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On a family of holomorphic self-maps of the unit disk

M. Ounaies and J.-M. Sac-Épée*

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Abstract

We give a characterization of the sets D_p ($1 < p < 2$) of complex numbers c such that $z \mapsto \frac{1+z}{2} + c \left(\frac{1-z}{2}\right)^p$ is a self-map of the closed unit disk and we show that these sets are increasing with respect to p .

1 Introduction

We will denote by \mathbb{D} the open unit disk, by $\overline{\mathbb{D}}$ the closed unit disk and by $C = \overline{\mathbb{D}} \setminus \mathbb{D}$ the unit circle. For $p \geq 0$ and $c \in \mathbb{C}$, let $f_{p,c}$ be the map, analytic over \mathbb{D} and continuous over $\overline{\mathbb{D}}$ defined by

$$f_{p,c}(z) = \frac{1+z}{2} + c \left(\frac{1-z}{2}\right)^p, \quad (1.1)$$

where the principal branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}^-$ is used to define the p 'th powers. Note that for $z \in \overline{\mathbb{D}} \setminus \{1\}$, $\Re\left(\frac{1-z}{2}\right) > 0$. In [3], Mortini and Rhin raised the question of determining precisely the set

$$D_p = \{c \in \mathbb{C} : f_{p,c} \text{ is a self-map of } \overline{\mathbb{D}}\}$$

for $p > 1$. They proved that $[0, 1] \subseteq D_p$ for $p \geq 1$.

The case $0 \leq p \leq 1$ was settled in [2], where the authors showed the following results:

- $D_0 = \{z \in \mathbb{C} : |z + \frac{1}{2}| \leq \frac{1}{2}\}$,
- $D_1 = [-1, 1]$,
- For all $0 < p < 1$, $D_p = \{-\left(\frac{1-a}{2}\right)^{1-p}, a \in \overline{\mathbb{D}}\}$.
- The regions $\{D_p\}_{0 \leq p < 1}$ are decreasing.

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In another paper (see [1]), Mortini and Sac-Épée showed that

$$D_2 = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{4} \right| \leq \frac{3}{4} \right\} \quad (1.2)$$

and they made the conjecture, based on numerical results, that the regions $\{D_p\}_{p>1}$ are increasing.

In the present work, we will focus on the case where $1 < p \leq 2$. In section 1, we will prove that the regions $\{D_p\}_{1 < p \leq 2}$ are increasing and in section 2, we will give a description of D_p for $1 < p < 2$.

Throughout this paper, we will use the function $g_{p,c}$ defined as follows for $p \geq 0$, $c \in \mathbb{C}$ and $t \in \mathbb{R}$,

$$g_{p,c}(t) = (1 + t^2)^p - |c + it(1 + it)^{p-1}|^2. \quad (1.3)$$

We will extensively make use of the next proposition which is contained in [1]. We include here a short proof for the sake of completeness.

Proposition 1.1. *Let $p \geq 0$. Then $c \in D_p$ if and only if for all $t \in \mathbb{R}$, $g_{p,c}(t) \geq 0$.*

Proof. Thanks to the maximum principle applied to the analytic function $f_{p,c}$ and the fact that 1 is a fixed point of $f_{p,c}$, we see that $c \in D_p$ if and only if for all $z \in C \setminus \{1\}$, $|f_{p,c}(z)| \leq 1$.

