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# ON THE INVERSION OF SUBSURFACE RESIDUAL STRESSES FROM SURFACE STRESS MEASUREMENTS

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## ABSTRACT

Matter removal is necessary in order to measure the stress tensor at an interior point of a solid by means of X-ray diffraction. This induces stress redistribution so that the measured stress is different from the original residual stress. This paper addresses the problem of reconstructing the original stress field from the measurements. A method of solution is described for the reconstruction problem, which leads to explicit inversion formulae for the case of some simple geometries. Closed-form expressions are obtained for the two- and three-dimensional half-space. For bodies of arbitrary shape, a numerical algorithm is proposed.

## INTRODUCTION

Residual stresses play a significant role in the mechanical behavior of structures. For example, they can decrease or increase the apparent fatigue strength of a mechanical component. They can be a simple by-product of the manufacturing process (for example, in a drawn wire) or they can be induced on purpose by means of a special treatment (for example, shot-peening or rolling).

One experimental method has proved to be particularly efficient to measure residual stresses: that is, X-ray diffraction. This technique allows the measurement of an average of all components of the stress tensor at the surface of a polycrystalline metal. The average is taken over a small material volume. The technique applies provided certain material conditions (such as small grain size, little plastic deformation, no texture) are fulfilled. The major sources of experimental error are generally well known. An extensive presentation can be found in Prevey (1981). It was proved by Maeder *et al.* (1981) that this technique is accurate enough for the majority of engineering problems.

It is obvious that X-ray diffraction does not allow direct measurements at interior points of a solid. In practice, layers at the surface of the solid are removed to reach the considered points. To minimize the risk of modifying the plastic strain field<sup>†</sup> at the origin of the residual stress field, chemical etching is the preferred removal technique. The point is that, even if the plastic strain field is not modified by matter removal, there is an elastic redistribution of the residual stress field. Thus, the mea-

<sup>†</sup> Or, more generally, the inherent strain or stress-free strain field.

sured stress tensor after removal is generally different from the original residual stress tensor at the corresponding point. Therefore, the main problem in residual stress analysis by means of X-ray diffraction is: how can the original stress (before matter removal) be related to the surface measured stress (after matter removal)?

Exact expressions relating the original residual stress field to the surface stress measurements have been obtained only for the following cases:

- Solid cylindrical bar with rotationally symmetric stresses independent of the axial coordinate  $z$  [exact integral expression, Moore and Evans (1958)].
- Solid cylindrical bar with stresses independent of the  $z$ -coordinate but varying along the angular coordinate  $\theta$  [approximate integral expression for stresses varying “smoothly” with  $\theta$ , Moore and Evans (1958)].
- Hollow cylinder with rotationally symmetric stresses independent of the  $z$ -coordinate [exact integral expression, Moore and Evans (1958)].
- Flat plate with stresses varying only along the thickness of the plate [exact integral expression in the framework of plate theory, Moore and Evans (1958)].
- Hollow cylinder (tube) with rotationally symmetric stresses possibly varying along the  $z$ -coordinate [exact infinite series expression, Nishimura (1993)].

The cases treated have a special geometrical symmetry and the residual stress field depends only on one space variable, with the exception of Nishimura’s problem (which is two-dimensional).

This paper addresses the general problem of expressing the original stresses as functions of surface-measured stresses in the case of a solid of arbitrary shape containing an arbitrary three-dimensional distribution of residual stresses for which no *a priori* information is available. In Section 1, some classical questions regarding this problem are examined through simple illustrative examples. In Section 2, explicit inversion formulas for the two-dimensional and the general three-dimensional half-space are developed. In Section 3, the general problem for an arbitrary geometry is considered. The quantity of measurements necessary to reconstruct the original residual stress field in a given part of the solid is investigated, or, equivalently, the conditions under which the problem is mathematically well posed are given. Finally, an algorithm for the numerical reconstruction of the original stresses from the X-ray surface measurements is also proposed.

Similar problems (inverse problems connected to residual stresses measurements) have already been studied. Cheng and Finnie (1986) have inverted the original residual stresses in a plate from strain gauge records on one side of a plate when layers are removed on the other side. Gao and Mura (1989) have inverted the residual stress field outside the plastically strained zone from surface displacements.

## 1. GENERALITIES AND EXAMPLES

In the sequel,  $\Omega$  is a linear elastic solid with a residual stress field  $\sigma$  (the original stresses). Let  $\Omega_t$  be the solid after matter removal at instant  $t$  (fictitious time parameter) and  $\sigma^m$  be the stress field at the surface of  $\Omega_t$  (the measured stresses). Note that  $\sigma^m$  is a function of two spatial coordinates on the surface of  $\Omega_t$ , and of  $t$ .

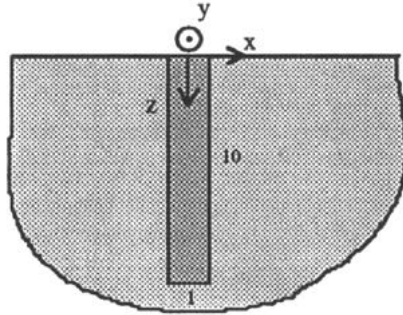


Fig. 1. Rectangular plastic inclusion in an elastic plane-strain half-space.

Our fundamental assumption is: *the relaxation of the remaining part  $\Omega_1$  of  $\Omega$  after the removal of  $\Omega \setminus \Omega_1$  is elastic.*

### 1.1. Stress redistribution

In this section, an example where matter removal induces a large stress redistribution is presented.

Let  $\Omega$  be a free elastic plane-strain half-space containing a plastic inclusion in Eshelby's sense.† The inclusion is infinite along  $y$  with size  $[-0.5, 0.5] \times [0, 10]$  in the  $(x, z)$  plane (see Fig. 1). This inclusion contains a homogeneous plastic strain field  $\mathbf{s}^P$  of the form:

$$\mathbf{s}^P = \begin{bmatrix} -\frac{\varepsilon^P}{2} & 0 & 0 \\ 0 & -\frac{\varepsilon^P}{2} & 0 \\ 0 & 0 & \varepsilon^P \end{bmatrix}$$

Let us suppose that infinite layers of uniform thickness are removed. Considering the thickness  $h$  of the removed layer as fictitious time parameter, it can be noted that  $\Omega_h$  is a half-space with a plastic inclusion of the same form. Closed-form formulae for the stress field induced by a rectangular inclusion in an elastic plane-strain half-space are presented in the Appendix. Therefore, the original stress field  $\sigma(x, z)$ , as well as the measured surface stress field  $\sigma^m(x, h)$  can be obtained from (40)–(42) of the Appendix. The functions  $\sigma_{xx}(0, 0, z)$  and  $\sigma_{xx}^m(0, 0, z)$  have been plotted on Fig. 2.

It can be observed that the two curves are very different. This illustrates that *matter removal may induce a strong stress redistribution and the measurements after removal have no reason to agree with the original stress distribution.*

### 1.2. Another example

Considering the previous example and Fig. 2, one could expect to find a method to compute the original stress evolution along  $z$  (dashed curve) just from the measured

† The inclusion has the same elastic moduli as the half-space.

