Generation of measures by statistics of rotations along sets of integers

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Let $S := (s_1 < s_2 < \dots)$ be a strictly increasing sequence of positive integers and denote $e(\beta) := e^{2\pi i\beta}$. We say S is *good* if for every real α the sequence $\left(\frac{1}{N}\sum_{n\le N} e(s_n\alpha)\right)_{N\in\mathbb{N}}$ of complex numbers is convergent. Equivalently, the sequence S is good if for every real α the sequence $(s_n\alpha)$ possesses an asymptotic distribution modulo 1. We are interested in finding out what the limit measure $\mu_{S,\alpha} := \lim_N \frac{1}{N}\sum_{n\le N}\delta_{s_n\alpha}$ can be. In this first paper on the subject, we investigate the case of a single irrational α. We show that if S is a good set then for every irrational α the limit measure $\mu_{S,\alpha}$ must be a continuous Borel probability measure. Using random methods, we show that the limit measure $\mu_{S,\alpha}$ can be any measure which is absolutely continuous with respect to the Haar-Lebesgue probability measure on $\mathbb{T}$. On the other hand, if ν is the uniform probability measure supported on the Cantor set, there are some irrational α so that for no good sequence S can we have the limit measure $\mu_{S,\alpha}$ equal ν. We leave open the question whether for any given singular, continuous Borel probability measure ν there is an irrational α and a good sequence S so that $\mu_{S,\alpha} = \nu$.

Contents

1 Acknowledgments

We thank Christophe Cuny for valuable remarks and his observation that in the second parts of theorems 10.3 and 11.3 it is enough to assume the positivity of $\|\|_1$ and there is no need to assume that the good sets and weights have mean.

2 Notations, definitions

Natural numbers $\mathbb{N}$ The set $\{1, 2, 3, \dots\}$ of natural numbers is denoted by $\mathbb{N}$.

Torus $\mathbb{T}$ We identify the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with the unit right-open interval,

$$\mathbb{T} := [0, 1) \tag{3.1}$$

Haar-Lebesgue measure λ We denote by λ the Haar-Lebesgue probability measure on the torus.

$\mathrm{e}(\theta)$, $\mathrm{e}_p(\theta)$ We use Weyl's notation[1]

[1] Weyl 1916.

$$\mathrm{e}(\theta) := \exp(2\pi i\theta), \text{ for } \theta \in \mathbb{T} \tag{3.2}$$

and

$$\mathrm{e}_p(\theta) := \exp(2\pi ip\theta), \text{ for } p \in \mathbb{Z}, \theta \in \mathbb{T} \tag{3.3}$$

so $\mathrm{e} = \mathrm{e}_1$.

$[N]$ We borrow the following convenient notation from combinatorists

$$[N] := \{1, 2, \dots, N\} \tag{3.4}$$

Nth initial segment $S(N)$ We denote by $S(N)$ the Nth initial segment of $S \subset \mathbb{N}$

$$S(N) := S \cap [N] \tag{3.5}$$

Counting measure $\#S$ We denote by $\#$ the counting measure on $\mathbb{N}$, so $\#S = \#(S) = \sum_{n\in\mathbb{N}} \mathbb{1}_S(n)$ is the number of elements in $S \subset \mathbb{N}$.

Arithmetic average $\mathbb{A}_S f$ For a finite set S and for a function f defined on S, we denote by $\mathbb{A}_S f$ the *arithmetic average* of f on S,

$$\mathbb{A}_S f := \frac{1}{\#S} \sum_{s\in S} f(s) \tag{3.6}$$

σ-average $\mathbb{A}^\sigma_S f$ If σ is a non-identically 0, finite measure on a set S, we then denote by $\mathbb{A}^\sigma_S f$ the *σ-average* of f on S,

$$\mathbb{A}^\sigma_S f := \frac{1}{\sigma(S)} \int_S f \, \mathrm{d}\sigma. \tag{3.7}$$

So $\mathbb{A}^\sigma$ is simply the normalized integral with respect to the measure σ, and hence the integral of f with respect to σ has to make sense. We usually encounter this notation for $S \subset \mathbb{N}$, and in this case it takes the form

$$\mathbb{A}_S^\sigma f = \frac{1}{\sigma(S)} \sum_{s \in S} \sigma(s) f(s). \tag{4.1}$$

2.1 *Good set*

In this paper we are interested in those sets $S \subset \mathbb{N}$ for which the averages $\mathbb{A}_{s \in S(N)} \mathrm{e}(s\alpha)$ converge for every $\alpha \in \mathbb{T}$ as $N \to \infty$.

4.1 Definition ▶ Good set

Let $S \subset \mathbb{N}$ be an infinite set.
We say that S is a *good set* if for every $\alpha \in \mathbb{T}$ the limit

$$\lim_N \mathbb{A}_{s \in S(N)} \mathrm{e}(s\alpha) \tag{4.2}$$

exists.

We see immediately that the set $\mathbb{N}$ is good. Since the work of Weyl,[2] we know that if S is the set of kth powers, $S = \left\{ n^k \,\middle|\, n \in \mathbb{N} \right\}$ for a $k \in \mathbb{N}$, then S is good. That the set of primes is good follows from the work of de la Vallée Poussin and Vinogradov.[3] In both cases, for irrational α the limit in eq. (4.2) is 0.

We find that it's more intuitive to describe the meaning of the limit in eq. (4.2) in terms of Borel probability measures on $\mathbb{T}$. To do this, we need some more notations.

Nth average measure $\mathbb{A}_{s \in S(N)} \delta_s$ We call $\mathbb{A}_{s \in S(N)} \delta_s$, where δ_s is Dirac's delta function at s the *Nth average measure along* S.

Nth transform measure $\mathbb{A}_{s \in S(N)} \delta_{s\alpha}$ We call $\mathbb{A}_{s \in S(N)} \delta_{s\alpha}$ the *Nth transform measure along* S.

$\nu(\phi)$ It is often convenient for us to use the functional notation for the integral with respect to a Borel measure ν on $\mathbb{T}$, so for a ν-integrable $\mathbb{T} \to \mathbb{C}$ function ϕ we denote

$$\nu(\phi) := \int_{\mathbb{T}} \phi \, \mathrm{d}\nu \tag{4.3}$$

With this notation, for a Borel measurable set $B \subset \mathbb{T}$ we have $\nu(B) = \nu(\mathbb{1}_B)$.

For a fixed α, the transform measure $\mathbb{A}_{s \in S(N)} \delta_{s\alpha}$ is a Borel probability measure on the torus. Notice that if the set S is good, that is the limit in eq. (4.2) exists for every $\alpha \in \mathbb{T}$, then, by Weierstrass's theorem

[2] Weyl 1916, Satz 9; Kuipers and Niederreiter 1974, Theorem 3.2.

[3] The case of rational α is equivalent with the prime number theorem for arithmetic progressions which is proved in Vallée Poussin 1896; the case of irrational α follows from the corresponding uniform distribution result of Vinogradov, see Vinogradow 1937.

on the density of trigonometric polynomials in the space of continuous functions on $\mathbb{T}$, for every continuous function ϕ defined on $\mathbb{T}$ the limit $\lim_N \mathbb{A}_{s\in S(N)}\phi(s\alpha)$ exists for every $\alpha \in \mathbb{T}$. This means that for a fixed $\alpha \in \mathbb{T}$, the sequence $\left(\mathbb{A}_{s\in S(N)}\delta_{s\alpha}\right)_N$ of transform measures converges weakly to a Borel probability measure on $\mathbb{T}$.

Limit transform measure $\mu_{S,\alpha}$ For a good set S and $\alpha \in \mathbb{T}$, we define the *limit measure $\mu_{S,\alpha}$ along S at α* by ,

$$\mu_{S,\alpha} := \lim_N \mathbb{A}_{s\in S(N)}\delta_{s\alpha} \tag{5.1}$$

where the limit is in the weak sense, that is, for every continuous $\mathbb{T} \to \mathbb{C}$ function ϕ we have

$$\mu_{S,\alpha}(\phi) = \lim_N \mathbb{A}_{s\in S(N)}\phi(s\alpha) \tag{5.2}$$

Let us record what we have established as a proposition.

5.1 Proposition ▶ Good set in terms of limit measures

Let $S \subset \mathbb{N}$ be an infinite set.
Then S is a good set if and only if the weak-limit measure $\mu_{S,\alpha} := \lim_N \mathbb{A}_{s\in S(N)}\delta_{s\alpha}$ exists for every $\alpha \in \mathbb{T}$.

Note that if α is irrational and $\lim_N \mathbb{A}_{s\in S(N)} e(ps\alpha) = 0$ for every nonzero $p \in \mathbb{Z}$, then the limit measure $\mu_{S,\alpha}$ is λ.

Our main object in this paper will be to try to figure out what the limit transform measure $\mu_{S,\alpha}$ can be, and for this it's convenient to introduce the following definition.

5.2 Definition ▶ Representable measure at α

Let S be a good set, and let ν be a finite Borel measure on $\mathbb{T}$.
We say that S *represents* ν *at* $\alpha \in \mathbb{T}$ if $\mu_{S,\alpha} = \frac{1}{\nu(\mathbb{T})}\nu$.
We say ν *is representable at* α if there is a good set which represents ν at α.

It's possible to reformulate the concept of a good set in terms of dynamical systems[4]. By the spectral theorem, we have the following proposition.

[4] By a *dynamical system*, we mean a probability space $(X, \mathfrak{p})$ equipped with a measurable, measure preserving transformation T of X.

5.3 Proposition ▶ Good set in terms of a dynamical systems

Let $S \subset \mathbb{N}$ be infinite.
Then the set S is good if and only if in every dynamical system $(X, \mathfrak{p}, T)$ for every $f \in L^2(X)$ the sequence $\left(\mathbb{A}_{s\in S(N)} f \circ T^s\right)_{N\in\mathbb{N}}$ is convergent in L^2-norm.

Mean $\mathbf{M}(f)$ The *mean* $\mathbf{M}(f)$ of $f \in \mathbb{C}^{\mathbb{N}}$ is the limit of the sequence $\left(\mathbb{A}_{[N]}f\right)_{N\in\mathbb{N}}$ as $N \to \infty$ if the limit exists,

$$\mathbf{M}(f) := \lim_N \mathbb{A}_{[N]}f \tag{6.1}$$

If f is the indicator of a set $S \subset \mathbb{N}$, we then often write $\mathbf{M}(S)$ in place of $\mathbf{M}(\mathbb{1}_S)$. Of course, $\mathbf{M}(S)$ is the density of S.

Relative mean $\mathbf{M}_R(f)$ Let $R \subset \mathbb{N}$ be an infinite set. The *relative mean* $\mathbf{M}_R(f)$ *of* $f \in \mathbb{C}^{\mathbb{N}}$ *in* R is the limit of the sequence $\left(\mathbb{A}_{R(N)}f\right)_{N\in\mathbb{N}}$ as $N \to \infty$ if the limit exists,

$$\mathbf{M}_R(f) := \lim_N \mathbb{A}_{R(N)}f \tag{6.2}$$

If f is the indicator of a set $S \subset R$, we then may write $\mathbf{M}_R(S)$ in place of $\mathbf{M}_R(\mathbb{1}_S)$. Of course, $\mathbf{M}_R(S)$ is the relative density of S in R.

Sequences with mean $\mathcal{M}$ We denote by $\mathcal{M}$ the collection of $f \in \mathbb{C}^{\mathbb{N}}$ for which the mean exists and is finite

$$\mathcal{M} := \{\, f \,|\, f \in \mathbb{C}^{\mathbb{N}}, \mathbf{M}(f) \text{ exists and is finite} \,\} \tag{6.3}$$

Weights, $\mathcal{W}$*, good weights,* $\mu_{w,\alpha}$ The sequence $w \in \mathbb{R}^{\mathbb{N}}$ is called a *weight* if w is unsigned and $\sum_n w(n) = \infty$. A weight can be considered a measure on $\mathbb{N}$ and in that case for $S \subset \mathbb{N}$ we may briefly write $w(S)$ in place of $\sum_{s\in S} w(s)$.

The set of all weights is denoted by $\mathcal{W}$,

$$\mathcal{W} := \{\, w \,|\, w \in \mathbb{R}^{\mathbb{N}}, w \geq 0, \sum_n w(n) = \infty \,\} \tag{6.4}$$

Let $R \subset \mathbb{N}$ be an infinite set. We often consider R as a strictly increasing sequence (r_n) of integers, in which case we sometimes write $w(n)$ instead of $w(r_n)$, so in this way we view w as supported on $\mathbb{N}$.

For a weight w supported on R we say w is *good* if the weak limit of the sequence $\left(\mathbb{A}^w_{n\in[N]}\delta_{r_n\beta}\right)_N$ of measures exists for every $\beta \in \mathbb{T}$. We denote this limit by $\mu_{w,\beta}$,

$$\mu_{w,\beta} := \lim_N \mathbb{A}^w_{n\in[N]}\delta_{r_n\beta} = \lim_N \frac{1}{w([N])} \sum_{n\in[N]} w(n)\delta_{r_n\beta} \tag{6.5}$$

In the special case of a good set S, we have $\mu_{S,\alpha} = \mu_{\mathbb{1}_S,\alpha}$ since the weighted averages with weight $w := \mathbb{1}_S$ correspond to the averages along S.

Let ν be a Borel probability measure on $\mathbb{T}$ and let $\alpha \in \mathbb{T}$.

We say the weight w represents ν at α if w is good and $\mu_{w,\alpha} = \nu$.

Seminorms $\|\|\|_{\mathbf{1}}$, $\|\|\|_{\mathbf{1},R}$ We define the Besicovitch type seminorm $\|\|\|_{\mathbf{1}}$ for all complex valued sequences by

$$\|f\|_{\mathbf{1}} := \limsup_N \mathbb{A}_{[N]}|f|, f \in \mathbb{C}^{\mathbb{N}} \tag{7.1}$$

The number **1** in the subscript of $\|\|\|_{\mathbf{1}}$ expresses the similarity of this norm to the L^1 norm.

For a set $S \subset \mathbb{N}$, we may use the notation $\|S\|_{\mathbf{1}}$ instead of $\|\mathbb{1}_S\|_{\mathbf{1}}$. It is the upper density of S.

For an infinite set $R \subset \mathbb{N}$ we define the relative **1**-norm $\|\|\|_{\mathbf{1},R}$ by

$$\|f\|_{\mathbf{1},R} := \limsup_N \mathbb{A}_{R(N)}|f|, f \in \mathbb{C}^R \tag{7.2}$$

If the set R is given as a strictly increasing sequence (r_n) and for an f defined on R we define F by $F(n) := f(r_n)$, then $\|f\|_{\mathbf{1},R} = \|F\|_{\mathbf{1}}$.

Seminorms $\|\|\|_{\mathbf{M}}$ We define the **M**-seminorm $\|\|\|_{\mathbf{M}}$ for all complex valued sequences by

$$\|f\|_{\mathbf{M}} := \limsup_N \left|\mathbb{A}_{[N]} f\right|, f \in \mathbb{C}^{\mathbb{N}} \tag{7.3}$$

Variation distance $\|\nu_1 - \nu_2\|_{\mathbf{V}}$ For finite Borel measures ν_1, ν_2 on $\mathbb{T}$ we denote by $\|\nu_1 - \nu_2\|_{\mathbf{V}}$ their *variation distance*.

$$\|\nu_1 - \nu_2\|_{\mathbf{V}} := \sup_{B \in \mathcal{B}} |\nu_1(B) - \nu_2(B)| \tag{7.4}$$

where $\mathcal{B}$ is the family of Borel subsets of $\mathbb{T}$. Note that we have

$$\|\nu_1 - \nu_2\|_{\mathbf{V}} = \sup_{\phi \in \mathcal{C}_+} |\nu_1(\phi) - \nu_2(\phi)| \tag{7.5}$$

where $\mathcal{C}_+$ denotes the set of $[0,1]$-valued continuous functions on $\mathbb{T}$,

$$\mathcal{C}_+ := \{\, \phi \,|\, \phi : \mathbb{T} \to [0,1],\ \text{continuous} \,\} \tag{7.6}$$

We summarize our notations in table 1.

