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A NOTE ON SOME EXAMPLES OF NSOP1 THEORIES

Yvon BOSSUT

Abstract : We present here some known and some new examples of non-simple NSOP1 theories and some behaviour that Kim-forking can exhibit in these theories, in particular that Kim-forking after forcing base monotonicity can or can not satisfy extension (on arbitrary sets). This study is based on the results of Chernikov, Ramsey, Dobrowolski and Granger.¹

1 Context :

NSOP1 theories have recently been studied as a generalisation of simple theories. Kim and Pillay [9] have shown that simplicity can be characterised in terms of the existence of an independence notion satisfying some properties, which then implies that this independence relation is forking independence. Chernikov and Ramsey [3, Theorem 5.8] have shown a similar result for NSOP1 theories and Kim-forking independence, another relation of independence which is defined as 'generic forking independence'. One question we can ask is the relation between these two notions of independence in the context of NSOP1 theories.

In this note we will present some known examples of NSOP1 theories as well as generalisations of these examples and we will compute, whenever possible, forking independence and Kim-independence in these theories. We will show that the relation between these two notions can vary among non-simple NSOP1 theories, and we will also show that Kim-forking may or may not satisfy strong local character. We shall write \downarrow^d for dividing independence, \downarrow^f for forking independence and \downarrow^K for Kim-independence. By algebraic independence we mean $A \downarrow_C^a B := \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$. The main notions we shall be looking at are the two following definitions which were introduced by H. Adler in [1] in his axiomatic approach of independence relations. They were also studied in the NSOP1 context and also more generally by C. D'Elbée in [5] and [4].

Definition 1.1. *Forcing base monotonicity : Let \downarrow be an independence relation defined on algebraically closed sets A, B, C such that $C \subseteq A, B$. We define \downarrow^m as the weakest independence relation that implies \downarrow and satisfies base monotonicity, meaning : $A \downarrow_C^m B$ iff $\text{acl}(AB') \downarrow_{\text{acl}(CB')}^m B$ for all $B' \subseteq B$.*

Definition 1.2. *Forcing extension : Let \downarrow be an independence relation defined on algebraically closed sets A, B, C such that $C \subseteq A, B$. We define \downarrow^* as the weakest independence relation that implies \downarrow and satisfies extension, meaning : $A \downarrow_C^* B$ iff for all $B' \supseteq B$ there exist $A' \equiv_B A$ such that $A \downarrow_C \text{acl}(B')$.*

2 Parametrized Fraïssé limits

In this section we will continue the work of Chernikov and Ramsey in [3, Section 6.3].

Definition 2.1. *Suppose \mathbf{K} is a class of finite structures. We say \mathbf{K} has the Strong Amalgamation Property (SAP) if given $A, B, C \in \mathbf{K}$ and embeddings $e : A \rightarrow B$ and $f : A \rightarrow C$ there is a $D \in \mathbf{K}$ and embeddings $g : B \rightarrow D$ and $h : C \rightarrow D$ so that $g \circ e = h \circ f$ and, moreover, $\text{im}(g) \cap \text{im}(h) = \text{im}(g \circ e)$.*

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Let T be the theory obtained as the Fraïssé limit of a Fraïssé class \mathbf{K} with SAP in a finite relational language \mathcal{L} . We now consider a new language \mathcal{L}_{pfc} with two sorts P (parameters) and O (objects), such that for any n -ary relation symbol $R(\bar{y}) \in \mathcal{L}$ we have an $(n+1)$ -ary relation symbol $R_x(\bar{y})$ where x is a variable of sort P and the n variables \bar{y} belong to the sort O .

If M is an \mathcal{L}_{pfc} structure we shall write $M = (A, D)$ with $O(M) = A$ and $P(M) = D$. Given an \mathcal{L}_{pfc} -structure $M = (A, D)$ and $d \in D$ we shall write A_d the \mathcal{L} -structure where $R \in \mathcal{L}$ is interpreted as R_d in A . We consider the class of finite \mathcal{L}_{pfc} -structures defined as : $\mathbb{K}_{pfc} = \{M = (A, D) : A_d \in \mathbf{K} \text{ for all } d \in D\}$.

Fact 2.2. *Ramsey and Chernikov have proven in [3] that \mathbf{K}_{pfc} is a Fraïssé class that satisfies SAP, that the theory of its limit T_{pfc} is non-simple and NSOP1, has Q.E. in \mathcal{L}_{pfc} and that if $(A, D) \models T_{pfc}$ then $A_d \models T$ for all $d \in D$. Moreover in this theory for any set A $A = \text{acl}(A)$.*

We place ourselves in a monster model $\mathbb{M} = (\mathbb{A}, \mathbb{D})$ of the theory T_{pfc} . When A is a set of objects and d a parameter we will write $\langle A \rangle_d$ for the isomorphism type of A in \mathbb{A}_d . We will use the following two lemmas from [3] to show that in this theory $\downarrow^{K^m} = \downarrow^f = \downarrow^d$ and that if T is supersimple then Kim-forking satisfies strong local character in T_{pfc} . If d and d' are parameters and $q(x)$ is a partial type in \mathbb{A}_d we will write $q_{d'}(x)$ to denote the partial type in \mathbb{M} where we replace every occurrence of $R \in \mathcal{L}$ in q by $R_{d'}$.

