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***Abstract*—This paper deals with the stabilization of a class of linear infinite-dimensional systems with unbounded control operators and subject to a boundary disturbance. We assume that there exists a linear feedback law that makes the origin of the closed-loop system globally asymptotically stable in the absence of disturbance. To achieve our objective, we follow a sliding mode strategy and we add another term to this controller in order to reject the disturbance. We prove the existence of solutions to the closed-loop system and its global asymptotic stability, while making sure the disturbance is rejected.**

I. Introduction

This paper is concerned with the stabilization of a class of linear infinite-dimensional systems with unbounded control operators and subject to a boundary disturbance (see e.g, [4], [10], [25] for a review on this class of system). To be more precise, we aim to design a sliding mode control (SMC) and a super-twisting control (STC) [12], [46], [52] for the stabilization of boundary or pointwise control for linear partial differential equations (PDEs).

The boundary control of systems described by partial differential equations has received a lot of attention for decades. It continues to be an important research focus today because its application in many important engineering systems is natural (see e.g., [3]). Such a problem has been studied in [9], [16], [44], [45] in the controllability context, in [7], [8], [23], [47], [51], [58] in terms of stabilization.

In this paper, as mentioned earlier, we focus on the case where infinite-dimensional systems are subject to a disturbance. Therefore, we are not only interested in the stabilization, but also in the rejection of this disturbance. This might be interpreted as a regulation problem. In the case where the disturbance is constant, one can follow a proportional integral (PI) strategy, which is quite well-known in the linear finite-dimensional context, but which is still nowadays an active topic when dealing with PDEs (see e.g., [1], [24], [28], [40], [49]). For more complicated cases, i.e. when the disturbance is time-dependant, one may apply the celebrated internal-model approach [11], [40], which consists, roughly speaking, in adding the dynamics of the disturbance in the loop of the controller. This method needs therefore the knowledge of the dynamics of the controller. The active disturbance rejection control (ADRC) is another powerful method to deal with disturbances since only the bound of the disturbance is needed. It was initially proposed in [21] in the context of finite dimensional systems. The main idea of the ADRC is to build an observer to estimate the disturbance. Then, the disturbance is compensated in a feedback-law by its estimated value. Recently, this approach has been successfully applied to systems described by one-dimensional partial differential equations (PDEs) [5], [18]–[20], [31], [56], [57] . Our strategy, based on SMC controllers, require also only the bound of the disturbance, at the price of assuming that the disturbance matches with the control (i.e., the control and the disturbance are located at the same place).

SMC strategy has been proved to be efficient for robust control of nonlinear systems of ordinary differential equations (ODEs) [12], [46], [52], [55]. Such controllers allow to force, thanks to discontinuous terms, the trajectories of the system to reach in a finite time a manifold, called the sliding manifold, and to evolve on it, this manifold being defined from control objectives. Basically, the design of the control is split into two steps: firstly, a sliding variable is selected such that, once this variable equals zero, global asymptotic stability is ensured; secondly, a discontinuous feedback-law is designed such that the trajectory reaches the sliding manifold, that is defined thanks to the sliding variable. On this sliding manifold, the disturbance is rejected. The generalization of the SMC procedure to the PDEs case is not new. In [38], [39], a definition of equivalent control (which is the control applied to the system after reaching the sliding manifold, to ensure that the trajectories stays on the surface thereafter) for systems governed by semilinear differential equations in Banach spaces has been proposed. One can refer also to [26], [27] where differential inclusions and viability theory are combined to design sliding mode controllers for semilinear differential equations in Banach spaces. We also mention the use of spectral reduction methods in [34]. In the last decade, a backstepping strategy has been used to select a sliding variable [17], [31], [42], [48], [53]. We also refer to these recent papers [2], [30], in which the sliding variable is derived from the gradient of some well-known Lyapunov functional in the hyperbolic context [3, Section 2.1.2]. Note also that the SMC feedback-law is discontinuous, which creates chattering phenomena when implementing the control numerically. Therefore, in practical control cases, it is important to reduce this phenomena by providing continuous or smooth controller.

The second-order sliding mode (see e.g, [46, Chapter 4]) was created to address control chattering. Second-order sliding mode controllers have the capability to drive both the sliding variable and its derivative to zero, thereby significantly reducing chattering phenomena. We refer to these papers [35]–[37], [41], [42], in which the second-order sliding mode controllers is addressed for distributed parameters systems. Based on second-order sliding mode techniques, the super twisting algorithm has been developed for systems whose sliding variable admits a relative degree (see [46, Definition 1.6]) equal to 1. The essential feature of the super twisting control is to require only the measurement of the sliding

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variable to guarantee the convergence in finite time to zero of the sliding variable and its derivative. Moreover, the super twisting feedback-law is continuous with respect to the state, and this drastically attenuates the chattering phenomenon.

