 with mass $m \in \mathbb{R}^+$ is the self-adjoint realization in $\mathbf{L}^2(\mathbb{R}^3, \mathbb{C}^4)$ of the differential operator

$$\begin{aligned} D_0 &= -i\hbar c \underline{\alpha} \cdot \nabla + mc^2 \beta \\ &= \hbar c \sum_{j=1}^3 \alpha_j P_j + mc^2 \beta, \end{aligned} \quad (3.0.1)$$

where $m > 0$ is the mass of a particle, $c > 0$ is the speed of light, \hbar is the reduced Planck constant, $\underline{\alpha}$ is the transposed vector $(\alpha_1, \alpha_2, \alpha_3)^T$ of three 4×4 matrices, and ∇ is the Gradient on \mathbb{R}^3 . In particular, D_0 is a matrix of operators of order 4×4 . This operator appeared in the Dirac equation (3.0.2) describes the behavior of elementary particles of halfentier spins represented by ψ in (3.0.2) : We study properties of relativistic massive charged particles with spin $1/2$ (e.g., electron, positron, (anti-)muon, (anti-)taun, . . .). We follow the Dirac formalism, see [D]. Because of the spin, the configuration space of the particle is vector valued. The free Dirac equation is given by :

$$i\hbar \frac{d}{dt} \psi(t, x) = D_0 \psi(t, x). \quad (3.0.2)$$

We shall consider external fields that are functions $V(Q)$ defined on \mathbb{R}^3 with values in the space of self-adjoint 4×4 matrices and tend to 0 at infinity. In particular, the full Dirac operator $D = D_0 + V(Q)$ will always be self-adjoint with domain $\mathcal{H}^1(\mathbb{R}^3; \mathbb{C}^4)$ and the essential spectrum of D will be the one of D_0 , that is $]-\infty; -mc^2] \cup [mc^2; +\infty[$, by Weyl's theorem. We also renormalize and consider $\hbar = c = 1$. We will be interested in the dirac operator with Von-Neumann potential $D = D_0 + V(Q)$, where $V(Q)$ is the matrix of operators of order 4×4 on \mathbb{R}^3 satisfying the following

Assumption 3.1 . Let $r_0, r_1, r_2 \in]0; 1]$, $q \in \mathbb{R}$ and let k be a strictly positive real. Let $\kappa \in C_c^\infty(\mathbb{R}; \mathbb{R})$ with $\kappa = 1$ on $[-1; 1]$ and $0 \leq \kappa \leq 1$. We consider V_{sr}, V_{lr} are two matrices of order 4×4 such that, they exist $C > 0, r_0, r_1, r_2 > 0$, such that for all $x \in \mathbb{R}^3$,

$$|\langle x \rangle^{r_0} V_{l,r}(x)| < C, |\langle x \rangle^{1+r_1} \nabla V_{l,r}(x)| < C, |\langle x \rangle^{1+r_2} V_{s,r}(x)| < C,$$

and

$$W(x) = q(1 - \kappa(|x|)) \frac{\sin(k|x|)}{|x|} \mathbb{1}_4, \quad (3.0.3)$$

with real q . Let $V = V_{sr} + V_{lr} + W$.

We can not begin to study the L.A.P without starting by defining the conjugate operator which we will note A_D later of the Dirac operator "D" that we will work with throughout this paper.

Definition 3.2 Let \tilde{A}_D be the self-adjoint realization of the operator $(F(P) \cdot Q + Q \cdot F(P))/2$ in $\mathbf{L}^2(\mathbb{R}^3)$ where $F = (F_j)_{j=1,2,3}$ is a vector of functions, with $F_j(\xi) = \mu^2(\xi) \frac{\theta \circ \mu(\xi)}{|\xi^2|} \xi_j$, and $\mu(\xi) = \sqrt{\xi^2 + m^2}$. We define A_D by

$$A_D := \Pi_+(P) \tilde{A}_D \Pi_+(P) + \Pi_-(P) \tilde{A}_D \Pi_-(P)$$

where $\Pi_\pm(\xi)$ is given by, for $\xi \in \mathbb{R}^3$:

$$\Pi_\pm(\xi) = \frac{1}{2} \pm \frac{1}{2\sqrt{\xi^2 + m^2}} (\alpha \cdot \xi + m\beta) = \frac{1}{2} \pm \frac{1}{2\sqrt{\xi^2 + m^2}} \left(\sum_{j=1}^3 \xi_j \alpha_j + m\beta \right). \quad (3.0.4)$$

Remark 3.3 The choice of the conjugate operator A_D is taken in [BMP].

