Super-Quantum Correlations: How to Interpret the No-Signaling Condition?

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Abstract

This article deals with the question of the maximal correlation degree of two intelligent machines that cannot exchange any signals. After reminding the reader of the incorrectness of the mainstream statistical interpretation of the "no-signaling" condition, its informational meaning is explored. It is emphasized that if Pawlowski *et al.*'s Information Causality Principle correctly expresses (and generalizes) the no-signaling condition, its application is, for now, based on a specific scenario (suggested by van Dam) and a no less specific (and simplified) relationship between mutual information and correlators. A more general informational interpretation of the no-signaling condition from which the Tsirelson bound can be derived is then formulated in terms of correlational independence.

Keywords: Super-quantum correlations; no-signaling condition; Tsirelson bound

Introduction

The no-signaling condition (NS) obviously is a condition asserting the impossibility of any exchange of signals or information between physical systems. However, this informational condition is currently reduced to the only statistical independence of the observables that can be respectively measured in these systems. As explained in this article, such a statistical interpretation of the no-signaling condition is far too weak to really express the *informational independence* of all possible events occurring, respectively, in each of these systems.

Section 1 will remind the reader of the incorrectness of this mainstream statistical interpretation of the no-signaling condition. In section 2, its informational interpretation by Pawlowski et *al*'s Information Causality Principle will be considered. It will be emphasized that

the derivation of the Tsirelson bound they propose lacks of generality. Section 3 will propose a more general informational formulation of the no-signaling condition in terms of correlational independence that allows a purely informational derivation of the Tsirelson bound (section 4).

1. Incorrectness of the mainstream statistical interpretation of the (NS) condition

One refers to the usual bipartite (Alice and Bob) Bell game where the question of super-quantum correlations is generally discussed –see, for example Popescu[1], Barrett *et al.*[2] or Gisin[3]. In this bipartite game, which can be seen as modeling the possible connection between two intelligent machines, the incomes chosen by Alice and Bob are noted as x and y respectively, and the corresponding outcomes, noted as a and b respectively, can take the values 0 or 1. A "box" is an abstract description of the relation between inputs and corresponding outputs, that is, it is the given of the conditional probabilities P(a,b/x,y) for all combinations of a, b, x and y. In the current literature, the no-signaling condition, called (NS) in the following, is interpreted as a condition of statistical independence. This condition asserts that the probability that Alice obtains a particular outcome "a" is independent of Bob's choice, that is, of the value of y, and vice versa[1-5]. This statistical condition is formulated as follows:

$(NS)_{stat}$ For all possible actions x, x', y, y' and for all possible outcomes a and b:

$$\Sigma_b P(a, b / x, y) = \Sigma_b P(a, b / x, y') \quad \text{and} \quad \Sigma_a P(a, b / x, y) = \Sigma_a P(a, b / x', y),$$

which can be more simply written as:

$$P(a / x, y) = P(a / x) \text{ and } P(b / x, y) = P(b / y) \text{ for all a, b, x and y.}$$

As shown by Popescu and Rohrlich, $(NS)_{stat}$ allows the "theoretical" existence of super-quantum correlations, that is, correlations whose CHSH degree is greater than the Tsirelson bound[6]:

(SQ) $$|R| > 2\sqrt{2},$$

where the Bell number R is defined as[7]:

$$R = C_{00} + C_{01} + C_{10} - C_{11}$$

and the correlators are defined as the expectation values of the products of the respective outcomes a_i and b_j of the conjunction of actions "Alice chooses x and Bob chooses y":

$$C_{xy} = \Sigma_{i,j}\ p(a_i, b_j / x, y)\ a_i b_j$$

In particular, Popescu and Rohrlich have shown that PR-boxes, defined as:

$$a \oplus b = x.y, \text{ where "}\oplus\text{" is the addition modulo 2,}$$

or, equivalently, in terms of probabilities of outcomes:

(PR) $$P(a, b / x, y) = ½ \quad \text{if } a \oplus b = x.y \text{ is realized}$$
$$= 0 \quad \text{otherwise}$$

satisfy the condition $(NS)_{stat}$ *and* violates maximally (SQ) -that is, for PR-boxes $|R| = 4$.