In this paper, a strategy different from the ones that have been mentioned earlier is proposed in order to design "classical" sliding mode controls and super-twisting sliding mode controls for general linear infinite-dimensional systems. The control design methodology is essentially based on a very particular selection of the sliding surface, which is assumed to be defined using an eigenfunction of the the adjoint operator of the closed-loop system without disturbance. Such a sliding variable allows to directly use well-known results on the stabilization of abstract infinite-dimensional systems with unbounded control operators in the absence of disturbance [13], [47], [51] together with well-known results about the finite-time convergence of the sliding variable in the context of the finite dimension [33], [43], [52]. For the proof of a well-posedness in this case, the classical Filippov regularization method [15] for discontinuous ODEs is adapted and combined with the concept of the mild solutions of evolution equations. In comparison with [26], [27], [38], [39], the approach proposed in this document allows to define explicitly and systematically the sliding variable for a large class of linear infinite-dimensional systems. Furthermore, altering the sliding surface could potentially address an output regulation issue. Nevertheless, it is beyond the scope of the current paper.

This paper is organized as follows. Section II presents a class of linear infinite-dimensional system with an unbounded control operator, the sliding mode based control law, the super-twisting based control law and the main results of the paper. Section III contains the proofs of the main results. Section IV introduces an illustrative example. Finally, Section V collects some remarks and introduces some future research lines to be followed.

Notation: Let $c \in \mathbb{C}$, $\mathfrak{Re}(c)$ (resp. $\mathfrak{Im}(c)$) denotes the real part (resp. the imaginary part) of c. The set of non-negative real numbers is denoted in this paper by $\mathbb{R}_+$. When a function f only depends on the time variable t (resp. on the space variable x), its derivative is denoted by $\dot{f}$ (resp. f'). Given two vector spaces E and F, $\mathcal{L}(E,F)$ denotes the space of linear continuous applications from E into F. If E is a normed vector space, we denote by $\|\cdot\|_E$ the norm on E. We denote by E' the dual space of E, that is, the space of all continuous linear functionals on E and we denote by $\langle\cdot,\cdot\rangle_{E,E'}$ the dual product on $E \times E'$. We denote by $C(E;F)$ the space of continuous functions from the space E to the space F. Throughout the paper, the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

II. Main results

A. Problem Statement

$(H, \langle\cdot,\cdot\rangle_H)$ denotes a Hilbert space over the field $\mathbb{K}$ and the corresponding norm is denoted by $\|\cdot\|_H$. In this paper we are interested in the stabilization (at the origin) problem for the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}z = Az + B(u+d), \\ z(0) = z_0, \end{cases} \tag{1}$$

where $z(t) \in H$ is the state, $u(t) \in \mathbb{K}$ is the control input and $d(t) \in \mathbb{K}$ is an unknown disturbance. In system (1), $A : D(A) \subseteq H \to H$ is a linear, densely defined operator in H and $B \in \mathcal{L}(\mathbb{K}, D(A^*)')$, with A^* the adjoint operator of A. Our objective is to provide a design method so that system (1) is globally stabilized despite the disturbance d. To do so, we will follow the sliding mode strategy.

This strategy can be applied thanks to the following set of assumptions.

Assumption 1: The following statements hold.

(i) The operator $A : D(A) \subseteq H \to H$ generates a strongly continuous semigroup, that is denoted by $(\mathbb{T}(t))_{t\geq 0}$.

(ii) The operator B is admissible[1] for $(\mathbb{T}(t))_{t\geq 0}$.

(iii) There exists an operator $L : D(L) \to \mathbb{K}$ such that the operator

$$\begin{cases} A_L = A + BL, \\ D(A_L) = \{z \in H; (A+BL)z \in H\}, \end{cases} \tag{2}$$

is the infinitesimal generator of a strongly continuous semigroup $(\mathbb{S}(t))_{t\geq 0}$ on H and the origin of the following system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}z = (A+BL)z, \\ z(0) = z_0, \end{cases} \tag{3}$$

is globally asymptotically stable. Moreover, the point spectrum of A_L is non-empty.

Items (i) and (ii) allow to state the well-posedness of system (1) in H with $u \in L^2_{loc}([0,+\infty),\mathbb{K})$. Finally, Item (iii) of Assumption 1 refers to a stabilizability property of system (1), needed to ensure that, without disturbance, the system can be stabilized.

The disturbance d is not supposed to be known entirely, but we assume the knowledge of its bound.

Assumption 2: The unknown disturbance d is supposed to be uniformly bounded measurable, i.e $|d(t)| \leq K_d$ for some $K_d > 0$ and for all $t \geq 0$.

Remark 1: Note that Item (iii) of Assumption 1 has been proven in [51, Theorem 2.1] in the case where the pair (A,B) is exactly controllable in time T.

Our goal is to find a state feedback control u which allows to reject the disturbance and to globally asymptotically stabilize the system (1) around 0. Precisely, we are looking for a sliding manifold on which the system (1) becomes the system (3) in a finite time. According to the item (iii) of Assumption 1, we know that 0 is globally asymptotically stable for the system (3). The next section will provide a definition of this sliding manifold (and its related sliding variable), the associated sliding mode controllers and the associated super-twisting controllers.

