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Conditions for uniform ensemble output controllability, and obstruction to uniform ensemble controllability

Baparou Danhane¹ Jérôme Lohéac¹ Marc Jungers¹

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Abstract

We consider the linear time-invariant system $\dot{x}(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t)$ with output $y(t, \theta) = C(\theta)x(t, \theta)$ where A , B and C are continuous matrices with respect to the constant parameter θ , which belongs to some compact set \mathcal{P} . Given any continuous initial state datum $\theta \mapsto x_0(\theta)$ and any continuous output function $\theta \mapsto y_1(\theta)$, we investigate the existence of a θ -independent open loop control u such that x_0 is steered, in finite time, arbitrarily closed to y_1 with the uniform norm. When $C(\theta)$ is the identity matrix, this notion is usually termed to be *uniform ensemble controllability*. In this paper, we extend some result on uniform ensemble controllability to the case where $C(\theta)$ is not the identity matrix. We will also give some obstructions to ensemble controllability when the parameter set admits an interior point.

keywords: Control of parameter dependent systems, ensemble controllability, output controllability, average controllability, linear control system.

Contents

1	Introduction	2
2	Preliminaries and facts on uniform ensemble output controllability	4
2.1	Preliminaries	4
2.2	Uniform ensemble output controllability and polynomial interpolation	5
3	Contributions to uniform ensemble controllability of the pair (A, B)	7
4	Uniform ensemble output controllability	13
4.1	Direct consequences of the restriction lemma	13
4.2	Results on particular structures	16
5	More general results	21
5.1	Uniform ensemble output controllability of parallel systems	22
5.2	Diagonalization of the state matrix A into Jordan blocks form	26
5.3	Uniform ensemble output controllability of cascade systems	29
6	Application of uniform ensemble output controllability to averaged controllability	31
7	Conclusion	35

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1 Introduction

The concept of ensemble controllability [2, 22, 28, 31, 10] needs no more publicity nowadays with regard to its practical importance. The aim being to study the controllability of systems for which the natural dynamic depends on some parameters. This class of systems arises for instance in the transport of quantum particles where the goal is to develop external excitation that can simultaneously steer the ensemble of systems with variation in their internal parameters from an initial state to a desired final state [27]. One can also see [6] where it is used to study the controllability of the Bloch equation, for an ensemble of non-interacting half-spins, in a static magnetic field, with dispersion in the Larmor frequency. Other real life practical example can be found as for instance in non-holonomic systems theory where ensemble controllability is used to derive an approximate steering algorithm for a non-holonomic unicycle in the presence of model perturbation [7].

In linear time invariant system theory, the ensemble controllability is a topic of intensive research as it can be seen from the above references. From the generic notion of ensemble controllability, many other notions emerged as the one of uniform ensemble controllability [22, 27, 28], uniform ensemble reachability, L^q -ensemble reachability [16]. In the above mentioned non-exhaustive references, authors derived necessary and/or sufficient conditions for system $(A(\theta), B(\theta))$ to be uniformly ensemble controllable (UEC). Even if this topic is studied actively, it is far from being well understood. From these references, especially [10] and [16], it could be seen that the topology of \mathcal{P} plays a crucial role in the controllability issue. In [16], it has been shown that for single input systems, uniform ensemble controllability cannot hold if the compact set \mathcal{P} is homeomorphic to some non-empty compact subset of \mathbb{R}^d for $d \geq 2$. In [10], it has been shown that any real-analytic linear system is not L^r -controllable (ensemble controllable), for $2 \leq r \leq \infty$, if its parameterization space \mathcal{P} contains an open set included in \mathbb{R}^d for $d \geq 2$. To the best of our knowledge, the question of uniform ensemble controllability when \mathcal{P} has interior points, the system $(A(\theta), B(\theta))$ is not real-analytic and the number of input greater than one is still open.

It should also be noted that in [26, 38], under the assumption that system $(A(\theta), B(\theta))$ is UEC, authors present methods to compute suitable open-loop control functions. In [39], feedback method for ensemble reachability of $(A(\theta), B(\theta))$ is considered, and in [37], a problem of optimal control of an ensemble of affine-control systems is considered.

A particularity of uniform ensemble controllability is that all the state variables are controlled. But in some situations, controlling all the state variables may not be of interest. For instance, let us consider the motion of two cars given by Figure 1. For the sake of simplicity, the motion will be supposed linear and the frictions $f_i(\theta) = -\alpha_i(\theta)v_i$, i.e., proportional to the speed v_i of the cars with coefficients α_i varying continuously with respect to the parameter θ living in some compact subset of the complex plan. The variables u_1 and u_2 can be seen as the driving forces. Applying the fundamental principle of dynamics, the state of this system contains at least four variables which are the two positions and two speeds. Instead of controlling the whole state of the system, one might, for instance, want to control the difference between the positions, y_d , or the difference in velocities or both, which is actually a linear combination of the state variables and not the state itself. For that purpose, we introduce in this paper a new notion of ensemble controllability termed *uniform ensemble output controllability* (UEOC), which is an extension of the notion of uniform ensemble controllability discussed for instance in [16, Section 3]. Instead of considering the pair $(A(\theta), B(\theta))$, we will consider the triplet $(A(\theta), B(\theta), C(\theta))$ where $(A(\theta), B(\theta))$

describes the evolution of the system state and the matrix C stands for the output matrix. In the case whereby C is the identity matrix, in which case the output is nothing else than the state, the notion of *uniform ensemble output controllability* and the one of *uniform ensemble controllability* overlap. Based on this last observation, *uniform ensemble controllability* of the pair $(A(\theta), B(\theta))$ will be investigated as a particular case of *uniform ensemble output controllability*. For instance, it will be shown that if the spectrum of $A(\theta)$ meets \mathbb{R} for any $\theta \in \mathcal{P}$ or if the entries of A are differentiable, it is impossible to have *uniform ensemble controllability* if \mathcal{P} has interior points with the topology induced by \mathbb{C} . Necessary and sufficient conditions will be given for the *uniform ensemble output controllability* of $(A(\theta), B(\theta), C(\theta))$ as well in some particular cases.

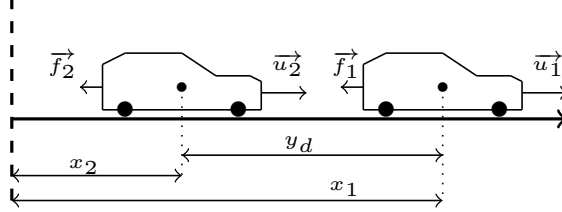


Figure 1: System of two cars.

The paper is organized as follows: In Section 2, we introduce the notion of *uniform ensemble output controllability* and link this notion to polynomial approximation. Also, a link between this notion and *uniform ensemble controllability* is given. Some structural results are also given in this section. The Section 3 is dedicated to our contributions to *uniform ensemble controllability* of the pair (A, B) . In this section, we mainly show, through Theorems 3.1 and 3.8, that if the pair (A, B) is regular enough or if the spectrum of $A(\theta)$ intercepts \mathbb{R} for any $\theta \in \mathcal{P}$, the *uniform ensemble controllability* of (A, B) cannot hold if the interior of \mathcal{P} with respect to the topology of \mathbb{C} is not empty. In Section 4, direct consequences of Lemma 2.5 on finite dimensional systems and holomorphic systems are given in the case of the *uniform ensemble output controllability*. The above mentioned lemma states how *uniform ensemble output controllability* behaves under restrictions of the parameters space \mathcal{P} . The *uniform ensemble output controllability* of particular systems are also investigated in this section. The Section 5 is dedicated to more general results on UEOC. In Section 6, we give an application of UEOC to averaged controllability. A conclusion is given in Section 7. Finally, we prove two technical lemmas in Appendix A, and recall some known and used results in Appendix B.

The following notations will be used in the paper. We denote by \mathbb{N} the set of natural numbers, including 0. Given $k \in \mathbb{N}$, the set $\mathbb{N}_{<k}$ (resp. $\mathbb{N}_{\leq k}$) stands for $\{0, 1, \dots, k-1\}$ (resp. $\{0, 1, \dots, k\}$). The set of real (resp. complex) numbers is denoted by \mathbb{R} (resp. \mathbb{C}). For X , any of the above introduced sets, X^* stands for $X \setminus \{0\}$. Given a set X , $\#X$ is its cardinal, and given $k, \ell \in \mathbb{N}^*$, X_ℓ^k (resp. X^k when $\ell = 1$ and X_ℓ when $k = 1$) stands for the set of k rows and ℓ columns matrices with entries in X . A topological space X is said *contractible* if the identity map is homotopic to a constant map, that is there exists a continuous map $H : [0, 1] \times X \rightarrow X$ such that $H(0, x) = x$ and $H(1, x) = c$ for some $c \in X$. For more details on the notion of contractibility, we refer to [43, Chapter 1]. A set $\Omega \subset \mathbb{C}$ is said *non-separating* if its complement in \mathbb{C} , $\mathbb{C} \setminus \Omega$, is connected. The set $\bar{\Omega}$ stands for the closure of Ω . For a complex number z , $|z|$ is the modulus of z , and \bar{z} its complex conjugate. For a linear operator F , we denote by F^* its adjoint, and by F^\dagger its Moore-Penrose pseudo inverse when it makes sense. When F is a matrix with entries in \mathbb{C} , denoting by F^\top the transpose of F , we have $F^* = \bar{F}^\top$. Also, $\text{Im } F$ is the space generated by the columns of F , $\text{rk } F$ is the dimension of $\text{Im } F$, and $\ker F$ the null space of F . The set $\mathbb{X}(\mathcal{P}) = \mathcal{C}(\mathcal{P}; \mathbb{C})$ stands for the space

of complex valued continuous functions on \mathcal{P} . For any function $f = (f_1 \dots f_k)^\top \in \mathbb{X}(\mathcal{P})^k$, we define $\|f\|_\infty := \max\{\|f_k\|_\infty, k \in \mathbb{N}_{\leq k}^*\}$ where $\|f_k\|_\infty = \sup_{\theta \in \mathcal{P}} |f_k(\theta)|$. For a one variable complex polynomial p , $d^\circ(p)$ stands for its degree and p' for its derivative.

2 Preliminaries and facts on uniform ensemble output controllability

2.1 Preliminaries

Let \mathcal{P} be a compact subset of \mathbb{C} and consider for every constant $\theta \in \mathcal{P}$ the system described by

$$\dot{x}(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t), \quad (2.1a)$$

$$y(t, \theta) = C(\theta)x(t, \theta). \quad (2.1b)$$

In (2.1), we assume that $A \in \mathbb{X}(\mathcal{P})_n^n$, $B \in \mathbb{X}(\mathcal{P})_m^n$ and $C \in \mathbb{X}(\mathcal{P})_n^q$. The input u is independent of the parameter θ , and \dot{x} is the derivative of x with respect to the time variable t . Given an initial state datum $x_0 : \theta \mapsto x_0(\theta)$ and an input u , the solution of (2.1) at time t , when it exists, will be denoted by $y(t, \theta; x_0, u)$.

The main problem can be formulated as follows:

Given any continuous state datum x_0 and any continuous output function y_1 , does there exist a finite time $T > 0$ and a θ -independent input $u \in \mathcal{U}(T)$ such that the output solution of (2.1) satisfies $y(T, \theta; x_0, u) = y_1(\theta)$ for every $\theta \in \mathcal{P}$?

In this problem, $\mathcal{U}(T)$ stands for a set of controls for which the equation $y(T, \cdot; x_0, u) = y_1$ makes sense. This problem is called, *exact controllability* in [11, 45], *complete controllability* in [42], or *exact simultaneous controllability* in [29].

- When $\#\mathcal{P} < \infty$.

If C is the identity matrix, then the uniform ensemble output controllability problem is reduced to the classical problem of controlling a parallel connection of finitely many linear systems. In this case, the output space of the system, which is its state space, remains finite dimensional and this case is well understood from the literature, see e.g. [17]. When the matrix C is different from the identity matrix, we similarly fall in a classical output controllability of a finite dimensional system. For more information about output controllability for linear parameter independent time invariant systems, we refer to our previous work [14], see also [15] for time variant systems.

- When $\#\mathcal{P} = \infty$.

In this case, with C the identity matrix, it is well-known that the problem of exact controllability in finite time cannot be achieved [13, Theorem 4.1.5], [29, Proposition 3.1] and [44, Theorem 3.3.1]. For C different from the identity matrix, following the same arguments, it can be deduced that *exact controllability* of system (2.1) cannot hold in finite time.

Hence, instead of trying to reach exactly the desired output function in finite time, we will try to get arbitrarily close to the target function. Let us therefore state clearly what will be considered as the definition of *uniform ensemble output controllability* in this paper.

Definition 2.1. The system (2.1) is said to be *uniformly ensemble output controllable* (UEOC) if for any continuous initial state datum $x_0 \in \mathbb{X}(\mathcal{P})^n$, any final output datum $y_1 \in \mathbb{X}(\mathcal{P})^q$ and any $\varepsilon > 0$, there exist a time $T > 0$ and a control $u \in L^1([0, T]; \mathbb{C}^m)$, independent of θ , such that

$$\|y(T, \cdot; x_0, u) - y_1\|_\infty \leq \varepsilon.$$

The terminology *uniform* in the above definition is used to emphasize the fact that the set of output functions is the continuous one endowed with the uniform norm. This definition actually makes sense since for any continuous initial state datum x_0 and any θ -independent input $u \in L^1([0, T]; \mathbb{C}^m)$, the output trajectory is continuous with respect to the time t and the parameter θ .

2.2 Uniform ensemble output controllability and polynomial interpolation

Matrices A , B and C induce the multiplication operators $\mathfrak{M}_A : \mathbb{X}(\mathcal{P})^n \rightarrow \mathbb{X}(\mathcal{P})^n$, $\mathfrak{M}_B : \mathbb{C}^m \rightarrow \mathbb{X}(\mathcal{P})^n$ and $\mathfrak{M}_C : \mathbb{X}(\mathcal{P})^n \rightarrow \mathbb{X}(\mathcal{P})^q$ defined respectively by

$$\begin{aligned} (\mathfrak{M}_A f)(\theta) &= A(\theta)f(\theta), & (\mathfrak{M}_B v)(\theta) &= B(\theta)v & \text{and} & & (\mathfrak{M}_C f)(\theta) &= C(\theta)f(\theta) \\ & & & & & & & (\theta \in \mathcal{P}, f \in \mathbb{X}(\mathcal{P})^n, v \in \mathbb{C}^m). \end{aligned} \quad (2.2)$$

Since the matrices A , B and C are continuous on \mathcal{P} which is compact, they are uniformly bounded and by [20, Proposition 2.1], the linear operators \mathfrak{M}_A , \mathfrak{M}_B and \mathfrak{M}_C are bounded.

With these operators, system (2.1) takes the form

$$\dot{x}(t, \cdot) = \mathfrak{M}_A x(t, \cdot) + \mathfrak{M}_B u(t), \quad (2.3a)$$

$$y(t, \cdot) = \mathfrak{M}_C x(t, \cdot). \quad (2.3b)$$

Since the linear operators \mathfrak{M}_A , \mathfrak{M}_B and \mathfrak{M}_C are bounded, we deduce from [44, Theorem 7.1.1] that the uniform ensemble output controllability of system (2.3) depends neither on the controllability time nor on the chosen initial state datum and is equivalent to

$$\overline{\sum_{k \in \mathbb{N}} \text{Im} \{ \mathfrak{M}_C \mathfrak{M}_A^k \mathfrak{M}_B \}} = \mathbb{X}(\mathcal{P})^q. \quad (2.4)$$

In the subsequent analysis, we then take without loss of generality $x_0 \equiv 0$. Also, from what precedes, one can see that the UEOC of system (2.1) does not depend on the regularity of the input. If the system is UEOC with controls in $L^1([0, T]; \mathbb{C}^m)$, then it is also UEOC with controls in $L^2([0, T]; \mathbb{C}^m)$. Therefore, when the analysis becomes easier or more elegant with input in $L^2([0, T]; \mathbb{C}^m)$, then $L^2([0, T]; \mathbb{C}^m)$ will be used as a set of admissible controls.

Defining $CA^k b_j : \mathbb{C} \rightarrow \mathbb{X}(\mathcal{P})^q$ by $(CA^k b_j v)(\theta) = C(\theta)A(\theta)^k b_j(\theta)v$ for every $v \in \mathbb{C}$ and every $\theta \in \mathcal{P}$, where $b_j(\theta)$ is the j -th column of $B(\theta)$, we infer from (2.4) that the system (A, B, C) is uniform output ensemble controllable if and only if the set

$$\mathcal{R}(A, B, C) := \text{span} \{ CA^k b_j \mid k \in \mathbb{N}, j \in \mathbb{N}_{\leq m}^* \}$$

is dense in $\mathbb{X}(\mathcal{P})^q$. This is equivalent, in terms of polynomial approximation, to the fact that for every $y \in \mathbb{X}(\mathcal{P})^q$ and every $\varepsilon > 0$ there exist m complex polynomials p_1, \dots, p_m such that

$$\left\| \sum_{j=1}^m C p_j(A) b_j - y \right\|_{\infty} \leq \varepsilon. \quad (2.5)$$

To digress a bit, let us note that criteria (2.4) and (2.5) remain true in the case of uniform ensemble output controllability of discrete-time systems. Needless to say that for discrete-time systems, the minimal controllability time cannot be zero.

As consequences of this approximation result, that is (2.5), we have the Propositions 2.2 to 2.4. The Proposition 2.2 mainly states that the uniform ensemble output controllability property in

real Banach space is preserved under complexification. The proof of this proposition will not be provided here since it can be easily deduced from [16, Lemma 1]. The Proposition 2.3 gives some invariant properties on uniform ensemble output controllability and Proposition 2.4 links the notion of uniform ensemble output controllability to the one of uniform ensemble controllability.

Proposition 2.2. *For $k \in \mathbb{N}^*$, let us denote by $\mathcal{X}_{\mathbb{R}}(\mathcal{P})^k = \mathcal{C}(\mathcal{P}, \mathbb{R})^k$ the set of continuous k values real functions. Denote also by $\mathcal{X}_{\mathbb{C}}(\mathcal{P})^k$ its complexification that is $\mathcal{X}_{\mathbb{C}}(\mathcal{P})^k := \{f + ig \mid f, g \in \mathcal{X}_{\mathbb{R}}(\mathcal{P})^k\}$ endowed with the smallest reasonable complexification norm¹ $\|f + ig\|_{\mathcal{X}_{\mathbb{C}}(\mathcal{P})^k} := \sup_{t \in [0, 2\pi]} \|f \cos(t) - g \sin(t)\|_{\infty}$. Assume that $(A, B, C) \in \mathcal{X}_{\mathbb{R}}(\mathcal{P})_n^n \times \mathcal{X}_{\mathbb{R}}(\mathcal{P})_m^n \times \mathcal{X}_{\mathbb{R}}(\mathcal{P})_n^q$. (A, B, C) is UEOC on $\mathcal{X}_{\mathbb{C}}(\mathcal{P})^q$ if and only if it is UEOC² on $\mathcal{X}_{\mathbb{R}}(\mathcal{P})^q$.*

Proposition 2.3. (i) *The system (A, B, C) is UEOC if and only if for any invertible matrix $T \in \mathbb{X}(\mathcal{P})_n^n$, the system $(T^{-1}AT, T^{-1}B, CT)$ is UEOC.*

(ii) *The system (A, B, C) is UEOC if and only if for any invertible matrix $Q \in \mathbb{X}(\mathcal{P})_q^q$, the system (A, B, QC) is UEOC.*

(iii) *Let φ be a continuous one-to-one map on \mathcal{P} . The system (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$ if and only if $(A \circ \varphi^{-1}, B \circ \varphi^{-1}, C \circ \varphi^{-1})$ is UEOC on $\mathbb{X}(\varphi(\mathcal{P}))^q$.*

Proof. (i) follows from the fact that for any continuous invertible matrix $T \in \mathbb{X}(\mathcal{P})_q^q$, $\mathcal{R}(T^{-1}AT, T^{-1}B, CT) = \mathcal{R}(A, B, C)$. For (ii), it suffices to note that, for any linear continuous invertible map Q , the linear bounded map $\mathfrak{M}_Q : \mathbb{X}(\mathcal{P})^q \rightarrow \mathbb{X}(\mathcal{P})^q$ defined for every $f \in \mathbb{X}(\mathcal{P})^q$ and $\theta \in \mathcal{P}$ by $(\mathfrak{M}_Q f)(\theta) = Q(\theta)f(\theta)$ is an isomorphism on $\mathbb{X}(\mathcal{P})^q$. Finally, for (iii), we observe that if φ is a continuous one-to-one map on \mathcal{P} , then $\varphi : \mathcal{P} \rightarrow \varphi(\mathcal{P})$ is a homeomorphism. Assume that (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$. Let f be an element of $\mathbb{X}(\varphi(\mathcal{P}))^q$ and $\varepsilon > 0$. Since $f \in \mathbb{X}(\varphi(\mathcal{P}))^q$, we have $f \circ \varphi \in \mathbb{X}(\mathcal{P})^q$. Using the fact that by assumption (A, B, C) is uniform ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$, we deduce the existence of m complex polynomials p_1, \dots, p_m such that

$$\sup_{\theta \in \mathcal{P}} \left| \sum_{k=1}^m C(\theta) p_k(A(\theta)) b_k(\theta) - f \circ \varphi(\theta) \right| \leq \varepsilon$$

and it follows that

$$\sup_{z \in \varphi(\mathcal{P})} \left| \sum_{k=1}^m C(\varphi^{-1}(z)) p_k(A(\varphi^{-1}(z))) b_k(\varphi^{-1}(z)) - f(z) \right| \leq \varepsilon.$$

Therefore, the system $(A \circ \varphi^{-1}, B \circ \varphi^{-1}, C \circ \varphi^{-1})$ is UEOC on $\mathbb{X}(\varphi(\mathcal{P}))^q$. The converse implication follows the same pattern. \square

Considering (2.4) and (2.5) it becomes obvious that if C is the identity matrix, then UEC and UEOC coincide. More generally, the relationship between uniform ensemble controllability and uniform ensemble output controllability can be summarized in the following proposition.

