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SOME MOTIVIC PROPERTIES OF GUSHEL-MUKAI SIXFOLDS

MICHELE BOLOGNESI AND ROBERT LATERVEER

ABSTRACT. Gushel-Mukai sixfolds are an important class of so-called Fano-K3 varieties. In this paper we show that they admit a multiplicative Chow-Künneth decomposition modulo algebraic equivalence and that they have the Franchetta property. As side results, we show that double EPW sextics and cubics have the Franchetta property, modulo algebraic equivalence, and some vanishing results for the Chow ring of Gushel-Mukai sixfolds.

1. INTRODUCTION

Fano varieties of K3 type, sometimes dubbed as FK3, are a special class of smooth projective varieties. Roughly speaking, their name comes from the fact that they have a Hodge-theoretical *heart* that it is very similar to the one of a K3 surface. Of course, the quintessential example of a FK3 variety is the celebrated cubic fourfold, the zero locus of a cubic polynomial in a five-dimensional projective space. More precisely:

Definition 1.1. Let H be a Hodge structure of weight k and of level $\lambda(H)$. We say that H is of K3 type if:

- (1) $\lambda(H) = 2$;
- (2) $h^{\frac{k-2}{2}, \frac{k+2}{2}} = 1$.

Definition 1.2. Let X be a smooth Fano variety. We define X to be of K3 type (FK3) if $H^*(X, \mathbb{C})$ contains at least one sub-Hodge structure of K3 type, and for every k there is at most one K3 structure.

One reason FK3 varieties are of interest is the (expected) link with HyperKähler varieties; for more on this cf. the nice recent overview [15] and the references given there.

Let Y be a smooth projective variety over \mathbb{C} , and let $\mathrm{CH}^i(Y)_{\mathbb{Q}}$ denote the Chow groups of Y . The intersection product defines a ring structure on the Chow ring of Y . The Chow ring of K3 surfaces has a particular structure.

Theorem 1.3. (*Beauville-Voisin* [3]) *Let S be a K3 surface. The \mathbb{Q} -subalgebra*

$$R^*(S) := \langle \mathrm{CH}^1(S), c_j(S) \rangle \subset \mathrm{CH}^*(S)$$

injects into cohomology under the cycle class map.

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Eventually Beauville [2] has conjectured that for certain classes of varieties, the Chow ring should admit a multiplicative splitting. This is the case for example for K3 surfaces and for abelian varieties. More recently Shen and Vial [54] have introduced the notion of multiplicative Chow–Künneth decomposition (MCK decomposition), which presents a more concrete realization of Beauville’s ”splitting property conjecture”.

The class of varieties admitting an MCK is still not completely understood. For instance, hyperelliptic curves admit an MCK but a very general curve of genus $g > 2$ does not have such a decomposition. In [36], the second named author has conjectured that a FK3 should always admit such a multiplicative Chow–Künneth decomposition. This is verified for certain FK3 varieties, for instance for cubic fourfolds [18] and for most of the varieties on the Fatighenti–Mongardi list [35], [36], [42]. Cubic fourfolds, in particular, admit also a Chow–Künneth decomposition that displays nicely the associated K3 surface, if it exists [7, 6, 1].

In this paper we go a little further along this path. A Gushel–Mukai variety (GM for short) is a smooth intersection of a linear subspace, a quadric and the cone over the Grassmannian $\text{Gr}(2, 5)$ inside \mathbb{P}^{10} . It is well-known that GM fourfolds and GM sixfolds are FK3. Our first main result is the following.

Theorem (=Theorem 6.3). *Let X be a smooth GM sixfold, then X admits an MCK decomposition, modulo algebraic equivalence.*

Of course the existence of an MCK decomposition modulo rational equivalence would imply our Theorem 6.3, and it seems likely that all GM varieties (of any dimension) admit an MCK decomposition modulo rational equivalence (cf. Remark 6.6 below for the obstacle to proving this for GM sixfolds).

Our proof of Theorem 6.3 employs instances of the *Franchetta property*. This property is motivated by a conjectural property of K3 surfaces (the so-called Franchetta conjecture, as proposed by O’Grady [48], cf. also [51]). Given any smooth family of projective varieties $\mathcal{X} \rightarrow \mathcal{F}$ with smooth \mathcal{F} , we say that $\mathcal{X} \rightarrow \mathcal{F}$ satisfies the Franchetta property if every cycle $z \in \text{CH}^*(\mathcal{X})$ which is fiberwise homologically trivial is also fiberwise rationally equivalent to zero.

GM sixfolds also share some Chow-theoretical properties with another class of very famous FK3 varieties, that is the Debarre–Voisin 20-fold. In fact, using methods very close to those developed in [61], we show that for all GM sixfolds X , we have

$$(1.4) \quad \text{CH}_0^{\text{hom}}(X) = 0; \quad \text{CH}_1^{\text{hom}}(X) = 0.$$

More generally, we also have the following.

Proposition (=Corollary 5.11). *Let X be a GM sixfold. Then*

$$\text{CH}_{\text{hom}}^i = 0 \quad \text{if } i \neq 4.$$

Moreover, the non-vanishing Chow group is generated by the so called σ -planes, which are special 2-dimensional cycles inside X . As a consequence of Corollary 5.11, we also show (Corollary 5.12) that GM varieties that are dual or period partners have isomorphic Chow motives.

Finally, by imitating constructions already made for hyperelliptic curves, K3 surfaces, and cubic hypersurfaces, we introduce the *tautological ring* of the m -th

self-product of a GM sixfold as the subgroup of algebraic cycles (up to algebraic equivalence) generated by the algebraic cycles of the Grassmannian $\text{Gr}(2, 5)$ and the diagonal (see Definition 7.1). We denote by $R^*(X^m)$ the tautological ring. We show that this ring enjoys properties similar to those known for the other varieties mentioned here above.

Theorem (=Theorem 7.2). Let X be a GM sixfold. The cycle class map induces injections

$$R^*(X^m) \hookrightarrow H^*(X^m, \mathbb{Q}),$$

for all $m \leq 45$.

The cycle class map induces injections

$$R^*(X^m) \hookrightarrow H^*(X^m, \mathbb{Q}),$$

for all m if and only if X is Kimura finite-dimensional.

As is well known, one can associate a HyperKähler fourfold and a HyperKähler sixfold to a GM sixfold (or fourfold). These are named respectively double EPW sextic and double EPW cube (see [48, 47, 50, 49, 24] for detailed constructions). As a consequence of our results, we can show that the Franchetta property holds also for double EPW sextics and for double EPW double cubes, but only modulo algebraic equivalence.

Theorem (=Corollary 8.2). (1) Let Y be a very general double EPW sextic, and \mathcal{M}_{EPW6} the moduli space of such varieties. Then we have

$$\text{GDB}_{\mathcal{M}_{EPW6}}^*(Y) \hookrightarrow H^*(Y, \mathbb{Q});$$

(2) Let Z be a very general double EPW cube, and \mathcal{M}_{EPW3} the moduli space of such varieties. Then we have

$$\text{GDB}_{\mathcal{M}_{EPW3}}^*(Z) \hookrightarrow H^*(Z, \mathbb{Q}).$$

Here the “generically defined cycles” $\text{GDB}_{\mathcal{M}_{EPW6}}^*(Y)$ and $\text{GDB}_{\mathcal{M}_{EPW3}}^*(Z)$ are defined as the algebraic cycles that exist relatively over the moduli space \mathcal{M}_{EPW6} , resp. \mathcal{M}_{EPW3} , modulo algebraic equivalence (cf. Definition 6.1 below). This adds to the existing results of families of HyperKähler varieties satisfying the Franchetta property [16], [17].

Conventions. In this note, the word *variety* will refer to a reduced irreducible scheme of finite type over \mathbb{C} . A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

All cycle class groups will be with rational coefficients: we denote by $\text{CH}_j(Y)$ (resp. $B_j(Y)$) the Chow group of j -dimensional cycles on Y with \mathbb{Q} -coefficients modulo rational equivalence (resp. algebraic equivalence); for Y smooth of dimension n the notations $\text{CH}_j(Y)$ and $\text{CH}^{n-j}(Y)$ (resp. $B_j(Y)$ and $B^{n-j}(Y)$) are used interchangeably. The notations $\text{CH}_{\text{hom}}^j(Y)$ and $\text{CH}_{\text{AJ}}^j(Y)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles. For a morphism $f: X \rightarrow Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f .

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [53], [45]) will be denoted \mathcal{M}_{rat} .