B. Sliding manifold

Let $\varphi \in D(A_L^*) := \{\varphi \in H \mid \exists c > 0, \forall \phi \in D(A_L), |\langle\varphi, A_L(\phi)\rangle_H| \leq c\|\phi\|_H\}$[2] be an eigenfunction of the adjoint operator of A_L such that $B^*\varphi \neq 0$ and λ the eigenvalue

[1] See e.g [50, Section 4.2]

[2] See e.g [50, Page 53]

associated with φ. We introduce the following sliding manifold Σ

$$\Sigma := \{z \in H \mid \langle \varphi, z\rangle_H = 0\}.$$

Its related sliding variable $\sigma : \mathbb{R}_+ \to \mathbb{K}$ is defined by

$$\sigma(t) := \langle \varphi, z(t)\rangle_H \tag{4}$$

for any solution z of (1). This sliding variable represents the scalar product between the state and an eigenfunction of A_L^*.

Remark 2: According to Item (iii) of Assumption 1, one can always select an eigenfunction φ of A_L^* such that $B^*\varphi \neq 0$. Let us explain how one select such an eigenfunction in the context of finite dimension. In this case, the (iii) of Assumption 1 stipulates that the pair (A, B) is stabilizable i.e., there exists a matrix L such that the eigenvalues of $A_L := A+BL$ belong to $\mathbb{C}_-$. For this , we distinguish the two possible cases.

a) First case: the matrix $L \neq 0$.: In this case, there exists an eigenvalue λ of $A_L^\top$ such that $\ker\left(A_L^\top - \lambda I\right) \not\subset B^\perp$, where $B^\perp$ is the set of orthogonal vectors to B and I is the identity matrix. Indeed, let us assume that,

$$\text{for all eigenvalue } \lambda \text{ of } A_L^\top \text{ we have } \ker\left(A_L^\top - \lambda I_d\right) \subset B^\perp. \tag{5}$$

Then, for all eigenvalue λ of $A_L^\top$ and for all $\varphi \in \ker\left(A_L^\top - \lambda I_d\right)$, we have

$$\begin{cases} A_L^\top \varphi = \lambda\varphi \\ B^\top \varphi = 0. \end{cases} \tag{6}$$

Since $A_L^\top = A^\top + L^\top B^\top$, then, from (6), we obtain for all eigenvalue λ of $A_L^\top$ and for all $\varphi \in \ker\left(A_L^\top - \lambda I_d\right)$

$$A_L^\top \varphi = \lambda\varphi \Leftrightarrow A^\top \varphi = \lambda\varphi. \tag{7}$$

Thus, $A_L^\top$ and $A^\top$ have the same spectrum, which is impossible. As a result, the assumption in (5) is not correct.

b) Second case: the matrix $L = 0$.: In this case, the matrix A is Hurwitz. Then, either there is an eigenvector φ of $A_L^\top$ such that $B^\top\varphi \neq 0$, or there is no such eigenvector. If such a vector does not exists and if the pair (A, B) is stabilizable, then we find ourselves in the previous case. As a consequence, in both cases, one can always select an eigenvector φ such that $B^\top\varphi \neq 0$. Since we are talking about eigenvalues and eigenfuntions, the argument is the same in the context of infinite dimension.

In this paper, we are interested in the design of a sliding mode controller and a super twisting controller. In the following section, we begin with the design of the sliding mode control.

1) Sliding mode Control: Since $B^*\varphi \neq 0$ then, we consider the sliding mode controller u defined by, for a.e. $t \geq 0$,

$$u(t) = Lz(t) - \frac{1}{B^*\varphi}\left(\lambda\sigma(t) + K\text{sign}(\sigma(t))\right), \tag{8}$$

where σ is given in (4), K is a positive constant that will be chosen later. Moreover, the set-valued function sign is defined by

$$\text{sign}(s) = \begin{cases} \frac{s}{|s|} & \text{if } s \neq 0, \\ \{\kappa \in \mathbb{K} \mid |\kappa|_{\mathbb{K}} \leq 1\} & \text{if } s = 0. \end{cases}$$

Note that, since $B^* \in \mathcal{L}(D(A^*), \mathbb{K})$, then $B^*\varphi$ is a scalar. Then, the closed$-$loop system (1)$-$(8) can be written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z \in A_L z + B\left(d - \dfrac{1}{B^*\varphi}\left(\lambda\sigma(t) + K\text{sign}(\sigma(t))\right)\right), \\ z(0) = z_0. \end{cases} \tag{9}$$

The solutions of (9) are understood in the sense of [27, Definition 2.1.]. However, to simplify the reading of this paper, we provide the definition in the context of our system.

Definition 1: Let $z_0 \in H$. We say that the map $z : [0,\infty) \to H$ is a mild solution of (9), if $z \in C([0,\infty); H) \cap \mathcal{H}^1_{loc}([0,\infty); D(A^*)')$ such that, for all $t \in [0,\infty)$,

$$z(t) = \mathbb{S}(t)z_0 + \int_0^t \mathbb{S}(t-s)Bh(s)ds, \tag{10}$$

where $(\mathbb{S}(t))_{t\geq 0}$ is the strongly continuous semigroup generated by the operator A_L and $h : [0,\infty) \to \mathbb{K}$ is in $L^2_{loc}([0,\infty);\mathbb{K})$ and satisfies, for a.e $t \geq 0$,

$$h(t) \in -\frac{1}{B^*\varphi}\left(\lambda\sigma(t) + K\text{sign}(\sigma(t))\right) + d(t) \tag{11}$$

with σ given in (4).

