Generation of measures on the torus with good sequences of integers

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Let $S := (s_1 < s_2 < \dots)$ be a strictly increasing sequence of positive integers and denote $\mathrm{e}(\beta) := e^{2\pi i\beta}$. We say S is *good* if for every real α the limit $\lim_N \frac{1}{N}\sum_{n\le N} \mathrm{e}(s_n\alpha)$ exists. By the Riesz representation theorem, a sequence S is good iff for every real α the sequence $(s_n\alpha)$ possesses an asymptotic distribution modulo 1. Another characterization of a good sequence follows from the spectral theorem: the sequence S is good iff in any probability measure preserving system $(X, \mathfrak{m}, T)$ the limit $\lim_N \frac{1}{N}\sum_{n\le N} f(T^{s_n}x)$ exists in L^2-norm for $f \in L^2(X)$.

Of these three characterization of a good set, the one about limit measures is the most suitable for us, and we are interested in finding out what the limit measure $\mu_{S,\alpha} := \lim_N \frac{1}{N}\sum_{n\le N} \delta_{s_n\alpha}$ on the torus can be. In this first paper on the subject, we investigate the case of a single irrational α. We show that if S is a good set then for every irrational α the limit measure $\mu_{S,\alpha}$ must be a continuous Borel probability measure. Using random methods, we show that the limit measure $\mu_{S,\alpha}$ can be any measure which is absolutely continuous with respect to the Haar-Lebesgue probability measure on the torus. On the other hand, if ν is the uniform probability measure supported on the Cantor set, there are some irrational α so that for no good sequence S can we have the limit measure $\mu_{S,\alpha}$ equal ν. We leave open the question whether for any continuous Borel probability measure ν on the torus there is an irrational α and a good sequence S so that $\mu_{S,\alpha} = \nu$.

Contents

1 Introduction, main results

Throughout the paper we will use the *arithmetic average operator* $\mathbb{A}$: for a finite index set S, a vector space V and a $S \to V$ function f we define $\mathbb{A}_S f(s)$

$$\mathbb{A}_S f(s) = \mathbb{A}_{s\in S} f(s) := \frac{1}{\#S} \sum_{s\in S} f(s) \tag{2.1}$$

where $\#S$ denotes the number of elements in S.

We use the convention that if an interval appears as an index set in a summation then we consider only the integers in the interval. For example, $\sum_{n\in[0,N)} a_n = \sum_{n\in\{0,1,\dots,N-1\}} a_n$.

We also use Weyl's notation $\mathrm{e}(\beta) := e^{2\pi i\beta}$. Note that $\mathrm{e}^p(\beta) = \mathrm{e}(p\beta)$ for every integer p.

We denote by $\mathbb{T}$ the torus $\mathbb{R}/\mathbb{Z}$ and we represent it as the unit closed interval $[0,1]$ with $0 = 1$.

1.1 Good sequences, main question

2.1 Definition ▶ Good sequence

We say that a sequence $S = (s_n)_{n\in\mathbb{N}}$ of integers is *good* if the limit $\lim_N \mathbb{A}_{n\in[1,N]} \mathrm{e}(s_n\alpha)$ exists for every real number α.

Good sequences have been extensively studied in many parts of mathematics, such as in number theory and ergodic theory.

In this paper we restrict our attention to strictly increasing sequences S of positive integers in which case we can and will consider S as a subset of $\mathbb{N}$, and we'll use the concept of good sequence and good set interchangeably.

Among the wellknown good sequences are the full set $\mathbb{N}$ of positive integers[1] , the sequence $(n^2)_{n\in\mathbb{N}}$ of squares[2] and the sequence $(p_n)_{n\in\mathbb{N}}$ of primes[3] where p_n denotes the nth prime number. For these sequences the limits $\lim_N \mathbb{A}_{n\in[1,N]}\, \mathrm{e}(s_n\alpha)$ are as follows

[1] Weyl 1916.
[2] Weyl 1916.
[3] Vinogradow 1937.

$$\begin{aligned}
\lim_N \mathbb{A}_{n\in[0,N)}\, \mathrm{e}(n\alpha) &= \begin{cases} 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 0 \end{cases} \\
\lim_N \mathbb{A}_{n\in[0,N)}\, \mathrm{e}(n^2\alpha) &= \begin{cases} \mathbb{A}_{b\in[1,q]}\, \mathrm{e}\left(b^2\frac{a}{q}\right) & \text{if } \alpha = \frac{a}{q},\ \gcd(a,q) = 1 \\ 0 & \text{if } \alpha \text{ is irrational} \end{cases} \\
\lim_N \mathbb{A}_{n\in[1,N]}\, \mathrm{e}(p_n\alpha) &= \begin{cases} \mathbb{A}_{\substack{b\in[1,q] \\ \gcd(b,q)=1}}\, \mathrm{e}(b/q) & \text{if } \alpha = \frac{a}{q},\ \gcd(a,q) = 1 \\ 0 & \text{if } \alpha \text{ is irrational} \end{cases}
\end{aligned} \tag{3.1}$$

In case of a good sequence $S = (s_n)$ and a fixed α, the existence of $\lim_N \mathbb{A}_{n\in[1,N]}\, \mathrm{e}(s_n p\alpha)$ for every $p \in \mathbb{Z}$ implies, by uniform approximation of a continuous $\mathbb{T} \to \mathbb{C}$ function by trigonometric polynomials, that for every continuous $\mathbb{T} \to \mathbb{C}$ function ϕ the limit $\lim_N \mathbb{A}_{n\in[1,N]}\phi(s_n\alpha)$ exists. By the Riesz representation theorem, this implies that the weak limit $\lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ of discrete measures $\mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ on $\mathbb{T}$ exists.

By this argument, the existence of $\lim_N \mathbb{A}_{n\in[1,N]}\, \mathrm{e}(s_n\alpha)$ for every α implies the existence of the limit measure $\lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ for every α. Denote the Haar-Lebesgue probability measure on the torus $\mathbb{T}$ by λ and recall that the Fourier coefficients $\lambda(\mathrm{e}^p)$ of λ satisfy

$$\lambda(\mathrm{e}^p) = \begin{cases} 1 & \text{for } p = 0 \\ 0 & \text{for } p \in \mathbb{Z},\ p \neq 0 \end{cases} \tag{3.2}$$

where for a given measure ν and ν-integrable function ϕ, we use[4] the functional notation $\nu(\phi)$ for the integral of ϕ with respect to ν,

[4] and will use troughout the paper

$$\nu(\phi) = \int \phi \,\mathrm{d}\nu \tag{3.3}$$

For our three good sets the limit measures are as follows.

$$\begin{aligned}
\lim_N \mathbb{A}_{n\in[1,N]}\delta_{n\alpha} &= \begin{cases} \mathbb{A}_{b\in[1,q]}\delta_{b/q} & \text{if } \alpha = a/q,\ \gcd(a,q)=1 \\ \lambda & \text{if } \alpha \text{ is irrational} \end{cases} \\
\lim_N \mathbb{A}_{n\in[1,N]}\delta_{n^2\alpha} &= \begin{cases} \mathbb{A}_{b\in[1,q]}\delta_{b^2\frac{a}{q}} & \text{if } \alpha = \frac{a}{q},\ \gcd(a,q)=1 \\ \lambda & \text{if } \alpha \text{ is irrational} \end{cases} \\
\lim_N \mathbb{A}_{n\in[1,N]}\delta_{p_n\alpha} &= \begin{cases} \mathbb{A}_{\substack{b\in[1,q] \\ \gcd(b,q)=1}}\delta_{b/q} & \text{if } \alpha = \frac{a}{q},\ \gcd(a,q)=1 \\ \lambda & \text{if } \alpha \text{ is irrational} \end{cases}
\end{aligned} \tag{4.1}$$

What we see in these three examples is that in case of irrational α the limit measure is the Haar-Lebesgue measure λ and in case of rational $\alpha = a/q$, $\gcd(a,q) = 1$, the limit measure is supported on a subset of the qth roots of unity and appears to be quite uniform on its support. In case of irrational α, the simplest question is if it's possible that the limit measure is not λ. In case of rational α, we can ask if the limit measure always has to show some kind of uniformity.

Let us consider a good sequence $S = (s_n)$. The existence of the limit $\lim_N \mathbb{A}_{n\in[1,N]} \mathrm{e}(s_n\alpha)$ for every α implies that the weak limit $\lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ of discrete measures $\mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ on $\mathbb{T}$ exists for every α. Let us denote this weak limit measure by $\mu_{S,\alpha}$,

$$\mu_{S,\alpha} := \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha} \tag{4.2}$$

The main question we want to investigate in this paper is

4.1 Question ▶ Main question

What can the limit measure $\mu_{S,\alpha}$ be? Can it be any Borel probability measure on $\mathbb{T}$?

1.2 *Main results*

As we stated earlier, we try to answer question 4.1 for strictly increasing sequences, and unless we say otherwise, we assume from now on that $S = (s_n)$ is a strictly increasing sequence of positive integers which we often consider as a subset of $\mathbb{N}$.

Our first observation is that the answer to question 4.1 will depend on α. If α is a rational number, say, $\alpha = \frac{a}{q}$ with $\gcd(a,q) = 1$, then the limit measure is clearly supported on the set

$$\mathbb{T}_q := \{\, b/q \,:\, b \in [1,q] \,\} \tag{4.3}$$

of qth roots of unity. So the question is if the limit measure $\mu_{S,a/q}$ can be any probability measure supported on $\mathbb{T}_q$? The answer is yes. First a terminology.

5.1 Definition ► Representable measure at α

Let S be a good set, and let ν be a nonzero, finite Borel measure on $\mathbb{T}$.
We say that S *represents* ν *at* $\alpha \in \mathbb{T}$ if $\mu_{S,\alpha} = \frac{1}{\nu(\mathbb{T})}\nu$.
We say ν *is representable at* α if there is a good set which represents ν at α.

5.2 Theorem ► Every probability measure on $\mathbb{T}_q$ can be represented

Let q and a be positive integers with $\gcd(a,q) = 1$, and let ν be a probability measure supported on the set $\mathbb{T}_q$ of qth roots of unity.
Then ν can be represented at $\frac{a}{q}$, that is, there is a good set S so that $\mu_{S,\frac{a}{q}} = \nu$.

Before discussing the limit measure $\mu_{S,\alpha}$ for irrational α, let us note the following fact which will help us appreciate the concept of a good set.

Suppose we are given an irrational number $\alpha \in \mathbb{T}$ and a Borel probability measure ν on $\mathbb{T}$. We claim that there exists a sequence (x_n) in $\mathbb{T}$ with asymptotic distribution ν, i. e. such that $\lim_N \mathbb{A}_{n\in[1,N]}\delta_{x_n} = \nu$. Considering such a sequence and using the density of the sequence $(n\alpha)_n$ in $\mathbb{T}$, we can select a strictly increasing sequence (s_n) of integers so that $\lim_n(s_n\alpha - x_n) = 0 \mod 1$, and we have $\lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha} = \nu$. Taking $S = \{\, s_n : n \in \mathbb{N} \,\}$, we could say that $\mu_{S,\alpha} = \nu$, but nothing insures us that the set S is good.

There are different ways to prove the preceding claim. For example we can pick the numbers x_n randomly and independently with law ν, and the strong law of large numbers asserts that the sequence (x_n) has, almost surely, the right asymptotic distribution.

It is particularly simple to get a point-mass as a limit measure. For example, to get the Dirac measure at $1/2$, so $\nu = \delta_{1/2}$, take a strictly increasing sequence (s_n) of natural numbers so that $s_n\alpha$ converges to $1/2 \mod 1$, and let $S := \{\, s_n : n \in \mathbb{N} \,\}$. In contrast to this example, for good sets we have a dramatic departure from the case of rational α.

5.3 Theorem ► $\mu_{S,\alpha}$ is continuous for irrational α

Only continuous measures can be represented at an irrational number.
To spell this out, let $S = (s_n)$ be a good sequence and α be an irrational number.
Then the limit Borel probability measure $\mu_{S,\alpha} = \lim_N \mathbb{A}_{n\in[1,N]}\delta_{s_n\alpha}$ is a continuous measure.

The obvious question in turn is if any given continuous Borel probability measure can be represented at any irrational number. The answer is no, as the next result shows.

6.1 Theorem ▶ Some continuous measures cannot be represented at every irrational point

Let ν be a Borel probability measure on $\mathbb{T}$ so that its Fourier coefficients do not converge to 0, so

$$\limsup_{p\to\infty} |\mu(e^p)| > 0 \tag{6.1}$$

Then there is a set $A \subset \mathbb{T}$ of full Lebesgue measure so that ν cannot be represented at any $\alpha \in A$.

Since a measure ν is called a *Rajchman* measure[5] if its Fourier coefficients vanish at infinity, that is, $\lim_p \nu(e^p) = 0$, we can rephrase theorem 6.1 by saying that if ν is representable at every irrational α then it must be a Rajchman measure. A well known non-Rajchman continuous measure is the uniform measure on the triadic Cantor set.

While theorem 6.1 doesn't exclude the possibility that $A = \mathbb{T}$, that is, a non-Rajchman measure cannot be represented anywhere, Christophe Cuny and François Parreau[6] constructed a non-Rajchman measure which is representable at uncountably many α's. Nevertheless, the following question remains open.

[5] Lyons 1995.

[6] Parreau and Cuny 2022.

6.2 Question ▶ Is every continuous measure representable somewhere?

Let ν be a continuous Borel probability measure on $\mathbb{T}$.
Is there an irrational α so that ν is representable at α?

The next result says that if ν is absolutely continuous with respect to the Lebesgue probability measure λ on the torus $\mathbb{T}$, then it can be represented at every irrational α.

6.3 Theorem ▶ Absolutely continuous measures are representable at every irrational point

Let ν be a Borel probability measure on $\mathbb{T}$ which is absolutely continuous with respect to the Lebesgue probability measure on $\mathbb{T}$. Let α be an irrational number.
Then ν is representable at α.

Our proof of theorem 6.3 is flexible and enables us to show a more general result, namely it turns out that a given absolutely continuous measure can be represented by a good subset of any given good set, provided it doesn't increase too fast, it is sublacunary. For a given set $R \subset \mathbb{N}$ let $R(N)$ denote the Nth initial segment of R,

$$R(N) := R \cap [1, N] \tag{6.2}$$

We say R is *sublacunary*[7] if it satisfies the growth condition

[7] Traditionally, (r_n) is called lacunary if it satisfies $\liminf_n \frac{r_{n+1}}{r_n} > 1$, and such a sequence satisfies $\#R(N) = O(\log N)$. Traditionally, a sublacunary sequence is one that satisfies $\lim_n \frac{r_{n+1}}{r_n} = 1$ and such a sequence satisfies $\lim_N \frac{\#R(N)}{\log N} = \infty$. Our definion of a sublacunary sequence in eq. (7.1) describes sequences which satisfy $\liminf_n \frac{r_{n+1}}{r_n} = 1$ but may not satisfy $\lim_n \frac{r_{n+1}}{r_n} = 1$.

$$\lim_N \frac{\#R(N)}{\log N} = \infty \tag{7.1}$$

In case we consider the sequence (r_n) instead of the set R, it's more useful to write eq. (7.1) in the form

$$\lim_N \frac{N}{\log r_N} = \infty \tag{7.2}$$

7.1 Theorem ▶ Absolutely continuous measures can be represented by subsets of a good set

Let R be a sublacunary good set. Let α be an irrational number, and let the Borel probability measure ν be absolutely continuous with respect to $\mu_{R,\alpha}$.
Then there is a good set $S \subset R$ which represents ν at α.