Remark 3.4 By the Mourre commutator method with A_D as conjugate operator, one has the following Theorem , which is a consequence of the much more general Theorem 7.6.8 in [ABG]:

Theorem 3.5 [BMP] Consider the above operator D with $q = 0$ (i.e. without the oscillating part of the potential). Then the point spectrum of D is locally finite in $] -\infty; mc^2[\cup]mc^2; +\infty[$. Furthermore, for any $s > \frac{1}{2}$ and any compact interval $I \subset] -\infty; mc^2[\cup]mc^2; +\infty[$, that does not intersect the point spectrum of D ,

$$\sup_{\mathcal{R}z \in I, \mathcal{I}z \neq 0} \|\langle A_D \rangle^{-s} (D - z)^{-1} \langle A_D \rangle^{-s}\| < +\infty. \quad (3.0.5)$$

Remark 3.6 In that case, it turns out that the spectrum of D , is noted by $\sigma(D)$, is purely absolutely continuous in I (cf., Theorem XIII.20 in [RS4]). Notice that (3.0.5) holds true for $s = 0$ if and only if $I \cap \sigma(D) = \emptyset$, where $\sigma(D)$ the spectrum of D .

Proposition 3.7 We have $D_0 \in \mathcal{C}^2(A_D)$. Moreover, under Assumption 3.1, the form $[V, iA_D]$, defined on $\mathcal{D}(A_D)$, extends to a bounded operator from $\mathcal{H}^1(\mathbb{R}^3)^4$ in $\mathcal{H}^{-1}(\mathbb{R}^3)^4$. In particular, $D \in \mathcal{C}^1(A_D)$.

Proof : In [BMP], it is shown that $D_0, V_{s,r}(Q), V_{l,r}(Q) \in \mathcal{C}^1(A_D)$. Since $W(Q)$ is bounded, and the quadratic form, $[W(Q), iA_D]$ is bounded on $\mathcal{D}(A_D)^2$ we obtain $W(Q) \in \mathcal{C}^1(A_D)$, and we concludes $D \in \mathcal{C}^1(A_D)$. \blacksquare

Theorem 3.8 Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ be a function supported in a small interval in $]m, \sqrt{m^2 + k^2}/4[$ or in $] -\sqrt{m^2 + k^2}/4, -m[$. there exist a constant $c > 0$ and a compact operator K on $\mathcal{L}^2(\mathbb{R}^3)^4$ such that

$$\theta(D)[D, iA_D]\theta(D) \geq c \theta(D)^2 + K.$$

Proof : Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ be a function supported in a small interval around λ in $]m, \sqrt{m^2 + k^2}/4[$ or in $] -\sqrt{m^2 + k^2}/4, -m[$, we have

$$\begin{aligned} \theta(D)[D, iA_D]\theta(D) &= (\theta(D) - \theta(D_0))[D, iA_D](\theta(D) - \theta(D_0)) \\ &\quad + (\theta(D) - \theta(D_0))[D, iA_D]\theta(D_0) + \theta(D_0)[D, iA_D](\theta(D) - \theta(D_0)) \\ &\quad + \theta(D_0)[D, iA_D]\theta(D_0) \\ &= M_1 + M_2 + M_3 + M_4. \end{aligned}$$

Let us first treat the term M_4 ,

$$\begin{aligned} M_4 &= \theta(D_0)[D_0, iA_D]\theta(D_0) + \theta(D_0)[V_{l,r}(Q), iA_D]\theta(D_0) \\ &\quad + \theta(D_0)[V_{s,r}(Q), iA_D]\theta(D_0) + \theta(D_0)[W(Q), iA_D]\theta(D_0) \\ &= M_{4,1} + M_{4,2} + M_{4,3} + M_{4,4}. \end{aligned}$$

Let's start by $M_{4,1}$, the idea of the study this term is in [BMP]. By the proposition 4.1 and 4.2 in the section of appendix, we have

$$\begin{aligned} M_{4,1} &= \theta(D_0)[\mu(P)(\Pi_+ - \Pi_-)(P), i\Pi_+(P)\tilde{A}_D\Pi_+(P) + \Pi_-(P)\tilde{A}_D\Pi_-(P)]\theta(D_0) \\ &= \theta(D_0)(\Pi_+(P)[\mu(P), i\tilde{A}_D]\Pi_+(P) - \Pi_-(P)[\mu(P), i\tilde{A}_D]\Pi_-(P))\theta(D_0) \\ &= \theta(D_0)D_0\theta \circ \mu(P)\theta(D_0) \\ &\geq c \theta(D_0)^2, \end{aligned}$$

where c is a positive constant. Second, let's treat $M_{4,2}$. According to the article [BMP] one can write A_D in the following form

$$\begin{aligned} A_D &= F(P).Q + \frac{i}{2}f(P) + i\alpha\beta G(P) \\ &= \sum_{j=1}^3 F_j(P)Q_j + \frac{i}{2}f(P) + i\sum_{j=1}^3 \alpha_j\beta G_j(P) \end{aligned}$$

where

$$f(\xi) = 2(\theta \circ \mu)(\xi) + \mu(\xi)\theta' \circ \mu(\xi) + \mu^2(\xi)(\theta \circ \mu)(\xi) \frac{1}{|\xi|^2}, \quad (3.0.6)$$

and

$$G_j(\xi) = \frac{m\xi_j}{\mu^2(\xi)}\theta \circ \mu(\xi), j = 1, 2, 3. \quad (3.0.7)$$