However, as shown by Uzan[8-9], counter-examples of boxes that satisfy the statistical condition $(NS)_{stat}$ *and* allow the exchange of signals informing one of the parties of the choice of the other can easily be found. For example, in a deterministic box, for which x_i determines the outcome a_i for Alice and y_j determines the outcome b_j for Bob, the fact that Bob is informed about Alice's

choice *does not change at all the conditional probability distribution* that defines this deterministic box:

(D) $$P(a_i, b_j / x_{i'}, y_{j'}) = \delta_{i\,i'} \cdot \delta_{j\,j'}.$$

In this situation, $(NS)_{stat}$ is satisfied since $p(a_i/x_{i'},y) = p(a_i / x_{i'}) = \delta_{ii'}$, which is independent of y, and $p(b_j/x,y_{j'}) = p(b_j / y_{j'}) = \delta_{j\,j'}$, which is independent of x. However, (NS) is obviously not satisfied since Bob is informed about Alice's choice. This communication between Alice and Bob has no influence on the *conditional* probabilities of (D) that regard the probability of obtaining a result *if* they chose to draw such or such inputs (which are the values of x and y).

Let us emphasize that this exchange of information between Alice and Bob *does not define a distinct communication channel* beside the one defined by this deterministic box (that is, by the given of the conditional probabilities P (a, b/ x, y) above, in (D)): the definition (D) correctly characterizes such a deterministic box where Alice and Bob exchange information about their inputs. This exchange of information can be regarded as an integral part of this deterministic box since it is taken into account in its very definition –nothing more needs to be added to this definition to say that Alice and Bob can communicate about their choice of input. Also note that even in the case Bob *explicitly uses the information sent by Alice about her input,* for example by drawing y = x, this protocol still remains compatible with the definition (D). This deterministic box satisfies $(NS)_{stat}$ even though Bob is informed about Alice's choice of input and can use this information for selecting his input. *The definition (D) does not impose that an exchange of information between Alice and Bob is impossible.*

Another paradigmatic example of such a signalling box that nevertheless satisfies the statistical condition $(NS)_{stat}$ is nothing but… the PR-box defined above. The definition of this box, which satisfies $(NS)_{stat}$ (see above), *does not at all prevent Bob to be explicitly informed about Alice's choice and even to choose his income according to the latter*, since this exchange of information is quite compatible with the conditional probability distribution (PR) that defines the PR-box. Indeed, as was the case for the deterministic box above, this exchange of information between Alice and Bob and the fact that Bob's choice is made according to Alice's choice is *independent* of the definition of this PR-box, which is the given of the conditional probabilities (PR). *A PR-box does not prevent Alice and Bob to exchange information about their input.*

Thus, it can be concluded that *the statistical condition $(NS)_{stat}$ is too weak to characterize (NS), which denotes the absence of any exchange of information between the parties, and, in particular, any exchange of "significant" information that could be used by the parties to choose their input.* Consequently, the mainstream results about super-quantum correlations is skewed by this statistical misinterpretation: if $(NS)_{stat}$ allows the existence of super-quantum correlations that satisfy (SQ), it could be quite wrong that the stronger no-signaling condition (NS) allows their existence –which is actually the case (see below)! We then have to reject this statistical interpretation of (NS) -unless we use $(NS)_{stat}$ *only as a necessary (but not sufficient) condition of (NS)*.

2. The no-signaling condition as an informational constraint

(NS) obviously is a condition of *informational independence* asserting that what occurs in Alice's subsystem cannot be known by Bob, whatever he does –and symmetrically, that what occurs in Bob's subsystem cannot be known by Alice, whatever she does. Along this line of thought, Pawlowski *et al.*[10] provided such a purely informational formulation of the no-signaling condition. These authors have suggested that (NS) could be reasonably interpreted by the Information Causality Principle according to which "Bob can gain no more than m (classical) bits of information about Alice's data if she sends him only m bits", with the additive constraint that m = 0 (no information is sent by Alice).

This informational condition is then applied by considering the amount of Shannon information that can be exchanged between the two parties in a specific scenario designed by van Dam[11]: Alice is endowed with a list of N random and independent bits $\{a_0, \ldots, a_k \ldots, a_{N-1}\}$ and sends m classical bits to Bob, with the help of which Bob tries to guess the value of the bth bit in Alice's list. If Bob can perfectly determine this bth bit when Alice sends him the sequence $\{a_0, \ldots, a_{m-1}\}$ and $0 \leq b \leq m-1$, it is not the case for $b > m-1$. In order to evaluate his chance of success, these authors then compute the Shannon mutual information between Alice's data and Bob's output, denoted as β, when Alice sends him this message of m classical bits:

$$I \equiv \sum_{k=0 \text{ to } N} I\,(a_k : \beta \,/\, b = k),$$

and they assert, in agreement with their Information Causality Principle, that this mutual information must be smaller than m: $I \leq m$. *In this scenario,* they compute Bob's probabilities of success for guessing specific bits in Alice's data, probabilities which are involved in the CHSH correlation factor *and* in the mutual information I. They can show that, *always in this scenario*, the Information Causality principle is violated as soon as the Tsirelson bound is.