Proposition 2.4. *If $\text{rk } C(\theta) = q$ for every $\theta \in \mathcal{P}$, and if the pair (A, B) is UEC then the system (A, B, C) is UEOC.*

Moreover, if $C(\theta)$ is invertible for every θ in \mathcal{P} , then there is equivalence between UEOC of (A, B, C) and UEC of (A, B) .

¹See [33] for more details.

²UEOC on $\mathcal{X}_{\mathbb{R}}(\mathcal{P})^q$, means that for every f in $\mathcal{X}_{\mathbb{R}}(\mathcal{P})^q$ and every $\varepsilon > 0$, there exist a time $T > 0$ and a real valued input function $u \in L^1([0, T]; \mathbb{R}^m)$ such that $\|y(T, \cdot; 0, u) - y_1\|_{\infty} \leq \varepsilon$.

Proof. To begin with, note that since $\text{rk } C(\theta) = q$ for every $\theta \in \mathcal{P}$ and C is continuous, the application $C^\dagger : \theta \mapsto C(\theta)^* (C(\theta)C(\theta)^*)^{-1}$ exists and is continuous on \mathcal{P} . The matrix C^\dagger is actually the Moore-Penrose pseudo-inverse of the matrix C . We now write $\|\sum_{k=1}^m Cp_k(A)b_k - f\|_\infty \leq \|C\|_\infty \|\sum_{k=1}^m p_k(A)b_k - C^\dagger f\|_\infty$ and observe that for every $f \in \mathbb{X}(\mathcal{P})^q$, the function $C^\dagger f$ belongs to $\mathbb{X}(\mathcal{P})^n$. To end the proof of this item, we use the assumption that (A, B) is uniformly ensemble controllable to deduce that for every $\varepsilon > 0$, there exists m complex polynomial p_1, \dots, p_m satisfying $\|\sum_{k=1}^m Cp_k(A)b_k - C^\dagger f\|_\infty \leq \varepsilon/\|C\|_\infty$.

In addition, if $C(\theta)$ is invertible for every θ in \mathcal{P} , it suffices to note that $C^\dagger = C^{-1}$ and the necessity is obtained by observing that $\|\sum_{k=1}^m p_k(A)b_k - f\|_\infty \leq \|C^{-1}\|_\infty \|\sum_{k=1}^m Cp_k(A)b_k - Cf\|_\infty$. \square

From these links, it follows that understanding the notion of uniform ensemble controllability will contribute to understand the one of uniform ensemble output controllability. Hence, in the next section, necessary conditions for uniform ensemble controllability are given, to this end we will use the following lemma. This lemma shows how uniform ensemble output controllability behaves under restrictions on the parameter set. This lemma is actually a necessary condition for UEOC, and can be easily derived from [16, Lemma 2].

Lemma 2.5 (Restriction). *Let \mathcal{P} and $\widehat{\mathcal{P}}$ be two compact sets of \mathbb{C} such that $\widehat{\mathcal{P}} \subset \mathcal{P}$, and let $A \in \mathbb{X}(\mathcal{P})_n^n$, $B \in \mathbb{X}(\mathcal{P})_m^n$ and $C \in \mathbb{X}(\mathcal{P})_n^q$, with $n, m, q \in \mathbb{N}^*$. If (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$ then (A, B, C) is UEOC on $\mathbb{X}(\widehat{\mathcal{P}})^q$.*

Proof. Let us define the restriction map $R : f \in \mathbb{X}(\mathcal{P}) \mapsto f|_{\widehat{\mathcal{P}}} \in \mathbb{X}(\widehat{\mathcal{P}})$. It is trivial that R is a linear bounded and unitary operator. Furthermore, from [36, Theorem 20.4], this operator is onto. Now assume that (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$, let $\widehat{f} \in \mathbb{X}(\widehat{\mathcal{P}})^q$ and choose $\varepsilon > 0$. Since R is onto, there exists $f \in \mathbb{X}(\mathcal{P})^q$ such that $Rf = \widehat{f}$. Using the UEOC of (A, B, C) on $\mathbb{X}(\mathcal{P})^q$ and the fact that $f \in \mathbb{X}(\mathcal{P})^q$, we deduce from (2.5) that there exist m complex polynomials p_1, \dots, p_m such that $\|\sum_{k=1}^m Cp_k(A)b_k - f\|_\infty \leq \varepsilon$. To end the proof, observe that

$$\left\| \sum_{k=1}^m R Cp_k(RA)Rb_k - \widehat{f} \right\|_\infty = \left\| R \left(\sum_{k=1}^m Cp_k(A)b_k - f \right) \right\|_\infty \leq \left\| \sum_{k=1}^m Cp_k(A)b_k - f \right\|_\infty \leq \varepsilon.$$

\square

Remark 2.6. An easy consequence of Lemma 2.5 is that if (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$, then for every $\theta \in \mathcal{P}$, the system $(A(\theta), B(\theta), C(\theta))$ is state to output controllable (see [14, Definition 2.2]). In Section 4.1, we will give some other direct consequences of Lemma 2.5.

3 Contributions to uniform ensemble controllability of the pair (A, B)

As mentioned in the introduction, it has been shown in [16] that for $m = 1$, i.e., single input system, a necessary condition for uniform ensemble controllability is that the compact set \mathcal{P} must be of empty interior with respect to the topology of \mathbb{C} . In [10], the author proved the same negative result for real-analytic system $(A(\theta), B(\theta))$ regardless to the number of input.

Here, we consider the case $m \geq 1$ and show that if the pair $(A(\theta), B(\theta))$ is regular enough or if the spectrum of $A(\theta)$ intercepts the real axis for every $\theta \in \mathcal{P}$ then the above condition on the topology of \mathcal{P} is still necessary.

Let's start with the regular case.

Theorem 3.1. *Assume that \mathcal{P} is not of empty interior with respect to the topology of \mathbb{C} . Assume also that A and B are $2m - 1$ times differentiable in the interior of \mathcal{P} . Then the pair (A, B) cannot be UEC on $\mathbb{X}(\mathcal{P})^n$.*

The proof we propose for this theorem is a proof by contraposition. The strategy of this proof is as follows.

- (a) First, we assume that (A, B) is UEC with \mathcal{P} of non-empty interior.
- (b) Secondly, we show that there exists a closed ball included in \mathcal{P} on which the matrix A admits an eigenvalue λ and a left eigenvector v associated to λ both having the same regularity as A . This will be done in Lemma A.1.
- (c) Thirdly, we show that by reducing the size of the ball mentioned in the above item, λ can be chosen one-to-one. This is done in Lemma 3.2.
- (d) Finally, we reduce the problem of uniform ensemble controllability to the case $n = 1$, link the obtained system's uniform ensemble controllability problem to a density result and show that this density cannot hold.

To begin with, let us give the following lemma

Lemma 3.2. *Assume that \mathcal{P} is not of empty interior in \mathbb{C} and that (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$. If A is differentiable in the interior of \mathcal{P} , then there exist θ_0 interior to \mathcal{P} and $\delta_0 > 0$ such that, A admits a differentiable and injective eigenvalue λ on $B_{\theta_0}(\delta_0)$.*

Before giving the proof of this Lemma, let us recall that a necessary condition for the pair (A, B) to be uniformly ensemble controllable on $\mathbb{X}(\mathcal{P})^n$ is that for distinct complex numbers $\theta_1, \dots, \theta_k \in \mathcal{P}$,

$$\sigma(A(\theta_1)) \cap \dots \cap \sigma(A(\theta_k)) = \emptyset, \quad \forall k > m. \quad (3.1)$$

In the above, $\sigma(A(\theta))$ stands for the spectrum of the matrix $A(\theta)$. This remark is actually [16, Proposition 4] which follows directly from Lemma 2.5 and the Hautus test given in [21, Theorem 1].

Proof. According to Lemma A.1, since A is \mathcal{C}^1 in the interior of \mathcal{P} , there exist θ_0 interior to \mathcal{P} and $\delta_0 > 0$ such that A admits a differentiable eigenvalue λ on $B_{\theta_0}(\delta_0)$.

Using (3.1) we will show that by shifting θ_0 inside $B_{\theta_0}(\delta_0)$ and reducing δ_0 , if necessary, λ is injective on $B_{\theta_0}(\delta_0)$.

Let $\Upsilon_{\theta_0}^{\delta_0} := \{(\Re(\theta), \Im(\theta)) \in \mathbb{R} \times \mathbb{R} \mid \theta \in B_{\theta_0}(\delta_0)\}$, with $\Re(z)$ and $\Im(z)$ the real and imaginary parts of $z \in \mathbb{C}$, and define $\tilde{\lambda} : \Upsilon_{\theta_0}^{\delta_0} \mapsto \mathbb{R}^2$ such that $\tilde{\lambda} = (\lambda_1, \lambda_2)$, with $\lambda_1 = \Re(\lambda)$ and $\lambda_2 = \Im(\lambda)$.

The Jacobian of $\tilde{\lambda}$ is given by

$$J_{\tilde{\lambda}}(\theta_1, \theta_2) = \begin{pmatrix} \frac{\partial \lambda_1}{\partial \theta_1}(\theta_1, \theta_2) & \frac{\partial \lambda_1}{\partial \theta_2}(\theta_1, \theta_2) \\ \frac{\partial \lambda_2}{\partial \theta_1}(\theta_1, \theta_2) & \frac{\partial \lambda_2}{\partial \theta_2}(\theta_1, \theta_2) \end{pmatrix}.$$

Let us assume by contradiction that λ is not injective on any open set of $B_{\theta_0}(\delta_0)$. Then for every $(\theta_1, \theta_2) \in \Upsilon_{\theta_0}^{\delta_0}$, we have $\det J_{\tilde{\lambda}}(\theta_1, \theta_2) = 0$. Since, by assumption, (A, B) is UEC, according to (3.1), A cannot admit a constant eigenvalue on a non-zero measure set. Therefore, using the fact that $J_{\tilde{\lambda}}$ is continuous and non invertible, we deduce the existence of $\tilde{\theta}_0 \in B_{\theta_0}(\delta_0)$ and $\tilde{\varepsilon} > 0$ such that $\Upsilon_{\tilde{\theta}_0}^{\tilde{\varepsilon}} \subset \Upsilon_{\theta_0}^{\delta_0}$ and $\text{rk } J_{\tilde{\lambda}} = 1$ on $\Upsilon_{\tilde{\theta}_0}^{\tilde{\varepsilon}}$. Hence, using Theorem B.1 there exists a non-zero continuous function $\alpha : \Upsilon_{\tilde{\theta}_0}^{\tilde{\varepsilon}} \rightarrow \mathbb{R}^2$ such that $J_{\tilde{\lambda}}(\theta_1, \theta_2)\alpha(\theta_1, \theta_2) = 0$ for every $(\theta_1, \theta_2) \in \Upsilon_{\tilde{\theta}_0}^{\tilde{\varepsilon}}$. Therefore, we can

construct inside $\Upsilon_{\theta_0}^{\delta_0}$ a path γ such that $\gamma(0) = (\Re(\tilde{\theta}_0), \Im(\tilde{\theta}_0))$ and $\dot{\gamma}(t) = \alpha(\gamma(t))$ thanks to the Cauchy–Peano’s Theorem. From

$$\frac{d}{dt} \left(\tilde{\lambda}(\gamma(t)) \right) = J_{\tilde{\lambda}}(\gamma(t)) \dot{\gamma}(t) = J_{\tilde{\lambda}}(\gamma(t)) \alpha(\gamma(t)) = 0,$$

we deduce that $\tilde{\lambda}$, and therefore λ , is constant along the flow of γ and this is in contradiction with (3.1). \square

The result of Lemma 3.2 leads to the following remark.

Remark 3.3. From the above lemma, observe that if \mathcal{P} was assumed to be homeomorphic to a non-empty interior compact subset of \mathbb{R}^d for $d > 2$, then it is not possible to have the UEC of (A, B) if A is differentiable. Indeed, in this case, Lemma 3.2 would imply the existence of a continuous and injective map from \mathbb{R}^d into \mathbb{R}^2 . This would contradict [8, Theorem 20.2, p.242] which states that there does not exist a continuous and injective map from κ -sphere (seen as a sub-manifold of $\mathbb{R}^{\kappa+1}$) into \mathbb{R}^κ for every $\kappa \in \mathbb{N}^*$. This theorem is usually termed as Borsuk-Ulam Theorem.

For every $\Omega \subset \mathbb{C}$, we denote by $\mathcal{H}(\Omega)$ the set of continuous functions defined on Ω that are holomorphic in the interior of Ω . In particular, if Ω is open, then, $\mathcal{H}(\Omega)$ is the set of holomorphic function defined on Ω . The following result will be useful for the proof of Theorem 3.1.

Lemma 3.4. *Let $D \subset \mathbb{C}$ be a compact with nonempty interior, $k \in \mathbb{N}^*$ and f_1, \dots, f_k be k continuous functions defined on D and of class \mathcal{C}^{2k+1} in the interior of D . Assume that for every $\ell \in \mathbb{N}_{\leq k}^*$, there does not exist a nonempty open $\mathcal{O} \subset D$ such that $f_\ell|_{\mathcal{O}} \in \mathcal{H}(\mathcal{O}) + \sum_{j=1, s \neq \ell}^k f_j|_{\mathcal{O}} \mathcal{H}(\mathcal{O})$. Then for every*

$$\varphi \in \overline{\mathcal{H}(D) + f_1 \mathcal{H}(D) + \dots + f_k \mathcal{H}(D)},$$

there exists a closed ball D_0 included in D on which φ is of class \mathcal{C}^{k+1} . Moreover, we have $\varphi|_{D_0} \in \mathcal{H}(D_0) + f_1|_{D_0} \mathcal{H}(D_0) + \dots + f_k|_{D_0} \mathcal{H}(D_0)$.

Proof. We give a proof in the case $k = 1$. The general case can be done following the same arguments.

Let us set $\mathcal{R}(D, f_1) := \mathcal{H}(D) + f_1 \mathcal{H}(D)$. By assumption, f_1 is three times differentiable and is not holomorphic in any open subset of D . Take $\varphi \in \mathcal{R}(D, f_1)$. By definition, there exists a sequence $(\varphi_n)_n$ in $\mathcal{R}(D, f_1)$ that converges uniformly on D to φ . Exploiting the fact that for every n , φ_n belongs to $\mathcal{R}(D, f_1)$, we deduce that for every n , φ_n is three times differentiable and takes the form $\varphi_n = \varphi_n^0 + f_1 \varphi_n^1$ where $(\varphi_n^0, \varphi_n^1) \in \mathcal{H}(D) \times \mathcal{H}(D)$. Setting $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ and $L = (\bar{\partial} f_1) \bar{\partial}^2 - (\bar{\partial}^2 f_1) \bar{\partial}$, we have in the interior of D

$$\varphi_n = \varphi_n^0 + f_1 \varphi_n^1 \iff \bar{\partial} \varphi_n = \varphi_n^1 \bar{\partial} f_1.$$

This is because $\bar{\partial} \varphi_n^0 = \bar{\partial} \varphi_n^1 = 0$. Since f_1 is not holomorphic in any open subset of D , there exists an open ball $\mathcal{O} \subset D$ such that for every $z \in \mathcal{O}$, $\bar{\partial} f_1(z) \neq 0$. We therefore have $\frac{1}{\bar{\partial} f_1} \bar{\partial} \varphi_n = \varphi_n^1$ on \mathcal{O} .

Applying $\bar{\partial}$ to this relation, we get

$$(\bar{\partial} f_1) \bar{\partial}^2 \varphi_n - (\bar{\partial}^2 f_1) \bar{\partial} \varphi_n = 0,$$

that is to say that

$$L \varphi_n = 0 \quad \text{on} \quad \mathcal{O}. \tag{3.2}$$

From the above, it should also be noticed that since f_1 is not holomorphic in any open subset of D , if $L\varphi_n = 0$ in \mathcal{O} then for every compact set $K \subset \mathcal{O}$, we have $\varphi_n|_K = h_n^0 + f_1|_K h_n^1$ for some functions $h_n^0, h_n^1 \in \mathcal{H}(K)$.

From (3.2), we deduce that for any infinitely differentiable function ψ supported in \mathcal{O} ,

$$\langle L\varphi_n, \psi \rangle = \langle \varphi_n, L^*\psi \rangle = 0, \quad \text{where } L^* = (\partial\bar{f}_1)\partial^2 + 3(\partial^2\bar{f}_1)\partial + 2\partial^3\bar{f}_1. \quad (3.3)$$

Using the fact that $(\varphi_n)_n$ converges uniformly to φ , we deduce from (3.3) that $\langle \varphi, L^*\psi \rangle = 0$ in the weak sense. Observing first that the coefficients of the uniformly elliptic operator L are differentiable, and second that the coefficients of the adjoint operator L^* are continuous and third that the function φ belongs to $L^\infty(\overline{\mathcal{O}})$, we infer from [9, Theorem 2.2] that for any compact at a positive distance from the complementary of $\overline{\mathcal{O}}$, the function φ is three times differentiable. Therefore, there exists an open subset of $\overline{\mathcal{O}}$ on which $L\varphi = 0$ in the classical sense. Bearing in mind the observation made below (3.2), it follows that there exists a compact set $K \subset \mathcal{O}$ on which φ takes the form $h_0|_K + f_1|_K h_1|_K$ for some $h_0, h_1 \in \mathcal{H}(K)$. \square

Theorem 2.2 of [9] shows that Lemma 3.4 remains true if the functions f_1, \dots, f_k are of the class $\mathcal{C}^{2k,1}$, that is to say that these functions belong to the space of continuous functions whose derivatives up to $2k$ are continuous and the derivative of order $2k+1$ is essentially bounded. Indeed, to apply [9, Theorem 2.2] we only need the coefficient of the adjoint operator L^* to be essentially bounded.

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. Assume that \mathcal{P} is not of empty interior in \mathbb{C} and that (A, B) is of class \mathcal{C}^{2m-1} in the interior of \mathcal{P} . Let us further assume that (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$.

From Lemma A.1, there exist θ_0 interior to \mathcal{P} and $\delta_0 > 0$ such that on $B_{\theta_0}(\delta_0)$, the matrix A admits a \mathcal{C}^{2m-1} eigenvalue λ with a \mathcal{C}^{2m-1} left eigenvector v associated to λ .

- *Reduction to the case $n = 1$.*

Multiplying the state equation on the left side by v , we get, for every $\theta \in B_{\theta_0}(\delta_0)$,

$$v(\theta)\dot{x}(t, \theta) = v(\theta)A(\theta)x(t, \theta) + v(\theta)B(\theta)u(t).$$

Using the fact that $v(\theta)A(\theta) = \lambda(\theta)v(\theta)$ on $B_{\theta_0}(\delta_0)$, setting $y(t, \theta) = v(\theta)x(t, \theta) \in \mathbb{C}$ and $v(\theta)B(\theta) = (\tilde{b}_1(\theta) \dots \tilde{b}_m(\theta)) \in \mathbb{C}_m$, the above equation becomes

$$\dot{y}(t, \theta) = \lambda(\theta)y(t, \theta) + v(\theta)B(\theta)u(t). \quad (3.4)$$

Using Proposition 2.4 and the fact that (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$, we infer that system (3.4) is uniformly ensemble controllable on $\mathbb{X}(K)$, for every compact set $K \subset B_{\theta_0}(\delta_0)$.

In the following, we chose a compact set $K \subset B_{\theta_0}(\delta_0)$ with nonempty interior, and we set $b(\theta) = (b_1 \dots b_m(\theta)) = v(\theta)B(\theta)$. Observe that b is of class \mathcal{C}^{2m-1} on K .

- *Link with Lemma 3.4.*

To begin with, note that from Proposition 2.3–(iii), one can consider without loss of generality λ to be the identity map. Indeed, since λ is continuous and one-to-one, $\lambda : \mathcal{P} \rightarrow \lambda(\mathcal{P})$ is a homeomorphism. Therefore, system (λ, b) is UEC on $\mathbb{X}(K)$ if and only if $(i_d, b \circ \lambda^{-1})$ is UEC on $\mathbb{X}(\lambda(K))$ where i_d stands for the identity map on $\lambda(K)$. Since K is compact with non-empty interior, $\lambda(K)$ is also compact with non-empty interior. Furthermore, thanks to Remark A.2, we have that $b \circ \lambda^{-1}$ is of class \mathcal{C}^{2m-1} on $\lambda(K')$, for some compact set $K' \subset K$ of nonempty interior. Therefore, we can assume without loss of generality that λ is the identity map on K .