Now, let z be the mild solution of (29). Since $\varphi \in D(A_L^*)$, and using Item (ii) of Assumption 1, then according to [50, Remark 4.2.6], we obtain for that, every $t \geq 0$,

$$\begin{aligned} \langle \varphi, z(t) - z_0\rangle_H &= \int_0^t \Big(\langle A_L^*\varphi, z(s)\rangle_H + B^*\varphi h(s)ds \\ &= \int_0^t \Big(\lambda\langle \varphi, z(s)\rangle_H + B^*\varphi h(s)\Big)ds, \end{aligned} \tag{12}$$

because $A_L^*\varphi = \lambda\varphi$. Then, using (4), one has, for all $t \geq 0$,

$$\sigma(t) - \sigma(0) = \int_0^t \big(\lambda\sigma(s) + B^*\varphi h(s)\big)ds. \tag{13}$$

As a consequence, σ is a Carathéodory solution to

$$\begin{cases} \dot\sigma(t) = \lambda\sigma(t) + B^*\varphi h(t), \\ \sigma(0) = \langle \varphi, w_0\rangle_H. \end{cases} \tag{14}$$

Since $h \in -\frac{1}{B^*\varphi}\Big(\lambda\sigma + K\,\text{sign}(\sigma)\Big) + d$, then σ is a Filippov solution [15, Chapter 2] of

$$\begin{cases} \dot\sigma(t) \in B^*\varphi\Big(d - \dfrac{K}{B^*\varphi}\,\text{sign}(\sigma(t))\Big), \quad t \in \mathbb{R}_+, \\ \sigma(0) = \langle \varphi, w_0\rangle_H. \end{cases} \tag{15}$$

Then, the following holds, for all $t \geq 0$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\sigma(t)|^2 &= \mathfrak{Re}\big(\bar\sigma(t)\dot\sigma(t)\big) \\ &= \mathfrak{Re}\Big(\bar\sigma(t)B^*\varphi\big(d(t) - \tfrac{K}{B^*\varphi}\text{sign}(\sigma(t))\big)\Big) \\ &\leq -(K - |B^*\varphi|K_d)|\sigma(t)| \end{aligned} \tag{16}$$

as long as $\sigma(t) \neq 0$, where $\bar\sigma$ denotes the complex-conjugate of σ. Therefore, separating variables and integrating inequality (16) over the time interval $0 \leq s \leq t$, we obtain

$$|\sigma(t)| \leq |\sigma(0)| - (K - |B^*\varphi|K_d)t. \tag{17}$$

Thus, we make the following assumption about the constant K.

Assumption 3: The constant K is chosen such that $\frac{K}{|B^*\varphi|} > K_d$, where K_d is defined in Assumption 2.

Then, there exists a finite time $t_r > 0$, for which we know a bound that will be given later on, such that $\sigma(t) = 0$ for any $t > t_r$. This means that the system (9) reaches the sliding manifold Σ in finite time t_r and remains on it. Since $\sigma(t) = 0$ for any $t > t_r$, then $\dot{\sigma}(t) = 0$ for any $t > t_r$. Thus, from (15), we have $0 \in d(t) - \frac{K}{B^*\varphi}\text{sign}(\sigma(t))$ for any $t > t_r$. Since for any $t > t_r$ $\text{sign}(\sigma(t)) = \text{sign}(0) = \{\kappa \in \mathbb{K} \mid |\kappa|_{\mathbb{K}} \leq 1\}$, which is closed, bounded, and upper semi-continuous, then according to [14, Page 78] there exists a measurable function $\hat{d}(t) \in \frac{K}{B^*\varphi}\text{sign}(\sigma(t))$ such that

$$\forall t \geq t_r,\, d(t) = \hat{d}(t). \tag{18}$$

Therefore, for all $t \geq t_r$, $\hat{d}(t)$ is an estimate of $d(t)$. As a consequence, the system (9) can be rewritten as (3) on the sliding surface, which is globally asymptotically stable around $(0, 0)$ from the item (iii) of Assumption 1.

The next section focuses on the design of the super twisting control.

2) Super twisting control: In this section, we make the following assumption about the disturbance.