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2. GM VARIETIES

Definition 2.1 (Debarre–Kuznetsov [9]). A Gushel–Mukai variety is a variety X obtained as a smooth dimensionally transverse intersection

$$X = \text{CGr}(2, V_5) \cap \mathbb{P}(W) \cap Q \subset \mathbb{P}^{10},$$

where $\text{CGr}(2, V_5)$ is the cone over the Grassmannian (of 2-dimensional subspaces in a fixed 5-dimensional vector space V_5), and $\mathbb{P}(W)$ and Q are a linear subspace resp. a quadric.

A Gushel–Mukai variety is called *special* if $\mathbb{P}(W)$ contains the vertex of the cone $\text{CGr}(2, V_5)$, and *ordinary* if it is not special.

Remark 2.2. For simplicity, we have restricted to *smooth* Gushel–Mukai varieties (for the general set-up including singularities, cf. [9]). Gushel–Mukai varieties have dimension at most 6. Examples of Gushel–Mukai varieties are: Clifford-general curves of genus 6; Brill–Noether general polarized $K3$ surfaces of degree 10 (i.e. genus 6); smooth prime Fano threefolds of degree 10 (i.e. genus 6) and index 2.

There exists an intrinsic characterization of Gushel–Mukai varieties [9, Theorem 2.3].

In this note, we will mainly be interested in Gushel–Mukai sixfolds. Here, the following is known:

Proposition 2.3 (Debarre–Kuznetsov). *Let X be a Gushel–Mukai variety of dimension 6. Then*

- (1) X is rational;
- (2) X is special, and hence X is a double cover

$$X \xrightarrow{2:1} \text{Gr}(2, V_5)$$

branched along an ordinary Gushel–Mukai fivefold $X_0 := \text{Gr}(2, V_5) \cap Q_0$;

- (3) *the Hodge diamond of X is*

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & 0 & & 0 \\
 & & & & & 0 & 1 & & 0 \\
 & & & & 0 & 0 & 0 & & 0 \\
 & & & 0 & 0 & 0 & 2 & & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 0 & & 0 & 0 & 1 & 22 & 1 & & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
 & & & 0 & 0 & 2 & 0 & & 0 & 0 \\
 & & & & 0 & 0 & 0 & & 0 & 0 \\
 & & & & & 0 & 1 & & 0 & 0 \\
 & & & & & & 0 & & 0 & 0 \\
 & & & & & & & & & 1
 \end{array}$$

Proof. Point (1) is [9, Proposition 4.2]. Point (2) is obvious. Point (3) is [12, Corollary 4.5]. \square

It is also worth mentioning the semi-orthogonal decomposition of the derived category of coherent sheaves given by Kuznetsov and Perry in [29]. The map $X \rightarrow \text{Gr}(2, V_5)$ endows X with a rank 2 vector bundle \mathcal{U} .

Proposition 2.4. *If $n = \dim(X)$ then there is a semi-orthogonal decomposition*

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X(1)^*, \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^*(n-3) \rangle.$$

The subcategory \mathcal{A}_X is defined as the left orthogonal to the rest of the semi-orthogonal decomposition, and if $n = 6$ is a K3 category.

2.1. Lagrangian data, EPW sextics, etc. Let X be a smooth n -dimensional Gushel-Mukai variety. The variety X is an intersection of quadrics. Let us denote by V_6 the 6-dimensional vector space of quadrics in the ideal of X . Inside V_6 , we can also identify our space V_5 to the hyperplane given by the space of Plücker quadrics $\text{CGr}(2, V_5)$. We will call the *Plücker point* of X the point $p_X \in \mathbb{P}(V_6^*)$ that corresponds to the hyperplane $V_5 \subset V_6$. The wedge product on $\bigwedge^3 V_6$ induces a symplectic form with values in $\det(V_6)$. In particular, there exists a subspace $A_X \subset \bigwedge^3 V_6$, lagrangian w.r.t. this symplectic form, associated to X [10, Thm. 2.9 and 3.6]. The triple (V_6, V_5, A_X) is called a *Lagrangian data* for X . If $n \geq 3$ the Lagrangian space A_X has no purely decomposable vectors [10, Thm. 3.14], and the vector space $A_X \cap \bigwedge^3 V_5$ has dimension $5 - n$ (resp. $6 - n$) if X is ordinary (resp. special).

As it is explained in [10, Sect. 3], it is also possible to construct a Gushel-Mukai variety starting from a Lagrangian data (V_6, V_5, A) , whenever A has no decomposable vectors. More precisely, Debarre and Kuznetsov, generalizing a construction due to Iliev and Manivel [25] show that, up to isomorphism, there is a unique ordinary (resp. special) Gushel-Mukai variety with associated Lagrangian data (V_6, V_5, A) of dimension $5 - t$ (resp. $6 - t$), with $l = \dim(A \cap \bigwedge^3 V_5)$.

Thanks to the work of O'Grady, [47], [48], [49], [50] one can also associate a HyperKähler manifold of dimension four to any Lagrangian space $A \subset \bigwedge^3 V_5$ with no decomposable vectors. More precisely, from the Lagrangian data (V_6, V_5, A) O'Grady constructs a normal integral sextic hypersurface $Y_A \subset \mathbb{P}(V_6)$. Then, the Lagrangian data defines some sublocus $Y_A^{\geq 2}$ inside Y_A , and the double cover \tilde{Y}_A of Y_A ramified along $Y_A^{\geq 2}$ is a smooth HyperKähler fourfold, named *double EPW sextic*. Double EPW sextics form a locally complete, 20-dimensional family of HyperKähler fourfolds of K3^2 type (i.e. deformation equivalent to the Hilbert scheme of length 2, dimension 0, subschemes of a K3 surface).

2.2. Generalized duals, generalized period partners.

Definition 2.5. Let X_1 and X_2 be two smooth GM varieties. X_1 and X_2 are called period partners (resp. generalized period partners) if $\dim(X_1) = \dim(X_2)$ (resp. $\dim(X_1) \equiv \dim(X_2) \pmod{2}$) and there exists an isomorphism $\varphi : V_6(X_1) \rightarrow V_6(X_2)$ between the 6-dimensional parts of the Lagrangian datas, such that $(\bigwedge^3 \varphi)(A_{X_1}) = A_{X_2}$.

Smooth GM varieties of the same dimension are period partners if and only if they lie in the same fiber of a certain period map defined on the moduli space of GM varieties. There exists also a notion of duality for GM varieties, which is similar to the notion of period partnership.

Definition 2.6. Two smooth GM varieties X_1 and X_2 are called dual (resp. generalized dual) if $\dim(X_1) = \dim(X_2)$ (resp. $\dim(X_1) \equiv \dim(X_2) \pmod{2}$) and there exists an isomorphism $\varphi : V_6(X_1) \rightarrow V_6(X_2)^*$ such that $(\bigwedge^3 \varphi)(A(X_1)) = A(X_2)^\perp$.

Generalized period partners and generalized duals are related on the level of derived categories:

Theorem 2.7. (*Kuznetsov–Perry* [30]) *Let X_1 and X_2 be GM varieties such that $A(X_1)$ and $A(X_2)$ contain no decomposable vectors. If X_1 and X_2 are generalized period partners or generalized duals there is an equivalence of categories*

$$\mathcal{A}_{X_1} \xrightarrow{\sim} \mathcal{A}_{X_2} .$$

Proof. This was conjectured in [29, Conjecture 3.7], and is proven in [30, Corollary 6.5]. \square

2.3. Abel–Jacobi isomorphism. We finish this section by recalling an Abel–Jacobi result from [10], of which we will make essential use in the body of the paper.

Let X be a general GM sixfold. Call $F_2^\sigma(X)$ the smooth, 4-dimensional subvariety of the affine Grassmannian $G(3, 11)$, parametrizing projective σ -planes (as defined in [10, Sect. 4.1]) contained in X . We will often simply denote it as F . The universal σ -plane $\mathcal{L}_2^\sigma(X) := \{(x, P) | x \in P\} \subset X \times G(3, 11)$ over F gives rise to an incidence correspondence. Moreover, as shown in [10, Section 5.2], F has a structure of \mathbb{P}^1 -bundle over a threefold \tilde{Y}_{A, V_5} , which is in turn a divisor inside the associated EPW double sextic \tilde{Y}_A . This gives rise to a diagram like the following:

$$\begin{array}{ccccc} & \mathcal{L}_2^\sigma(X) & & & \\ & \swarrow q & & \searrow p & \\ X & & & & F \xrightarrow{\tilde{\sigma}} \tilde{Y}_{A, V_5} \hookrightarrow \tilde{Y}_A \end{array}$$

In view of this diagram, we define a correspondence Γ_0 from X to the double EPW sextic \tilde{Y}_A as

$$\Gamma_0 := \Gamma_\iota \circ \Psi_{\tilde{\sigma}} \circ \Gamma_p \circ {}^t\Gamma_q \in \text{CH}^4(X \times \tilde{Y}_A) ,$$

where $\Psi_{\tilde{\sigma}}$ denotes the correspondence which acts as intersection with the relatively ample line bundle of the \mathbb{P}^1 -bundle $\tilde{\sigma}$ followed by push-forward along $\tilde{\sigma}$.