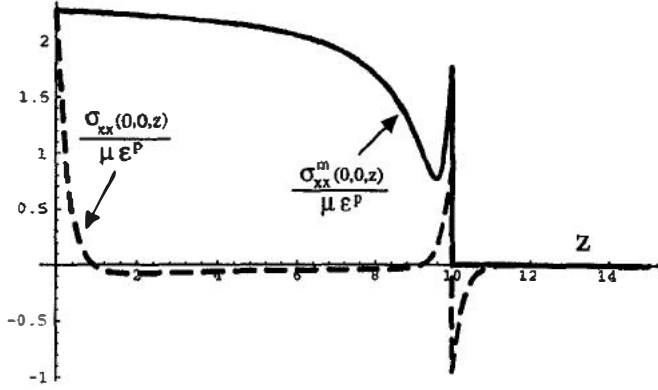


Fig. 2. Distributions of the original stress component  $\sigma_{xx}$  along the  $z$ -axis and of the surface stress  $\sigma_{xx}^m$  after matter removal of thickness  $z$ .

stress evolution along  $z$  (unbroken curve). Another simple example will show that it is impossible.

Let us consider the elastic plane-strain half-space with plastic inclusions of the same geometry as previously :  $[-5.5, -4.5] \times [0, 10]$  and  $[4.5, 5.5] \times [0, 10]$  (see Fig. 3).

$\varepsilon_1^P$  and  $\varepsilon_2^P$  denote the plastic strain fields (still assumed homogeneous) corresponding to inclusions 1 and 2. These plastic strain tensors have the form

$$\varepsilon_1^P = \begin{bmatrix} -\frac{\varepsilon^P}{2} & 0 & 0 \\ 0 & -\frac{\varepsilon^P}{2} & 0 \\ 0 & 0 & \varepsilon^P \end{bmatrix}, \quad \varepsilon_2^P = -\varepsilon_1^P.$$

The closed-form expression of the stress distribution can be obtained from (40)–(42) of the Appendix and the superposition principle. The stress tensor vanishes at the origin but  $\sigma_{xz}$  does not vanish on the  $z$ -axis for strictly positive  $z$ . Applying the same

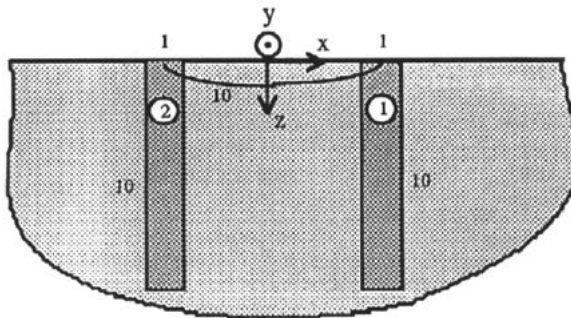


Fig. 3. Two rectangular plastic inclusions in the elastic plane-strain half-space.

matter removal process as in the first example, the measured stress distribution vanishes identically on the  $z$ -axis.

This example proves that *there is not enough information in the measured stress distribution along a line to reconstruct the initial stress distribution along this line.*

### 1.3. Is it necessary to remove matter?

There is another question of interest : is the matter removal absolutely necessary? In other terms, is it possible to reconstruct the interior stress field from only original stresses on the surface of  $\Omega$ ? It is well known that the answer is no. However, a proof through a simple example shall be given.

Let  $\Omega$  be a free elastic ball of radius  $r_0$  with Lamé constants  $\lambda$  and  $\mu$ . We consider a given plastic strain tensor field  $\varepsilon^p$  whose expression in the spherical coordinate system  $(r, \theta, \phi)$  is

$$\varepsilon^p(r) = \begin{bmatrix} -2\varepsilon^p(r) & 0 & 0 \\ 0 & \varepsilon^p(r) & 0 \\ 0 & 0 & \varepsilon^p(r) \end{bmatrix} \quad \text{with} \quad \varepsilon^p(r) = \frac{(r_0 - r)r}{r_0^2} \varepsilon^p, \quad (1)$$

where  $\varepsilon^p$  is a constant. It can easily be checked that the corresponding displacement field  $\mathbf{u}$  is given by

$$\begin{aligned} u_r(r, \theta, \phi) &= \frac{2\mu\varepsilon^p}{\lambda + 2\mu} \frac{(r_0 - r)^2 r}{r_0^2}, \\ u_\theta(r, \theta, \phi) &= u_\phi(r, \theta, \phi) = 0 \end{aligned} \quad (2)$$

and that the corresponding residual stress field is

$$\begin{aligned} \sigma_{rr}(r, \theta, \phi) &= \frac{2\mu(3\lambda + 2\mu)\varepsilon^p}{\lambda + 2\mu} \left( \frac{r_0 - r}{r_0} \right)^2, \\ \sigma_{\theta\theta}(r, \theta, \phi) &= \sigma_{\phi\phi}(r, \theta, \phi) = \frac{2\mu(3\lambda + 2\mu)\varepsilon^p}{\lambda + 2\mu} \frac{(r_0 - r)(r_0 - 2r)}{r_0^2}, \\ \sigma_{r\theta}(r, \theta, \phi) &= \sigma_{r\phi}(r, \theta, \phi) = \sigma_{\theta\phi}(r, \theta, \phi) = 0. \end{aligned} \quad (3)$$

It can be remarked that this residual stress field vanishes ( $\sigma = 0$ ) on the free surface ( $r = r_0$ ) whereas it does not vanish identically inside the ball (in fact, in this example, the displacement field also vanishes on the free surface). This stress field is an example of a nonidentically vanishing residual stress field which is identically zero on the free surface.

The consequence is : *X-ray analysis of subsurface residual stresses with no a priori information on the residual stress field is necessarily destructive.*

In other terms, there is not enough information in the surface residual stress field to evaluate the stress field inside the solid.

## 2. THE HALF-SPACE PROBLEM

In this section,  $\Omega$  is a free elastic half-space with rectangular coordinate system  $(x, y, z)$ . Infinite layers of uniform thickness  $h$  are removed. Therefore,  $\Omega_h$  is also a free half-space. The surface stress distribution  $\sigma^m(x, y, h)$  is measured on the free surface of  $\Omega_h$ . The two-dimensional case first and then the general three-dimensional problem are presented.

### 2.1. The two-dimensional case

In this section, all the fields are independent of  $y$ . Therefore,

$$\begin{aligned}\sigma^m &= \sigma^m(x, z), \\ \sigma &= \sigma(x, z).\end{aligned}\tag{4}$$

Let us consider three elastic problems (see Fig. 4).

• **Problem 1 (equilibrium of  $\Omega$ ):** the free half-space with plastic strain field  $\epsilon^p(x, y, z)$  (which is not known) and free surface. The stress field  $\sigma$  satisfies:

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \end{cases}\tag{5}$$

$$\sigma_{xz}(x, 0) = 0, \quad \sigma_{yz}(x, 0) = 0, \quad \sigma_{zz}(x, 0) = 0.$$

• **Problem 2 (equilibrium of  $\Omega_h$ ):** the free half-space with the plastic strain field  $\epsilon^p(x, y, z)$  ( $z \geq h$ ). The stress field on the free surface is  $\sigma^m(x, y, h)$ .

• **Problem 3 (auxiliary problem):** the elastic half-space with no plastic strain field and surface forces equal to  $\sigma(x, y, h) \cdot \mathbf{n}$ . The stress field is denoted by  $\sigma^h(x, y, z)$  ( $z \geq h$ ).

From the superposition principle, it follows that

$$\sigma(x, y, h) = \sigma^m(x, y, h) + \sigma^h(x, y, h).\tag{6}$$

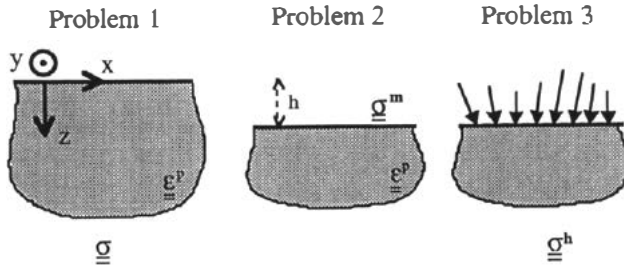


Fig. 4. Description of the three elastic problems.