Assumption 4: The disturbance $d(\cdot)$ is globally Lipschitz over $\mathbb{R}_+$ and there exists a known positive constant C such that, for a.e $t \in \mathbb{R}_+$,

$$|\dot{d}(t)| \leq C. \tag{19}$$

We assume that $\mathbb{K} = \mathbb{R}$. We do not treat the complex case, since we are not aware whether there exist super-twisting controllers for system whose state is in $\mathbb{C}$. We consider the super twisting controller u defined by, for all $t \geq 0$,

$$\begin{cases} u(t) = Lz(t) \\ + \dfrac{1}{B^*\varphi}\Big(- \lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}}\text{sign}(\sigma(t)) + v(t)\Big), \\ \dot{v}(t) \in -\beta\text{sign}(\sigma(t)), \end{cases} \tag{20}$$

where σ is given in (4), α and β are positive constants which will be chosen later. Then, according to the following transformation

$$w(t) = B^*\varphi d(t) + v(t), \tag{21}$$

the closed$-$loop system (1)$-$(20) can be written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}z = A_L z + B\Bigg(\dfrac{1}{B^*\varphi}\Big(- \lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}}\text{sign}(\sigma(t)) \\ + w(t)\Big)\Bigg), \\ \dot{w}(t) \in B^*\varphi\dot{d}(t) - \beta\text{sign}(\sigma(t)), \\ z(0) = z_0 \in H, w(0) = w_0 \in \mathbb{R}. \end{cases} \tag{22}$$

The solutions of (22) are understood in the sense of [27, Definition 2.1.].

3) Main results: Before presenting the results of this paper, we present the following definition of the equilibrium point of systems (9) and (22).

Definition 2:

1) We say that $\tilde{z} \in H$ is an equilibrium point of system (9) , if $\tilde{z} \in D(A_L)$ and there exists $\tilde{z}^* \in [-K_d, K_d] - \frac{K}{B^*\varphi}\left(\langle\varphi, \tilde{z}\rangle_H + \text{sign}(\langle\varphi, \tilde{z}\rangle_H)\right)$ such that
$$A_L\tilde{z} + B\tilde{z}^* = 0. \tag{23}$$
2) We say that $(\tilde{z}, \tilde{w}) \in H \times \mathbb{R}$ is an equilibrium point of system (22), if $(\tilde{z}, \tilde{w}) \in D(A_L) \times \mathbb{R}$ such that
$$A_L\tilde{z} + B\Bigg(\frac{1}{B^*\varphi}\Big(- \lambda\langle\varphi, \tilde{z}\rangle_H + \tilde{w} \\ -\alpha|\langle\varphi, \tilde{z}\rangle_H|^{\frac{1}{2}}\text{sign}\big(\langle\varphi, \tilde{z}\rangle_H\big)\Big)\Bigg) = 0 \tag{24}$$
and
$$0 \in B^*\varphi[-C, C] - \beta\text{sign}\big(\langle\varphi, \tilde{z}\rangle_H\big). \tag{25}$$

Remark 3: One can check that $0 \in H$ (resp. $(0, 0) \in H \times \mathbb{R}$) is the unique equilibrium point of (9) (resp. (22)).

The main results of this paper can be formulated as follows:

Theorem 1 (Existence of solutions):

1) Assume that Assumption 1, Assumption 2 and Assumption 3 are satisfied. For any initial condition $z_0 \in H$, the system (9) admits a mild solution.
2) Assume that Assumption 1, Assumption 4 and
$$\beta > |B^*\varphi|C \quad \text{and } \alpha > \sqrt{\beta + |B^*\varphi|C}, \tag{26}$$
are satisfied. For any initial condition $z_0 \in H$ and $w_0 \in \mathbb{R}$, the system (22) admits a mild solution.

The next result of this paper is stated as follows:

Theorem 2 (Global asymptotic stability):

1) Assume that Assumption 1, Assumption 2 and Assumption 3 are satisfied. For any initial condition $z_0 \in H$, $0 \in H$ is globally asymptotically stable for (9).
2) Assume that Assumption 1, Assumption 4 and
$$\beta > |B^*\varphi|C \quad \text{and } \alpha > \sqrt{\beta + |B^*\varphi|C}, \tag{27}$$
are satisfied. For any initial condition $(z_0, w_0) \in H \times \mathbb{R}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H \times \mathbb{R}$ is globally asymptotically stable for (22).

III. Proof of Theorem 1 and Theorem 2

A. Proof of Theorem 1

The proof of Theorem 1 is divided into two parts. In the first part, the proof of the Theorem 1 is presented in the case of system (9). The second part deals with the proof of Theorem 1 in the case of system (22).

Let us start the proof of the first part.

1) Sliding mode control: We consider the following ODE

$$\begin{cases} \dot{\gamma}(t) \in B^*\varphi\Big(d - \dfrac{K}{B^*\varphi}\text{sign}(\gamma(t))\Big), \quad t \in \mathbb{R}_+, \\ \gamma(0) = \gamma_0 \in \mathbb{R}, \end{cases} \tag{28}$$

The system (28) is understood in the sense of Filippov [15]. In the next lemma, we state that there exists a unique solution to (28) and that (28) is stabilized in finite-time.

Lemma 1: Assume that Assumption 2 holds. Then, the ODE (28) admits a unique Filippov solution. Moreover, there exists $t_r > 0$ such that, for any Filippov solution γ of (28),

$$\gamma(t) = 0, \, \forall \, t \geq t_r,$$

with

$$t_r \leq \frac{|\gamma(0)|}{K - K_d|B^*\varphi|}.$$

Lemma 1 is an immediate consequence of the general Filippov theory [15, Chapter 2] (for the real case), [54, Theorem 2.8] (for the complex case), when applied to the particular case of (28). Finite-time stability can be deduced easily by Lyapunov arguments (given in Section II).