3 *Main results*

To appreciate the concept of a good set, note the following. Suppose we are given an irrational number $\alpha \in \mathbb{T}$ and let ν be any Borel probability measure on $\mathbb{T}$. Then we can always find a set $S \subset \mathbb{N}$ so that $\lim_N \mathbb{A}_{s \in S(N)} \delta_{s\alpha} = \nu$. To do this, take an iid sequence (X_n) of $\mathbb{T}$-valued random variables, each with law ν. Then the sequence $(\mathrm{e}_p \circ X_n)$ is also an iid sequence of random variables

Table 1: Notations

Symbol	Definition	Parameters	Name
$\mathbb{N}$	$\{1,2,3,\dots\}$		Natural numbers
$\mathbb{T}$	$[0,1)$		torus
λ	$\lambda(\mathbb{T})$		Haar-Lebesgue measure on $\mathbb{T}$
$\mathrm{e}(\theta)$	$\exp(2\pi i\theta)$	$\theta \in \mathbb{T}$	
$\mathrm{e}_p(\theta)$	$\mathrm{e}(p\theta)$	$p \in \mathbb{Z}$	
$[N]$	$\{1,2,\dots,N\}$	$N \in \mathbb{N}$	
$S(N)$	$S \cap [N]$	$S \subset \mathbb{N}$	initial segment of S
$\#S(N)$	$\sum_{s\in S(N)} 1$	$S \subset \mathbb{N}$	counting function of S
$\mathbb{A}_S f$	$\frac{1}{\#S}\sum_{s\in S} f(s)$	set S is finite	average of f on S
$\mathbb{A}^\sigma_S f$	$\frac{1}{\sigma(S)}\int_S f \,\mathrm{d}\sigma$	σ is a finite measure on set S	σ-average of f on set S
$\mu_{S,\alpha}$	$\lim_N \mathbb{A}_{s\in S(N)}\delta_{s\alpha}$	$S \subset \mathbb{N}$, $\alpha \in \mathbb{T}$	limit transfer measure of S at α
$\mu_{w,\alpha}$	$\lim_N \mathbb{A}^w_{n\in[N]}\delta_{r_n\beta}$	weight w on (r_n), $\alpha \in \mathbb{T}$	limit transfer measure of w at α
$\nu(\phi)$	$\int_{\mathbb{T}} \phi \,\mathrm{d}\nu$		
$\mathbf{M}(f)$	$\lim_N \mathbb{A}_{[N]} f$	$f \in \mathbb{C}^{\mathbb{N}}$	mean of f
$\mathbf{M}_R(f)$	$\lim_N \mathbb{A}_{R(N)} f$	$f \in \mathbb{C}^{\mathbb{N}}$, $R \subset \mathbb{N}$	relative mean of f
$\mathcal{M}$	$\{\, f \mid f \in \mathbb{C}^{\mathbb{N}}, \mathbf{M}(f) \text{ exists and is finite} \,\}$		sequences with mean
$\mathcal{W}$	$\{\, w \mid w \in \mathbb{R}^{\mathbb{N}}, w \geq 0, \sum_n w(n) = \infty \,\}$		set of weights
$\|f\|_{\mathbf{1}}$	$\limsup_N \mathbb{A}_{[N]}\lvert f\rvert$	$f \in \mathbb{C}^{\mathbb{N}}$	**1**-seminorm
$\|f\|_{\mathbf{1},R}$	$\limsup_N \mathbb{A}_{R(N)}\lvert f\rvert$	$R \subset \mathbb{N}$, $f \in \mathbb{C}^R$	relative **1**-seminorm
$\|f\|_{\mathbf{M}}$	$\limsup_N \left\lvert \mathbb{A}_{[N]} f \right\rvert$	$f \in \mathbb{C}^{\mathbb{N}}$	**M**-seminorm
$\mathcal{C}_+$	$\{\, \phi \mid \phi : \mathbb{T} \to [0.1],\ \text{continuous} \,\}$		
$\|\nu_1 - \nu_2\|_{\mathbf{V}}$	$\sup_{\phi\in\mathcal{C}_+}(\nu_1(\phi) - \nu_2(\phi))$	ν_i finite Borel measures on $\mathbb{T}$	variation distance

for every $p \in \mathbb{Z}$, and by the strong law or large numbers we have $\lim_N \mathbb{A}_{n\in[N]} \mathrm{e}_p \circ X_n = \nu(\mathrm{e}_p)$ with full probability for every $p \in \mathbb{Z}$. This implies that there is an ω so that $\lim_N \mathbb{A}_{n\in[N]}\delta_{X_n(\omega)} = \nu$. Then, using the density of the sequence $(n\alpha)_n \pmod 1$, we can select a strictly increasing sequence (s_n) of integers so that $\lim_n (s_n\alpha - X_n(\omega)) = 0 \pmod 1$. We finally take $S := \{\, s_n \mid n \in \mathbb{N} \,\}$.

It is particularly simple to get a point-mass as a limit measure. For example, to get the Dirac measure at $1/2$, so $\mu = \delta_{1/2}$, take a strictly increasing sequence (s_n) of natural numbers so that $s_n\alpha$ converges to $1/2 \mod 1$, and let $S := \{\, s_n \mid n \in \mathbb{N} \,\}$. In contrast to this example, for good sets we have,

9.1 Theorem ► Only continuous measures can be represented at irrational points

Let $S \subset \mathbb{N}$ be a good set and α be an irrational number.
Then the limit Borel probability measure

$$\mu_{S,\alpha} = \lim_N \mathbb{A}_{s\in S(N)} \delta_{s\alpha} \tag{9.1}$$

is a continuous measure.
In other words, only continuous measures can be represented at an irrational number.

The obvious question in turn is if every continuous Borel probability measure can be represented at a given irrational number. The answer is no, as the next result shows.

9.2 Theorem ► Some continuous measures cannot be represented at every irrational point

Let ν be a Borel probability measure on $\mathbb{T}$ so that its Fourier coefficients do not converge to 0, so

$$\limsup_{p\to\infty} \left|\mu\left(e_p\right)\right| > 0 \tag{9.2}$$

Then there is a set $A \subset \mathbb{T}$ of full Lebesgue measure so that ν cannot be represented at any $\alpha \in A$.

Since a measure ν is called a *Rajchman* measure[5] if $\lim_p \mu\, e_p = 0$, we can rephrase theorem 9.2 that if ν is representable at every irrational α then it must be a Rajchman measure.[6] A well known non-Rajchman continuous measure is the uniform measure on the triadic Cantor set.

The following questions remain open.

9.3 Question ► Is every continuous measure representable somewhere?

Let ν be a continuous Borel probability measure on $\mathbb{T}$.
Is there an irrational α so that ν is representable at α?

9.4 Question ► Is a Rajchman measure representable at every α?

Let ν be a Rajchman probability measure on $\mathbb{T}$ and let α be irrational.
Is ν representable at α?

These questions can also be addressed for particular examples of singular measures such as the uniform probability measure on the triadic Cantor set (a non-Rajchman measure) or its modification by Menshov who gave the first example of a singular Rajchman mea-

[5] Lyons 1995.

[6] After discussions on the content of the present paper, Christophe Cuny and François Parreau constructed a non-Rajchman measure which is representable at uncountably many α's. This appears in the preprint Cuny and Parreau 2022.

sure.[7]

[7] Menchoff 1916; see also Lyons 1995.

The next result says that the answer to question 9.4 is *yes* if ν is absolutely continuous with respect to the Lebesgue measure on the torus $\mathbb{T}$.

10.1 Theorem ► Absolutely continuous measures are representable at every irrational point

Let ν be a Borel probability measure on $\mathbb{T}$ which is absolutely continuous with respect to the Lebesgue probability measure on $\mathbb{T}$.
Let α be an irrational number.
Then ν is representable at α.

Our proof of theorem 10.1 is flexible and enables us to show a more general result.

10.2 Theorem ► Absolutely continuous measures can be represented by subsets of a good set

Let R be a good set which is "sublacunary", that is, it satisfies the growth condition

$$\lim_N \frac{\log N}{\#R(N)} = 0. \tag{10.1}$$

Let α an irrational number, and let the Borel probability measure ν be absolutely continuous with respect to $\mu_{R,\alpha}$.
Then there is a good set $S \subset R$ which represents ν at α.

As a consequence of theorem 10.2, every measure which is absolutely continuous with respect to the Lebesgue measure can be represented at any given irrational α by a subset of the primes, squares, or $\{ \lfloor n^2 \log n \rfloor \mid n \in \mathbb{N} \}$.

In fact, the proof of theorem 10.2 reveals a close connection between the Radon-Nikodym derivative ρ of ν with respect to $\mu_{R,\alpha}$ and the mean of the set S representing ν.

10.3 Theorem ► Connection between $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_{R,\alpha}}$, $\mathbf{M}_R(S)$ and $\|S\|_{\mathbf{1},R}$

Let R be a sublacunary good set.

1. For an irrational α let the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ with $\mu_{R,\alpha}(\rho) = 1$ be bounded so $\|\rho\|_{L^\infty(\mu_{R,\alpha})} < \infty$.
   Then there is a good set $S \subset R$ representing the measure $\rho \cdot \mu_{R,\alpha}$ at α and satisfying $\mathbf{M}_R(S) = \frac{1}{\|\rho\|_{L^\infty(\mu_{R,\alpha})}}$.
2. Let S be a good subset of R with positive upper density in R, so $\|S\|_{\mathbf{1},R} > 0$.
   Then for every irrational β the limit measure $\mu_{S,\beta}$ is absolutely continuous with respect to $\mu_{R,\beta}$. Furthermore, the Radon-Nikodym derivative $\rho_\beta := \frac{\mathrm{d}\mu_{S,\beta}}{\mathrm{d}\mu_{R,\beta}}$ is a bounded function satisfying $\|\rho_\beta\|_{L^\infty(\mu_{R,\beta})} \leq \frac{1}{\|S\|_{\mathbf{1},R}}$.

Item 2 of theorem 10.3 has the following consequence.

11.1 Corollary ▶ If the RN derivative ρ is unbounded, then $\mathbf{M}_R(S) = 0$

Let R be a good set and α an irrational number. Suppose the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ with $\mu_{R,\alpha}(\rho) = 1$ is unbounded, and that the good set $S \subset R$ represents the measure $\rho \cdot \mu_{R,\alpha}$ at α. Then S must have 0 mean in R, $\mathbf{M}_R(S) = 0$.

In contrast to good sets, the representation of absolutely continuous measures by weights can always be accomplished by weights with positive, finite mean. In fact, the representing weight has an additional property.

11.2 Definition ▶ Integrable weight

We call the weight w *integrable* if it can be approximated arbitrary closely in the seminorm $\|\|_{\mathbf{1}}$ by bounded, good weights: for every $\epsilon > 0$ there is a good weight v with $\|v\|_\infty < \infty$ so that $\|v - w\|_{\mathbf{1}} < \epsilon$.

11.3 Theorem ▶ Representation by weights

Let R be a good set.

1. For an irrational α let the unsigned function $\rho \in L^1(\mu_{R,\alpha})$ satisfy $\mu_{R,\alpha}(\rho) = 1$.

   Then there is an integrable weight w on R with $\mathbf{M}_R(w) = 1$ which represents the measure $\rho \cdot \mu_{R,\alpha}$ at α. If $\rho \in L^\infty(\mu_{R,\alpha})$ then the representing weight w can also satisfy $\|\rho\|_{L^\infty(\mu_{R,\alpha})} = \|w\|_\infty$.

2. Let w be a good, integrable weight supported on R which satisfies $\|w\|_{\mathbf{1},R} > 0$.

   Then for every β the limit measure $\mu_{w,\beta}$ is absolutely continuous with respect to $\mu_{R,\beta}$.

In contrast to the case of irrational points, representation of measures at rational points is completely resolved. Consider the rational number $\alpha = \frac{a}{q}$ where $q \in \mathbb{N}$ and $a \in [q]$ and let S be a good set. Then, as can be seen readily, the limit probability measure $\mu_{S,\frac{a}{q}}$ is supported on the set $\mathbb{T}_q$ of qth roots of unity

$$\mathbb{T}_q := \left\{ \frac{j}{q} \,\middle|\, j \in [0, q-1] \right\}. \tag{11.1}$$

The next result says that this limit measure $\mu_{S,\frac{a}{q}}$ can be any probability measure supported on $\mathbb{T}_q$.

12.1 Theorem ▶ Every probability measure on $\mathbb{T}_q$ can be represented

Let q and a be positive integers with $\gcd(a,q) = 1$, and let ν be a probability measure supported on $\mathbb{T}_q$.
Then ν can be represented at $\frac{a}{q}$, that is, there is a good set S so that $\mu_{S,\frac{a}{q}} = \nu$.

The techniques developed in this paper allow one to address the *simultaneous* representability of probability measures at several different points of the torus, and we plan to explore this in a future work. But which family $\{ \mu_\alpha \,|\, \alpha \in \mathbb{T} \}$ of measures can be represented by a single good set remains open even if we restrict the family to absolutely continuous measures with respect to the Lebesgue probability measure λ. What we can say at this point is that for a given good set S, the set of $\alpha \in \mathbb{T}$ where the limit measure $\mu_{S,\alpha}$ is not the Lebesgue measure is small: it is both of first Baire category and of 0 measure under every Rajchman measure[8] on $\mathbb{T}$.

[8] Lyons 1985, Theorem 3; see also Lyons 1995.

4 *Basic example for representation*

In this section we want to work out a rather simple but instructive example, which will then motivate and form the basis of many of our constructions later on. When we are done with presenting this example, we in fact proved theorem 10.2 in case the Radon-Nikodym derivative is the indicator of a Jordan measurable set.

Let α be irrational and let $I \subset \mathbb{T}$ be an interval. We want to show that if a probability measure ν is absolutely continuous with respect to λ with the Radon-Nikodym derivative equal $\mathbb{1}_I$, the indicator of I, then there is a set S which represents ν at α. Probably the simplest way[9] to define such a set S is by taking

$$S = \{ n \,|\, n \in \mathbb{N}, n\alpha \in I \} \tag{12.1}$$

[9] We could also define such a set by taking $\{ n \,|\, n \in \mathbb{N}, n^2\alpha \in I \pmod 1 \}$ or $\{ p \,|\, p \in \Pi, p\alpha \in I \pmod 1 \}$ where Π is the set of primes. Here is a dynamically generated good set: in a dynamical system let A be a set with positive measure and consider $\{ n \,|\, n \in \mathbb{N}, T^n x \in A \}$. By the Wiener-Wintner theorem, this set is good for a.e. x.