To study $M_{4,2}$, this amounts studying the following three terms $\theta(D_0)[V_{1,r}(Q), f(P)]\theta(D_0)$, $\theta(D_0)[V_{1,r}(Q), F_j(P)Q_j]\theta(D_0)$, and $\theta(D_0)[V_{1,r}(Q), \alpha_j\beta G_j(P)]\theta(D_0)$. We refer to proposition 4.3 of Appendix for this proof. To treat the term $M_{4,3}$, it suffice to study $\theta(D_0)Q.V_{s,r}(Q)F(P)\theta(D_0)$. Indeed, we know that $\theta(D_0)\langle Q \rangle^{-r_2}$ is a compact operator, and $\langle Q \rangle^{r_2}Q.V_{s,r}(Q)F(P)\theta(D_0)$ is a bounded operator on $\mathcal{L}^2(\mathbb{R}^3)^4$ then $M_{4,3}$ is a compact operator, the same for $M_{4,4}$, it suffice to show that $\theta(D_0)WQ.F(P)\theta(D_0)$ is a compact operator. In fact,

$$\begin{aligned}\theta(D_0)WQ.F(P)\theta(D_0) &= \theta(D_0) \sin(k|Q|)\langle Q \rangle^{-1}Q.F(P)\theta(D_0) \\ &= \theta(D_0) \sin(k|Q|)\langle x \rangle^{-1} \sum_{j=1}^3 Q_j F_j(P)\theta(D_0) \\ &= \theta(D_0) \sin(k|Q|)\theta(D_0)\langle Q \rangle^{-1} \sum_{j=1}^3 Q_j F_j(P) \\ &\quad + \theta(D_0) \sin(k|Q|) \sum_{j=1}^3 [\langle Q \rangle^{-1}Q_j F_j(P), \theta(D_0)].\end{aligned}$$

$\theta(D_0)\langle Q \rangle^{-1}$ and by the lemma 3.9, $\theta(D_0) \sin(k|Q|)\theta(D_0)$ are a compacts operators on $\mathcal{L}^2(\mathbb{R}^3)^4$, and for $j \in \{1, 2, 3\}$, $\langle Q \rangle^{-1}Q_j F_j(P)$, $\sin(k|Q|) \sum_{j=1}^3 \langle Q \rangle [\langle Q \rangle^{-1}Q_j F_j(P), \theta(D_0)]$ are a bounded operators on $\mathcal{L}^2(\mathbb{R}^3)^4$, we conclude that $M_{4,4}$ is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$.

M_2 and M_3 are treated the same way. Let's study $M_2 = (\theta(D) - \theta(D_0))[D, iA_D]\theta(D_0)$. By the Helffer-Sjöstrand formula, $(\theta(D) - \theta(D_0))\langle Q \rangle^\epsilon \langle D_0 \rangle^\delta$ is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$ for $0 \leq \epsilon < r_0, 0 \leq \delta < 1$, and $\langle Q \rangle^{-\epsilon} \langle D_0 \rangle^{-\delta} [D, iA_D]\theta(D_0)$ is a bounded operator on $\mathcal{L}^2(\mathbb{R}^3)^4$, we obtain M_2 is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$. For the same reason M_4 is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$, we conclude there exists a constante $c > 0$, and a compact operator K' on $\mathcal{L}^2(\mathbb{R}^3)^4$ such that,

$$\begin{aligned}\theta(D)[D, iA_D]\theta(D) &\geq c \theta(D_0)^2 + K' \\ &\geq c (\theta(D_0) - \theta(D) + \theta(D))^2 + K' \\ &\geq c (\theta(D))^2 + (\theta(D_0) - \theta(D))\theta(D) + \theta(D)(\theta(D_0) - \theta(D)) \\ &\quad + (\theta(D_0) - \theta(D))^2 + K'.\end{aligned}$$

Again by the Helffer-Sjöstrand formula, $\theta(D_0) - \theta(D)$ is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$, then $(\theta(D_0) - \theta(D))\theta(D)$, $\theta(D)(\theta(D_0) - \theta(D))$ and $(\theta(D_0) - \theta(D))^2$ are a compacts operators $\mathcal{L}^2(\mathbb{R}^3)^4$, we conclude the Mourre estimate, i.e., there exist a positive constant $c > 0$ and a compact operator K on $\mathcal{L}^2(\mathbb{R}^3)^4$ such that,

$$\theta(D)[D, iA_D]\theta(D) \geq c \theta(D)^2 + K.$$

■

Lemma 3.9 Let $\theta \in C_c^\infty(\mathbb{R})$ be a function supported in a small interval around λ in $]m, \sqrt{m^2 + k^2}/4[$ or in $] -\sqrt{m^2 + k^2}/4, -m[$. We have for all $0 \leq \delta < 1$, $\theta(D_0) \sin(k|Q|)\theta(D_0)\langle Q \rangle^\delta$ is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$,