The solution of the auxiliary problem is straightforward provided the stress field produced by a surface line loading of an elastic half-space (Boussinesq's problem) is given. This can be found for example in Johnson (1985) or deduced from the half-space Green's function (Mindlin, 1936) after having taken carefully the limit in sense of generalized functions. For a surface line loading  $\mathbf{f}$  of the form

$$f_x(x, y, z) = f_x \delta(x) \delta(z),$$

$$f_y(x, y, z) = 0,$$

$$f_z(x, y, z) = f_z \delta(x) \delta(z),$$

where  $\delta(\ )$  is Dirac's generalized function, the stress field at the surface of the half-space is

$$\sigma_{xx}(x, 0) = -\frac{2f_x}{\pi} \text{P.v.} \left( \frac{1}{x} \right) - f_z \delta(x),$$

$$\sigma_{xz}(x, 0) = -f_x \delta(x),$$

$$\sigma_{zz}(x, 0) = -f_z \delta(x), \quad (7)$$

where  $\text{P.v.}(1/x)$  is the generalized function: principal value of  $1/x$ . Using equation (6), this result leads to

$$\sigma_{xx}(x, z) = \sigma_{xx}^m(x, z) + \frac{2}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{xz}(x', z)}{x-x'} dx' + \sigma_{zz}(x, z),$$

$$\sigma_{xy}(x, z) = \sigma_{xy}^m(x, z) + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{yz}(x', z)}{x-x'} dx',$$

$$\sigma_{yy}(x, z) = \sigma_{yy}^m(x, z) + \frac{2\nu}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{xz}(x', z)}{x-x'} dx' + 2\nu \sigma_{zz}(x, z). \quad (8)$$

Therefore,  $\sigma$  is the solution of the system:

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \\ \sigma_{xx}(x, z) = \sigma_{xx}^m(x, z) + \frac{2}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{xz}(x', z)}{x-x'} dx' + \sigma_{zz}(x, z), \\ \sigma_{xy}(x, z) = \sigma_{xy}^m(x, z) + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{yz}(x', z)}{x-x'} dx', \\ \sigma_{yy}(x, z) = \sigma_{yy}^m(x, z) + \frac{2\nu}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \frac{\sigma_{xz}(x', z)}{x-x'} dx' + 2\nu \sigma_{zz}(x, z), \end{array} \right. \quad (9)$$

with the following boundary conditions:

$$\sigma_{xz}(x, 0) = 0, \quad \sigma_{yz}(x, 0) = 0, \quad \sigma_{zz}(x, 0) = 0. \quad (10)$$

System (9) can be solved explicitly by use of the one-dimensional Fourier transform. Let us define it by

$$\tilde{f}(s) = \int_{-\infty}^{+\infty} f(x') e^{-2i\pi x's} dx'. \quad (11)$$

The application of the Fourier transform in the sense of generalized functions to system (9) gives the following system of ordinary differential equations:

$$\begin{cases} \frac{\partial \tilde{\sigma}_{xz}}{\partial z}(s, z) = -4\pi|s|\tilde{\sigma}_{xz}(s, z) - 2i\pi s\tilde{\sigma}_{zz}(s, z) - 2i\pi s\tilde{\sigma}_{xx}^m(s, z), \\ \frac{\partial \tilde{\sigma}_{zz}}{\partial z}(s, z) = -2i\pi s\tilde{\sigma}_{xz}(s, z), \\ \tilde{\sigma}_{xx}(s, z) = \tilde{\sigma}_{xx}^m(s, z) - 2i \operatorname{sgn}(s)\tilde{\sigma}_{xz}(s, z) + \tilde{\sigma}_{zz}(s, z). \end{cases} \quad (12)$$

The corresponding boundary conditions, deduced from (10), are

$$\tilde{\sigma}_{xz}(s, 0) = \tilde{\sigma}_{zz}(s, 0) = 0.$$

The unique solution of system (12) is found to be

$$\begin{aligned} \tilde{\sigma}_{xx}(s, z) &= \tilde{\sigma}_{xx}^m(s, z) - \int_0^z 4\pi|s| e^{-2\pi|s|(z-z')} \tilde{\sigma}_{xx}^m(s, z') dz' \\ &\quad + \int_0^z 4\pi^2 s^2 (z-z') e^{-2\pi|s|(z-z')} \tilde{\sigma}_{xx}^m(s, z') dz', \\ \tilde{\sigma}_{xz}(s, z) &= - \int_0^z 2i\pi s e^{-2\pi|s|(z-z')} \tilde{\sigma}_{xx}^m(s, z') dz' \\ &\quad + \int_0^z 4i\pi^2 s|s|(z-z') e^{-2\pi|s|(z-z')} \tilde{\sigma}_{xx}^m(s, z') dz', \\ \tilde{\sigma}_{zz}(s, z) &= - \int_0^z 4\pi^2 s^2 (z-z') e^{-2\pi|s|(z-z')} \tilde{\sigma}_{xx}^m(s, z') dz'. \end{aligned} \quad (13)$$

Use of the inverse Fourier transform gives

$$\begin{aligned} \sigma_{xx}(x, z) &= \sigma_{xx}^m(x, z) + \frac{2}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{(x-x')^4 - 3(x-x')^2(z-z')^2}{[(x-x')^2 + (z-z')^2]^3} \\ &\quad \times [\sigma_{xx}^m(x', z') - \sigma_{xx}^m(x, z')] dx' dz', \\ \sigma_{xz}(x, z) &= \frac{4}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{(x-x')^3(z-z') - (x-x')(z-z')^3}{[(x-x')^2 + (z-z')^2]^3} \\ &\quad \times [\sigma_{xx}^m(x', z') - \sigma_{xx}^m(x, z')] dx' dz', \end{aligned}$$

$$\sigma_{zz}(x, z) = \frac{2}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{3(x-x')^2(z-z')^2 - (z-z')^4}{[(x-x')^2 + (z-z')^2]^3} \times [\sigma_{xx}^m(x', z') - \sigma_{xx}^m(x, z')] dx' dz'. \quad (14)$$

All the integrals in (14) are regular improper integrals.

To complete the determination of  $\sigma$ , the remaining equations of system (9) can also be solved using the same technique and one obtains:

$$\sigma_{xy}(x, z) = \sigma_{xy}^m(x, z) + \frac{1}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{(x-x')^2 - (z-z')^2}{[(x-x')^2 + (z-z')^2]^2} \times [\sigma_{xy}^m(x', z') - \sigma_{xy}^m(x, z')] dx' dz',$$

$$\sigma_{yy}(x, z) = \sigma_{yy}^m(x, z) + \frac{2\nu}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{(x-x')^2 - (z-z')^2}{[(x-x')^2 + (z-z')^2]^2} \times [\sigma_{xx}^m(x', z') - \sigma_{xx}^m(x, z')] dx' dz',$$

$$\sigma_{yz}(x, z) = \frac{2}{\pi} \int_0^z \int_{-\infty}^{+\infty} \frac{(x-x')(z-z')}{[(x-x')^2 + (z-z')^2]^2} [\sigma_{xy}^m(x', z') - \sigma_{xy}^m(x, z')] dx' dz', \quad (15)$$

where  $\nu$  is Poisson's ratio of the material. It is clear that (14) and (15) allow us to compute  $\sigma$  from  $\sigma^m$ . As an illustration, let us come back to the example of Section 1.1 and compute  $\sigma$  on the  $z$ -axis from  $\sigma^m$  using (14). Results are reported in Fig. 5. It should be emphasized that to reconstruct  $\sigma(0, 0, z)$ , not only  $\sigma^m(0, 0, z)$  (the plotted curve) but the whole field  $\sigma^m(x', y', z')$  ( $z' \leq z$ ) has been used.