There are two things to verify. First, that S is indeed a good set, and to do that, we need to show that the weak limit $\mu_{S,\beta} = \lim_N \mathbb{A}_{s\in S(N)} \delta_{s\beta}$ exists for every β. Second, we then have to verify that $\mu_{S,\alpha} = \frac{1}{\lambda(I)} \mathbb{1}_I \cdot \lambda$. The second one, in fact, is almost instantaneous to do since it follows from the uniform distribution of $(n\alpha)_{n\in\mathbb{N}} \pmod 1$. To see how it follows, it's enough to show that for every interval $J \subset \mathbb{T}$ we have $\mu_{S,\alpha}(J) = \lambda\left(\mathbb{1}_J \cdot \frac{1}{\lambda(I)} \mathbb{1}_I\right)$, that is

$$\lim_N \mathbb{A}_{s\in S(N)} \mathbb{1}_J(s\alpha) = \lambda\left(\mathbb{1}_J \cdot \frac{1}{\lambda(I)} \mathbb{1}_I\right) \tag{12.2}$$

The right hand side is $\frac{1}{\lambda(I)} \lambda(J \cap I)$. The left hand side can be written

as

$$\lim_N \mathbb{A}_{s\in S(N)} \mathbb{1}_J(s\alpha) = \lim_N \frac{N}{\#S(N)} \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\mathbb{1}_J(n\alpha)$$

since $\lim_N \frac{\#S(N)}{N} = \lambda(I)$ by the uniform distribution of $(n\alpha)_{n\in\mathbb{N}}$ (mod 1),

$$= \frac{1}{\lambda(I)} \lim_N \mathbb{A}_{n\in[N]} \mathbb{1}_{I\cap J}(n\alpha)$$

again by the unifom distribution of $(n\alpha)_{n\in\mathbb{N}} \pmod 1$

$$= \frac{1}{\lambda(I)} \lambda(I \cap J).$$

To show that the weak limit $\mu_{S,\beta} = \lim_N \mathbb{A}_{s\in S(N)} \delta_{s\beta}$ exists for every β, it's enough to show that $\lim_N \mathbb{A}_{s\in S(N)} \mathrm{e}(s\beta)$ exists for every β. Since

$$\mathbb{A}_{s\in S(N)} \mathrm{e}(s\beta) = \frac{N}{\#S(N)} \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) \tag{13.1}$$

and since $\lim_N \frac{\#S(N)}{N} = \lambda(I)$, it's enough to show that the limit $\lim_N \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta)$ exists for every $\beta \in \mathbb{T}$. To see this, first note that if we replace $\mathbb{1}_I$ by the character e_k the limit of $\mathbb{A}_{n\in[N]} \mathrm{e}_k(n\alpha)\, \mathrm{e}(n\beta) = \mathbb{A}_{n\in[N]} \mathrm{e}(n(k\alpha+\beta))$ as $N \to \infty$ exists and is as follows

$$\lim_N \mathbb{A}_{n\in[N]} \mathrm{e}_k(n\alpha)\, \mathrm{e}(n\beta) = \begin{cases} 1 & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.2}$$

From this we get that if we replace $\mathbb{1}_I$ by a trigonometric polynomial ϕ, the limit of $\mathbb{A}_{n\in[N]} \phi(n\alpha)\, \mathrm{e}(n\beta)$ exists and can be given explicitly as[10]

$$\lim_N \mathbb{A}_{n\in[N]} \phi(n\alpha)\, \mathrm{e}(n\beta) = \begin{cases} \lambda(\phi\, \mathrm{e}_k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.3}$$

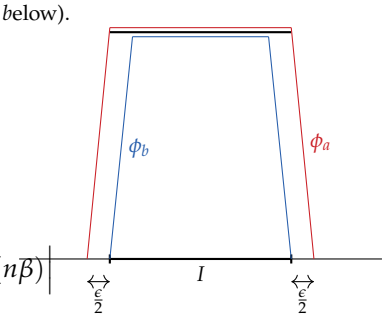
Using Weierstrass' theorem on being able to uniformly approximate a continuous function by trigonometric polynomials, we can verify that in eq. (13.3) we can take ϕ to be any continuous function.

Now, for a given $\epsilon > 0$ let us choose unsigned continuous functions ϕ_a, ϕ_b so that $\phi_b \le \mathbb{1}_I \le \phi_a$ and $\lambda(\phi_a - \phi_b) < \epsilon$ (see fig. 1). We then have

$$\begin{aligned} \left|\mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) - \mathbb{A}_{n\in[N]} \phi_b(n\alpha)\, \mathrm{e}(n\beta)\right| &= \left|\mathbb{A}_{n\in[N]} \Big(\mathbb{1}_I(n\alpha) - \phi_b(n\alpha)\Big)\, \mathrm{e}(n\beta)\right| \\ &\le \mathbb{A}_{n\in[N]} \Big(\mathbb{1}_I(n\alpha) - \phi_b(n\alpha)\Big) \\ &\le \mathbb{A}_{n\in[N]} \Big(\phi_a(n\alpha) - \phi_b(n\alpha)\Big). \end{aligned}$$

[10] Notice that in eq. (13.3) $\lambda(\phi\, \mathrm{e}_k)$ is the kth Fourier coefficient of ϕ.

Figure 1: Approximating the indicator $\mathbb{1}_I$ of the interval I by continuous functions ϕ_a (from *a*bove) and ϕ_b (from *b*elow).



It follows, since the sequence $(n\alpha)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 and $\lambda(\phi_a - \phi_b) < \epsilon$, that

$$\limsup_N \left| \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) - \mathbb{A}_{n\in[N]} \phi_b(n\alpha)\, \mathrm{e}(n\beta) \right| < \epsilon. \tag{14.1}$$

Denoting $L(\epsilon, \beta) := \lim_N \mathbb{A}_{n\in[N]} \phi_b(n\alpha)\, \mathrm{e}(n\beta)$, we can rewrite eq. (14.1) as

$$\limsup_N \left| \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) - L(\epsilon, \beta) \right| < \epsilon. \tag{14.2}$$

Since

$$\lim_{\epsilon\to 0} L(\epsilon, \beta) = \begin{cases} \lambda(\mathbb{1}_I\, \mathrm{e}_k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise,} \end{cases} \tag{14.3}$$

the limit of $\mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta)$ as $N \to \infty$ exists and is given by

$$\lim_N \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta) = \begin{cases} \lambda(\mathbb{1}_I\, \mathrm{e}_k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise.} \end{cases} \tag{14.4}$$

We finally get, since $\mu_{S,\beta}(\mathrm{e}) = \lim_N \mathbb{A}_{s\in S(N)}\, \mathrm{e}(n\beta) = \frac{1}{\lambda(I)} \lim_N \mathbb{A}_{n\in[N]} \mathbb{1}_I(n\alpha)\, \mathrm{e}(n\beta)$,

$$\mu_{S,\beta}(\mathrm{e}) = \begin{cases} \frac{1}{\lambda(I)}\lambda(\mathbb{1}_I\, \mathrm{e}_k) & \text{if } \beta = -k\alpha \pmod 1 \\ 0 & \text{otherwise.} \end{cases} \tag{14.5}$$

The above shows that $\mu_{S,\beta}(\mathrm{e})$ can be nonzero only if β is an integer multiple of α, and we recognize $\lambda(\mathbb{1}_I\, \mathrm{e}_k)$ as the kth Fourier coefficient of the function $\mathbb{1}_I$, that is, $\frac{1}{\lambda(I)}\lambda(\mathbb{1}_I\, \mathrm{e}_k)$ is the kth Fourier coefficient of the measure $\frac{1}{\lambda(I)}\mathbb{1}_I\lambda$.

One can rather easily extend this example in two ways. First, the proof can be repeated almost verbatim for the case when we take any Jordan measurable set B in place of the interval I. Indeed, all we need to remark is that a set B is Jordan measurable iff, for every given $\epsilon > 0$, its indicator function $\mathbb{1}_B$ can be approximated by a pair of unsigned, continuous functions ϕ_a and ϕ_b so that $\phi_b \le \mathbb{1}_B \le \phi_a$ and $\lambda(\phi_a - \phi_b) < \epsilon$. This approximability by continuous functions both below and above is also equivalent with saying that the boundary of the set has zero Lebesgue measure.

14.1 Definition ▶ ν-Riemann integrability

Let ν be a finite Borel measure on $\mathbb{T}$ and let ϕ be a Borel measurable $\mathbb{T} \to \mathbb{C}$ function.
We call the function ϕ *ν-Riemann* integrable if it's continuous at ν-almost every point.
We call the Borel measurable set B *ν-Jordan measurable* if its indicator function $\mathbb{1}_B$ is ν-Riemann integrable.

As it is well known, the equivalence of approximability by continuous functions and the boundary having zero measure carries over to the setting of any finite Borel measure on the torus. We can thus extend the example to the setting when the Lebesgue measure is replaced by an arbitrary finite Borel measure.

We record our findings in the following result.

15.1 Proposition ► The Radon-Nikodym derivative can be the indicator of a Jordan measurable set

Let R be a good set, α be an irrational number and let $B \subset \mathbb{T}$ be $\mu_{R,\alpha}$-Jordan measurable with $\mu_{R,\alpha}(B) > 0$.
Then the measure $\mathbb{1}_B \mu_{R,\alpha}$, which is absolutely continuous with respect to $\mu_{R,\alpha}$, can be represented at α by the good set S defined by

$$S := \{ r \,|\, r \in R, r\alpha \in B \} \tag{15.1}$$

so we have $\mu_{S,\alpha} = \frac{1}{\mu_{R,\alpha}(B)} \mathbb{1}_B \mu_{R,\alpha}$. We also have $\mu_{R,\alpha}(B) = \mathbf{M}_R(S)$.

Let us go back to trying to represent measures which are absolutely continuous with respect to the Lebesgue measure λ. New ideas are needed to cover the case when we want to represent the measure $\mathbb{1}_B \lambda$ when B is a Borel set which is not Jordan measurable. What is the new difficulty? We'd like to think that we could just again take the "visit set" $S = \{ n \,|\, n \in \mathbb{N}, n\alpha \in B \}$, but this is not the case anymore. Indeed, take B to be an open set with $\lambda(B) < 1$ and containing all integer multiples of our irrational α. This open set is not Jordan measurable anymore. The set S cannot represent the measure $\mathbb{1}_B \lambda$ anymore since $S = \mathbb{N}$. In fact, we show in section 12.3 that for any given irrational α, one can construct an open set B so that the visit set of B doesn't even have mean. So we definitely need new ideas.

We also need new ideas even for the case when we try to represent a measure which is absolutely continuous with respect to the Lebesgue measure with a Radon-Nikodym derivative which is not an indicator function. We need these new ideas even if this Radon-Nikodym derivative is a continuous function.

5 *Proof of theorem 10.2 for indicators*

Strictly speaking, we have already begun the proof of theorem 10.2 in the previous section, when we proved that at an irrational number every measure with Jordan measurable Radon-Nikodym derivative can be represented. Our fixed set up in this section is that we are given a good "base" set $R \subset \mathbb{N}$ and an irrational number α. Since the set R is fixed throughout the section, we suppress the set R from our

notations, so we write

$$\mu_\beta := \mu_{R,\beta}, \text{ for every } \beta \tag{16.1}$$

$$\mathbf{M} := \mathbf{M}_R \tag{16.2}$$

When we consider the elements of R arranged in the increasing sequence (r_n), we identify a subset S of R with the index set $\{\, n \,|\, r_n \in S \,\}$, and we denote this index set also by S.

The sets we consider in this section have positive mean in R. For such a set S, the non-normalized averages $\mathbb{A}_{n\in[N]}\mathbb{1}_S(n)\delta_{r_n\beta}$ are easier to handle than the normalized ones $\mathbb{A}_{n\in S(N)}\delta_{r_n\beta}$. The convergence or divergence properties of the two averages are equivalent since they are connected as

$$\lim_N \mathbb{A}_{n\in[N]}\mathbb{1}_S(n)\delta_{r_n\beta} = \mathbf{M}(S)\lim_N \mathbb{A}_{n\in S(N)}\delta_{r_n\beta} \tag{16.3}$$

Since our focus is to widen the class of the Radon-Nikodym derivatives with respect to the base limit measure μ_α, the following definition will simplify our language.

16.1 Definition ► Representing a function, a Borel set

Let $\rho \in L^1(\mathbb{T}, \mu_\alpha)$ be unsigned and $\mu_\alpha(\rho) > 0$.
We say that the good set $S \subset \mathbb{N}$ *represents* ρ *at* α if the weak limit of the sequence $\left(\mathbb{A}_{n\in[N]}\mathbb{1}_S(n)\delta_{r_n\alpha}\right)_N$ is the measure $\rho\mu_\alpha$.
If ρ is the indicator of a Borel measurable set $B \subset \mathbb{T}$, we then say $S \subset \mathbb{N}$ *represents* B *at* α.

In section 4 we proved that if B is μ_α-Jordan measurable, then it can be represented by the set S_B defined by

$$S_B = \{\, n \,|\, r_n\alpha \in B \,\} \tag{16.4}$$

and we have the relation

$$\mathbf{M}(S_B) = \mu_\alpha(B) \tag{16.5}$$

We also indicated that this definition of S_B may not give a good set if B is not Jordan measurable. The idea of extending the representation to any Borel measurable set is via a limit procedure. To explain what we mean by "a limit procedure", consider the case when B is an open set, and write it as a disjoint union of open intervals, $B = \cup_j I_j$. Defining $B_k := \cup_{j\in[k]} I_j$ for every $k \in \mathbb{N}$, each B_k is Jordan measurable and the sequence (B_k) increases monotonically to B. We have $\lim_k \mu_\alpha(B_k) = \mu_\alpha(B)$. Denoting $S_k := S_{B_k}$, the sequence (S_k) also increases to a set $S \subset \mathbb{N}$, but $\mathbf{M}(S)$ not only may not be equal $\lim_k \mathbf{M}(S_k)$ but $\mathbf{M}(S)$ may not even exist. The limit procedure which is suitable for our purposes is determined by the seminorm $\|f\|_{\mathbf{1}}$.

Our main tools will be two lemmas. The first one is modeled after a result of Marcinkiewicz[11] on the completeness of Besicovitch spaces.

[11] Marcinkiewicz 1939.

17.1 Lemma ▶ Cauchy sequence of sets is convergent in the seminorm $\|\|\|_{\mathbf{1}}$

Let (S_k) be a Cauchy sequence of subsets of $\mathbb{N}$ with respect to the seminorm $\|\|\|_{\mathbf{1}}$, so they satisfy

$$\lim_k \sup_{l\geq k} \left\| \mathbb{1}_{S_l} - \mathbb{1}_{S_k} \right\|_{\mathbf{1}} = 0 \tag{17.1}$$

Then there is a set $S \subset \mathbb{N}$ satisfying

$$\lim_k \left\| \mathbb{1}_{S_k} - \mathbb{1}_{S} \right\|_{\mathbf{1}} = 0 \tag{17.2}$$

Proof. We construct the set S by pasting together finite pieces of the S_k. More precisely, we recursively define a fast enough increasing sequence $N_1 < N_2 < \dots$ of indices, and then we define S to be equal S_k on the interval $(N_k, N_{k+1}]$

$$\mathbb{1}_S := \sum_k \mathbb{1}_{S_k} \mathbb{1}_{(N_k, N_{k+1}]} \tag{17.3}$$

For the recursive definition of the (N_k), define first the sequence (ϵ_k) by

$$\epsilon_k := 2 \sup_{l\geq k} \left\| \mathbb{1}_{S_l} - \mathbb{1}_{S_k} \right\|_{\mathbf{1}} \tag{17.4}$$

We can assume, without loss of generality, that $\epsilon_k > 0$ for every k, since $\epsilon_k = 0$ for some k would imply $\left\| \mathbb{1}_{S_l} - \mathbb{1}_{S_k} \right\|_{\mathbf{1}} = 0$ for $l \geq k$ hence we could take $S = S_k$.