We will see that the proof of theorem 7.1 reveals a close connection between the Radon-Nikodym derivative ρ of ν with respect to $\mu_{R,\alpha}$ and the relative mean[8] of the set S representing ν. For a given $R \subset \mathbb{N}$ and $S \subset R$, the *relative mean* $\mathbf{M}_R(S)$ of S in R is defined by

$$\mathbf{M}_R(S) := \lim_N \frac{\#S(N)}{\#R(N)} \tag{7.3}$$

provided the limit on the right exists. The *relative upper mean* $\overline{\mathbf{M}}_R(S)$ of S in R is defined by

$$\overline{\mathbf{M}}_R(S) := \limsup_N \frac{\#S(N)}{\#R(N)} \tag{7.4}$$

In case $R = \mathbb{N}$, we suppress the base set in our notation, and we write $\mathbf{M}(S)$ for $\mathbf{M}_{\mathbb{N}}(S)$ and $\overline{\mathbf{M}}(S)$ for $\overline{\mathbf{M}}_{\mathbb{N}}(S)$.

As a consequence of theorem 7.1, every measure which is absolutely continuous with respect to the Lebesgue measure can be represented at any given irrational α by a subset of the primes, squares, or $\{ \lfloor n^2 \log n \rfloor : n \in \mathbb{N} \}$.

[8] The usual terminology is relative *density* instead of relative mean, but we will use the more general concept of the *mean of a $R \to \mathbb{C}$ function* in section 1.3 and we prefer to use a single terminology and notation for economical reasons.

7.2 Theorem ▶ Connection between $\frac{d\nu}{d\mu_{R,\alpha}}$, $\mathbf{M}_R(S)$ and $\overline{\mathbf{M}}_R(S)$

Let R be a sublacunary good set.

a) For an irrational α let the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ with $\mu_{R,\alpha}(\rho) = 1$ be bounded so $\|\rho\|_{L^\infty(\mu_{R,\alpha})} < \infty$.
Then there is a good set $S \subset R$ representing the measure $\rho \cdot \mu_{R,\alpha}$ at α and satisfying $\mathbf{M}_R(S) = \frac{1}{\|\rho\|_{L^\infty(\mu_{R,\alpha})}}$.

b) Let S be a good subset of R with positive upper density in R, so $\overline{\mathbf{M}}_R(S) > 0$.
Then for every irrational β the limit measure $\mu_{S,\beta}$ is absolutely continuous with respect to $\mu_{R,\beta}$. Furthermore, the Radon-Nikodym derivative $\rho_\beta := \frac{d\mu_{S,\beta}}{d\mu_{R,\beta}}$ is a bounded function satisfying $\|\rho_\beta\|_{L^\infty(\mu_{R,\beta})} \le \frac{1}{\overline{\mathbf{M}}_R(S)}$.

We see that theorem 7.1 gives a *full characterization* of the limit measure for sets with positive upper mean[9], giving an exact relationship between the upper mean of the set and the bound of the RN derivative: On the one hand if $\overline{\mathbf{M}}(S) > 0$, the limit measure $\mu_{S,\beta}$ for every β must be absolutely continuous with respect to λ with bounded RN derivative ρ_β satisfying $\|\rho_\beta\|_{L^\infty(\lambda)} \leq \frac{1}{\overline{\mathbf{M}}(S)}$. On the other hand, any Borel probability measure ν which is absolutely continuous with respect to λ with bounded, nonzero RN derivative ρ is representable at any irrational α with a set of positive mean satisfying $\mathbf{M}(S) = \frac{1}{\|\rho\|_{L^\infty(\lambda)}}$.

[9] so now $R = \mathbb{N}$

Theorem 7.2 (b) has the following consequence.

8.1 Corollary ▶ If the RN derivative ρ is unbounded, then $\mathbf{M}_R(S) = 0$

Let R be a good set and α an irrational number. Suppose the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ with $\mu_{R,\alpha}(\rho) = 1$ is unbounded, and that the good set $S \subset R$ represents the measure $\rho \cdot \mu_{R,\alpha}$ at α.
Then S must have 0 mean in R, so $\mathbf{M}_R(S) = 0$.

1.3 *Weighted averages*

Our results in theorems 7.1 and 6.3 will be consequences, via a random procedure, of results on weighted averages.

We need to fix some terminology and notation. We define the Besicovitch type seminorm $\|\|_{\mathbf{1}}$ for all complex valued sequences $f \in \mathbb{C}^{\mathbb{N}}$ by

$$\|f\|_{\mathbf{1}} := \limsup_N \mathbb{A}_{[1,N]}|f|, \quad f \in \mathbb{C}^{\mathbb{N}} \tag{8.1}$$

The number $\mathbf{1}$ in the subscript of $\|\|_{\mathbf{1}}$ expresses the similarity of this norm to the L^1 norm.

For a set $S \subset \mathbb{N}$, we may use the notation $\|S\|_{\mathbf{1}}$ instead of $\|\mathbb{1}_S\|_{\mathbf{1}}$, though in this case we do not get a new concept, since $\|S\|_{\mathbf{1}} = \overline{\mathbf{M}}(S)$.

For an infinite set $R \subset \mathbb{N}$ we define the relative $\mathbf{1}$-norm $\|f\|_{\mathbf{1},R}$ of a complex valued $R \to \mathbb{C}$ function by

$$\|f\|_{\mathbf{1},R} := \limsup_N \mathbb{A}_{R(N)}|f|, \quad f \in \mathbb{C}^R \tag{8.2}$$

If the set R is given as a strictly increasing sequence (r_n) and for an $f \in \mathbb{C}^R$ we define F by $F(n) := f(r_n)$, then $\|f\|_{\mathbf{1},R} = \|F\|_{\mathbf{1}}$.

Let $R \subset \mathbb{N}$ be an infinite set. The $R \to \mathbb{R}$ function w is called a *R-weight* if w is unsigned, so $w \geq 0$, and $\sum_{r\in R} w(r) = \infty$. We may refer to an R-weight as "a weight supported on R".

An R-weight w can be considered a measure on the set R and in that case for $S \subset R$ we may briefly write $w(S)$ for the sum $\sum_{s\in S} w(s)$.

For a finite set $S \subset \mathbb{N}$ let σ be a real valued, unsigned function defined on S. We can consider σ a measure on S, and as such, we assume $\sigma(S) >$

0. For a vector space V and $S \to V$ function f, define the σ-weighted average $\mathbb{A}^\sigma_S f$ of f on S by

$$\mathbb{A}^\sigma_S f = \mathbb{A}^\sigma_{s\in S} f(s) := \frac{1}{\sigma(S)} \sum_{s\in S} \sigma(s) f(s) \tag{9.1}$$

9.1 Definition ▶ Good weights and represented measures by them

Let $R \subset \mathbb{N}$ be infinite. Let w be an R-weight.
We say w is a *good R-weight* if the weak limit $\lim_N \mathbb{A}^w_{r\in R(N)} \delta_{r\beta}$ exists for every $\beta \in \mathbb{T}$. We denote this limit by $\mu_{w,\beta}$,

$$\mu_{w,\beta} := \lim_N \mathbb{A}^w_{r\in R(N)} \delta_{r\beta} \tag{9.2}$$

Let ν be a Borel probability measure on $\mathbb{T}$ and let $\alpha \in \mathbb{T}$.
We say the *R-weight w represents ν at α* if w is good and $\mu_{w,\alpha} = \nu$.

Note the following characterization of good weights: The R-weight w is good iff the limit $\lim_N \mathbb{A}^w_{r\in R(N)} \mathrm{e}(r\alpha)$ exists for every α.

In the special case of a good set $S \subset \mathbb{N}$, we have $\mu_{S,\alpha} = \mu_{\mathbb{1}_S,\alpha}$ since the weighted averages with weight $w := \mathbb{1}_S$ correspond to the averages along S.

In contrast to good sets, the representation of absolutely continuous measures by weights can always be accomplished by weights with positive, finite mean. In fact, the representing weight has an additional property.

Note the following form of the definition of the limit measure $\mu_{w,\alpha}$ when we consider R as the strictly increasing sequence (r_n): $\mu_{w,\alpha} = \lim_N \mathbb{A}^w_{n\in[1,N]} \delta_{r_n\beta}$, so now we have $\mathbb{A}^w_{n\in[1,N]} \delta_{r_n\beta} = \frac{1}{\sum_{n\in[1,N]} w(r_n)} \sum_{n\in[1,N]} w(r_n) \delta_{r_n\alpha}$.

9.2 Definition ▶ Integrable weight

Let $R \subset \mathbb{N}$ be infinite.
We call the R-weight w *integrable* if it can be approximated arbitrary closely in the seminorm $\|\|_{\mathbf{1},R}$ by bounded, good weights: for every $\epsilon > 0$ there is a good R-weight v with $\|v\|_\infty < \infty$ so that $\|v - w\|_{\mathbf{1},R} < \epsilon$.

9.3 Theorem ▶ Representation by weights

Let R be a good set.

a) For an irrational α let the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ satisfy $\mu_{R,\alpha}(\rho) = 1$.

   Then there is an integrable R-weight w with $\mathbf{M}_R(w) = 1$ which represents the measure $\rho \cdot \mu_{R,\alpha}$ at α. If $\rho \in L^\infty(\mu_{R,\alpha})$ then the R-weight w representing the measure $\rho \cdot \mu_{R,\alpha}$ can also satisfy $\|\rho\|_{L^\infty(\mu_{R,\alpha})} = \|w\|_\infty$.

b) Let w be a good, integrable R-weight which satisfies $\|w\|_{\mathbf{1},R} > 0$.
Then for every β the limit measure $\mu_{w,\beta}$ is absolutely continuous with respect to $\mu_{R,\beta}$.

1.4 *Applications in ergodic theory*

Besides the intrinsic interest of our main question, question 4.1, there may be several applications of studying limit measures. One major application is in ergodic theory.

Recall that a measure preserving dynamical system is a probability space $(X, \mathfrak{m})$, where $\mathfrak{m}(X) = 1$, equipped with a measurable, measure preserving transformation T of X. By the spectral theorem, a good set has the following characterization: the sequence $S = (s_n)$ of positive integers is good iff the limit $\lim_N \mathbb{A}_{n\in[1,N]} f(T^{s_n}x)$ exists in $L^2(X)$-norm in any measure preserving dynamical system $(X, \mathfrak{m}, T)$ for any $f \in L^2(X)$.

This means that our work in describing the possible limit measures in case of a good set yields an identification of the limit in mean ergodic theorems. Identification of the limit is often the crucial step in some applications, and here we just mention two of these, recurrence and almost sure convergence. In case of studying recurrence, the identification of the limit readily tells us whether a given set is a set of recurrence. In case of trying to see if some ergodic averages converge almost everywhere, after the identification of the L^2-limit, we usually want to see if there is some kind of rate with which the averages converge to the L^2-limit. For example, this is the case when one proves that the ergodic averages along the squares converge almost surely. The application of the circle method here is exactly a quantitative expression of how the averages converge in L^2-norm.

1.5 *Future work*

The techniques developed in this paper allow one to address the *simultaneous* representability of probability measures at several different points of the torus, and we plan to explore this in a future work. But which family $\{\nu_\alpha : \alpha \in \mathbb{T}\}$ of measures can be represented by a single good set remains open even if we restrict the family to absolutely continuous measures with respect to the Lebesgue probability measure λ. What we can say at this point is that for a given good set S, the set of $\alpha \in \mathbb{T}$ where the limit measure $\mu_{S,\alpha}$ is not the Lebesgue measure is small: it is both of first Baire category and of 0 measure under every Rajchman measure[10] on $\mathbb{T}$, so $\nu\{\alpha : \mu_{S,\alpha} \neq \lambda\} = 0$ for every Rajchman measure ν.

[10] Lyons 1985, Theorem 3; see also Lyons 1995.

1.6 *Summary of notation*

We realize that we use quite extensive notation, many of which are new, so we give a summary of our notations in table 1.

Table 1: Notations

Symbol	Definition	Parameters	Name
$\mathbb{N}$	$\{1,2,3,\dots\}$		Natural numbers
$\mathbb{T}$			torus
λ			Haar-Lebesgue measure on $\mathbb{T}$
$\mathrm{e}(\theta)$	$\exp(2\pi i\theta)$	$\theta \in \mathbb{T}$	
$\mathrm{e}^p(\theta)$	$\mathrm{e}(p\theta)$	$p \in \mathbb{Z}$	
$S(N)$	$S \cap [1,N]$	$S \subset \mathbb{N}$	initial segment of S
$\#S(N)$	$\sum_{s\in S(N)} 1$	$S \subset \mathbb{N}$	counting function of S
$\mathbb{A}_S f$	$\frac{1}{\#S}\sum_{s\in S} f(s)$	set S is finite	average of f on S
$\mathbb{A}_S^w f$	$\frac{1}{w(S)}\sum_{s\in S} w(s)f(s)$	w is a weight on S	w-average of f on set S
$\mu_{S,\alpha}$	$\lim_N \mathbb{A}_{s\in S(N)}\delta_{s\alpha}$	$S \subset \mathbb{N}$, $\alpha \in \mathbb{T}$	limit measure of S at α
$\mu_{w,\alpha}$	$\lim_N \mathbb{A}^w_{s\in S(N)}\delta_{s\alpha}$	weight w on S, $\alpha \in \mathbb{T}$	limit measure of w at α
$\nu(\phi)$	$\int_{\mathbb{T}} \phi \,\mathrm{d}\nu$		
$\mathbf{M}(f)$	$\lim_N \mathbb{A}_{[1,N]} f$	$f \in \mathbb{C}^{\mathbb{N}}$	mean of f
$\mathbf{M}_R(f)$	$\lim_N \mathbb{A}_{R(N)} f$	$R \subset \mathbb{N}$, $f \in \mathbb{C}^R$	relative mean of f
$\mathcal{M}$	$\{ f : f \in \mathbb{C}^{\mathbb{N}}, \mathbf{M}(f) \text{ exists and is finite} \}$		sequences with mean
$\overline{\mathbf{M}}(f)$	$\limsup_N \lvert\mathbb{A}_{[1,N]} f\rvert$	$f \in \mathbb{C}^{\mathbb{N}}$	upper mean
$\overline{\mathbf{M}}_R(f)$	$\limsup_N \lvert\mathbb{A}_{R(N)} f\rvert$	$R \subset \mathbb{N}$, $f \in \mathbb{C}^R$	relative upper mean
$\lVert f\rVert_{\mathbf{1}}$	$\limsup_N \mathbb{A}_{[1,N]}\lvert f\rvert$	$f \in \mathbb{C}^{\mathbb{N}}$	$\mathbf{1}$-seminorm
$\lVert f\rVert_{\mathbf{1},R}$	$\limsup_N \mathbb{A}_{R(N)}\lvert f\rvert$	$R \subset \mathbb{N}$, $f \in \mathbb{C}^R$	relative $\mathbf{1}$-seminorm
$\mathcal{C}_+$	$\{ \phi : \phi : \mathbb{T} \to [0,1], \text{ continuous } \}$		
$\lVert\nu_1 - \nu_2\rVert_{\mathbf{V}}$	$\sup_{\phi\in\mathcal{C}_+} \lvert\nu_1(\phi) - \nu_2(\phi)\rvert$	ν_i finite Borel measures on $\mathbb{T}$	variation distance

2 *Basic example for representation*

In this section we want to work out a rather simple but instructive example, which will then motivate and form the basis of many of our constructions later on. When we are done with presenting this example, we in fact proved theorem 7.1 in case the Radon-Nikodym derivative is the indicator of a Jordan measurable set.

Let α be irrational and let $I \subset \mathbb{T}$ be an interval. We want to show that if a probability measure ν is absolutely continuous with respect to λ with the Radon-Nikodym derivative equal $\mathbb{1}_I$, the indicator of I, then there is a set S which represents ν at α. Probably the simplest way[11] to define such a set S is by taking

$$S = \{ n : n \in \mathbb{N}, n\alpha \in I \} \tag{11.1}$$

[11] We could also define such a set by taking $\{ n : n \in \mathbb{N}, n^2\alpha \in I \pmod 1 \}$ or $\{ p : p \in \mathcal{P}, p\alpha \in I \pmod 1 \}$ where $\mathcal{P}$ is the set of primes.