Lemma 2.3. *[3, Lemma 6.13] Suppose $\{p_i : i < \alpha\} \subseteq B$ is a collection of distinct parameters and $(q^i : i < \alpha)$ is a sequence of non-algebraic \mathcal{L} -types over $C \subseteq A$ (possibly with repetition), where q^i is considered as a type in \mathbb{A}_{p_i} . Then the \mathcal{L}_{pfc} -type $\bigcup_{i < \alpha} q_{p_i}^i$ is consistent.*

Lemma 2.4. *[3, Lemma 6.14] Suppose $A, B, C \subseteq \mathbb{A}$ are small sets of objects, $F \subseteq \mathbb{B}$ is a small set of parameters, $A \cap B \subseteq C$ and $b_0, b_1 \in \mathbb{B}$ are such that $b_0 \equiv_{(C, F)} b_1$. Then there is some $b \in \mathbb{B}$ so that $b \equiv_{(AC, F)} b_0$ and $b \equiv_{(BC, F)} b_1$ (all in \mathcal{L}_{pfc}).*

Definition 2.5. *Let $(C, F) \subseteq (A, D), (B, E)$ be \mathcal{L} -structures. We define $(A, D) \downarrow_{(C, F)}^{pfc} (B, E)$ by $(A, D) \cap (B, E) = (C, F)$ and $A \downarrow_C^p B$ for all $p \in F$, where \downarrow^p denotes forking independence inside \mathbb{A}_p (so inside the simple theory T).*

Now if we force base monotonicity for this independence relation \downarrow^{pfc} , since the second condition already satisfies this property we get : for \mathcal{L} -structures $(C, F) \subseteq (A, D), (B, E)$ we have $(A, D) \downarrow_{(C, F)}^{pfc^m} (B, E)$ iff $(A, D) \cap (B, E) = (C, F)$ and $A \downarrow_C^p B$ for all $p \in E$, i.e., we now take parameters inside of the structure on the right rather than just in the base.

Lemma 2.6. *If $(C, F) \subseteq (A, D), (B, E)$ then $(A, D) \downarrow_{(C, F)}^d (B, E)$ if and only if $(A, D) \downarrow_{(C, F)}^{pfc^m} (B, E)$.*

Proof. $[\Rightarrow]$: If $(A, D) \downarrow_{(C, F)}^d (B, E)$ then since the theory T_{pfc} has trivial algebraic closure and since dividing independence implies algebraic independence we get that $(A, D) \cap (B, E) = (C, F)$. Now consider $p \in E$, we want to show that $A \downarrow_C^p B$. Let $(B_i)_{i < \omega}$ be an indiscernible sequence in \mathbb{A}_p with $B_0 = B$, we want to find A' such that $A'B_i \equiv_C AB$ for all $i < \omega$ in \mathbb{A}_p .

Let $q(x, B_0) := \text{tp}(B_{<\omega} \setminus B_0, B_0)$ and $\{p_i : i < \alpha\}$ an enumeration of E . Assume that $q(x, B_0)$ is non-algebraic (else the sequence would be constant and $A \downarrow_C^p B$), by Lemma 2.3 we can find $(B'_i)_{i < \omega} \models \bigcup_{i < \alpha} q_{p_i}(x, B)$, with $B'_0 = B$, so $(B'_i)_{i < \omega}$ is a C -indiscernible sequence in $\mathbb{A}_{p'}$ for all $p' \in E$.

By Q.E. the sequence $(B'_i, E)_{i < \omega}$ is (C, E) -indiscernible in \mathbb{M} . By hypothesis $(A, D) \downarrow_{(C, E)}^d (B, E)$, so there is $(A', D') \equiv_{C, E} (A, D)$ such that $(A'B'_i, D'E) \equiv_{C, E} (AB, DE)$ for all $i < \omega$, so in particular $A'B'_i \equiv_C AB$ in \mathbb{A}_p for all $i < \omega$, and since $(B'_i)_{i < \omega} \equiv_C (B_i)_{i < \omega}$ we can find A'' such that $A''B_i \equiv_C AB$ in \mathbb{A}_p for all $i < \omega$, and $A \downarrow_C^p B$.

[\Leftarrow] : Assume $(A, D) \downarrow_{(C,F)}^{pfc^m} (B, E)$. Let $(B_i, E_i)_{i < \omega}$ be a (C, F) -indiscernible sequence. Define $B^* = B_0 \cap B_1 (= \bigcap_{i < \omega} B_i)$ the set of parameters that repeats along the sequence, $E^* = E_0 \cap E_1$, $B'_i = B_i \setminus B^*$, $E'_i = E_i \setminus E^*$, $A' = A \setminus C$ and $D' = D \setminus F$. Let $(e^j)_{j < \kappa}$ enumerate E'_0 , $(e'_i)_{j < \kappa}$ the corresponding enumeration of E'_i and $(d^j)_{j < \lambda}$ enumerate D' . By Q.E. we want to find (\tilde{A}, \tilde{D}) such that if $(\tilde{d}^j)_{j < \lambda}$ enumerate $\tilde{D} \setminus F$:

- 1 $\tilde{A}B_i \equiv_C AB$ inside of \mathbb{A}_p for all $p \in E^*$ and for all $i < \omega$,
- 2 $\langle \tilde{A}B_i \rangle_{e_i^j} \equiv_C \langle AB \rangle_{e^j}$ for all $j < \kappa$ and for all $i < \omega$,
- 3 $\langle \tilde{A}B_i \rangle_{\tilde{d}^j} \equiv_C \langle AB \rangle_{d^j}$ for all $j < \lambda$ and for all $i < \omega$.

1 - If $p \in E^*$ by hypothesis we have that $A \downarrow_C^p B$ and that $(B_i)_{i < \omega}$ is an indiscernible sequence over C in \mathbb{A}_p , so by the chain condition in \mathbb{A}_p there is \tilde{A}_p such that $\tilde{A}_p B_i \equiv_C AB$ inside of \mathbb{A}_p for all $i < \omega$.

2 - Let $j < \kappa$, $q^j(x, y) := \text{tp}_{\mathbb{A}_{e^j}}(AB/C)$. We can assume that $q^j(x, B)$ is a non algebraic type, else we would have that $A = C = B_i$ for all $i < \omega$ and then we already have our objects fixed. By indiscernibility we now have that $q^j(x, B_i)$ is a non algebraic type for all $i < \omega$. Using Lemma 2.3 we can find \tilde{A} satisfying the conditions 1 and 2.