In the first step of the recursion, let $N_1 = 1$.

In the second step, let $N_2 > N_1$ to be large enough to satisfy

$$\frac{N_1}{N_2} < \epsilon_1 \tag{17.5}$$

$$\mathbb{A}_{[N]} \left| \mathbb{1}_{S_1} - \mathbb{1}_{S_2} \right| < \epsilon_1 \text{ for every } N \geq N_2 \tag{17.6}$$

and

$$\mathbb{A}_{[N]} \left| \mathbb{1}_{S_1} - \mathbb{1}_{S_3} \right| < \epsilon_1 \text{ for every } N \geq N_2 \tag{17.7}$$

Complete the second step of the recursion by defining S to be equal S_1 on the interval $(N_1, N_2]$. Let $k > 2$ and assume that we have defined $N_1 < N_2 < \dots < N_{k-1}$ and S to be equal S_j on the interval $(N_j, N_{j+1}]$ for $j \in [k-2]$. For step k of the recursion let $N_k > N_{k-1}$ be

large enough to satisfy

$$\frac{1}{N_k}\sum_{n\in[N_{k-1}]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right|<\epsilon_j,\ \text{for every } j\in[k-2] \tag{18.1}$$

$$\mathbb{A}_{[N]}\left|\mathbb{1}_{S_j}-\mathbb{1}_{S_{k-1}}\right|<\epsilon_j \text{ for every } N\geq N_k, j\in[k-2] \tag{18.2}$$

and

$$\mathbb{A}_{[N]}\left|\mathbb{1}_{S_j}-\mathbb{1}_{S_k}\right|<\epsilon_j \text{ for every } N\geq N_k, j\in[k-2] \tag{18.3}$$

Complete the kth step of the recursion by defining S to be equal S_{k-1} on the interval $(N_{k-1}, N_k]$.

Let us fix j and let N be large enough so that for some $k\geq j+2$ we have

$$N_k\leq N<N_{k+1} \tag{18.4}$$

We want to show that

$$\mathbb{A}_{[N]}\left|\mathbb{1}_{S_j}-\mathbb{1}_S\right|<3\epsilon_j \tag{18.5}$$

Let us estimate $\mathbb{A}_{[N]}\left|\mathbb{1}_{S_j}-\mathbb{1}_S\right|$ as,

$$\mathbb{A}_{[N]}\left|\mathbb{1}_{S_j}-\mathbb{1}_S\right|=\frac{1}{N}\sum_{n\in[N_{k-1}]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right| \tag{18.6}$$

$$+\frac{1}{N}\sum_{n\in(N_{k-1},N_k]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right| \tag{18.7}$$

$$+\frac{1}{N}\sum_{n\in(N_k,N]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right| \tag{18.8}$$

We can estimate the term in eq. (18.6), using eq. (18.1) and that $N\geq N_k$, as

$$\frac{1}{N}\sum_{n\in[N_{k-1}]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right|<\epsilon_j \tag{18.9}$$

For the term in eq. (18.7) we have

$$\frac{1}{N}\sum_{n\in(N_{k-1},N_k]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right|<\epsilon_j \tag{18.10}$$

This follows from eq. (18.2) since $S=S_{k-1}$ on the interval $(N_{k-1}, N_k]$.

For the term in eq. (18.8) we have

$$\frac{1}{N}\sum_{n\in(N_k,N]}\left|\mathbb{1}_{S_j}(n)-\mathbb{1}_S(n)\right|<\epsilon_j \tag{18.11}$$

This follows from eq. (18.3) since $S=S_k$ on the interval $(N_k, N]$.

Putting the estimates in eqs. (18.9) to (18.11) together we obtain eq. (18.5). □

The second lemma shows that the family $\mathcal{M}$ of sequences with mean is closed with respect to the seminorm $\|\|\|_{\mathbf{M}}$ defined in eq. (7.3).

19.1 Lemma ▶ $\mathcal{M}$ is closed with respect to the seminorm $\|\|\|_{\mathbf{M}}$

Let (f_j) be a sequence from $\mathcal{M}$. Suppose that (f_j) converges to $f \in \mathbb{C}^{\mathbb{N}}$ in the seminorm $\|\|\|_{\mathbf{M}}$, so

$$\lim_j \left\|f_j - f\right\|_{\mathbf{M}} = 0 \tag{19.1}$$

Then $f \in \mathcal{M}$ and

$$\mathbf{M}(f) = \lim_j \mathbf{M}(f_j) \tag{19.2}$$

Proof. First note that, as a consequence of eq. (19.2), the sequence (f_j) is a Cauchy sequence, meaning that for a given $\epsilon > 0$ there is J so that

$$\left\|f_j - f_J\right\|_{\mathbf{M}} < \epsilon \text{ for every } j \geq J \tag{19.3}$$

Since $\left|\mathbf{M}(f_j) - \mathbf{M}(f_J)\right| = \left|\mathbf{M}(f_j - f_J)\right| = \left\|f_j - f_J\right\|_{\mathbf{M}}$ we see,

$$\left|\mathbf{M}(f_j) - \mathbf{M}(f_J)\right| < \epsilon \text{ for every } j \geq J \tag{19.4}$$

so the sequence $\mathbf{M}(f_j)$ of means is a Cauchy sequence of numbers. Denote $L := \lim_j \mathbf{M}(f_j)$. We want to show that $\mathbf{M}(f) = L$. For a given $\epsilon > 0$, choose a j so that $\left|\mathbf{M}(f_j) - L\right| < \epsilon$ and $\left\|f - f_j\right\|_{\mathbf{M}} < \epsilon$. We then have, for an arbitrary N,

$$\left|\mathbb{A}_{[N]} f - L\right| \leq \left|\mathbb{A}_{[N]}(f - f_j)\right| + \left|\mathbb{A}_{[N]} f_j - L\right| \tag{19.5}$$

Taking $\limsup_N$ of both sides, we get

$$\limsup_N \left|\mathbb{A}_{[N]} f - L\right| \leq \left\|f - f_j\right\|_{\mathbf{M}} + \left|\mathbf{M}(f_j) - L\right| \tag{19.6}$$

Since $\left\|f - f_j\right\|_{\mathbf{M}} < \epsilon$ and $\left|\mathbf{M}(f_j) - L\right| < \epsilon$, we get $\limsup_N \left|\mathbb{A}_{[N]} f - L\right| < 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\lim_N \left|\mathbb{A}_{[N]} f - L\right| = 0$ which means $\mathbf{M}(f) = L = \lim_j \mathbf{M}(f_j)$.

□

How do we now show that every open set can be represented? Let $B \subset \mathbb{T}$ be open with positive μ_α measure, let $B = \cup_j I_j$ be its decomposition into pairwise disjoint open intervals I_j and set $B_k := \cup_{j \in [k]} I_j$. Since $\mu_\alpha(B) > 0$, we have $\mu_\alpha(B_k) > 0$ for large enough k. For simplicity, we assume that $\mu_\alpha(B_k) > 0$ for every k. The sets B_k increase to B monotonically, hence, in particular, we have $\lim_k \mu_\alpha(B_k \triangle B) = 0$. According to proposition 15.1, the set B_k can be represented by the set S_k defined by

$$S_k := \{\, n \,|\, r_n \alpha \in B_k \,\} \tag{19.7}$$

and we have $\mathbf{M}(S_k) = \mu_\alpha(B_k)$. Since for every k, l the set $B_k \triangle B_l$ is Jordan measurable, we also have

$$\mathbf{M}\left|\mathbb{1}_{S_k} - \mathbb{1}_{S_l}\right| = \mathbf{M}(S_k \triangle S_l) = \mu_\alpha(B_k \triangle B_l) \tag{20.1}$$

Since (B_k) is a Cauchy sequence, so $\lim_k \sup_{l \geq k} \mu_\alpha(B_k \triangle B_l) = 0$, the isometry in eq. (20.1) implies that (S_k) is also a Cauchy sequence, so we have $\lim_k \sup_{l \geq k} \mathbf{M}(S_k \triangle S_l) = 0$. According to lemma 17.1, there is a set S to which the (S_k) converges, so $\lim_k \left\|\mathbb{1}_{S_k} - \mathbb{1}_S\right\|_{\mathbf{1}} = 0$, and by lemma 19.1, $\mathbf{M}(S) = \lim_k \mathbf{M}(S_k) > 0$. We want to show that the set S is good and it represents B at α. To this end, let $\beta \in \mathbb{T}$ be arbitrary and define the sequences f_k^β and f^β by

$$f_k^\beta(n) \coloneqq \mathbb{1}_{S_k}(n)\, \mathrm{e}(r_n\beta) \text{ for } n \in \mathbb{N} \tag{20.2}$$

$$f^\beta(n) \coloneqq \mathbb{1}_S(n)\, \mathrm{e}(r_n\beta) \text{ for } n \in \mathbb{N} \tag{20.3}$$

Since each set S_k is good with $\mathbf{M}(S_k) > 0$, we have $f_k^\beta \in \mathcal{M}$ for every k, β. The fact that for every β, the sequence (f_k^β) converges to f^β in the norm $\|\|_{\mathbf{M}}$ follows from the uniform estimate

$$\left\|f_k^\beta - f^\beta\right\|_{\mathbf{M}} \leq \left\|\mathbb{1}_{S_k} - \mathbb{1}_S\right\|_{\mathbf{1}} \text{ for every } \beta \tag{20.4}$$

By lemma 19.1, $f^\beta \in \mathcal{M}$ and

$$\mathbf{M}(f^\beta) = \lim_k \mathbf{M}(f_k^\beta) \tag{20.5}$$

so, in particular, S is good. Let now $\beta = \alpha$. Since the sequence (B_k) converges to B in $L^1(\mu_\alpha)$-norm we have

$$\lim_k \mu_\alpha\left(\mathrm{e}_p\, \mathbb{1}_{B_k}\right) = \mu_\alpha\left(\mathrm{e}_p\, \mathbb{1}_B\right) \text{ for every } p \in \mathbb{Z} \tag{20.6}$$

Since $\mathbf{M}(f_k^{p\alpha}) = \mu_\alpha\left(\mathrm{e}_p\, \mathbb{1}_{B_k}\right)$ and, by eq. (20.5), $\lim_k \mathbf{M}(f_k^{p\alpha}) = \mathbf{M}(f^{p\alpha})$, eq. (20.6) implies that

$$\mathbf{M}(f^{p\alpha}) = \mu_\alpha\left(\mathrm{e}_p\, \mathbb{1}_B\right) \text{ for every } p \in \mathbb{Z} \tag{20.7}$$

This is equivalent with saying that S represents B at α.

We record the general idea we used as item 2 in proposition 21.1 below.

21.1 Proposition ▶ Limit of good sets with positive mean is good

Let (S_k) be a sequence of good subsets of R with mean which converge to $S \subset R$ in $\|\|\|_{\mathbf{1}}$-seminorm, that is, $\lim_k \|S_k \triangle S\|_{\mathbf{1}} = 0$. Assume that $\limsup_k \mathbf{M}(S_k) > 0$.
Then we have the following.

1. $\lim_k \mathbf{M}(S_k)$ exists and $\mathbf{M}(S) = \lim_k \mathbf{M}(S_k) > 0$.
2. S is a good set.
3. The sequence $\left(\mu_{S_k,\beta}\right)_k$ of limit measures converge to $\mu_{S,\beta}$ in variation distance and uniformly in β,
$$\lim_k \sup_\beta \left\|\mu_{S_k,\beta} - \mu_{S,\beta}\right\|_{\mathbf{V}} = 0 \tag{21.1}$$
4. Let ν be a Borel measure on $\mathbb{T}$.
If for some α, $\mu_{S_k,\alpha}$ is absolutely continuous with respect to ν with Radon-Nikodym derivative ρ_k for every k, then $\mu_{S,\alpha}$ is also absolutely continuous with respect to ν with Radon-Nikodym derivative ρ which satisfies
$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{21.2}$$

Proof. The proof of item 1 follows from the triangle inequality for the $\|\|\|_{\mathbf{1}}$-seminorm, since we then have

$$\begin{aligned} |\mathbf{M}(S_k) - \mathbf{M}(S)| &= |\|S_k\|_{\mathbf{1}} - \|S\|_{\mathbf{1}}| \\ &\leq \|S_k \triangle S\|_{\mathbf{1}} \end{aligned}$$

and just use the assumption that $\lim_k \|S_k \triangle S\|_{\mathbf{1}} = 0$.

The argument we gave just before the enunciation of our proposition proves that S is a good set.

For the proof of item 3 note that in the argument preceding our proposition we proved that the sequence $\left(\mu_{S_k,\beta}\right)_k$ of measures converges weakly to $\mu_{S,\beta}$ for every β but an estimate similar to eq. (20.4) enables us to draw the stronger conclusion of eq. (21.1).

The following lemma gives us the estimates we need.

22.1 Lemma ▶ $\|\|\|_{\mathbf{1}}$ dominates $\|\|\|_{\mathbf{V}}$ and $\|\|\|_{L^1}$

Let v_1, v_2 be good weights on $R = (r_n)$. Assume that

$$\max\{\|v_1\|_{\mathbf{1}}, \|v_2\|_{\mathbf{1}}\} > 0 \tag{22.1}$$

Then we have the following.

(a)

$$\sup_{\beta} \left\|\mu_{v_1,\beta} - \mu_{v_2,\beta}\right\|_{\mathbf{V}} \leq \frac{2}{\max\{\|v_1\|_{\mathbf{1}}, \|v_2\|_{\mathbf{1}}\}} \|v_1 - v_2\|_{\mathbf{1}} \tag{22.2}$$

(b) If, for some α, the limit measures $\mu_{v_1,\alpha}$ and $\mu_{v_2,\alpha}$ are absolutely continuous with respect to a Borel measure ν on $\mathbb{T}$ with Radon-Nikodym derivatives ρ_1 and ρ_2, respectively, then

$$\|\rho_1 - \rho_2\|_{L^1(\nu)} \leq \frac{4}{\max\{\|v_1\|_{\mathbf{1}}, \|v_2\|_{\mathbf{1}}\}} \|v_1 - v_2\|_{\mathbf{1}} \tag{22.3}$$

Proof. To prove item a, that is, the inequality in eq. (22.2), fix β and $\phi \in \mathcal{C}_+$, so ϕ is a continuous $\mathbb{T} \to \mathbb{C}$ function with $0 \leq \phi \leq 1$. We can assume without loss of generality that $\max\{\|v_1\|_{\mathbf{1}}, \|v_2\|_{\mathbf{1}}\} = \|v_1\|_{\mathbf{1}}$. Let $(N_l)_l$ be a strictly increasing sequence of indices so that

$$\lim_l \mathbb{A}_{[N_l]} v_1 = \|v_1\|_{\mathbf{1}} \tag{22.4}$$

Let us estimate as

$$\begin{aligned}
&\left|\mathbb{A}^{v_1}_{n\in[N_l]}\phi(r_n\beta) - \mathbb{A}^{v_2}_{n\in[N_l]}\phi(r_n\beta)\right| \\
&= \left|\frac{1}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{n\in[N_l]}v_1(n)\phi(r_n\beta) - \frac{1}{\mathbb{A}_{[N_l]}v_2}\mathbb{A}_{n\in[N_l]}v_2(n)\phi(r_n\beta)\right|
\end{aligned}$$

adding $0 = -\frac{1}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{n\in[N_l]}v_2(n)\phi(r_n\beta) + \frac{1}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{n\in[N_l]}v_2(n)\phi(r_n\beta)$ inside the absolute value and using the triangle inequality,

$$\begin{aligned}
&\leq \frac{1}{\mathbb{A}_{[N_l]}v_1}\left|\mathbb{A}_{n\in[N_l]}v_1(n)\phi(r_n\beta) - \mathbb{A}_{n\in[N_l]}v_2(n)\phi(r_n\beta)\right| \\
&\quad + \left|\frac{1}{\mathbb{A}_{[N_l]}v_1} - \frac{1}{\mathbb{A}_{[N_l]}v_2}\right|\left|\mathbb{A}_{n\in[N_l]}v_2(n)\phi(r_n\beta)\right| \\
&\leq \frac{1}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{[N_l]}|v_1 - v_2| + \frac{\mathbb{A}_{[N_l]}|v_1 - v_2|}{\mathbb{A}_{[N_l]}v_1\mathbb{A}_{[N_l]}v_2}\mathbb{A}_{[N_l]}v_2 \\
&= \frac{2}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{[N_l]}|v_1 - v_2|
\end{aligned}$$

so we have

$$\left|\mathbb{A}^{v_1}_{n\in[N_l]}\phi(r_n\beta) - \mathbb{A}^{v_2}_{n\in[N_l]}\phi(r_n\beta)\right| \leq \frac{2}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{[N_l]}|v_1 - v_2| \tag{22.5}$$

Since $\lim_l \mathbb{A}^{v_i}_{n\in[N_l]}\phi(r_n\beta) = \mu_{v_i,\beta}(\phi)$, $\lim_l \mathbb{A}_{[N_l]}v_1 = \|v_1\|_{\mathbf{1}}$ and $\limsup_l \frac{2}{\mathbb{A}_{[N_l]}v_1}\mathbb{A}_{[N_l]}|v_1 - v_2| \leq \frac{2}{\|v_1\|_{\mathbf{1}}}\|v_1 - v_2\|_{\mathbf{1}}$, we get

$$\left|\mu_{v_1,\beta}(\phi) - \mu_{v_2,\beta}(\phi)\right| \leq \frac{2}{\|v_1\|_{\mathbf{1}}}\|v_1 - v_2\|_{\mathbf{1}} \tag{23.1}$$

which is independent of β and $\phi \in \mathcal{C}_+$, proving eq. (22.2).