There are two things to verify. First, that S is indeed a good set, and to do that, we need to show that the weak limit $\mu_{S,\beta} = \lim_N \mathbb{A}_{s\in S(N)}\delta_{s\beta}$ exists for every β. Second, we then have to verify that $\mu_{S,\alpha} = \frac{1}{\lambda(I)}\mathbb{1}_I \cdot \lambda$. The second one, in fact, is almost instantaneous to do since it follows from the uniform distribution of $(n\alpha)_{n\in\mathbb{N}} \pmod 1$. To see how it follows, it's enough to show that for every interval $J \subset \mathbb{T}$ we have $\mu_{S,\alpha}(J) = \lambda\left(\mathbb{1}_J \cdot \frac{1}{\lambda(I)}\mathbb{1}_I\right)$, that is

$$\lim_N \mathbb{A}_{s\in S(N)}\mathbb{1}_J(s\alpha) = \frac{1}{\lambda(I)}\lambda(J\cap I) \tag{12.1}$$

The left hand side can be written as

$$\lim_N \mathbb{A}_{s\in S(N)}\mathbb{1}_J(s\alpha) = \lim_N \frac{N}{\#S(N)}\mathbb{A}_{n\in[1,N]}\mathbb{1}_I(n\alpha)\mathbb{1}_J(n\alpha)$$

since $\lim_N \frac{\#S(N)}{N} = \lambda(I)$ by the uniform distribution of $(n\alpha)_{n\in\mathbb{N}} \pmod 1$,

$$= \frac{1}{\lambda(I)}\lim_N \mathbb{A}_{n\in[1,N]}\mathbb{1}_{I\cap J}(n\alpha)$$

again by the unifom distribution of $(n\alpha)_{n\in\mathbb{N}} \pmod 1$

$$= \frac{1}{\lambda(I)}\lambda(I\cap J)$$

To show that the weak limit $\mu_{S,\beta} = \lim_N \mathbb{A}_{s\in S(N)}\delta_{s\beta}$ exists for every β, it's enough to show that $\lim_N \mathbb{A}_{s\in S(N)}\,\mathrm{e}(s\beta)$ exists for every β. Since

$$\mathbb{A}_{s\in S(N)}\,\mathrm{e}(s\beta) = \frac{N}{\#S(N)}\mathbb{A}_{n\in[1,N]}\mathbb{1}_I(n\alpha)\,\mathrm{e}(n\beta) \tag{12.2}$$

and since $\lim_N \frac{\#S(N)}{N} = \lambda(I)$, it's enough to show that the limit $\lim_N \mathbb{A}_{n\in[1,N]}\mathbb{1}_I(n\alpha)\,\mathrm{e}(n\beta)$ exists for every $\beta \in \mathbb{T}$. To see this, first note that if we replace $\mathbb{1}_I$ by the character e^k the limit of $\mathbb{A}_{n\in[1,N]}\,\mathrm{e}^k(n\alpha)\,\mathrm{e}(n\beta) = \mathbb{A}_{n\in[1,N]}\,\mathrm{e}(n(k\alpha+\beta))$ as $N\to\infty$ exists and is as follows

$$\lim_N \mathbb{A}_{n\in[1,N]}\,\mathrm{e}^k(n\alpha)\,\mathrm{e}(n\beta) = \begin{cases} 1 & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise} \end{cases} \tag{12.3}$$

From this we get that if we replace $\mathbb{1}_I$ by a trigonometric polynomial ϕ, the limit of $\mathbb{A}_{n\in[1,N]}\phi(n\alpha)\,\mathrm{e}(n\beta)$ exists and can be given explicitly as[12]

$$\lim_N \mathbb{A}_{n\in[1,N]}\phi(n\alpha)\,\mathrm{e}(n\beta) = \begin{cases} \lambda\left(\phi\,\mathrm{e}^k\right) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise} \end{cases} \tag{12.4}$$

[12] Notice that in eq. (12.4) $\lambda(\phi\,\mathrm{e}^k)$ is the kth Fourier coefficient of ϕ.

Using Weierstrass' theorem on being able to uniformly approximate a continuous function by trigonometric polynomials, we can verify that in eq. (12.4) we can take ϕ to be any continuous function.

Now, to go from continuous functions to the indicator $\mathbb{1}_I$ of any interval I, it is enough to know that the indicator $\mathbb{1}_I$ can be sandwiched between two unsigned continuous functions whose integrals (with respect to λ) are arbitrarily close. We thus have

$$\lim_N \mathbb{A}_{n\in[1,N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) = \begin{cases} \lambda(\mathbb{1}_I\, \mathrm{e}_k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.1}$$

We finally get, since $\mu_{S,\beta}(\mathrm{e}) = \frac{1}{\lambda(I)} \lim_N \mathbb{A}_{n\in[1,N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta)$,

$$\mu_{S,\beta}(\mathrm{e}) = \begin{cases} \frac{1}{\lambda(I)}\lambda(\mathbb{1}_I\, \mathrm{e}^k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise} \end{cases} \tag{13.2}$$

The above shows that $\mu_{S,\beta}(\mathrm{e})$ can be nonzero only if β is an integer multiple of α, and we recognize $\lambda\left(\mathbb{1}_I\, \mathrm{e}^k\right)$ as the kth Fourier coefficient of the function $\mathbb{1}_I$, that is, $\frac{1}{\lambda(I)}\lambda\left(\mathbb{1}_I\, \mathrm{e}^k\right)$ is the kth Fourier coefficient of the measure $\frac{1}{\lambda(I)}\mathbb{1}_I\lambda$.

One can rather easily extend this example in two ways. First, the proof can be repeated almost verbatim for the case when we take any Jordan measurable set B in place of the interval I. Indeed, all we need to remark is that a set B is Jordan measurable iff, for every given $\epsilon > 0$, its indicator function $\mathbb{1}_B$ can be sandwiched between two unsigned, continuous functions ϕ_a and ϕ_b so that $\phi_b \leq \mathbb{1}_B \leq \phi_a$ and $\lambda(\phi_a - \phi_b) < \epsilon$. Another way of expressing that the indicator of a set can be sandwiched between two continuous functions is that the boundary of the set has zero Lebesgue measure.

13.1 Definition ▶ ν-Riemann integrability

Let ν be a finite Borel measure on $\mathbb{T}$ and let ϕ be a Borel measurable $\mathbb{T} \to \mathbb{C}$ function.
We call the function ϕ *ν-Riemann* integrable if it's continuous at ν-almost every point.
We call the Borel measurable set B *ν-Jordan measurable* if its indicator function $\mathbb{1}_B$ is ν-Riemann integrable.

As it is well known, the equivalence of approximability by continuous functions and the boundary having zero measure carries over to the setting of any finite Borel measure on the torus. We can thus extend the example to the setting when the Lebesgue measure is replaced by an arbitrary finite Borel measure.

We record our findings in the following result.

14.1 Proposition ▶ The Radon-Nikodym derivative can be the indicator of a Jordan measurable set

Let R be a good set, α be an irrational number and let $B \subset \mathbb{T}$ be $\mu_{R,\alpha}$-Jordan measurable with $\mu_{R,\alpha}(B) > 0$.
Then the measure $\mathbb{1}_B \mu_{R,\alpha}$, which is absolutely continuous with respect to $\mu_{R,\alpha}$, can be represented at α by the good set S defined by

$$S := \{ r : r \in R, r\alpha \in B \} \tag{14.1}$$

so we have $\mu_{S,\alpha} = \frac{1}{\mu_{R,\alpha}(B)} \mathbb{1}_B \mu_{R,\alpha}$. We also have $\mu_{R,\alpha}(B) = \mathbf{M}_R(S)$.

Let us go back to trying to represent measures which are absolutely continuous with respect to the Lebesgue measure λ. New ideas are needed to cover the case when we want to represent the measure $\mathbb{1}_B \lambda$ when B is a Borel set which is not Jordan measurable. What is the new difficulty? We'd like to think that we could just again take the "visit set" $S = \{ n : n \in \mathbb{N}, n\alpha \in B \}$, but this is not the case anymore. Indeed, take B to be an open set with $\lambda(B) < 1$ and containing all integer multiples of our irrational α. This open set is not Jordan measurable anymore. The set S cannot represent the measure $\mathbb{1}_B \lambda$ anymore since $S = \mathbb{N}$. In fact, we show in section 10.3 that for any given irrational α, one can construct an open set B so that the visit set of B doesn't even have mean. So we definitely need new ideas.

We also need new ideas even for the case when we try to represent a measure which is absolutely continuous with respect to the Lebesgue measure with a Radon-Nikodym derivative which is not an indicator function. We need these new ideas even if this Radon-Nikodym derivative is a continuous function.

3 *Proof of theorem 7.1 for indicators*

Strictly speaking, we have already begun the proof of theorem 7.1 in the previous section, when we proved that at an irrational number every measure with Jordan measurable Radon-Nikodym derivative can be represented. Our fixed set up in this section is that we are given a good "base" set $R \subset \mathbb{N}$ and an irrational number α. Since the set R is fixed throughout the section, we suppress the set R from our notation for the limit measure,

$$\mu_\beta := \mu_{R,\beta}, \text{ for every } \beta \tag{14.2}$$

Since our focus is to widen the class of the Radon-Nikodym derivatives with respect to the base limit measure μ_α, the following definition will simplify our language.

15.1 Definition ▶ Representing a function, a Borel set

Let $\rho \in L^1(\mathbb{T}, \mu_\alpha)$ be unsigned and $\mu_\alpha(\rho) > 0$.
We say that the good set $S \subset R$ *represents* ρ *at* α if it represents the measure $\rho \cdot \mu_\alpha$, that is, $\mu_{S,\alpha} = \frac{1}{\mu_\alpha(\rho)} \rho \cdot \mu_\alpha$.
If ρ is the indicator of a Borel measurable set $B \subset \mathbb{T}$, we then say S *represents* B *at* α.

The sets $S \subset R$ we consider in this section have positive mean in R. For such a set, the non-normalized averages $\mathbb{A}_{n\in[1,N]} \mathbb{1}_S(r_n)\delta_{r_n\beta}$ are easier to handle than the normalized ones $\mathbb{A}_{s\in S(N)}\delta_{s\beta}$. The convergence or divergence properties of the two averages are identical since they are connected by

$$\lim_N \mathbb{A}_{n\in[1,N]} \mathbb{1}_S(r_n)\delta_{r_n\beta} = \mathbf{M}_R(S) \lim_N \mathbb{A}_{s\in S(N)}\delta_{s\beta} \tag{15.1}$$

as can be seen from writing $\mathbb{A}_{s\in S(N)}\delta_{s\beta} = \frac{\#R(N)}{\#S(N)} \mathbb{A}_{r\in R(N)} \mathbb{1}_S(r)\delta_{r\beta}$ and noting that $\lim_N \frac{\#S(N)}{\#R(N)} = \mathbf{M}_R(S)$ and $\lim_N \mathbb{A}_{r\in R(N)} \mathbb{1}_S(r)\delta_{r\beta} = \lim_N \mathbb{A}_{n\in[1,N]} \mathbb{1}_S(r_n)\delta_{r_n\beta}$.

In section 2 we proved that if B is μ_α-Jordan measurable, then it can be represented by the set S_B defined by

$$S_B = \{ r_n \,:\, r_n\alpha \in B \} \tag{15.2}$$

and we have the relation

$$\mathbf{M}_R(S_B) = \mu_\alpha(B) \tag{15.3}$$

We also indicated that this definition of S_B may not give a good set if B is not Jordan measurable. The idea of extending the representation to any Borel measurable set is via a limit procedure. To explain what we mean by "a limit procedure", consider the case when B is an open set, and write it as a disjoint union of open intervals, $B = \cup_j I_j$. Defining $B_k := \cup_{j\in[1,k]} I_j$ for every $k \in \mathbb{N}$, each B_k is Jordan measurable and the sequence (B_k) increases monotonically to B. We have $\lim_k \mu_\alpha(B_k) = \mu_\alpha(B)$. Denoting $S_k := S_{B_k}$, the sequence (S_k) also increases to a set $S \subset R$, but $\mathbf{M}_R(S)$ not only may not be equal $\lim_k \mathbf{M}_R(S_k)$ but $\mathbf{M}_R(S)$ may not even exist[13]. The limit procedure which is suitable for our purposes is determined by the seminorm $\|f\|_{\mathbf{1}}$ which is defined by

$$\|f\|_{\mathbf{1}} := \limsup_N \mathbb{A}_{[1,N]} |f(n)|, \quad f \in \mathbb{C}^{\mathbb{N}} \tag{15.4}$$

[13] See also section 10.3.

Our main tools will be two lemmas. The first one is modeled after a result of Marcinkiewicz[14] on the completeness of Besicovitch spaces.

[14] Marcinkiewicz 1939.

16.1 Lemma ▶ Cauchy sequence is convergent in the seminorm $\|\|_1$

For each $k \in \mathbb{N}$, let $f_k \in \mathbb{C}^{\mathbb{N}}$. Suppose that (f_k) is a Cauchy sequence in the seminorm $\|\|_1$, so we have

$$\lim_k \sup_{l \geq k} \|f_l - f_k\|_1 = 0 \tag{16.1}$$

Then there is $f \in \mathbb{C}^{\mathbb{N}}$ satisfying

$$\lim_k \|f_k - f\|_1 = 0 \tag{16.2}$$

The f in eq. (16.2) is pasted together from the f_k's in the following way: there are indices $N_1 < N_2 < \dots$ so that $f = f_k$ on the interval $(N_k, N_{k+1}]$,

$$f = \sum_k f_k \cdot \mathbb{1}_{(N_k, N_{k+1}]} \tag{16.3}$$

16.2 Remark ▶ f inherits properties of (f_k)

Since f is pasted together from the f_k's the way we can see it in eq. (16.3), f inherits some common properties the f_k may have. For example

a) If $f_k \geq 0$ for every k then $f \geq 0$.

b) If $|f_k| \leq c$ for a constant c for every k then $|f| \leq c$.

c) If each f_k is $0-1$ valued then so is f.

d) If each f_k is a weight, then the construction can be adjusted so that f also becomes a weight.

Only remark 16.2 (d) requires some explanation since we need to have $\sum_{n \in \mathbb{N}} f(n) = \infty$. For this, we observe a flexibility in the choice of the sequence $N_1 < N_2 < \dots$ in the upcoming proof of lemma 16.1. Namely the sequence (N_k) is defined recursively, and once $N_1 < N_2 < \cdots < N_{k-1}$ are chosen, the index N_k, $N_k > N_{k-1}$, is chosen "large enough" to satisfy some criteria. So it can always be chosen to be "even larger" to satisfy additional criteria. For our case the single additional criterion is to ensure $\sum_{n \in (N_{k-1}, N_k]} f_{k-1}(n) > 1$, which is possible since f_{k-1} is assumed to be a weight, so $\sum_{n \in (N_{k-1}, \infty)} f_{k-1}(n) = \infty$.

Proof of lemma 16.1. For the recursive definition of the (N_k), define first the sequence (ϵ_k) by

$$\epsilon_k := 2 \sup_{l \geq k} \|f_l - f_k\|_1 \tag{16.4}$$

We can assume, without loss of generality, that $\epsilon_k > 0$ for every k, since

$\epsilon_k = 0$ for some k would imply $\|f_l - f_k\|_{\mathbf{1}} = 0$ for $l \geq k$ hence we could take $f = f_k$.

In the first step of the recursion, let $N_1 = 1$.