Proposition 2.8. [10, Thm. 5.19] *Let X be a general GM sixfold. The correspondence Γ_0 induces an isomorphism:*

$$(\Gamma_0)_* : H^6(X, \mathbb{Z})_0 \xrightarrow{\sim} H^2(\tilde{Y}_A, \mathbb{Z})_0 ,$$

where $H^*(\)_0$ denotes primitive cohomology. This isomorphism is compatible with the Beauville–Bogomolov form on $H^2(\tilde{Y}_A, \mathbb{Q})$.

Proof. This is [10, Theorem 5.19]. Note that for X general the assumption (10) of loc. cit. will be satisfied. Note also that the vanishing cohomology $H^6(X, \mathbb{Z})_{00}$ of loc. cit. coincides with the primitive cohomology $H^6(X, \mathbb{Z})_0$ in view of [10, Lemma 3.8]. \square

3. MCK DECOMPOSITION

Definition 3.1 (Murre [44]). Let X be a smooth projective n -dimensional variety. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi^0 + \pi_X^1 + \cdots + \pi^{2n} \quad \text{in } \text{CH}^n(X \times X) ,$$

such that the cycles π^i are mutually orthogonal idempotents and π^i acts on cohomology as a projector on $H^i(X, \mathbb{Q})$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 3.2. According to Murre’s conjectures [44], [26], any smooth projective variety should have a CK decomposition.

Definition 3.3 (Shen–Vial [54]). Let X be a smooth projective variety of dimension n . Let $\Delta_X^{sm} \in \text{CH}^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X .$$

An *MCK decomposition* is a CK decomposition $\{\pi^i\}$ of X that is *multiplicative*. This means that it satisfies

$$(3.4) \quad \pi^k \circ \Delta_X^{sm} \circ (\pi^i \times \pi^j) = 0 \quad \text{in } \text{CH}^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k .$$

Here $\pi^i \times \pi^j$ is by definition $(p_{13})^*(\pi^i) \cdot (p_{24})^*(\pi^j) \in \text{CH}^{2n}(X^4)$, where $p_{rs}: X^4 \rightarrow X^2$ is the projection on r th and s th factors.

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

Remark 3.5. We observe that the vanishing (3.4) is always true modulo homological equivalence. In fact, this is due to the fact the cup product in cohomology respects the grading.

The small diagonal (considered as a correspondence from $X \times X$ to X) induces the following *multiplication morphism*

$$\Delta_X^{sm}: h(X) \otimes h(X) \rightarrow h(X) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

Suppose now that X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\text{rat}} .$$

This decomposition is by definition multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \rightarrow h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

Suppose X has an MCK decomposition. Then, by setting

$$\text{CH}_{(j)}^i(X) := (\pi^{2i-j})_* \text{CH}^i(X) ,$$

one obtains a bigraded ring structure on the Chow ring. That is, the intersection product sends $\text{CH}_{(j)}^i(X) \otimes \text{CH}_{(j')}^{i'}(X)$ to $\text{CH}_{(j+j')}^{i+i'}(X)$.

It is reasonable to expect that for any X with an MCK decomposition, one has

$$\mathrm{CH}_{(j)}^i(X) \stackrel{??}{=} 0 \quad \text{for } j < 0, \quad \mathrm{CH}_{(0)}^i(X) \cap \mathrm{CH}_{\mathrm{hom}}^i(X) \stackrel{??}{=} 0.$$

This is strictly related to Murre's conjectures B and D, that were formulated for any CK decomposition [44].

Having an MCK decomposition is a severely restrictive property, and it is closely related to Beauville's "splitting property" conjecture [2]. Let us give a short list of examples: hyperelliptic curves have an MCK decomposition [54, Example 8.16], but the very general curve of genus ≥ 3 does not [18, Example 2.3]. In dimension two, a smooth quartic in \mathbb{P}^3 has an MCK decomposition, but a very general surface of degree ≥ 7 in \mathbb{P}^3 should not have one [18, Proposition 3.4]. For a discussion in greater detail, and further examples of varieties with an MCK decomposition, the reader may check [54, Section 8], as well as [58], [55], [19], [32], [33], [34], [35], [36], [37], [38], [39], [40], [18], [41], [42], [43].

4. FRANCHETTA PROPERTY

Definition 4.1. Let $\mathcal{Y} \rightarrow \mathcal{B}$ be a smooth projective morphism, where \mathcal{Y}, \mathcal{B} are smooth quasi-projective varieties. We say that $\mathcal{Y} \rightarrow \mathcal{B}$ has the *Franchetta property in codimension j* if the following holds: for every $\Gamma \in \mathrm{CH}^j(\mathcal{Y})$ such that the restriction $\Gamma|_{Y_b}$ is homologically trivial for all $b \in \mathcal{B}$, the restriction $\Gamma|_b$ is zero in $\mathrm{CH}^j(Y_b)$ for all $b \in \mathcal{B}$.

We say that $\mathcal{Y} \rightarrow \mathcal{B}$ has the *Franchetta property* if $\mathcal{Y} \rightarrow \mathcal{B}$ has the Franchetta property in codimension j for all j .

This property is studied in [4], [16], [17].

Definition 4.2. Given a family $\mathcal{Y} \rightarrow \mathcal{B}$ as above, with $Y := Y_b$ a fiber, we write

$$\mathrm{GDCH}_{\mathcal{B}}^j(Y) := \mathrm{Im}\left(\mathrm{CH}^j(\mathcal{Y}) \rightarrow \mathrm{CH}^j(Y)\right)$$

for the subgroup of *generically defined cycles*. In a context where it is clear to which family we are referring, the index \mathcal{B} will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that $\mathrm{GDCH}_{\mathcal{B}}^*(Y)$ injects into cohomology, under the cycle class map.

4.1. Franchetta for X .

Definition 4.3. Let

$$\bar{\mathcal{B}} := \mathbb{P}H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2))$$

denote the linear system of quadric sections of the cone \mathcal{C} over the Grassmannian $\mathrm{Gr}(2, 5)$, and let $\mathcal{B} \subset \bar{\mathcal{B}}$ denote the Zariski open subset parametrizing smooth GM sixfolds. We will denote by $\mathcal{X} \rightarrow \mathcal{B}$ the universal family of GM varieties over \mathcal{B} .

Proposition 4.4. *Let $\mathcal{X} \rightarrow \mathcal{B}$ be the universal family of GM sixfolds, as in Definition 4.3. The Franchetta property holds for $\mathcal{X} \rightarrow \mathcal{B}$, that is*

$$\mathrm{GDCH}_{\mathcal{B}}^*(X) \hookrightarrow H^*(X, \mathbb{Q}).$$

Proof. Let $\bar{\mathcal{X}} \rightarrow \bar{\mathcal{B}}$ denote the universal family of all (possibly singular and degenerate) sections. We observe that the line bundle $\mathcal{O}_{\mathcal{C}}(2)$ is base point free, and so

$\bar{\mathcal{X}} \rightarrow \mathcal{C}$ has the structure of a projective bundle. Now using the projective bundle formula (and reasoning as in [51] and [16]), this implies that

$$\mathrm{GDCH}_{\mathcal{B}}^*(X) = \mathrm{Im}\left(\mathrm{CH}^*(\mathcal{C}) \rightarrow \mathrm{CH}^*(X)\right),$$

for any GM sixfold. But X being smooth by definition, X must be contained in the punctured cone $\mathcal{C}^0 := \mathcal{C} \setminus \{\text{summit}\}$, and so the above implies

$$\mathrm{GDCH}_{\mathcal{B}}^*(X) = \mathrm{Im}\left(\mathrm{CH}^*(\mathcal{C}^0) \rightarrow \mathrm{CH}^*(X)\right).$$

The punctured cone \mathcal{C}^0 is an \mathbb{A}^1 -fibration over the Grassmannian $\mathrm{Gr}(2, 5)$, and so the above boils down to

$$\mathrm{GDCH}_{\mathcal{B}}^*(X) = \mathrm{Im}\left(\mathrm{CH}^*(\mathrm{Gr}(2, 5)) \rightarrow \mathrm{CH}^*(X)\right).$$