To prove item 3 observe that, since $\mu_{v_i,\alpha} = \rho_i\nu$ and $\|\rho_1\nu - \rho_2\nu\|_{\mathbf{V}} = \frac{1}{2}\|\rho_1 - \rho_2\|_{L^1(\nu)}$, we have

$$\|\mu_{v_1,\alpha} - \mu_{v_2,\alpha}\|_{\mathbf{V}} = \frac{1}{2}\|\rho_1 - \rho_2\|_{L^1(\nu)} \tag{23.2}$$

and then use eq. (22.2). □

Now, using eq. (22.2) with $v_1 = \mathbb{1}_{S_k}$ and $v_2 = \mathbb{1}_S$, we get

$$\sup_\beta \left\|\mu_{S_k,\beta} - \mu_{S,\beta}\right\|_{\mathbf{V}} \leq \frac{2}{\max\{\|S_k\|_{\mathbf{1}}, \|S\|_{\mathbf{1}}\}}\|S_k\triangle S\|_{\mathbf{1}} \tag{23.3}$$

Using the assumption that $\lim_k \|S_k\triangle S\|_{\mathbf{1}} = 0$ and that, by item 1, we have $\lim_k \|S_k\|_{\mathbf{1}} = \lim_k \mathbf{M}(S_k) = \mathbf{M}(S) = \|S\|_{\mathbf{1}} > 0$, we get eq. (21.1).

For the proof of item 4, by item 1, we can assume, without loss of generality that $\mathbf{M}(S_k) > 0$ for every k. Using eq. (22.3) with $v_1 = S_k$ and $v_2 = S_l$ we get

$$\|\rho_l - \rho_k\|_{L^1(\nu)} \leq \frac{4}{\max\{\|S_k\|_{\mathbf{1}}, \|S_l\|_{\mathbf{1}}\}}\|S_k\triangle S_l\|_{\mathbf{1}} \tag{23.4}$$

This implies, since the sequence (S_k) is convergent in $\|\|_{\mathbf{1}}$-seminorm and hence is Cauchy, that the sequence (ρ_k) is Cauchy in $L^1(\nu)$-norm. Since $L^1(\nu)$ is complete and $\nu(\rho_k) = 1$ for every k, there is a $\rho \in L^1(\nu)$ with $\nu(\rho) = 1$ so that

$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{23.5}$$

Since $\|\rho_k - \rho\|_{L^1(\nu)} = 2\|\rho_k\nu - \rho\nu\|_{\mathbf{V}}$ and $\rho_k\nu = \mu_{S_k,\alpha}$, we get

$$\lim_k \left\|\mu_{S_k,\alpha} - \rho\nu\right\|_{\mathbf{V}} = 0 \tag{23.6}$$

But by item 3 we also have $\lim_k \left\|\mu_{S_k,\alpha} - \mu_{S,\alpha}\right\|_{\mathbf{V}} = 0$ hence we must have $\mu_{S,\alpha} = \rho\nu$.

□

We can use proposition 21.1 in an argument similar to the one we used to show that any open set can be represented at α to prove that if a G_δ set B has positive μ_α-measure then it can be represented at α. Only the initial setup of the proof is different. This time let (B_k) be a decreasing sequence of open sets which converges to B. Let $S_k \subset R$

represent B_k at α. We again have the isometry eq. (20.1) from which everything follows: the existence of a good set S which represents B at α and $\mathbf{M}(S) = \mu_\alpha(B)$.

Since every Borel measurable set differs from a G_δ set on a set of μ_α-measure zero, we in fact showed that every Borel set of positive μ_α-measure can be represented. So we proved the following more precise version of theorem 10.2 for the case when the Radon-Nikodym derivative of a measure with respect to μ_α is an indicator.

24.1 Proposition ▶ Theorem 10.2 for indicator

Let $R \subset \mathbb{N}$ be a good set, α be an irrational number, and let B be a Borel set with $\mu_\alpha(B) > 0$.
Then B can be represented at α by a set $S \subset R$ which satisfies

$$\mathbf{M}_R(S) = \mu_\alpha(B) > 0 \tag{24.1}$$

6 *Measures that cannot be represented at every irrational α*

For this section, we suspend the proof of theorem 10.2 just to see how proposition 24.1 can be used to prove theorem 9.1. We will also prove theorem 9.2.

6.1 *Proof of theorem 9.1*

In this section we want to prove that if the Borel probability measure ν has a point-mass at a point $\gamma \in \mathbb{T}$ and α is irrational then ν cannot be represented at α.

The proof is by contradiction: let us assume that for some $\gamma \in \mathbb{T}$, $\nu(\{\gamma\}) > 0$ and that ν can be represented by the set R at α, so $\mu_{R,\alpha} = \nu$. Then the Dirac mass δ_γ is absolutely continuous with respect to $\mu_{R,\alpha}$ with Radon-Nikodym derivative equal $\frac{1}{\nu(\gamma)}\mathbb{1}_{\{\gamma\}}$. By proposition 24.1 there is a good set $S \subset R$ which represents δ_γ at α, so $\mu_{S,\alpha} = \delta_\gamma$. Let us define the function $\phi : \mathbb{T} \to \mathbb{C}$ as

$$\phi(\beta) \coloneqq \mu_{S,\beta}(\mathrm{e}) \tag{24.2}$$

Then, by the definition of $\mu_{S,\beta}(\mathrm{e})$, ϕ is the limit of the sequence (ϕ_N) of continuous functions defined by $\phi_N(\beta) \coloneqq \mathbb{A}_{n\in[N]}\, \mathrm{e}(s_n\beta)$ where (s_n) is the elements of S arranged in increasing order. Since for every $p \in \mathbb{Z}$ we have $\mu_{S,p\alpha}(\mathrm{e}) = \mu_{S,\alpha}(\mathrm{e}_p)$ and $\mu_{S,\alpha}(\mathrm{e}_p) = \mathrm{e}_p(\gamma)$, we have

$$|\phi| = 1 \text{ on the dense set } \{\, p\alpha \mid p \in \mathbb{Z} \,\} \tag{24.3}$$

By Weyl's theorem,[12] $\phi = 0$ on a set of full Lebesgue measure, so, as

[12] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

a consequence,

$$\phi = 0 \text{ on a dense set.} \tag{25.1}$$

By Baire's theorem,[13] eqs. (25.1) and (24.3) together are impossible to hold simultaneously for the limit of continuous functions.

[13] Baire 1995, Page 83.

6.2 *Proof of theorem 9.2*

So in this section we want to prove that if ν is a Borel probability measure on $\mathbb{T}$ with $\limsup_{p\to\infty}|\nu(\mathrm{e}_p)| > 0$ then there is an irrational α where ν cannot be represented. In fact the set of such α's is of full Lebesgue measure.

From the assumption that $\limsup_{p\to\infty}|\nu(\mathrm{e}_p)| > 0$ it follows that there is an $\epsilon > 0$ and a infinite sequence $p_1 < p_2 < \dots$ of indices so that

$$|\nu\,\mathrm{e}_{p_k}| > \epsilon \text{ for } k \in \mathbb{N} \tag{25.2}$$

By Weyl's result,[14] the set $A \subset \mathbb{T}$ defined by

[14] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

$$A := \{\, \alpha \mid (p_k\alpha)_{k\in\mathbb{N}} \text{ is uniformly distributed} \pmod 1 \,\} \tag{25.3}$$

has full λ measure. We want to show that A is a subset of those α's at which the measure ν cannot be represented.

Let $\alpha \in A$, and suppose the measure ν can be represented at α, say, by the set $S = (s_n)$, that is, $\mu_{S,\alpha} = \nu$. Let us define the function $\phi : \mathbb{T} \to \mathbb{C}$ as

$$\phi(\beta) := \mu_{S,\beta}(\mathrm{e}) \tag{25.4}$$

Then, by the definition of $\mu_{S,\beta}(\mathrm{e})$, ϕ is the limit of the sequence (ϕ_N) of continuous functions defined by $\phi_N(\beta) := \mathbb{A}_{n\in[N]}\,\mathrm{e}(s_n\beta)$. Since for every $p \in \mathbb{Z}$ we have $\mu_{S,p\alpha}(\mathrm{e}) = \mu_{S,\alpha}(\mathrm{e}_p)$ and $\mu_{S,\alpha}(\mathrm{e}_p) = \nu(\mathrm{e}_p)$, by eq. (25.2) we have

$$|\mu_{S,p_k\alpha}(\mathrm{e})| > \epsilon \text{ for every } k \in \mathbb{N} \tag{25.5}$$

Since $\alpha \in A$, the sequence $(p_k\alpha)$is uniformly distributed $\mod 1$, hence dense in $\mathbb{T}$. So we have that

$$|\phi| > \epsilon \text{ on the dense set } \{\, p_k\alpha \mid k \in \mathbb{N} \,\}. \tag{25.6}$$

By Weyl's theorem,[15] $\phi = 0$ on a set of full Lebesgue measure, so, as a consequence

[15] Weyl 1916, Satz 21; Kuipers and Niederreiter 1974, Theorem 4.1.

$$\phi = 0 \text{ on a dense set} \tag{25.7}$$

By Baire's theorem,[16] eqs. (25.6) and (25.7) cannot be true together for the limit of continuous functions.

[16] Baire 1995, Page 83.

7 *Representing by weights*

In this section, we fix the good set R and the irrational number α, and we continue in the tradition of section 5 suppressing the set R in our notation, so $\mu_\alpha = \mu_{R,\alpha}$ and $\mathbf{M} = \mathbf{M}_R$.

In trying to extend the class of representable functions ρ from indicators, we first consider an easier problem. Instead of representing by sets, we represent by weights, that is, unsigned sequences w.

26.1 Definition ▶ Function represented by a weight

Let ρ be an unsigned $L^1(\mathbb{T}, \mu_\alpha)$ function with $\mu_\alpha(\rho) > 0$.
We say the weight w *represents* ρ *at* α if w is good and we have

$$\lim_N \mathbb{A}_{n\in[N]} w(n)\delta_{r_n\alpha} = \rho\mu_\alpha \tag{26.1}$$

Note that in this case we have $\mathbf{M}(w) = \mu_\alpha(\rho)$ and w represents the measure $\frac{1}{\mu_\alpha(\rho)}\rho\mu_\alpha$ at α.

In section 4 we have already seen that if ρ is an unsigned continuous function with $\mu_\alpha(\rho) > 0$ then the weight w defined by

$$w(n) \coloneqq \rho(r_n\alpha) \tag{26.2}$$

is good, unsigned and it represents ρ at α. Since every unsigned μ_α-integrable function can be approximated arbitrary closely by unsigned continuous functions in $L^1(\mathbb{T}, \mu_\alpha)$-norm, the proof of item 1 of theorem 11.3 requires only an approximation argument similar to what we had in section 5. We restate item 1 of theorem 11.3 in the following form for the readers convenience.

26.2 Proposition ▶ Any integrable function is representable with weights

Let ρ be an unsigned function from $L^1(\mathbb{T}, \mu_\alpha)$ with $\mu_\alpha(\rho) > 0$.
Then there is a weight w which represents ρ at α. In particular, we have

$$\mathbf{M}(w) = \mu_\alpha(\rho) \tag{26.3}$$

Furthermore, if ρ is a bounded function then the weight w can be chosen to be bounded.

The following analog of lemma 17.1 for weights[17] is the main ingredient for proving proposition 26.2.

[17] Marcinkiewicz 1939.

27.1 Lemma ► Cauchy sequence of weights is convergent in the seminorm $\|\|_{\mathbf{1}}$

Let (w_k) be a Cauchy sequence of weights with respect to the seminorm $\|\|_{\mathbf{1}}$, so they satisfy

$$\lim_k \sup_{l \geq k} \|w_l - w_k\|_{\mathbf{1}} = 0 \tag{27.1}$$

Then there is a weight w satisfying

$$\lim_k \|w_k - w\|_{\mathbf{1}} = 0 \tag{27.2}$$

Furthermore, if the sequence (w_k) is uniformly bounded, so $\sup_k \|w_k\|_\infty < \infty$, then we can also have $\|w\|_\infty \leq \sup_k \|w_k\|_\infty$.

The proof of lemma 27.1 is the same as the proof of lemma 17.1: we recursively define a fast enough increasing sequence $N_1 < N_2 < \dots$ of indices, and then we define w to be equal w_k on the interval $(N_k, N_{k+1}]$

$$w := \sum_k w_k \mathbb{1}_{(N_k, N_{k+1}]} \tag{27.3}$$

The details of the proof are safely left for the reader. The form of w in eq. (27.3) guarantees that w is unsigned, since each w_k is an unsigned sequence, and if the (w_k) is uniformly bounded, then w is a bounded weight.

Lemma 27.1, combined with lemmas 19.1 and 22.1, gives the following analog of proposition 21.1

28.1 Proposition ▶ Limit of good weights with positive mean is good

Let (w_k) be a sequence of good weights with mean which converge to the weight w in $\|\|_{\mathbf{1}}$-seminorm, so $\lim_N \|w_k - w\|_{\mathbf{1}} = 0$. Assume that $\limsup_k \mathbf{M}(w_k) > 0$.
Then we have the following.