## 2.2. The three-dimensional problem

In this section  $\sigma^m(x, y, z)$  is an arbitrary three-dimensional field. To proceed as previously, the auxiliary problem must be solved first. In order to do so, the expression

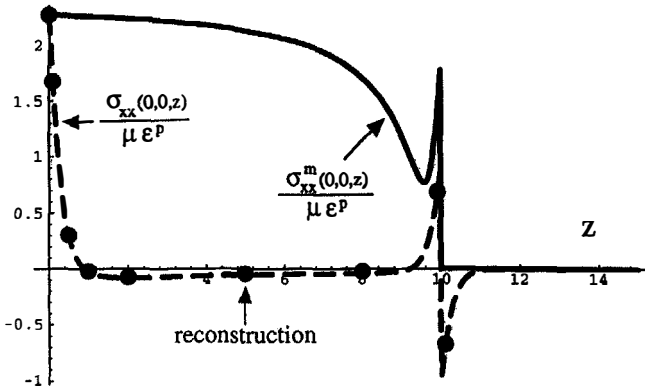


Fig. 5. Original stress distribution  $\sigma_{xx}$ , surface stress  $\sigma_{xx}^m$  after matter removal and reconstruction of  $\sigma_{xx}$  along the  $z$ -axis.

of the surface stress produced by a point force loading  $\mathbf{f}$  at the surface of a half-space is needed. General expressions can be found, for example, in Mindlin (1936).

After having taken the limit in the sense of generalized functions, we get :

$$\begin{aligned}
\sigma_{xx}(x, y, 0) &= -\frac{f_x}{\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{x}{\sqrt{x^2+y^2^3}} \right) + 3\nu \text{P.v.} \left( \frac{x^3}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad - \frac{f_y}{\pi} \left[ 3\nu \text{P.v.} \left( \frac{x^2 y}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad + \frac{f_z}{2\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \right] - \frac{1+2\nu}{2} f_z \delta(x) \delta(y), \\
\sigma_{yy}(x, y, 0) &= -\frac{f_x}{\pi} \left[ 3\nu \text{P.v.} \left( \frac{xy^2}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad - \frac{f_y}{\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{y}{\sqrt{x^2+y^2^3}} \right) + 3\nu \text{P.v.} \left( \frac{y^3}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad + \frac{f_z}{2\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \right] - \frac{1+2\nu}{2} f_z \delta(x) \delta(y), \\
\sigma_{zz}(x, y, 0) &= -f_z \delta(x) \delta(y), \\
\sigma_{xy}(x, y, 0) &= -\frac{f_x}{2\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{y}{\sqrt{x^2+y^2^3}} \right) + 6\nu \text{P.v.} \left( \frac{x^2 y}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad - \frac{f_y}{2\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{x}{\sqrt{x^2+y^2^3}} \right) + 6\nu \text{P.v.} \left( \frac{xy^2}{\sqrt{x^2+y^2^5}} \right) \right] \\
&\quad + \frac{f_z}{\pi} \left[ (1-2\nu) \text{P.v.} \left( \frac{xy}{(x^2 + y^2)^2} \right) \right], \\
\sigma_{xz}(x, y, 0) &= -f_x \delta(x) \delta(y), \\
\sigma_{yz}(x, y, 0) &= -f_y \delta(x) \delta(y).
\end{aligned} \tag{16}$$

As previously, we can derive a system of equations fulfilled by the components of the original stress field  $\sigma$  :

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0,
\end{aligned}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0,$$

$$\begin{aligned} \sigma_{xx}(x, y, z) &= \sigma_{xx}^m(x, y, z) + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{x-x'}{\sqrt{(x-x')^2 + (y-y')^2}^3} \right. \\ &\quad \left. + 3\nu \frac{(x-x')^3}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \sigma_{xz}(x', y', z) \, dx' \, dy' \\ &\quad + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ 3\nu \frac{(x-x')^2(y-y')}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \sigma_{yz}(x', y', z) \, dx' \, dy' \\ &\quad - \frac{1}{2\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{(x-x')^2 - (y-y')^2}{((x-x')^2 + (y-y')^2)^2} \right] \\ &\quad \times \sigma_{zz}(x', y', z) \, dx' \, dy' + \frac{1+2\nu}{2} \sigma_{zz}(x, y, z), \end{aligned}$$

$$\begin{aligned} \sigma_{yy}(x, y, z) &= \sigma_{yy}^m(x, y, z) \\ &\quad + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ 3\nu \frac{(x-x')(y-y')^2}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \sigma_{xz}(x', y', z) \, dx' \, dy' \\ &\quad + \frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{y-y'}{\sqrt{(x-x')^2 + (y-y')^2}^3} + 3\nu \frac{(y-y')^3}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \\ &\quad \times \sigma_{yz}(x', y', z) \, dx' \, dy' \\ &\quad - \frac{1}{2\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{(y-y')^2 - (x-x')^2}{((x-x')^2 + (y-y')^2)^2} \right] \\ &\quad \times \sigma_{zz}(x', y', z) \, dx' \, dy' + \frac{1+2\nu}{2} \sigma_{zz}(x, y, z), \end{aligned}$$

$$\begin{aligned} \sigma_{xy}(x, y, z) &= \sigma_{xy}^m(x, y, z) \\ &\quad + \frac{1}{2\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{y-y'}{\sqrt{(x-x')^2 + (y-y')^2}^3} + 6\nu \frac{(x-x')^2(y-y')}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \\ &\quad \times \sigma_{xz}(x', y', z) \, dx' \, dy' \\ &\quad + \frac{1}{2\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{x-x'}{\sqrt{(x-x')^2 + (y-y')^2}^3} + 6\nu \frac{(x-x')(y-y')^2}{\sqrt{(x-x')^2 + (y-y')^2}^5} \right] \\ &\quad \times \sigma_{yz}(x', y', z) \, dx' \, dy' \end{aligned}$$

$$-\frac{1}{\pi} \text{P.v.} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1-2\nu) \frac{(x-x')(y-y')}{((x-x')^2 + (y-y')^2)^2} \right] \sigma_{zz}(x', y', z) dx' dy', \quad (17)$$

with the following boundary conditions:

$$\sigma_{xz}(x, 0) = \sigma_{yz}(x, 0) = \sigma_{zz}(x, 0) = 0. \quad (18)$$

Obtained by exactly the same technique as in Section 2.1, the solution of system (17) is:

$$\begin{aligned} \sigma_{xx}(x, y, z) = & \sigma_{xx}^m(x, y, z) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left\{ (1-2\nu) \left[ \frac{1}{R^3} - \frac{3}{R(R+z-z')^2} \right] \right. \\ & + \left[ \frac{15(x-x')^4}{R^7} - \frac{12(x-x')^2}{R^5} + \frac{6(x-x')^2}{R^3(R+z-z')^2} + \frac{12(x-x')^2}{R^2(R+z-z')^3} \right] \\ & - \left. \left( (x-x')^4 + 2\nu(x-x')^2(y-y')^2 \right) \right. \\ & \times \left. \left[ \frac{3}{R^5(R+z-z')^2} + \frac{6}{R^4(R+z-z')^3} + \frac{6}{R^3(R+z-z')^4} \right] \right\} \\ & \times [\sigma_{xx}^m(x', y', z') - \sigma_{xx}^m(x, y, z')] dx' dy' dz' \\ & + \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left\{ (1+2\nu) \left[ \frac{3(x-x')(y-y')}{R^3(R+z-z')^2} + \frac{6(x-x')(y-y')}{R^2(R+z-z')^3} \right] \right. \\ & + \left. \left[ \frac{15(x-x')^3(y-y')}{R^7} - \frac{6(x-x')(y-y')}{R^5} \right] \right. \\ & - \left. \left( (x-x')^3(y-y') + 2\nu(x-x')(y-y')^3 \right) \right. \\ & \times \left. \left[ \frac{3}{R^5(R+z-z')^2} + \frac{6}{R^4(R+z-z')^3} + \frac{6}{R^3(R+z-z')^4} \right] \right\} \\ & \times [\sigma_{xy}^m(x', y', z') - \sigma_{xy}^m(x, y, z')] dx' dy' dz' \\ & + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left\{ (1-2\nu) \left[ \frac{3}{R(R+z-z')^2} \right] \right. \\ & + \left. \left[ \frac{15(x-x')^2(y-y')^2}{R^7} + \frac{3(z-z')^2}{R^5} - \frac{3}{R^3} \right] \right. \\ & - \left. \left( (x-x')^2(y-y')^2 + 2\nu(y-y')^4 \right) \right. \\ & \times \left. \left[ \frac{3}{R^5(R+z-z')^2} + \frac{6}{R^4(R+z-z')^3} + \frac{6}{R^3(R+z-z')^4} \right] \right\} \end{aligned}$$

$$+ 2\nu \left[ \frac{6(y-y')^2}{R^3(R+z-z')^2} + \frac{12(y-y')^2}{R^2(R+z-z')^3} \right] \left\{ [\sigma_{yy}^m(x', y', z') - \sigma_{yy}^m(x, y, z')] \right\} dx' dy' dz', \quad (19)$$

$$\begin{aligned} \sigma_{xy}(x, y, z) &= \sigma_{xy}^m(x, y, z) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left\{ \left[ \frac{1}{R^3} - \frac{3(z-z')^2}{R^5} \right] \right. \\ &+ 4\nu \left[ \frac{3(x-x')^2(y-y')^2}{R^5(R+z-z')^2} + \frac{6(x-x')^2(y-y')^2}{R^4(R+z-z')^3} + \frac{6(x-x')^2(y-y')^2}{R^3(R+z-z')^4} \right. \\ &+ \left. \frac{1}{R^3} - \frac{3}{R(R+z-z')^2} \right] \left\{ [\sigma_{xy}^m(x', y', z') - \sigma_{xy}^m(x, y, z')] \right\} dx' dy' dz' \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left\{ \left[ \frac{3(x-x')(y-y')}{R^5} \right] - 2\nu \left[ \frac{3(x-x')(y-y')}{R^3(R+z-z')^2} + \frac{6(x-x')(y-y')}{R^2(R+z-z')^3} \right] \right\} \\ &\times [(\sigma_{xx}^m(x', y', z') - \sigma_{xx}^m(x, y, z')) + (\sigma_{yy}^m(x', y', z') - \sigma_{yy}^m(x, y, z'))] dx' dy' dz' \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z 2\nu \left[ \frac{3(x-x')(y-y')}{R^5(R+z-z')^2} + \frac{6(x-x')(y-y')}{R^4(R+z-z')^3} + \frac{6(x-x')(y-y')}{R^3(R+z-z')^4} \right] \\ &\times [(x-x')^2(\sigma_{xx}^m(x', y', z') - \sigma_{xx}^m(x, y, z')) + (y-y')^2(\sigma_{yy}^m(x', y', z') \\ &- \sigma_{yy}^m(x, y, z'))] dx' dy' dz', \quad (20) \end{aligned}$$

$$\sigma_{yy}(x, y, z) = \sigma_{xx}(x, y, z) \quad \text{with all } x \text{ and } y \text{ interchanged,} \quad (21)$$

$$\begin{aligned} \sigma_{xz}(x, y, z) &= + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{15(x-x')^3(z-z')}{R^7} - \frac{6(x-x')(z-z')}{R^5} \right] \\ &\times [\sigma_{xx}^m(x', y', z') - \sigma_{xx}^m(x, y, z')] dx' dy' dz' \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{30(x-x')^2(y-y')(z-z')}{R^7} - \frac{3(y-y')(z-z')}{R^5} \right] \\ &\times [\sigma_{xy}^m(x', y', z') - \sigma_{xy}^m(x, y, z')] dx' dy' dz' \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{15(x-x')(y-y')^2(z-z')}{R^7} - \frac{3(x-x')(z-z')}{R^5} \right] \\ &\times [\sigma_{yy}^m(x', y', z') - \sigma_{yy}^m(x, y, z')] dx' dy' dz', \quad (22) \end{aligned}$$

$$\sigma_{yz}(x, y, z) = \sigma_{xz}(x, y, z) \quad \text{with all } x \text{ and } y \text{ interchanged,} \quad (23)$$

$$\begin{aligned} \sigma_{zz}(x, y, z) &= + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{15(x-x')^2(z-z')^2}{R^7} - \frac{3(z-z')^2}{R^5} \right] \\ &\times [\sigma_{xx}^m(x', y', z') - \sigma_{xx}^m(x, y, z')] dx' dy' dz' \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{15(x-x')(y-y')(z-z')^2}{R^7} [\sigma_{xy}^m(x', y', z') - \sigma_{xy}^m(x, y, z')] \right] dx' dy' dz' \\
& + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^z \left[ \frac{15(y-y')^2(z-z')^2}{R^7} - \frac{3(z-z')^2}{R^5} \right] \\
& \times [\sigma_{yy}^m(x', y', z') - \sigma_{yy}^m(x, y, z')] dx' dy' dz', \tag{24}
\end{aligned}$$

where

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

Formulae (19)–(24) are explicit inversion formulae for the general three-dimensional problem in the particular case of the half-space geometry. Simple inspection of these results leads to the following conclusions for the half-space geometry :

- To reconstruct the original stress field  $\sigma$ , one needs to know *the whole field*  $\sigma^m$  on *each intermediate surface*. This represents of course a large number of measurements but it is the nature of the mathematical problem itself.

- If the measured stress field  $\sigma^m$  appears to be independent of the spatial coordinate  $y$ , it can be seen from formulas (19)–(24) that then the original stress field is also independent of  $y$  and formulas of the previous section are recovered. Thus, the initial assumption (if  $\sigma^m$  is independent of  $y$  then  $\sigma$  is also independent of  $y$ ) of Section 2.1 is proved.

- If the measured stress  $\sigma^m$  appears to be independent of the two spatial coordinates  $x$  and  $y$ , there is *no stress redistribution* due to the removal of the layers (i.e.  $\sigma(x, y, x) \equiv \sigma^m(x, y, z)$ ).

### 3. THE GENERAL PROBLEM

#### 3.1. Description of the problem

In this section,  $\Omega_0$  is an elastic body of arbitrary shape supporting the original residual stress field  $\sigma$ . We suppose that matter removal is a continuous process depending on a fictitious time parameter  $t$  ( $0 \leq t \leq T$ ), generating a sequence of bodies  $\Omega_t$  of boundary  $\partial\Omega_t$ .  $S_t$  denotes the part of  $\partial\Omega_t$  which is exposed during this process ( $S_t$  is the complementary of  $\partial\Omega_t \cap \partial\Omega_0$  in  $\partial\Omega_t$ ) (see Fig. 6).