The proposition is now implied by the following lemma:

Lemma 4.5. *Let $r_X : \mathrm{CH}^*(\mathrm{Gr}(2, 5)) \rightarrow \mathrm{CH}^*(X)$ be the composition of pullback $\mathrm{CH}^*(\mathrm{Gr}(2, 5)) \rightarrow \mathrm{CH}^*(\mathcal{C})$ and restriction $\mathrm{CH}^*(\mathcal{C}) \rightarrow \mathrm{CH}^*(X)$. There is an injection into cohomology*

$$\mathrm{Im}(r_X) \hookrightarrow H^*(X, \mathbb{Q}).$$

To prove the lemma, we note that the Grassmannian has trivial Chow groups, i.e. $\mathrm{CH}^*(\mathrm{Gr}(2, 5)) \cong H^*(\mathrm{Gr}(2, 5), \mathbb{Q})$. Moreover, the natural map

$$H^i(\mathrm{Gr}(2, 5), \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q})$$

is bijective for $i < 6$, and injective for $i = 6$ [10, Proposition 3.4(b)]. The remaining cases $i > 6$ now readily follow by Poincaré duality. \square

4.2. Franchetta for $X \times X$ in codimension 6.

Proposition 4.6. *Let $\mathcal{X} \rightarrow \mathcal{B}$ be the universal family of GM sixfolds, as in Definition 4.3. The Franchetta property holds for $X \times X$ in codimension 6, that is*

$$\mathrm{GDCH}_{\mathcal{B}}^6(X \times X) \hookrightarrow H^{12}(X \times X, \mathbb{Q}).$$

Proof. Let $\bar{\mathcal{X}} \subset \bar{\mathcal{X}}$ be the Zariski closure, as in the proof of Proposition 4.4, and let us consider the fiber product $\bar{\mathcal{X}} \times_{\bar{\mathcal{B}}} \bar{\mathcal{X}}$. This comes naturally embedded in a diagram as follows

$$\begin{array}{ccc} \bar{\mathcal{X}} \times_{\bar{\mathcal{B}}} \bar{\mathcal{X}} & \xrightarrow{\sigma} & \mathcal{C} \times \mathcal{C} \\ \downarrow & & \\ \bar{\mathcal{B}} & & \end{array}$$

Let $\Delta_{\mathcal{C}} \subset \mathcal{C} \times \mathcal{C}$ denote the diagonal. The line bundle $\mathcal{O}_{\mathcal{C}}(2)$ is very ample and so separates 2 different points on \mathcal{C} ; this means that condition $(*_2)$ of [17, Definition 2.5] (cf. also [16, Definition 5.6]) is verified. Thus, the morphism σ displays $\bar{\mathcal{X}} \times_{\bar{\mathcal{B}}} \bar{\mathcal{X}}$ as a \mathbb{P}^r -bundle on $\mathcal{C} \times \mathcal{C}/\Delta_{\mathcal{C}}$, whereas it is a \mathbb{P}^{r+s} over $\Delta_{\mathcal{C}}$. Now, the *stratified projective bundle argument* [17, Proposition 2.6] implies there is an equality of subgroups:

$$(4.7) \quad \mathrm{GDCH}_{\mathcal{B}}^*(X \times X) = \langle \mathrm{CH}^*(\mathcal{C} \times \mathcal{C}), \Delta_X \rangle.$$

In particular, in codimension 6, we have

$$(4.8) \quad \text{GDCH}_{\mathcal{B}}^6(X \times X) = \mathbb{Q}[\Delta_X] + \text{CH}^*(\mathcal{C}) \otimes \text{CH}^*(\mathcal{C}).$$

Recall from the proof of Proposition 4.4 that X avoids the summit of the cone \mathcal{C} and so the image of $\text{CH}^*(\mathcal{C}) \rightarrow \text{CH}^*(X)$ coincides with the image of $r_X: \text{CH}^*(\text{Gr}(2,5)) \rightarrow \text{CH}^*(X)$. Thus, equality 4.8 boils down to

$$\text{GDCH}_{\mathcal{B}}^6(X \times X) = \mathbb{Q}[\Delta_X] + \text{Im } r_X \otimes \text{Im } r_X .$$

Now Lemma 4.5 (combined with the Künneth isomorphism in cohomology) implies that there is also an injection

$$\text{Im } r_X \otimes \text{Im } r_X \hookrightarrow H^*(X \times X, \mathbb{Q}).$$

However, the diagonal Δ_X is linearly independent of $\text{Im } r_X \otimes \text{Im } r_X$ inside $H^*(X \times X, \mathbb{Q})$. In fact, $H^{4,2}(X, \mathbb{Q})$ contains transcendental cycles on which Δ_X acts of course as the identity, whereas an easy argument show that decomposable correspondences $\text{CH}^*(X) \otimes \text{CH}^*(X)$ act as 0 on them. Hence we conclude. \square

5. VANISHING RESULTS FOR THE CHOW RING

Let CH_i^{hom} (resp. CH_{hom}^i) denote the group of dimension i (resp. codimension i) homologically trivial algebraic cycles. The first result of this paper concerns the vanishing of some of these groups for GM sixfolds.

Theorem 5.1. *Let $X \subset \mathbb{P}^{10}$ be a GM sixfold, then we have*

$$\text{CH}_0^{\text{hom}}(X) = 0; \quad \text{CH}_1^{\text{hom}}(X) = 0.$$

Proof. (of Theorem 5.1) First, we will prove the claimed vanishings for general GM sixfolds. In a second step, a classical spread argument will allow us to extend this to all GM sixfolds.

Let $h \in \text{CH}^1(\tilde{Y}_A)$ denote an ample class. We are going to use the Abel–Jacobi isomorphism of Proposition 2.8. By well-known properties of the Beauville–Bogomolov form, the compatibility in the statement of Proposition 2.8 amounts to the compatibility

$$(5.2) \quad \langle \alpha, \beta \rangle_X = \gamma \langle (\Gamma_0)_*(\alpha), h^2 \cdot (\Gamma_0)_*(\beta) \rangle_{\tilde{Y}_A}, \quad \gamma \in \mathbb{Q}^*$$

for $\alpha, \beta \in H^6(X, \mathbb{Q})_0$ (here $\langle \cdot, \cdot \rangle_M$ denotes cup product on the variety M). By taking a general codimension 2 linear section $g: S \hookrightarrow \tilde{Y}_A$ we get a map

$$g^*: H_{\text{tr}}^2(\tilde{Y}_A, \mathbb{Q}) \rightarrow H_{\text{tr}}^2(S, \mathbb{Q}),$$

which is an injection by the Lefschetz hyperplane Theorem. Of course S is a smooth surface. Denoting by Γ' the composition

$$\Gamma' := \Gamma_g \circ \Gamma_0 \in \text{CH}^4(X \times S),$$

we thus find that there is an injection

$$(5.3) \quad (\Gamma')_*: H_{\text{tr}}^6(X, \mathbb{Q}) \hookrightarrow H^2(S, \mathbb{Q})$$

which by (5.2) is compatible with cup-product. Applying [21, Lemma 2.9(ii)], this implies that the left-inverse to the injection (5.3) is given by a multiple of the transpose ${}^t\Gamma'$.

Observe that all the cohomology of X except for $H_{tr}^6(X, \mathbb{Q})$ is algebraic, and so the above gives an injection

$$\Gamma_*: H^*(X, \mathbb{Q}) \hookrightarrow H^2(S, \mathbb{Q}) \oplus \mathbb{Q}^r(*)$$

induced by some correspondence Γ (and with correspondence-induced left-inverse). In other words, we obtain a split injection of homological motives

$$(5.4) \quad \Gamma: h(X) \hookrightarrow h^2(S)(-2) \oplus \bigoplus_i \mathbb{L}(*) \text{ in } \mathcal{M}_{\text{hom}},$$

where the asterisk stands for some Tate twists. Let $\Delta_X \subset X \times X$ be the diagonal. Now, the injection (5.4) amounts to an equality of correspondences:

$$(5.5) \quad \Delta_X = c^t \Gamma \circ \Gamma \text{ in } H^{12}(X \times X, \mathbb{Q}), \quad c \in \mathbb{Q}^*.$$

Moreover, all the terms in equation (5.5) are generically defined over \mathcal{B} , hence by Proposition 4.6, the same holds on the level of Chow groups:

$$(5.6) \quad \Delta_X = c^t \Gamma \circ \Gamma \text{ in } \text{CH}^6(X \times X), \quad c \in \mathbb{Q}^*.$$

The upshot is that the injection (5.4) now holds also in \mathcal{M}_{rat} :

$$(5.7) \quad \Gamma: h(X) \hookrightarrow h^2(S)(-2) \oplus \bigoplus_i \mathbb{L}(*) \text{ in } \mathcal{M}_{\text{rat}}.$$

In turn, by taking Chow groups on both sides of Eq. (5.7), we obtain a split injection

$$(5.8) \quad \text{CH}_{\text{hom}}^j(X) \hookrightarrow \text{CH}^{j-2}(S)_{\text{hom}}, \quad \forall j.$$

As the right-hand side obviously vanishes for $j \geq 5$, this proves the required vanishings, for general GM sixfolds.