3 - Now that we have found our set of objects we use Lemma 2.4 to find the parameters \tilde{d}^j : Let $j < \lambda$, by induction on $n < \omega$ we show that there exist \tilde{d}_n^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_n^j} \equiv_{(C,F)} \langle AB \rangle_{d^j}$ for all $i \leq n$:

We have $\tilde{A}B_i \equiv_{(C,F)} AB$ for all $i < \omega$, so there is \tilde{d}_i^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_i^j} \equiv_{(C,F)} \langle AB \rangle_{d^j}$, and we can take $\tilde{d}_0^j = \tilde{d}_0^j$. Now for $n + 1$ assume we have \tilde{d}_n^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_n^j} \equiv_C \langle AB \rangle_{d^j}$ for all $i \leq n$. We have \tilde{d}_{n+1}^j such that $\langle \tilde{A}B_{n+1} \rangle_{\tilde{d}_{n+1}^j} \equiv_{(C,F)} \langle AB \rangle_{d^j}$. So $\tilde{d}_n^j \equiv_{(\tilde{A}B^*, F)} \tilde{d}_{n+1}^j$ and $\tilde{A}B_{n+1} \cap \tilde{A}B_{\leq n} = \tilde{A}B^*$. Using Lemma 2.4 we can find \tilde{d}_{n+1}^j such that $\tilde{d}_{n+1}^j \equiv_{(\tilde{A}B_0 B_{n+1}, F)} \tilde{d}_{n+1}^j$ and $\tilde{d}_{n+1}^j \equiv_{(\tilde{A}B_{\leq n} C, F)} \tilde{d}_n^j$, which is the parameter we were looking for.

By compactness there is a \tilde{d}^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}^j} \equiv_{(C,F)} \langle AB \rangle_{d^j}$ for all $i < \omega$, and this completes our proof of this direction. \square

Lemma 2.7. *If $(C, F) \subseteq (A, D), (B, E)$ then $(A, D) \downarrow_{(C,F)}^d (B, E)$ if and only if $(A, D) \downarrow_{(C,F)}^f (B, E)$, i.e. \downarrow^d satisfies extension and so the theory T_{pfc} satisfies existence.*

Proof. Let $(B, E) \subseteq (B', E')$, we want to find $(A', D') \equiv_{(B,E)} (A, D)$ such that $(A', D') \downarrow_{(C,F)}^d (B', E')$. By Q.E., if we fix $(d^j)_{j < \lambda}$ an enumeration of D and $(d'^j)_{j < \lambda}$ the corresponding enumeration of D' this means :

1. $A' \equiv_B A$ in \mathbb{A}_e for all $e \in E$,
2. $\langle A'B \rangle_{d'^j} \equiv_B \langle AB \rangle_{d^j}$ for all $j < \lambda$,
3. $D' \cap E' = F$,
4. $A' \downarrow_C^p B'$ for all $p \in E'$.

If $p \in E$ we have $A \downarrow_C^p B$ so by extension there is $A_p \equiv_B A$ in \mathbb{A}_p such that $A' \downarrow_C^p B'$. If $p \in E' \setminus E$ by existence there is A_p in \mathbb{A}_p such that $A' \downarrow_C^p B'$. In both those cases we can assume that $\text{tp}_{\mathbb{A}_p}(A_p/B')$ is non-algebraic, else we would have that $A = C = A'$ and the conditions 1,4 are already satisfied, so by Lemma 2.3 there is $A' \subseteq \mathbb{A}$ such that $A' \models \bigcup_{p \in E'} \text{tp}_{\mathbb{A}_p}(A_p/B')$, and this gives us our set of objects

satisfying 1 and 4. Now all is left is to find parameters d'^j outside of E' that satisfies 2, which we can always do using Lemma 2.4. \square

Proposition 2.8. *If $(C, F) \subseteq (A, D), (B, E)$ then $(A, D) \downarrow_{(C, F)}^{pfc} (B, E)$ if and only if $(A, D) \downarrow_{(C, F)}^K (B, E)$.*

Proof. This proof is similar to the one of Lemma 2.6 but slightly easier since Morley sequences can not overlap outside of the base, so there is no need for B^* in that case.

[\Rightarrow] : If $(A, D) \downarrow_{(C, F)}^K (B, E)$ then since the theory T_{pfc} has trivial algebraic closure and since Kim-independence implies algebraic independence we get that $D \cap E = F$. Now consider $p \in F$, we want to show that $A \downarrow_C^p B$. Let $(B_i, E_i)_{i < \omega}$ be a C -Morley sequence in \mathbb{M} with $(B_0, E_0) = (B, E)$, by hypothesis there is (A', D') such that $(A'B_i, D'E_i) \equiv_{(C, F)} (AB, DE)$ for all $i < \omega$, so in particular $A'B_i \equiv_C AB$ in \mathbb{A}_p for all $i < \omega$, and since $B_i \downarrow_C^p B_{< i}$ the sequence $(B_i)_{i < \omega}$ is C -Morley in \mathbb{A}_p so by Kim's lemma $A \downarrow_C^p B$.

[\Leftarrow] : Assume $(A, D) \downarrow_{(C, F)}^{pfc} (B, E)$. Let $(B_i, E_i)_{i < \omega}$ be a (C, F) -Morley sequence. Let $(e^j)_{j < \kappa}$ enumerate $E \setminus F$, $(e_i^j)_{j < \kappa}$ the corresponding enumeration of $E_i \setminus F$ and $(d^j)_{j < \lambda}$ enumerate $D \setminus F$. By Q.E. we want to find (\tilde{A}, \tilde{D}) such that if $(\tilde{d}^j)_{j < \lambda}$ enumerate $\tilde{D} \setminus F$:

- 1 $\tilde{A}B_i \equiv_C AB$ inside of \mathbb{A}_p for all $p \in F$ and for all $i < \omega$,
- 2 $\langle \tilde{A}B_i \rangle_{e_i^j} \equiv_C \langle AB \rangle_{e^j}$ for all $j < \kappa$ and for all $i < \omega$,
- 3 $\langle \tilde{A}B_i \rangle_{\tilde{d}^j} \equiv_C \langle AB \rangle_{d^j}$ for all $j < \lambda$ and for all $i < \omega$.