Let γ be the Filippov solution of (28) with initial condition $\gamma(0) = \langle \varphi, z_0 \rangle_H$. We consider the following system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\phi = A_L\phi + \dfrac{1}{B^*\varphi}B(\dot{\gamma} - \lambda\gamma), \\ \phi(0) = \phi_0 \in H. \end{cases} \tag{29}$$

If B is an admissible operator for $\mathbb{S}$ and $\dot{\gamma} - \lambda\gamma \in L^2_{loc}([0,\infty);\mathbb{K})$, then system (29) admits a unique mild solution, where $(\mathbb{S}(t))_{t\geq 0}$ is the strongly continuous semigroup associated with the operator A_L. This is what we will prove in the next Lemma, which says that there exists a unique solution in the sense of [50, Definition 4.1.5].

Lemma 2: For all $\phi_0 \in H$, the system (29) admits a unique mild solution $\phi \in C([0,\infty);H) \cap \mathcal{H}^1_{loc}([0,\infty);D(A^*)')$.

Proof: Let γ be a Filippov solution of (28). Then, according to Lemma 1, γ is absolutely continuous. Moreover, $\dot{\gamma}$ is bounded and measurable according to Assumption 2. Thus, we have $\dot{\gamma} - \lambda\gamma \in L^2_{loc}([0,\infty);\mathbb{K})$. On the other hand, according to the item (ii) of Assumption 1, B is admissible for $(\mathbb{T}(t))_{t\geq 0}$, then according to [22, Proposition 4.2], B is an admissible control operator for $(\mathbb{S}(t))_{t\geq 0}$. Then, according to [50, Proposition 4.2.5], the statement of Lemma 2 holds, achieving the proof. □

Now, the aim is to prove that the mild solution ϕ to (29) with initial condition z_0 is a mild solution to (9). To that end, we will show that the following function

$$y(t) = \langle \varphi, \phi(t) \rangle_H, \tag{30}$$

with ϕ the solution of (29), is equal to γ, for any $t > 0$.

Lemma 3: For all $z_0 \in H$, y is a Carathéodory solution to

$$\begin{cases} \dot{y}(t) = \lambda y + \dot{\gamma}(t) - \lambda\gamma(t), \quad \text{for a.e } t \geq 0, \\ y(0) = \langle \varphi, z_0 \rangle_H \end{cases} \tag{31}$$

i.e y is an absolutely continuous map such that, for all $t \geq 0$

$$y(t) - y(0) = \int_0^t \big(\lambda y(s) + \dot{\gamma}(s) - \lambda\gamma(s)\big)ds. \tag{32}$$

Proof: Let ϕ be the mild solution of (29). Since $\varphi \in D(A_L^*)$, and using Item (ii) of Assumption 1, then according to [50, Remark 4.2.6], we obtain for that, every $t \geq 0$,

$$\begin{aligned} \langle \varphi, \phi(t) - z_0 \rangle_H &= \int_0^t \Big(\langle A_L^*\varphi, \phi(s) \rangle_H + \frac{1}{B^*\varphi}B^*\varphi(\dot{\gamma}(s) \\ &\quad - \lambda\gamma(s)) \Big) ds \\ &= \int_0^t \Big(\lambda \langle \varphi, \phi(s) \rangle_H + \dot{\gamma}(s) - \lambda\gamma(s) \Big) ds, \end{aligned} \tag{33}$$

because $A_L^*\varphi = \lambda\varphi$. Then, using (30), one has, for all $t \geq 0$,

$$y(t) - y(0) = \int_0^t \big(\lambda y(s) + \dot{\gamma}(s) - \lambda\gamma(s)\big)ds. \tag{34}$$

This concludes the proof. □

We introduce the function g defined by $g(t) = y(t) - \gamma(t)$. From (28) and (31) with $\gamma(0) = \langle \varphi, z_0 \rangle_H$, g is solution of

$$\begin{cases} \dot{g}(t) = \lambda g(t) \\ g(0) = 0 \end{cases} \tag{35}$$

Thus, for any $t \in \mathbb{R}$, $g(t) = 0$. By definition of g, we deduce that, for any $t \in \mathbb{R}$, $y(t) = \gamma(t)$. Therefore, according to (28) we have, for a.e $t \geq 0$,

$$\frac{1}{B^*\varphi}\dot{\gamma}(t) \in -\frac{K}{B^*\varphi}\text{sign}(y(t)) + d(t). \tag{36}$$

Thus, according to Lemma 2 and (36), ϕ satisfies Definition 1. Then, we conclude that, for any Filippov solution γ of (28) with initial condition $\gamma(0) = \langle \varphi, z_0 \rangle_H$, the associated mild solution ϕ of (29) is a mild solution of (9). This concludes the proof of Theorem 1 in the case of system (9). The proof of Theorem 1 for the system (22) follows the same reasoning as before. Due to page limitations, we have omitted this proof. □

B. Proof of Theorem 2

Like the proof of Theorem 1, the proof of Theorem 2 is divided into two parts. In the first part, the proof of the Theorem 2 is presented in the case of the system (9). The second part deals with the proof of Theorem 2 in the case of system (22).