It remains to extend the theorem to *all* GM sixfolds. To this end, we remark that the vanishing

$$\text{CH}_0^{\text{hom}} = \text{CH}_1^{\text{hom}} = 0$$

is equivalent [31] to having a decomposition

$$(5.9) \quad \Delta_X = \gamma + \gamma' \text{ in } \text{CH}^6(X \times X),$$

where γ is a cycle supported on $W \times W$ for some codimension 2 closed subvariety $W \subset X$, and γ' is decomposable, *i.e.* $\gamma' \in \text{CH}^*(X) \otimes \text{CH}^*(X)$.

Let $\mathcal{X} \rightarrow \mathcal{B}$ be the universal family of GM sixfolds, as in Definition 4.3. By the above, we have thus obtained a decomposition 5.9 for $X = X_b$, where $b \in \mathcal{B}$ is general. The Hilbert schemes argument as detailed by Voisin [60, Proposition] allows to extend this to *all* GM sixfolds, *i.e.* the decomposition 5.9 actually holds for $X = X_b$ for all $b \in \mathcal{B}$. In view of the afore-mentioned equivalence, this proves the theorem. \square

Remark 5.10. Voisin proves in [61, Section 2] the triviality of certain Chow groups of the Debarre–Voisin 20fold (i.e., a Plücker hyperplane section of the Grassmannian $\mathrm{Gr}(3, 10)$), using her method of “spread” and the Abel–Jacobi isomorphism with the associated Debarre–Voisin HyperKähler fourfold. The proof of Theorem 5.1 given above is very close in spirit to Voisin’s argument; the Debarre–Voisin HyperKähler fourfold is replaced by the double EPW sextic, and Voisin’s “spread” argument is replaced by the Franchetta property Proposition 4.6.

We also remark that the analogous result $\mathrm{CH}_1^{\mathrm{hom}}(Y) = 0$ for GM *fivefolds* Y was proven in [38].

5.1. Some consequences.

Corollary 5.11. *Let X be a GM sixfold. Then*

$$\mathrm{CH}_{\mathrm{hom}}^i(X) = 0 \quad \forall i \neq 4 .$$

Moreover, $\mathrm{CH}^4(X)$ is generated by σ -planes.

Proof. The first statement follows from the vanishing $\mathrm{CH}_{\mathrm{hom}}^6(X) = \mathrm{CH}_{\mathrm{hom}}^5(X) = 0$ (Theorem 5.1) by the Bloch–Srinivas “decomposition of the diagonal” argument (cf. [31, Remark 1.8.1] for the precise result applied here).

For the second statement, we recall (cf. (5.8) above) that there is an injection

$$(\Gamma_0)_* : \mathrm{CH}_{\mathrm{hom}}^4(X) \hookrightarrow \mathrm{CH}_{\mathrm{hom}}^2(S) ,$$

with left-inverse induced by the transpose ${}^t\Gamma_0$, i.e.

$$({}^t\Gamma_0)_* : \mathrm{CH}_{\mathrm{hom}}^2(S) \rightarrow \mathrm{CH}_{\mathrm{hom}}^4(X)$$

is surjective. Unravelling the definition of the correspondence Γ_0 (cf. Subsection 2.3), this implies in particular that the universal σ -plane $\mathcal{L}_2^\sigma(X) \in \mathrm{CH}^4(X \times F)$ induces a surjection

$$({}^t\mathcal{L}_2^\sigma(X))_* : \mathrm{CH}_{\mathrm{hom}}^4(F) \rightarrow \mathrm{CH}_{\mathrm{hom}}^4(X) ,$$

i.e. $\mathrm{CH}_{\mathrm{hom}}^4(X)$ is generated by σ -planes. As $H^8(X, \mathbb{Q}) \cong \mathbb{Q}^2 \cong H^8(\mathrm{Gr}(2, 5), \mathbb{Q})$ is also generated by σ -planes, this proves the second statement. \square

Corollary 5.12. *Let X, X' be two equidimensional GM varieties, such that $A(X)$ and $A(X')$ do not contain decomposable vectors. Assume X and X' are period partners (resp. dual). Then there is an isomorphism of Chow motives*

$$h(X) \cong h(X') \quad \text{in } \mathcal{M}_{\mathrm{rat}} .$$

Proof. Recall from Proposition 2.4 that a GM variety X has a semi-orthogonal decomposition, where the only (possibly) non-exceptional object is the Kuznetsov component $Kuz(X) = \mathcal{A}_X$. Moreover, from Theorem 2.7 we know that period partners and dual GM varieties have equivalent Kuznetsov components of their semi-orthogonal decompositions.

The fact that $\mathcal{A}_X \cong \mathcal{A}_{X'}$ implies that there is an equivalence of categories $D^{\mathrm{b}}(X) \cong D^{\mathrm{b}}(X')$, induced by a Fourier–Mukai functor $\Phi_{\mathcal{E}}$, with kernel \mathcal{E} . Let $[\mathcal{E}] \in K_0(X \times X')$ be the alternate sum of cohomologies of \mathcal{E} . Then the FM equivalence induces a (a priori) non-graded isomorphism of Chow rings (see for example [5, Sect. 2.3]) via the following commutative diagram

$$\begin{array}{ccc}
D^{\mathbf{b}}(X) & \xrightarrow{\Phi_{\mathcal{E}}} & D^{\mathbf{b}}(X') \\
\downarrow & & \downarrow \\
\mathrm{CH}^*(X) & \xrightarrow{e} & \mathrm{CH}^*(X'),
\end{array}$$

where e is the correspondence obtained as $ch([\mathcal{E}] \cdot Td(X))$. Since this isomorphism is compatible with the cycle class map, we obtain in particular a (a priori non-graded) isomorphism

$$\mathrm{CH}_{hom}^*(X) \cong \mathrm{CH}_{hom}^*(X') .$$

But we know that X and X' have only one non-trivial Chow group (for GM varieties of dimension 6, this is Corollary 5.11, for GM varieties of dimension 5 this is proven in [38], and in smaller dimensions this is trivially true), and so there actually is a graded isomorphism

$$\mathrm{CH}_{hom}^i(X) \cong \mathrm{CH}_{hom}^i(X') .$$

Since X and X' also have isomorphic cohomology, it follows that there exists a graded and correspondence-induced isomorphism

$$\mathrm{CH}^*(X) \cong \mathrm{CH}^*(X') .$$

Applying [23, Lemma 1.1], this concludes the proof. \square

Remark 5.13. (1) If we do not assume that X and X' have the same dimension then we obtain the following relation

$$h(X) \oplus \mathbb{L}(\ast) \cong h(X') \oplus \mathbb{L}(\ast) \quad \text{in } \mathcal{M}_{\mathrm{rat}} ,$$

but the proof is more involved.

- (2) It would be interesting to prove Corollary 5.12 directly, without appealing to the deep and difficult derived categories result. We have not been able to do so.

6. MCK FOR X

The goal of this section is to prove that all GM sixfolds have a multiplicative Chow-Künneth decomposition *modulo algebraic equivalence*. As before, we write $\mathcal{X} \rightarrow \mathcal{B}$ for the universal family of GM sixfolds. We denote $B^*(X)$ the group of algebraic cycles (with \mathbb{Q} -coefficients) modulo algebraic equivalence. The following is the obvious adaptation of Definition 4.2 to algebraic equivalence:

Definition 6.1. Given a family $\mathcal{Y} \rightarrow \mathcal{B}$ as in Definition 4.2, with $Y := Y_b$ a fiber, we write

$$\mathrm{GDB}_{\mathcal{B}}^j(Y) := \mathrm{Im}\left(B^j(\mathcal{Y}) \rightarrow B^j(Y)\right)$$

for the subgroup of *generically defined cycles*. In a context where it is clear to which family we are referring, the index \mathcal{B} will often be suppressed from the notation.