Since system (3.4) is assumed to be UEC, Remark 2.6 ensures that $\text{rk } b(\theta) = 1$ for every $\theta \in K$. We can therefore choose $\theta_0 \in K$ and get the existence of $j_0 \in \mathbb{N}_{\leq m}^*$ such that $b_{j_0}(\theta_0) \neq 0$. By continuity of b_{j_0} , we deduce that there exists a compact neighborhood of θ_0 denoted by D included

in K on which b_{j_0} does not vanish. Now, consider the b_j 's for which the intersection of their support with D have interior point in \mathbb{C} and choose finally D to be the closure of a small ball contained in that intersection. By construction of D and the continuity of b_j 's, either $b_j(\theta) \neq 0$ for every $\theta \in D$ or vanishes identically. We assume without loss of generality that for every $k \in \mathbb{N}_{\leq m}^*$, $b_j(\theta) \neq 0$ for every $\theta \in D$. By Proposition 2.4 we can also assume that $b_1(\theta) = 1$ for every $\theta \in D$ and this is what will be done for the rest of this proof. Thanks to Lemma 2.5, we deduce that system (3.4) restricted to the parameter set D is UEC on $\mathbb{X}(D)$. Hence, as we have seen in Section 2 the UEC on $\mathbb{X}(D)$ of system (3.4) is equivalent to

$$\overline{\sum_{j=1}^m b_j \text{span} \{\theta \in D \mapsto \theta^k \mid k \in \mathbb{N}\}} = \mathbb{X}(D). \quad (3.5)$$

Using the fact that for every $k \in \mathbb{N}$ the maps $\theta \mapsto \theta^k$ are holomorphic over \mathbb{C} and that D is a compact with connected complement, we infer from (3.5) (with $b_1 = 1$) that

$$\overline{\mathcal{H}(D) + b_2\mathcal{H}(D) + \dots + b_m\mathcal{H}(D)} = \mathbb{X}(D). \quad (3.6)$$

Thanks to the restriction lemma, we can also assume that for every $\ell \in \mathbb{N}_{\leq m}^*$ there does not exist a nonempty open set $\mathcal{O} \subset D$ such that $b_\ell|_K \in \mathcal{H}(K) + \sum_{k=2, k \neq \ell}^m b_k \mathcal{H}(K)$.

- *Conclusion of the proof.*

The density relation (3.6) and Lemma 3.4 imply that every continuous function on D is locally a \mathcal{C}^m function which is obviously not true. \square

Remark 3.5. Observe through the proof of Theorem 3.1 that in the case $m = 1$, one needs not A to be differentiable to show that a necessary condition for (A, B) to be UEC is that \mathcal{P} must be of empty interior in \mathbb{C} . Indeed, the differentiability of A was used to show the injectivity of the selected eigenvalue. In the case $m = 1$ the injectivity is directly given by the restriction lemma and (3.1). It is therefore enough to use Lemma A.1 to reduce the UEC of (A, B) to the one of a scalar system, i.e., to the case $n = m = 1$, and notice that the UEC of (A, B) implies $\mathcal{H}(D) = \mathbb{X}(D)$ for some closed non-empty interior ball included in \mathcal{P} which is not possible. Recall that this result was already given in [16, Theorem 3].

Remark 3.6. One can also observe through this proof that one can still get relation (3.6) with the only assumption that A is $\mathcal{C}^1(\mathcal{P})$. But this time the functions b_k , $k \in \{2, 3, \dots, m\}$, will be only continuous. Note that if the space $\mathcal{H}(D) + b_2\mathcal{H}(D) + \dots + b_m\mathcal{H}(D)$ is a closed subspace of $\mathbb{X}(D)$, then it is easy to see that (3.6) cannot hold. Unfortunately $\mathcal{H}(D) + b_2\mathcal{H}(D) + \dots + b_m\mathcal{H}(D)$ is in general not closed in $\mathbb{X}(D)$. Taking for instance $m = 2$, $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $b_2(z) = \bar{z}$ for all $z \in D$, one can show, by applying for instance [46, Theorem 1.1], that $\mathcal{H}(D) + b_2\mathcal{H}(D)$ is not a closed subspace of $\mathbb{X}(D)$.

Remark 3.7. From Theorem 3.1, the following result can be deduced.

Let \mathcal{P} be a compact subset of \mathbb{C} . If \mathcal{P} has an interior point with respect to the topology of \mathbb{C} and if (A, B) is of class \mathcal{C}^{2m-1} in the interior of \mathcal{P} , then there exists a non-zero complex measure μ supported on \mathcal{P} such

$$\int_{\mathcal{P}} B(\theta)^* A(\theta)^{*k} d\mu(\theta) = 0, \quad \forall k \in \mathbb{N}.$$

To see this, let $T > 0$ and define, for every $u \in L^2([0, T]; \mathbb{C}^m)$, the end-point map Φ_T by

$$(\Phi_T u)(\theta) = \int_0^T e^{(T-t)A(\theta)} B(\theta) u(t) dt \quad (\theta \in \mathcal{P}).$$

As defined, one can see that Φ_T is a linear and continuous map from $L^2([0, T]; \mathbb{C}^m)$ into $\mathbb{X}(\mathcal{P})$. Also, its dual Φ_T^* is a linear and continuous map from the space of Radon measures $\mathcal{M}(\mathcal{P})$ into $L^2([0, T]; \mathbb{C}^m)$ and is given, for every $\mu \in \mathcal{M}(\mathcal{P})$, by

$$(\Phi_T^* \mu)(t) = \int_{\mathcal{P}} B(\theta)^* e^{(T-t)A(\theta)^*} d\mu(\theta) \quad (t \in [0, T]).$$

From [45, Theorem 11.2.1], it follows that the UEC of (A, B) is equivalent to the unique continuation property

$$\forall t \in [0, T], \quad (\Phi_T^* \mu)(t) = 0 \quad \implies \quad \mu = 0. \quad (3.7)$$

In (3.7), $\mu = 0$ has to be understood in the sense that for every $f \in \mathbb{X}(\mathcal{P})^n$,

$$\langle \mu, f \rangle_{\mathcal{M}(\mathcal{P}), \mathbb{X}(\mathcal{P})^n} = \int_{\mathcal{P}} f d\mu = 0.$$

Since we have shown in Theorem 3.1 that if \mathcal{P} has interior point, the pair (A, B) cannot be uniformly ensemble controllable, it follows from (3.7) that there exists a non-zero complex measure μ such that for every $t \in [0, T]$, $(\Phi_T^* \mu)(t) = 0$. The expected result is obtained by evaluating the successive derivatives of $(\Phi_T^* \mu)(t) = 0$ at time T .

The following theorem states that if A admits a continuous real eigenvalue on \mathcal{P} (in particular Hermitian matrices with continuous eigenvalues and left eigenvectors), then a necessary condition for uniform ensemble controllability is that \mathcal{P} has to be of empty interior in \mathbb{C} .

Theorem 3.8. *Assume \mathcal{P} is not of empty interior in \mathbb{C} , and for every $\theta \in \mathcal{P}$, $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$. Then (A, B) cannot be uniformly ensemble controllable on $\mathbb{X}(\mathcal{P})^n$.*

Let us give a proof to Theorem 3.8.

Proof. The proof of this theorem follows the same pattern as the one of Theorem 3.1 but is shorter. Indeed, we will show that under the assumption of Theorem 3.8, if (A, B) is uniformly ensemble controllable on $\mathbb{X}(\mathcal{P})^n$ then there exists a continuous and injective map from \mathbb{R}^2 into \mathbb{R} which is in contradiction with the Borsuk-Ulam Theorem mentioned in Remark 3.3.

- *Continuous real eigenvalue selection.*

Since \mathcal{P} is not of empty interior in \mathbb{C} and A is continuous and satisfies $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$ for every $\theta \in \mathcal{P}$, we deduce from Lemma A.3 that there exist θ_0 interior to \mathcal{P} and $\delta_0 > 0$ such that on $B_{\theta_0}(\delta_0)$, A admits a continuous real eigenvalue selection λ .

- *Injectivity of λ .*

Now that we have a continuous real eigenvalue λ on $B_{\theta_0}(\delta_0)$, we show by contradiction that if (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$, there exists $\delta_1 \leq \delta_0$ such that λ is one-to-one on $B_{\theta_0}(\delta_1)$.

Let assume that (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$ and that for every $\delta_1 \leq \delta_0$, λ is not one-to-one on $B_{\theta_0}(\delta_1)$. Since λ is not one-to-one on $B_{\theta_0}(\delta_0)$ there exist two complex numbers $\theta_1 \neq \theta_2$ in $B_{\theta_0}(\delta_0)$ such that $\lambda(\theta_1) = \lambda(\theta_2)$.

If $\lambda(\theta_1) \neq \lambda(\theta_0)$, we can construct $m + 1$ different smooth curves γ_k connecting θ_0 to θ_1 (since $B_{\theta_0}(\delta_0)$ is an open path-connected set). Using the Intermediate Value Theorem on functions $\lambda \circ \gamma_k$, we would deduce the existence of $m + 1$ distinct complex numbers $\tilde{\theta}_0, \dots, \tilde{\theta}_m$ in $B_{\theta_0}(\delta_0)$ such that $\lambda(\tilde{\theta}_0) = \dots = \lambda(\tilde{\theta}_m)$ and this would contradict (3.1).

Assume now that $\lambda(\theta_1) = \lambda(\theta_2) = \lambda(\theta_0)$. Here, either $\theta_1 = \theta_0$ or $\theta_2 = \theta_0$ or θ_1 and θ_2 are distinct from θ_0 . We assume without loss of generality that $\theta_1 \neq \theta_0$. Choose δ_1 small enough such $B_{\theta_0}(\delta_1)$ does not contain θ_1 . Since λ is not one-to-one on $B_{\theta_0}(\delta_1)$, there exist θ_2 and θ_3 such that $\lambda(\theta_2) = \lambda(\theta_3)$. If $\lambda(\theta_0) \neq \lambda(\theta_2)$, then use the previous $m + 1$ curves argument. Otherwise, continue the processes until you construct $m + 1$ distinct complex numbers θ_k in \mathcal{P} such that $\lambda(\theta_0) = \dots = \lambda(\theta_m)$. This will contradict (3.1). All in all, we have deduced that there exists $\delta_1 \leq \delta_0$ such that λ is one-to-one on $B_{\theta_0}(\delta_1)$.

- *Conclusion of the proof.*

In fine, if (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$, there exist $\theta_0 \in \mathcal{P}$, $\delta_0 > 0$ and a continuous and one-to-one function $\lambda : B_{\theta_0}(\delta_0) \rightarrow \mathbb{R}$. This leads to a contradiction, since there does not exist one-to-one function from \mathbb{C} to \mathbb{R} . \square

Remark 3.9. It can be seen from the proof of Theorem 3.8 that its conclusion remains true if \mathcal{P} is homeomorphic to any non-empty interior compact of \mathbb{R}^d , for $d \geq 2$. Indeed, if \mathcal{P} admits a non-empty interior in \mathbb{R}^d , there exist $(\theta_3, \dots, \theta_d) \in \mathbb{R}^{d-2}$ and $\hat{\mathcal{P}} \subset \mathbb{R}^2$ with non-empty interior such that $\hat{\mathcal{P}} \times \{(\theta_3, \dots, \theta_d)\} \subset \mathcal{P}$. Using Lemma 2.5, if the pair (A, B) is UEC on $\mathbb{X}(\mathcal{P})^n$, then it is UEC on $\mathbb{X}(\hat{\mathcal{P}})^n$, which is impossible, according to Theorem 3.8. Bearing in mind Borsuk-Ulam Theorem ([8, Theorem 20.2, p.242]), the above observation could also be deduced from the proof of Theorem 3.8.

Remark 3.10. In Theorem 3.8, one can also relax the assumption $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$ to $\sigma(A(\theta)) \cap \mathcal{M} \neq \emptyset$, where $\mathcal{M} \subset \mathbb{C}$ is a continuous manifold of dimension one.

To end this section dedicated to uniform ensemble controllability, we note that in general the matrix A is just continuous and admits no real eigenvalue. In this case, the uniform ensemble controllability with \mathcal{P} of nonempty interior in \mathbb{C} is open.

4 Uniform ensemble output controllability

4.1 Direct consequences of the restriction lemma

The first consequence of Lemma 2.5 is Remark 2.6. Indeed, from Lemma 2.5, it follows that a necessary condition for uniform ensemble output controllability is that for every $\theta \in \mathcal{P}$, the system $(A(\theta), B(\theta), C(\theta))$ has to be *state to output controllable*, see [14, Definition 2.2]. This criterion can be checked by using the extended Kalman rank condition given in [25, Theorem III], that is

$$\text{rk} \begin{pmatrix} C(\theta)B(\theta) & C(\theta)A(\theta)B(\theta) & \dots & C(\theta)A(\theta)^{n-1}B(\theta) \end{pmatrix} = q, \quad (4.1)$$

or the extended Hautus Test given in [14, Lemma 4.2], that is

$$\text{rk} C(\theta) = q \quad \text{and} \quad \text{Im} C(\theta)^* \cap \bigoplus_{\lambda \in \sigma(A(\theta))} E_{\theta, \lambda} = \{0\}, \quad (4.2)$$

where $E_{\theta, \lambda} = \{z \in \ker(A(\theta)^* - \bar{\lambda}I_n)^{n_{\theta, \lambda}} \mid B(\theta)^* (A(\theta)^* - \bar{\lambda}I_n)^l z = 0, \forall l \in \mathbb{N}_{< n_{\theta, \lambda}}\}$, with $n_{\theta, \lambda}$, the algebraic multiplicity of λ in the minimal polynomial of $A(\theta)$.

Assume now that the parameter set \mathcal{P} is reduced to $\{\theta_1, \theta_2, \dots, \theta_N\}$ for some positive integer N . In that case, system (2.1) becomes a finite dimensional system given, for all $k \in \mathbb{N}_{\leq N}^*$, by $\dot{x}(\cdot, \theta_k) = A(\theta_k)x(\cdot, \theta_k) + B(\theta_k)u$ and $y(\cdot, \theta_k) = C(\theta_k)x(\cdot, \theta_k)$.

By setting $x(t) = \begin{pmatrix} x(t, \theta_1)^\top & \dots & x(t, \theta_N)^\top \end{pmatrix}^\top \in \mathbb{C}^{Nn}$ and similarly $y(t) = \begin{pmatrix} y(t, \theta_1)^\top & \dots & y(t, \theta_N)^\top \end{pmatrix}^\top \in \mathbb{C}^{Nq}$, the system (2.1) becomes

$$\dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}u(t), \quad (4.3a)$$

$$y(t) = \mathbb{H}x(t), \quad (4.3b)$$

where the matrices $\mathbb{A} \in \mathbb{C}_{Nn}^{Nn}$, $\mathbb{B} \in \mathbb{C}_m^{Nn}$ and $\mathbb{H} \in \mathbb{C}_{Nn}^{Nq}$ are given by

$$\mathbb{A} = \begin{pmatrix} A(\theta_1) & & & \\ & \ddots & & \\ & & & A(\theta_N) \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} B(\theta_1) \\ \vdots \\ B(\theta_N) \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} C(\theta_1) & & & \\ & \ddots & & \\ & & & C(\theta_N) \end{pmatrix}.$$

Note that system (4.3) is a finite dimensional parallel system and the UEOC of this system is equivalent to its state to output controllability. To check if system (4.3) is state to output controllable, one can apply criterion (4.1) or (4.2). Here, we are going to use (4.2) since it directly gives information on the controllability of the above system based on the spectrum of the matrix $A(\theta)$.

Proposition 4.1. *Let $\mathcal{P} = \{\theta_1, \theta_2, \dots, \theta_N\}$ and assume that*

- (i) *for every $k \neq l$, $\sigma(A(\theta_k)) \cap \sigma(A(\theta_l)) = \emptyset$,*
- (ii) *for every $k \in \mathbb{N}_{\leq N}^*$, the system $(A(\theta_k), B(\theta_k), C(\theta_k))$ is state to output controllable.*

Then system (4.3) is state to output controllable.

Proof. Using (4.2), the state to output controllability of (4.3) is equivalent to

$$\text{rk}(\mathbb{H}) = Nq \quad \text{and} \quad \text{Im} \mathbb{H}^* \cap \left(\bigoplus_{\lambda \in \sigma(\mathbb{A})} E_\lambda \right) = \{0\}, \quad (4.4)$$

where $E_\lambda = \left\{ z \in \ker(\mathbb{A}^* - \bar{\lambda}I_{nN})^{n_\lambda} \mid (\mathbb{A}^* - \bar{\lambda}I_{nN})^l z \in \ker \mathbb{B}^*, \forall l \in \mathbb{N}_{< n_\lambda} \right\}$, and n_λ is the algebraic multiplicity of λ in the minimal polynomial of \mathbb{A} .

Since $\mathbb{A} = \text{diag}(A(\theta_k))_k$, we have $\sigma(\mathbb{A}) = \bigcup_{k=1}^N \sigma(A(\theta_k))$ and $n_\lambda = \max_{k \in \mathbb{N}_{\leq N}^*} n_{\theta_k, \lambda}$, where $n_{\theta_k, \lambda}$ is the algebraic multiplicity of λ in the minimal polynomial of $A(\theta_k)$.

From $\mathbb{A} = \text{diag}(A(\theta_k))_k$, we deduce that $(\mathbb{A}^* - \bar{\lambda}I_{nN})^l = \text{diag} \left((A(\theta_k)^* - \bar{\lambda}I_n)^l \right)_k$ for any $l \in \mathbb{N}$ and any $\lambda \in \mathbb{C}$. This also shows that

$$\ker(\mathbb{A}^* - \bar{\lambda}I_{nN})^{n_\lambda} = \prod_{k=1}^N \ker(A(\theta_k)^* - \bar{\lambda}I_n)^{n_{\theta_k, \lambda}} \quad (4.5)$$

and that every $z \in \ker(\mathbb{A}^* - \bar{\lambda}I_{nN})^{n_\lambda}$ can be written as $z = (z_1^\top \dots z_N^\top)^\top$, where each z_k belongs to $\ker(A(\theta_k)^* - \bar{\lambda}I_n)^{n_{\theta_k, \lambda}}$.

Also, note that $(\mathbb{A}^* - \bar{\lambda}I_{nN})^l z \in \ker \mathbb{B}^*$ for all $l \in \mathbb{N}_{< n_\lambda}$ is equivalent to

$$\sum_{k=1}^N B(\theta_k)^* (A(\theta_k)^* - \bar{\lambda}I_n)^l z_k = 0 \quad (l \in \mathbb{N}_{< n_\lambda}). \quad (4.6)$$

From (4.5) and (4.6), we deduce that for every $\lambda \in \sigma(\mathbb{A})$,

$$E_\lambda = \left\{ (z_1, z_2, \dots, z_N) \in \prod_{k=1}^N \ker(A(\theta_k)^* - \bar{\lambda}I_n)^{n_{\theta_k, \lambda}} \mid \sum_{k=1}^N B(\theta_k)^* (A(\theta_k)^* - \bar{\lambda}I_n)^l z_k = 0, \forall l \in \mathbb{N}_{< n_\lambda} \right\}. \quad (4.7)$$

Using assumption (i), we deduce that for every $\lambda \in \sigma(\mathbb{A})$, there exists a unique $k_0 \in \mathbb{N}_{\leq N}^*$ such that $\lambda \in \sigma(A(\theta_{k_0}))$. It follows from (4.7) that

$$E_\lambda = \dots \times \{0\} \times E_{\theta_{k_0}, \lambda} \times \{0\} \times \dots, \quad (4.8)$$

where at the k_0 -th position, we have

$$E_{\theta_{k_0}, \lambda} := \left\{ z \in \ker (A(\theta_{k_0})^* - \bar{\lambda}I_n)^{n_{\theta_{k_0}, \lambda}} \mid B(\theta_{k_0})^* (A(\theta_{k_0})^* - \bar{\lambda}I_n)^l z = 0, \forall l \in \mathbb{N}_{< n_{\theta_{k_0}, \lambda}} \right\}.$$

From (4.8), we deduce that

$$\bigoplus_{\lambda \in \sigma(\mathbb{A})} E_\lambda = \prod_{k=1}^N \bigoplus_{\lambda \in \sigma(A(\theta_k))} E_{\theta_k, \lambda}. \quad (4.9)$$

Finally, note that since $\mathbb{H} = \text{diag}(C(\theta_k))_k$, $\text{rk}(\mathbb{H}) = Nq$ if and only if $\text{rk}(C(\theta_k)) = q$ for every $k \in \mathbb{N}_{\leq N}^*$ and $\text{Im } \mathbb{H}^* = \text{Im } C(\theta_1)^* \times \cdots \times \text{Im } C(\theta_N)^*$. Therefore, the state to output controllability of system (4.3) is equivalent to the fact that for every $k \in \mathbb{N}_{\leq N}^*$,

$$\text{rk}(C(\theta_k)) = q \quad \text{and} \quad \text{Im } C(\theta_k)^* \cap \bigoplus_{\lambda \in \sigma(A(\theta_k))} E_{\theta_k, \lambda} = \{0\}. \quad (4.10)$$

We end the proof by noticing that assumption (ii) implies (4.10). \square

Remark 4.2. From the above proof, it can be seen that the equality of the output subsystems dimension, that is the fact that y_{θ_k} belongs to \mathbb{C}^q for every $k \in \mathbb{N}_{\leq N}^*$ is not relevant. The result still holds even if every $y_{\theta_k} \in \mathbb{C}^{q_k}$ with potentially different q_k 's.