1 - If $p \in F$ by hypothesis we have that $A \downarrow_C^p B$ and that $(B_i)_{i < \omega}$ is a Morley sequence over C in \mathbb{A}_p by Lemma 2.7, so by the chain condition in \mathbb{A}_p there is \tilde{A}_p such that $\tilde{A}_p B_i \equiv_C AB$ inside of \mathbb{A}_p for all $i < \omega$.

2 - Let $j < \kappa$, $q^j(x, y) := \text{tp}_{\mathbb{A}_{e^j}}(AB/C)$. We can assume that $q^j(x, B)$ is a non algebraic type, else we would have that $A = B = C$ and we already have our objects fixed. Then by indiscernibility we have that $q^j(x, B_i)$ is a non algebraic type for all $i < \omega$. Using Lemma 2.3 we can find \tilde{A} satisfying the conditions 1 and 2.

3 - Now that we have found our set of objects we use Lemma 2.4 to find the parameters \tilde{d}^j : Let $j < \lambda$, by induction on $n < \omega$ we show that there exist \tilde{d}_n^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_n^j} \equiv_{(C, F)} \langle AB \rangle_{d^j}$ for all $i \leq n$:

We have $\tilde{A}B_i \equiv_{(C, F)} AB$ for all $i < \omega$, so there is \tilde{d}_i^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_i^j} \equiv_{(C, F)} \langle AB \rangle_{d^j}$, and we can take $\tilde{d}_0^j = \tilde{d}_0^j$. Now for $n + 1$ assume we have \tilde{d}_n^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}_n^j} \equiv_C \langle AB \rangle_{d^j}$ for all $i \leq n$. We have \tilde{d}_{n+1}^j such that $\langle \tilde{A}B_{n+1} \rangle_{\tilde{d}_{n+1}^j} \equiv_{(C, F)} \langle AB \rangle_{d^j}$. So $\tilde{d}_n^j \equiv_{(\tilde{A}C, F)} \tilde{d}_{n+1}^j$ and $\tilde{A}B_{n+1} \cap \tilde{A}B_{\leq n} = \tilde{A}C$. Using Lemma 2.4 we can find \tilde{d}_{n+1}^j such that $\tilde{d}_{n+1}^j \equiv_{(\tilde{A}B_{n+1}, F)} \tilde{d}_{n+1}^j$ and $\tilde{d}_{n+1}^j \equiv_{(\tilde{A}B_{\leq n}, F)} \tilde{d}_n^j$, which is the parameter we were looking for.

By compactness there is a \tilde{d}^j such that $\langle \tilde{A}B_i \rangle_{\tilde{d}^j} \equiv_{(C, F)} \langle AB \rangle_{d^j}$ for all $i < \omega$, and this completes our proof of this direction. \square

Corollary 2.9. *If T is supersimple, (A, D) is a finite tuple and (B, E) is any small set then there is a finite $(B_0, E_0) \subseteq (B, E)$ such that $(A, D) \downarrow_{(B_0, E_0)}^K (B, E)$.*

Remark 2.10. *By [2, Proposition 3.6] we know that inside of any non-simple NSOP1-theory we can find arbitrary long sequences of Kim-forking extensions, i.e. for any κ cardinal there is a finite tuple d and an infinite (non-continuous) increasing sequence $(E_i)_{i < \kappa}$ such that $d \not\downarrow_{E_i}^K E_{i+1}$ for all $i < \kappa$. With this in mind this last property tells us, as we already know, that Kim-forking does not satisfy base monotonicity : we have in this case a strong local character, in the sense of a finite $E_i^0 \subseteq E_i$ such that $d \downarrow_{E_i^0}^K E_i$ for every $i < \kappa$, but no chain local character as in simple theories.*

To sum up, T_{pfc} is a non-simple NSOP1 theory such that forking = dividing for complete types over arbitrary sets and forcing base monotonicity on Kim-independence gives us forking independence.

3 Vector spaces of infinite dimension over an algebraically closed field with a non degenerate bilinear form

In this section we will do similar computation as in the previous one and show that this time \downarrow^{K^m} does not satisfy extension in the theory of vector spaces of infinite dimension over an algebraically closed field with a non-degenerate symmetric bilinear form. This theory, denoted sT_∞^{ACF} was studied in [7] and [6] and is NSOP1 non-simple. We will write $\langle A \rangle$ to denote the linear span of A with respect to the field of the monster model, and $K(A)$ for the field sort of A . We will use the following result :

Proposition 3.1. *[6, Proposition 8.12] Let $A, B, E \subseteq \mathbb{C}$ be small algebraically closed sets with $E \subseteq A, B$. Then $A \downarrow_E^K B$ if and only if $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \downarrow_{K(E)}^f K(B)$ (in ACF).*

We will now explain what is \downarrow^{K^m} in this theory and show that this independence relation does not satisfy the extension property, so in particular that it is not forking in sT_∞^{ACF} .

Let $A, B, E \subseteq \mathbb{C}$ be small algebraically closed sets with $E \subseteq A, B$. Then $A \downarrow_E^{K^m} B$ if and only if for all \tilde{B} algebraically closed such that $E \subseteq \tilde{B} \subseteq B$ we have $A\tilde{B} \downarrow_{\tilde{B}}^K B$. Using Proposition 3.1, $A \downarrow_E^{K^m} B$ if and only if $A \downarrow_E^K B$ and for all \tilde{B} algebraically closed such that $E \subseteq \tilde{B} \subseteq B$ we have $K(\text{acl}(A\tilde{B})) \downarrow_{K(\tilde{B})}^f K(B)$, since the condition in the vector sort already satisfies base monotonicity.