As a preparatory result, we first establish a “Franchettina property” for the square of a GM sixfold:

Proposition 6.2. *Let X be a GM sixfold. The cycle class map induces a split injection*

$$\mathrm{GDB}_{\mathbb{B}}^*(X \times X) \hookrightarrow H^*(X \times X, \mathbb{Q}).$$

Proof. Recall that we have already proven (Equality 4.7) that

$$\begin{aligned} \mathrm{GDCH}_{\mathbb{B}}^*(X \times X) &= \langle \mathrm{Im} r_X \otimes \mathrm{Im} r_X, \Delta_X \rangle \\ &= \langle c_1, c_2, \Delta_X \rangle. \end{aligned}$$

Here $c_1 \in \mathrm{CH}^1(X)$, $c_2 \in \mathrm{CH}^2(X)$ denote the image under r_X of the first (resp, second) Chern class of the tautological bundle Q on the Grassmannian $\mathrm{Gr}(2, 5)$, and we use the well-known fact that $\mathrm{CH}^*(\mathrm{Gr}(2, 5))$ is generated as a \mathbb{Q} -algebra by the first and second Chern classes of Q [14].

It is not hard to see that the intersection product $\Delta_X \cdot c_1$ is contained in $\mathrm{Im} r_X \otimes \mathrm{Im} r_X$. Indeed, the excess intersection formula [22, Theorem 6.3], applied to the inclusion morphism of X into the punctured cone \mathcal{C}^0 , gives that

$$\Delta_X \cdot c_1 \in \mathrm{Im} \left(\mathrm{CH}^*(\mathcal{C}^0 \times \mathcal{C}^0) \rightarrow \mathrm{CH}^*(X \times X) \right) = \mathrm{Im} r_X \otimes \mathrm{Im} r_X,$$

cf. [17, Equation (13)].

It remains to show that $\Delta_X \cdot c_2$ is also contained in $\mathrm{Im} r_X \otimes \mathrm{Im} r_X$, modulo algebraic equivalence. Once this is proven, we find an equality of \mathbb{Q} -algebras

$$\mathrm{GDB}_{\mathbb{B}}^*(X \times X) = \mathrm{Im} r_X \otimes \mathrm{Im} r_X \oplus \mathbb{Q}[\Delta_X],$$

and so the combination of Lemma 4.5 and Proposition 4.6 implies that $\mathrm{GDB}_{\mathbb{B}}^*(X \times X)$ injects into cohomology.

We first observe that $\Delta_X \cdot c_2$ is contained in $\mathrm{Im} r_X \otimes \mathrm{Im} r_X$, modulo *homological* equivalence. Indeed, the correspondence $\Delta_X \cdot c_2$ acts on $H^*(X, \mathbb{Q})$ as cupping with c_2 . As the primitive cohomology of X is concentrated in degree 6, the correspondence $\Delta_X \cdot c_2$ acts as 0 on the primitive cohomology. The algebraic part of $H^*(X, \mathbb{Q})$ can be expressed in terms of c_1 and c_2 , and so there exists some $p \in \mathrm{Im} r_X \otimes \mathrm{Im} r_X$ such that p and $\Delta_X \cdot c_2$ act in the same way on $H^*(X, \mathbb{Q})$. Manin's identity principle (plus the Künneth formula in cohomology) then implies that

$$\Delta_X \cdot c_2 = p \text{ in } H^{16}(X \times X, \mathbb{Q}).$$

Now we want to upgrade to algebraic equivalence. As we have seen before, there is an injection

$$B^8(X \times X) \hookrightarrow B^4(S \times S) \oplus \mathbb{Q}^r,$$

where S is the surface of Theorem 5.1. Passing to $B_{hom}^8(X \times X)$ a very similar inclusion holds:

$$B_{hom}^8(X \times X) \hookrightarrow B_{hom}^4(S \times S).$$

But $B_{hom}^4(S \times S) = 0$ (homological and algebraic equivalence coincide for zero-cycles), and so $B_{hom}^8(X \times X) = 0$. This proves that $\Delta_X \cdot c_2$ lies in $\mathrm{Im} r_X \otimes \mathrm{Im} r_X$ modulo algebraic equivalence, as claimed. This ends the proof. \square

We are now in position to prove the main result of this section.

Theorem 6.3. *Let X be a smooth GM sixfold, then X admits an MCK decomposition modulo algebraic equivalence.*

Proof. Set $i < 6$ a positive integer, and let $\pi^i \in \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(X)$ be a projector on $H^i(X, \mathbb{Q})$ that is decomposable (i.e. $\pi^i \in \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(X)$). We define $\pi^i = {}^t\pi^{12-i}$ for $i > 6$ and $\pi^6 = \Delta_X - \sum_{i \neq 6} \pi^i$. Then we have a CK decomposition

$$h(X) = h^0(X) \oplus \cdots \oplus h^{12},$$

with the property that

$$(6.4) \quad h^i(X) = (X, \pi^i, 0) = \bigoplus \mathbf{L}(*), \quad \forall i \neq 6.$$

Now we define a submotive of $h^6(X)$ as follows. We will call *tautological cohomology* the subgroup $H_{\mathrm{taut}}^6(X, \mathbb{Q})$ of $H^6(X, \mathbb{Q})$ obtained as the image of the natural map $H^6(\mathcal{C}^0, \mathbb{Q}) \rightarrow H^6(X, \mathbb{Q})$. We will also denote *primitive cohomology* the complement $H_{\mathrm{prim}}^6(X, \mathbb{Q})$ of $H_{\mathrm{taut}}^6(X, \mathbb{Q})$ inside $H^6(X, \mathbb{Q})$. As before, we can define a projector π_{taut}^6 on $H_{\mathrm{taut}}^6(X, \mathbb{Q})$ that is decomposable, and so

$$(6.5) \quad h_{\mathrm{taut}}^6(X) = (X, \pi_{\mathrm{taut}}^6, 0) = \bigoplus \mathbf{L}(*).$$

Finally, we define a projector π_{prim}^6 on $H_{\mathrm{prim}}^6(X, \mathbb{Q})$ by setting $\pi_6 = \pi_{\mathrm{taut}}^6 \oplus \pi_{\mathrm{prim}}^6$.

Let us now prove that the CK decomposition $\{\pi^i\}$ is MCK, modulo algebraic equivalence. Let us first consider a triple $\{\pi^i, \pi^j, \pi^k\} \neq \{\pi_{\mathrm{prim}}^6, \pi_{\mathrm{prim}}^6, \pi_{\mathrm{prim}}^6\}^1$. We have

$$\begin{aligned} \Gamma_{ijk} &:= \pi^k \circ \Delta_X^{\mathrm{sm}} \circ (\pi^i \times \pi^j) \\ &= ({}^t\pi^i \times {}^t\pi^j \times \pi^k)_* \Delta_X^{\mathrm{sm}} \\ &= (\pi^{12-i} \times \pi^{12-j} \times \pi^k)_* \Delta_X^{\mathrm{sm}} \in B^*(h^i \otimes h^j \otimes h^k), \end{aligned}$$

where the second equality is thanks to Lieberman's lemma [45, Lemma 2.1.3]. We remark moreover that this cycle Γ_{ijk} is generically defined (with respect to \mathcal{B}), i.e. it lies in $\mathrm{GDB}_{\mathcal{B}}^*(X \times X \times X)$.