Proof : Let $\chi_1 \in C_c^\infty(\mathbb{R})$ be a function such that

$$\chi_1 = \begin{cases} 0 & \text{on } [-1; 1], \\ 1 & \text{on } \mathbb{R} \setminus [-2; 2]. \end{cases}$$

$$\begin{aligned}\theta(\alpha.P + m\beta) \sin(k|Q|)\theta(\alpha.P + m\beta) \\ &= \theta(\alpha.P + m\beta)\chi_1(Q) \sin(k|Q|)\theta(\alpha.P + m\beta) + K'_1 \\ &= \theta(\alpha.P + m\beta)\chi_1(Q) \sin(k|Q|)\{\theta(\alpha.\xi + m\beta)\}^w + K'_2 \\ &= \{\theta(\alpha.\xi + m\beta)\}^w \left(\frac{e_+ - e_-}{2i}\right) \{\theta(\alpha.\xi + m\beta)\}^w + K'_3.\end{aligned}$$

Since θ supported on the right of m , we find

$$\begin{aligned}&= \{\theta(\sqrt{m^2 + \xi^2})\Pi_+^1(\xi)\}^w \left(\frac{e_+ - e_-}{2i}\right) \{\theta(\sqrt{m^2 + \xi^2})\Pi_+^1(\xi)\}^w + K'_3, \\ &= \{\theta(\sqrt{m^2 + \xi^2})\}^w \{\Pi_+^1(\xi)\}^w \left(\frac{e_+ - e_-}{2i}\right) \{\theta(\sqrt{m^2 + \xi^2})\}^w \{\Pi_+^1(\xi)\}^w + K'_4.\end{aligned}$$

Applying the lemma 4.3 in [GJ] with $a(x, \xi) = \theta(\sqrt{m^2 + \xi^2})$, we find

$$\begin{aligned}
&= \frac{1}{2i} \{\Pi_+^1(\xi)\}^w \{\theta(\sqrt{m^2 + \xi^2})\}^w (a_\pm^w e_\pm + e_\pm b_\pm^w + c_\pm^w e_\pm) \{\Pi_+^1(\xi)\}^w + K'_4 \\
&= \frac{1}{2i} \{\Pi_+^1(\xi)\}^w \{\theta(\sqrt{m^2 + \xi^2})\}^w a_\pm^w e_\pm \{\Pi_+^1(\xi)\}^w + K'_5 \\
&= \frac{1}{2i} \{\Pi_+^1(\xi)\}^w \{\theta(\sqrt{m^2 + \xi^2}) a_\pm\}^w e_\pm \{\Pi_+^1(\xi)\}^w + K'_6 \\
&= \frac{1}{2i} \{\Pi_+^1(\xi)\}^w \{\theta(\sqrt{m^2 + \xi^2}) \chi_1 a_\pm\}^w e_\pm \{\Pi_+^1(\xi)\}^w + K'_7 \\
&= \frac{1}{2i} \{\Pi_+^1(\xi)\}^w \{\theta(\sqrt{m^2 + \xi^2}) \theta(\sqrt{m^2 + (\xi \mp kx/|x|)^2}) \chi_1\}^w e_\pm \{\Pi_+^1(\xi)\}^w + K'_8
\end{aligned}$$

where $(K'_j)_{j=1, \dots, 8}$ are of the following form $B'_1 b_1^{w}$ or $b_2^{w} B'_2$ with $b'_1, b'_2 \in S(\langle x \rangle^{-1} \langle \xi \rangle^{-1}, g)$ and B'_1, B'_2 are a bounded operators on $\mathcal{L}^2(\mathbb{R}^3)^4$, therefore for all $1 \leq j \leq 6, 0 \leq \delta < 1$ we have $K_j \langle x \rangle^\delta$ is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$.

As in the proof of the lemma 4.3 in [GJ], we show under the new conditions we took on θ that

$$\theta(\sqrt{m^2 + \xi^2}) \theta(\sqrt{m^2 + (\xi \mp kx/|x|)^2}) = 0.$$

Let's study the support of the function $\theta(\sqrt{m^2 + \xi^2}) \theta(\sqrt{m^2 + (\xi \mp kx/|x|)^2})$.

Let $\xi \in \text{supp}(\theta(\sqrt{m^2 + \xi^2})) \cap \text{supp}(\theta(\sqrt{m^2 + (\xi \mp kx/|x|)^2}))$, i.e.

$$\lambda - \epsilon < \sqrt{m^2 + \xi^2} < \lambda + \epsilon \quad \text{and} \quad \lambda - \epsilon < \sqrt{m^2 + (\xi \mp kx/|x|)^2} < \lambda + \epsilon,$$

then

$$(\lambda - \epsilon)^2 < m^2 + \xi^2 < (\lambda + \epsilon)^2 \quad \text{et} \quad (\lambda - \epsilon)^2 < m^2 + (\xi \mp kx/|x|)^2 < (\lambda + \epsilon)^2$$