Now lets consider $E = (\text{Lin}_{K(E)}((e_i)_{i < \omega}), K(E)) \models sT_\infty^{ACF}$ a small model. Let a, b_0, b_1 be vectors linearly independent over $\langle E \rangle$ such that $a, b_0, b_1 \in E^\perp$, $[a, a] = [b_0, b_0] = [b_1, b_1] = 0$ and $[a, b_0] = [a, b_1] = [b_0, b_1] = 1$. Define $A = (\text{Lin}_{K(E)}((e_i)_{i < \omega}, a), K(E))$ and $B = (\text{Lin}_{K(E)}((e_i)_{i < \omega}, b_0, b_1), K(E))$. Then B is a model of sT_∞^{ACF} and $K(AB) = K(E)$ where $K(AB)$ is the field sort of the structure generated by A and B . We will show in the next section that we can always find such vectors (Lemma 4.1). A and B are algebraically closed and contain E . By Proposition 3.1 we have that $A \downarrow_E^K B$, and since $K(AB) = K(E)$ if \tilde{B} is such that $E \subseteq \tilde{B} \subseteq B$ then $A\tilde{B} \downarrow_{\tilde{B}}^K B$, so $A \downarrow_E^{K^m} B$.

Now let us consider $\alpha \in \mathbb{K}$ transcendental over $K(E)$ and $B' := (\text{Lin}_{K(E)[\alpha]}((e_i)_{i < \omega}, b_0, b_1), K(E)[\alpha])$. We show that there is no $A' \equiv_B A$ such that $A' \downarrow_E^{K^m} B'$, so that \downarrow^{K^m} does not satisfy extension. If there were such an A' , setting $\tilde{B}' = (\text{Lin}_{K(E)}((e_i)_{i < \omega}, \alpha \cdot b_0, \alpha^{-1} \cdot b_1), K(E))$ we would have $A'\tilde{B}' \downarrow_{\tilde{B}'}^K B'$, so in particular for the field sort $[a', \alpha \cdot b_0] \downarrow_{K(E)}^f \alpha$ so $\alpha \downarrow_{K(E)}^f \alpha$ and $a \in K(E)$, a contradiction. Also in this case the structures B' and \tilde{B}' are models.

This shows us that in an NSOP1 theory with existence we do not necessary have that \downarrow^{K^m} is equal to \downarrow^f over models which answers a question of N. Ramsey. However it does not tell us that \downarrow^d does not coincide with \downarrow^f in this theory since in this example $A \not\downarrow_E^d B$, as we will now show :

Let us consider a tuple $(\alpha_i)_{1 \leq i < \omega}$ algebraically independent over $K(E)$, $B_0 = B$ and the sequence $B_i = (\text{Lin}_{K(E)}((e_i)_{i < \omega}, \alpha_i \cdot b_0, \alpha_i^{-1} \cdot b_1), K(E))$. This sequence is indiscernible over E : set $\alpha_0 = 1$, then if $i_0 < \dots < i_n$ we have that $B_{i_0} \dots B_{i_n} = (\text{Lin}_{K(E)}(\alpha_1/\alpha_0, \dots, \alpha_n/\alpha_0)((e_i)_{i < \omega}, \alpha_{i_0} \cdot b_0, \alpha_{i_0}^{-1} \cdot b_1), K(E)(\alpha_1/\alpha_0, \dots, \alpha_n/\alpha_0))$. The type of A over B_0 is inconsistent along that sequence, since we would then have $[a', \alpha_i \cdot b_0] = 1$ for all $i < \omega$. In the end we have $A \downarrow_E^{K^m} B$ and $A \not\downarrow_E^d B$, and it remains unknown if $\downarrow^f = \downarrow^d$ in this theory.

4 Generalisation of the previous theory

In this section we will give a generalisation of Proposition 3.1. We consider theories of vector spaces of infinite dimension over a field with a non degenerate bilinear form, show that under an additional technical condition this theory is NSOP1 if and only if the theory of the field is NSOP1, and compute what Kim-forking is over models, and over algebraically closed set if we assume existence.

Let us consider an NSOP1 theory of fields T_K with Q.E. in the language \mathcal{L}_K and let sT_∞^K be the two sorted theory of infinite dimensional vector spaces over a model of T_K with a non degenerate symmetric bilinear form (written $[,]$) such that (\dagger) : if \bar{b} is a finite tuple there is $u \in \bar{b}^\perp \setminus \{0\}$ such that $[u, u] = 0$. This last condition (\dagger) is a sufficient condition for this theory to have Q.E.. (\dagger) is in particular satisfied when every element of the field is a square. Let $\mathbb{M} = (\mathbb{V}, \mathbb{K})$ be a monster model of sT_∞^K .

Lemma 4.1. *Let $(\lambda_i)_{i < n}$ be scalars, $(b_i)_{i < n}$ be linearly independent vectors in \mathbb{M} , $\alpha \in \mathbb{K}$ and \bar{e} be a finite tuple of vectors. Then there is $x \in \mathbb{V}$ such that $[x, x] = \alpha$, $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$.*