In the second step, let $N_2 > N_1$ to be large enough to satisfy

$$\frac{N_1}{N_2} < \epsilon_1 \tag{17.1}$$

$$\mathbb{A}_{[1,N]}|f_1 - f_2| < \epsilon_1 \text{ for every } N \geq N_2 \tag{17.2}$$

and

$$\mathbb{A}_{[1,N]}|f_1 - f_3| < \epsilon_1 \text{ for every } N \geq N_2 \tag{17.3}$$

Complete the second step of the recursion by defining f to be equal f_1 on the interval $(N_1, N_2]$. Let $k > 2$ and assume that we have defined $N_1 < N_2 < \cdots < N_{k-1}$ and f to be equal f_j on the interval $(N_j, N_{j+1}]$ for $j \in [1, k-2]$. For step k of the recursion let $N_k > N_{k-1}$ be large enough to satisfy

$$\frac{1}{N_k} \sum_{[1,N_{k-1}]} |f_j - f| < \epsilon_j, \text{ for every } j \in [1, k-2] \tag{17.4}$$

$$\mathbb{A}_{[1,N]}|f_j - f_{k-1}| < \epsilon_j \text{ for every } N \geq N_k, j \in [1, k-2] \tag{17.5}$$

and

$$\mathbb{A}_{[1,N]}|f_j - f_k| < \epsilon_j \text{ for every } N \geq N_k, j \in [1, k-2] \tag{17.6}$$

Complete the kth step of the recursion by defining f to be equal f_{k-1} on the interval $(N_{k-1}, N_k]$.

Let us fix j and let N be large enough so that for some $k \geq j+2$ we have

$$N_k \leq N < N_{k+1} \tag{17.7}$$

We want to show that

$$\mathbb{A}_{[1,N]}|f_j - f| < 3\epsilon_j \tag{17.8}$$

Let us estimate $\mathbb{A}_{[1,N]}|f_j - f|$ as,

$$\mathbb{A}_{[1,N]}|f_j - f| = \frac{1}{N} \sum_{[1,N_{k-1}]} |f_j - f| \tag{17.9}$$

$$+ \frac{1}{N} \sum_{(N_{k-1},N_k]} |f_j - f| \tag{17.10}$$

$$+ \frac{1}{N} \sum_{(N_k,N]} |f_j - f| \tag{17.11}$$

We can estimate the term in eq. (17.9), using eq. (17.4) and that $N \geq N_k$, as

$$\frac{1}{N} \sum_{[1,N_{k-1}]} |f_j - f| < \epsilon_j \tag{17.12}$$

For the term in eq. (17.10) we have

$$\frac{1}{N}\sum_{(N_{k-1},N_k]}|f_j - f| < \epsilon_j \tag{18.1}$$

This follows from eq. (17.5) since $f = f_{k-1}$ on the interval $(N_{k-1}, N_k]$.

For the term in eq. (17.11) we have

$$\frac{1}{N}\sum_{(N_k,N]}|f_j - f| < \epsilon_j \tag{18.2}$$

This follows from eq. (17.6) since $f = f_k$ on the interval $(N_k, N]$.

Putting the estimates in eqs. (18.1), (18.2) and (17.12) together we obtain eq. (17.8). □

The second lemma shows that the family $\mathcal{M}$ of sequences f for which $\mathbf{M}(f) = \lim_N \mathbb{A}_{[1,N]}f$ exists is closed with respect to the upper mean $\overline{\mathbf{M}}()$ defined by

$$\overline{\mathbf{M}}(f) := \limsup_N \left|\mathbb{A}_{n\in[1,N]}f(n)\right|, \quad f \in \mathbb{C}^{\mathbb{N}} \tag{18.3}$$

18.1 Lemma ▶ $\mathcal{M}$ is closed with respect to $\overline{\mathbf{M}}()$

Let (f_j) be a sequence from $\mathcal{M}$. Suppose that (f_j) converges to $f \in \mathbb{C}^{\mathbb{N}}$ in the seminorm $\overline{\mathbf{M}}()$, so

$$\lim_j \overline{\mathbf{M}}(f_j - f) = 0 \tag{18.4}$$

Then $f \in \mathcal{M}$ and

$$\mathbf{M}(f) = \lim_j \mathbf{M}(f_j) \tag{18.5}$$

Proof. First note that, as a consequence of eq. (18.4), the sequence (f_j) is a Cauchy sequence, meaning that for a given $\epsilon > 0$ there is J so that

$$\overline{\mathbf{M}}(f_j - f_J) < \epsilon \text{ for every } j \geq J \tag{18.6}$$

Since $\left|\mathbf{M}(f_j) - \mathbf{M}(f_J)\right| = \left|\mathbf{M}(f_j - f_J)\right| = \overline{\mathbf{M}}(f_j - f_J)$ we see,

$$\left|\mathbf{M}(f_j) - \mathbf{M}(f_J)\right| < \epsilon \text{ for every } j \geq J \tag{18.7}$$

so the sequence $\mathbf{M}(f_j)$ of means is a Cauchy sequence of numbers. Denote $L := \lim_j \mathbf{M}(f_j)$. We want to show that $\mathbf{M}(f) = L$. For a given $\epsilon > 0$, choose a j so that $\left|\mathbf{M}(f_j) - L\right| < \epsilon$ and $\overline{\mathbf{M}}(f - f_j) < \epsilon$. We then have, for an arbitrary N,

$$\left|\mathbb{A}_{[1,N]}f - L\right| \leq \left|\mathbb{A}_{[1,N]}(f - f_j)\right| + \left|\mathbb{A}_{[1,N]}f_j - L\right| \tag{18.8}$$

Taking $\limsup_N$ of both sides, we get

$$\limsup_N \left|\mathbb{A}_{[1,N]}f - L\right| \leq \overline{\mathbf{M}}(f - f_j) + \left|\mathbf{M}(f_j) - L\right| \tag{18.9}$$

Since $\overline{\mathbf{M}}(f - f_j) < \epsilon$ and $\left|\mathbf{M}(f_j) - L\right| < \epsilon$, we get $\limsup_N \left|\mathbb{A}_{[1,N]} f - L\right| < 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\lim_N \left|\mathbb{A}_{[1,N]} f - L\right| = 0$ which means $\mathbf{M}(f) = L = \lim_j \mathbf{M}(f_j)$.

□

How do we now show that every open set can be represented? Let $B \subset \mathbb{T}$ be open with positive μ_α measure, let $B = \cup_j I_j$ be its decomposition into pairwise disjoint open intervals I_j and set $B_k := \cup_{j\in[1,k]} I_j$. Since $\mu_\alpha(B) > 0$, we have $\mu_\alpha(B_k) > 0$ for large enough k. For simplicity, we assume that $\mu_\alpha(B_k) > 0$ for every k. The sets B_k increase to B monotonically, hence, in particular, we have $\lim_k \mu_\alpha(B_k \triangle B) = 0$. According to proposition 14.1, the set B_k can be represented by the set $S_k \subset R$ defined by

$$S_k := \{\, r_n \,:\, r_n\alpha \in B_k \,\} \tag{19.1}$$

and we have $\mathbf{M}_R(S_k) = \mu_\alpha(B_k)$. Since for every k, l the set $B_k \triangle B_l$ is Jordan measurable, we also have

$$\mathbf{M}_R(S_k \triangle S_l) = \mu_\alpha(B_k \triangle B_l) \tag{19.2}$$

For each k let us define the sequence f_k by

$$f_k(n) := \mathbb{1}_{S_k}(r_n), \qquad n \in \mathbb{N} \tag{19.3}$$

We have

$$\mathbf{M}(f_k) = \mathbf{M}_R(S_k) \text{ for every } k \in \mathbb{N} \tag{19.4}$$

and we can rewrite eq. (19.2) as

$$\mathbf{M}|f_k - f_l| = \mu_\alpha(B_k \triangle B_l) \tag{19.5}$$

Since (B_k) is a Cauchy sequence, so $\lim_k \sup_{l\geq k} \mu_\alpha(B_k \triangle B_l) = 0$, eq. (19.5) implies that (f_k) is also a Cauchy sequence in $\|\|_{\mathbf{1}}$, so we have $\lim_k \sup_{l\geq k} \mathbf{M}|f_k - f_l| = 0$. Since $\mathbf{M}|f_k - f_l| = \|f_k - f_l\|_{\mathbf{1}}$, according to lemma 16.1, there is f to which the (f_k) converges, that is, so $\lim_k \|f_k - f\|_{\mathbf{1}} = 0$, and by lemma 18.1, $\mathbf{M}(f) = \lim_k \mathbf{M}(f_k)$. Since $\mathbf{M}(f_k) = \mu_\alpha(B_k)$ and $\lim_k \mu_\alpha(B_k) = \mu_\alpha(B)$, we have $\mathbf{M}(f) = \mu_\alpha(B) > 0$. According to remark 16.2 (c) f is $0-1$ valued hence we can define a set $S \subset R$ by its indicator as

$$\mathbb{1}_S(r_n) := f(n), \qquad n \in \mathbb{N} \tag{19.6}$$

We have

$$\mathbf{M}_R(S) = \mathbf{M}(f) \tag{19.7}$$

We want to show that S is good and it represents B at α. To this end, let $\beta \in \mathbb{T}$ be arbitrary and define the sequences f_k^β and f^β by

$$f_k^\beta(n) := f_k(n)\, \mathrm{e}(r_n\beta) \text{ for } n \in \mathbb{N} \tag{19.8}$$

$$f^\beta(n) := f(n)\, \mathrm{e}(r_n\beta) \text{ for } n \in \mathbb{N} \tag{19.9}$$

Since $\mathbf{M}(f) = \lim_k \mathbf{M}(f_k) > 0$ and $\mathbf{M}(f_k) = \mathbf{M}_R(S_k)$, we have $\mathbf{M}(S) > 0$. It follows that, by eq. (15.1), to show that S is good, it's enough to show that $\mathbf{M}(f^\beta)$ exists for every β and to show that S represents B at α it's enough to show that $\mathbf{M}(f^{p\alpha}) = \mu_\alpha(e^p \mathbb{1}_B)$ for every $p \in \mathbb{Z}$.

Let us first show that $\mathbf{M}(f^\beta)$ exists for every β. Since each set S_k is good with $\mathbf{M}(S_k) > 0$, we have, as a consequence of eq. (15.1), that $f_k^\beta \in \mathcal{M}$ for every k, β. The fact that for every β, the sequence (f_k^β) converges to f^β in the norm $\overline{\mathbf{M}}()$ follows from the uniform estimate

$$\overline{\mathbf{M}}(f_k^\beta - f^\beta) \leq \|f_k - f\|_{\mathbf{1}} \text{ for every } \beta \tag{20.1}$$

By lemma 18.1, $f^\beta \in \mathcal{M}$ and

$$\mathbf{M}(f^\beta) = \lim_k \mathbf{M}(f_k^\beta) \tag{20.2}$$

Let us now show that S represents B at α, that is, $\mathbf{M}(f^{p\alpha}) = \mu_\alpha(e^p \mathbb{1}_B)$ for every $p \in \mathbb{Z}$. Since the sequence (B_k) converges to B in $L^1(\mu_\alpha)$-norm we have

$$\lim_k \mu_\alpha(e^p \mathbb{1}_{B_k}) = \mu_\alpha(e^p \mathbb{1}_B) \text{ for every } p \in \mathbb{Z} \tag{20.3}$$

Since $\mathbf{M}(f_k^{p\alpha}) = \mu_\alpha(e^p \mathbb{1}_{B_k})$ and, by eq. (20.2), $\lim_k \mathbf{M}(f_k^{p\alpha}) = \mathbf{M}(f^{p\alpha})$, eq. (20.3) implies that $\mathbf{M}(f^{p\alpha}) = \mu_\alpha(e^p \mathbb{1}_B)$.

We record the general idea we used as proposition 20.1 (b) below.

20.1 Proposition ▶ Limit of good sets with positive mean is good

Let (S_k) be a sequence of good subsets of R with mean which converge to $S \subset R$ in $\|\|\|_{\mathbf{1},R}$-seminorm, that is, $\lim_k \|S_k \triangle S\|_{\mathbf{1},R} = 0$. Assume that $\limsup_k \mathbf{M}_R(S_k) > 0$.
Then we have the following.

a) $\lim_k \mathbf{M}_R(S_k)$ exists and $\mathbf{M}_R(S) = \lim_k \mathbf{M}_R(S_k) > 0$.

b) S is a good set.

c) The sequence $(\mu_{S_k,\beta})_k$ of limit measures converge to $\mu_{S,\beta}$ in variation distance and uniformly in β,

$$\lim_k \sup_\beta \|\mu_{S_k,\beta} - \mu_{S,\beta}\|_{\mathbf{V}} = 0 \tag{20.4}$$

d) Let ν be a Borel measure on $\mathbb{T}$.

If for some α, $\mu_{S_k,\alpha}$ is absolutely continuous with respect to ν with Radon-Nikodym derivative ρ_k for every k, then $\mu_{S,\alpha}$ is also absolutely continuous with respect to ν with Radon-Nikodym derivative ρ which satisfies

$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{20.5}$$

Proof. The proof of proposition 20.1 (a) follows from the triangle inequality for the $\|\|\|_{\mathbf{1}}$-seminorm, since we then have

$$\begin{aligned}|\mathbf{M}_R(S_k) - \mathbf{M}_R(S)| &= \Big|\|S_k\|_{\mathbf{1},R} - \|S\|_{\mathbf{1},R}\Big| \\ &\leq \|S_k \triangle S\|_{\mathbf{1},R}\end{aligned}$$

and just use the assumption that $\lim_k \|S_k \triangle S\|_{\mathbf{1},R} = 0$.

The argument we gave just before the enunciation of our proposition proves that S is a good set.

For the proof of proposition 20.1 (c) note that in the argument preceding our proposition we proved that the sequence $(\mu_{S_k,\beta})_k$ of measures converges weakly to $\mu_{S,\beta}$ for every β but an estimate similar to eq. (20.1) enables us to draw the stronger conclusion of eq. (20.4).

The following lemma gives us the estimates we need.