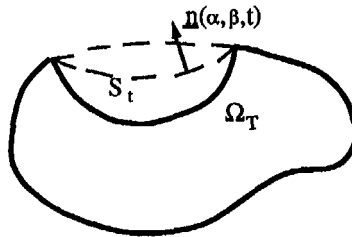


Fig. 6. Geometry and notations.

Let  $(\alpha, \beta, \gamma)$  be a curvilinear coordinate system on  $\Omega_0$  such that :

- The coordinate lines parametrized by  $\gamma$  are perpendicular to the surfaces  $S_t$  ( $0 \leq t \leq T$ ).
- On each surface  $S_t$ , the coordinate  $\gamma$  remains constant and equal to  $t$ .
- The coordinate lines  $\gamma$  are everywhere perpendicular to the others ( $\alpha$  and  $\beta$ ).

$\mathbf{n}(\alpha, \beta, t)$  denotes the outward unit normal to  $\partial\Omega_t$ ; this vector field is defined all over  $\Omega_0 \setminus \Omega_T$  as well as on  $\partial\Omega_0$ .

The measured stress field  $\sigma^m$  is assumed to be known on each surface  $S_t$  ( $0 \leq t \leq T$ ) and such that

$$\forall t \in [0, T] \quad \sigma^m(\alpha, \beta, t) \cdot \mathbf{n}(\alpha, \beta, t) = 0. \quad (25)$$

The fundamental hypothesis is, as in the half-space problem, that *the stress relaxation in  $\Omega_t$  due to the removal of  $\Omega_0 \setminus \Omega_t$  is elastic.*

As previously, three elastic problems at instant  $t$  are considered (see Fig. 7) :

- Problem 1 : equilibrium of  $\Omega_0$ . The stress field is  $\sigma(\alpha, \beta, \gamma)$ , fulfilling the equilibrium equation

$$\operatorname{div} \sigma(\alpha, \beta, \gamma) = 0. \quad (26)$$

- Problem 2 : equilibrium of  $\Omega_t$ . The stress field on the free surface  $S_t$  is  $\sigma^m(\alpha, \beta, t)$ .

- Problem 3 : auxiliary problem. Tractions  $\sigma(\alpha, \beta, t) \cdot \mathbf{n}(\alpha, \beta, t)$  are imposed on the boundary of  $\Omega_t$ , supposed to be initially at rest.  $\sigma^t(\alpha, \beta, \gamma)$  ( $\gamma \geq t$ ) denotes the resulting stress field in  $\Omega_t$ , fulfilling

$$\operatorname{div} \sigma^t(\alpha, \beta, \gamma) = 0 \quad \gamma \geq t. \quad (27)$$

Problem 3 is purely elastic, whereas problems 1 and 2 involve the inelastic strain field (assumed to be identical in both cases as a consequence of the elastic relaxation hypothesis) which is the source of the stress field. From the superposition principle, it is readily seen that

$$\forall t \in [0, T] \quad \sigma(\alpha, \beta, t) = \sigma^m(\alpha, \beta, t) + \sigma^t(\alpha, \beta, t). \quad (28)$$

For the half-space problem, the key point was the explicit solution of the auxiliary problem. In the general problem, such an explicit solution is not available. However, the splitting in three elastic problems will help to prove the following results :

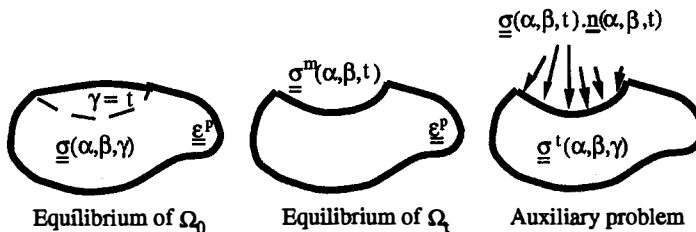


Fig. 7. Definition of problems 1, 2 and 3.

- for a given measured stress field  $\sigma^m$ , there corresponds at most one initial stress field  $\sigma$  (uniqueness);
- provided two conditions are fulfilled (to be detailed in the sequel), any  $\sigma^m$  field in  $\Omega_0 \setminus \Omega_T$  is derived from an initial stress field  $\sigma$  (existence).

### 3.2. Existence and uniqueness proof

The point of departure of the proof is (28). In this equation, the stress field  $\sigma'$  is defined on the basis of the unknown original stress field  $\sigma$ . It will now be proved that the stress field  $\sigma'$  is fully determined by the measured stress field  $\sigma^m$ .

In order to provide an efficient use of the equilibrium equation, we shall develop its expression in the curvilinear coordinate system. Let us recall that, in  $\Omega_0 \setminus \Omega_T$ ,  $\mathbf{n}(\alpha, \beta, \gamma)$  is the normalized vector associated with the  $\gamma$  coordinate and let us denote by  $\text{div}_2$  the two-dimensional divergence operator relative to coordinates  $\alpha$  and  $\beta$  when the third coordinate  $\gamma$  is fixed. Using the fact that the coordinate  $\gamma$  is perpendicular to the others, the equilibrium equation may be written in the form :

$$\text{div } \sigma(\alpha, \beta, \gamma) = \text{div}_2 \sigma + \frac{1}{h} \frac{\partial}{\partial \gamma} (\sigma \cdot \mathbf{n}) + \mathbf{D} : \sigma = 0, \quad (29)$$

where  $h(\alpha, \beta, \gamma)$  is a scalar function and  $\mathbf{D}(\alpha, \beta, \gamma)$  is a third rank tensor.  $h$  and  $\mathbf{D}$  are determined by the choice of the coordinate system.  $h(\alpha, \beta, \gamma)$  is nothing but the local normal speed of the matter removal process.

Writing (26) and (27) in the form of (29) and using (28) and (25), it is readily seen that

$$\frac{\partial}{\partial \gamma} (\sigma \cdot \mathbf{n})(\alpha, \beta, \gamma = t) = \frac{\partial}{\partial \gamma} (\sigma' \cdot \mathbf{n})(\alpha, \beta, \gamma = t) - h \text{div } \sigma^m(\alpha, \beta, t). \quad (30)$$

Differentiating (28), we have

$$\frac{\partial}{\partial \gamma} (\sigma \cdot \mathbf{n})(\alpha, \beta, \gamma = t) = \frac{\partial}{\partial t'} \Big|_{t'=t} (\sigma' \cdot \mathbf{n})(\alpha, \beta, t) + \frac{\partial}{\partial \gamma} (\sigma' \cdot \mathbf{n})(\alpha, \beta, \gamma = t). \quad (31)$$

The combination of (30) and (31) gives

$$\forall t \in [0, T] \quad \frac{\partial}{\partial t'} \Big|_{t'=t} (\sigma' \cdot \mathbf{n})(\alpha, \beta, t) = -h \text{div } \sigma^m(\alpha, \beta, t). \quad (32)$$

Now let  $t$  be any point in the interval  $[0, T]$  and  $t'$  any real number in  $[0, t]$ . Clearly, we have

$$\Omega_t \subseteq \Omega_{t'}. \quad (33)$$

By definition,  $\sigma'(\alpha, \beta, \gamma)$  is the stress field resulting in  $\Omega_{t'}$  with surface traction equal to  $\sigma(\alpha, \beta, t') \cdot \mathbf{n}(\alpha, \beta, t')$  on  $S_{t'}$  and to 0 on  $\partial\Omega_{t'} \setminus S_{t'}$ .  $\sigma'(\alpha, \beta, \gamma)$  in  $\Omega_t$  may be seen as the stress field induced by surface tractions equal to  $\sigma'(\alpha, \beta, t) \cdot \mathbf{n}(\alpha, \beta, t)$  on  $S_t$  and to 0 on  $\partial\Omega_t \setminus S_t$ . This field may now be differentiated with respect to  $t'$ . This yields

$$\left. \frac{\partial}{\partial t'} \right|_{t'=t} \boldsymbol{\sigma}'(\alpha, \beta, \gamma)$$

and is the stress field resulting in  $\Omega_t$  with surface tractions equal to

$$\left. \frac{\partial}{\partial t'} \right|_{t'=t} (\boldsymbol{\sigma}' \cdot \mathbf{n})(\alpha, \beta, t)$$

on  $S_t$  and to 0 on  $\partial\Omega_t \setminus S_t$ . Recalling (32), these surface tractions are completely determined by the field  $\boldsymbol{\sigma}^m$ .