Proof. We begin by showing that there is $x \in \mathbb{V}$ such that $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$, for this we do not need the assumption (\dagger) . What we want to show is that given $\bar{b} = b_0, \dots, b_n$ l.i. the linear function $\psi_{\bar{b}} : x \rightarrow ([x, b_i])_{i < n}$ is surjective, the other condition follows then from the fact that its kernel has infinite dimension. Now let us prove surjectivity by induction on $n < \omega$. For $n = 0$ it is trivial, now assume that it holds for $n > 0$. We want to find $b'_0 \in (b_1, \dots, b_n)^\perp \setminus b_0^\perp$. By hypothesis there is b'_i for $1 \leq i \leq n$ such that $[b'_i, b_j] = \delta_{i,j}$ for all $1 \leq i, j \leq n$. With this for all $u \in \mathbb{V}$ we can write $u = \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i + (u - \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i)$, so we have $\mathbb{V} = \bigoplus_{1 \leq i \leq n} \mathbb{K} \cdot b'_i + (b_1, \dots, b_n)^\perp$, we will write $u_0 = u - \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i$. Now if such a b'_0 does not exist we would have that $(b_1, \dots, b_n)^\perp \subseteq b_0^\perp$, so using the previous decomposition, for all $u \in \mathbb{V}$, $[u, b_0] = \sum_{1 \leq i \leq n} [b'_i, b_0] \cdot [u, b_i] + [u_0, b_0] = \sum_{1 \leq i \leq n} [b_i, b_0] \cdot [u, b_i]$, which by non-degeneracy contradicts the fact that $b_0 \notin \langle b_1, \dots, b_n \rangle$.

We now use our assumption (\dagger) to choose the value of the quadratic form. Choose $x \in \mathbb{V}$ such that $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$. By what we have just shown and (\dagger) there are x', x'' l.i. such that $x', x'' \notin \langle \bar{e}, x \rangle$, $x', x'' \in (\bar{b}, x)^\perp$, $[x', x'] = 0$ and $[x', x''] = 1$. Then the element $x + (\alpha - [x, x] - [x'', x'']) \cdot x' + x''$ corresponds to what we are looking for.

We now show that when every element of the field is a square the condition (\dagger) is satisfied. Let \bar{b} be a finite tuple of vectors and $x \in \bar{b}^\perp \setminus \langle \bar{b} \rangle$. By the previous point we can find $y \in \bar{b}^\perp \setminus \langle \bar{b}, x \rangle$ such that $[x, y] = 0$. Then, if $[x, x] = 0$ or $[y, y] = 0$ we have the element we are looking for, and else if $\lambda^2 = [x, x]$ and $\mu^2 = -[y, y]$ then $\mu \cdot x + \lambda \cdot y$ satisfies our conditions. \square

Corollary 4.2. *The theory sT_∞^K has Q.E. in the language $\mathcal{L} = \mathcal{L}_K \cup \{[,], \cdot, +\} \cup \{\theta_n : n < \omega\}$ where $\theta_n(e_1, \dots, e_n, \sum_{i < n} \lambda_i e_i) = (\lambda_i)_{i < n}$ when e_1, \dots, e_n is linearly independent. In particular the field is stably embedded.*

Proof. For this we show that finite partial isomorphism inside of \mathbb{M} have the back and forth property. Consider two finitely generated structures A and B in \mathbb{M} such that $\varphi : A \rightarrow B$ is an isomorphism. We show that we can extend it to any element a inside of \mathbb{M} . Let a_1, \dots, a_{n-1} be a base of A and $b_i := \varphi(a_i)$. Let $\varphi_K : K(A) \rightarrow K(B)$ be the induced field isomorphism. By Q.E. in the field sort we can extend φ_K to any element of \mathbb{K} , so if $a \in \mathbb{K}$ we can extend φ to Aa . If $a \in \langle A \rangle$ and $a = \sum_i \lambda_i \cdot a_i$ it is clear that extending φ to the structure generated by Aa is equivalent to extending φ to $A(\lambda_i)_{i < n}$, which we can do by Q.E. in the field sort. Now if $a \notin \langle A \rangle$ let us consider an extension φ'_K of φ_K to the field sort of the structure generated by Aa (which is just $K(A)([a, a], [a, a_i])_{i < n}$). Now using Lemma 4.1 we can

find $b \notin \langle B \rangle$ such that $[b, b] = \varphi'_K$ and $[b, b_i] = \varphi'_K([a, a_i])$ for all $i < n$. Then it is clear that sending a to b extends the isomorphism φ . \square

In this context having Q.E. in the language \mathcal{L} depends on the theory of the field (having squares for example) and on the bilinear form (being positively defined in a RCF for example), the (\dagger) assumption allows us to keep full generality on the theory of the field.

Definition 4.3. For $E \subseteq A, B$ algebraically closed subsets of \mathbb{M} and $E \models sT_\infty^K$, we define the relation $A \downarrow_E^\infty B$ by : $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \downarrow_{K(E)}^K K(B)$ in the field theory.

Proposition 4.4. sT_∞^K is NSOP1 and $\downarrow^\infty = \downarrow^K$ over models.

Proof. We will now show that this relation satisfies the conditions of [8, Theorem 9.1]. Clearly it satisfies symmetry, monotonicity and existence over models. For strong finite character if $A \downarrow_E^\infty B$ and $K(A) \downarrow_{K(E)}^K K(B)$ then we can take the formula in $\text{tp}(A/B)$ that Kim-forks over E . If $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$ let $(e_i)_{i < \alpha}$ be a base of $V(E)$ (it is then also a base of $\langle E \rangle$), $(a_i)_{i < \beta}$ completing it into a base of $V(A)$ and $(b_i)_{i < \beta}$ completing it into a base of $V(B)$. Then we have finite tuples \bar{e} , \bar{a} and \bar{b} in these bases and an element $a \in \langle \bar{a} \rangle$ such that $a \in \langle \bar{e}\bar{b} \rangle \setminus \langle \bar{e} \rangle$, and the formula expressing this satisfies the condition.