Remark 4.3. The condition (i) of Proposition 4.1 is not necessary. One can get convinced by considering the state to output controllable system given by the following matrices,

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbb{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Another immediate consequence of Lemma 2.5, is the following result which is a necessary condition for (2.1) to be uniform ensemble output controllable in the case of holomorphic systems.

Theorem 4.4. *Assume that the interior of \mathcal{P} with respect to \mathbb{C} is not empty. If there exists a non-empty open subset \mathcal{V} of \mathcal{P} on which matrices A , B and C are holomorphic, then system (2.1) cannot be uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$.*

Proof. Assume that there exists a non-empty open subset \mathcal{V} of \mathcal{P} on which matrices A , B and C are holomorphic and that system (2.1) is UEOC on $\mathbb{X}(\mathcal{P})^q$. Let \mathcal{P}_1 be a compact subset of \mathbb{C} included in \mathcal{V} . By Lemma 2.5, system (2.1) is UEOC on $\mathbb{X}(\mathcal{P}_1)^q$. Therefore, from (2.5), for every function $f \in \mathbb{X}(\mathcal{P}_1)^q$, there exists a sequence of complex polynomials $(p_1^k, \dots, p_m^k)_k$ such that $\left\| \sum_{j=1}^m C p_j^k(A) b_j - f \right\|_\infty$ goes to zero when k goes to infinity. In other words, there exists a sequence of polynomials $(P^k)_k$ depending on the entries of the matrices A , B and C such that $\|P^k - f\|_\infty \rightarrow 0$ when $k \rightarrow +\infty$. But, since matrices A , B and C are holomorphic on \mathcal{V} , then the sequence $(P^k)_k$ is a sequence of holomorphic functions and since it converges uniformly on \mathcal{P}_1 towards f , then f is necessarily holomorphic on \mathcal{V} . This yields a contradiction. \square

Remark 4.5. Note that the result of this theorem relies mainly on the fact that if a sequence of holomorphic functions converges uniformly on all compacts of an open set, then the limit function is not only continuous but also holomorphic on that open set, see [32, Theorem 1.4]. In opposition, for real analysis, uniform convergence does not preserve differentiability. Hence, for a real analytic

system defined on a compact subset of \mathbb{R} , UEOC can hold even if the compact set is not of empty interior with respect to the topology of \mathbb{R} . The following system illustrates this fact.

$$\mathcal{P} = [-1, 1], \quad A(\theta) = \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}, \quad B(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C(\theta) = (1 \quad 1).$$

It is immediate that matrices A , B and C are analytic on \mathbb{R} and that \mathcal{P} does not have an empty interior with respect to the topology of \mathbb{R} . Using Theorem 4.9, given below, this system is UEOC on $\mathcal{C}(\mathcal{P}, \mathbb{R})$.

4.2 Results on particular structures

To begin with, let us consider the simplest systems, that is, the case whereby for three continuous functions λ , b and c defined from \mathcal{P} into \mathbb{C} , $(A, B, C) = (\lambda, b, c)$. We have the following theorem.

Theorem 4.6. *Let \mathcal{P} be a compact subset of \mathbb{C} with connected complement. The scalar system (λ, b, c) is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})$ if and only if*

- (i) *for every $\theta \in \mathcal{P}$, the system $(\lambda(\theta), b(\theta), c(\theta))$ is state to output controllable;*
- (ii) *λ is one-to-one;*
- (iii) *\mathcal{P} has no interior point with respect to the topology of \mathbb{C} .*

Proof. For the proof, it suffices to note that thanks to Proposition 2.4 system (λ, b, c) is UEOC if and only if for every $\theta \in \mathcal{P}$, $c(\theta) \neq 0$ and (λ, b) UEC, [16, Proposition 3]. Therefore, the necessity of these conditions, follows from [16, Theorem 3].

For the sufficient part, one can follow the proof of [16, Proposition 3]. The only argument that seems to fall down is the fact that we cannot use [35, Proposition 4.2.8] since $\lambda(\mathcal{P})$ is not contractible. Nevertheless, if \mathcal{P} is compact with connected complement in \mathbb{C} , then the connected components of \mathcal{P} are simply connected since $\mathbb{C} \setminus \mathcal{P}$ is path connected. Therefore, if λ is continuous and one-to-one on \mathcal{P} , then $\lambda(\mathcal{P})$ is of connected complement. Indeed, one can assume without loss of generality that \mathcal{P} is simply connected. If \mathcal{P} is not simply connected, write \mathcal{P} as a disjoint union of its connected components and use [16, Lemma 8]. Since λ is continuous, $\lambda(\mathcal{P})$ is compact. Let γ be a closed curve in $\lambda(\mathcal{P})$. Since λ is a homeomorphism from \mathcal{P} to $\lambda(\mathcal{P})$, $\lambda^{-1}(\gamma)$ is a closed curve in \mathcal{P} . Using the fact that \mathcal{P} is simply connected, we deduce that for any point $\theta_0 \in \mathcal{P}$, there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow \mathcal{P}$ such that $H(0, t) = \lambda^{-1} \circ \gamma(t)$ and $H(1, t) = \theta_0$. It follows that $\lambda \circ H(0, t) = \gamma(t)$ and $\lambda \circ H(1, t) = \lambda(\theta_0)$. We then deduce that any closed curve in $\lambda(\mathcal{P})$ can be continuously deformed into a single point. Therefore, $\lambda(\mathcal{P})$ is compact and simply connected. Thus, $\lambda(\mathcal{P})$ is of connected complement in \mathbb{C} . We can then conclude as in [16, Proposition 3], using Mergelyan's Theorem (see Theorem B.2). \square

For scalar systems with multiple inputs, the injectivity of λ in Theorem 4.6 fails to be necessary. For a more intuitive justification of this failure, let us first give the following sufficient condition of uniform ensemble output controllability for scalar systems with multiple inputs. Note that in contrast to linear parameter independent systems, it is reasonable to consider systems with $m > n$ since the state space is of infinite dimension.

Theorem 4.7. *Let \mathcal{P} be a compact and contractible subset of \mathbb{C} , $b = (b_1 \quad \dots \quad b_m)$ where b_k 's are scalar continuous functions, and let c be a scalar continuous function. Assume that*

- (i) *for every $\theta \in \mathcal{P}$, the system $(\lambda(\theta), b(\theta), c(\theta))$ is state to output controllable;*
- (ii) *λ is one-to-one;*

(iii) \mathcal{P} has no interior point with respect to the topology of \mathbb{C} .

Then (λ, b, c) is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})$.

Proof. To begin with, note that, since λ is assumed to be continuous and one-to-one, using Proposition 2.3, we can assume without loss of generality that λ is the identity map of \mathcal{P} . Also, from assumption (i) and Proposition 2.4, we can assume that $c(\theta) = 1$ for every $\theta \in \mathcal{P}$. Therefore, we will show that for every $f \in \mathbb{X}(\mathcal{P})$ and every $\varepsilon > 0$, there exist m complex polynomials p_1, \dots, p_m such that

$$\left\| \sum_{k=1}^m p_k(i_d) b_k - f \right\|_{\infty} = \sup_{\theta \in \mathcal{P}} \left| \sum_{k=1}^m p_k(\theta) b_k(\theta) - f(\theta) \right| \leq \varepsilon. \quad (4.11)$$

Let $f \in \mathbb{X}(\mathcal{P})$ and $\varepsilon > 0$. The assumption (i) ensures that $\text{rk } b(\theta) = 1$ for every $\theta \in \mathcal{P}$. In addition, since b is continuous on \mathcal{P} , \mathcal{P} compact and contractible, we deduce from Theorem B.1 that there exist m continuous functions χ_1, \dots, χ_m defined on \mathcal{P} such that $\chi_1 b_1 + \dots + \chi_m b_m = 1$. Thus, $f = f \chi_1 b_1 + \dots + f \chi_m b_m$. We now observe that for every $k \in \{1, \dots, m\}$, the function $f \chi_k$ belongs to $\mathbb{X}(\mathcal{P})$. Since \mathcal{P} is compact and contractible, we infer from [35, Proposition 4.2.8] that $\mathbb{C} \setminus \mathcal{P}$ is connected. Using assumption (iii) and applying Mergelyan's Theorem, we deduce by setting $\beta = \sum_{k=1}^m \sup_{\theta \in \mathcal{P}} |b_k(\theta)| > 0$ that for $k \in \{1, \dots, m\}$, there exists a complex polynomial p_k such that

$$\sup_{\theta \in \mathcal{P}} |p_k(\theta) - \chi_k(\theta) f(\theta)| \leq \frac{\varepsilon}{\beta}.$$

This leads to the following inequalities,

$$\begin{aligned} \sup_{\theta \in \mathcal{P}} \left| \sum_{k=1}^m p_k(\theta) b_k(\theta) - f(\theta) \right| &= \sup_{\theta \in \mathcal{P}} \left| \sum_{k=1}^m p_k(\theta) b_k(\theta) - \sum_{k=1}^m f(\theta) \chi_k(\theta) b_k(\theta) \right| \\ &\leq \sum_{k=1}^m \sup_{\theta \in \mathcal{P}} |(p_k(\theta) - f(\theta) \chi_k(\theta)) b_k(\theta)| \\ &\leq \sum_{k=1}^m \sup_{\theta \in \mathcal{P}} |b_k(\theta)| \sup_{\theta \in \mathcal{P}} |p_k(\theta) - f(\theta) \chi_k(\theta)| \\ &\leq \frac{\varepsilon}{\beta} \sum_{k=1}^m \sup_{\theta \in \mathcal{P}} |b_k(\theta)| = \varepsilon. \end{aligned}$$

□

Remark 4.8. • Thanks to Lemma 2.5, assumption (i) is a necessary condition.

- As mentioned earlier, assumption (ii) is not necessary unless $m = 1$. In the case $m = 1$, the necessity can be directly inferred from Lemma 2.5 and the Hautus Test [21, Theorem 1]. One can also see [16, Theorem 3]. To see that the injectivity of λ is not a necessary condition when $m > 1$, one can consider [16, Example 3]. More precisely, assuming that (iii) is fulfilled, we can, without loss of generality, use polynomials instead of continuous functions on \mathcal{P} for the controllability issue since any continuous function can be uniformly approximated by polynomials. To show the UEOC of the considered system, it would therefore be necessary and sufficient to prove that for every polynomial p and every $\varepsilon > 0$, there exist m complex polynomials p_k such that $\sup_{\theta \in \mathcal{P}} |\sum_{k=1}^m p_k(\lambda(\theta)) b_k(\theta) - p(\theta)| \leq \varepsilon$. The non necessity of condition (ii) follows from the fact that in some situations, it is possible to find such polynomials even if λ is not injective by solving the polynomial equation $\sum_{k=1}^m b_k p_k \circ \lambda - p = 0$. One can consider $\mathcal{P} = [-1, 1]$, $\lambda(\theta) = \theta^2$ and $B(\theta) = (1 \ \theta \ \dots)$. With this example, λ is clearly not injective on \mathcal{P} but for any fixed

polynomial p , it is not difficult to see that there exist two complex polynomials p_1 and p_2 such that $p_1(\theta^2) + \theta p_2(\theta^2) = p(\theta)$.

Note however, that even if the injectivity of λ is not required for $m > 1$, λ still have to be such that (3.1) holds, i.e., for every $\theta_0, \dots, \theta_m$ two by two distinct parameters, there exist $k, l \in \mathbb{N}_{\leq m}$ such that $\lambda(\theta_k) \neq \lambda(\theta_l)$.

- The necessity of assumption (iii) is still unclear to us except the case whereby (λ, b) is regular enough, in which case, the necessity follows from Theorem 3.1.

Let us consider now the case where A is a Jordan block, B and C respectively are constant column and row vectors.

Theorem 4.9. *Let \mathcal{P} be a compact subset of \mathbb{C} with connected complement. The system*

$$A(\theta) = \begin{pmatrix} \lambda(\theta) & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda(\theta) \end{pmatrix} \in \mathbb{C}_n^n, \quad B(\theta) \equiv B = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n$$

and $C(\theta) \equiv C = (c_1 \ \dots \ c_n) \in \mathbb{C}_n \quad (\theta \in \mathcal{P}). \quad (4.12)$

is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})$ if and only if

- (i) for every $\theta \in \mathcal{P}$, the system $(A(\theta), B, C)$ is state to output controllable;
- (ii) λ is one-to-one on \mathcal{P} ;
- (iii) \mathcal{P} is of empty interior, with respect to the topology of \mathbb{C} .

Proof. We give the proof for $n = 2$, i.e., with $A(\theta) = \begin{pmatrix} \lambda(\theta) & 1 \\ 0 & \lambda(\theta) \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $C = (c_1 \ c_2)$. The general case being similar.

- *Necessity.*

It suffices to note that setting the constants $\alpha_0 = c_1 b_1 + c_2 b_2 \in \mathbb{C}$ and $\alpha_1 = c_1 b_2 \in \mathbb{C}$, we have

$$\begin{aligned} \text{span} \{ CA^k B \mid k \in \mathbb{N} \} &= \text{span} \{ \alpha_0, \alpha_0 \lambda^k + k \alpha_1 \lambda^{k-1} \mid k \in \mathbb{N}^* \} \\ &\subset \text{span} \{ \alpha_0 \lambda^k \mid k \in \mathbb{N} \} + \text{span} \{ k \alpha_1 \lambda^{k-1} \mid k \in \mathbb{N}^* \} \\ &= \text{span} \{ \alpha_0 \lambda^k \mid k \in \mathbb{N} \} + \text{span} \{ \alpha_1 \lambda^k \mid k \in \mathbb{N} \} \\ &= \begin{cases} \{0\} & \text{if } \alpha_0 = \alpha_1 = 0, \\ \text{span} \{ \lambda^k \mid k \in \mathbb{N} \} & \text{otherwise.} \end{cases} \end{aligned}$$

From the above, if system (4.12) is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})$ then we have

$$\overline{\text{span} \{ CA^k B, k \in \mathbb{N} \}} = \mathbb{X}(\mathcal{P})$$

which is possible if α_0 and α_1 are not both null, that is if (i) is fulfilled, and if $\overline{\text{span} \{ \lambda^k \mid k \in \mathbb{N} \}} = \mathbb{X}(\mathcal{P})$, that is, if conditions (ii) and (iii) are satisfied.

- *Sufficiency.*

Thanks to (ii) and Proposition 2.3, we assume without loss of generality that $\lambda = i_d$. We now aim to show that for every $f \in \mathbb{X}(\mathcal{P})$ and every $\varepsilon > 0$, there exists a complex polynomial p such that

$$\sup_{\theta \in \mathcal{P}} |Cp(A(\theta))B - f(\theta)| = \sup_{\theta \in \mathcal{P}} |\alpha_1 p'(\theta) + \alpha_0 p(\theta) - f(\theta)| \leq \varepsilon. \quad (4.13)$$

Let $f \in \mathbb{X}(\mathcal{P})$ and $\varepsilon > 0$. Since the compact set \mathcal{P} has connected complement and no interior point, we deduce from Theorem B.2 that there exists a complex polynomial q such that

$$\sup_{\theta \in \mathcal{P}} |q(\theta) - f(\theta)| \leq \varepsilon/2. \quad (4.14)$$

To end the proof, it suffices to show that there exists a complex polynomial p such that

$$\sup_{\theta \in \mathcal{P}} |\alpha_0 p(\theta) + \alpha_1 p'(\theta) - q(\theta)| \leq \varepsilon/2. \quad (4.15)$$

Let us write $q(z) = \sum_{k=0}^N q_k z^k$. Recall that thanks to the necessary condition (i), α_0 or α_1 is different from zero. If $\alpha_0 \neq 0$, one can check that the polynomial p defined by $p(z) = \sum_{k=0}^N p_k z^k$, with $p_N = q_N/\alpha_0$, $p_{k-1} = (q_{k-1} - \alpha_1 k p_k)/\alpha_0$ for all $k \in \mathbb{N}_{\leq N}^*$ satisfies (4.15). If $\alpha_0 = 0$, then $\alpha_1 \neq 0$ and the polynomial $p = Q/\alpha_1$, with Q any polynomial such that $Q' = q$ satisfies (4.15). \square

It should be noticed through the above proof that the conditions listed in Theorem 4.9 are still, mutatis mutandis, necessary and sufficient even in multi-input case as far as the input and the output matrices are respectively a constant matrix and a single constant output row vector.

For the sufficiency of conditions in Theorem 4.9, one can see that we have solved explicitly the polynomial differential equation

$$\alpha_1 p' + \alpha_0 p = q. \quad (4.16)$$

This was simple to do since the complex numbers α_0 and α_1 are constants and at least one of them is not zero. In the case whereby B and C depend on θ , α_0 and α_1 in (4.15) become $\alpha_0(\theta) = c_1(\theta)b_1(\theta) + c_2(\theta)b_2(\theta)$ and $\alpha_1(\theta) = c_1(\theta)b_2(\theta)$. That is to say that α_0 and α_1 are in general non-constant complex continuous functions on \mathcal{P} . Nevertheless, one should note that if there exists a polynomial p satisfying (4.16) on \mathcal{P} then (4.15) is fulfilled. The dependency of α_0 and α_1 on θ makes the analysis more difficult, and some ingredients and particular cases are discussed in the following remarks.

Remark 4.10. If α_0 and α_1 are polynomials, one can see by setting $d_0 = d^\circ(\alpha_0)$, $d_1 = d^\circ(\alpha_1)$, $d_p = d^\circ(p)$ and $d_q = d^\circ(q)$, that given d_q , the number of equations and unknowns in (4.16) are respectively $\max\{d_0 + d_p, d_1 + d_p - 1, d_q\} + 1$ and $d_p + 1$. Therefore, for (4.16) to admit a polynomial solution, it is necessary that $\max\{d_0 + d_p, d_1 + d_p - 1, d_q\} \leq d_p$, that is $\max\{d_0, d_1 - 1, d_q - d_p\} \leq 0$. We infer that for the polynomial equation (4.16) to admit a solution, d_0 has to be equal to zero and $d_1 \in \{0, 1\}$. In general, one can see that under these necessary conditions on d_0 and d_1 , equation (4.16) admits a polynomial solution for every polynomial q with $\alpha_0(z) = \varsigma_0$, $\alpha_1(\theta) = \varsigma_{1,1}\theta + \varsigma_{1,0}$ for every $(\varsigma_0, \varsigma_{1,1}, \varsigma_{1,0}) \in \mathbb{C}^3$, if and only if $\varsigma_0 \notin -\varsigma_{1,1}\mathbb{N}$.

In the above remark, the case whereby $a \in -b\mathbb{N}$ can be handled in some particular cases. It is for instance, the case if the parameterization set \mathcal{P} is a real interval, see Remark 4.12. Before showing this, let us introduce the following remark in which α_0 and α_1 do not need to be necessarily polynomials.

Remark 4.11. Let us consider the case where $\mathcal{P} = [\theta_0, \theta_1] \subset \mathbb{R}$ with $\theta_0 < \theta_1$ and assume that the continuous function α_1 does not vanish on \mathcal{P} . In this case, one can solve the ordinary differential equation $h' = -\frac{\alpha_0}{\alpha_1}h + \frac{q}{\alpha_1}$. Therefore, there exists a differentiable solution h of (4.16), but h might

not be polynomial.

Now, choose for $\eta > 0$, a polynomial \tilde{p} satisfying $\|h' - \tilde{p}\|_\infty \leq \eta$, and set $p(\theta) = h(\theta_0) + \int_{\theta_0}^\theta \tilde{p}(z) dz$ for any $\theta \in \mathcal{P}$. We have $\|p - h\|_\infty \leq \eta(\theta_1 - \theta_0)$. In addition, we have

$$\begin{aligned} \|\alpha_1 p' + \alpha_0 p - q\|_\infty &\leq \|\alpha_0\|_\infty \|p - h\|_\infty + \|\alpha_1\|_\infty \|\tilde{p} - h'\|_\infty + \|\alpha_1 h' + \alpha_0 h - q\|_\infty \\ &\leq \|\alpha_0\|_\infty \eta(\theta_1 - \theta_0) + \|\alpha_1\|_\infty \eta. \end{aligned}$$

Hence, choosing $\eta > 0$ small enough, we get $\|\alpha_1 p' + \alpha_0 p - q\|_\infty \leq \varepsilon$.