1. $\lim_k \mathbf{M}(w_k)$ exists and $\lim_k \mathbf{M}(w_k) = \mathbf{M}(w) > 0$.
2. w is a good weight.
3. The sequence $\left(\mu_{w_k,\beta}\right)_k$ of limit measures converge to $\mu_{w,\beta}$ in variation distance and uniformly in β,
$$\lim_k \sup_\beta \left\|\mu_{w_k,\beta} - \mu_{w,\beta}\right\|_{\mathbf{V}} = 0 \tag{28.1}$$
4. Let ν be a Borel measure on $\mathbb{T}$.
If for some α, $\mu_{w_k,\alpha}$ is absolutely continuous with respect to ν with Radon-Nikodym derivative ρ_k for every k then $\mu_{w,\alpha}$ is also absolutely continuous with respect to ν with Radon-Nikodym derivative ρ which satisfies
$$\lim_k \|\rho_k - \rho\|_{L^1(\nu)} = 0 \tag{28.2}$$

With this proposition, we can complete the proof of proposition 26.2 exactly as we proved proposition 24.1, using a sequence (ρ_k) of unsigned continuous functions that converge to ρ in $L^1(\mu_\alpha)$-norm. We need to remark only that if ρ is a bounded function, then the sequence (ρ_k) of continuous functions can be chosen to be uniformly bounded.

8 *Proof of theorem 10.2 for bounded ρ*

In this section, we still are working with a fixed good set R of positive integers, an irrational number α, but now we also fix a bounded Borel measurable, unsigned function ρ with $\mu_\alpha(\rho) > 0$. We proved in section 7 that ρ can be represented at α by a good, bounded weight w. In this section we will show that there is a good set $S \subset R$ which also represents ρ at α, hence proving theorem 10.2 for bounded ρ. It follows from the definition of representation that if the good weight w represents ρ then so does the weight cw for every positive constant c. In particular, we can assume that the weight w representing ρ is bounded by 1. We will show that then there is a set $S \subset R$ so that

$$\lim_N \sup_\beta \left| \mathbb{A}_{n\in[N]} \mathbb{1}_S(n)\, \mathrm{e}(r_n\beta) - \mathbb{A}_{n\in[N]} w(n)\, \mathrm{e}(r_n\beta) \right| = 0 \tag{28.3}$$

The "construction" of S satisfying eq. (28.3) is done randomly. Our random method requires that we limit the growth of the set R; we need to assume that R is *sublacunary*.[18]

[18] Jones, Lacey, and Wierdl 1999, Theorem B.

29.1 Definition ▶ Sublacunary set and weight

The set R is *sublacunary* if its counting function $\#R(N)$ satisfies

$$\lim_N \frac{\log N}{\#R(N)} = 0 \tag{29.1}$$

The weight w supported on the set R is *sublacunary* if it satisfies

$$\lim_N \frac{\log N}{w(R(N))} = 0 \tag{29.2}$$

Our main tool in this section is the following.

29.2 Proposition ▶ There is a set representing the same measures as a bounded weight

Let $w = (w(r))_{r\in R}$ be a bounded, sublacunary weight supported on R.
Then there is a set $S \subset R$ so that

$$\lim_N \max_{\beta\in\mathbb{T}} \left| \mathbb{A}_{s\in S(N)} \, \mathrm{e}(s\beta) - \mathbb{A}^w_{r\in R(N)} \, \mathrm{e}(r\beta) \right| = 0 \tag{29.3}$$

As a consequence, if the weight w is good then so is the S and we have

$$\mu_{S,\beta} = \mu_{w,\beta} \text{ for every } \beta \tag{29.4}$$

Proof. Since we can always assume that the bound of the weight w is 1, proposition 29.2 follows from the following lemma.

29.3 Lemma ▶ Random selection of a good set

let σ be a weight on R bounded by 1. We assume that for a constant $b > 0$ we have

$$\liminf_N \frac{\sigma\big(R(N)\big)}{\log N} > b \tag{29.5}$$

Let (Ω, P) be a probability space and and let $(X_r)_{r\in R}$ be a sequence of totally independent $\Omega \to \{0,1\}$ random variables indexed by R and with distribution $P(X_r = 1) = \sigma(r)$.
Then we have

$$P\left\{ \omega \;\middle|\; \sup_N \max_{\beta\in\mathbb{T}} \frac{\left|\sum_{r\in R(N)} \Big(X_r(\omega) - \sigma(r)\Big) \, \mathrm{e}(r\beta)\right|}{\sqrt{(\log N)\sigma\big(R(N)\big)}} < \infty \right\} = 1 \tag{29.6}$$

To see that proposition 29.2 indeed follows from lemma 29.3, let $\sigma = \frac{w}{\|w\|_\infty}$, so σ is bounded by 1. Here we make a bit more complicated argument than needed to show that there is a rate of convergence in eq. (29.3).

The sublacunarity assumption on w implies that σ is sublacunary. We then have, as a consequence of eq. (29.6), that there is an Ω_1 with $P(\Omega_1) = 1$ so that for every $\omega \in \Omega_1$ there is a finite positive constant C_ω with

$$\max_{\beta\in\mathbb{T}} \left| \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega)\, \mathrm{e}(r\beta) - \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} \sigma(r)\, \mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\log N}{\sigma\big(R(N)\big)}} \tag{30.1}$$

For $\beta = 0$, we then have

$$\left| \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega) - 1 \right| \le C_\omega \sqrt{\frac{\log N}{\sigma\big(R(N)\big)}} \tag{30.2}$$

This implies that if we replace $\sigma(R(N))$ by $\sum_{r\in R(N)} X_r(\omega)$ in $\frac{1}{\sigma(R(N))} \sum_{r\in R(N)} X_r(\omega)\, \mathrm{e}(r\beta)$ we make a $O\left(\sqrt{\frac{\log N}{\sigma\big(R(N)\big)}}\right)$ error, hence eq. (30.1) implies

$$\max_{\beta\in\mathbb{T}} \left| \frac{1}{\sum_{r\in R(N)} X_r(\omega)} \sum_{r\in R(N)} X_r(\omega)\, \mathrm{e}(r\beta) - \frac{1}{\sigma(R(N))} \sum_{r\in R(N)} \sigma(r)\, \mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\log N}{\sigma\big(R(N)\big)}} \tag{30.3}$$

Defining $S_\omega \subset R$ by

$$S_\omega := \{\, r \,|\, r \in R, X_r(\omega) = 1 \,\} \tag{30.4}$$

we can write eq. (30.3) as

$$\max_{\beta\in\mathbb{T}} \left| \mathbb{A}_{s\in S_\omega(N)}\, \mathrm{e}(s\beta) - \mathbb{A}^{\sigma}_{r\in R(N)}\, \mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\log N}{\sigma\big(R(N)\big)}} \text{ for every } \omega \in \Omega_1 \tag{30.5}$$

Since σ is a constant multiple of w, we can replace σ by w in eq. (30.5),

$$\max_{\beta\in\mathbb{T}} \left| \mathbb{A}_{s\in S_\omega(N)}\, \mathrm{e}(s\beta) - \mathbb{A}^{w}_{r\in R(N)}\, \mathrm{e}(r\beta) \right| \le C_\omega \sqrt{\frac{\|w\|_\infty \log N}{w\big(R(N)\big)}} \text{ for every } \omega \in \Omega_1 \tag{30.6}$$

Since $\lim_N \frac{\|w\|_\infty \log N}{w\big(R(N)\big)} = 0$, due to the sublacunarity assumption on the weight w, we get eq. (29.3) if we take $S = S_\omega$ for any $\omega \in \Omega_1$.

Proof of lemma 29.3. To see clearly what we need to do, denote

$$Z_N(\beta) := \sum_{r\in R(N)} \Big(X_r(\omega) - \sigma(r) \Big)\, \mathrm{e}(r\beta)$$

and

$$t_N := c \cdot \sqrt{(\log N)\sigma\big(R(N)\big)}$$

where we'll choose the constant c appropriately later. By the Borel-Cantelli lemma, it's enough to prove

$$\sum_N P\left(\max_{\beta\in\mathbb{T}} |Z_N(\beta)| \geq t_N\right) < \infty \tag{31.1}$$

The first idea in proving eq. (31.1) is that we do not have to take the maximum over all $\beta \in \mathbb{T}$, but over a finite subset B of $\mathbb{T}$ which contains N^3 elements[19]. Since the degree of the trigonometric polynomial $Z_N(\beta)$ is at most N, we can readily see that $\sup_{\beta\in\mathbb{T}} |Z_N'(\beta)| \leq N^2 \sup_{\beta\in\mathbb{T}} |Z_N(\beta)|$. It follows that if we take $B_N \subset \mathbb{T}$ to be an arithmetic progression with $|B_N| = N^3$ then

[19] In fact, we can take a set B with as few elements as $10N$, but in our applications, $10N$ won't improve anything over N^3.

$$\max_{\beta\in\mathbb{T}} |Z_N(\beta)| \leq 2 \max_{\beta\in B_N} |Z_N(\beta)| \tag{31.2}$$

Hence we have

$$P\left(\max_{\beta\in\mathbb{T}} |Z_N(\beta)| \geq t_N\right) \leq P\left(\max_{\beta\in B_N} |Z_N(\beta)| \geq t_N/2\right) \tag{31.3}$$

Using the union estimate, we get

$$P\left(\max_{\beta\in B_N} |Z_N(\beta)| \geq t_N/2\right) \leq N^3 \max_{\beta\in B_N} P(|Z_N(\beta)| \geq t_N/2) \tag{31.4}$$

Thus eq. (31.1) follows from

$$\sum_N N^3 \max_{\beta\in\mathbb{T}} P(|Z_N(\beta)| \geq t_N/2) < \infty \tag{31.5}$$

This follows if we prove

$$P\Big(|Z_N(\beta)| \geq t_N/2\Big) < \frac{2}{N^5} \text{ for every } \beta \in \mathbb{T} \tag{31.6}$$

To prove eq. (31.6), we use Bernstein-Chernoff exponential estimate.[20] This estimate says that if Y_k, $k \in [K]$, are totally independent, mean zero, complex valued random variables with $|Y_k| \leq 1$, then

[20] Tao and Vu 2010, Exercise 1.3.4 with $t = \lambda\sigma$.

$$P\left(\left|\sum_{k\in[K]} Y_k\right| \geq t\right) \leq 4 \max\left\{\exp\left(-\frac{t^2/8}{\sum_{k\in[K]} \mathbb{E}|Y_k|^2}\right), \exp(-t/3)\right\} \text{ for every } t > 0. \tag{31.7}$$

Take $K = \#R(N)$ and $Y_r(\beta) := (X_r - \sigma(r))e(r\beta)$ for $r \in R(N)$. Then $|Y_r(\beta)| \leq 1$ so the Y_r satisfy the assumption in Bernstein's inequality, hence, with $t = t_N/2$, we get the estimate

$$P\Big(|Z_N(\beta)| \geq t_N/2\Big) \leq 4 \max\left\{\exp\left(-\frac{t_N^2/32}{\sum_{r\in R(N)} \mathbb{E}|Y_r|^2}\right), \exp(-t_N/6)\right\} \tag{31.8}$$

Since $\mathbb{E}|Y_r(\beta)|^2 = \sigma(r)(1-\sigma(r))$ we have

$$\sum_{r\in R(N)} \mathbb{E}|Y_r(\beta)|^2 \leq \sigma\big(R(N)\big) \tag{32.1}$$

Using that $t_N = c\cdot\sqrt{(\log N)\sigma\big(R(N)\big)}$, we get

$$\frac{t_N^2/32}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2} = \frac{(c^2/32)(\log N)\sigma\big(R(N)\big)}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2}$$

using the estimate in eq. (32.1)

$$\begin{aligned}&\geq \frac{(c^2/32)(\log N)\sigma\big(R(N)\big)}{\sigma\big(R(N)\big)}\\ &= (c^2/32)(\log N)\end{aligned}$$

hence

$$\exp\left(-\frac{t_N^2/32}{\sum_{r\in R(N)}\mathbb{E}|Y_r(\beta)|^2}\right) \leq e^{-(c^2/32)(\log N)} \tag{32.2}$$

In order to get $e^{-(c^2/32)(\log N)} \leq N^{-5} = e^{-5\log N}$, we need to have $c^2/32 \geq 5$, so it enough to have, since $\sqrt{160} < 13$,

$$c \geq 13. \tag{32.3}$$

We also have

$$t_N/6 = (c/6)\cdot\sqrt{(\log N)\sigma\big(R(N)\big)}$$

by the assumption in eq. (29.5) for all large enough N

$$\geq (c/6)\sqrt{b}\log N$$

It follows that

$$\exp(-t_N/6) \leq e^{-(c/6)\sqrt{b}\log N} \tag{32.4}$$

We again need to have $e^{-(c/6)\sqrt{b}\log N} \leq N^{-5} = e^{-5\log N}$ which poses the requirement $(c/6)\sqrt{b} \geq 5$, that is,

$$c \geq \frac{30}{\sqrt{b}} \tag{32.5}$$

Thus choosing the constant c large enough to satisfy both eqs. (32.3) and (32.5), the estimate in eq. (31.8) implies the one in eq. (31.6). □

□

8.1 *Notes to lemma 29.3*

The type of method we used in lemma 29.3 to estimate trigonometric polynomials goes back to Salem-Zygmund.[21] Recent developments have been given for example by Weber[22] and by Cohen-Cuny.[23]

[21] Salem and Zygmund 1954, Chapter IV.
[22] Weber 2000.
[23] Cohen and Cuny 2006.

9 *Absolute continuity and positive mean*

The general theme of this section is that if a good set or weight has positive mean then it can represent only an absolutely continuous measure. To be specific, we want to prove item 2 of theorems 10.3 and 11.3.

Our standing assumption is that R is a sublacunary good set, and hence we suppress it in our notation: We write μ_α instead of $\mu_{R,\alpha}$, $\mathbf{M}$ instead of $\mathbf{M}_R$ and $\|S\|_{\mathbf{1}}$ instead of $\|S\|_{\mathbf{1},R}$. For a given f defined on R, if we consider R as the strictly increasing sequence of integers (r_n) and define F by $F(n) := f(r_n)$ then we actually have $M_R(f) = \mathbf{M}(F)$ and $\|f\|_{\mathbf{1},R} = \|F\|_{\mathbf{1}}$.

9.1 *Proof of Item 2 of theorem 10.3*

Item 1 of theorem 10.3 says that if ρ is an unsigned $L^\infty(\mu_\alpha)$ function with $\mu_\alpha(\rho) > 0$ and α is an irrational number then ρ can be represented at α with a good set $S \subset R$ satisfying $\mathbf{M}(S) = \frac{\mu_\alpha(\rho)}{\|\rho\|_{L^\infty(\mu_\alpha)}}$. We have proved this in section 8.

Item 2 of theorem 10.3 says that the converse is also true: if the good set $S \subset R$ satisfies $\|S\|_{\mathbf{1}} > 0$ then the limit measure $\mu_{S,\beta}$ is absolutely continuous with respect to μ_β with a bounded Radon-Nikodym derivative ρ_β which must satisfy

$$\left\|\rho_\beta\right\|_{L^\infty(\mu_\beta)} \leq \frac{1}{\|S\|_{\mathbf{1}}} \text{ for every } \beta \tag{33.1}$$

This is what we intend to prove now. Since $\beta \in \mathbb{T}$ is fixed, we suppress it in our notation, so for example we write μ for μ_β and μ_S for $\mu_{S,\beta}$. Let $S \subset R$ be such that $\|S\|_{\mathbf{1}} > 0$. Let us first show that for every β, the limit measure μ_S is absolutely continuous with respect to μ.