We will now show that \downarrow^∞ satisfies amalgamation over models. Let $E \subseteq A_0, A_1, B_0, B_1$ with E a model, $A_0 \equiv_E A_1$, $A_j \downarrow_E^\infty B_j$ for $j = 0, 1$ and $B_0 \downarrow_E^\infty B_1$. Let $(e_i)_{i < \alpha}$ be a base of $V(E)$, complete it by $(b_i^j)_{i < \beta_j}$ into a base of $V(B_j)$ for $j = 0, 1$, so $(e_i)_{i < \alpha} (b_i^0)_{i < \beta_0} (b_i^1)_{i < \beta_1}$ is a base of $B_0 B_1$. On the other side let $(a_i^0)_{i < \gamma}$ complete $(e_i)_{i < \alpha}$ into a base of A_0 . Now let $a_0 = k_0 v_0$ be a tuple enumerating A_0 , $A_0 \equiv_E A_1$ so we can find a tuple $a_1 = k_1 v_1$ enumerating A_1 such that $\varphi : a_0 \rightarrow a_1$ is an E -elementary embedding. Define $(a_i^1)_{i < \gamma} = \varphi((e_i)_{i < \alpha})$, since φ is elementary over E we have $\varphi([e_i, a_j^0]) = [e_i, a_j^1]$ and $\varphi([a_i^0, a_j^0]) = [a_i^1, a_j^1]$ for all i, j .

We have that $\text{tp}(k_0/K(E)) = \text{tp}(k_1/K(E))$, $k_j \downarrow_{K(E)}^K K(B_j)$ for $j = 0, 1$ and $K(B_0) \downarrow_{K(E)}^K K(B_1)$. By applying the independence theorem for Kim-forking in T_K we find $k \models \text{tp}(k_0/K(B_0)) \cup \text{tp}(k_1/K(B_1))$ such that $k \downarrow_{K(E)}^K K(B_0)K(B_1)$. Using extension we can assume that $k \downarrow_{K(E)}^K K(B_0 B_1)$. We have two partial embeddings $\varphi_0 : k_0 \rightarrow k$ and $\varphi_1 : k_1 \rightarrow k$ which are elementary over $K(B_0)$ and $K(B_1)$ respectively, by construction we have that $\varphi_0 = \varphi_1 \circ \varphi$. We will now define our structure A . Let $(a_i)_{i < \gamma}$ be a tuple of l.i. vectors not in $\langle (e_i)_{i < \alpha} (b_i^0)_{i < \beta_0} (b_i^1)_{i < \beta_1} \rangle = \langle B_0 B_1 \rangle$ such that :

$$\begin{aligned} [e_i, a_j] &= \varphi_0([e_i, a_j^0]) = \varphi_1([e_i, a_j^1]) \text{ for all } i < \alpha, j < \gamma, \\ [a_i, a_j] &= \varphi_0([a_i^0, a_j^0]) = \varphi_1([a_i^1, a_j^1]) \text{ for all } i, j < \gamma, \\ [a_i, b_j^0] &= \varphi_0([a_i^0, b_j^0]) \text{ for all } i < \gamma, j < \beta_0, \\ [a_i, b_j^1] &= \varphi_1([a_i^1, b_j^1]) \text{ for all } i < \gamma, j < \beta_1. \end{aligned}$$

This is possible by Lemma 4.1 and compactness. Then the structure $A = (\text{Lin}_k((e_i)_{i < \alpha} (a_i)_{i < \gamma}), k)$ satisfies $A \downarrow_E^\infty B_0 B_1$ and $A \equiv_{B_j} A_j$ by construction and Q.E..

By [8, Theorem 9.1] we have that sT_∞^K is NSOP1 and $\downarrow^\infty \implies \downarrow^K$ over models. For the converse, it is clear that if $K(A) \downarrow_{K(E)}^K K(B)$ then $A \downarrow_E^K B$ (looking at coheir Morley sequence and using Kim's lemma for Kim-forking for example). Now if $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$, let $a \in \langle \bar{a} \rangle$ such that $a \in \langle B \rangle \setminus \langle E \rangle$ with \bar{a} a finite tuple in A . Now if $(B_i)_{i < \omega}$ is a coheir Morley sequence over E with $B_0 = B$ we have $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$, so if $q(x, y) := \text{tp}(\bar{a}, B)$ then $\bigcup_{i < \omega} q(x, B_i)$ is inconsistent and $A \not\downarrow_E^K B$. We will give more general argument when generalising this to algebraically closed sets. \square

We get in particular that if T_K is a theory of fields, the theory sT_∞^K is NSOP1 if and only if the theory T_K is, and sT_∞^K is also always non-simple, as we can construct infinite decreasing sequences of type definable groups of unbounded index using orthogonality. We now assume that the theory of the field T_K has existence, so we can define \downarrow^∞ on arbitrary sets as previously. We now show that in this case some properties pass from \downarrow^K in T_K to \downarrow^∞ .

Proposition 4.5. *If Kim-forking in T_K satisfies strong local character, i.e. if for all finite tuple $\bar{a} \in \mathbb{K}$ and for all $B \subseteq \mathbb{K}$ small set there is a finite B_0 such that $\bar{a} \downarrow_{B_0}^K B$, then \downarrow^∞ also does. This is the case when T_K is supersimple for example.*

Proof. Let $A = (\text{Lin}_{K(A)}(\bar{a}), K(A))$ be a finitely generated structure and $B = (\text{Lin}_{K(B)}((b_i)_{i < \beta}), K(B))$. Since \bar{a} is a finite tuple we can find a finite $\bar{b} \subseteq \{b_i : i < \beta \text{ such that } \langle \bar{a} \rangle \cap \langle (b_i)_{i < \beta} \rangle \subseteq \langle \bar{b} \rangle\}$. We then have that $\langle \bar{a}\bar{b} \rangle \cap \langle (b_i)_{i < \beta} \rangle = \langle \bar{b} \rangle$. We now consider the structure A_0 generated by $\bar{a}\bar{b}$. We have $\langle A_0 \rangle = \langle \bar{a}\bar{b} \rangle$, and the field sort is finitely generated, since $K(A)$ is finitely generated and by adding \bar{b} we only have to add a finite number of images by the bilinear form and coefficients from linearly dependent tuples. So by hypothesis there is a finitely generated $E_0 \subseteq K(B)$ such that $K(A_0) \downarrow_{E_0}^K B$, and by setting B_0 to be the substructure of B generated by (\bar{b}, E_0) we get that $A \downarrow_{B_0}^\infty B$. \square