If α_1 vanishes on \mathcal{P} (but is not identically null on \mathcal{P}), the solvability of (4.16) is not clear, unless α_0 and α_1 are polynomials. This situation is discussed in the next remark.

Remark 4.12. We also assume, as in Remark 4.11, that $\mathcal{P} = [\theta_0, \theta_1] \subset \mathbb{R}$, and we continue Remark 4.10 in the case $\varsigma_0 = -\varsigma_{1,1} N$ for some $N \in \mathbb{N}$. In this case, it is not possible to get directly a polynomial satisfying (4.16) with $q(\theta) = \theta^N$ but thanks to the margin in (4.15), we can show that there exists a polynomial satisfying (4.15). If α_1 does not vanish on $[\theta_0, \theta_1]$, then one can use Remark 4.11 to conclude. But, when α_1 vanishes, one cannot use Remark 4.11. The idea is then to perturb α_1 by some η such that $\alpha_1 + \eta$ does not vanish, use Remark 4.11 with α_1 replaced by $\alpha_1 + \eta$, and finally pass to the limit $|\eta| \rightarrow 0$.

We illustrate this strategy with $N = 1$, $\alpha_0 = 1$ and $\alpha_1 = -\theta$ (and $0 \in [\theta_0, \theta_1]$). The other situations can be computed similarly. For every $\eta \in \mathbb{R}^*$, we set $\tilde{\alpha}_1(\theta) = -\theta + i\eta$, so that $\tilde{\alpha}_1$ does not vanish on $[\theta_0, \theta_1]$. Solving the ordinary differential equation $h'_\eta = -\frac{\alpha_0}{\tilde{\alpha}_1} h_\eta + \frac{q}{\tilde{\alpha}_1} = \frac{h_\eta - \theta}{\theta - i\eta}$, we obtain the particular solution³ $h_\eta(\theta) = -(\theta - i\eta) \log(\theta - i\eta) + \theta$, for which we have $h'_\eta(\theta) = -\log(\theta - i\eta)$. We now have,

$$|\alpha_1(\theta)h'_\eta(\theta) + \alpha_0(\theta)h_\eta(\theta) - q(\theta)| \leq |(\alpha_1 + i\eta)h'_\eta(\theta) + \alpha_0(\theta)h_\eta(\theta) - q(\theta)| + |\eta h'_\eta(\theta)| = |\eta h'_\eta(\theta)|.$$

That is to say, $|\alpha_1(\theta)h'_\eta(\theta) + \alpha_0(\theta)h_\eta(\theta) - q(\theta)| \leq |\eta \log(\theta - i\eta)|$. It is then easy to see that $\sup_{\theta \in [\theta_0, \theta_1]} |\eta \log(\theta - i\eta)| \rightarrow 0$ as $|\eta| \rightarrow 0$. In particular, for every $\varepsilon > 0$, there exist $\eta = \eta(\varepsilon) \neq 0$ such that $\sup_{\theta \in [\theta_0, \theta_1]} |\alpha_1(\theta)h'_\eta(\theta) + \alpha_0(\theta)h_\eta(\theta) - q(\theta)| \leq \varepsilon/2$. We then conclude as in Remark 4.11 by approximating h_η by a polynomial p , such that h'_η is also approximated by p' .

To conclude this paragraph, we consider a particular case where matrices B and C are constant and where A is linear with respect to the parameter. The following theorem extends [28, Theorem 1] and [40, Theorem 5] to the case of UEOC.

Theorem 4.13. *Let \mathcal{P} be a compact subset of \mathbb{C} with empty interior and connected complement. The system $(\theta A, B, C)$ where matrices A , B and C are constant is uniformly ensemble output controllable if*

$$\text{rk}(CA^k B) = q, \quad \forall k \in \mathbb{N}. \quad (4.17)$$

Moreover, (4.17) is necessary if

$$0 \in \mathcal{P}, \quad A \ker C \subset \ker C \quad \text{and} \quad \text{Im } C^* \cap \ker(A^*)^n = \{0\}. \quad (4.18)$$

It has to be mentioned that condition (4.17) is not necessary. This can be seen by considering for a non-zero complex number θ the system

$$\mathcal{P} = \{\theta\}, \quad A(\theta) = \theta \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C = (0 \quad 1).$$

It is straightforward that \mathcal{P} satisfies the assumptions of Theorem 4.13 and that UEOC here is nothing else than the state to output controllability of the above system. Through (4.1), it can be seen that the above system is state to output controllable but $\text{rk}(CB) = 0$.

³We use \log to denote the principal branch of the complex logarithm.

Proof. Let $T > 0$. From our discussion of Remark 3.5, the UEOC of $(\theta A, B, C)$ is equivalent to the unique continuation property

$$\forall t \in [0, T], (\Phi_T^* \mu)(t) = \int_{\mathcal{P}} B^* e^{\bar{\theta}(T-t)A^*} C^* d\mu(\theta) = 0 \implies \mu = 0, \quad (4.19)$$

where μ belongs to $\mathcal{M}(\mathcal{P})$. Assume now that (4.17) holds and let $\mu \in \mathcal{M}(\mathcal{P})$ such that

$$(\Phi_T^* \mu)(t) = 0, \quad \forall t \in [0, T]. \quad (4.20)$$

Computing the derivatives of (4.20) with respect to t and evaluating the resulting equation at time T , we get

$$B^* (A^*)^k C^* \int_{\mathcal{P}} \bar{\theta}^k d\mu(\theta) = 0 \quad (k \in \mathbb{N}). \quad (4.21)$$

From (4.17), we infer that

$$\int_{\mathcal{P}} \theta^k d\bar{\mu}(\theta) = 0 \quad (k \in \mathbb{N}) \quad (4.22)$$

and therefore that for every polynomial p ,

$$\int_{\mathcal{P}} p(\theta) d\bar{\mu}(\theta) = 0. \quad (4.23)$$

To finish the proof, note that thanks to the assumptions on \mathcal{P} and to Mergelyan's Theorem (see Theorem B.2), every $f \in \mathbb{X}(\mathcal{P})^q$ can be uniformly approximated by polynomials on \mathcal{P} . This ensures that $\bar{\mu} = 0$.

Let us now prove the necessity of (4.17) for UEOC when (4.18) holds.

Assume that the system $(\theta A, B, C)$ is UEOC and that (4.18) holds. Assume also, by contradiction, that (4.17) does not hold, i.e., there exists $\kappa \in \mathbb{N}$ such that $\text{rk}(CA^\kappa B) < q$.

Since the system is UEOC and 0 belongs to \mathcal{P} , then $\kappa \neq 0$ because, using Lemma 2.5 together with Proposition 4.1 (with $\mathcal{P} = \{0\}$), we necessarily have $\ker B^* \cap \text{Im } C^* = \{0\}$. Using the fact that $\text{rk}(CA^\kappa B) < q$, we deduce the existence of a non-zero vector $\eta \in \mathbb{C}^q$ such that $\eta^* CA^\kappa B = 0$, that is $(A^\kappa)^* v \in \ker B^*$ where $v = C^* \eta \in \text{Im } C^*$. We have $v \neq 0$, since the system is assumed to be uniform ensemble output controllable, recall that this implies $\text{rk } C = q$. If $(A^\kappa)^* v = 0$ then we end up with a contradiction since $v \in \ker(A^\kappa)^* \cap \text{Im } C^*$. If $(A^\kappa)^* v \neq 0$, we deduce from $\ker B^* \cap \text{Im } C^* = \{0\}$ and $(A^\kappa)^* v \in \ker B^*$ that $(A^\kappa)^* v \notin \text{Im } C^*$. Therefore, $\text{Im } C^*$ cannot be stable by A^* . \square

5 More general results

In this section, we give more abstract contributions to uniform ensemble output controllability. Before stating our results, we need to introduce the spectrum, $\sigma_{\mathbb{X}^n}(\mathfrak{M}_A)$, of the multiplication operator $\mathfrak{M}_A : \mathbb{X}(\mathcal{P})^n \rightarrow \mathbb{X}(\mathcal{P})^n$ associated to A . Recall that \mathfrak{M}_A has been defined in (2.2). To this end, we introduce as in [16] the set-valued *spectral map* $\text{spec } A : \mathcal{P} \rightsquigarrow \mathbb{C}$, defined for every $\theta \in \mathcal{P}$ by

$$\text{spec } A(\theta) = \sigma(A(\theta)).$$

It has been shown in [34, Lemma 7.1] (see also [20]) that the closure in \mathbb{C} of the image of the above *spectral map* is the spectrum of the multiplication operator \mathfrak{M}_A . More precisely, we have

$$\sigma_{\mathbb{X}^n}(\mathfrak{M}_A) = \overline{\bigcup_{\theta \in \mathcal{P}} \sigma(A(\theta))}.$$

5.1 Uniform ensemble output controllability of parallel systems

In this paragraph we give some results on UEOC of parallel systems.

Theorem 5.1. *Let \mathcal{P} be a compact subset of \mathbb{C} , $N, m, \kappa \in \mathbb{N}^*$, $n_1, \dots, n_N \in \mathbb{N}^*$, and $q_1, \dots, q_\kappa \in \mathbb{N}^*$. For every $i \in \mathbb{N}_{\leq N}^*$ we consider $A_i \in \mathbb{X}(\mathcal{P})_{n_i}^{n_i}$, $B_i \in \mathbb{X}(\mathcal{P})_m^{n_i}$, and for every $i \in \mathbb{N}_{\leq \kappa}^*$ and every $j \in \{i, \dots, N\}$, we consider $C_{i,j} \in \mathbb{X}(\mathcal{P})_{n_j}^{q_i}$. Let us consider the parallel system*

$$\left(\left(\begin{array}{ccc} A_1 & & \\ & \ddots & \\ & & A_N \end{array} \right), \left(\begin{array}{c} B_1 \\ \vdots \\ B_N \end{array} \right), \left(\begin{array}{cccc} C_{1,1} & \cdots & \cdots & C_{1,N} \\ & \ddots & & \vdots \\ & & C_{\kappa,\kappa} & \cdots \\ & & & C_{\kappa,N} \end{array} \right) \right) \quad (5.1)$$

and assume that

- (i) for every $k \in \mathbb{N}_{\leq \kappa-1}^*$ the system $(A_k, B_k, C_{k,k})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_k}$;
- (ii) there exists $r \in \{\kappa, \dots, N\}$ such that the system $(A_r, B_r, C_{\kappa,r})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_\kappa}$;
- (iii) for every $k \in \mathbb{N}_{\leq N}^*$, $\sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k})$ is non-separating with finitely many connected components;
- (iv) for every $k, \ell \in \mathbb{N}_{\leq N}^*$ such that $k \neq \ell$, $\sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k}) \cap \sigma_{\mathbb{X}^{n_\ell}}(\mathfrak{M}_{A_\ell}) = \emptyset$.

Then the parallel system (5.1) is UEOC on $\mathbb{X}(\mathcal{P})^q$, with $q = q_1 + \dots + q_\kappa$.

The criteria in the above theorem are sufficient and not necessary. One can see from Remark 4.2 that even for finite dimensional systems, the spectral disjointness condition (iv) is not necessary. Also, in some cases it might happen that two or many blocks of operator A share the same eigenvalue. In the application of this theorem, one can, thanks to Proposition 2.3, perform a permutation in the state space variable in order to gather together the blocks that share the same eigenvalue.

To make the proof easier, we first consider the following two lemmas. More precisely, in Lemma 5.2, we consider the case $\kappa = N$ and in Lemma 5.3, we consider the case $\kappa = 1$. These two lemmas are obviously particular cases of Theorem 5.1.

Lemma 5.2. *With the notations and assumptions introduced in Theorem 5.1, with $\kappa = N$, the parallel system*

$$\left(\left(\begin{array}{ccc} A_1 & & \\ & \ddots & \\ & & A_N \end{array} \right), \left(\begin{array}{c} B_1 \\ \vdots \\ B_N \end{array} \right), \left(\begin{array}{ccc} C_{1,1} & \cdots & C_{1,N} \\ & \ddots & \vdots \\ & & C_{N,N} \end{array} \right) \right) \quad (5.2)$$

is UEOC on $\mathbb{X}(\mathcal{P})^q$.

Proof. The proof of this lemma follows the same arguments as the ones of [16, Theorem 1] and is also done for the case $N = 2$ and $m = 1$, $B_i = b_i$. The proof of the general case follows the same arguments.

Let $(y_1, y_2) \in \mathbb{X}(\mathcal{P})^{q_1} \times \mathbb{X}(\mathcal{P})^{q_2}$ and take any $\varepsilon > 0$. The aim is to show that there exists a complex polynomial p such that

$$\left\| \begin{pmatrix} C_{1,1}p(A_1)b_1 + C_{1,2}p(A_2)b_2 - y_1 \\ C_{2,2}p(A_2)b_2 - y_2 \end{pmatrix} \right\|_\infty \leq \varepsilon. \quad (5.3)$$

From assumption (i), there exists a complex polynomial p_2 such that

$$\|C_{2,2}p_2(A_2)b_2 - y_2\|_\infty \leq \varepsilon.$$

Using the fact that any connected component of a compact set is compact, applying [16, Lemmas 7 and 8], and using assumptions (ii)–(iii), we get the existence of two disjoint compact sets K_1 and K_2 which do not separate the plane and properly contain $\sigma_{\mathbb{X}^{n_1}}(\mathfrak{M}_{A_1})$ and $\sigma_{\mathbb{X}^{n_2}}(\mathfrak{M}_{A_2})$ respectively. From [33] (see also [16, Lemma 9]), considering, for $\ell \in \{1, 2\}$, the functions $h_\ell : K_1 \cup K_2 \rightarrow \mathbb{C}$ defined by

$$h_1(z) = \begin{cases} 1 & \text{if } z \in K_1, \\ 0 & \text{if } z \in K_2, \end{cases} \quad \text{and} \quad h_2(z) = \begin{cases} 0 & \text{if } z \in K_1, \\ 1 & \text{if } z \in K_2, \end{cases}$$

there exist, for every $\tilde{\varepsilon} > 0$, two polynomials π_1 and π_2 such that

$$|\pi_1(z) - h_1(z)| < \tilde{\varepsilon} \quad \text{and} \quad |\pi_2(z) - h_2(z)| < \tilde{\varepsilon} \quad (z \in K_1 \cup K_2). \quad (5.4)$$

From (5.4), we have

$$\begin{aligned} |\pi_1(z) - 1| < \tilde{\varepsilon} & \quad \text{and} \quad |\pi_2(z)| < \tilde{\varepsilon} & (z \in K_1), \\ |\pi_1(z)| < \tilde{\varepsilon} & \quad \text{and} \quad |\pi_2(z) - 1| < \tilde{\varepsilon} & (z \in K_2). \end{aligned} \quad (5.5)$$

At this stage, the only completely known polynomial is p_2 . The polynomials π_1 and π_2 have to be adjusted by the choice of $\tilde{\varepsilon}$ in order to get (5.3). Setting therefore $\tilde{y}_1 = y_1 - C_{1,2}p_2(A_2)b_2$, we have by assumption (i) the existence of a complex polynomial p_1 , depending on p_2 and y_1 , such that

$$\|C_{1,1}p_1(A_1)b_1 - \tilde{y}_1\|_\infty \leq \frac{\varepsilon}{2} \iff \|C_{1,1}p_1(A_1)b_1 - y_1 + C_{1,2}p_2(A_2)b_2\|_\infty \leq \frac{\varepsilon}{2}.$$

We now choose, as in [16], $p(z) = \pi_1(z)p_1(z) + \pi_2(z)p_2(z)$ and show that a suitable choice of $\tilde{\varepsilon}$ can be made in (5.5) so that p satisfies (5.3). For more details on the construction and properties of p , we refer to [3].

By adding and subtracting $C_{1,1}p_1(A_1)b_1$ and $C_{1,2}p_2(A_2)b_2$ and using triangular inequality, we have

$$\begin{aligned} \|C_{11}p(A_1)b_1 + C_{12}p(A_2)b_2 - y_1\|_\infty &\leq \|C_{1,1}\pi_1(A_1)p_1(A_1)b_1 - C_{1,1}p_1(A_1)b_1\|_\infty \\ &\quad + \|C_{1,1}\pi_2(A_1)p_2(A_1)b_1\|_\infty \\ &\quad + \|C_{1,2}\pi_1(A_2)p_1(A_2)b_2\|_\infty \\ &\quad + \|C_{1,2}p_2(A_2)\pi_2(A_2)b_2 - C_{1,2}p_2(A_2)b_2\|_\infty \\ &\quad + \|C_{1,1}p_1(A_1)b_1 - y_1 + C_{1,2}p_2(A_2)b_2\|_\infty. \end{aligned}$$

Let us first consider the first term on the right-hand side of this inequality. We have,

$$\|C_{1,1}\pi_1(A_1)p_1(A_1)b_1 - C_{1,1}p_1(A_1)b_1\|_\infty \leq \|C_{1,1}\|_\infty \|(\pi_1(A_1) - I_{n_1})p_1(A_1)\|_\infty \|b_1\|_\infty.$$

Using the Dunford-Taylor formula, that is,

$$g(A_k(\theta)) = \frac{1}{2i\pi} \int_{\gamma_k} g(z)(zI_{n_k} - A_k(\theta))^{-1} dz \quad (k \in \{1, 2\}),$$

where g is any complex polynomial and γ_k consists of the union of finitely closed polygonal grid curves with positive direction contained in $K_k \setminus \sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k})$ (see [24, Chapter 1]), and (5.5), we have

$$\begin{aligned} \|(\pi_1(A_1) - I_{n_1})p_1(A_1)\|_\infty &\leq \frac{1}{2\pi} \int_{\gamma_1} |(\pi_1(z) - 1)||p_1(z)|||(zI_{n_1} - \mathfrak{M}_{A_1})^{-1}| dz \\ &\leq \frac{\tilde{\varepsilon}L_1}{2\pi} \sup_{z \in \text{Im}(\gamma_1)} (|p_1(z)|||(zI_{n_1} - \mathfrak{M}_{A_1})^{-1}|). \end{aligned}$$

In what precedes (and follows), L_k stands for the length of the curve γ_k for $k \in \{1, 2\}$. From the above, we deduce by setting for $k, \ell \in \{1, 2\}$, $\beta_k = \|b_k\|_\infty$, $\nu_{k,\ell} = \|C_{k,\ell}\|_\infty$ and $\alpha_{k,\ell} = \sup_{z \in \gamma_k} (|p_\ell(z)| \|zI_{n_k} - \mathfrak{M}_{A_k}\|^{-1})$,

$$\begin{aligned} \|C_{1,1}\pi_1(A_1)p_1(A_1)b_1 - C_{1,1}p_1(A_1)b_1\|_\infty &\leq \frac{\tilde{\varepsilon}\nu_{1,1}\beta_1\alpha_{1,1}L_1}{2\pi}, \\ \|C_{1,1}\pi_2(A_1)p_2(A_1)b_1\|_\infty &\leq \frac{\tilde{\varepsilon}\alpha_{1,2}\beta_1\nu_{1,1}L_1}{2\pi}, \\ \|C_{1,2}\pi_1(A_2)p_1(A_2)b_2\|_\infty &\leq \frac{\tilde{\varepsilon}\alpha_{2,1}\beta_2\nu_{1,2}L_2}{2\pi}, \\ \|C_{1,2}p_2(A_2)\pi_2(A_2)b_2 - C_{1,2}p_2(A_2)b_2\|_\infty &\leq \frac{\tilde{\varepsilon}\alpha_{2,2}\beta_2\nu_{1,2}L_2}{2\pi}. \end{aligned}$$

Finally, we have

$$\|C_{1,1}p(A_1)b_1 + C_{1,2}p(A_2)b_2 - y_1\|_\infty \leq \frac{\tilde{\varepsilon}}{2\pi} (\beta_1\nu_{1,1}L_1(\alpha_{1,1} + \alpha_{1,2}) + \beta_2\nu_{1,2}L_2(\alpha_{2,1} + \alpha_{2,2})) + \frac{\varepsilon}{2}.$$

We then choose $\tilde{\varepsilon} < \pi\varepsilon (\beta_1\nu_{1,1}L_1(\alpha_{1,1} + \alpha_{1,2}) + \beta_2\nu_{1,2}L_2(\alpha_{2,1} + \alpha_{2,2}))^{-1} := \tilde{\varepsilon}_1$ to get

$$\|C_{1,1}p(A_1)b_1 + C_{1,2}p(A_2)b_2 - y_1\|_\infty \leq \varepsilon.$$

To end this proof, let us show that $\tilde{\varepsilon}$ in (5.5) can also be chosen such a way that the polynomial p satisfies

$$\|C_{2,2}p(A_2)b_2 - y_2\|_\infty \leq \frac{\varepsilon}{2}.$$

Again, by adding and subtracting $C_{2,2}p_1(A_2)p_1(A_2)b_2$ and using the triangular inequality, we get

$$\begin{aligned} \|C_{2,2}p(A_2)b_2 - y_2\|_\infty &\leq \|C_{2,2}\pi_1(A_2)p_1(A_2)b_2\|_\infty + \|C_{2,2}p_2(A_2)b_2 - y_2\|_\infty \\ &\quad + \|C_{2,2}\pi_2(A_2)p_2(A_2)b_2 - C_{2,2}p_2(A_2)b_2\|_\infty. \end{aligned}$$