Being the solution of an elastic Neumann problem, the stress field

$$\left. \frac{\partial}{\partial t'} \right|_{t'=t} \boldsymbol{\sigma}'(\alpha, \beta, \gamma) \quad (0 \leq t \leq \gamma \leq T) \quad (34)$$

exists and is unique if and only if the resultant of the surface tractions resultant as well as their resultant moment vanish. Therefore, for a given field  $\boldsymbol{\sigma}^m$  of measured stress such as (25) in  $\Omega_0 \setminus \Omega_T$ , the field

$$\left. \frac{\partial}{\partial t'} \right|_{t'=t} \boldsymbol{\sigma}'(\alpha, \beta, \gamma) \quad (0 \leq t \leq \gamma \leq T) \quad (35)$$

exists and is unique if and only if:

$$\forall t \in [0, T] \int_{S_t} h(\alpha, \beta, t) \operatorname{div} \boldsymbol{\sigma}^m(\alpha, \beta, t) \, dS_t = 0, \quad (36)$$

$$\forall t \in [0, T] \int_{S_t} OM(\alpha, \beta, t) \wedge (h(\alpha, \beta, t) \operatorname{div} \boldsymbol{\sigma}^m(\alpha, \beta, t)) \, dS_t = 0, \quad (37)$$

where  $O$  is any fixed point in the space and  $M(\alpha, \beta, \gamma)$  is the current point of integration. Moreover, this field is fully determined by the field  $\boldsymbol{\sigma}^m$ .

The existence and uniqueness of  $\boldsymbol{\sigma}$  in  $\Omega_0 \setminus \Omega_T$  under the conditions (36) and (37) is, then, a consequence of (28), which may be written as:

$$\forall \gamma \in [0, T] \quad \boldsymbol{\sigma}(\alpha, \beta, \gamma) = \boldsymbol{\sigma}^m(\alpha, \beta, \gamma) + \int_0^\gamma \left. \frac{\partial}{\partial t'} \right|_{t'=t} \boldsymbol{\sigma}'(\alpha, \beta, \gamma) \, dt. \quad (38)$$

Let us make some comments on the above result. It is clear that (36) and (37) will not help for a potential numerical solution of the problem. However, the problem itself may be slightly modified to make these conditions vanish. Indeed, if instead of letting the body  $\Omega_0$  be free during the matter removal process, a small part of its boundary (if possible far from the removed part) is kept fixed, one may assume that the stress relaxation will be only slightly influenced. Considering this hypothesis, the above proof may be rewritten using practically the same arguments and, in this case, existence and uniqueness may be proved without the conditions (36) and (37). The reason is that, in the case of mixed elastic problem (displacement prescribed on a part of the boundary and tractions on the remaining part), no condition, as in a Neumann

problem, has to be fulfilled by the surface traction distribution to ensure the existence of the solution.

### 3.3. Some comments about the numerical solving of the problem

A numerical algorithm may be immediately derived from the previous proof.

The matter removal process is supposed to be discretized in  $N$  steps, corresponding to  $N$  intermediate surfaces  $S_i$  and  $N$  elastic bodies  $\Omega_i$ .

An obvious method to find a numerical solution is to compute, from the measurements  $\sigma^m$ , a numerical approximation of

$$-h(\alpha, \beta, t) \operatorname{div} \sigma^m(\alpha, \beta, t) \quad (39)$$

and to solve the  $N$  elastic auxiliary problems (by a Finite Element Method or by a Boundary Element Method) with surface tractions  $-h(\alpha, \beta, t_i) \operatorname{div} \sigma^m(\alpha, \beta, t_i)$  and geometries  $\Omega_i$ . A numerical integration in (38) will provide an evaluation of the stress field  $\sigma$ .

Clearly, the numerical computation of  $\operatorname{div} \sigma^m$  from the discretized field  $\sigma^m$  will be an important source of numerical error and a suitable numerical procedure will probably be necessary in order to stabilize the algorithm.

## CONCLUSION

The aim of this paper was to investigate the problem of inverting a residual stress field from surface stress measurements after matter removal. A mathematical formalism for the general problem has been described. This has led to an explicit inversion formula for the half-space geometry. For other geometries, a numerical algorithm has been derived.

From a mathematical point of view, it appears that the reconstruction of the original stress field requires the knowledge of the surface stress distribution all over each intermediate surface. From a practical point of view, one can only expect to measure approximately surface stresses at a finite number of points. The inspection of explicit formulas (19)–(24) shows that small errors on measurements have little influence on the reconstruction of the original stress. Moreover, the far field measurements can also be neglected. In both cases, an estimate of the error can be obtained.

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## APPENDIX

In this Appendix, we give closed-form expressions of the stress field produced by a plastic inclusion in a plane-strain, isotropic, linear elastic half-space.

Geometry of the problem and notations are described in Fig. A1.

In this inclusion, we consider a given plastic strain field  $\varepsilon^p$  of the form :

$$\varepsilon^p(x, z) = \begin{bmatrix} -\frac{\varepsilon^p}{2} & 0 & 0 \\ 0 & -\frac{\varepsilon^p}{2} & 0 \\ 0 & 0 & \varepsilon^p \end{bmatrix} \chi(x, z),$$

where  $\chi(x, z)$  is the characteristic function of the inclusion (it vanishes outside the inclusion and is equal to one inside).

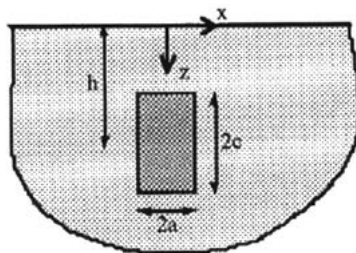


Fig. A1. Plastic inclusion in the plane-strain half-space.

Integrations on  $y$  from  $-\infty$  to  $+\infty$  are performed on Green's function of the elastic half-space (Mindlin, 1936) in order to get the two-dimensional Green's function. It is then easy to get the stress field in the whole half-space (see, for example, Mura, 1982). The stress tensor field is found to be :