For all $z \in C \setminus \{1\}$, we write $z = -e^{-2i\theta}$ with $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and we put $t = \tan \theta$. Thus we have

$$\begin{aligned} \frac{1-z}{2} &= \frac{1+e^{-2i\theta}}{2} = \cos \theta e^{-i\theta} = (1+it)^{-1}, \\ \frac{1+z}{2} &= \frac{1-e^{-2i\theta}}{2} = i \sin \theta e^{-i\theta} = it(1+it)^{-1}, \\ \frac{1+z}{2} + c \left(\frac{1-z}{2} \right)^p &= (1+it)^{-p} (c + it(1+it)^{p-1}). \end{aligned}$$

With this setting, $c \in D_p$ if and only if for all $t \in \mathbb{R}$,

$$(1+t^2)^{-\frac{p}{2}} |c + it(1+it)^{p-1}| \leq 1$$

which is equivalent to $g_{p,c}(t) \geq 0$. ■

We will denote by

$$D_p^+ = D_p \cap \{z \in \mathbb{C} : \Im z \geq 0\}; \quad D_p^- = D_p \cap \{z \in \mathbb{C} : \Im z \leq 0\}.$$

Remark 1.2. Clearly, for all $c \in \mathbb{C}$ and for all $t \in \mathbb{R}$, $g_{p,c}(t) = g_{p,\bar{c}}(-t)$, thus

$$c \in D_p^+ \iff \bar{c} \in D_p^-.$$

From now on for a complex number $c \neq 0$, we will denote by $\theta_c \in]-\pi, \pi]$ the principal argument of c .

2 The regions $\{D_p\}_{1 < p \leq 2}$ are increasing.

Theorem 2.1. *If $1 < p < q \leq 2$, then $D_p \subset D_q$.*

Before proving this theorem, we need two lemmas.

Lemma 2.2. *Let $1 < p < 2$. Then $c \in D_p^+$ if and only if the two following conditions are satisfied:*

- a) $\theta_c \in [0, (p-1)\frac{\pi}{2}]$;
- b) For all $t \in [0, \tan(\frac{\theta_c}{p-1})]$, $g_{p,c}(t) \geq 0$.

In particular, $D_p \cap \mathbb{R} = [0, 1]$.

Proof. Recall that by definition (1.3), $g_{p,c}(t) = (1+t^2)^p - |c+it(1+it)^{p-1}|^2$. Let us prove the following formula:

$$g_{p,c}(t) = (1+t^2)^{p-1} - |c|^2 - 2t(1+t^2)^{\frac{p-1}{2}} |c| \sin(\theta_c - (p-1) \arctan t). \quad (2.1)$$

Note that

$$|c+it(1+it)^{p-1}|^2 = |c|^2 + t^2(1+t^2)^{p-1} + 2t(1+t^2)^{\frac{p-1}{2}} |c| \cos(\theta_c - (p-1) \arctan t - \frac{\pi}{2}).$$

In the previous line, the expression inside the cosine comes from the fact that $(p-1) \arctan t$ is the principal argument of $(1+it)^{p-1}$, so that $(p-1) \arctan t + \frac{\pi}{2}$ is the principal argument of $i(1+it)^{p-1}$.

Thus

$$\begin{aligned} g_{p,c}(t) &= (1+t^2)(1+t^2)^{p-1} - |c+it(1+it)^{p-1}|^2 \\ &= (1+t^2)^{p-1} - |c|^2 - 2t(1+t^2)^{\frac{p-1}{2}} |c| \cos(\theta_c - (p-1) \arctan t - \frac{\pi}{2}), \end{aligned}$$

and (2.1) follows. Assume that $c \in D_p^+$. Then by definition of D_p^+ , $0 \leq \theta_c \leq \pi$. Since $1 < p < 2$, we deduce that

$$-\frac{\pi}{2} \leq -(p-1)\frac{\pi}{2} \leq \theta_c - (p-1)\frac{\pi}{2} \leq \theta_c \leq \pi.$$

Now we multiply (2.1) by t^{-p} for $t > 0$, and we observe that

$$t^{-p}g_{p,c}(t) = t^{-(2-p)}(t^{-2}+1)^{p-1} - |c|^2 t^{-p} - 2(t^{-2}+1)^{\frac{p-1}{2}} |c| \sin(\theta_c - (p-1) \arctan t).$$

We have

$$\lim_{t \rightarrow +\infty} t^{-p}g_{p,c}(t) = -2|c| \sin\left(\theta_c - (p-1)\frac{\pi}{2}\right). \quad (2.2)$$

By Proposition 1.1, $\sin(\theta_c - (p-1)\frac{\pi}{2}) \leq 0$ which implies condition a).