Using the same arguments as those used for the first component of (5.3), we have

$$\begin{aligned} \|C_{2,2}\pi_1(A_2)p_1(A_2)b_2\|_\infty &\leq \frac{\tilde{\varepsilon}\alpha_{2,1}\beta_2\nu_{2,2}L_2}{2\pi}, \\ \|C_{2,2}\pi_2(A_2)p_2(A_2)b_2 - C_{2,2}p_2(A_2)b_2\|_\infty &\leq \frac{\tilde{\varepsilon}\alpha_{2,2}\beta_2\nu_{2,2}L_2}{2\pi}, \end{aligned}$$

and therefore,

$$\|C_{2,2}p(A_2)b_2 - y_2\|_\infty \leq \frac{\tilde{\varepsilon}}{2\pi} \beta_2\nu_{2,2}L_2(\alpha_{2,1} + \alpha_{2,2}) + \frac{\varepsilon}{2}.$$

We then choose $\tilde{\varepsilon}_2 < \pi\varepsilon (L_2\beta_2\nu_{2,2}(\alpha_{2,1} + \alpha_{2,2}))^{-1} := \tilde{\varepsilon}_2$ to get

$$\|C_{2,2}p(A_2)b_2 - y_2\|_\infty \leq \varepsilon.$$

We finally get (5.3), by choosing $\tilde{\varepsilon} \in (0, \min\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\})$ in (5.5). \square

Lemma 5.3. *Using the notations of Theorem 5.1, with $\kappa = 1$, assume that*

- (ii') *there exists $r \in \mathbb{N}_{\leq N}^*$ such that (A_r, B_r, C_r) is UEOC on $\mathbb{X}(\mathcal{P})^q$,*
- (iii') *the sets $\sigma_{\mathbb{X}^{n_r}}(\mathfrak{M}_{A_r})$ and $\bigcup_{\substack{k=1 \\ k \neq r}}^N \sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k})$ are non-separating, with finitely many connected components;*

$$(iv') \quad \sigma_{\mathbb{X}^{n_r}}(\mathfrak{M}_{A_r}) \cap \bigcup_{\substack{k=1 \\ k \neq r}}^N \sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k}) = \emptyset.$$

Then the system

$$\left(\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}, (C_1 \quad \cdots \quad C_N) \right)$$

is UEOC on $\mathbb{X}(\mathcal{P})^q$

Let us first mention that in Lemma 5.3, condition (iv') is not necessary. To see that, let us go back to the example of Figure 1. For the sake of simplicity, we take $m_1 = m_2 = 1$, $f_1 = -\theta v_1$, $f_2 = -2\theta v_2$ where v_k stands for the speed of the k -th car and θ living in $\mathcal{P} = [0, 1]$. Let us set as output $y(t) = v_1(t) - v_2(t)$. Setting $x = (x_1, v_1, x_2, v_2)^\top$, $u = (u_1, u_2)^\top$, the state of the system coupled with the output variable y yields (2.1) with

$$\mathcal{P} = [0, 1], \quad A(\theta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2\theta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = (0 \quad 1 \quad 0 \quad -1). \quad (5.6)$$

On this simple example with two blocks in matrix A , one can see that the spectral disjointness condition (iv') is not fulfilled. Keeping in mind the Hautus Test consequence (3.1), it can also be readily seen that the system $(A(\theta), B)$ is not uniformly ensemble controllable since $0 \in \sigma(A(\theta))$ for every $\theta \in \mathcal{P}$. But performing a permutation in the state space variable, one can see that the uniform ensemble output controllability of (5.6) is equivalent to the one of $(\tilde{\theta}\tilde{A}, \tilde{B}, \tilde{C})$ where $\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{C} = (1 \quad -1)$. Therefore, the uniform ensemble output controllability of system (5.6) follows from Theorem 4.13.

This example also gives a contrast between UEC and UEOC. Here, we can see that the system is UEOC while the matrix $A(\theta)$ admits a constant eigenvalue. This is an obstruction to UEC.

Proof of Lemma 5.3. The proof is done for $m = 1$, that is the case where the operator B is a column vector that will be denoted by b . In addition, we assume without loss of generality that (A_1, b_1, C_1) is UEOC on $\mathbb{X}(\mathcal{P})^q$. The aim is to show that for every $y_1 \in \mathbb{X}(\mathcal{P})^q$ and every $\varepsilon > 0$, there exists a complex polynomial p such that

$$\left\| C_1 p(A_1) b_1 + \tilde{C} p(\tilde{A}) \tilde{b} - y_1 \right\|_\infty \leq \varepsilon, \quad (5.7)$$

where

$$\tilde{A} = \begin{pmatrix} A_2 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b_2 \\ \vdots \\ b_N \end{pmatrix}, \quad \tilde{C} = (C_2 \quad \cdots \quad C_N).$$

Since (A_1, b_1, C_1) is UEOC by assumption, there exists a complex polynomial p_1 such that

$$\|C_1 p_1(A_1) b_1 - y_1\|_\infty \leq \frac{\varepsilon}{2}.$$

Using (iii') and (iv') and following the proof of Lemma 5.2, one can construct two disjoint non-separating compact sets K_1 and K_2 strictly containing $\sigma_{\mathbb{X}^{n_1}}(\mathfrak{M}_{A_1})$ and $\sigma_{\mathbb{X}^{\tilde{n}}}(\mathfrak{M}_{\tilde{A}}) =$

$\bigcup_{k=2}^N \sigma_{\mathbb{X}^{n_k}}(\mathfrak{M}_{A_k})$ respectively, where $\tilde{n} = \sum_{k=2}^N n_k$. Design now π_1 as in (5.5) and choose $p = p_1 \pi_1$. To finish the proof, write

$$\begin{aligned} \left\| C_1 p(A_1) b_1 + \tilde{C} p(\tilde{A}) \tilde{b} - y_1 \right\|_{\infty} &\leq \|C_1 p_1(A_1) \pi_1(A_1) b_1 - C_1 p_1(A_1) b_1\|_{\infty} \\ &\quad + \left\| \tilde{C} p_1(\tilde{A}) \pi_1(\tilde{A}) \tilde{b} \right\|_{\infty} + \|C_1 p_1(A_1) b_1 - y_1\|_{\infty} \end{aligned}$$

and use the same arguments as in the proof of Lemma 5.2 to design a suitable $\tilde{\varepsilon} > 0$ for (5.5) and therefore an adequate approximation of h_1 , the indicator function of K_1 , so that the polynomial p satisfies (5.7). \square

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We take $N = 3$ and $\kappa = 2$ in which case (5.1) becomes

$$\left(\left(\begin{array}{cc} A_1 & \\ & A_2 \\ & & A_3 \end{array} \right), \left(\begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} \right), \left(\begin{array}{ccc} C_{1,1} & C_{1,2} & C_{1,3} \\ & C_{2,2} & C_{2,3} \end{array} \right) \right). \quad (5.8)$$

The general case follows using the same arguments. Set now

$$\tilde{A}_2 = \begin{pmatrix} A_2 & \\ & A_3 \end{pmatrix}, \tilde{B}_2 = \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}, \tilde{C}_{1,2} = (C_{1,2} \quad C_{1,3}) \text{ and } \tilde{C}_{2,2} = (C_{2,2} \quad C_{2,3}).$$

With these new notations, (5.8) becomes

$$\left(\left(\begin{array}{cc} A_1 & \\ & \tilde{A}_2 \end{array} \right), \left(\begin{array}{c} B_1 \\ \tilde{B}_2 \end{array} \right), \left(\begin{array}{cc} C_{1,1} & \tilde{C}_{1,2} \\ & \tilde{C}_{2,2} \end{array} \right) \right).$$

Using assumptions (ii)–(iv) and Lemma 5.3, we deduce that the system $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_{2,2})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_2}$.

Finally, note that $\sigma_{\mathbb{X}^{n_1+n_2}}(\mathfrak{M}_{\tilde{A}_2}) = \sigma_{\mathbb{X}^{n_2}}(\mathfrak{M}_{A_2}) \cup \sigma_{\mathbb{X}^{n_3}}(\mathfrak{M}_{A_3})$ has finitely many connected components, does not separate the plane, thanks to [16, Lemma 7], and does not intersect $\sigma_{\mathbb{X}^{n_1}}(\mathfrak{M}_{A_1})$, by assumption. Therefore, since the subsystems $(A_1, B_1, C_{1,1})$ and $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_{2,2})$ are UEOC, the UEOC of (5.8) follows from Lemma 5.2. \square

5.2 Diagonalization of the state matrix A into Jordan blocks form

From Theorem 5.1 and keeping in mind Proposition 2.3, it seems natural to look for the situation whereby the system $(A(\theta), B(\theta), C(\theta))$ can take the form (5.1). To this end, we introduce as in [16] the following definitions and notations.

A set valued map $\Gamma : \mathcal{P} \rightsquigarrow \mathbb{C}$ is said to be a *partial spectral map*, if for every $\theta \in \mathcal{P}$, $\Gamma(\theta) \subset \sigma(A(\theta))$, and is said to be continuous if it is with respect to the Hausdorff metric. A *single-valued partial spectral map* will be referred to as an *eigenvalue selection* and will be denoted by $\lambda : \mathcal{P} \rightarrow \mathbb{C}$. Two partial spectral maps Γ_1 and Γ_2 are said to be *pointwise disjoint* if $\Gamma_1(\theta) \cap \Gamma_2(\theta) = \emptyset$ for every $\theta \in \mathcal{P}$ and *strictly disjoint* if $\Gamma_1(\mathcal{P}) \cap \Gamma_2(\mathcal{P}) = \emptyset$. Finitely many *continuous partial spectral maps* $\Gamma_1, \dots, \Gamma_N$ are called a *continuous spectral decomposition* of A if for every $\theta \in \mathcal{P}$,

$$\bigcup_{k=1}^N \Gamma_k(\theta) = \sigma(A(\theta)).$$

Continuous spectral decomposition always exists. In fact, one can take for instance the trivial decomposition $\Gamma(\theta) := \sigma(A(\theta))$. If $\Gamma_1, \dots, \Gamma_N$ are single-valued, then the spectral decomposition will be said to be *single valued* and if they are *pairwise pointwise disjoint* (resp. *pairwise strictly disjoint*), then the spectral decomposition will be said to be *pointwise disjoint* (resp. *strictly disjoint*).

The following result has been given in [16, Proposition 2] and this proposition is proven in [16, Section 6.2]. In this proposition, the parameter set \mathcal{P} will be required not only to be compact but also to be contractible. For a better understanding of these assumptions on \mathcal{P} , we refer to [19].

Proposition 5.4. *Assume that \mathcal{P} is compact and contractible and that there exists a continuous spectral decomposition $\Gamma_1, \dots, \Gamma_N$ of A which is pointwise disjoint. Then, there exists a continuous family of regular matrices $T(\theta)$ such that*

$$T(\theta)^{-1}A(\theta)T(\theta) = \begin{pmatrix} A_1(\theta) & & \\ & \ddots & \\ & & A_N(\theta) \end{pmatrix}, \quad (5.9)$$

where, for every $k \in \mathbb{N}_{\leq N}^*$, A_k are continuous matrices and for every $\theta \in \mathcal{P}$, $\sigma(A_k(\theta)) = \Gamma_k(\theta)$.

Unless otherwise stated in this paragraph, we assume that A admits a continuous spectral decomposition $\Gamma_1, \dots, \Gamma_N$ which is pointwise disjoint. With the assumption made in Proposition 5.4 on the set \mathcal{P} , the matrix A is continuously transformed in the block diagonal form as in Lemma 5.3. The matrix B is transformed in $T^{-1}B$ and can be expressed as the one of Lemma 5.3. Indeed, let us define $\Pi_k := (0 \ \dots \ 0 \ I_{n_k} \ 0 \ \dots \ 0) \in \mathbb{C}_n^{n_k}$, where n_k is the number of rows of the Jordan block A_k . We then have $A_k(\theta) = \Pi_k T(\theta)^{-1} A(\theta) T(\theta) \Pi_k^*$, and set $B_k(\theta) := \Pi_k T(\theta)^{-1} B(\theta)$ and $C_k(\theta) := C(\theta) T(\theta) \Pi_k^*$. Performing this transformation, the following result follows from Lemma 5.3.

Theorem 5.5. *Assume that \mathcal{P} is compact and contractible and that the operator A admits a continuous strictly disjoint spectral decomposition $\Gamma_1, \dots, \Gamma_N$. Assume also that for every $k \in \mathbb{N}_{\leq N}^*$ the image of the partial spectral map Γ_k , that is $\Gamma_k(\mathcal{P})$, does not separate the plane. If there exists an integer $r \in \mathbb{N}_{\leq N}^*$ such that the subsystem (A_r, B_r, C_r) , where A_r, B_r, C_r are defined above, is UEOC then system (2.1) is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$.*

Proof. Note that under the assumptions of the above proposition, all the assumptions of Lemma 5.3 are satisfied. Using the fact that there exists an integer $r \in \mathbb{N}_{\leq N}^*$ such that the subsystem (A_r, B_r, C_r) is UEOC on $\mathbb{X}(\mathcal{P})^q$, we deduce from Lemma 5.3 that the system $(T^{-1}AT, TB, CT)$ where T is the continuous invertible transformation given by (5.9) is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$. Since $(T^{-1}AT, TB, CT)$ is obtained by applying to system (2.1) an invertible transformation in the state space variable and the uniform output ensemble controllability is preserved under such a transformation (see Proposition 2.3), system (A, B, C) is therefore uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$. \square

Theorem 5.5 provides conditions which are sufficient, but not necessary. For instance, let us consider the following simple example.

Example 5.6.

$$\mathcal{P} = [1, 2], \quad A(\theta) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & -2\theta & 1 \\ 0 & 0 & -2\theta \end{pmatrix}, \quad B = \begin{pmatrix} \theta \\ 3\theta \\ \theta \end{pmatrix}, \quad C = \begin{pmatrix} \theta & 1 & 2\theta \\ 2\theta & 1 & 2 \end{pmatrix}.$$

The block diagonal elements of A which are $A_1 = \theta$ and $A_2 = \begin{pmatrix} -2\theta & 1 \\ 0 & -2\theta \end{pmatrix}$ satisfy the spectral disjointness condition. But setting $B_1 = \theta$, $B_2 = \begin{pmatrix} 3\theta \\ \theta \end{pmatrix}$, $C_1 = \begin{pmatrix} \theta \\ 2\theta \end{pmatrix}$, $C_2 = \begin{pmatrix} 1 & 2\theta \\ 1 & 2 \end{pmatrix}$, we have $C = (C_1, C_2)$ and one can readily see that none of the subsystems (A_k, B_k, C_k) is uniformly ensemble output controllable. Nevertheless, by the end of this paragraph, see Example 5.9, we will see that the above system is UEOC.

Observe through (4.1) that, for finite dimensional systems, the state to output controllability property is preserved under a left multiplication of the output variable equation by an invertible matrix. This observation still holds in the case whereby the parameter set \mathcal{P} is a compact continuum as it was shown in Proposition 2.3. Therefore, the following theorem follows immediately from Theorem 5.1.

Theorem 5.7. *Assume that \mathcal{P} is compact and contractible and that A admits a continuous spectral decomposition $\Gamma_1, \dots, \Gamma_N$ which is strictly disjoint. Also, assume that for every $k \in \mathbb{N}_{\leq N}^*$ the image of the partial spectral map Γ_k , that is $\Gamma_k(\mathcal{P})$, does not separate the plane. Finally, assume that there exists a continuous invertible transformation M such that*

$$MCT = \begin{pmatrix} C_{1,1} & \cdots & \cdots & \cdots & C_{1,N} \\ & \ddots & & & \vdots \\ & & C_{\kappa,\kappa} & \cdots & C_{\kappa,N} \end{pmatrix}, \quad (5.10)$$

for some $\kappa \in \mathbb{N}_{\leq N}^*$, and where T is the linear continuous and invertible transformation given in Proposition 5.4.

The system (A, B, C) is UEOC on $\mathbb{X}(\mathcal{P})^q$ if

(i) for every $\ell \in \mathbb{N}_{\leq \kappa-1}^*$ the system $(A_\ell, B_\ell, C_{\ell,\ell})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_\ell}$;

(ii) there exists $r \in \{\kappa, \dots, N\}$ such that the system $(A_r, B_r, C_{\kappa,r})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_\kappa}$.

In the above items, we recall that $A_k(\theta) = \Pi_k T(\theta)^{-1} A(\theta) T(\theta) \Pi_k^*$ and $B_k(\theta) := \Pi_k T(\theta)^{-1} B(\theta)$.

Remark 5.8. Note that if the first q columns of the matrix CT are linearly independent for every $\theta \in \mathcal{P}$, then the linear continuous invertible transformation M exists. One can just consider M to be the inverse of the $q \times q$ matrix obtained by extracting the q first columns of CT . But the continuous and invertible transformation M does not exist in general. Indeed, for $\mathcal{P} = [-1, 1]$, let us consider the matrix CT to be

$$(CT)(\theta) = \begin{cases} \begin{pmatrix} f(\theta) \cos(\theta) & 1 & 0 \\ f(\theta) \sin(\theta) & 0 & 1 \end{pmatrix} & \text{if } \theta \leq 0, \\ \begin{pmatrix} f(\theta) \sin(\theta) & 1 & 0 \\ f(\theta) \cos(\theta) & 0 & 1 \end{pmatrix} & \text{if } \theta > 0, \end{cases}$$

where f is a continuous function which vanishes only at $\theta = 0$. As defined above, one can see that the matrix CT is continuous on \mathcal{P} , its first column vanishes in 0 and for every $\theta \in \mathcal{P}$, we have $\text{rk}(CT(\theta)) = 2$. Also, all the matrices M such that MCT takes the form of (5.10) are of the form,

$$M(\theta) = \begin{cases} \begin{pmatrix} \star & \star \\ -h(\theta) \sin(\theta) & h(\theta) \cos(\theta) \end{pmatrix} & \text{if } \theta < 0, \\ \begin{pmatrix} \star & \star \\ -g(\theta) \cos(\theta) & g(\theta) \sin(\theta) \end{pmatrix} & \text{if } \theta > 0. \end{cases}$$

A necessary condition for M to be continuous on \mathcal{P} is that M has to be continuous in 0. This implies $h(0) = g(0) = 0$. But $h(0) = g(0) = 0$ leads to the non invertibility of M for $\theta = 0$. All in all, the continuity of M and its invertibility are not compatible.

To end this paragraph, let us show that the system given in Example 5.6 is uniformly ensemble output controllable.

Example 5.9 (Example 5.6 continued). Setting $M = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, we have $MC = \begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{2,2} \end{pmatrix}$, with $C_{1,1} = \theta$, $C_{1,2} = (1 \ 2\theta)$ and $C_{2,2} = (-1 \ 2(1-2\theta))$. Observe now that the system (A, B, MC) takes the form of (5.2) with $N = 2$ and

$$\begin{aligned} (A_1, b_1, C_{1,1}) &= (\theta, \theta, \theta), \\ (A_2, b_2, C_{2,2}) &= \left(\begin{pmatrix} -2\theta & 1 \\ 0 & -2\theta \end{pmatrix}, \begin{pmatrix} 3\theta \\ \theta \end{pmatrix}, (-1 \ 2(1-2\theta)) \right). \end{aligned}$$

From Theorem 4.6 we deduce that $(A_1, b_1, C_{1,1})$ is uniformly ensemble output controllable. Using Remark 4.11, one can also check that $(A_2, b_2, C_{2,2})$ is uniformly ensemble output controllable. Furthermore, the spectrum of A_1 and A_2 which are respectively $[1, 2]$ and $[-2, -4]$ are compact, connected, strictly disjoint and do not separate the plane. It therefore follows from Theorem 5.7 that the system considered in Example 5.6 is uniformly ensemble output controllable.

Let us illustrate numerically this fact. To this end, we set $x_0(\theta) = (\theta \ -\theta \ \sqrt{\theta})^\top$, $y_1(\theta) = (0 \ 0)^\top$ and $T = 1$. For the numerical approximation, we follow the strategy proposed in [26, Section 2.2], and the dynamical system is solved using Crank–Nicolson method with a time step of 10^{-3} . It appears that 4 values of θ are enough to get a final error less than $\varepsilon := 10^{-1}$. To obtain these three values, we have used a *greedy search*. More precisely, assume that at step k the set of parameters used is P_k , then the corresponding ensemble control u_k is construct using parameters in P_k . At step $k + 1$, we consider $P_{k+1} = P_k \cup \{\theta_{k+1}\}$, with $\theta_{k+1} \in \operatorname{argmax}_{\theta \in P} \|y_\theta^1 - y(T, \theta; x_0, u_k)\|$. For the example considered, we have obtained $P_4 = \{1, 1.363, 1.803, 2\}$. On Figure 2, the output at final time is displayed, and on Figure 3, the obtained control is displayed. On Figures 4 and 5, we have respectively displayed times trajectories for the output and the state, for some values of $\theta \in P$.