By subtracting these two inequalities, we find a situation similar to the one it appeared in the case of Schrödinger in [GJ],

$$-4\lambda\epsilon < \mp 2k\xi \frac{x}{|x|} + k^2 < 4\lambda\epsilon,$$

is equivalent to

$$\frac{-4\lambda\epsilon}{k} < \mp |\xi| \frac{\xi}{|\xi|} \frac{x}{|x|} + \frac{k}{2} < \frac{4\lambda\epsilon}{k}.$$

Suppose that $|\xi| < k/2$, we obtain

$$0 < -|\xi| + \frac{k}{2} < \frac{4\lambda\epsilon}{k},$$

then

$$0 < \frac{k}{4\lambda} \left(\frac{k}{2} - |\xi| \right) < \epsilon.$$

Now to not have ξ in the intersection supports it suffice to take θ supported in $]\lambda - \epsilon_1, \lambda + \epsilon_1[$ where $0 < \epsilon_1 < \frac{k}{4\lambda} \left(\frac{k}{2} - |\xi| \right)$. \blacksquare

3.1 Limiting Absorption Principle

We inspired from the method given by [GJ1], of course by adapting in our case.

Theorem 3.10 Let I be an open interval in $]m, \sqrt{m^2 + k^2/4}[$ or in $] -\sqrt{m^2 + k^2/4}, -m[$ such that $I \cap \sigma_{\text{disc}}(D) = \emptyset$ and I' a closed sub-interval of I . We have,

$$\sup_{\mathcal{R}z \in I', \text{Im}z \neq 0} \|\langle A_D \rangle^{-s} (D - z)^{-1} \langle A_D \rangle^{-s}\| < \infty.$$

Proof : Let $\theta, \chi, \tau \in C_c^\infty(]m, \sqrt{m^2 + k^2/4}[)$ such that $\tau\chi = \chi$, $\chi\theta = \theta$ and $\theta = 1$ near to I . Let $1/2 < s < 1$. We have $\chi(D) \in C^1(A_D)$. We define as in [GJ1] the map $\psi : \mathbb{R} \rightarrow \mathbb{R}$, $t \mapsto \psi(t) := \int_{-\infty}^t \langle u \rangle^{-2s} du$, $\psi \in S^0$ in particular is bounded. Let $R \geq 1$. $D\tau(D)$ is a bounded operator in $C^1(A_D)$. Suppose F the following expression, we have

$$\begin{aligned}
F &:= \theta(D)[D, i\psi(A_D/R)]\theta(D) \\
&= \theta(D)[D\tau(D), i\psi(A_D/R)]\theta(D).
\end{aligned}$$

This equality of operators by the property $\tau\theta = \theta$. By the Helffer-Sjöstrand formula we have

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} [D\tau(D), iA_D/R] \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) \chi(D) (z - A_D/R)^{-1} [D\tau(D), iA_D/R] \\
&\quad (z - A_D/R)^{-1} \chi(D) \theta(D) dz \wedge d\bar{z} \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) ((z - A_D/R)^{-1} \chi(D) + [\chi(D), (z - A_D/R)^{-1}]) \\
&\quad [D\tau(D), iA_D/R] (\chi(D) (z - A_D/R)^{-1} + [(z - A_D/R)^{-1}, \chi(D)]) \theta(D) dz \wedge d\bar{z} \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \chi(D) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \\
&\quad [(z - A_D/R)^{-1}, \chi(D)] \theta(D) dz \wedge d\bar{z} \\
&\quad + \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) [\chi(D), (z - A_D/R)^{-1}] [D\tau(D), iA_D/R] \chi(D) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) [\chi(D), (z - A_D/R)^{-1}] [D\tau(D), iA_D/R] \\
&\quad [(z - A_D/R)^{-1}, \chi(D)] \theta(D) dz \wedge d\bar{z},
\end{aligned}$$

therefore

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \chi(D) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + A_1^D + A_2^D + A_3^D,
\end{aligned}$$

with

$$\begin{aligned}
A_1^D &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \\
&\quad [(z - A_D/R)^{-1}, \chi(D)] \theta(D) dz \wedge d\bar{z} \\
&= \theta(D) \langle A_D/R \rangle^{-s} \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \langle A_D/R \rangle^s (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \\
&\quad [(z - A_D/R)^{-1}, \chi(D)] \langle A_D/R \rangle^s dz \wedge d\bar{z} \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \langle A_D/R \rangle^{-s} \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \langle A_D/R \rangle^s (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \\
&\quad (z - A_D/R)^{-1} [A_D/R, \chi(D)] (z - A_D/R)^{-1} \langle A_D/R \rangle^s dz \wedge d\bar{z} \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \langle A_D/R \rangle^{-s} R^{-2} \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \langle A_D/R \rangle^s (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D] \\
&\quad (z - A_D/R)^{-1} [A_D, \chi(D)] (z - A_D/R)^{-1} \langle A_D/R \rangle^s dz \wedge d\bar{z} \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \langle A_D/R \rangle^{-s} R^{-2} B \langle A_D/R \rangle^{-s} \theta(D),
\end{aligned}$$

where B is a bounded operator independently of R on $\mathcal{L}^2(\mathbb{R}^3)^4$, indeed, $D\tau(D), \chi(D)$ in $C^1(A_D)$, and for all $0 \leq s \leq 1$ we have $\langle A_D/R \rangle^s (z - A_D/R)^{-1}$ is a bounded operator independently of R on $\mathcal{L}^2(\mathbb{R}^3)^4$. The same argument for A_2^D and A_3^D allows us to write F in the following form

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} \chi(D) [D\tau(D), iA_D/R] \chi(D) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} R^{-2} B_1 \langle A_D/R \rangle^{-s} \theta(D)
\end{aligned}$$

where B_1 is a bounded operator independently of R on $\mathcal{L}^2(\mathbb{R}^3)^4$.