Let us start the proof of the first part.

1) Sliding-mode control: Let us consider z a mild solution of (9) with initial condition $z_0 \in H$. Since σ is a Filippov solution of (28) with initial condition $\langle \varphi, z_0 \rangle_H$, then from Lemma 1, there exists a finite time t_r such that

$$\sigma(t) = 0 \text{ for any } t > t_r.$$

Therefore, $\dot{\sigma}(t) = 0$ for any $t > t_r$. As a consequence, from (14), for any $t > t_r$, $h(t) = 0$. Thus, for any $t > t_r$, the system (9) is equivalent to the system (3) and hence is asymptotically stable in H from the item (iii) of Assumption 1. Therefore, to conclude the proof of Theorem 2 in the case of system (9), it is just necessary to prove that the solution z of system (9) depends continuously on initial conditions on the time interval $[0, t_r]$. For this purpose, we consider z a mild solution of (9)

with initial condition $z_0 \in H$ on the interval $[0, t_r]$. Then, using the Definition 1, there exists $C_0 > 0$ such that, for all $t \in [0, t_r]$, we have

$$\|z(t)\|_H \leq C_0\|z_0\|_H + \left\|\int_0^t \mathbb{S}(t-s)Bh(s)ds\right\|_H. \quad (37)$$

Since $(\mathbb{S}(t))_{t\geq 0}$ is exponentially stable and B is an admissible operator for $(\mathbb{S}(t))_{t\geq 0}$, then according to [50, Proposition 4.3.3], there exists $C_1 > 0$ independent of t_r such that, for all $t \in [0, t_r]$

$$\|z(t)\|_H \leq C_1\bigg(\|z_0\|_H + \|h\|_{L^2(0,t_r)}\bigg). \quad (38)$$

Moreover, since $h \in -\frac{1}{B^*\varphi}\big(\lambda\sigma + K\text{sign}(\sigma)\big) + d$, then according to Assumption 2, h is bounded. Therefore, there exists $C_2 > 0$ such that

$$\|h\|_{L^2(0,t_r)} \leq C_2 t_r^{\frac{1}{2}}. \quad (39)$$

Moreover, according to Lemma1, $t_r \leq \frac{|\langle\varphi,z_0\rangle_H|}{K-|B^*\varphi|\|d\|_{L^\infty(\mathbb{R}_+)}}$. Thus, using Cauchy-Schwarz's inequality, we have

$$t_r \leq \frac{\|\varphi\|_H}{K - |B^*\varphi|\|d\|_{L^\infty(\mathbb{R}_+)}}\|z_0\|_H. \quad (40)$$

As a consequence, according to (38), (39) and (40), there exists $C_3 > 0$ (independent of t_r) such that for all $t \in [0, t_r]$,

$$\|z(t)\|_H \leq C_3\bigg(\|z_0\|_H + \sqrt{\|z_0\|_H}\bigg). \quad (41)$$

According to [32, Definition 2.3], this concludes the proof of Lyapunov stability of the system (9) over the time interval $[0, t_r]$. □

Remark 4: In contrast with many stabilization techniques, we do not need here to compute time-derivative of Lyapunov functionals for the infinite-dimensional system. More precisely, classical techniques rely on the existence of strong solutions for which one computes time derivative of a suitable Lyapunov functional, and one concludes then on the stability for weak solution by a density argument.
The proof of Theorem 2 for the system (22) follows the same reasoning as before. Due to page limitations, we have omitted this proof

IV. Illustrative example: Transport equation

Consider the following system,

$$\begin{cases} z_t(t,x) + z_x(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z(t,0) = u(t) + d(t), \quad t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \quad (42)$$

where $u(t) \in \mathbb{R}$ is the control input and $d(t) \in \mathbb{R}$ is an unknown disturbance.

This equation can be written in an abstract way as in (1) if one sets $H = L^2(0, L)$,

$$\begin{aligned} A : D(A) \subset L^2(0,L) &\to L^2(0,L), \\ z &\mapsto -z', \end{aligned} \quad (43)$$

where

$$D(A) := \{z \in \mathcal{H}^1(0,1) \mid z(0) = 0\}, \quad (44)$$

and the control operator B is the delta function in $\mathcal{L}(\mathbb{R}, D(A)')$ defined as follow

$$\langle\varphi, Bu\rangle_{D(A),D(A)'} = \varphi(0)u \quad (45)$$

for all $u \in \mathbb{R}$ and $\varphi \in D(A)$, where $\langle\cdot,\cdot\rangle_{D(A),D(A)'}$ is the dual product. The adjoint operator of A is

$$\begin{aligned} A^* : D(A^*) \subset H &\to H, \\ z &\mapsto z', \end{aligned} \quad (46)$$

with $D(A^*) := \{z \in \mathcal{H}^1(0,1) \mid z(1) = 0\}$. The adjoint of operator of B is defined as follow

$$\begin{aligned} B^* : D(A^*) &\to \mathbb{R} \\ \varphi &\mapsto \varphi(0). \end{aligned} \quad (47)$$