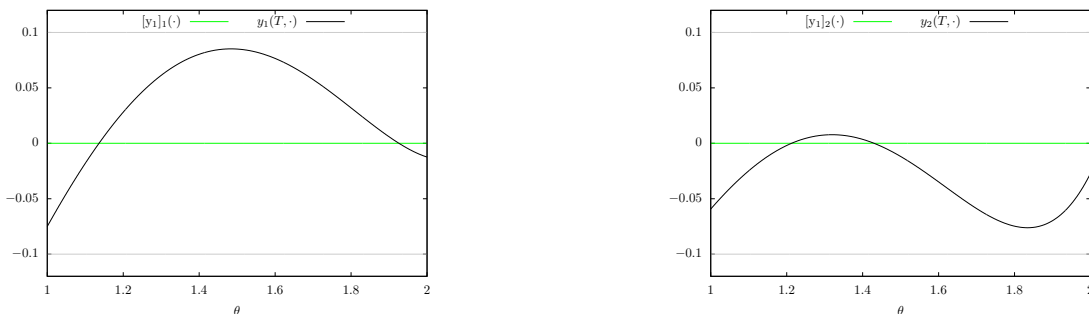


Figure 2: Output at final time for the ensemble controllability problem described in Example 5.9.

5.3 Uniform ensemble output controllability of cascade systems

Let us conclude Section 5, with another particular structure which is cascade systems. For this structure, we have the following general result.

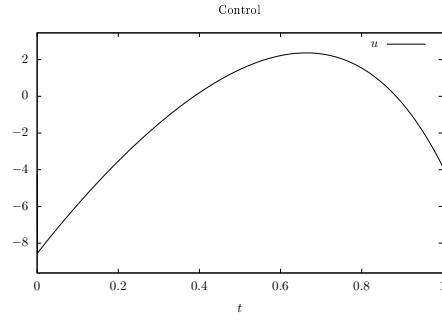


Figure 3: Control obtained for the ensemble controllability problem described in Example 5.9.

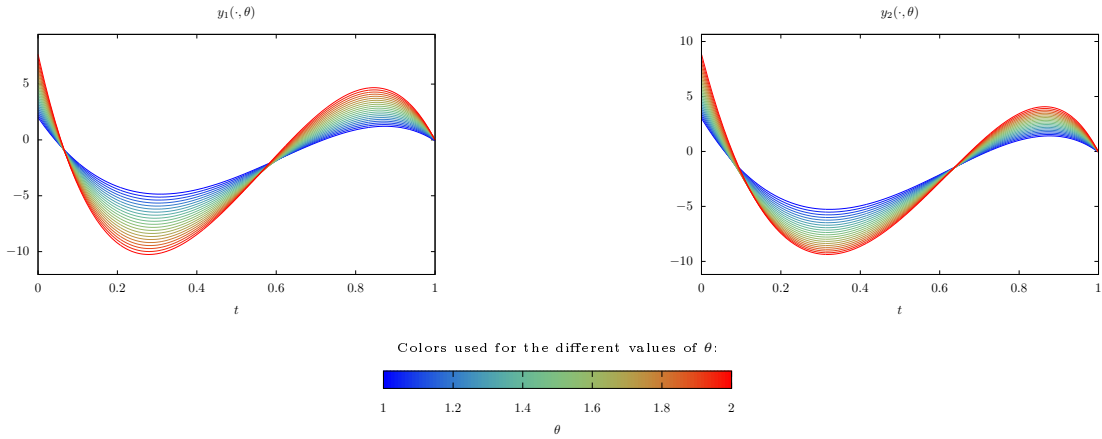


Figure 4: Time evolution of the output for some values of the parameter θ . The ensemble controllability problem is described in Example 5.9.

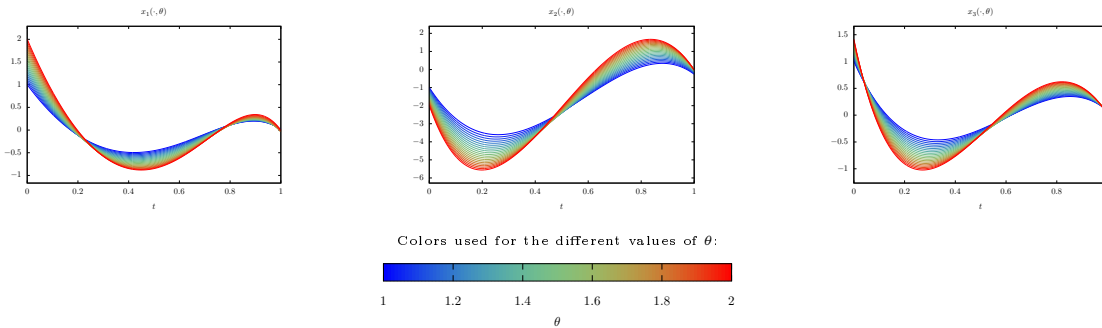


Figure 5: Time evolution of the state for some values of the parameter θ . The ensemble controllability problem is described in Example 5.9.

Theorem 5.10. *Let \mathcal{P} be a compact subset of \mathbb{C} , $N \in \mathbb{N}^*$, $m_1, \dots, m_n \in \mathbb{N}^*$, $n_1, \dots, n_N \in \mathbb{N}^*$, and $q_1, \dots, q_N \in \mathbb{N}^*$. For every $i \in \mathbb{N}_{\leq N}^*$ and every $j \in \{1, \dots, N\}$, we consider $A_{i,j} \in \mathbb{X}(\mathcal{P})_{n_j}^{n_i}$, $B_{i,j} \in \mathbb{X}(\mathcal{P})_{m_j}^{n_i}$ and $C_{i,j} \in \mathbb{X}(\mathcal{P})_{n_j}^{q_i}$.*

Assume that the triples $(A_{k,k}, B_{k,k}, C_{k,k})$ are UEOC on $\mathbb{X}(\mathcal{P})^{q_k}$ for every $k \in \mathbb{N}_{\leq N}^*$. Then the cascade system

$$\left(\left(\begin{array}{ccc} A_{1,1} & \cdots & A_{1,N} \\ & \ddots & \vdots \\ & & A_{N,N} \end{array} \right), \left(\begin{array}{ccc} B_{1,1} & \cdots & B_{1,N} \\ & \ddots & \vdots \\ & & B_{N,N} \end{array} \right), \left(\begin{array}{ccc} C_{1,1} & \cdots & C_{1,N} \\ & \ddots & \vdots \\ & & C_{N,N} \end{array} \right) \right)$$

is UEOC on $\mathbb{X}(\mathcal{P})^q$, with $q = \sum_{k=1}^N q_k$.

Proof. The proof is done for $N = 2$. The general case follows the same argument. When $N = 2$, the system (2.1) takes the form

$$\dot{x}_1(t, \theta) = A_{1,1}(\theta)x_1(t, \theta) + B_{1,1}(\theta)u_1(t) + A_{1,2}(\theta)x_2(t, \theta) + B_{1,2}(\theta)u_2(t), \quad (5.11a)$$

$$\dot{x}_2(t, \theta) = A_{2,2}(\theta)x_2(t, \theta) + B_{2,2}(\theta)u_2(t), \quad (5.11b)$$

$$y_1(t, \theta) = C_{1,1}(\theta)x_1(t, \theta) + C_{1,2}(\theta)x_2(t, \theta), \quad (5.11c)$$

$$y_2(t, \theta) = C_{2,2}(\theta)x_2(t, \theta). \quad (5.11d)$$

Denote by φ_k , $k = 1, 2$ the output solution of the system

$$\begin{aligned} \dot{x}_k(t, \theta) &= A_{k,k}(\theta)x_k(t, \theta) + B_{k,k}(\theta)u_k, \\ \varphi_k(t, \theta) &= C_{k,k}(\theta)x_k(t, \theta). \end{aligned}$$

Let $(f_1, f_2) \in \mathbb{X}(\mathcal{P})^{q_1} \times \mathbb{X}(\mathcal{P})^{q_2}$ and take any $\varepsilon > 0$. We aim to show that there exists a time $T > 0$ and a control u such that the output solution (y_1, y_2) of (5.11) satisfies $\max\{\|y_1(T, \cdot; 0, u) - f_1\|_\infty, \|\varphi_2(T, \cdot; 0, u) - f_2\|_\infty\} \leq \varepsilon$.

Since the system $(A_{2,2}, B_{2,2}, C_{2,2})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_2}$, for every time $T > 0$, there exists a θ -independent control u_2 such that

$$\|\varphi_2(T, \cdot; 0, u_2) - f_2\|_\infty \leq \varepsilon.$$

From (5.11c), we have

$$y_1(T, \theta; 0, u_1) = \varphi_1(T, \theta; 0, u_1) + \tilde{f}(\theta),$$

with

$$\tilde{f}(\theta) = C_{1,1}(\theta) \int_0^T e^{(T-t)A_{1,1}(\theta)} (A_{1,2}(\theta)x_2(t, \theta) + B_{1,2}(\theta)u_2(t)) dt + C_{1,2}(\theta)x_2(T, \theta).$$

Since $(A_{1,1}, B_{1,1}, C_{1,1})$ is UEOC on $\mathbb{X}(\mathcal{P})^{q_1}$, then for the function $f_1 - \tilde{f}$, there exists a θ -independent input u_1 such that

$$\left\| \varphi_1(T, \cdot; 0, u_1) - f_1 + \tilde{f} \right\|_\infty \leq \varepsilon.$$

We end the proof by taking the control $u = (u_1^\top \quad u_2^\top)^\top$. \square

6 Application of uniform ensemble output controllability to averaged controllability

In this section, we link the notion of uniform ensemble output controllability to the one of averaged controllability. As far as we know, the notion of averaged controllability has been introduced in [47]. In the above mentioned paper, the author considers a pair $(A(\theta), B(\theta))$ where $\theta \in (0, 1)$, and

investigates conditions under which, given a control time $T > 0$ any initial state $x_0 \in \mathbb{R}^n$ (which can depend on θ) and any $x_1 \in \mathbb{R}^n$, there exists a θ -independent control u such that the state of this system satisfies $\int_0^1 x(T, \theta) d\theta = x_1$. The author shows, in particular, that this control task can be achieved if and only if $\text{rk} \left(\int_0^1 A(\theta)^k B(\theta) d\theta, k \in \mathbb{N} \right) = n$. Also, the averaged controllability of some class of PDE has been investigated in the same paper. Many other results on PDE have been obtained and can be found for instance in [4, 5, 12, 30]. Dealing with the numerical aspect of the averaged controllability, we refer to [1, 26].

In this paper, the following definition borrowed from [16] will be considered.

Definition 6.1. Let μ be a Borel probability measure on \mathcal{P} . The system (2.1) is said to be averaged controllable, if for any $y_1 \in \mathbb{C}^q$, there exist a time T and a control $u \in L^1([0, T]; \mathbb{C}^m)$ such that

$$\int_{\mathcal{P}} y(T, \theta; 0, u) d\mu(\theta) = y_1,$$

where $y(T, \cdot; 0, u) \in \mathbb{X}(\mathcal{P})^q$ is the output solution of (2.1) with 0 as initial state data and u as input.

It has been shown in [16] that a necessary and sufficient condition for averaged controllability is

$$\text{rk} \left(\int_{\mathcal{P}} C(\theta) A(\theta)^k b_\ell(\theta) d\mu(\theta), k \in \mathbb{N}, \ell \in \mathbb{N}_{\leq m}^* \right) = q. \quad (6.1)$$

Moreover, it was shown in [16, Proposition 9 and Corollary 5] that if the pair $(A(\theta), B(\theta))$ is uniform ensemble controllable, and if there exist $\theta_1, \dots, \theta_k$ in the support of the measure μ such that $\text{rk} (C(\theta_1) \dots C(\theta_k)) = q$ then system (2.1) is averaged controllable.

As a contribution to this topic, we have the following simple result which do not require uniform ensemble controllability of the pair $(A(\theta), B(\theta))$.

Proposition 6.2. *The system (2.1) is averaged controllable if it is uniformly ensemble output controllable on $\mathbb{X}(\mathcal{P})^q$.*

Proof. To begin with, note that since the output space is of finite dimension, there is equivalence between averaged controllability and the fact for any $y_1 \in \mathbb{C}^q$ and any $\varepsilon > 0$, there exist a time T and an input $u \in L^1([0, T]; \mathbb{C}^m)$ such that

$$\left| \int_{\mathcal{P}} y(T, \theta; 0, u) d\mu(\theta) - y_1 \right| \leq \varepsilon.$$

This last property is termed as approximate averaged controllability in [16].

To end the proof, it suffices to write

$$\left| \int_{\mathcal{P}} y(T, \theta; 0, u) d\mu(\theta) - y_1 \right| = \left| \int_{\mathcal{P}} (y(T, \theta; 0, u) - y_1) d\mu(\theta) \right| \leq \|y(T, \cdot; 0, u) - y_1\|_{\infty}.$$

□

Going back to the cars example given by (5.6), one can see that with the criteria given in [16, Proposition 9 and Corollary 5], nothing can be inferred on the averaged controllability of this system since it is not uniformly ensemble controllable. But with Proposition 6.2, we can infer that this system is averaged controllable because it is uniformly ensemble output controllable.

Example 6.3 (Averaged controllability for Example 5.6). As stated in Example 5.9, the system given in Example 5.6 is UEOR. Hence, according to Proposition 6.2, it is also averaged controllable.

We numerically illustrate this fact by taking the same parameters as the ones used in Example 5.6, and we consider the measure μ given by $d\mu(\theta) = d\theta$. For the numerical experiment, we use the strategy proposed in [26, Section 2.1]. In particular, we use the Crank–Nicolson method for the numerical resolution of the dynamical system (the time step is set to 10^{-3}), and the mid-point rule for the computation of the average (the step is also set to 10^{-3}). On Figure 6, we display the values of $\theta \mapsto y(T, \theta)$, on Figure 7, we display the computed control. Finally, on Figures 8 and 9, we give some time trajectories for some values of θ for the output and the state respectively. Let us mention that the norm of the control obtained for the averaged control problem is smaller than the one obtained for the ensemble control problem described in Example 5.9. Let us also mention that we could have used the strategy proposed in [29] to get a control that both solves the averaged and the ensemble control problem.

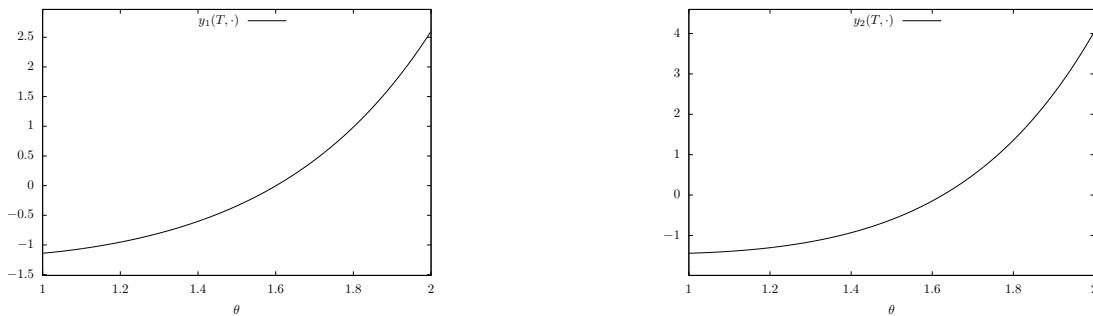


Figure 6: Output at final time for the averaged controllability problem described in Example 6.3.

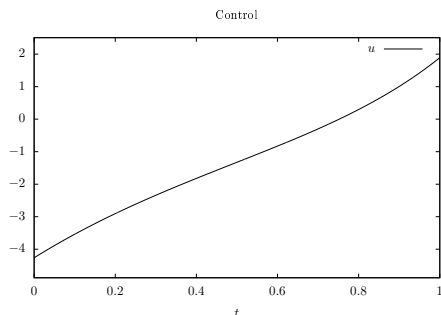


Figure 7: Control obtained for the averaged controllability problem described in Example 6.3.

To conclude this section, let us consider the case whereby $\mu = (1/N) \sum_{k=1}^N \delta_{\theta_k}$ where for every k , $\theta_k \in \mathcal{P}$ and δ_{θ_k} stands for the Dirac measure located in θ_k . In this case $\int_{\mathcal{P}} y(T, \theta; 0, u) d\mu(\theta) = (1/N) \sum_{k=1}^N C(\theta_k) x(T, \theta_k; 0, u)$ and the averaged controllability of system (2.1) with this measure is equivalent to the state to output controllability of the finite dimensional system (4.3), with matrices \mathbb{A} , \mathbb{B} and \mathbb{H} given by

$$\mathbb{A} = \begin{pmatrix} A(\theta_1) & & \\ & \ddots & \\ & & A(\theta_N) \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} B(\theta_1) \\ \vdots \\ B(\theta_N) \end{pmatrix}, \quad \mathbb{H} = (C(\theta_1) \quad \dots \quad C(\theta_N)).$$

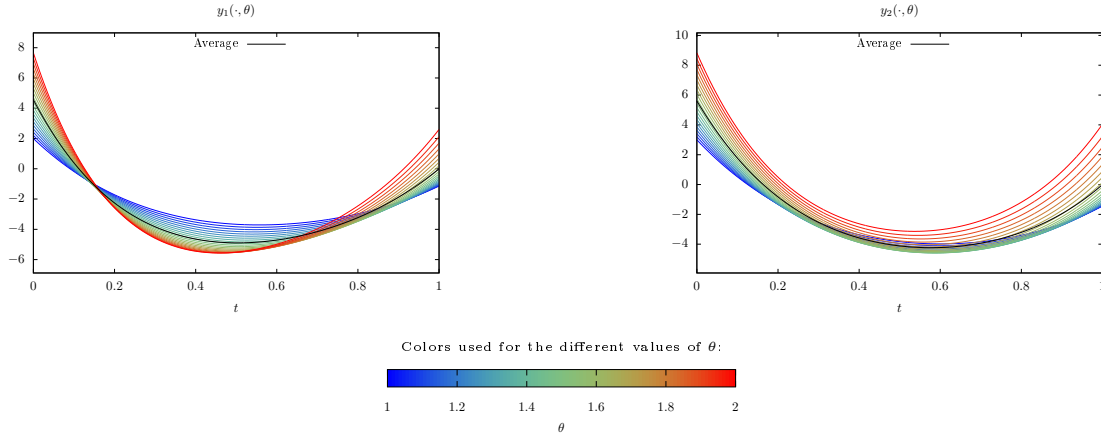


Figure 8: Time evolution of the average of the outputs, and of some outputs for different values of the parameter θ . The averaged controllability problem is described in Example 6.3.

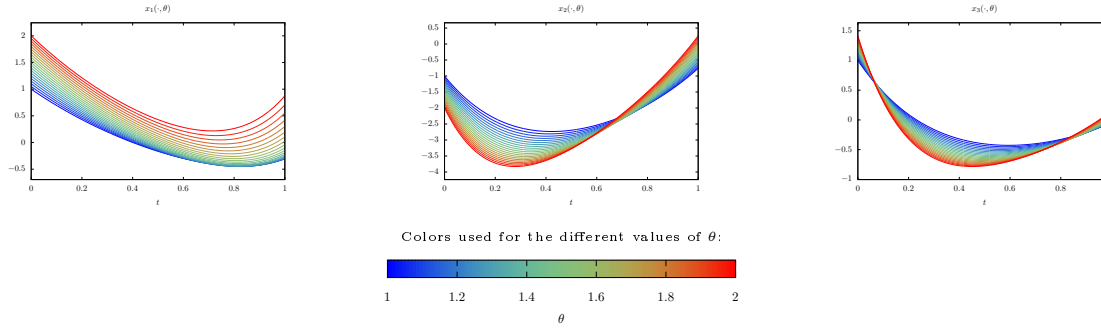


Figure 9: Time evolution of some states for different values of the parameter θ . The averaged controllability problem is described in Example 6.3.

Let us set

$$\mathbb{W} = \text{Im} \begin{pmatrix} C(\theta_1)^* \\ \vdots \\ C(\theta_N)^* \end{pmatrix} \cap \prod_{k=1}^N \left(\bigoplus_{\lambda \in \sigma(A(\theta_k))} E_{\theta_k, \lambda} \right),$$

where $E_{\theta_k, \lambda}$ is defined just below equation (4.8). The following proposition can be deduced from the proof of Proposition 4.1.

Proposition 6.4. *Assume that the following assumptions hold:*

- (i) for every $k \neq l$, $\sigma(A(\theta_k)) \cap \sigma(A(\theta_l)) = \emptyset$,
- (ii) $\text{rk} \begin{pmatrix} C(\theta_1) & \dots & C(\theta_N) \end{pmatrix} = q$ and $\mathbb{W} = \{0\}$.

Then system (2.1) is averaged controllable with probability measure $\mu = \frac{1}{N} \sum_{k=1}^N \delta_{\theta_k}$.

The following corollary is a direct consequence of the above proposition.