21.1 Lemma ▶ $\|\|\|_{\mathbf{1},R}$ dominates $\|\|\|_{\mathbf{V}}$ and $\|\|\|_{L^1}$

Let v_1, v_2 be good R-weights. Assume that

$$\max\Big\{\|v_1\|_{\mathbf{1},R}, \|v_2\|_{\mathbf{1},R}\Big\} > 0 \tag{21.1}$$

Then we have the following.

a)

$$\sup_\beta \big\|\mu_{v_1,\beta} - \mu_{v_2,\beta}\big\|_{\mathbf{V}} \leq \frac{2}{\max\Big\{\|v_1\|_{\mathbf{1},R}, \|v_2\|_{\mathbf{1},R}\Big\}} \|v_1 - v_2\|_{\mathbf{1},R} \tag{21.2}$$

b) If, for some α, the limit measures $\mu_{v_1,\alpha}$ and $\mu_{v_2,\alpha}$ are absolutely continuous with respect to a Borel measure ν on $\mathbb{T}$ with Radon-Nikodym derivatives ρ_1 and ρ_2, respectively, then

$$\|\rho_1 - \rho_2\|_{L^1(\nu)} \leq \frac{4}{\max\Big\{\|v_1\|_{\mathbf{1},R}, \|v_2\|_{\mathbf{1},R}\Big\}} \|v_1 - v_2\|_{\mathbf{1},R} \tag{21.3}$$

Proof. To prove lemma 21.1 (a), that is, the inequality in eq. (21.2), fix β and $\phi \in \mathcal{C}_+$, so ϕ is a continuous $\mathbb{T} \to \mathbb{C}$ function with $0 \leq \phi \leq 1$. We can assume without loss of generality that $\max\Big\{\|v_1\|_{\mathbf{1},R}, \|v_2\|_{\mathbf{1},R}\Big\} = \|v_1\|_{\mathbf{1},R}$. Let $(N_l)_l$ be a strictly increasing sequence of indices so that

$$\lim_l \mathbb{A}_{[1,N_l]} v_1 = \|v_1\|_{\mathbf{1},R} \tag{21.4}$$

Let us estimate as

$$\begin{aligned}
&\left|\mathbb{A}^{v_1}_{n\in[1,N_l]}\phi(r_n\beta) - \mathbb{A}^{v_2}_{n\in[1,N_l]}\phi(r_n\beta)\right| \\
&\quad= \left|\frac{1}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{n\in[1,N_l]}v_1(r_n)\phi(r_n\beta) - \frac{1}{\mathbb{A}_{[1,N_l]}v_2}\mathbb{A}_{n\in[1,N_l]}v_2(r_n)\phi(r_n\beta)\right|
\end{aligned}$$

adding $0 = -\frac{1}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{n\in[1,N_l]}v_2(r_n)\phi(r_n\beta) + \frac{1}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{n\in[1,N_l]}v_2(r_n)\phi(r_n\beta)$ inside the absolute value and using the triangle inequality,

$$\begin{aligned}
&\leq \frac{1}{\mathbb{A}_{[1,N_l]}v_1}\left|\mathbb{A}_{n\in[1,N_l]}v_1(r_n)\phi(r_n\beta) - \mathbb{A}_{n\in[1,N_l]}v_2(r_n)\phi(r_n\beta)\right| \\
&\quad+ \left|\frac{1}{\mathbb{A}_{[1,N_l]}v_1} - \frac{1}{\mathbb{A}_{[1,N_l]}v_2}\right|\left|\mathbb{A}_{n\in[1,N_l]}v_2(r_n)\phi(r_n\beta)\right| \\
&\leq \frac{1}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{[1,N_l]}|v_1 - v_2| + \frac{\mathbb{A}_{[1,N_l]}|v_1 - v_2|}{\mathbb{A}_{[1,N_l]}v_1\mathbb{A}_{[1,N_l]}v_2}\mathbb{A}_{[1,N_l]}v_2 \\
&= \frac{2}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{[1,N_l]}|v_1 - v_2|
\end{aligned}$$

so we have

$$\left|\mathbb{A}^{v_1}_{n\in[1,N_l]}\phi(r_n\beta) - \mathbb{A}^{v_2}_{n\in[1,N_l]}\phi(r_n\beta)\right| \leq \frac{2}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{[1,N_l]}|v_1 - v_2| \quad (22.1)$$

Since $\lim_l \mathbb{A}^{v_i}_{n\in[1,N_l]}\phi(r_n\beta) = \mu_{v_i,\beta}(\phi)$, $\lim_l \mathbb{A}_{[1,N_l]}v_1 = \|v_1\|_{\mathbf{1},R}$ and $\limsup_l \frac{2}{\mathbb{A}_{[1,N_l]}v_1}\mathbb{A}_{[1,N_l]}|v_1 - v_2| \leq \frac{2}{\|v_1\|_{\mathbf{1},R}}\|v_1 - v_2\|_{\mathbf{1},R}$, we get

$$\left|\mu_{v_1,\beta}(\phi) - \mu_{v_2,\beta}(\phi)\right| \leq \frac{2}{\|v_1\|_{\mathbf{1},R}}\|v_1 - v_2\|_{\mathbf{1},R} \quad (22.2)$$

which is independent of β and $\phi \in \mathcal{C}_+$, proving eq. (21.2).

To prove lemma 21.1 (b), observe first that, since $\mu_{v_i,\alpha} = \rho_i\nu$ and ρ_i are probability densities with respect to ν, we have $\|\rho_1\nu - \rho_2\nu\|_{\mathbf{V}} = \frac{1}{2}\|\rho_1 - \rho_2\|_{L^1(\nu)}$. It follows that

$$\|\mu_{v_1,\alpha} - \mu_{v_2,\alpha}\|_{\mathbf{V}} = \frac{1}{2}\|\rho_1 - \rho_2\|_{L^1(\nu)} \quad (22.3)$$

and now just use eq. (21.2). □

Now, let us come back to the proof of proposition 20.1. Using eq. (21.2) with $v_1 = \mathbb{1}_{S_k}$ and $v_2 = \mathbb{1}_S$, we get

$$\sup_\beta \left\|\mu_{S_k,\beta} - \mu_{S,\beta}\right\|_{\mathbf{V}} \leq \frac{2}{\max\left\{\|S_k\|_{\mathbf{1},R}, \|S\|_{\mathbf{1},R}\right\}}\|S_k\triangle S\|_{\mathbf{1},R} \quad (22.4)$$

Using the assumption that $\lim_k \|S_k\triangle S\|_{\mathbf{1},R} = 0$ and that, by proposition 20.1 (a), we have $\lim_k \|S_k\|_{\mathbf{1},R} = \lim_k \mathbf{M}_R(S_k) = \mathbf{M}_R(S) = \|S\|_{\mathbf{1},R} > 0$, we get eq. (20.4).

For the proof of proposition 20.1 (d), by proposition 20.1 (a), we can assume, without loss of generality that $\mathbf{M}_R(S_k) > 0$ for every k. Using eq. (21.3) with $v_1 = \mathbb{1}_{S_k}$ and $v_2 = \mathbb{1}_{S_l}$ we get

$$\|\rho_l - \rho_k\|_{L^1(\nu)} \leq \frac{4}{\max\left\{\|S_k\|_{\mathbf{1},R}, \|S_l\|_{\mathbf{1},R}\right\}} \|S_k \triangle S_l\|_{\mathbf{1},R} \tag{23.1}$$

This implies, since the sequence (S_k) is convergent in $\|\|_{\mathbf{1},R}$-seminorm and hence is Cauchy, that the sequence (ρ_k) is Cauchy in $L^1(\nu)$-norm. Since $L^1(\nu)$ is complete and $\nu(\rho_k) = 1$ for every k, there is a $\rho \in L^1(\nu)$ with $\nu(\rho) = 1$ so that

$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{23.2}$$

Since $\|\rho_k - \rho\|_{L^1(\nu)} = 2\|\rho_k \nu - \rho\nu\|_{\mathbf{V}}$ and $\rho_k \nu = \mu_{S_k,\alpha}$, we get

$$\lim_k \left\|\mu_{S_k,\alpha} - \rho\nu\right\|_{\mathbf{V}} = 0 \tag{23.3}$$

But by proposition 20.1 (c) we also have $\lim_k \left\|\mu_{S_k,\alpha} - \mu_{S,\alpha}\right\|_{\mathbf{V}} = 0$ hence we must have $\mu_{S,\alpha} = \rho\nu$.

□

We can use proposition 20.1 in an argument similar to the one we used to show that any open set can be represented at α to prove that if a G_δ set B has positive μ_α-measure then it can be represented at α. Only the initial setup of the proof is different. This time let (B_k) be a *decreasing* sequence of open sets which converges to B. Let $S_k \subset R$ represent B_k at α. We again have the isometry eq. (19.2) from which everything follows: the existence of a good set S which represents B at α and $\mathbf{M}(S) = \mu_\alpha(B)$.

Since every Borel measurable set differs from a G_δ set on a set of μ_α-measure zero, we in fact showed that every Borel set of positive μ_α-measure can be represented. So we proved the following more precise version of theorem 7.1 for the case when the Radon-Nikodym derivative of a measure with respect to μ_α is an indicator.

23.1 Proposition ▶ Theorem 7.1 for indicators

Let $R \subset \mathbb{N}$ be a good set, α be an irrational number, and let B be a Borel set with $\mu_\alpha(B) > 0$.
Then B can be represented at α by a set $S \subset R$ which satisfies

$$\mathbf{M}_R(S) = \mu_\alpha(B) > 0 \tag{23.4}$$

4 *Measures that cannot be represented at every irrational α*

For this section, we suspend the proof of theorem 7.1 just to see how proposition 23.1 can be used to prove theorem 5.3. We will also prove theorem 6.1.

4.1 *Proof of theorem 5.3*

In this section we want to prove that if the Borel probability measure ν has a point-mass at a point $\gamma \in \mathbb{T}$ and α is irrational then ν cannot be represented at α.

The proof is by contradiction: let us assume that for some $\gamma \in \mathbb{T}$, $\nu(\{\gamma\}) > 0$ and that ν can be represented by the set R at α, so $\mu_{R,\alpha} = \nu$. Then the Dirac mass δ_γ is absolutely continuous with respect to $\mu_{R,\alpha}$ with Radon-Nikodym derivative equal $\frac{1}{\nu(\gamma)}\mathbb{1}_{\{\gamma\}}$. By proposition 23.1 there is a good set $S \subset R$ which represents δ_γ at α, so $\mu_{S,\alpha} = \delta_\gamma$. Let us define the function $\phi : \mathbb{T} \to \mathbb{C}$ as

$$\phi(\beta) := \mu_{S,\beta}(\mathrm{e}) \tag{24.1}$$

Then, by the definition of $\mu_{S,\beta}(\mathrm{e})$, ϕ is the limit of the sequence (ϕ_N) of continuous functions defined by $\phi_N(\beta) := \mathbb{A}_{n\in[1,N]}\, \mathrm{e}(s_n\beta)$ where (s_n) is the elements of S arranged in increasing order. Since for every $p \in \mathbb{Z}$ we have $\mu_{S,p\alpha}(\mathrm{e}) = \mu_{S,\alpha}(\mathrm{e}^p)$ and $\mu_{S,\alpha}(\mathrm{e}^p) = \mathrm{e}^p(\gamma)$, we have

$$|\phi| = 1 \text{ on the dense set } \{\, p\alpha \,:\, p \in \mathbb{Z} \,\} \tag{24.2}$$

By Weyl's theorem,[15] $\phi = 0$ on a set of full Lebesgue measure, so, as a consequence,

$$\phi = 0 \text{ on a dense set.} \tag{24.3}$$

By Baire's theorem,[16] eqs. (24.2) and (24.3) together are impossible to hold simultaneously for the limit of continuous functions.

[15] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

[16] Baire 1905, Page 83.

4.2 *Proof of theorem 6.1*

So in this section we want to prove that if ν is a Borel probability measure on $\mathbb{T}$ with $\limsup_{p\to\infty}|\nu(\mathrm{e}^p)| > 0$ then there is an irrational α where ν cannot be represented. In fact the set of such α's is of full Lebesgue measure.

From the assumption that $\limsup_{p\to\infty}|\nu(\mathrm{e}^p)| > 0$ it follows that there is an $\epsilon > 0$ and a infinite sequence $p_1 < p_2 < \dots$ of indices so that

$$|\nu\, \mathrm{e}^{p_k}| > \epsilon \text{ for } k \in \mathbb{N} \tag{24.4}$$

By Weyl's result,[17] the set $A \subset \mathbb{T}$ defined by

$$A := \left\{ \alpha \,:\, \overline{\{\, p_k\alpha \,:\, k \in \mathbb{N} \,\}} \text{ has nonempty interior} \pmod 1 \right\} \tag{24.5}$$

has full λ measure. We want to show that A is a subset of those α's at which the measure ν cannot be represented.

Let $\alpha \in A$, and suppose the measure ν can be represented at α, say, by the set $S = (s_n)$, that is, $\mu_{S,\alpha} = \nu$. Let us define the function $\phi : \mathbb{T} \to \mathbb{C}$ as

$$\phi(\beta) := \mu_{S,\beta}(\mathrm{e}) \tag{24.6}$$

[17] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

Then, by the definition of $\mu_{S,\beta}(\mathrm{e})$, ϕ is the limit of the sequence (ϕ_N) of continuous functions defined by $\phi_N(\beta) := \mathbb{A}_{n\in[1,N]}\,\mathrm{e}(s_n\beta)$. Since for every $p \in \mathbb{Z}$ we have $\mu_{S,p\alpha}(\mathrm{e}) = \mu_{S,\alpha}(\mathrm{e}^p)$ and $\mu_{S,\alpha}(\mathrm{e}^p) = \nu(\mathrm{e}^p)$, by eq. (24.4) we have

$$|\mu_{S,p_k\alpha}(\mathrm{e})| > \epsilon \text{ for every } k \in \mathbb{N} \tag{25.1}$$

By the definition of ϕ, we can write the above as

$$|\phi| > \epsilon \text{ on the set } \{\, p_k\alpha \,:\, k \in \mathbb{N} \,\} \tag{25.2}$$

Since $\alpha \in A$, the set $\{\, p_k\alpha \,:\, k \in \mathbb{N} \,\}$ is dense in a nondegenerate interval $I \subset \mathbb{T}$.

By Weyl's theorem,[18] $\phi = 0$ on a set U of full Lebesgue measure

$$\phi = 0 \text{ on } U \tag{25.3}$$

Since both $\{\, p_k\alpha \,:\, k \in \mathbb{N} \,\}$ and U are dense in the interval I, by Baire's theorem,[19] eqs. (25.2) and (25.3) cannot be true together for the limit ϕ of continuous functions.

[18] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

[19] Baire 1905, Page 83.

5 *Representing by weights*

In this section, we fix the good set[20] R and the irrational number α, and we continue in the tradition of section 3 suppressing the set R in our notation for the limit measure, so $\mu_\alpha = \mu_{R,\alpha}$.

[20] Note that we make no further assumption on R, such as sublacunarity

In trying to extend the class of representable functions ρ from indicators, we first consider an easier problem. Instead of representing by sets, we represent by R-weights.

25.1 Definition ▶ Function represented by a weight

Let ρ be an unsigned $L^1(\mathbb{T}, \mu_\alpha)$ function with $\mu_\alpha(\rho) > 0$.
We say the R-weight w *represents* ρ *at* α if w is good and it represents the measure $\rho \cdot \mu_\alpha$, that is, $\mu_{w,\alpha} = \frac{1}{\mu_\alpha(\rho)}\rho \cdot \mu_\alpha$.

The R-weights w we consider in this section have positive mean in R, so $\mathbf{M}_R(w) > 0$. For such a weight, the non-normalized averages $\mathbb{A}_{n\in[1,N]}w(r_n)\delta_{r_n\beta}$ are easier to handle than the normalized ones $\mathbb{A}^w_{n\in[1,N]}\delta_{r_n\beta}$. The convergence or divergence properties of the two averages are identical since they differ only by the nonzero factor $\mathbf{M}_R(w)$,

$$\lim_N \mathbb{A}_{n\in[1,N]}w(r_n)\delta_{r_n\beta} = \mathbf{M}_R(w)\lim_N \mathbb{A}^w_{n\in[1,N]}\delta_{r_n\beta} \tag{25.4}$$

as can be seen from writing $\mathbb{A}^w_{n\in[1,N]}\delta_{r_n\beta} = \frac{N}{\sum_{n\in[1,N]} w(r_n)}\mathbb{A}_{n\in[1,N]}w(r_n)\delta_{r\beta}$.

In section 2 we have already seen that if ρ is an unsigned continuous function with $\mu_\alpha(\rho) > 0$ then the R-weight w defined by

$$w(r_n) := \rho(r_n\alpha) \tag{25.5}$$

is good, unsigned and it represents ρ at α. Since every unsigned μ_α-integrable function can be approximated arbitrary closely by unsigned continuous functions in $L^1(\mathbb{T}, \mu_\alpha)$-norm, the proof of theorem 9.3 (a) requires only an approximation argument similar to what we had in section 3. We restate theorem 9.3 (a) in the following form for the readers convenience.

26.1 Proposition ▶ Any integrable function is representable with weights

Let ρ be an unsigned function from $L^1(\mathbb{T}, \mu_\alpha)$ with $\mu_\alpha(\rho) > 0$.
Then there is an R-weight w which represents ρ at α. In particular, we have

$$\mathbf{M}_R(w) = \mu_\alpha(\rho) \tag{26.1}$$

Furthermore, if ρ is a bounded function then the representing R-weight w can be chosen to be bounded.

The proof of proposition 20.1 can be easily adjusted to obtain the following analog for weights.