This will follow if we show that for every Borel set B we have

$$\mu_S(B) \leq \frac{1}{\|S\|_{\mathbf{1}}}\mu(B) \tag{33.2}$$

To see this, it's enough to show that for every unsigned, continuous function ϕ on $\mathbb{T}$ we have

$$\mu_S(\phi) \leq \frac{1}{\|S\|_{\mathbf{1}}}\mu(\phi) \tag{33.3}$$

Let ϕ be such a function and let $N_1 < N_2 < \dots$ be a sequence of

indices for which $\lim_k \mathbb{A}_{n\in[N_k]}\mathbb{1}_S(n) = \|S\|_{\mathbf{1}}$. We can then estimate as

$$\begin{aligned}\mu_S(\phi) &= \lim_N \mathbb{A}_{n\in S(N)}\phi(r_n\beta)\\ &= \lim_k \mathbb{A}_{n\in S(N_k)}\phi(r_n\beta)\\ &= \lim_k \frac{1}{\mathbb{A}_{n\in[N_k]}\mathbb{1}_S(n)}\mathbb{A}_{n\in[N_k]}\mathbb{1}_S(n)\phi(r_n\beta)\\ &\le \limsup_k \frac{1}{\mathbb{A}_{n\in[N_k]}\mathbb{1}_S(n)}\mathbb{A}_{n\in[N_k]}\phi(r_n\beta)\end{aligned}$$

since $\lim_k \frac{1}{\mathbb{A}_{n\in[N_k]}\mathbb{1}_S(n)} = \frac{1}{\|S\|_{\mathbf{1}}}$ and $\lim_N \mathbb{A}_{n\in[N]}\phi(r_n\beta)$ exists,

$$\begin{aligned}&= \frac{1}{\|S\|_{\mathbf{1}}}\lim_N \mathbb{A}_{n\in[N]}\phi(r_n\beta)\\ &= \frac{1}{\|S\|_{\mathbf{1}}}\cdot\mu(\phi)\end{aligned}$$

proving eq. (33.3).

Now, inequality $\mu\big(\rho_\beta\mathbb{1}_B\big) \le \frac{1}{\|S\|_{\mathbf{1}}}\mu(B)$ applied to the Borel set $B = \left\{\rho_\beta > \frac{1}{\|S\|_{\mathbf{1}}}\right\}$ readily gives eq. (33.1).

9.2 *Proof of item 2 of theorem 11.3*

So we need to prove that if the good weight w has positive $\mathbf{1}$-norm and it is integrable, that is, it can be approximated arbitrary closely by bounded, good weights in $\|\|_{\mathbf{1}}$-seminorm, then for every irrational β the limit measure $\mu_{w,\beta}$ is absolutely continuous with respect to μ_β.

Let (w_k) be a sequence of good, bounded weights which converges to w in $\|\|_{\mathbf{1}}$-seminorm, $\lim_k\|w_k - w\|_{\mathbf{1}} = 0$. Since $|\|w_k\|_{\mathbf{1}} - \|w\|_{\mathbf{1}}| \le \|w_k - w\|_{\mathbf{1}}$, we have $\lim_k\|w_k\|_{\mathbf{1}} = \|w\|_{\mathbf{1}} > 0$, and hence we can assume without loss of generality that $\|w_k\|_{\mathbf{1}} > 0$ for every k. That for every k the measure $\mu_{w_k,\beta}$ is absolutely continuous with respect to μ_β for every β follows from

$$\mu_{w_k,\beta}(B) \le \frac{\|w_k\|_\infty}{\|w_k\|_{\mathbf{1}}}\mu_\beta(B) \text{ for every Borel set } B \tag{34.1}$$

The proof of this inequality is almost identical to the proof of the inequality in eq. (33.2), hence we omit it.

Now the rest of the proof of theorem 11.3 follows from lemma 22.1.

10 *Proof of theorem 10.2 for unbounded ρ*

In this section we work with a fixed, sublacunary good set $R \subset \mathbb{N}$ which we view as a sequence (r_n) arranged in increasing order. As a consequence, we omit R from our notation, so we write μ_β instead

of $\mu_{R,\beta}$ and $\mathbf{M}(w)$instead of $\mathbf{M}_R(w)$. We also fix an irrational number α. Let $\rho \in L^1(\mu_\alpha)$. We want to find a sublacunary good set $S \subset R$ which represents ρ at α. According to proposition 26.2 there is a good weight w on R which represents ρ at α. Since this weight w has positive relative mean with respect to R, it's a sublacunary weight. The problem is that, as per construction, w is not a bounded weight if ρ is unbounded, hence we cannot use our proposition 29.2 to construct the desired set S.

Our main job in this section hence will be to construct a good weight v satisfying the following properties

- v is bounded by 1;
- v is sublacunary;
- v represents the same measure at every β as w, so $\mu_{v,\beta} = \mu_{w,\beta}$ for every β.

Once we have such a good weight v, we can use proposition 29.2 to "construct" the desired good set S.

The weight v will be of the form σw where the weight σ is a *decreasing* weight, that is, $\sigma(n) \geq \sigma(n+1)$ for every $n \in \mathbb{N}$. That a weight v of this form represents the same measures everywhere is a consequence of the following general but probably familiar result—our main new tool in this section.

First recall the definition of a dissipative sequence of measures on $\mathbb{N}$.

35.1 Definition ► Dissipative sequence of measures

Let $(v_N)_{N\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{N}$.
We say, the sequence $(v_N)_{N\in\mathbb{N}}$ is *dissipative* if

$$\lim_N \frac{v_N(j)}{v_N(\mathbb{N})} = 0, \text{ for every } j \in \mathbb{N} \tag{35.1}$$

36.1 Proposition ▶ Decreasing weights preserve limits

Let w be a weight, $(\sigma_N)_{N\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{N}$ and let $\mathbf{x} = (x_n)$ be a sequence from a normed space $(X, \|\|)$. Denoting $v_N := \sigma_N \cdot w$, we assume the following

1. Each σ_N has finite support.
2. The sequence (v_N) is dissipative.
3. For each N the measure σ_N is decreasing, $\sigma_N(1) \geq \sigma_N(2) \geq \dots$.
4. The sequence $\left(\mathbb{A}^w_{n\in[N]} x_n\right)_N$ converges to some $y \in X$,

$$\lim_N \mathbb{A}^w_{n\in[N]} x_n = y \tag{36.1}$$

Then, the sequence $\left(\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j\right)_N$ of averages converge to the same limit as the w-weighted averages,

$$\lim_N \mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j = y \tag{36.2}$$

At the heart of this result is the following quantitative estimate: For a given $\epsilon > 0$, if K is such that $\left\|\mathbb{A}^w_{n\in[j]} x_n - y\right\| < \epsilon$ for $j \geq K$ then we have

$$\left\|\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y\right\| \leq \epsilon + \max_{j\in[K]} \left\|\mathbb{A}^w_{n\in[j]} x_n - y\right\| \cdot \frac{v_N([K])}{v_N(\mathbb{N})} \tag{36.3}$$

for every $N \geq K$.

Note that the estimate in eq. (36.3) indeed implies the conclusion of the proposition in eq. (36.2). To see this, let $N \to \infty$ in eq. (36.3). Then, since (v_N) is a dissipative sequence so $\lim_N \frac{v_N([K])}{v_N(\mathbb{N})} = 0$, we get that $\limsup_N \left\|\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y\right\| \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\lim_N \left\|\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y\right\| = 0$.

Proof of proposition 36.1. The main idea of the proof is to write $\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j$ as an average of the w-averages with respect to another measure q_N on $\mathbb{N}$

$$\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j = \mathbb{A}^{q_N}_{j\in\mathbb{N}} \mathbb{A}^w_{n\in[j]} x_n \text{ for all } N \tag{36.4}$$

These measures q_N will also satisfy

$$q_N(\mathbb{N}) = v_N(\mathbb{N}) \text{ for every } N \in \mathbb{N} \tag{36.5}$$

The measure q_N appears during performing summation by parts:

setting $\sigma_N(0) := 0$, $w(0) := 0$ and $x_0 := 0$, we have

$$\begin{aligned}
\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j &= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \sigma_N(j) w(j) x_j \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \sigma_N(j) \left(\sum_{n\in[j]} w(n)x_n - \sum_{n\in[j-1]} w(n)x_n \right) \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \sum_{n\in[j]} w(n)x_n \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([j]) \cdot \mathbb{A}^{w}_{n\in[j]} x_n
\end{aligned}$$

Thus, defining the measure q_N by

$$q_N(j) := \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([j]), \text{ for } j \in \mathbb{N} \tag{37.1}$$

we get the identity in eq. (36.4) once we show that q_N really is a measure satisfying eq. (36.5). That $q_N(j)$ is unsigned follows from the assumption that the sequence $(\sigma_N(j))_{j\in\mathbb{N}}$ is decreasing for fixed N. That $q_N(\mathbb{N}) = v_N(\mathbb{N})$ follows by setting $x_j = 1$ for every j in the summation by parts argument above since then we get exactly $q_N(\mathbb{N}) = v_N(\mathbb{N})$:

$$\begin{aligned}
1 &= \mathbb{A}^{v_N}_{j\in\mathbb{N}} 1 \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} \Big(\sigma_N(j) - \sigma_N(j+1)\Big) \cdot w([j]) \cdot \mathbb{A}^{w}_{n\in[j]} 1 \\
&= \frac{1}{v_N(\mathbb{N})} \sum_{j\in\mathbb{N}} q_N(j) \cdot 1 \\
&= \frac{1}{v_N(\mathbb{N})} \cdot q_N(\mathbb{N})
\end{aligned}$$

Using the now obvious identity $y = \mathbb{A}^{q_N}_{j\in\mathbb{N}} y$ together with eq. (36.4), we can now write $\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y$ as

$$\mathbb{A}^{v_N}_{j\in\mathbb{N}} x_j - y = \mathbb{A}^{q_N}_{j\in\mathbb{N}} \Big(\mathbb{A}^{w}_{n\in[j]} x_n - y \Big) \tag{37.2}$$

Let $\epsilon > 0$. Since we assumed $\lim_N \mathbb{A}^{w}_{n\in[N]} x_n = y$, there is an $K = K(\epsilon)$ so that

$$\left\| \mathbb{A}^{w}_{n\in[j]} x_n - y \right\| < \epsilon, \text{ for } j \geq K \tag{37.3}$$

Splitting the summation on j in $\mathbb{A}^{q_N}_{j\in\mathbb{N}} \Big(\mathbb{A}^{w}_{n\in[j]} x_n - y \Big)$ into two parts at

K and using the triangle inequality, we get the estimate

$$\left\| \mathbb{A}^{q_N}_{j\in\mathbb{N}} \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| \leq \left\| \frac{1}{q_N(\mathbb{N})} \sum_{j\in[K]} q_N(j) \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| + \left\| \frac{1}{q_N(\mathbb{N})} \sum_{j>K} q_N(j) \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| \quad (38.1)$$

We can estimate the first term as

$$\left\| \frac{1}{q_N(\mathbb{N})} \sum_{j\in[K]} q_N(j) \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| \leq \max_{j\in[K]} \left\| \mathbb{A}^{w}_{n\in[j]} x_n - y \right\| \cdot \frac{q_N([K])}{q_N(\mathbb{N})} \quad (38.2)$$

Using the definition of $q_N(j)$ as given in eq. (37.1), we can estimate $q_N([K])$ as

$$\begin{aligned} q_N([K]) &= \sum_{j\in[K]} \Big(\sigma_N(j) - \sigma_N(j+1) \Big) \cdot w([j]) \\ &= \sum_{j\in[K]} \sigma_N(j) \Big(w([j]) - w([j-1]) \Big) - \sigma_N(K+1) w([K]) \\ &= \sum_{j\in[K]} \sigma_N(j) w(j) - \sigma_N(K+1) w([K]) \\ &= \sum_{j\in[K]} v_N(j) - \sigma_N(K+1) w([K]) \\ &\leq v_N([K]) \end{aligned}$$

Using this estimate and that $q_N(\mathbb{N}) = v_N(\mathbb{N})$ in eq. (38.2) we get

$$\left\| \frac{1}{q_N(\mathbb{N})} \sum_{j\in[K]} q_N(j) \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| \leq \max_{j\in[K]} \left\| \mathbb{A}^{w}_{n\in[j]} x_n - y \right\| \cdot \frac{v_N([K])}{v_N(\mathbb{N})} \quad (38.3)$$

The second term in eq. (38.1) can be estimated, using eq. (37.3), as

$$\left\| \frac{1}{q_N(\mathbb{N})} \sum_{j>K} q_N(j) \left(\mathbb{A}^{w}_{n\in[j]} x_n - y \right) \right\| \leq \epsilon \quad (38.4)$$

Putting the estimates in eqs. (38.3) and (38.4) into eq. (38.1) and using the identity in eq. (37.2) we get eq. (36.2). □

39.1 Corollary ▶ Decreasing weights preserve limit measures of weights

Let w and σ be weights. Denoting $v \coloneqq \sigma \cdot w$, we assume the following

1. $v(\mathbb{N}) = \infty$.
2. The weight σ is decreasing $\sigma(1) \geq \sigma(2) \geq \dots$.
3. The weight w is good.

Then the weight v is good and represents the same measures everywhere as w,

$$\mu_{v,\beta} = \mu_{w,\beta} \text{ for every } \beta. \tag{39.1}$$

Proof. We need to show that for a given β we have

$$\lim_N \mathbb{A}^v_{n\in[N]} \, \mathrm{e}(r_n\beta) = \mu_{w,\beta}(\mathrm{e}) \tag{39.2}$$

to do this, use proposition 36.1 with σ_N as σ restricted to the set $[N]$,

$$\sigma_N(n) \coloneqq \sigma(n)\mathbb{1}_{[N]}(n) \tag{39.3}$$

and (x_n) defined by

$$x_n \coloneqq \mathrm{e}(r_n\beta) \tag{39.4}$$

□

Let us now go back to our good weight w which represents ρ at α. Since we assumed that R is sublacunary and $\mathbf{M}(w) > 0$, the weight w is also sublacunary. Recall that we obtained w as the limit of a sequence (w_k) of bounded good weights. In fact, we pasted w together from the w_k piece by piece in a sense that after choosing indices $N_1 < N_2 < \dots$, we define w to be equal w_k on the interval $(N_k, N_{k+1}]$

$$w \coloneqq \sum_k w_k \mathbb{1}_{(N_k,N_{k+1}]}. \tag{39.5}$$

In order to obtain a good weight v which is bounded by 1 and would represent the same measures as w, we could do the following. Define σ by

$$\sigma \coloneqq \frac{1}{\max_{j\in[k]} \|w_j\|_\infty} \cdot \mathbb{1}_{(N_k,N_{k+1}]} \tag{39.6}$$

Then σ is a decreasing and the weight $v \coloneqq \sigma w$ is bounded by 1. The remaining issue is that v may not be sublacunary. But in the recursive process of choosing the indices (N_k) if we choose N_k large enough compared to N_{k-1} we can ensure that v is sublacunary. Here is what we need to do for this.

We set $F(N) \coloneqq \log(r_N)$. Since the weight w is sublacunary and have positive mean, we have

$$\lim_N \frac{w([N])}{F(N)} = \infty \tag{40.1}$$

So now, if we make sure that our weight v, bounded by 1, satisfies

$$\lim_N \frac{v([N])}{F(N)} = \infty \tag{40.2}$$

then it will be a sublacunary weight.

We want to show that we can choose the indices N_k so that we will have eq. (40.2).