The necessity of condition b) is obvious by Proposition 1.1.

Reciprocally, assume that conditions a) and b) are satisfied.

Let $t \geq \tan(\frac{\theta_c}{p-1})$. Then $-\frac{\pi}{2} \leq \theta_c - (p-1) \arctan t \leq 0$ which means that

$$g_{p,c}(t) \geq (1+t^2)^{p-1} - |c|^2 \geq 1 - |c|^2 = g_{p,c}(0) \geq 0.$$

Now we only need to verify that for $t < 0$, $g_{p,c}(t) \geq 0$.
If $t < 0$, a simple computation shows that

$$g_{p,c}(t) - g_{p,c}(-t) = -4t(1+t^2)^{\frac{p-1}{2}}|c|\sin\theta_c \cos((p-1)\arctan t).$$

Since $-\frac{\pi}{2} < (p-1)\arctan t < 0$, we have $\cos((p-1)\arctan t) \geq 0$. Thus

$$g_{p,c}(t) > g_{p,c}(-t) \geq 0.$$

Hence $g_{p,c}(t) \geq 0$ for all $t \in \mathbb{R}$ and so, by Proposition 1.1, $c \in D_p^+$. ■

Remark 2.3. In the case where $p = 2$, (2.2) is replaced by

$$\lim_{t \rightarrow +\infty} t^{-p}g_{2,c}(t) = 1 + 2\Re c,$$

and we retrieve the fact that $D_2 \subset \{\Re z \geq -\frac{1}{2}\}$ but we don't get the necessity of condition (a). However, it is easy to deduce from the proof of Lemma 2.2 the following:

Assume that $\theta_c \in [0, \frac{\pi}{2}]$, then $c \in D_2$ if and only if for all $t \in [0, \tan \theta_c]$, $g_{2,c}(t) \geq 0$.

Lemma 2.4. *Let $1 < p < q \leq 2$ and $c \in \mathbb{C}$ such that $\theta_c \in [0, (p-1)\frac{\pi}{2}]$. Then for all $t \in [0, \tan(\frac{\theta_c}{q-1})]$,*

$$g_{q,c}(t) \geq (1+t^2)^{q-p}g_{p,c}(t).$$

Proof. For all $t \in \mathbb{R}$ we have

$$\begin{aligned} g_{q,c}(t) - (1+t^2)^{q-p}g_{p,c}(t) &= ((1+t^2)^{q-p} - 1)|c|^2 + \\ &+ 2t(1+t^2)^{\frac{q-1}{2}}|c| \left((1+t^2)^{\frac{q-p}{2}} \sin(\theta_c - (p-1)\arctan t) - \sin(\theta_c - (q-1)\arctan t) \right). \end{aligned}$$

Now for all $0 \leq t \leq \tan(\frac{\theta_c}{q-1})$,

$$0 \leq \theta_c - (q-1)\arctan t \leq \theta_c - (p-1)\arctan t \leq \theta_c \leq \frac{\pi}{2}$$

and consequently

$$\begin{aligned} 0 \leq \sin(\theta_c - (q-1)\arctan t) &\leq \sin(\theta_c - (p-1)\arctan t) \\ &\leq (1+t^2)^{\frac{q-p}{2}} \sin(\theta_c - (p-1)\arctan t). \end{aligned}$$

The desired result follows immediately. ■

Proof of Theorem 2.1. Consider p and q such that $1 < p < q \leq 2$ and let $c \in D_p$. In view of Remark 1.2, we may assume that $c \in D_p^+$. Then by Lemma 2.2 we have

$$0 \leq \theta_c \leq (p-1)\frac{\pi}{2} \leq (q-1)\frac{\pi}{2}.$$

Let $t \in [0, \tan(\frac{\theta_c}{q-1})]$. Then by Proposition 1.1 and Lemma 2.4, we have

$$g_{q,c}(t) \geq (1+t^2)^{q-p}g_{p,c}(t) \geq 0.$$

Now we use Lemma 2.2 (case $q < 2$) and Remark 2.3 (case $q = 2$) to conclude that $c \in D_q^+$. ■