26.2 Proposition ▶ Limit of good weights with positive mean is good

Let (w_k) be a sequence of good R-weights with mean which converge to the R-weight w in $\|\|_{\mathbf{1},R}$-seminorm, so $\lim_N \|w_k - w\|_{\mathbf{1},R} = 0$.
Assume that $\limsup_k \mathbf{M}_R(w_k) > 0$.
Then we have the following.

a) $\lim_k \mathbf{M}_R(w_k)$ exists and $\lim_k \mathbf{M}_R(w_k) = \mathbf{M}_R(w) > 0$.

b) w is a good R-weight.

c) The sequence $(\mu_{w_k,\beta})_k$ of limit measures converge to $\mu_{w,\beta}$ in variation distance and uniformly in β,

$$\lim_k \sup_\beta \|\mu_{w_k,\beta} - \mu_{w,\beta}\|_{\mathbf{V}} = 0 \tag{26.2}$$

d) Let ν be a Borel measure on $\mathbb{T}$.

If for some α, $\mu_{w_k,\alpha}$ is absolutely continuous with respect to ν with Radon-Nikodym derivative ρ_k for every k then $\mu_{w,\alpha}$ is also absolutely continuous with respect to ν with Radon-Nikodym derivative ρ which satisfies

$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{26.3}$$

With this proposition, we can complete the proof of proposition 26.1 exactly as we proved proposition 23.1, using a sequence (ρ_k) of unsigned continuous functions that converge to ρ in $L^1(\mu_\alpha)$-norm. We need to

remark only that if ρ is a bounded function, then the sequence (ρ_k) of continuous functions can be chosen to be uniformly bounded.

6 *Proof of theorem 7.1 for bounded ρ*

In this section, we still are working with a fixed good set R of positive integers, an irrational number α, but now we also fix a bounded Borel measurable, unsigned function ρ with $\mu_\alpha(\rho) > 0$. We proved in section 5 that ρ can be represented at α by a good, bounded R-weight w. In this section we will show that there is a good set $S \subset R$ which also represents ρ at α, hence proving theorem 7.1 for bounded ρ. It follows from the definition of representation that if the good R-weight w represents ρ then so does the R-weight cw for every positive constant c. In particular, we can assume that the R-weight w representing ρ is bounded by 1. We will show that then there is a set $S \subset R$ so that

$$\lim_N \sup_\beta \left| \mathbb{A}_{n\in[1,N]} \mathbb{1}_S(r_n)\, \mathrm{e}(r_n\beta) - \mathbb{A}_{n\in[1,N]} w(r_n)\, \mathrm{e}(r_n\beta) \right| = 0 \tag{27.1}$$

The "construction" of S satisfying eq. (27.1) is done randomly. Our random method requires that we limit the growth of the set R; we need to assume that R is *sublacunary*.[21] We need the concept of a sublacunary weight.

[21] Jones, Lacey, and Wierdl 1999, Theorem B.

27.1 Definition ▶ Sublacunary weight

The R-weight w is called *sublacunary* if it satisfies

$$\lim_N \frac{w(R(N))}{\log N} = \infty \tag{27.2}$$

We often consider the sequence (r_n) instead of the set R in which case we can use the following more convenient version of eq. (27.2).

$$\lim_N \frac{\sum_{n\in[1,N]} w(r_n)}{\log r_{N+1}} = \infty \tag{27.3}$$

Our main tool in this section is the following.

27.2 Proposition ▶ There is a set representing the same measures as a bounded weight

Let w be a bounded, sublacunary R-weight.
Then there is a set $S \subset R$ so that

$$\lim_N \max_{\beta\in\mathbb{T}} \left| \mathbb{A}_{s\in S(N)}\, \mathrm{e}(s\beta) - \mathbb{A}^w_{r\in R(N)}\, \mathrm{e}(r\beta) \right| = 0 \tag{27.4}$$

As a consequence, if the R-weight w is good then so is the set S and we have

$$\mu_{S,\beta} = \mu_{w,\beta} \text{ for every } \beta \tag{27.5}$$

Proof. Since we can always assume that the bound of the R-weight w is 1, proposition 27.2 follows from the following lemma.

28.1 Lemma ▶ Random selection of a good set

Let σ be an R-weight bounded by 1. We assume that for a constant $b > 0$ we have

$$\liminf_N \frac{\sigma(R(N))}{\log N} > b \tag{28.1}$$

Let (Ω, P) be a probability space and and let $(X_r)_{r\in R}$ be a sequence of totally independent $\Omega \to \{0,1\}$ random variables indexed by R and with distribution $P(X_r = 1) = \sigma(r)$ (so $P(X_r = 0) = 1 - \sigma(r)$). Then we have

$$P\left\{ \omega \,:\, \sup_N \max_{\beta\in\mathbb{T}} \frac{\left|\sum_{r\in R(N)} \left(X_r(\omega) - \sigma(r)\right) \mathrm{e}(r\beta)\right|}{\sqrt{(\log N)\sigma(R(N))}} < \infty \right\} = 1 \tag{28.2}$$

To see that proposition 27.2 indeed follows from lemma 28.1, let $\sigma = \frac{w}{\|w\|_\infty}$, so σ is bounded by 1. Here we make a bit more complicated argument than needed to show that there is a rate of convergence in eq. (27.4).

The sublacunarity assumption on w implies that σ is sublacunary. We then have, as a consequence of eq. (28.2), that there is a measurable subset Ω_1 of Ω with $P(\Omega_1) = 1$ so that for every $\omega \in \Omega_1$ there is a finite positive constant C_ω with

$$\max_{\beta\in\mathbb{T}} \left| \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega)\,\mathrm{e}(r\beta) - \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} \sigma(r)\,\mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\log N}{\sigma(R(N))}} \tag{28.3}$$

For $\beta = 0$, we then have

$$\left| \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega) - 1 \right| \le C_\omega \sqrt{\frac{\log N}{\sigma(R(N))}} \tag{28.4}$$

This implies that if we replace $\sigma(R(N))$ by $\sum_{r\in R(N)} X_r(\omega)$ in $\frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega)\,\mathrm{e}(r\beta)$ we make a $O\left(\sqrt{\frac{\log N}{\sigma(R(N))}}\right)$ error, hence eq. (28.3) implies

$$\max_{\beta\in\mathbb{T}} \left| \frac{1}{\sum_{r\in R(N)} X_r(\omega)} \sum_{r\in R(N)} X_r(\omega)\,\mathrm{e}(r\beta) - \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} \sigma(r)\,\mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\log N}{\sigma(R(N))}} \tag{28.5}$$

Defining $S_\omega \subset R$ by

$$S_\omega := \{\, r \,:\, r \in R, X_r(\omega) = 1 \,\} \tag{28.6}$$

we can write eq. (28.5) as

$$\max_{\beta\in\mathbb{T}}\left|\mathbb{A}_{s\in S_\omega(N)}\,\mathrm{e}(s\beta)-\mathbb{A}^{\sigma}_{r\in R(N)}\,\mathrm{e}(r\beta)\right|\le C_\omega\sqrt{\frac{\log N}{\sigma\big(R(N)\big)}}\ \text{for every } \omega\in\Omega_1 \tag{29.1}$$

Since σ is a constant multiple of w, we can replace σ by w in eq. (29.1),

$$\max_{\beta\in\mathbb{T}}\left|\mathbb{A}_{s\in S_\omega(N)}\,\mathrm{e}(s\beta)-\mathbb{A}^{w}_{r\in R(N)}\,\mathrm{e}(r\beta)\right|\le C_\omega\sqrt{\frac{\|w\|_\infty\log N}{w\big(R(N)\big)}}\ \text{for every } \omega\in\Omega_1 \tag{29.2}$$

Since $\lim_N \frac{\|w\|_\infty \log N}{w\big(R(N)\big)} = 0$, due to the sublacunarity assumption on the R-weight w, we get eq. (27.4) if we take $S = S_\omega$ for any $\omega\in\Omega_1$.

Proof of lemma 28.1. To see clearly what we need to do, denote

$$Z_N(\beta) := \sum_{r\in R(N)}\Big(X_r(\omega)-\sigma(r)\Big)\,\mathrm{e}(r\beta)$$

and

$$t_N := c\cdot\sqrt{(\log N)\sigma\big(R(N)\big)}$$

where we'll choose the constant c appropriately later. By the Borel-Cantelli lemma, it's enough to prove

$$\sum_N P\left(\max_{\beta\in\mathbb{T}}|Z_N(\beta)|\ge t_N\right)<\infty \tag{29.3}$$

The first idea in proving eq. (29.3) is that we do not have to take the maximum over all $\beta\in\mathbb{T}$, but over a finite subset B of $\mathbb{T}$ which contains N^3 elements[22]. Since the degree of the trigonometric polynomial $Z_N(\beta)$ is at most N, we can readily see that $\sup_{\beta\in\mathbb{T}}|Z_N'(\beta)|\le N^2\sup_{\beta\in\mathbb{T}}|Z_N(\beta)|$. It follows that if we take $B_N\subset\mathbb{T}$ to be an arithmetic progression with $|B_N|=N^3$ then

$$\max_{\beta\in\mathbb{T}}|Z_N(\beta)|\le 2\max_{\beta\in B_N}|Z_N(\beta)| \tag{29.4}$$

[22] In fact, we can take a set B with as few elements as $10N$, but in our applications, $10N$ won't improve anything over N^3.

Hence we have

$$P\left(\max_{\beta\in\mathbb{T}}|Z_N(\beta)|\ge t_N\right)\le P\left(\max_{\beta\in B_N}|Z_N(\beta)|\ge t_N/2\right) \tag{29.5}$$

Using the union estimate, we get

$$P\left(\max_{\beta\in B_N}|Z_N(\beta)|\ge t_N/2\right)\le N^3\max_{\beta\in B_N}P(|Z_N(\beta)|\ge t_N/2) \tag{29.6}$$

Thus eq. (29.3) follows from

$$\sum_N N^3\max_{\beta\in B_N}P(|Z_N(\beta)|\ge t_N/2)<\infty \tag{29.7}$$

This follows if we prove

$$P\Big(|Z_N(\beta)| \geq t_N/2\Big) < \frac{2}{N^5} \text{ for every } \beta \in \mathbb{T} \tag{30.1}$$

To prove eq. (30.1), we use the Bernstein-Chernoff exponential estimate.[23] This estimate says that if Y_k, $k \in [1,K]$, are totally independent, mean zero, complex valued random variables with $|Y_k| \leq 1$, then

[23] Tao and Vu 2006, Exercise 1.3.4 with $t = \lambda\sigma$.

$$P\left(\left|\sum_{k\in[1,K]} Y_k\right| \geq t\right) \leq 4\max\left\{\exp\left(-\frac{t^2/8}{\sum_{k\in[1,K]}\mathbb{E}|Y_k|^2}\right), \exp(-t/3)\right\} \text{ for every } t > 0 \tag{30.2}$$

Take $K = \#R(N)$ and $Y_r(\beta) := (X_r - \sigma(r))e(r\beta)$ for $r \in R(N)$. Then $|Y_r(\beta)| \leq 1$ so the Y_r satisfy the assumption in Bernstein's inequality, hence, with $t = t_N/2$, we get the estimate

$$P\Big(|Z_N(\beta)| \geq t_N/2\Big) \leq 4\max\left\{\exp\left(-\frac{t_N^2/32}{\sum_{r\in R(N)}\mathbb{E}|Y_r|^2}\right), \exp(-t_N/6)\right\} \tag{30.3}$$

Since $\mathbb{E}|Y_r(\beta)|^2 = \sigma(r)(1-\sigma(r))$ we have

$$\sum_{r\in R(N)} \mathbb{E}|Y_r(\beta)|^2 \leq \sigma\big(R(N)\big) \tag{30.4}$$

Using that $t_N = c\cdot\sqrt{(\log N)\sigma\big(R(N)\big)}$, we get

$$\frac{t_N^2/32}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2} = \frac{(c^2/32)(\log N)\sigma\big(R(N)\big)}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2}$$

using the estimate in eq. (30.4)

$$\begin{aligned} &\geq \frac{(c^2/32)(\log N)\sigma\big(R(N)\big)}{\sigma\big(R(N)\big)} \\ &= (c^2/32)(\log N) \end{aligned}$$

hence

$$\exp\left(-\frac{t_N^2/32}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2}\right) \leq e^{-(c^2/32)(\log N)} \tag{30.5}$$

In order to get $e^{-(c^2/32)(\log N)} \leq N^{-5} = e^{-5\log N}$, we need to have $c^2/32 \geq 5$, so it enough to have, since $\sqrt{160} < 13$,

$$c \geq 13 \tag{30.6}$$

We also have

$$t_N/6 = (c/6)\cdot\sqrt{(\log N)\sigma\big(R(N)\big)}$$

by the assumption in eq. (28.1) for all large enough N

$$\geq (c/6)\sqrt{b}\log N$$

It follows that

$$\exp(-t_N/6) \leq e^{-(c/6)\sqrt{b}\log N} \tag{31.1}$$

We again need to have $e^{-(c/6)\sqrt{b}\log N} \leq N^{-5} = e^{-5\log N}$ which poses the requirement $(c/6)\sqrt{b} \geq 5$, that is,

$$c \geq \frac{30}{\sqrt{b}} \tag{31.2}$$

Thus choosing the constant c large enough to satisfy both eqs. (31.2) and (30.6), the estimate in eq. (30.3) implies the one in eq. (30.1). □

□

6.1 *Notes to lemma 28.1*

The type of method we used in lemma 28.1 to estimate trigonometric polynomials goes back to Salem-Zygmund.[24] Recent developments have been given for example by Weber[25] and by Cohen-Cuny.[26]

[24] Salem and Zygmund 1954, Chapter IV.

[25] Weber 2000.

[26] Cohen and Cuny 2006.

7 *Absolute continuity and positive mean*

The general theme of this section is that if a good set or weight has positive mean then it can represent only an absolutely continuous measure. To be specific, we want to prove theorems 7.2 (b) and 9.3 (b).

Our standing assumption is that R is a sublacunary good set, and hence we suppress it in our notation for the limit measure, so we write μ_α instead of $\mu_{R,\alpha}$.

7.1 *Proof of theorem 7.2 (b)*

Theorem 7.2 (a) says that if ρ is an unsigned $L^\infty(\mu_\alpha)$ function with $\mu_\alpha(\rho) > 0$ and α is an irrational number then ρ can be represented at α with a good set $S \subset R$ satisfying $\mathbf{M}_R(S) = \frac{\mu_\alpha(\rho)}{\|\rho\|_{L^\infty(\mu_\alpha)}}$. We have proved this in section 6.

Theorem 7.2 (b) says that the converse is also true: if the good set $S \subset R$ satisfies $\|S\|_{\mathbf{1},R} > 0$ then the limit measure $\mu_{S,\beta}$ is absolutely continuous with respect to μ_β with a bounded Radon-Nikodym derivative ρ_β which must satisfy

$$\|\rho_\beta\|_{L^\infty(\mu_\beta)} \leq \frac{1}{\|S\|_{\mathbf{1},R}} \text{ for every } \beta \tag{31.3}$$

This is what we intend to prove now. Since $\beta \in \mathbb{T}$ is fixed, we suppress it in our notation, so for example we write μ for μ_β and μ_S for $\mu_{S,\beta}$. Let

$S \subset R$ be such that $\|S\|_{\mathbf{1},R} > 0$. Let us first show that for every β, the limit measure μ_S is absolutely continuous with respect to μ.