These authors also claim that this result remains true for any no-signaling box. To show this, they use the fact that, as shown by Masanes *et al.*[12], the local outcomes of any no-signaling box are uniformly random and that their conditional probabilities can be expressed as the following function of the Bell number (and then in terms of the correlators)–with the above notations:

$$P\,(a \oplus b = x.y \,/\, x, y) = \tfrac{1}{2}\,(1 + R \,/\, 4)$$

This relationship between the conditional probabilities, which are involved in the computation of the mutual information I, and the correlators C_{xy}, which are involved in the Bell number R, is then used to derive the Tsirelson bound for any no-signaling box. However, this more general derivation (for any no-signaling box) still lacks generality since the mutual information I is still computed *for the specific scenario suggested by van Dam,* the consequence of which being that the mutual information I can be written as a specific function of the correlators C_{xy} which can be minored by an expression that directly involves (the square of) the Bell number R (see the supplementary materials of this reference[10], page 8). However, it is not at all certain that this

specific minoration of the mutual information I, which is essential to derive the Tsirelson bound, generally holds. In the general case the relationship between mutual information and correlators looks more complex and might even reverse this order[13]. Pawlowski *et al.*'s derivation is then based on a specific scenario and a specific relationship between correlators and mutual information.

Could this derivation be made more general, that is, without appealing to a specific scenario and to a particular relationship between mutual information and correlators? As will be shown in the following, a more general derivation of the Tsirelson bound from the no-signaling condition can be proposed if the latter informational condition is expressed by the *choice-independence of the correlators between one party's choice of action and all the possible activities of the other.* This derivation, which is made in terms of correlators, is not based on a specific scenario of information exchange.

3. A more general, informational interpretation of (NS)

What we want to express is the informational independence of the events occurring in the two subsystems: by her activity of choosing an input and observing the outcome, Alice cannot obtain any information about Bob's activity, and vice-versa. However, this independence cannot be reduced (as is the case in $(NS)_{stat}$) to the only independence of Alice's conditional probabilities of outcomes and Bob's choice of input (the value of y), and vice versa. As reminded in section 1, this condition of statistical independence *is not sufficient*: it does not forbid the two parties to communicate.

In order to forbid any communication capable of informing Alice about Bob's choice of action, we have to set a more constraining condition bearing on *the independence of Alice's all possible activities and Bob's choice of input –and symmetrically, between Bob's all possible activities and Alice's choice of input.* This condition can be expressed in terms of the relevant *correlators:* In order to impose the condition according to which Alice's choice A_0 or A_1 (for the two possible inputs, noted as 0 or 1) does not have any influence on Bob's activity -and vice versa- we have to assert that the correlator of Alice's activities and Bob's choice of input does not depend on the latter and that the correlator of Bob's activities and Alice's choice of input does not depend on the latter. This condition can simply be expressed by the equality of the correlators of Bob's activity and Alice's *all possible choices* –and symmetrically, by the equality of Alice's activity and Bob's all possible choices.

In order to formulate this constraint, let's consider the correlators between Alice's choice A_0 and Bob's all possible activities, that is, when Bob chooses B_0 and obtains the outcome $b_0{}^j$ *or* when Bob chooses B_1 and obtains the outcome $b_1{}^k$. The respective correlators between these two exclusive activities of Bob and Alice's choice A_0 are, respectively, C_{00} and C_{01}. As will be

shown hereafter, the correlation degree between Alice's choice A_0 and Bob's *all possible activities* (that is, whatever Bob's choice and whatever Bob's results) can be defined by the product of these correlators: $C(A_0, B) = C_{00}.C_{01}$.

This definition merely extends that of the correlators for two random variables: $C(A_0, B)$ is computed as the expected value of the product of possible outcomes of the event relating the (sequential) realization of the two following independent events "Alice choose A_0 and Bob chooses B_0" and "Alice choose A_0 and Bob chooses B_1". These two joint events, which are incompatible (since B_0 and B_1 cannot be chosen at the same time), can be understood as occurring sequentially, say at times t1 and t2 > t1 respectively, and, as mentioned above, *they are independent* (because A_x and A_y are, like B_x and B_y, randomly chosen). If the outcomes of these sequential joint events are respectively noted as $(a_0^i, b_0^j)_{t1}$ and $(a_0^{i'}, b_1^k)_{t2}$, their expectation value, which provides an evaluation of the correlation degree between Alice's choice A_0 and Bob's all possible activities, can thus be computed as:

$$C(A_0, B) = \Sigma_{i,i',j,k}\, p((a_0^i, b_0^j)_{t1}, (a_0^{i'}, b_1^k)_{t2}/(x_0, y_0)_{t1}, (x_0, y_1)_{t2})\, a_0^i b_0^j a_0^{i'} b_1^k$$

where the joint sequential probability $p((a_0^i, b_0^j)_{t1}, (a_0^{i'}, b_1^k)_{t2}/(x_0, y_0)_{t1}, (x_0, y_1)_{t2})$ denotes the probability of the conjunction of events "Alice draws x_0 at time t1 and obtains a_0^i while Bob draws y_0 and obtains b_0^j" and "Alice draws x_0 at time t2 and obtains $a_0^{i'}$ while Bob draws y_1 and obtains b_0^k". Since these two joint events are independent, the probability of their conjunction can be factorized:

$$p((a_0^i, b_0^j)_{t1}, (a_0^{i'}, b_1^k)_{t2}/(x_0, y_0)_{t1}, (x_0, y_1)_{t2}) = p(a_0^i, b_0^j/x_0, y_0) \,.\, p(a_0^{i'}, b_1^k/x_0, y_1).$$

(Note that in the above expression the mention of time has been deleted since it has no other significance than that of remembering that these events cannot be realized at the same time).
The associated "composed" correlator $C(A_0, B)$ can thus be written as:

$$C(A_0, B) = (\Sigma_{i,j}\, p(a_0^i, b_0^j/x_0, y_0)\, a_0^i b_0^j)\,(\Sigma_{i',k}\, p(a_0^{i'}, b_1^k/x_0, y_1)\, a_0^{i'} b_1^k) = C_{00}\, C_{01}.$$

A similar reasoning shows that the correlator of A_1 and Bob's all possible activities is:

$$C(A_1, B) = C_{10}.C_{11}.$$

The condition that Bob's all possible activities are performed independently of Alice's choice of action (x = 0 or 1), which is expressed as $C(A_0, B) = C(A_1, B)$, can thus be written as:

$$C_{00} \,.\, C_{01} = C_{10} \,.\, C_{11}$$

Similarly, the condition that Alice's all possible activities are performed independently of Bob's choice of action (y = 0 or 1) is: $C_{00} \,.\, C_{10} = C_{01} \,.\, C_{11}$
Consequently, the no-signaling condition (NS) can be express as follows:

(NS) $$C_{00} \,.\, C_{01} - C_{10} \,.\, C_{11} = C_{00} \,.\, C_{10} - C_{01} \,.\, C_{11} = 0.$$

Note that this condition means that the absolute values of the correlators are equal two by two: $|C_{00}| = |C_{11}|$ and $|C_{01}| = |C_{10}|$. However, the expression of (NS) written above will be more useful below for deriving the Tsirelson bound.

4. The Tsirelson bound derived from the no-signaling condition

Let's consider the square of the CHSH correlation factor R defined above:

$$R^2 = (C_{00} + C_{01})^2 + (C_{10} - C_{11})^2 + 2\,(C_{00} + C_{01})\,(C_{10} - C_{11}).$$

Since the absolute values of the correlators C_{xy} are all bounded by 1, their squares are also bounded by 1 and then:

$$R^2 \leq 4 + 2\,(C_{00}\,C_{01} - C_{10}\,C_{11} + C_{00}\,C_{10} - C_{01}\,C_{11} + C_{01}\,C_{10} - C_{00}\,C_{11})$$

Using now the no-signaling condition (NS) expressed above, the combination of the four first products between brackets is equal to zero:

$$C_{00}\,C_{01} - C_{10}\,C_{11} + C_{00}\,C_{10} - C_{01}\,C_{11} = 0.$$

Consequently, we have: $R^2 \leq 4 + 2\,(C_{01}\,C_{10} - C_{00}\,C_{11})$.

Taking now into account that $|C_{xy}| \leq 1$, one has $|C_{01}\,C_{10} - C_{00}\,C_{11}| \leq 2$ and then $R^2 \leq 8$. Consequently the Tsirelson bound is satisfied: $|R| \leq 2\sqrt{2}$.

The no-signaling condition, if correctly understood as an informational constraint asserting the informational independence of the events occurring in the two subsystems is thus sufficient to derive the Tsirelson bound. Of course, this result corroborates Pawlowski's derivation, which is based on a specific (van Dam's) scenario and then on a particular relationship between mutual information and correlators (see section 2), but it has been shown here more generally. The non-signaling condition has been expressed in terms of correlators, as the independence of the activities of each of the parties and the choice of action of the other. This very general result shows that, unless non-local communication (conflicting with quantum theory) would be possible, the maximal correlation degree of two intelligent machines that cannot exchange any signal is the Tsirelson bound.

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