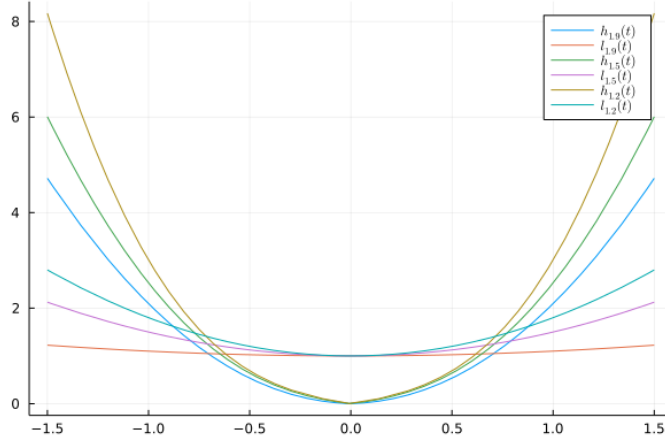


Figure 1: $h_p(t) = (p-1)t \tan((2-p) \arctan t^{-1} + \arctan(pt))$
 $l_p(t) = 1 + (2-p)t^2$.

3 Description of D_p , $1 < p < 2$.

We know from Lemma 2.2 that $D_p \cap \mathbb{R} = [0, 1]$. Our next theorem gives an explicit expression of the remaining elements of D_p .

Theorem 3.1. *Let $1 < p < 2$. Then $c \in D_p$ if and only if either $c \in [0, 1]$ or c is of the form*

$$c = (1 + it)^{p-2}(1 - ipt)^{-1}(\alpha + i(p-1)t)$$

where $t \neq 0$ and α are such that

$$(p-1)t \tan((2-p) \arctan t^{-1} + \arctan(pt)) \leq \alpha \leq 1 + (2-p)t^2. \quad (3.1)$$

In order to carry out the proof of this theorem, we need two preliminary lemmas.

Lemma 3.2.

Let $1 < p \leq 2$ and assume that $\theta_c \in [0, (p-1)\frac{\pi}{2}]$. Then the following properties are satisfied:

- (i) $g_{p,c}(t) \rightarrow +\infty$ as $|t| \rightarrow +\infty$,
- (ii) $g_{p,c}$ has a unique critical point $t_{p,c}$ on \mathbb{R} ,
- (iii) If $\theta_c = 0$, then $t_{p,c} = 0$ and if $\theta_c > 0$, then $0 < t_{p,c} < \tan\left(\frac{\theta_c}{p-1}\right)$.

Obviously, (i) and (ii) imply that $g_{p,c}$ attains its minimum at $t_{p,c}$.

Proof. Property (i) follows from (2.1) and

$$\sin\left(\theta_c - (p-1)\frac{\pi}{2}\right) \leq 0, \quad \sin\left(\theta_c + (p-1)\frac{\pi}{2}\right) \geq 0.$$

Recall that by definition (1.3), for all $t \in \mathbb{R}$,

$$g_{p,c}(t) = (1 + t^2)^p - |c + it(1 + it)^{p-1}|^2.$$

To prove (ii) and (iii), let us compute the derivative of $g_{p,c}$:

$$\begin{aligned}
g'_{p,c}(t) &= 2(pt(1+t^2)^{p-1} + \Re(i(c+it(1+it)^{p-1})(1-it)^{p-2}(1-ipt))) \\
&= 2(1+t^2)^{p-2} (pt(1+t^2) + \Re(i(c+it(1+it)^{p-1})(1+it)^{2-p}(1-ipt))) \\
&= 2(1+t^2)^{p-2} (pt(1+t^2) - \Im(c(1+it)^{2-p}(1-ipt)) - t\Re((1+it)(1-ipt))) \\
&= 2(1+t^2)^{p-2} ((p-1)t - \Im(c(1+it)^{2-p}(1-ipt))) \\
&= 2(1+t^2)^{p-2} \left((p-1)t + |c|(1+t^2)^{\frac{2-p}{2}} (1+p^2t^2)^{\frac{1}{2}} \sin(\phi_{p,c}(t)) \right),
\end{aligned} \tag{3.2}$$

where we have denoted by

$$\phi_{p,c}(t) = \arctan(pt) - (2-p)\arctan t - \theta_c.$$

Observe that whenever $t < 0$, one has the inequalities

$$-\pi < \arctan(pt) - \theta_c \leq \phi_{p,c}(t) \leq (p-1)\arctan t - \theta_c < 0,$$

thus $g'_{p,c}(t) < 0$.

Now we compute the derivative of $\phi_{p,c}$:

$$\phi'_{p,c}(t) = (p-1) \frac{2 + p(p-1)t^2}{(1+t^2)(1+p^2t^2)} \geq 0.$$

Note that for all $t \geq 0$,

$$-\frac{\pi}{2} \leq -\theta_c \leq \phi_{p,c}(t) \leq \arctan(pt) \leq \frac{\pi}{2}.$$

We conclude that the function $t \mapsto \sin(\phi_{p,c}(t))$ is increasing on $[0, +\infty[$ and so is the function $t \mapsto (1+t^2)^{2-p}g'_{p,c}$. Therefore, $g'_{p,c}$ has at most one zero in $[0, +\infty[$.

Now let us prove the existence of a zero $t_{p,c}$ of $g'_{p,c}$.

If $\theta_c = 0$ then obviously $g'_{p,c}(0) = 0$ and we put $t_{p,c} = 0$.

If $\theta_c \in]0, (p-1)\frac{\pi}{2}[$, then $g'_{p,c}(0) = -2\Im c < 0$. Since $g_{p,c}(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, the derivative $g'_{p,c}$ has to vanish at some point $t_{p,c} > 0$.

Besides, if $\theta_c \in]0, (p-1)\frac{\pi}{2}[$, let us put $T = \tan\left(\frac{\theta_c}{p-1}\right)$. Then by (2.1),

$$g_{p,c}(T) = (1+T^2)^{p-1} - |c|^2 > 1 - |c|^2 = g_{p,c}(0).$$

We deduce that $0 < t_{p,c} < T$. ■

Remark 3.3. In the case where $p = 2$, we may solve explicitly the equation $g'_{2,c}(t) = 0$, which gives

$$t_{2,c} = \frac{\Im c}{1 + 2\Re c}$$

Lemma 3.4. *Let $1 < p \leq 2$.*

(i) *Let $t \in \mathbb{R}$. Then $g'_{p,c}(t) = 0$ if and only if c is of the form*

$$c = (1+it)^{p-2}(1-ipt)^{-1}(\alpha + i(p-1)t), \quad \alpha \in \mathbb{R}. \tag{3.3}$$

(ii) For all $t > 0$, $0 < (2-p) \arctan t^{-1} + \arctan(pt) < \frac{\pi}{2}$.

(iii) Assume that c is of the form (3.3) with $t > 0$ and $\alpha > 0$. Then $\theta_c \geq 0$ and we have the following equivalences:

$$g_{p,c}(t) \geq 0 \iff \alpha \leq 1 + (2-p)t^2; \quad (3.4)$$

$$\theta_c \leq (p-1)\frac{\pi}{2} \iff (p-1)t \tan((2-p) \arctan t^{-1} + \arctan(pt)) \leq \alpha. \quad (3.5)$$

Proof. (i) is an obvious consequence of formula (3.2).