According to the proof of [3, Theorem A.1], it generates a C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ of contractions in $L^2(0, L)$ and we can prove that the operator B is admissible for the semigroup $(\mathbb{T}(t))_{t\geq 0}$. Thus, the operators A and B satisfy the items (i) and (ii) Assumption 1. Moreover, [3, Theorem 2.1], the origin of

$$\begin{cases} z_t(t,x) + z_x(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z(t,0) = az(t,1), \quad t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \quad (48)$$

is globally exponentially stable in H. As a consequence, Item (iii) of Assumption 1 holds, where the operator A_L is defined as follow

$$\begin{cases} A_L z = -z', \\ D(A_L) = \big\{z \in H^1(0,L) \mid z(0) = az(1)\big\} \end{cases} \quad (49)$$

and its adjoint is defined as follow

$$\begin{cases} A_L^* z = z', \\ D(A_L^*) = \big\{z \in H^1(0,L) \mid z(1) = az(0)\big\} \end{cases} \quad (50)$$

with $0 < a < 1$. Therefore, a direct computation gives that the eigenpairs $(\lambda, \varphi_\lambda)$ of A_L^* satisfy

$$\begin{cases} \varphi_\lambda(x) = k\exp(\lambda x),\ k \in \mathbb{R} \setminus \{0\} \\ \lambda = \ln a. \end{cases} \quad (51)$$

Then, the sliding variable and the feedback law under consideration are as follows

$$\begin{aligned} \sigma(t) &= \int_0^L z(t,x)\varphi_\lambda(x)dx \quad \text{and} \\ u(t) &= az(t,1) - \frac{1}{k}\big(\lambda\sigma(t) + K\text{sign}(\sigma(t))\big). \end{aligned} \quad (52)$$

Thus, if d satisfies the conditions in Assumption 2 and K is chosen as in Assumption 3, we can conclude that the origin of

$$\begin{cases} z_t(t,x) + z_x(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z(t,0) \in az(t,1) - \frac{1}{k}\big(\lambda\sigma(t) + K\text{sign}(\sigma(t))\big) + d(t), \\ \quad t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \quad (53)$$

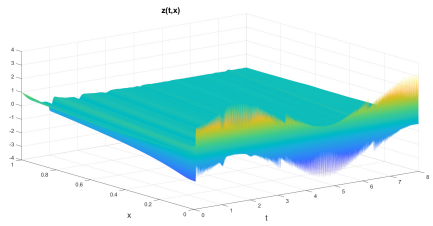

Fig. 1: Solution z versus time t (sec) and position x with the sliding mode control.

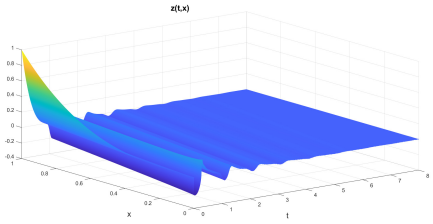

Fig. 2: Solution z versus time t (sec) and position x with the super-twisting control.

is globally asymptotically stable in H. On the other hand the super-twisting control under consideration is as follows

$$\begin{cases} u(t) = az(t,1) + \dfrac{1}{k}\Big(-\lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}}\mathrm{sign}(\sigma(t)) + v(t)\Big), \\ \dot{v}(t) \in -\beta\mathrm{sign}(\sigma(t)). \end{cases} \tag{54}$$

Therefore, if d satisfies the conditions in Assumption 4, β and α are chosen as in (27) then, we can conclude that the origin of

$$\begin{cases} z_t(t,x) + z_x(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z_x(t,1) = \dfrac{1}{k}\Big(-\lambda\sigma(t) - \alpha|\sigma(t)|^{\frac{1}{2}}\mathrm{sign}(\sigma(t)) + v(t)\Big) \\ \quad + az(t,1) + d(t), \quad t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \tag{55}$$

is globally asymptotically stable in H.

Using the finite difference method [29], we performed some numerical simulations. We choose $k = 1$, $a = 0.5$, $K = 2.5$, $z_0(x) = x^3$ and $d(t) = 2\sin(t)$.

In Figure 1, the stabilization of z from (53) is illustrated, and the chattering phenomenon is observed.

Figures 2 is obtained with the same settings as the Figures 1 with $\beta = 2.5$ and $\alpha = 2.2$. It must be noted that, thanks to the use of super twisting algorithm, the chattering has been removed (super twisting is continuous) whereas the stabilization is kept.

V. Conclusion

In this paper, we have proposed a design method based on sliding mode control for the stabilization of class of linear infinite-dimensional systems with unbounded control operators and subject to a boundary disturbance. The existence of solutions of the closed-loop system has been proved as well as the disturbance rejection and the asymptotic stability of the closed-loop control system. We further have extended the super-twisting method for the same class of linear infinite-dimensional systems.

Future works will consider the case where the operator A in (1) is nonlinear, for which many notions will need to be adapted such as the controllability or the admissibility. It might also be interesting to investigate the case where the disturbance does not match with the control as it has been done for ODEs in [6].

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