$$\begin{aligned}
\sigma_{xx}(x, z) = & \mu \varepsilon^p \chi(x, z) + \frac{\mu \varepsilon^p}{4\pi(1-\nu)} \left[ 4\nu \left( \arctan \left( \frac{x-a}{z-h-c} \right) - \arctan \left( \frac{x+a}{z-h-c} \right) \right. \right. \\
& - \arctan \left( \frac{x-a}{z-h+c} \right) + \arctan \left( \frac{x+a}{z-h+c} \right) \left. \right) - 2(1-\nu) \left( \arctan \left( \frac{z-h-c}{x-a} \right) \right. \\
& - \arctan \left( \frac{z-h-c}{x+a} \right) - \arctan \left( \frac{z-h+c}{x-a} \right) + \arctan \left( \frac{z-h+c}{x+a} \right) \left. \right) \\
& + 4\nu(3-4\nu) \left( \arctan \left( \frac{x-a}{z+h+c} \right) - \arctan \left( \frac{x+a}{z+h+c} \right) - \arctan \left( \frac{x-a}{z+h-c} \right) \right. \\
& + \arctan \left( \frac{x+a}{z+h-c} \right) \left. \right) - 2(3-9\nu+8\nu^2) \left( \arctan \left( \frac{z+h+c}{x-a} \right) - \arctan \left( \frac{z+h+c}{x+a} \right) \right. \\
& - \arctan \left( \frac{z+h-c}{x-a} \right) + \arctan \left( \frac{z+h-c}{x+a} \right) \left. \right) - 3 \left( \frac{(x-a)(z-h-c)}{(x-a)^2 + (z-h-c)^2} \right. \\
& - \frac{(x+a)(z-h-c)}{(x+a)^2 + (z-h-c)^2} - \frac{(x-a)(z-h+c)}{(x-a)^2 + (z-h+c)^2} + \frac{(x+a)(z-h+c)}{(x+a)^2 + (z-h+c)^2} \left. \right) \\
& - (5-4\nu) \left( \frac{(x-a)z}{(x-a)^2 + (z+h+c)^2} - \frac{(x+a)z}{(x+a)^2 + (z+h+c)^2} - \frac{(x-a)z}{(x-a)^2 + (z+h-c)^2} \right. \\
& + \frac{(x+a)z}{(x+a)^2 + (z+h-c)^2} \left. \right) + 9 \left( \frac{(x-a)(h+c)}{(x-a)^2 + (z+h+c)^2} - \frac{(x+a)(h+c)}{(x+a)^2 + (z+h+c)^2} \right. \\
& - \frac{(x-a)(h-c)}{(x-a)^2 + (z+h-c)^2} + \frac{(x+a)(h-c)}{(x+a)^2 + (z+h-c)^2} \left. \right) - 12 \left( \frac{(x-a)(h+c)(z+h+c)z}{((x-a)^2 + (z+h+c)^2)^2} \right. \\
& - \frac{(x+a)(h+c)(z+h+c)z}{((x+a)^2 + (z+h+c)^2)^2} - \frac{(x-a)(h-c)(z+h-c)z}{((x-a)^2 + (z+h-c)^2)^2} + \left. \frac{(x+a)(h-c)(z+h-c)z}{((x+a)^2 + (z+h-c)^2)^2} \right) \left. \right], \quad (40)
\end{aligned}$$

$$\begin{aligned}
\sigma_{xz}(x, z) = & \frac{\mu \varepsilon^p}{4\pi(1-\nu)} \left[ -3 \left( \frac{(z-h-c)^2}{(x-a)^2 + (z-h-c)^2} - \frac{(z-h-c)^2}{(x+a)^2 + (z-h-c)^2} \right. \right. \\
& - \frac{(z-h+c)^2}{(x-a)^2 + (z-h+c)^2} + \left. \frac{(z-h+c)^2}{(x+a)^2 + (z-h+c)^2} \right) - (5-4\nu) \left( \frac{z^2}{(x-a)^2 + (z+h+c)^2} \right. \\
& - \frac{z^2}{(x+a)^2 + (z+h+c)^2} - \frac{z^2}{(x-a)^2 + (z+h-c)^2} + \left. \frac{z^2}{(x+a)^2 + (z+h-c)^2} \right) \\
& + 4(1+\nu) \left( \frac{z(h+c)}{(x-a)^2 + (z+h+c)^2} - \frac{z(h+c)}{(x+a)^2 + (z+h+c)^2} - \frac{z(h-c)}{(x-a)^2 + (z+h-c)^2} \right. \\
& + \left. \frac{z(h-c)}{(x+a)^2 + (z+h-c)^2} \right) + 3 \left( \frac{(h+c)^2}{(x-a)^2 + (z+h+c)^2} - \frac{(h+c)^2}{(x+a)^2 + (z+h+c)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{(h-c)^2}{(x-a)^2+(z+h-c)^2} + \frac{(h-c)^2}{(x+a)^2+(z+h-c)^2} - 12 \left( \frac{(h+c)(z+h+c)^2 z}{((x-a)^2+(z+h+c)^2)^2} \right. \\
& \left. - \frac{(h+c)(z+h+c)^2 z}{((x+a)^2+(z+h+c)^2)^2} - \frac{(h-c)(z+h-c)^2 z}{((x-a)^2+(z+h-c)^2)^2} + \frac{(h-c)(z+h-c)^2 z}{((x+a)^2+(z+h-c)^2)^2} \right), \quad (41)
\end{aligned}$$

$$\begin{aligned}
\sigma_{zz}(x, z) = & -2\mu\epsilon^P\chi(x, z) + \frac{\mu\epsilon^P}{4\pi(1-\nu)} \left[ 4(1-\nu) \left( \arctan\left(\frac{x-a}{z-h-c}\right) - \arctan\left(\frac{x+a}{z-h-c}\right) \right. \right. \\
& - \arctan\left(\frac{x-a}{z-h+c}\right) + \arctan\left(\frac{x+a}{z-h+c}\right) \Big) - 2\nu \left( \arctan\left(\frac{z-h-c}{x-a}\right) \right. \\
& - \arctan\left(\frac{z-h-c}{x+a}\right) - \arctan\left(\frac{z-h+c}{x-a}\right) + \arctan\left(\frac{z-h+c}{x+a}\right) \Big) \\
& + 4(1-\nu) \left( \arctan\left(\frac{x-a}{z+h+c}\right) - \arctan\left(\frac{x+a}{z+h+c}\right) - \arctan\left(\frac{x-a}{z+h-c}\right) \right. \\
& + \arctan\left(\frac{x+a}{z+h-c}\right) \Big) - 2\nu \left( \arctan\left(\frac{z+h+c}{x-a}\right) - \arctan\left(\frac{z+h+c}{x+a}\right) \right. \\
& - \arctan\left(\frac{z+h-c}{x-a}\right) + \arctan\left(\frac{z+h-c}{x+a}\right) \Big) + 3 \left( \frac{(x-a)(z-h-c)}{(x-a)^2+(z-h-c)^2} \right. \\
& - \frac{(x+a)(z-h-c)}{(x+a)^2+(z-h-c)^2} - \frac{(x-a)(z-h+c)}{(x-a)^2+(z-h+c)^2} + \frac{(x+a)(z-h+c)}{(x+a)^2+(z-h+c)^2} \Big) \\
& + 3 \left( \frac{(x-a)(h+c)}{(x-a)^2+(z+h+c)^2} - \frac{(x+a)(h+c)}{(x+a)^2+(z+h+c)^2} - \frac{(x-a)(h-c)}{(x-a)^2+(z+h-c)^2} \right. \\
& + \frac{(x+a)(h-c)}{(x+a)^2+(z+h-c)^2} \Big) + (5-4\nu) \left( \frac{(x-a)z}{(x-a)^2+(z+h+c)^2} - \frac{(x+a)z}{(x+a)^2+(z+h+c)^2} \right. \\
& - \frac{(x-a)z}{(x-a)^2+(z+h-c)^2} + \frac{(x+a)z}{(x+a)^2+(z+h-c)^2} \Big) + 12 \left( \frac{(x-a)(h+c)(z+h+c)z}{((x-a)^2+(z+h+c)^2)^2} \right. \\
& \left. - \frac{(x+a)(h+c)(z+h+c)z}{((x+a)^2+(z+h+c)^2)^2} - \frac{(x-a)(h-c)(z+h-c)z}{((x-a)^2+(z+h-c)^2)^2} + \frac{(x+a)(h-c)(z+h-c)z}{((x+a)^2+(z+h-c)^2)^2} \right). \quad (42)
\end{aligned}$$