This will follow if we show that for every Borel set B we have

$$\mu_S(B) \leq \frac{1}{\|S\|_{\mathbf{1},R}} \mu(B) \tag{32.1}$$

To see this, it's enough to show that for every unsigned, continuous function ϕ on $\mathbb{T}$ we have

$$\mu_S(\phi) \leq \frac{1}{\|S\|_{\mathbf{1},R}} \mu(\phi) \tag{32.2}$$

Let ϕ be such a function and let $N_1 < N_2 < \dots$ be a sequence of indices for which $\lim_k \mathbb{A}_{r\in R(N_k)} \mathbb{1}_S(r) = \|S\|_{\mathbf{1},R}$. We can then estimate as

$$\begin{aligned}
\mu_S(\phi) &= \lim_N \mathbb{A}_{s\in S(N)} \phi(s\beta) \\
&= \lim_k \mathbb{A}_{s\in S(N_k)} \phi(s\beta) \\
&= \lim_k \frac{1}{\mathbb{A}_{r\in R(N_k)} \mathbb{1}_S(r)} \mathbb{A}_{r\in R(N_k)} \mathbb{1}_S(r)\phi(r\beta) \\
&\leq \limsup_k \frac{1}{\mathbb{A}_{r\in R(N_k)} \mathbb{1}_S(r)} \mathbb{A}_{r\in R(N_k)} \phi(r\beta)
\end{aligned}$$

since $\lim_k \frac{1}{\mathbb{A}_{n\in[1,N_k]} \mathbb{1}_S(n)} = \frac{1}{\|S\|_{\mathbf{1},R}}$ and $\lim_N \mathbb{A}_{r\in R(N_k)} \phi(r\beta)$ exists,

$$= \frac{1}{\|S\|_{\mathbf{1},R}} \lim_N \mathbb{A}_{r\in R(N_k)} \phi(r\beta)$$

since $\lim_N \mathbb{A}_{r\in R(N_k)} \phi(r\beta) = \mu(\phi)$,

$$= \frac{1}{\|S\|_{\mathbf{1},R}} \cdot \mu(\phi)$$

proving eq. (32.2).

Now, inequality $\mu(\rho_\beta \mathbb{1}_B) \leq \frac{1}{\|S\|_{\mathbf{1},R}} \mu(B)$ applied to the Borel set $B = \left\{\rho_\beta > \frac{1}{\|S\|_{\mathbf{1},R}}\right\}$ readily gives eq. (31.3).

7.2 *Proof of theorem 9.3 (b)*

Since the good set R is fixed, we suppress it in our notation for the limit measures, so we write μ_α instead of $\mu_{R,\alpha}$.

In this section, we need to prove that if the good R-weight w has positive relative $\mathbf{1}$-norm and it is integrable, that is, it can be approximated arbitrary closely by bounded, good R-weights in $\|\|_{\mathbf{1},R}$-seminorm, then for every irrational β the limit measure $\mu_{w,\beta}$ is absolutely continuous with respect to μ_β.

Let (w_k) be a sequence of good, bounded R-weights which converges to w in $\|\|_{\mathbf{1},R}$-seminorm, $\lim_k \|w_k - w\|_{\mathbf{1},R} = 0$. Since $\left| \|w_k\|_{\mathbf{1},R} - \|w\|_{\mathbf{1},R} \right| \leq$

$\|w_k - w\|_{\mathbf{1},R}$, we have $\lim_k \|w_k\|_{\mathbf{1},R} = \|w\|_{\mathbf{1},R} > 0$, and hence we can assume without loss of generality that $\|w_k\|_{\mathbf{1},R} > 0$ for every k. That for every k the measure $\mu_{w_k,\beta}$ is absolutely continuous with respect to μ_β for every β follows from

$$\mu_{w_k,\beta}(B) \leq \frac{\|w_k\|_\infty}{\|w_k\|_{\mathbf{1},R}} \mu_\beta(B) \text{ for every Borel set } B \tag{33.1}$$

The proof of this inequality is almost identical to the proof of the inequality in eq. (32.1), hence we omit it.

Now the rest of the proof of theorem 9.3 follows from lemma 21.1.

8 *Proof of theorem 7.1 for unbounded ρ*

In this section we again work with a fixed, sublacunary good set $R \subset \mathbb{N}$ which we view as a sequence (r_n) arranged in increasing order. We omit R from our notation for the limit measures, so we write μ_β instead of $\mu_{R,\beta}$. We also fix an irrational number α. Let $\rho \in L^1(\mu_\alpha)$. We want to find a good set[27] $S \subset R$ which represents ρ at α. According to proposition 26.1 there is a good R-weight w which represents ρ at α. Since this weight w has positive relative mean with respect to R, it's a sublacunary weight. The problem is that, as per construction, w is not a bounded weight if ρ is unbounded, hence we cannot use our proposition 27.2 to construct the desired set S.

[27] Which can be shown to be sublacunary as a consequence of the sublacunarity of the weight v below.

Our main job in this section hence will be to construct a good R-weight v satisfying the following properties

- v is bounded by 1;
- v is sublacunary;
- v represents the same measure at every β as w, so $\mu_{v,\beta} = \mu_{w,\beta}$ for every β.

Once we have such a good weight v, we can use proposition 27.2 to "construct" the desired good set S.

The weight v will be of the form $\sigma \cdot w$ where the weight σ is a *decreasing* weight, that is, $\sigma(r_n) \geq \sigma(r_{n+1})$ for every $n \in \mathbb{N}$. That a weight v of this form represents the same measures everywhere is a consequence of a general but probably familiar result—our main new tool in this section. Not to get bugged down with unnecessary notation, we will state the result for weights with the reindexing $w(n) = w(r_n)$ with which R weights become $\mathbb{N}$-weights.

First recall the definition of a dissipative sequence of measures on $\mathbb{N}$.

34.1 Definition ▶ Dissipative sequence of measures

Let $(\nu_N)_{N\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{N}$.
We say, the sequence $(\nu_N)_{N\in\mathbb{N}}$ is *dissipative* if

$$\lim_N \frac{\nu_N(j)}{\nu_N(\mathbb{N})} = 0, \text{ for every } j \in \mathbb{N} \tag{34.1}$$

34.2 Proposition ▶ Decreasing weights preserve limits

Let w be a weight, $(\sigma_N)_{N\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{N}$ and let $\mathbf{x} = (x_n)$ be a sequence from a normed space $(X, \|\|)$. Denoting $\nu_N := \sigma_N \cdot w$, we assume the following

a) Each σ_N has finite support.

b) The sequence (ν_N) is dissipative.

c) For each N the measure σ_N is decreasing, $\sigma_N(1) \geq \sigma_N(2) \geq \dots$.

d) The sequence $\left(\mathbb{A}^w_{n\in[1,N]} x_n\right)_N$ converges to some $y \in X$,

$$\lim_N \mathbb{A}^w_{n\in[1,N]} x_n = y \tag{34.2}$$

Then, the sequence $\left(\mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j\right)_N$ of averages converge to the same limit as the w-weighted averages,

$$\lim_N \mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j = y \tag{34.3}$$

At the heart of this result is the following quantitative estimate: For a given $\epsilon > 0$, if K is such that $\left\|\mathbb{A}^w_{n\in[1,j]} x_n - y\right\| < \epsilon$ for $j \geq K$ then we have

$$\left\|\mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j - y\right\| \leq \epsilon + \max_{j\in[1,K]} \left\|\mathbb{A}^w_{n\in[1,j]} x_n - y\right\| \cdot \frac{\nu_N([1,K])}{\nu_N(\mathbb{N})} \tag{34.4}$$

for every $N \geq K$.

Note that the estimate in eq. (34.4) indeed implies the conclusion of the proposition in eq. (34.3). To see this, let $N \to \infty$ in eq. (34.4). Then, since (ν_N) is a dissipative sequence so $\lim_N \frac{\nu_N([1,K])}{\nu_N(\mathbb{N})} = 0$, we get that $\limsup_N \left\|\mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j - y\right\| \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\lim_N \left\|\mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j - y\right\| = 0$.

Proof of proposition 34.2. The main idea of the proof is to write $\mathbb{A}^{\nu_N}_{j\in\mathbb{N}} x_j$ as

an average of the w-averages with respect to another measure q_N on $\mathbb{N}$

$$\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j = \mathbb{A}^{q_N}_{j\in\mathbb{N}} \mathbb{A}^{w}_{n\in[1,j]} x_n \text{ for all } N \tag{35.1}$$

These measures q_N will also satisfy

$$q_N(\mathbb{N}) = v_N(\mathbb{N}) \text{ for every } N \in \mathbb{N} \tag{35.2}$$

The measure q_N appears during performing summation by parts: setting $\sigma_N(0) := 0$, $w(0) := 0$ and $x_0 := 0$, we have

$$\begin{aligned}
\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j &= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \sigma_N(j) w(j) x_j \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \sigma_N(j) \left(\sum_{n\in[1,j]} w(n) x_n - \sum_{n\in[j-1]} w(n) x_n \right) \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \sum_{n\in[1,j]} w(n) x_n \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([1,j]) \cdot \mathbb{A}^{w}_{n\in[1,j]} x_n
\end{aligned}$$

Thus, defining the measure q_N by

$$q_N(j) := \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([1,j]), \text{ for } j \in \mathbb{N} \tag{35.3}$$

we get the identity in eq. (35.1) once we show that q_N really is a measure satisfying eq. (35.2). That $q_N(j)$ is unsigned follows from the assumption that the sequence $(\sigma_N(j))_{j\in\mathbb{N}}$ is decreasing for fixed N. That $q_N(\mathbb{N}) = v_N(\mathbb{N})$ follows by setting $x_j = 1$ for every j in the summation by parts argument above since then we get exactly $q_N(\mathbb{N}) = v_N(\mathbb{N})$:

$$\begin{aligned}
1 &= \mathbb{A}^{v_N}_{j\in\mathbb{N}} 1 \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([1,j]) \cdot \mathbb{A}^{w}_{n\in[1,j]} 1 \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} q_N(j) \cdot 1 \\
&= \frac{1}{v_N(\mathbb{N})} \cdot q_N(\mathbb{N})
\end{aligned}$$

Using the now obvious identity $y = \mathbb{A}^{q_N}_{j\in\mathbb{N}} y$ together with eq. (35.1), we can now write $\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y$ as

$$\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y = \mathbb{A}^{q_N}_{j\in\mathbb{N}} \Big(\mathbb{A}^{w}_{n\in[1,j]} x_n - y \Big) \tag{35.4}$$

Let $\epsilon > 0$. Since we assumed $\lim_N \mathbb{A}^{w}_{n\in[1,N]} x_n = y$, there is an $K = K(\epsilon)$ so that

$$\left\| \mathbb{A}^{w}_{n\in[1,j]} x_n - y \right\| < \epsilon, \text{ for } j \geq K \tag{35.5}$$

Splitting the summation on j in $\mathbb{A}^{q_N}_{j\in\mathbb{N}}\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)$ into two parts at K and using the triangle inequality, we get the estimate

$$\left\|\mathbb{A}^{q_N}_{j\in\mathbb{N}}\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| \leq \left\|\frac{1}{q_N(\mathbb{N})}\sum_{j\in[1,K]} q_N(j)\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| + \left\|\frac{1}{q_N(\mathbb{N})}\sum_{j>K} q_N(j)\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| \tag{36.1}$$

We can estimate the first term as

$$\left\|\frac{1}{q_N(\mathbb{N})}\sum_{j\in[1,K]} q_N(j)\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| \leq \max_{j\in[1,K]}\left\|\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right\| \cdot \frac{q_N([1,K])}{q_N(\mathbb{N})} \tag{36.2}$$

Using the definition of $q_N(j)$ as given in eq. (35.3), we can estimate $q_N([1,K])$ as

$$\begin{aligned} q_N([1,K]) &= \sum_{j\in[1,K]}\Big(\sigma_N(j) - \sigma_N(j+1)\Big)\cdot w([1,j]) \\ &= \sum_{j\in[1,K]}\sigma_N(j)\Big(w([1,j]) - w([j-1])\Big) - \sigma_N(K+1)w([1,K]) \\ &= \sum_{j\in[1,K]}\sigma_N(j)w(j) - \sigma_N(K+1)w([1,K]) \\ &= \sum_{j\in[1,K]} v_N(j) - \sigma_N(K+1)w([1,K]) \\ &\leq v_N([1,K]) \end{aligned}$$

Using this estimate and that $q_N(\mathbb{N}) = v_N(\mathbb{N})$ in eq. (36.2) we get

$$\left\|\frac{1}{q_N(\mathbb{N})}\sum_{j\in[1,K]} q_N(j)\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| \leq \max_{j\in[1,K]}\left\|\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right\| \cdot \frac{v_N([1,K])}{v_N(\mathbb{N})} \tag{36.3}$$

The second term in eq. (36.1) can be estimated, using eq. (35.5), as

$$\left\|\frac{1}{q_N(\mathbb{N})}\sum_{j>K} q_N(j)\left(\mathbb{A}^{w}_{n\in[1,j]}x_n - y\right)\right\| \leq \epsilon \tag{36.4}$$

Putting the estimates in eqs. (36.3) and (36.4) into eq. (36.1) and using the identity in eq. (35.4) we get eq. (34.4). □

36.1 Corollary ▶ Decreasing weights preserve limit measures of weights

Let w and σ be R-weights. Denoting $v := \sigma \cdot w$, we assume the following

a) $v(R) = \infty$.

b) The R-weight σ is decreasing $\sigma(r_1) \geq \sigma(r_2) \geq \dots$.

c) The R-weight w is good.

Then v is a good R-weight and it represents the same measures everywhere as w,

$$\mu_{v,\beta} = \mu_{w,\beta} \text{ for every } \beta \tag{37.1}$$

Proof. We need to show that for a given β we have

$$\lim_N \mathbb{A}^v_{n\in[1,N]} \mathrm{e}(r_n\beta) = \mu_{w,\beta}(\mathrm{e}) \tag{37.2}$$

to do this, use proposition 34.2 with σ_N defined by

$$\sigma_N(n) := \sigma(r_n)\mathbb{1}_{[1,N]}(n) \tag{37.3}$$

and (x_n) defined by

$$x_n := \mathrm{e}(r_n\beta) \tag{37.4}$$

□

Let us now go back to our good R-weight w which represents ρ at α. Since we now consider R as the sequence (r_n), its sublacunarity assumption is expressed more conveniently as

$$\lim_N \frac{N}{\log r_N} = \infty \tag{37.5}$$

as we noted in eq. (7.2). Since the weight w satisfies $\mathbf{M}_R(w) > 0$, eq. (37.5) implies that w is also sublacunary. Writing $\frac{N+1}{\log r_{N+1}} = \frac{N+1}{N} \cdot \frac{N}{\log r_{N+1}}$ we see that eq. (37.5) implies

$$\lim_N \frac{N}{\log r_{N+1}} = \infty \tag{37.6}$$

According to the proof of lemma 16.1, we obtained w as the limit of a sequence (w_k) of bounded good weights by pasting the w_k together piece by piece in a sense that after choosing indices $N_1 < N_2 < \dots$, we define w to be equal w_k on the interval $(N_k, N_{k+1}]$

$$w(r_n) := \sum_k w_k(r_n)\mathbb{1}_{(N_k,N_{k+1}]}(n) \tag{37.7}$$

Now, in order to obtain a good weight v which is bounded by 1 and would represent the same measures as w, we could do the following. Define σ by

$$\sigma(r_n) := \frac{1}{\max_{j\in[1,k]} \|w_j\|_\infty} \cdot \mathbb{1}_{(N_k,N_{k+1}]}(n) \tag{37.8}$$

Then σ is decreasing and $v := \sigma w$ is bounded by 1. The remaining issue is to ensure that v is sublacunary, and to do that it's enough to ensure

$$\lim_N \frac{\sum_{n\in[1,N]} v(r_n)}{\log r_{N+1}} = \infty \tag{38.1}$$

as we noted in eq. (27.3). This would also ensure that both σ and v are weights. It turns out that in the recursive process of choosing the indices (N_k) if we choose N_k large enough compared to N_{k-1} we can ensure that v is sublacunary. We want to show that we can choose the indices N_k so that we will have eq. (38.1). Let us note that in the proof of lemma 16.1 the choice of N_k is flexible, since it just has to be large enough to staisfy some criteria. So we now add one additional criterion, namely we want to choose N_k large enough to also satisfy

$$\frac{N}{\max_{j\in[1,k]} \|w_j\|_\infty} > k \log r_{N+1} \text{ for every } N \geq N_k \tag{38.2}$$

This is possible because of the sublacunarity condition in eq. (37.6), and eq. (38.2) ensures the sublacunarity of v, that is, eq. (38.1).