In view of (6.4) and (6.5), we can now write

$$\Gamma_{ijk} \in \bigoplus B^*(X \times X),$$

and since the cycle Γ_{ijk} is generically defined it actually lies in $\bigoplus \mathrm{GDB}_{\mathcal{B}}^*(X \times X)$. But the cycle Γ_{ijk} is homologically trivial (Remark 3.5), and so an application of the Franchettina property (Proposition 6.2) gives the vanishing

$$\Gamma_{ijk} = 0 \text{ in } \bigoplus B^*(X \times X).$$

It only remains to treat the ‘‘correspondence of the beast’’

$$\Gamma_{666} := \pi_{\mathrm{prim}}^6 \circ \Delta_X^{\mathrm{sm}} \circ (\pi_{\mathrm{prim}}^6 \times \pi_{\mathrm{prim}}^6) \in B^{12}(X \times X \times X).$$

To this end, we consider the involution $\sigma \in \mathrm{Aut}(X)$ coming from the double cover $X \rightarrow \mathrm{Gr}(2, 5)$. This involution gives rise to a splitting of the motive of X

$$h(X) = h(X)^+ \oplus h(X)^- \text{ in } \mathcal{M}_{\mathrm{rat}},$$

where $h(X)^+$ (resp. $h(X)^-$) is defined by the projector $1/2(\Delta_X + \Gamma_\sigma)$ (resp. the projector $1/2(\Delta_X - \Gamma_\sigma)$). It is readily seen that there is equality

$$h(X)^- = h_{\mathrm{tr}}^6(X) := (X, \pi_{\mathrm{tr}}^6, 0) \text{ in } \mathcal{M}_{\mathrm{rat}}$$

¹Remark the highly non-incidental presence of the number of the beast here

(indeed, the double cover $X \rightarrow \text{Gr}(2, 5)$ induces isomorphisms $h(X)^+ \cong h(\text{Gr}(2, 5)) \cong \sum_{j \neq 6} h^j(X) \oplus h_{\text{alg}}^6(X)$). On the other hand, the intersection product map

$$(\Delta_X^{sm})_*: h(X)^- \otimes h(X)^- \rightarrow h(X) \text{ in } \mathcal{M}$$

factors over $h(X)^+$, and so the map

$$h(X)^- \otimes h(X)^- \xrightarrow{\Delta_X^{sm}} h(X) \rightarrow h(X)^- \text{ in } \mathcal{M}$$

is zero. This is equivalent to the vanishing

$$\pi_{\text{prim}}^6 \circ \Delta_X^{sm} \circ (\pi_{\text{prim}}^6 \times \pi_{\text{prim}}^6) = 0 \text{ in } \text{CH}^{12}(X \times X \times X),$$

and so a fortiori the correspondence of the beast vanishes modulo algebraic equivalence:

$$\Gamma_{666} = 0 \text{ in } B^{12}(X \times X \times X).$$

This closes the proof. \square

Remark 6.6. For GM *fivefolds* a stronger result is true: these varieties have an MCK decomposition *modulo rational equivalence* [38]. It seems likely that GM varieties (of any dimension) admit an MCK decomposition (modulo rational equivalence); we have not been able to prove this.

Inspection of the above proof gives the following: to prove that GM sixfolds have an MCK decomposition *modulo rational equivalence*, it would suffice to prove that for a very general GM sixfold X one has

$$\Delta_X \cdot c_2 \in \text{Im } r_X \otimes \text{Im } r_X \subset \text{CH}^*(X \times X),$$

where c_2 and r_X are as above. We have proven this modulo algebraic equivalence, but we don't know how to upgrade to rational equivalence.

7. THE TAUTOLOGICAL RING

Definition 7.1. Let X be a GM sixfold, and $m \in \mathbb{N}$. We define the *tautological ring* as the \mathbb{Q} -subalgebra

$$\begin{aligned} R^*(X^m) &:= \left\langle (p_i)^* \text{Im}(B^*(\text{Gr}(2, 5)) \rightarrow B^*(X)), (p_{ij})^* \Delta_X \right\rangle \\ &= \left\langle (p_i)^*(c_1), (p_i)^*(c_2), (p_{ij})^*(\Delta_X) \right\rangle \\ &= \left\langle (p_i)^* B^1(X), (p_i)^* B^2(X), (p_{ij})^* \Delta_X \right\rangle \subset B^*(X). \end{aligned}$$

(Here $p_i : X^m \rightarrow X$ and $p_{ij} : X^m \rightarrow X^2$ denote the various projection morphisms from X^m to X resp. to X^2 .)

This tautological ring is similar to tautological subrings of the Chow ring that have been studied for hyperelliptic curves in [56], [57], for K3 surfaces in [59], [63], and for cubic hypersurfaces in [17].

Theorem 7.2. *Let X be a GM sixfold. The cycle class map induces injections*

$$R^*(X^m) \hookrightarrow H^*(X^m, \mathbb{Q})$$

for all $m \leq 45$.

Moreover, the cycle class map induces injections

$$R^*(X^m) \hookrightarrow H^*(X^m, \mathbb{Q})$$

for all m if and only if X is Kimura finite-dimensional [28].

Proof. First, let us introduce some notations. Let $h \in B^1(X)$ be the hyperplane section class, and let $c \in B^2(X)$ denote the class induced by $c_2(Q) \in B^2(\text{Gr}(2, 5))$ where Q is the tautological bundle on the Grassmannian. Note that we have

$$B^1(X) = \text{CH}^1(X) = \mathbb{Q}h, \quad B^2(X) = \text{CH}^2(X) = \mathbb{Q}h^2 \oplus \mathbb{Q}c$$

(cf. Corollary 5.11). We define

$$o := \frac{1}{10}h^6 \in B^6(X)$$

to be the degree 1 zero-cycle proportional to $h^6 \in B_0(X)$. Note that $B_0(X) \cong \mathbb{Q}$, and so any point on X represents the class o . For ease of notation, let us write

$$\begin{aligned} o_i &:= p_i^*o \in B^6(X), \\ h_i &:= p_i^*h \in B^1(X), \\ c_i &:= p_i^*c \in B^2(X), \end{aligned}$$

where $p_i : X^m \rightarrow X$ is the projection on the i -th factor. Let us also write

$$\tau := \pi_{\text{prim}}^6 \in B^6(X \times X)$$

where π_{prim}^6 is the projector on the primitive cohomology introduced in the proof of Theorem 6.3, and $\tau_{i,j} := p_{i,j}^*\tau$, where $p_{i,j} : X^m \rightarrow X \times X$ is the projection on the product of the i -th and j -th factors.

To prove Theorem 7.2, we first determine (just as in [63, Lemma 2.3]) the relations on the level of cohomology among the cycles involved:

Proposition 7.3. *The \mathbb{Q} -subalgebra $\bar{R}^*(X^m)$ of the cohomology algebra $H^*(X^m, \mathbb{Q})$ generated by $o_i, h_i, c_i, \tau_{j,k}$, $1 \leq i \leq m$, $1 \leq j < k \leq m$, is isomorphic to the free graded \mathbb{Q} -algebra generated by $o_i, h_i, c_i, \tau_{j,k}$, modulo the following relations:*

$$(7.4) \quad h_i \cdot o_i = c_i \cdot o_i = 0, \quad c_i^4 = 0, \quad c_i^2 = \lambda c_i \cdot h_i^2 + \mu h_i^4, \quad h_i^6 = \nu c_i \cdot h_i^4 = 10 o_i;$$

$$(7.5) \quad \tau_{i,j} \cdot o_i = 0, \quad \tau_{i,j} \cdot h_i = \tau_{i,j} \cdot c_i = 0, \quad \tau_{i,j} \cdot \tau_{i,j} = b_{\text{prim}} o_i \cdot o_j;$$

$$(7.6) \quad \tau_{i,j} \cdot \tau_{i,k} = \tau_{j,k} \cdot o_i;$$

$$(7.7) \quad \sum_{\sigma \in \mathfrak{S}_{b_{\text{prim}}+1}} \text{sgn}(\sigma) \prod_{i=1}^{b_{\text{prim}}+1} \tau_{i, b_{\text{prim}}+1+\sigma(i)} = 0.$$

Here $\lambda, \mu, \nu \in \mathbb{Q}$ are some constants, $b_{\text{prim}} := \dim H^6(X, \mathbb{Q})_{\text{prim}} = 22$ is the rank of the primitive part of $H^*(X, \mathbb{Q})$, and \mathfrak{S}_b denotes the symmetric group on b elements acting by permutations on the factors.

Proof. First, let us check that the above relations hold in $H^*(X^m, \mathbb{Q})$. The relations (7.4) take place in X and are clear. The relations (7.5) take place in X^2 : the relations $\tau_{i,j} \cdot o_i = 0$ and $\tau_{i,j} \cdot h_i = 0$ follow directly from (7.5), while the relation $\tau_{i,j} \cdot \tau_{i,j} = b_{\text{prim}} o_i \cdot o_j$ follows directly from the general fact that $\deg(\Delta_X \cdot \Delta_X) = \chi(Y)$, the topological Euler characteristic of X . The relation (7.6) takes place in X^3 and follows from the relations (7.4) and (7.5) together with the fact that the cohomology algebra $H^*(X^m, \mathbb{Q})$ is graded. Finally, the relation (7.7) takes place in $X^{2(b_{\text{prim}}+1)}$ and expresses the fact that the $(b_{\text{prim}}+1)$ -th exterior power of $H_{\text{prim}}^*(Y)$ vanishes.