Corollary 6.5. *Assume that the assumption (i) of Proposition 6.4 is fulfilled, and assume there exists $k_0 \in \mathbb{N}_{\leq N}^*$ such that $(A(\theta_{k_0}), B(\theta_{k_0}), C(\theta_{k_0}))$ is state to output controllable. Then system (2.1) is averaged controllable with probability measure $\mu = \frac{1}{N} \sum_{k=1}^N \delta_{\theta_k}$.*

Proof. It suffices to show that the assumption (ii) of Proposition 6.4 is satisfied. To this end, let us assume, without loss of generality, that $(A(\theta_1), B(\theta_1), C(\theta_1))$ is state to output controllable. In this case, we have $\text{rk } C(\theta_1) = q$ and the necessary condition $\text{rk} \begin{pmatrix} C(\theta_1) & \dots & C(\theta_N) \end{pmatrix} = q$ is satisfied. Hence, it remains to show that $\mathbb{W} = \{0\}$. Let us take $z = (z_1^\top \dots z_N^\top)^\top \in \mathbb{W}$. There exists $\eta \in \mathbb{C}^q$ such that $z_k = C(\theta_k)^* \eta \in \bigoplus_{\lambda \in \sigma(A(\theta_k))} E_{\theta_k, \lambda}$ for every $k \in \mathbb{N}_{\leq N}^*$. Since by assumption $(A(\theta_1), B(\theta_1), C(\theta_1))$ is state to output controllable, then $\text{Im}(C(\theta_1)^*) \cap \bigoplus_{\lambda \in \sigma(A(\theta_1))} E_{\theta_1, \lambda} = \{0\}$ which implies that $\eta = 0$ and that $\mathbb{W} = \{0\}$. \square

From the above proof, note that if $C(\theta_k) = p_{\theta_k} I_n$, where p_{θ_k} , satisfying $0 < p_{\theta_k} < 1$ and $\sum_{k=1}^N p_{\theta_k} = 1$, is the probability for having $\theta = \theta_k$, then we have

$$\mathbb{W} = \{0\} \iff \bigcap_{k=1}^N \left(\bigoplus_{\lambda \in \sigma(A(\theta_k))} E_{\theta_k, \lambda} \right) = \{0\}.$$

It follows that if at least one of the pairs $(A(\theta_k), B(\theta_k))$ is state controllable then system (2.1) with $\mu = \sum_{k=1}^N p_k \delta_{\theta_k}$ is averaged controllable.

7 Conclusion

In this paper, we have introduced the notion of uniform ensemble output controllability (UEOC) of the system $(A(\theta), B(\theta), C(\theta))$ which is a generalization of uniform ensemble controllability (UEC) of the pair $(A(\theta), B(\theta))$. We have also extended some results on UEC. In particular, we have shown that if the set of parameters θ has an interior point with respect to the topology of \mathbb{C} , then the pair $(A(\theta), B(\theta))$ cannot be uniformly ensemble controllable if A admits a real eigenvalue or if its components are differentiable. Also, some necessary and sufficient conditions have been given to ensure the UEC or UEOC in some particular case. Finally, an application to averaged controllability has been given.

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A Technical lemmas

In this Appendix, we state some results on the existence of a local continuous eigenvalue and eigenvector associated to matrix A . Since we were not able to locate these results in the literature, we provide them here.

Lemma A.1. *Let $\mathcal{P} \subset \mathbb{C}$ be a compact with non-empty interior, and assume for some $k \in \mathbb{N}$ that A is of class \mathcal{C}^k in the interior of \mathcal{P} . There exist an interior point θ_0 to \mathcal{P} and $\delta_0 > 0$ such that A admits an eigenvalue of class \mathcal{C}^k on $B_{\theta_0}(\delta_0)$, and there exists a left eigenvector of class \mathcal{C}^k associated to the selected eigenvalue.*

Proof. 1. *Existence of local continuous eigenvalue selection.*

Define for every $\theta \in \mathcal{P}$ and $\nu \in \mathbb{C}$,

$$f(\theta, \nu) = \det(A(\theta) - \nu I_n). \quad (\text{A.1})$$

By definition and using the continuity of A , it follows that $f(\theta, \nu)$ is continuous with respect to θ and holomorphic with respect to ν .

Since \mathcal{P} has interior point, let \mathcal{V} be a non-empty open subset of \mathcal{P} and set

$$k_0 = \min \{n_z \mid z \in \sigma(A(\theta)), \theta \in \mathcal{V}\},$$

where n_z , for every $z \in \sigma(A(\theta))$, is the algebraic multiplicity of z in the characteristic polynomial of $A(\theta)$. Since $\{n_z \mid z \in \sigma(A(\theta)), \theta \in \mathcal{V}\}$ is a finite subset of \mathbb{N}^* , k_0 exists and is positive. Let θ_0 be an element of \mathcal{V} in which k_0 is achieved. Let d be the distance⁴ between λ_0 and $\sigma(A(\theta_0)) \setminus \{\lambda_0\}$ and choose $\varepsilon_1 \in (0, d)$. Finally, let γ be a smooth closed curve inside $B_{\lambda_0}(\varepsilon_1)$, separating strictly λ_0 with the other zeros of $f(\theta_0, \cdot)$.

By the construction of γ , $f(\theta_0, \cdot)$ does not vanish on $\text{Im } \gamma$. Hence, there exists $\eta > 0$ such that

$$\min_{z \in \text{Im } \gamma} |f(\theta_0, z)| > 2\eta. \quad (\text{A.2})$$

Since $f(\theta, z)$ is continuous with respect to θ and holomorphic with respect to z , we infer that

$$\exists \delta_0 > 0 \text{ such that } \forall (\theta, z) \in B_{\theta_0}(\delta_0) \times \text{Im } \gamma, \quad |f(\theta_0, z) - f(\theta, z)| \leq \eta. \quad (\text{A.3})$$

It then follows from (A.2) and (A.3) that for every $\theta \in B_{\theta_0}(\delta_0)$, we have

$$|f(\theta, z)| = |f(\theta, z) - f(\theta_0, z) + f(\theta_0, z)| \geq |f(\theta_0, z)| - \eta > \eta, \quad \forall z \in \text{Im } \gamma. \quad (\text{A.4})$$

We infer from (A.3) and (A.4) that for every $\theta \in B_{\theta_0}(\delta_0)$ and every $z \in \text{Im } \gamma$,

$$|f(\theta_0, z) - f(\theta, z)| < |f(\theta, z)|.$$

Using Rouché's Theorem (see Theorem B.3), we infer that for every $\theta \in B_{\theta_0}(\delta_0)$, the function $f(\theta, \cdot)$ admits k_0 roots inside $\text{Im } \gamma$ counted with multiplicities. From the above, we deduce that $\#\sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon_1) = 1$ for every $\theta \in B_{\theta_0}(\delta_0)$. If not, there would exist on \mathcal{V} an eigenvalue of A with algebraic multiplicity strictly less than k_0 , and this would undermine the choice of θ_0 . Let us now define λ to be the map that associates every $\theta \in B_{\theta_0}(\delta_0)$ to the element of $\sigma(A(\theta))$ located in $B_{\lambda_0}(\varepsilon_1)$. By construction, for every $\theta \in B_{\lambda_0}(\varepsilon_1)$, the algebraic multiplicity of $\lambda(\theta)$ in the characteristic polynomial of $A(\theta)$ is exactly k_0 .

The selection λ is also continuous on $B_{\theta_0}(\delta_0)$. Indeed, let $\theta \in B_{\theta_0}(\delta_0)$ and $\varepsilon > 0$. By construction, $\lambda(\theta) \in \sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon_1)$. Applying Rouché's Theorem again, we deduce that there exists $\delta > 0$ such that for every $\tilde{\theta} \in B_{\theta}(\delta) \cap B_{\theta_0}(\delta_0)$, $\lambda(\tilde{\theta}) \in \sigma(A(\tilde{\theta})) \cap B_{\lambda(\theta)}(\varepsilon)$. It follows that for every $\tilde{\theta} \in B_{\theta}(\delta) \cap B_{\theta_0}(\delta_0)$, $|\lambda(\theta) - \lambda(\tilde{\theta})| \leq \varepsilon$ and therefore that λ is continuous on $B_{\theta_0}(\delta_0)$.

Until now, we have shown that the matrix A admits on $B_{\theta_0}(\delta_0)$ a continuous eigenvalue λ with constant algebraic multiplicity k_0 , that is, for every $\theta \in B_{\theta_0}(\delta_0)$,

$$\partial_\nu^\ell f(\theta, \lambda(\theta)) = 0, \quad \forall \ell \in \mathbb{N}_{< k_0} \quad \text{and} \quad \partial_\nu^{k_0} f(\theta, \lambda(\theta)) \neq 0. \quad (\text{A.5})$$

Observe that if $k = 0$ in Lemma A.1 then we would have done with the proof of the existence of a continuous eigenvalue.

Assume now that $k \geq 1$. In this case, since A is k times differentiable, then the function f defined

⁴We use the convention that the distance to an empty set is infinite.

by (A.1) is k times differentiable with respect to θ and holomorphic with respect to ν . Taking $\theta = \theta_0$ in (A.5), we deduce, by applying the implicit functions theorem to the function $\partial_\nu^{k_0-1} f$, that there exist $\varepsilon > 0$, a neighbourhood $\Gamma_{\lambda(\theta_0)}$ of $\lambda(\theta_0)$ and a differentiable function φ such that $\varphi(\theta_0) = \lambda(\theta_0)$ and

$$\forall(\theta, \lambda) \in B_{\theta_0}(\varepsilon) \times \Gamma_{\lambda(\theta_0)}, \quad \partial_\nu^{k_0-1} f(\theta, \lambda) = 0 \iff \lambda = \varphi(\theta). \quad (\text{A.6})$$

Relations (A.5) and (A.6) show that by reducing δ_0 , if necessary, λ is differentiable on $B_{\theta_0}(\delta_0)$. Let us set $g(\theta) = \partial_\nu^{k_0-1} f(\theta, \lambda(\theta))$ for every $\theta \in B_{\theta_0}(\delta_0)$. The function g is differentiable on $B_{\theta_0}(\delta_0)$ and since $g(\theta) = 0$ for every $\theta \in B_{\theta_0}(\delta_0)$ we have $\partial_\theta g(\theta) = 0$ leading to

$$\partial_\theta \partial_\nu^{k_0-1} f(\theta, \lambda(\theta)) + \lambda'(\theta) \partial_\nu^{k_0} f(\theta, \lambda(\theta)) = 0.$$

Using the fact that $\partial_\nu^{k_0} f(\theta, \lambda(\theta)) \neq 0$ for every $\theta \in B_{\theta_0}(\delta_0)$ (see (A.5)), it follows from the above identity that for every $\theta \in B_{\theta_0}(\delta_0)$,

$$\lambda'(\theta) = -\frac{\partial_\theta \partial_\nu^{k_0-1} f(\theta, \lambda(\theta))}{\partial_\nu^{k_0} f(\theta, \lambda(\theta))}. \quad (\text{A.7})$$

From (A.7) it can be seen that λ is also k times differentiable on $B_{\theta_0}(\delta_0)$.

2. Local continuous selection of a left eigenvector.

Let K be a closed ball included in $B_{\theta_0}(\delta_0)$. By construction of λ , the matrix $(A(\theta)^* - \overline{\lambda(\theta)} I_n)^{k_0}$ has constant rank $n - k_0$ on K and is continuous. Using (B.1.b) of Theorem B.1, we deduce that there exists a \mathcal{C}^k function v such that $v(\theta) \neq 0$ for every $\theta \in K$ satisfying $(A(\theta)^* - \overline{\lambda(\theta)} I_n)^{k_0} v(\theta) = 0$. If $k_0 = 1$, then v^* is a left eigenvector associated to λ . If $k_0 > 1$, choose the smallest ℓ such that $(A(\theta)^* - \overline{\lambda(\theta)} I_n)^\ell v(\theta) = 0$ for every $\theta \in K$. Setting $\zeta(\theta) = (A(\theta)^* - \overline{\lambda(\theta)} I_n)^{\ell-1} v(\theta)$, it follows that there exists $\theta_1 \in K$ such that $\zeta(\theta_1) \neq 0$. By the continuity of ζ , there exists $\delta_1 > 0$ such that for every $\theta \in B_{\theta_1}(\delta_1) \subset K$, $\zeta(\theta) \neq 0$. Therefore, ζ^* is a left eigenvector of class \mathcal{C}^k associated to λ on $B_{\theta_1}(\delta_1)$. \square

Let us give the following remark which is used in the proof of Theorem 3.1.

Remark A.2. One should also note that if λ is assumed to be one-to-one on $B_{\theta_0}(\delta_0)$, then $\lambda : B_{\theta_0}(\delta_0) \mapsto \lambda(B_{\theta_0}(\delta_0))$ is a homeomorphism. Therefore, we can always find an open set \mathcal{O} included in $\lambda(B_{\theta_0}(\delta_0))$ on which λ^{-1} is of class \mathcal{C}^k .

The following lemma is used in the proof of Theorem 3.8.

Lemma A.3. *Let $\mathcal{P} \subset \mathbb{C}$ be a compact with non-empty interior, and let A be a continuous $n \times n$ matrix on \mathcal{P} satisfying $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$ on some nonempty open subset $\mathcal{O} \subset \mathcal{P}$. Then there exist $\theta_0 \in \mathcal{O}$ and $\delta_0 > 0$ such that A admits a continuous real eigenvalue selection on $B_{\theta_0}(\delta_0)$.*

Proof. Define

$$k_0 = \min \{n_z \mid z \in \sigma(A(\theta)) \cap \mathbb{R}, \theta \in \mathcal{O}\},$$

where n_z is the algebraic multiplicity of z in the characteristic polynomial of $A(\theta)$. Since by assumption, $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$ for every $\theta \in \mathcal{P}$, and thanks to the fact that $\{n_z \mid z \in \sigma(A(\theta)) \cap \mathbb{R}, \theta \in \mathcal{O}\}$ is a finite subset of \mathbb{N}^* , k_0 exists and is positive. Let θ_0 be an element of \mathcal{O} in which k_0 is achieved, and let $\lambda_0 \in \sigma(A(\theta_0)) \cap \mathbb{R}$ such that $n_{\lambda_0} = k_0$. Setting $m_0 = \sum_{z \in \sigma(A(\theta_0)) \cap \mathbb{R}} n_z$, we have

$$m_0 \leq n \quad \text{and} \quad m_0 \geq k_0 \# (\sigma(A(\theta_0)) \cap \mathbb{R}).$$

Define d to be the distance⁵ between λ_0 and $\sigma(A(\theta_0)) \setminus \{\lambda_0\}$ and choose $\varepsilon \in (0, d)$.

Using Rouché's Theorem (see Theorem B.3), we deduce, as in the **1st step** of the proof of Lemma A.1, that there exists $\delta > 0$ such that for every $\theta \in B_{\theta_0}(\delta)$, $\sum_{z \in \sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon)} n_z = k_0$. Two situations can arise⁶.

(s₁) There exists $\delta_0 \leq \delta$ such that for every $\theta \in B_{\theta_0}(\delta_0)$, we have $\sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon) \cap \mathbb{R} \neq \emptyset$.

(s₂) There exists $\theta_0^1 \in B_{\theta_0}(\delta)$ such that $\sigma(A(\theta_0^1)) \cap B_{\lambda_0}(\varepsilon) \cap \mathbb{R} = \emptyset$.

If (s₁) occurs, proceeding as in the **1st step** of the proof of Lemma A.1, we can ensure that $\#(\sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon)) = 1$ for every $\theta \in B_{\theta_0}(\delta_0)$. We then define λ to be the map which associates each element of $B_{\theta_0}(\delta_0)$ to the unique element of $\sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon) \cap \mathbb{R}$. Finally, as it has been done in the **1st step** of the proof of Lemma A.1, one can show, by using Rouché's Theorem (see Theorem B.3), that λ is continuous on $B_{\theta_0}(\delta_0)$.

If (s₁) does not occur, then (s₂) occurs, we get by applying Rouché's Theorem, the existence of $\delta_1 > 0$ such that $B_{\theta_0^1}(\delta_1) \subset B_{\theta_0}(\delta)$ and for every $\theta \in B_{\theta_0^1}(\delta_1)$, $\sigma(A(\theta)) \cap B_{\lambda_0}(\varepsilon) \cap \mathbb{R} = \emptyset$.

In this case, set $\mathcal{P}_1 = B_{\theta_0^1}(\delta_1)$ and define

$$k_1 := \min \{n_z \mid z \in \sigma(A(\theta)) \cap \mathbb{R}, \theta \in \mathcal{P}_1\}.$$

Let θ_1 be an element of \mathcal{P}_1 in which k_1 is achieved.

From the above, we have by setting $m_1 = \sum_{\lambda \in \sigma(A(\theta_1)) \cap \mathbb{R}} n_\lambda$,

$$k_0 \leq k_1 \quad \text{and} \quad k_1 \#(\sigma(A(\theta_1)) \cap \mathbb{R}) \leq m_1 \leq n - k_0.$$

We then restart the process by making θ_1 play the role of θ_0 , \mathcal{P}_1 the role of $\mathcal{P}_0 := \mathcal{O}$ and k_1 the one of k_0 .

Iterating this process ℓ times, we get the existence of ℓ distinct complex numbers θ_j , ℓ sets \mathcal{P}_j satisfying $\mathcal{P}_{j+1} \subsetneq \mathcal{P}_j$, $\theta_j \in \mathcal{P}_j \setminus \mathcal{P}_{j+1}$, ℓ integers k_j such that $k_j \leq k_{j+1}$ and ℓ integers m_j such that $1 \leq k_j \#(\sigma(A(\theta_j)) \cap \mathbb{R}) \leq m_j \leq n - jk_0$. Since by assumption $\sigma(A(\theta)) \cap \mathbb{R} \neq \emptyset$ for every $\theta \in \mathcal{P}$, we necessarily end in (s₁) since the process has to stop at most after n iterations. Therefore, a continuous selection of real eigenvalue is possible.

To sum up, we have shown that there exist θ_0 interior to \mathcal{P} , $\delta > 0$ such that $B_{\theta_0}(\delta) \subset \mathcal{P}$ and on $B_{\theta_0}(\delta)$, A admits a continuous real eigenvalue λ (with constant algebraic multiplicity). \square

B A list of classical results

We recall here some fundamental mathematical theorems used in our paper.

Theorem B.1 ([19, Theorem 3.8], [41]). *Let \mathcal{X} be a topological space that is Hausdorff, paracompact and contractible. Let $M : \mathcal{X} \rightarrow \mathcal{Y}_\ell^k$ be a continuous map with the property that for some $r_0 \in \{0, \dots, \min\{k, \ell\}\}$, we have $\text{rk}(M(z)) = r_0$ for every $z \in \mathcal{X}$.*

(B.1.a) *There exist a continuous map $M_0 : \mathcal{X} \rightarrow U(k, \mathcal{Y})$ and a continuous map $N_0 : \mathcal{X} \rightarrow \mathcal{Y}_{r_0}^k$ such that for all $z \in \mathcal{X}$,*

$$M(z)M_0(z) = [N_0(z) \mid 0_{\ell-r_0}^k]. \quad (\text{B.1})$$

(B.1.b) *If \mathcal{X} has a structure of a real, finite-dimensional, differentiable manifold of class \mathcal{C}^ℓ and if $M \in \mathcal{C}^\ell$, $\ell \in \mathbb{N}$, then there exist \mathcal{C}^ℓ mappings $M_0 : \mathcal{X} \rightarrow U(k, \mathcal{Y})$ and $N_0 : \mathcal{X} \rightarrow \mathcal{Y}_{r_0}^k$ such that (B.1) holds.*

⁵We use the convention that the distance to an empty set is infinite.

⁶Note that (s₁) and (s₂) can occur simultaneously, but if (s₁) does not occur, then (s₂) occurs.

Let us point out that the contractibility of K is very important in this theorem. For instance, let us consider on $K = \mathcal{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{k=1}^3 x_k^2 = 1\}$ the unit outer normal vector $M(x) = (x_1 \ x_2 \ x_3)$. It can be easily seen that \mathcal{S}^2 is compact and simply connected, and for every $x \in \mathcal{S}^2$, $\text{rk } M(x) = 1$. But the matrix M_0 does not exist. Indeed, if M_0 existed, then its second and the third column would be in the tangent bundle of \mathcal{S}^2 and non-zero everywhere. But it is well-known that this is impossible (see e.g. [23, Corollary 2.4, p.134]).

Theorem B.2 ([36, Theorem 20.5]). *If K is a compact set in the plane whose complement is connected, and if f is a continuous complex function which is holomorphic in the interior of K , then for every $\varepsilon > 0$, there exists a complex polynomial p such that $|f(z) - p(z)| < \varepsilon$ for every $z \in K$.*

Theorem B.2 is due to Mergelyan and known as Mergelyan’s Theorem. Also, the ongoing discussion in [36] shows that if K has empty interior then the conclusion holds for every continuous function on K .

Theorem B.3 ([18, Rouché’s Theorem, p.229]). *Let D be a bounded domain with piecewise smooth boundary ∂D . Let f and g be holomorphic functions on $D \cup \partial D$. If $|f(z) - g(z)| < |f(z)|$ on ∂D , then f and g have the same number of zeros inside D , counting multiplicities.*

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