For $t > 0$, define $h_p(t) = (2-p) \arctan t^{-1} + \arctan(pt)$. Then the derivative

$$h'_p(t) = \frac{(p-1)(2+p(p-1)t^2)}{(1+t^2)(1+p^2t^2)}$$

is obviously positive, which implies (ii).

Finally, let us prove (iii). Assume that c is of the form (3.3) with $t > 0$ and $\alpha > 0$. The argument of c is given by

$$\theta_c = \arctan(pt) - (2-p) \arctan t + \arctan\left(\frac{(p-1)t}{\alpha}\right)$$

Clearly

$$\theta_c \geq \arctan(pt) - \arctan t > 0.$$

To prove (3.5), it is sufficient to observe that

$$\theta_c = (p-1)\frac{\pi}{2} + (2-p) \arctan t^{-1} + \arctan(pt) - \arctan\left(\frac{\alpha}{(p-1)t}\right).$$

$$(1+t^2)^{2-p}(1+p^2t^2)g_{p,c}(t) = (1+t^2)^2 - (\alpha + (p-1)t^2)^2.$$

Now since $c = (1+it)^{p-2}(1-ipt)^{-1}(\alpha + i(p-1)t)$, we have

$$\begin{aligned} c + it(1+it)^{p-1} &= (1+it)^{p-2}(1-ipt)^{-1}(\alpha + i(p-1)t + it(1+it)(1-ipt)) \\ &= (1+it)^{p-2}(1-ipt)^{-1}(\alpha + (p-1)t^2 + ipt(1+t^2)) \end{aligned}$$

Thus

$$\begin{aligned} |c + it(1+it)^{p-1}|^2 &= (1+t^2)^{p-2}(1+p^2t^2)^{-1}((\alpha + (p-1)t^2)^2 + p^2t^2(1+t^2)^2) \\ &= (1+t^2)^{p-2}(1+p^2t^2)^{-1}((\alpha + (p-1)t^2)^2 - (1+t^2)^2) + (1+t^2)^p. \end{aligned}$$

By the definition (1.3) of $g_{p,c}$,

$$\begin{aligned} g_{p,c}(t) &= (1+t^2)^p - |c + it(1+it)^{p-1}|^2 \\ &= (1+t^2)^{p-2}(1+p^2t^2)^{-1}((1+t^2)^2 - (\alpha + (p-1)t^2)^2). \end{aligned}$$

and we readily deduce (3.4). ■

Proof of Theorem 3.1.

Necessity:

Thank to Remark 1.2 we may assume that $c \in D_p^+$.

If $\theta_c = 0$, then by Lemma 2.2, $c \in [0, 1]$.

If $\theta_c > 0$, then by Lemma 2.2, $\theta_c \in]0, (p-1)\frac{\pi}{2}]$. Consider $t = t_{p,c} > 0$ given by Lemma 3.2. Lemma 3.4 gives a real number α such that

$$c = (1 + it)^{p-2}(1 - ipt)^{-1}(\alpha + i(p-1)t).$$

Thus we have

$$\begin{aligned} \alpha &= \Re(c(1 + it)^{2-p}(1 - ipt)) \\ &= |c|(1 + t^2)^{\frac{2-p}{2}}(1 + p^2t^2)^{\frac{1}{2}} \cos(\theta_c + (2-p)\arctan t - \arctan(pt)). \end{aligned}$$

Thanks to the inequalities

$$-\frac{\pi}{2} < -\arctan(pt) \leq \theta_c + (2-p)\arctan t - \arctan(pt) \leq \theta_c + (1-p)\arctan t < \theta_c \leq \frac{\pi}{2},$$

we deduce that $\alpha > 0$.

Condition (3.1) follows from Lemma 3.4.

Sufficiency:

We know from Lemma 2.2 that $D_p \cap \mathbb{R} = [0, 1]$.