Let $\epsilon = \frac{\tau_0}{2}$. By the lemma 3.11, $(\theta(D) - \theta(D_0)) \langle A_D \rangle^\epsilon$ is a compact. Let

$$G := \chi(D) [D\tau(D), iA_D/R] \chi(D),$$

we have

$$\begin{aligned}
G &= \chi(D)[D, iA_D/R]\chi(D) \\
&= \chi(D)[D, iA_D/R](\chi(D_0) + \chi(D) - \chi(D_0)) \\
&= \chi(D)[D, iA_D/R]\chi(D_0) + \chi(D)[D, iA_D/R](\chi(D) - \chi(D_0)) \\
&= \chi(D)[D, iA_D/R]\chi(D_0) + \chi(D)\tau(D)[D, iA_D/R](\chi(D) - \chi(D_0))\langle A_D \rangle^\epsilon R^{-1} \\
&\quad \langle A_D/R \rangle^\epsilon \langle A_D \rangle^{-\epsilon} \langle A_D/R \rangle^{-\epsilon} \\
&= \chi(D)[D, iA_D/R]\chi(D_0) + \chi(D)K_1 R^{-1} B_2 \langle A_D/R \rangle^{-\epsilon}
\end{aligned}$$

where B_2 is a uniformly bounded operator at R , and K_1 is a compact operator on $\mathcal{L}^2(\mathbb{R}^3)^4$.

$$\begin{aligned}
G &= (\chi(D_0) + \chi(D) - \chi(D_0))[D, iA_D/R]\chi(D_0) + \chi(D)K_1 R^{-1} B_2 \langle A_D/R \rangle^{-\epsilon} \\
&= \chi(D_0)[D, iA_D/R]\chi(D_0) + (\chi(D) - \chi(D_0))[D, iA_D/R]\chi(D_0) \\
&\quad + \chi(D)K_1 R^{-1} B_2 \langle A_D/R \rangle^{-\epsilon} \\
&= \chi(D_0)[D, iA_D/R]\chi(D_0) + \chi(D)K_1 R^{-1} B_2 \langle A_D/R \rangle^{-\epsilon} \\
&\quad + \langle A_D/R \rangle^{-\epsilon} \langle A_D/R \rangle^\epsilon \langle A_D \rangle^{-\epsilon} \langle A_D \rangle^\epsilon (\chi(D) - \chi(D_0))[D, iA_D/R] R^{-1} \chi(D_0) \\
&= \chi(D_0)[D, iA_D/R]\chi(D_0) + \chi(D)K_1 R^{-1} B_2 \langle A_D/R \rangle^{-\epsilon} \\
&\quad + \langle A_D/R \rangle^{-\epsilon} B_3 K_2 R^{-1} \chi(D_0).
\end{aligned}$$

Using the lemma 3.9, there exists a compact operator K_3 and a uniformly bounded operator B_4 at R such that

$$\chi(D_0)[W + V, iA_D]\chi(D_0) = R^{-1}\chi(D_0)K_3 B_4 \langle A_D/R \rangle^{-\epsilon} \chi(D_0),$$

therefore

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_1 + R^{-1} K_4) \langle A_D/R \rangle^{-s} \theta(D).
\end{aligned}$$

where K_4 is a compact such that, for some constant $c_1 > 0$, we have

$$\|K_4\| \leq c_1 (\|\chi(D)K_1\| + \|K_2\chi(D_0)\| + \|\chi(D_0)K_3\|),$$

Let $\tilde{\chi}$ be such that $\tilde{\chi}\chi = \tilde{\chi}$ and $\tilde{\chi}\theta = \theta$. We have

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-1} R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) \\
&\quad (z - A_D/R)^{-1} \theta(D) dz \wedge d\bar{z} \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_1 + R^{-1} K_4 \tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D).
\end{aligned}$$