Second, in order to check that these relations generate the relations among $o_i, h_i, \tau_{j,k}$, $1 \leq i \leq m$, $1 \leq j < k \leq m$, it is sufficient to check (just as in [63, §3]) that the pairing

$$\overline{R}^i(X^m) \times \overline{R}^{mn-i}(X^m) \rightarrow \overline{R}^{mn}(X^m) \simeq \mathbb{Q} o_1 \cdot o_2 \cdots o_m$$

is non-degenerate. Now, the argument given in [63, §3] adapts *mutatis mutandis* to our setting (cf. [17, Proof of Lemma 2.12] for details on the *mutatis mutandis*). This proves the proposition. \square

Now, let us end the proof of Theorem 7.2. In view of Lemma 7.3, it suffices to establish relations (7.4), (7.5), (7.6) and (7.7) modulo algebraic equivalence. The relations (7.4) hold true in $R^*(X)$ thanks to the Franchetta property for X (Proposition 4.4): indeed, the relations (7.4) hold in cohomology and involve only generically defined cycles, and so Proposition 4.4 guarantees that these relations are true modulo rational equivalence. The relations (7.5) hold true in $R^*(X^2)$ thanks to the Franchettina property for $X \times X$ (Proposition 6.2). (Note that the relations (7.4) in $R^*(X^2)$ follow also more simply from the fact that $B^{12}(X \times X) = \mathbb{Q}$.) The relation (7.6) takes place in $R^*(X^3)$ and, given the relations (7.4) and (7.5) in $R^*(\cdot)$, it follows from the MCK decomposition. More precisely, Theorem 6.3 guarantees that we have equality

$$\Delta_X^{\text{sm}} \circ (\pi_{\text{prim}}^6 \times \pi_{\text{prim}}^6) = \pi^{12} \circ \Delta_X^{\text{sm}} \circ (\pi_{\text{prim}}^6 \times \pi_{\text{prim}}^6) \quad \text{in } B^{12}(X^3),$$

which (using Lieberman's lemma) translates into

$$(\pi_{\text{prim}}^6 \times \pi_{\text{prim}}^6 \times \Delta_X)_* \Delta_X^{\text{sm}} = (\pi_{\text{prim}}^6 \times \pi_{\text{prim}}^6 \times \pi^{12})_* \Delta_Y^{\text{sm}} \quad \text{in } B^{12}(X^3),$$

which implies that

$$\tau_{13} \cdot \tau_{23} = \tau_{12} \cdot o_3 \quad \text{in } B^{12}(X^3).$$

Finally, the relation (7.7), which takes place in $R^*(X^{2(b_{\text{prim}}+1)})$, holds if and only if the motive of X is Kimura–O'Sullivan finite-dimensional. \square

A Franchettina property holds also for X^3 , and - as we will see in the next section - for related HyperKähler varieties.

Proposition 7.8. *There is an injection*

$$\text{GDB}_{\mathcal{B}}^*(X^3) \hookrightarrow H^*(X^3, \mathbb{Q}).$$

Proof. Once again we use a “stratified projective bundle” argument. Let $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{B}}$ denote the universal family of all quadric sections of the cone \mathcal{C} (see Proposition 4.4). The argument is similar to that of Proposition 4.6. Condition $(*_3)$ from [17, Definition 2.5] holds for $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2))$, and we have a natural diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\overline{\mathcal{B}}} \mathcal{X} \times_{\overline{\mathcal{B}}} \mathcal{X} & \longrightarrow & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \\ \downarrow & & \\ \overline{\mathcal{B}} & & \end{array}$$

Let $\Delta_{\mathcal{C}}^{\text{sm}}$ denote the small diagonal in $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$. The horizontal map displays $\mathcal{X} \times_{\overline{\mathcal{B}}} \mathcal{X} \times_{\overline{\mathcal{B}}} \mathcal{X}$ as a stratified projective bundle over $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$, where the strata are given by the various partial diagonals.

Since $(*_3)$ holds true, we can apply [17, Proposition 2.6], and obtain the following equality

$$\mathrm{GDB}_{\mathbb{B}}^*(X^3) = \langle c_1, c_2, \Delta_X \rangle,$$

Here as usual we denote by c_1, c_2 the two Chern classes of the tautological bundle of the Grassmannian, and c_i, Δ_X are considered as elements of $B^i(X^3)$ resp. of $B^6(X^3)$ via all possible pullbacks under the projections $X^3 \rightarrow X$ resp. $X^3 \rightarrow X^2$. Then, in order to conclude it is enough to invoke Theorem 7.2. \square

8. FAMILIES OF HYPERKÄHLER VARIETIES

We recall the following conjecture, known as “generalized Franchetta conjecture”:

Conjecture 8.1. *Let \mathcal{M} be the moduli stack of a locally complete family of polarized HyperKähler varieties, and let $\mathcal{X} \rightarrow \mathcal{M}$ be the universal family (considered as a stack). Then $\mathcal{X} \rightarrow \mathcal{M}$ has the Franchetta property, i.e. there are injections*

$$\mathrm{GDCH}_{\mathcal{M}}^*(X) \hookrightarrow H^*(X, \mathbb{Q})$$

for any fiber X .

This conjecture is studied for K3 surfaces in [51], and for higher-dimensional HyperKähler varieties in [4], [16], [17]. Here, we obtain results for new families by restricting to *algebraic equivalence*.

Let us recall that, associated to a given Lagrangian datum, there exists another family of HyperKähler varieties, this time of dimension 6. These are dubbed *double EPW cubes*; for a complete definition of the complicated geometric construction of such objects see [24].

Corollary 8.2. (1) *Let Y be a very general double EPW sextic, and \mathcal{M}_{EPW6} the moduli space of such varieties. Then we have*

$$\mathrm{GDB}_{\mathcal{M}_{EPW6}}^*(Y) \hookrightarrow H^*(Y, \mathbb{Q});$$

(2) *let Z be a very general double EPW cube, and \mathcal{M}_{EPW3} the moduli space of such varieties. Then we have*

$$\mathrm{GDB}_{\mathcal{M}_{EPW3}}^*(Z) \hookrightarrow H^*(Z, \mathbb{Q}).$$

Proof. The argument is basically the same for EPW double sextics and EPW double cubes. The only difference consists in taking X^2 for sextics and X^3 for cubes. We will then develop just the second case in detail.

In [52, 5.4.2], the authors construct certain HyperKähler sixfolds (that have the same numerical invariants as double EPW cubes) as moduli spaces $\mathcal{M}_{\sigma, \nu}(Kuz(X))$ of stable objects inside the Kuznetsov component of GM sixfolds. They also observe that a very general double EPW cube would coincide with one of these sixfolds (hence endowing the very general cube with a moduli space structure), once it is proven that X and the associated double EPW cube have the same period point. In the forthcoming preprint [27, Section 5 and Remark 5.7], the authors prove that the very general double EPW cube has the same period point as the double EPW sextic associated to the same Lagrangian data, and [52, Prop. 5.17] shows that the sextic and the GM variety have the same period point. Hence we can assume that the very general double EPW cube is a moduli space of stable objects. This means

that, if \mathcal{B} denotes the parameter space of GM sixfolds and \mathcal{M}_{EPW3} the moduli space of double EPW cubes, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{B}} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{M}_{EPW3}. \end{array}$$

Here \mathcal{Z} stands for the universal family of double EPW cubes, and $\mathcal{Z}_{\mathcal{B}}$ denotes the base change. The map on the lower level of the diagram is given by taking the moduli space of stable objects with a given Mukai vector.

A second important consequence of the modular nature of double EPW cubes is that, by [17, Theorem 1.1], we have a split injection

$$(8.3) \quad h(Z) \hookrightarrow \bigoplus_i h(X^3)(*) \quad \text{in } \mathcal{M}_{rat},$$

that embeds the Chow motive of Z inside some copies of the motive of the triple self-product of the GM sixfold (up to some twists). Moreover, this split injection is generically defined (with respect to the parameter space \mathcal{B} as above). It follows that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{GDB}_{\mathcal{B}}^*(Z) & \hookrightarrow & \bigoplus_i \mathrm{GDB}_{\mathcal{B}}^*(X^3) \\ \downarrow & & \downarrow \\ H^*(Z, \mathbb{Q}) & \hookrightarrow & \bigoplus_i H^*(X^3, \mathbb{Q}). \end{array}$$

Since the right-hand side vertical map is injective by Proposition 7.8, this proves (2).

To prove (1), we use that the very general double EPW sextic can be described as a moduli space of stable objects in the Kuznetsov component of a GM sixfold [52, Proposition 5.17]. Then, after replacing X^3 by X^2 the same argument applies. \square

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