That v represents the same measures as w at every β follows from corollary 36.1. As in the last step of our proof of theorem 7.1, we use proposition 27.2 to show the existence of a good set $S \subset R$ which represents the same measures as v at every β, hence at $\beta = \alpha$ we have $\mu_{S,\alpha} = \rho\mu_\alpha$.

9 *The limit measure at rational points*

In this section we want to prove theorem 5.2. The base set is $\mathbb{N}$ which we suppress in our notation, so we write μ_β instead of $\mu_{\mathbb{N},\beta}$.

Given the probability measure ν on $\mathbb{T}_q$ and the rational number $\frac{a}{q}$, $\gcd(a,q) = 1$, let us see what properties a good set S would need to have so that $\mu_{S,a/q} = \nu$.

Introducing the sets S_j by

$$S_j := \{\, s : s \in S, sa \equiv j \pmod q \,\}, \text{ for every } j \in [1,q] \tag{38.3}$$

let us write, using that the S_j are pairwise disjoint,

$$\begin{aligned}
\mathbb{A}_{s\in S(N)} &= \frac{1}{\#S(N)} \sum_{s\in S(N)} \delta_{sa/q} \\
&= \frac{1}{\#S(N)} \sum_{j\in[1,q]} \sum_{s\in S_j(N)} \delta_{j/q} \\
&= \sum_{j\in[1,q]} \frac{\#S_j(N)}{\#S(N)} \delta_{j/q}
\end{aligned}$$

If we make the assumption[28] that $\lim_N \frac{\#S_j(N)}{\#S(N)}$ exists for every j then, letting $N \to \infty$, we get

$$\mu_{S,a/q} = \sum_{j\in[1,q]} \delta_{j/q} \lim_N \frac{\#S_j(N)}{\#S(N)} \tag{39.1}$$

Since $\mu_{S,a/q}$ is supposed to be equal ν, we get

$$\lim_N \frac{\#S_j(N)}{\#S(N)} = \nu(j/q) \tag{39.2}$$

This gives us the idea how to construct S: we start out from the set R_j defined by

$$R_j := \{\, n \,:\, na \equiv j \pmod q \,\}, \text{ for every } j \in [1,q] \tag{39.3}$$

Note that R_j is a full residue class $\mod q$, namely, if j' denotes the unique solution to the congruence $j'a \equiv j \pmod q$, then R_j is the arithmetic progression $\{\, kq + j' \,:\, k \in \mathbb{N} \,\}$. Note that R_j is a good set, as are all arithmetic progressions. We clearly have

$$\mathbf{M}(R_j) = \frac{1}{q} \text{ for every } j \in [1,q] \tag{39.4}$$

Now what remains is to find a set $S_j \subset R_j$ with relative mean $\nu\left(\frac{j}{q}\right)$ and make sure that S_j is a good set. Let γ be an irrational number and consider

$$S_j := \left\{ r \,:\, r \in R_j, r\gamma \in \left[0, \nu\left(\frac{j}{q}\right)\right) \right\} \text{ for every } j \in [1,q] \tag{39.5}$$

Using proposition 14.1 with $\alpha = \gamma$ and $R = R_j$, we deduce that S_j is a good set with $\mathbf{M}_{R_j}(S_j) = \nu\left(\frac{j}{q}\right)$, as desired. We finally define S as

$$S := \bigcup_{j\in[1,q]} S_j \tag{39.6}$$

The set S is good since it's the finite union of pairwise disjoint good sets with mean. Indeed, we have $\mathbf{M}(S_j) = \frac{1}{q} \cdot \nu\left(\frac{j}{q}\right)$ and hence $\mathbf{M}(S) = \frac{1}{q}$.

10 *Examples*

10.1 *Two good sets, but their intersection has no mean.*

Here we construct randomly two good sets, R, S with $\mathbf{M}(R) = \mathbf{M}(S) = 1/2$ but $\mathbf{M}(R \cap S)$ doesn't exist.

Let (X_n) be a iid sequence of random variables on the probability space (Ω, P), modeling fair coin flipping, so with distribution $P(X_n = 1) =$

[28] In fact, the existence of $\lim_N \frac{\#S_j(N)}{\#S(N)}$ follows from S being a good set.

$P(X_n = 0) = 1/2$. Let us also consider another sequence of random variables (Y_n) defined by

$$Y_n = \begin{cases} X_n & \text{if } n \in [2^k, 2^{k+1}) \text{ for even } k \\ 1 - X_n & \text{if } n \in [2^k, 2^{k+1}) \text{ for odd } k \end{cases} \tag{40.1}$$

The (Y_n) is also an iid sequence with the same distribution as the (X_n). Define the sets R^ω, S^ω by $R^\omega := \{ n : X_n(\omega) = 1 \}$ and $S^\omega := \{ n : Y_n(\omega) = 1 \}$. By lemma 28.1 both R^ω and S^ω are good sets almost surely with $\mathbf{M}(R^\omega) = \mathbf{M}(S^\omega) = 1/2$. We claim that $\mathbf{M}(R^\omega \cap S^\omega)$ almost surely doesn't exists. To see this, denote $T^\omega := R^\omega \cap S^\omega$ and observe that if $\mathbf{M}(T^\omega)$ existed then $\lim_k \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k}$ would exist. But, denoting by O the odd numbers and by E the even numbers, we almost surely have

$$\lim_{k \in O} \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k} = 0$$
$$\lim_{k \in E} \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k} = \frac{1}{2}$$

10.2 *$R_1 \cup R_2$ and $R_1 \cap R_2$ have means but are not good*

Here is an example of two good sets R_1 and R_2 each with mean $2/3$, $\mathbf{M}(R_1 \cap R_2) = 1/2$ but $R_1 \cap R_2$ is not good and $\mathbf{M}(R_1 \cup R_2) = 5/6$ but $R_1 \cup R_2$ is not good.

Both sets will be defined in blocks of intervals. Partition $\mathbb{N}$ into a sequence of disjoint intervals I_n so that their lengths go to infinity but slower than the left endpoints go to infinity. For example, $I_n = [n^2, (n+1)^2)$ will do.

The first good set R_1 will contain all i***N***tegers from I_1, then only ***O***dd numbers from I_2 then ***E***ven numbers from I_3 then repeat this pattern for I_4, I_5, I_6 etc:

$$NOENOE\ldots \tag{40.2}$$

The set R_2 is defined similarly, except it will have one pattern in intervals $J_k := [3^k, 3^{k+1})$ for even k and another for odd k.

$$EONEON\ldots \text{ for even } k \tag{40.3}$$

$$ONEONE\ldots \text{ for odd } k \tag{40.4}$$

Both of these sets are good and they represent the same (uniform) measure at every β.

The intersection $R_1 \cap R_2$ has the patterns

$$EOEEOE\ldots \text{ for even } k \tag{40.5}$$

$$OOEOOE\ldots \text{ for odd } k \tag{40.6}$$

Clearly $\mathbf{M}(R_1 \cap R_2) = 1/2$ but the average of $\mathrm{e}(n/2)$ is different on J_k for even k from those on odd k: for even k the average will go to $1/3$ while for odd k it goes to $-1/3$.

As for the union $R_1 \cup R_2$, it has the patterns

$$NONNON\dots \text{ for even } k \tag{41.1}$$

$$NNENNE\dots \text{ for odd } k \tag{41.2}$$

Clearly $\mathbf{M}(R_1 \cup R_2) = 5/6$ but the average of $\mathrm{e}(n/2)$ is different on J_k for even k from those on odd k: for even k the average will go to $-1/3$ while for odd k it goes to $1/3$.

10.3 *Open set U with visit set $\{ n : n\alpha \in U \}$ not good*

The construction does not use at all the group structure or the dimensional properties of the torus. This can be extended in a general context of a sequence in a compact metric space with a non purely atomic asymptotic distribution.

Let α be an irrational number in the torus $\mathbb{T}$. We show that there exists an open subset U of the torus such that the sequence $\left(\mathbb{A}_{n\in[1,N]}\mathbb{1}_U(n\alpha)\right)_N$ does not converge when N goes to infinity.

We want to construct an open subset U of the torus and an increasing sequence of positive integers $(N_k)_{k\geq 0}$ such that the averages $\mathbb{A}_{n\in[1,N_{2k}]}\mathbb{1}_U(n\alpha)$, $k = 0, 1, 2, \dots$, with even indices are large whereas the averages $\mathbb{A}_{n\in[1,N_{2k+1}]}\mathbb{1}_U(n\alpha)$, $k = 0, 1, 2, \dots$ with odd indices are small.

The sequence (N_k) will be constructed by induction and each N_k will be associated to $\epsilon_k := 1/(2^{k+4}N_k)$. In this induction process, we construct also a sequence of open subsets $(U_k)_{k\geq 0}$.

We start with $N_0 > 1$ fixed and we define

$$U_0 := \bigcup_{n\in[1,N_0]} (n\alpha - \epsilon_0, n\alpha + \epsilon_0)$$

We have of course

$$\mathbb{A}_{n\in[1,N_0]}\mathbb{1}_{U_0}(n\alpha) = 1 \quad \text{and} \quad 0 < \lambda\big(\overline{U_0}\big) \leq 2N_0\epsilon_0$$

This is the initial step of our construction. In order to be understandable, let us describe the two next steps.

By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_1 > N_0$ such that

$$\mathbb{A}_{n\in[1,N_1]}\mathbb{1}_{\overline{U_0}}(n\alpha) \leq 2\lambda(\overline{U_0}) \leq 4(N_0\epsilon_0)$$

We fix such a N_1. To any $n \in [1, N_1]$ with $n\alpha \notin \overline{U_0}$ we associate a real δ_n that

$$0 < \delta_n \leq \epsilon_1 \quad \text{and} \quad (n\alpha - \delta_n, n\alpha + \delta_n) \cap U_0 = \emptyset$$

We define

$$U_1 := \bigcup_{\substack{n\in[1,N_1]\\ n\alpha\notin\overline{U_0}}} (n\alpha - \delta_n, n\alpha + \delta_n)$$

We have

$$\mathbb{A}_{n\in[1,N_1]}\mathbb{1}_{U_1}(n\alpha) \geq 1 - 4N_0\epsilon_0 \quad \text{and} \quad 0 < \lambda\big(\overline{U_1}\big) \leq 2N_1\epsilon_1$$

Note also that by construction $U_0 \cap U_1 = \emptyset$.

By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_2 > N_1$ such that

$$\mathbb{A}_{n\in[1,N_2]}\mathbb{1}_{\overline{U_1}}(n\alpha) \leq 2\lambda\big(\overline{U_1}\big) \leq 4(N_1\epsilon_1)$$

We fix such a N_2. To any $n \in [1, N_2]$ with $n\alpha \notin \overline{U_1}$ we associate a real δ_n satisfying

$$0 < \delta_n \leq \epsilon_2 \quad \text{and} \quad (n\alpha - \delta_n, n\alpha + \delta_n) \cap U_1 = \emptyset$$

Note that the values of the δ_n's are reinitialized.

We define

$$U_2 \coloneqq U_0 \cup \bigcup_{\substack{n\in[1,N_2]\\ n\alpha\notin\overline{U_1}}} (n\alpha - \delta_n, n\alpha + \delta_n)$$

We have

$$\mathbb{A}_{n\in[1,N_2]}\mathbb{1}_{U_2}(n\alpha) \geq 1 - 4N_1\epsilon_1 \quad \text{and} \quad \lambda\big(\overline{U_2}\big) \leq 2N_0\epsilon_0 + 2N_2\epsilon_2$$

Note also that by construction $U_2 \cap U_1 = \emptyset$ and $U_0 \subset U_2$.

Let us state now our induction hypothesis. Suppose that, for a fixed integer $k > 0$ we have already constructed two sequences

$$(U_\ell)_{0\leq\ell\leq k} \quad \text{and} \quad N_0 < N_1 < N_2 < \ldots < N_k$$

such that

- $U_0 \subset U_2 \subset U_4 \subset \ldots$ and $U_1 \subset U_3 \subset U_5 \subset \ldots$,
- If ℓ is even and ℓ' is odd, then U_ℓ and $U_{\ell'}$ are disjoint,
- Each U_ℓ is a finite union of open intervals,
- If $0 \leq 2\ell \leq k$, then

$$\lambda(U_{2\ell}) \leq 2(N_0\epsilon_0 + N_2\epsilon_2 + \ldots + N_{2\ell}\epsilon_{2\ell})$$

and

$$\mathbb{A}_{n\in[1,N_{2\ell}]}\mathbb{1}_{U_{2\ell}}(n\alpha) \geq 1 - 4(N_1\epsilon_1 + N_3\epsilon_3 + \ldots + N_{2\ell-1}\epsilon_{2\ell-1})$$

- If $1 \leq 2\ell + 1 \leq k$, then

$$\lambda(U_{2\ell+1}) \leq 2(N_1\epsilon_1 + N_3\epsilon_3 + \ldots + N_{2\ell+1}\epsilon_{2\ell+1})$$

and

$$\mathbb{A}_{n\in[1,N_{2\ell+1}]}\mathbb{1}_{U_{2\ell+1}}(n\alpha) \geq 1 - 4(N_0\epsilon_0 + N_2\epsilon_2 + \ldots + N_{2\ell}\epsilon_{2\ell})$$

Here begins the induction process. By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_{k+1} > N_k$ such that

$$\mathbb{A}_{n\in[1,N_{k+1}]}\mathbb{1}_{\overline{U_k}}(n\alpha) \leq 2\lambda\left(\overline{U_k}\right)$$

We fix such a N_{k+1}. To any $n \in [1, N_{k+1}]$ with $n\alpha \notin \overline{U_k}$ we associate a real δ_n that

$$0 < \delta_n \leq \epsilon_{k+1} \quad \text{and} \quad (n\alpha - \delta_n, n\alpha + \delta_n) \cap U_k = \emptyset$$

We define

$$U_{k+1} := U_{k-1} \cup \bigcup_{\substack{n\in[1,N_{k+1}]\\ n\alpha\notin\overline{U_k}}} (n\alpha - \delta_n, n\alpha + \delta_n)$$

Note that the values of δ_n's are reinitialized at each induction step.

The items of the induction hypothesis are now satisfied by the sequences $(U_\ell)_{0\leq\ell\leq k+1}$ and $(N_\ell)_{0\leq\ell\leq k+1}$.

We can consider these sequences as infinite, and we define $U := \bigcup_{k\geq 0} U_{2k}$.

Recalling our choice $N_k\epsilon_k = 2^{-k-4}$, we obtain

$$\begin{aligned}\mathbb{A}_{n\in[1,N_{2k}]}\mathbb{1}_U(n\alpha) &\geq \mathbb{A}_{n\in[1,N_{2k}]}\mathbb{1}_{U_{2k}}(n\alpha)\\ &\geq 1 - 4\sum_\ell N_{2\ell+1}\epsilon_{2\ell+1}\\ &= 5/6\end{aligned}$$

and

$$\begin{aligned}\mathbb{A}_{n\in[1,N_{2k+1}]}\mathbb{1}_U(n\alpha) &\leq \mathbb{A}_{n\in[1,N_{2k+1}]}\mathbb{1}_{U^c_{2k+1}}(n\alpha)\\ &\leq 4\sum_\ell N_{2\ell}\epsilon^{2\ell}\\ &= 1/3\end{aligned}$$

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