By commuting $R^{-1}\chi(D_0)D_0\theta \circ \mu(P)\chi(D_0)$ with $(z - A_D/R)^{-1}$, we obtain

$$\begin{aligned}
F &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) (z - A_D/R)^{-2} R^{-1} \\
&\quad \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) dz \wedge d\bar{z} \theta(D) \\
&\quad + \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \psi^{\mathbb{C}}(z) \theta(D) [(z - A_D/R)^{-1}, R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0)] \\
&\quad (z - A_D/R)^{-1} dz \wedge d\bar{z} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_1 + R^{-1} K_4 \tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \psi'(A_D/R) R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_4 + R^{-1} K_4) \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \langle A_D/R \rangle^{-2s} R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_4 + R^{-1} K_4 \tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D) \\
&= \theta(D) \langle A_D/R \rangle^{-s} R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0) \langle A_D/R \rangle^{-s} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} [\langle A_D/R \rangle^{-s}, R^{-1} \chi(D_0) D_0 \theta \circ \mu(P) \chi(D_0)] \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2} B_4 + R^{-1} K_4 \tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D).
\end{aligned}$$

Suppose $\varphi(t) = \langle t/R \rangle^{-s}$. By the d'Helffer-Sjöstrand formula, we find

$$\begin{aligned}
& [\langle A_D/R \rangle^{-s}, R^{-1}\chi(D_0)D_0\theta \circ \mu(P)\chi(D_0)] \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) [(z - A_D/R)^{-1}, R^{-1}\chi(D_0)D_0\theta \circ \mu(P)\chi(D_0)] dz \wedge d\bar{z} \\
&= R^{-2} \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - A_D/R)^{-1} [A_D, \chi(D_0)D_0\theta \circ \mu(P)\chi(D_0)] \\
&= R^{-2} B'
\end{aligned}$$

where B' is a bounded operator, then

$$\begin{aligned}
F &= \theta(D) \langle A_D/R \rangle^{-s} R^{-1}\chi(D_0)D_0\theta \circ \mu(P)\chi(D_0) \langle A_D/R \rangle^{-s} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2}B_5 + R^{-1}K_4\tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D) \\
&\geq aR^{-1}\theta(D) \langle A_D/R \rangle^{-s} \chi^2(D_0) \langle A_D/R \rangle^{-s} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2}B_5 + R^{-1}K_4\tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D),
\end{aligned}$$

where $a > 0$ is the lower support bound of θ . We define K_5 by $K_5 := \chi^2(D) - \chi^2(D_0)$. K_5 is a compact.

$$\begin{aligned}
F &\geq aR^{-1}\theta(D) \langle A_D/R \rangle^{-s} \chi^2(D) \langle A_D/R \rangle^{-s} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2}B_6 + R^{-1}K_4\tilde{\chi}(D) + R^{-1}K_5\tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D) \\
&\geq aR^{-1}\theta(D) \langle A_D/R \rangle^{-2s} \theta(D) \\
&\quad + \theta(D) \langle A_D/R \rangle^{-s} (R^{-2}B_6 + R^{-1}K_4\tilde{\chi}(D) + R^{-1}K_5\tilde{\chi}(D)) \langle A_D/R \rangle^{-s} \theta(D)
\end{aligned}$$

Decreasing the support of $\tilde{\chi}$ ensures that $\|K_4\tilde{\chi}(D)\| + \|K_5\tilde{\chi}(D)\| \leq a/2$. By choosing $R \gg 1$ such that

$$F \geq a/2 R^{-1} \theta(D) \langle A_D/R \rangle^{-s} \langle A_D/R \rangle^{-s} \theta(D).$$

■

Lemma 3.11 Let $\theta \in C_c^\infty(\mathbb{R})$. Let ϵ be such that $0 < \epsilon < r_0$, $(\theta(D) - \theta(D_0)) \langle A_D \rangle^\epsilon$ is a compact operator.

Proof : Inspired of [GJ1], we have

$$(\theta(D) - \theta(D_0)) \langle A_D \rangle^\epsilon = (\theta(D) - \theta(D_0)) \langle Q \rangle^\epsilon \langle D_x \rangle^\epsilon \langle D_x \rangle^{-\epsilon} \langle Q \rangle^{-\epsilon} (A_D + i)^\epsilon (A_D + i)^{-\epsilon} \langle A_D \rangle^\epsilon.$$

By the Helffer-Sjöstrand formula, $(\theta(D) - \theta(D_0)) \langle Q \rangle^\epsilon \langle D_x \rangle^\epsilon$ is a compact operator, $(A_D + i)^{-\epsilon} \langle A_D \rangle^\epsilon$ is a bounded. It suffice to show that $\langle D_x \rangle^{-\epsilon} \langle Q \rangle^{-\epsilon} (A_D + i)^\epsilon$ is a bounded $\mathcal{L}^2(\mathbb{R}^3)^4$. By the same argument in [GJ1] we show for $\epsilon \in \mathbb{N}$, $\langle D_x \rangle^{-\epsilon} \langle Q \rangle^{-\epsilon} (A_D + i)^\epsilon$ is a compact and by interpolation we obtain for all $\epsilon > 0$. ■