Assume that $c = (1 + it)^{p-2}(1 - ipt)^{-1}(\alpha + i(p-1)t)$ with $t \neq 0$ and α satisfying (3.1). Observe that $\alpha > 0$. Again thanks to Remark 1.2 we may assume that $t > 0$. Using lemma 3.4 we know that $g'_{p,c}(t) = 0$, $\theta_c \in [0, (p-1)\frac{\pi}{2}]$ and $g_{p,c}(t) \geq 0$.

By lemma 3.2, we necessarily have $t = t_{p,c}$ which means that the minimum of $g_{p,c}$ on \mathbb{R} is attained at t . Using Proposition 1.1, we deduce that $c \in D_p$. ■

Remark 3.5. In the case where $p = 2$, we may repeat the proof of Theorem 3.1 and use Remark 2.3 to obtain the following:

$$\begin{aligned} D_2 \cap \{\Re z \geq 0\} &= \{(1 - 2it)^{-1}(\alpha + it), (t, \alpha) \in \mathbb{R}^2 : 2t^2 \leq \alpha \leq 1\} \\ &= \left\{ z \in \mathbb{C} : \left| z - \frac{1}{4} \right| \leq \frac{3}{4} \right\} \cap \{\Re z \geq 0\}. \end{aligned}$$

The second equality comes from (1.2) but may also be proved directly by the following formal equivalences for $\alpha, t, a, b \in \mathbb{R}$ and $c = a + ib$:

$$\begin{aligned} c = \frac{\alpha + it}{1 - 2it} &\iff \frac{1}{2c + 1} = \frac{1 - 2it}{2\alpha + 1} \\ &\text{by taking real and imaginary parts, we obtain} \\ \frac{1}{2\alpha + 1} &= \frac{2a + 1}{|2c + 1|^2} \text{ and } \frac{t}{2\alpha + 1} = \frac{b}{|2c + 1|^2} \\ \iff \alpha &= \frac{2|c|^2 + a}{2a + 1} = 1 + 2\frac{|c - \frac{1}{4}|^2 - \frac{9}{16}}{2a + 1} \text{ and } t = \frac{b}{1 + 2a}. \end{aligned}$$

Note that

$$c = \frac{\alpha - 2t^2}{1 + 4t^2} + i\frac{t(1 + 2\alpha)}{1 + 4t^2}.$$

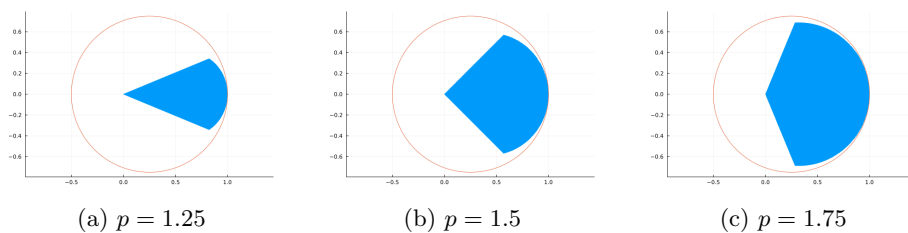


Figure 2: D_p for three particular values of p

4 Numerical illustration

For illustrating purposes, we use the characterization given by Theorem 3.1 to draw the sets D_p for three particular values of p (Figure 2). The circle with center $\frac{1}{4}$ and radius $\frac{3}{4}$, which is the boundary of D_2 , appears in red. We also provide a hyperlink to an animated image showing the evolution of the D_p 's for p varying from 1.01 to 1.99 with a step of 0.007 ([4]).

We may point out that our Figure 2 and Figure 2 in [1] show a discrepancy when p is close to 2. This is due to the fact that, while our graphs are based on theoretical results, those in [1] are obtained experimentally in the following way: many complex numbers c have been randomly drawn, and each of them has been tested and kept only if it survives a test consisting in verifying that $|f_{p,c}(z)| \leq 1$ for numerous random values of z (thanks to the maximum principle, one takes random z such that $|z| = 1$). The experimental results obtained in [1] for p close to 2 are not validated by the present theoretical study.

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