In the proof of lemma 27.1 (and in lemma 17.1) we can see that, during the recursive construction of the sequence (N_k) we can choose N_k arbitrary large compared to N_{k-1}. For our purposes, we just need to choose N_k large enough to satisfy the following additional criterion

$$\frac{N}{\max_{j\in[k]} \left\| w_j \right\|_\infty} > kF(N) \text{ for every } N \geq N_k \tag{40.3}$$

which is possible since the set R is sublacunary, since eq. (40.3) ensures that the weight $v = \sigma w$ is itself sublacunary.

That v represents the same measures as w at every β follows from corollary 39.1. As in the last step of our proof of theorem 10.2, we use proposition 29.2 to show the existence of a good set $S \subset R$ which represents the same measures as v at every β, hence at $\beta = \alpha$ we have $\mu_{S,\alpha} = \rho\mu_\alpha$.

11 *The limit measure at rational points*

In this section we want to prove theorem 12.1. The base set is $\mathbb{N}$ which we suppress in our notation, so we write μ_β instead of $\mu_{\mathbb{N},\beta}$.

Given the probability measure ν on $\mathbb{T}_q$ and the rational number $\frac{a}{q}$, $\gcd(a,q) = 1$, let us see what properties a good set S would need to have so that $\mu_{S,a/q} = \nu$.

Introducing the sets S_j by

$$S_j \coloneqq \{\, s \,|\, s \in S, sa \equiv j \pmod q \,\}, \text{ for every } 0 \leq j \leq q-1 \tag{40.4}$$

let us write, using that the S_j are pairwise disjoint,

$$\begin{aligned}
\mathbb{A}_{s\in S(N)} &= \frac{1}{\#S(N)} \sum_{s\in S(N)} \delta_{sa/q} \\
&= \frac{1}{\#S(N)} \sum_{0\leq j\leq q-1} \sum_{s\in S_j(N)} \delta_{j/q} \\
&= \sum_{0\leq j\leq q-1} \frac{\#S_j(N)}{\#S(N)} \delta_{j/q}
\end{aligned}$$

If we make the assumption[24] that $\lim_N \frac{\#S_j(N)}{\#S(N)}$ exists for every j then, letting $N \to \infty$, we get

$$\mu_{S,a/q} = \sum_{0 \le j \le q-1} \delta_{j/q} \lim_N \frac{\#S_j(N)}{\#S(N)} \tag{41.1}$$

Since $\mu_{S,a/q}$ is supposed to be equal ν, we get

$$\lim_N \frac{\#S_j(N)}{\#S(N)} = \nu(j/q) \tag{41.2}$$

This gives us the idea how to construct S: we start out from the set R_j defined by

$$R_j := \{\, n \mid na \equiv j \pmod q \,\}, \text{ for every } 0 \le j \le q-1 \tag{41.3}$$

Note that R_j is a full residue class $\mod q$, namely, if j' denotes the unique solution to the congruence $j'a \equiv j \pmod q$, then R_j is the arithmetic progression $\{\, kq + j' \mid k \in \mathbb{N} \,\}$. Note that R_j is a good set, as are all arithmetic progressions. We clearly have

$$\mathbf{M}(R_j) = \frac{1}{q} \text{ for every } 0 \le j \le q-1 \tag{41.4}$$

Now what remains is to find a set $S_j \subset R_j$ with relative mean $\nu\left(\frac{j}{q}\right)$ and make sure that S_j is a good set. Let γ be an irrational number and consider

$$S_j := \left\{ r \,\middle|\, r \in R_j, r\gamma \in \left[0, \nu\left(\frac{j}{q}\right)\right) \right\} \text{ for every } 0 \le j \le q-1 \tag{41.5}$$

Using proposition 15.1 with $\alpha = \gamma$ and $R = R_j$, we deduce that S_j is a good set with $\mathbf{M}_{R_j}(S_j) = \nu\left(\frac{j}{q}\right)$, as desired. We finally define S as

$$S := \bigcup_{0 \le j \le q-1} S_j \tag{41.6}$$

The set S is good since it's the finite union of pairwise disjoint good sets with mean. Indeed, we have $\mathbf{M}(S_j) = \frac{1}{q} \cdot \nu\left(\frac{j}{q}\right)$ and hence $\mathbf{M}(S) = \frac{1}{q}$.

12 *Examples*

12.1 *Two good sets, but their intersection has no mean.*

Here we construct randomly two good sets, R, S with $\mathbf{M}(R) = \mathbf{M}(S) = 1/2$ but $\mathbf{M}(R \cap S)$ doesn't exist.

Let (X_n) be a iid sequence of random variables on the probability space (Ω, P), modeling fair coin flipping, so with distribution $P(X_n =$

[24] In fact, the existence of $\lim_N \frac{\#S_j(N)}{\#S(N)}$ follows from S being a good set.

$1) = P(X_n = 0) = 1/2$. Let us also consider another sequence of random variables (Y_n) defined by

$$Y_n = \begin{cases} X_n & \text{if } n \in [2^k, 2^{k+1}) \text{ for even } k \\ 1 - X_n & \text{if } n \in [2^k, 2^{k+1}) \text{ for odd } k \end{cases} \tag{42.1}$$

The (Y_n) is also an iid sequence with the same distribution as the (X_n). Define the sets R^ω, S^ω by $R^\omega := \{ n \mid X_n(\omega) = 1 \}$ and $S^\omega := \{ n \mid Y_n(\omega) = 1 \}$. By lemma 29.3 both R^ω and S^ω are good sets almost surely with $\mathbf{M}(R^\omega) = \mathbf{M}(S^\omega) = 1/2$. We claim that $\mathbf{M}(R^\omega \cap S^\omega)$ almost surely doesn't exists. To see this, denote $T^\omega := R^\omega \cap S^\omega$ and observe that if $\mathbf{M}(T^\omega)$ existed then $\lim_k \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k}$ would exist. But, denoting by O the odd numbers and by E the even numbers, we almost surely have

$$\lim_{k \in O} \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k} = 0$$
$$\lim_{k \in E} \frac{T^\omega \cap [2^k, 2^{k+1})}{2^k} = \frac{1}{2}$$

12.2 $R_1 \cup R_2$ and $R_1 \cap R_2$ have means but are not good

Here is an example of two good sets R_1 and R_2 each with mean $2/3$, $\mathbf{M}(R_1 \cap R_2) = 1/2$ but $R_1 \cap R_2$ is not good and $\mathbf{M}(R_1 \cup R_2) = 5/6$ but $R_1 \cup R_2$ is not good.

Both sets will be defined in blocks of intervals . Partition $\mathbb{N}$ into a sequence of disjoint intervals I_n so that their lengths go to infinity but slower than the left endpoints go to infinity. For example, $I_n = [n^2, (n+1)^2)$ will do.

The first good set R_1 will contain all i***N***tegers from I_1, then only ***O***dd numbers from I_2 then ***E***ven numbers from I_3 then repeat this pattern for I_4, I_5, I_6 etc:

$$NOENOE\ldots \tag{42.2}$$

The set R_2 is defined similarly, except it will have one pattern in intervals $J_k := [3^k, 3^{k+1})$ for even k and another for odd k.

$$EONEON\ldots \text{ for even } k, \tag{42.3}$$

$$ONEONE\ldots \text{ for odd } k. \tag{42.4}$$

Both of these sets are good and they represent the same (uniform) measure at every β.

The intersection $R_1 \cap R_2$ has the patterns

$$EOEEOE\ldots \text{ for even } k, \tag{42.5}$$

$$OOEOOE\dots \text{ for odd } k, \tag{43.1}$$

Clearly $\mathbf{M}(R_1 \cap R_2) = 1/2$ but the average of $\mathrm{e}(n/2)$ is different on J_k for even k from those on odd k: for even k the average will go to $1/3$ while for odd k it goes to $-1/3$.

As for the union $R_1 \cup R_2$, it has the patterns

$$NONNON\dots \text{ for even } k, \tag{43.2}$$

$$NNENNE\dots \text{ for odd } k, \tag{43.3}$$

Clearly $\mathbf{M}(R_1 \cup R_2) = 5/6$ but the average of $\mathrm{e}(n/2)$ is different on J_k for even k from those on odd k: for even k the average will go to $-1/3$ while for odd k it goes to $1/3$.

12.3 *Open set U with visit set $\{\, n \mid n\alpha \in U \,\}$ not good*

Let α be an irrational number in the torus $\mathbb{R}/\mathbb{Z}$. We show that there exists an open subset U of the torus such that the sequence $\left(\mathbb{A}_{n\in[N]}\mathbb{1}_U(n\alpha)\right)_N$ does not converge when N goes to infinity.

The construction does not use at all the group structure or the dimensional properties of the torus. This can be extended in a general context of a sequence in a compact metric space with a non purely atomic asymptotic distribution.

We want to construct an open subset U of the torus and an increasing sequence of positive integers $(N_k)_{k\geq 0}$ such that the averages $\mathbb{A}_{n\in[N_{2k}]}\mathbb{1}_U(n\alpha)$, $k = 0, 1, 2, \dots$, with even indices are large whereas the averages $\mathbb{A}_{n\in[N_{2k+1}]}\mathbb{1}_U(n\alpha)$, $k = 0, 1, 2, \dots$ with odd indices are small.

The sequence (N_k) will be constructed by induction and each N_k will be associated to $\epsilon_k := 1/(2^{k+4}N_k)$. In this induction process, we construct also a sequence of open subsets $(U_k)_{k\geq 0}$.

We start with $N_0 > 1$ fixed and we define

$$U_0 := \bigcup_{n\in[N_0]} (n\alpha - \epsilon_0, n\alpha + \epsilon_0)$$

We have of course

$$\mathbb{A}_{n\in[N_0]}\mathbb{1}_{U_0}(n\alpha) = 1 \quad \text{and} \quad 0 < \lambda(U_0) \leq 2N_0\epsilon_0$$

This is the initial step of our construction. In order to be understandable, let us describe the two next steps.

By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_1 > N_0$ such that

$$\mathbb{A}_{n\in[N_0]}\mathbb{1}_{\overline{U_0}}(n\alpha) \leq 2\lambda(U_0)N_1 \leq 4(N_0\epsilon_0)N_1$$

We fix such a N_1. To any $n \in [N_1] \cap \overline{U_0}^c$ we associate a real δ_n that

$$0 < \delta_n \leq \epsilon_1 \quad \text{and} \quad (n\alpha - \delta_n, n\alpha + \delta_n) \cap U_0 = \emptyset.$$

We define

$$U_1 := \bigcup_{n\in[N_1]\cap\overline{U_0}^c} (n\alpha-\delta_n, n\alpha+\delta_n)$$

We have

$$\mathbb{A}_{n\in[N_1]}\mathbb{1}_{U_1}(n\alpha) \geq 1-4N_0\epsilon_0 \quad \text{and} \quad 0<\lambda(U_1)\leq 2N_1\epsilon_1$$

Note also that by construction $U_0\cap U_1=\emptyset$.

By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_2>N_1$ such that

$$\mathbb{A}_{n\in[N_2]}\mathbb{1}_{\overline{U_1}}(n\alpha) \leq 2\lambda(U_1)N_2 \leq 4(N_1\epsilon_1)N_2$$

We fix such a N_2. To any $n\in[N_2]\cap\overline{U_1}^c$ we associate a real δ_n that

$$0<\delta_n\leq\epsilon_2 \quad \text{and} \quad (n\alpha-\delta_n, n\alpha+\delta_n)\cap U_1=\emptyset$$

Note that the values of the δ_n's are reinitialized.

We define

$$U_2 := U_0 \cup \bigcup_{n\in[N_2]\cap\overline{U_1}^c} (n\alpha-\delta_n, n\alpha+\delta_n)$$

We have

$$\mathbb{A}_{n\in[N_2]}\mathbb{1}_{U_2}(n\alpha) \geq 1-4N_1\epsilon_1 \quad \text{and} \quad \lambda(U_2)\leq 2N_0\epsilon_0+2N_2\epsilon_2$$

Note also that by construction $U_2\cap U_1=\emptyset$ and $U_0\subset U_2$.

Let us state now our induction hypothesis. Suppose that, for a fixed integer $k>0$ we have already constructed two sequences

$$(U_\ell)_{0\leq\ell\leq k} \quad \text{and} \quad N_0<N_1<N_2<\ldots<N_k$$

such that

- $U_0\subset U_2\subset U_4\subset\ldots$ and $U_1\subset U_3\subset U_5\subset\ldots$,
- If ℓ is even and ℓ' is odd, then U_ℓ and $U_{\ell'}$ are disjoint,
- Each U_ℓ is a finite union of open intervals,
- If $0\leq 2\ell\leq k$, then

$$\lambda(U_{2\ell}) \leq 2(N_0\epsilon_0+N_2\epsilon_2+\ldots+N_{2\ell}\epsilon_{2\ell})$$

and

$$\mathbb{A}_{n\in[N_{2\ell}]}\mathbb{1}_{U_{2\ell}}(n\alpha) \geq 1-4(N_1\epsilon_1+N_3\epsilon_3+\ldots+N_{2\ell-1}\epsilon_{2\ell-1})$$

- If $1 \leq 2\ell + 1 \leq k$, then

$$\lambda(U_{2\ell+1}) \leq 2(N_1\epsilon_1 + N_3\epsilon_3 + \ldots + N_{2\ell+1}\epsilon_{2\ell+1})$$

and

$$\mathbb{A}_{n\in[N_{2\ell+1}]}\mathbb{1}_{U_{2\ell+1}}(n\alpha) \geq 1 - 4(N_0\epsilon_0 + N_2\epsilon_2 + \ldots + N_{2\ell}\epsilon_{2\ell})$$

Here begins the induction process. By the uniform distribution of the sequence $(n\alpha)_n$ in the torus, there exists a number $N_{k+1} > N_k$ such that

$$\mathbb{A}_{n\in[N_{k+1}]}\mathbb{1}_{\overline{U_k}}(n\alpha) \leq 2\lambda(U_k)N_{k+1}$$

We fix such a N_{k+1}. To any $n \in [N_{k+1}] \cap \overline{U_k}^c$ we associate a real δ_n that

$$0 < \delta_n \leq \epsilon_{k+1} \quad \text{and} \quad (n\alpha - \delta_n, n\alpha + \delta_n) \cap U_k = \emptyset$$

We define

$$U_{k+1} \coloneqq U_{k-1} \cup \bigcup_{n\in[N_{k+1}]\cap\overline{U_k}^c} (n\alpha - \delta_n, n\alpha + \delta_n)$$

Note that the values of δ_n's are reinitialized at each induction step.

The items of the induction hypothesis are now satisfied by the sequences $(U_\ell)_{0\leq\ell\leq k+1}$ and $(N_\ell)_{0\leq\ell\leq k+1}$.

We can consider these sequences as infinite, and we define $U \coloneqq \bigcup_{k\geq 0} U_{2k}$.

Recalling our choice $N_k\epsilon_k = 2^{-k-4}$, we obtain

$$\begin{aligned}\mathbb{A}_{n\in[N_{2k}]}\mathbb{1}_U(n\alpha) &\geq \mathbb{A}_{n\in[N_{2k}]}\mathbb{1}_{U_{2k}}(n\alpha)\\ &\geq 1 - 4\sum_\ell N_{2\ell+1}\epsilon_{2\ell+1}\\ &= 5/6\end{aligned}$$

and

$$\begin{aligned}\mathbb{A}_{n\in[N_{2k+1}]}\mathbb{1}_U(n\alpha) &\leq \mathbb{A}_{n\in[N_{2k+1}]}\mathbb{1}_{\overline{U_{2k+1}}^c}(n\alpha)\\ &\leq 4\sum_\ell N_{2\ell}\,\mathrm{e}_{2\ell}\\ &= 1/3\end{aligned}$$

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