4 Appendix

Proposition 4.1 Let $\Pi_+(P)$ and $\Pi_-(P)$ be given as in (3.0.4). $\Pi_+(P)$ and $\Pi_-(P)$ are orthogonal projectors, and we have

$$\mu(P)(\Pi_+ - \Pi_-)(P) = D_0.$$

Proof : We can first check that $\Pi_+(P)$ and $\Pi_-(P)$ are orthogonal projectors. Indeed,

$$\begin{aligned}
(\Pi_\pm(P))^2 &= \left(\frac{1}{2} \pm \frac{1}{2\sqrt{P^2+m^2}} (\alpha.P + m\beta) \right)^2 \\
&= \frac{1}{4} \pm \frac{1}{2\sqrt{P^2+m^2}} (\alpha.P + m\beta) + \left(\frac{1}{2\sqrt{P^2+m^2}} (\alpha.P + m\beta) \right)^2 \\
&= \frac{1}{4} \pm \frac{1}{2\sqrt{P^2+m^2}} (\alpha.P + m\beta) + \frac{1}{4} \\
&= \frac{1}{2} \pm \frac{1}{2\sqrt{P^2+m^2}} \\
&= \Pi_\pm(P)
\end{aligned}$$

and

$$\begin{aligned}\Pi_+(P)\Pi_-(P) &= \left(\frac{1}{2} + \frac{1}{2\sqrt{P^2+m^2}}(\alpha.P + m\beta)\right)\left(\frac{1}{2} - \frac{1}{2\sqrt{P^2+m^2}}(\alpha.P + m\beta)\right) \\ &= \frac{1}{4} - \left(\frac{1}{2\sqrt{P^2+m^2}}(\alpha.P + m\beta)\right)^2 \\ &= 0.\end{aligned}$$

We can also verify that D_0 is given by $\mu(P)(\Pi_+ - \Pi_-)(P)$, indeed,

$$\begin{aligned}\mu(P)(\Pi_+ - \Pi_-)(P) &= \sqrt{P^2 + m^2}\left(\frac{1}{2} + \frac{1}{2\sqrt{P^2+m^2}}(\alpha.P + m\beta) - \frac{1}{2} + \frac{1}{2\sqrt{P^2+m^2}}(\alpha.P + m\beta)\right) \\ &= \alpha.P + m\beta = D_0.\end{aligned}$$

■

Proposition 4.2 Let μ and F be given by the definition 3.2, we have

$$[\mu(P), iQ.F(P)] = \mu(P)\theta \circ \mu(P).$$

Proof :

$$\begin{aligned}[\mu(P), iQ.F(P)] &= \sum_{j=1}^3 [\mu(P), iQ_j F_j(P)] \\ &= \sum_{j=1}^3 [\mu(P), iQ_j] F_j(P) + Q_j [\mu(P), iF_j(P)] \\ &= [\mu(P), iQ].F(P) + Q.[\mu(P), iF(P)] \\ &= [\mu(P), iQ].F(P) \\ &= \mu'(P)[P, iQ].F(P) \\ &= \mu(P)\theta \circ \mu(P).\end{aligned}$$

■

Proposition 4.3 Let $\theta \in C_c^\infty(\mathbb{R})$ be a function supported in a small interval in $]m, \sqrt{m^2 + k^2/4}[$ or in $] -\sqrt{m^2 + k^2/4}, -m[$. Let f and G_j be given respectively by (3.0.6) and (3.0.7) and F_j is given by the definition 3.2, we have $\theta(D_0)[V_{l,r}(Q), f(P)]\theta(D_0)$, $\theta(D_0)[V_{l,r}(Q), F_j(P)Q_j]\theta(D_0)$, and $\theta(D_0)[V_{l,r}(Q), \alpha_j\beta G_j(P)]\theta(D_0)$ are compact operators on $\mathcal{L}^2(\mathbb{R}^3)^4$.

Proof : $\theta(D_0)\langle Q \rangle^{-r_0}$ and $\theta(D_0)\langle Q \rangle^{-r_0}$ are compact operators on $\mathcal{L}^2(\mathbb{R}^3)^4$, and for $j \in \{1, 2, 3\}$, $\langle Q \rangle^{r_0}[V_{l,r}(Q), \alpha_j\beta G_j(P)]\theta(D_0)$, and $\langle Q \rangle^{r_0}[V_{l,r}(Q), f(P)]\theta(D_0)$ are bounded operators on $\mathcal{L}^2(\mathbb{R}^3)^4$, and by the proposition 2.6. of [GJ1] with $A = P$, $\varphi = F_j$, $k = 1$ and $B = V_{l,r}$, we have $\langle Q \rangle^{r_1} I_1(F_j)\theta(D_0)$ is a bounded operator on $\mathcal{L}^2(\mathbb{R}^3)^4$, where $I_1(F_j)$ is given as in the proposition 2.6. of [GJ1], we obtain the result. ■

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AGM, Université de Cergy Pontoise, Site de Saint Martin, 2, rue Adolphe Chauvin, F-95000 Cergy-Pontoise, France.

E-mail address: aiman.mbarek@u-cergy.fr

LAMA, Université Paris-Est Créteil, Batiment P, 61 avenue du Général de Gaulle 94010 Créteil cedex France.

E-mail address : sana.benhafsia.1994@gmail.com