Lemma 4.6. *If $E \subseteq A, B$, then $A \downarrow_E^d B$ implies $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$.*

Proof. We can assume that A is a finite tuple. As previously let $E = (\text{Lin}_{K(E)}((e_i)_{i < \alpha}), K(E))$, $B = (\text{Lin}_{K(B)}((e_i)_{i < \alpha}(b_i)_{i < \beta}), K(B))$. Now let us define the sequence $B_j = (\text{Lin}_{K(B)}((e_i)_{i < \alpha}(b_i^j)_{i < \beta}), K(B))$, with $B_0 = B$, the tuple $(b_i^j)_{i < \beta, j < \omega}$ being linearly independent over $\langle E \rangle$, $[b_i^j, e_l] = [b_i, e_l]$ for all $i < \beta, j < \omega$, $[b_i^j, b_l^{j'}] = [b_i, b_l]$ for all $i, l < \beta, j < \omega$ and $[b_i^j, b_l^{j'}] = 0$ for all $i, l < \beta$ and $j \neq j' < \omega$. We can find such a sequence by Lemma 4.1 and compactness. By Q.E. it is an indiscernible sequence over E , so $\langle A \rangle \cap \langle B \rangle \subseteq \langle E \rangle$: else the type of A over B_0 would divide along that sequence since $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$ for all $i < \omega$. \square

Proposition 4.7. *Let A be a structure such that \downarrow^K in T_K satisfies the independence theorem over $K(A)$. If D^0, D^1 and B^0, B^1 are such that $B^0 \downarrow_A^\infty B^1$, $D^0 \equiv_A D^1$ and $D^j \downarrow_A^\infty B^j$ for $j = 0, 1$ then there is D' such that $D' \equiv_{AB^j} D^j$ for $j = 0, 1$ and $D' \downarrow_A^\infty B^0 B^1$.*

Proof. The proof is strictly similar to the one of amalgamation over models. \square

We will now additionally assume that the theory sT_∞^K has existence, which was only proven for ACF. This assumption allows us to use Kim-forking independence over arbitrary sets and its properties, here Kim's lemma for Kim-forking ([6, Theorem 3.5]).

Proposition 4.8. *If $E \subseteq A, B$ are algebraically closed, then $A \downarrow_E^K B$ if and only if $A \downarrow_E^\infty B$.*

Proof. $[\Rightarrow]$: If $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$, let B_i be a Morley sequence with $B_0 = B$, by Lemma 4.6 we have that $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$ for all $i < \omega$, so the type of A over B is inconsistent along that sequence, and so it Kim-divides over E . Now if $K(A) \not\downarrow_{K(E)}^K K(B)$, the induced structure in the field sort is the one of T_K since the field is stably embedded by Q.E., so $A \not\downarrow_E^K B$.

$[\Leftarrow]$: Let $A \downarrow_E^\infty B$, we show that $A \downarrow_E^K B$ using Kim's Lemma for Kim-forking. Let $(B_i)_{i < \omega}$ be an E -Morley sequence, with $B_0 = B$, $B_j = (\text{Lin}_{K(B_j)}((e_i)_{i < \alpha}(b_i^j)_{i < \beta}), K(B_j))$ for all $j < \omega$ and $A = (\text{Lin}_{K(A)}((e_i)_{i < \alpha}(a_i)_{i < \gamma}), K(A))$. We have that the sequence of the $K(B_j)$ is Morley over $K(E)$ and $K(A) \downarrow_{K(E)}^K K(B_0)$. Let k_a be a tuple enumerating $K(A)$ and $q(x, y) := \text{tp}(k_a K(B_0))$. Using the chain condition for Kim-forking we have a realisation $k' \models \bigcup_{j < \omega} q(x, K(B_j))$. As for amalgamation over models we have a partial E -elementary embedding $\varphi : k_a \rightarrow k'$ such that for all $j < \omega$ this embedding

extends into $\varphi_i : k_a K(B) \rightarrow k' K(B_i)$, also elementary over $K(E)$. Let us now define a structure A' . Let $(a'_i)_{i < \gamma}$ be linearly independent vectors outside of $\langle B_{< \omega} \rangle$ such that :

$$\begin{aligned} [e_i, a'_j] &= \varphi_0([e_i, a_j^0]) = \varphi_l([e_i, a_j]) \text{ for all } i < \alpha, j < \gamma \text{ and } l < \omega, \\ [a'_i, a'_j] &= \varphi_0([a_i, a_j]) = \varphi_l([a_i, a_j]) \text{ for all } i, j < \gamma \text{ and } l < \omega, \\ [a'_i, b'_j] &= \varphi_l([a_i, b_j]) \text{ for all } i < \gamma, j < \beta_0 \text{ and } l < \gamma. \end{aligned}$$

This is possible by Lemma 4.1 and compactness. Then by Q.E. the structure $A' = (\text{Lin}_{k'}((e_i)_{i < \alpha} (a'_i)_{i < \gamma}), k)$ is such that $A' B_j \equiv_E AB$ for all $j < \omega$, and so $A \downarrow_E^K B$. \square

$$(\text{Lin}_F(Ad), F) \downarrow_{(\text{Lin}_F(A\bar{d}), F)}^\Gamma (\text{Lin}_{FL}([d_i, b_j])_{i < n, j < m} (A\bar{b}\bar{d}), FL([d_i, b_j])_{i < n, j < m}),$$

5 Proving existence for simple fields

We will now assume that T_K is a simple theory of fields (see [10] for properties of simple fields). We will generalise the notion of Gamma-forking that Granger developed in [7] for the case of ACF and also give complete proofs. In the case of Proposition 5.4 it is a straightforward adaptation of [7, Proposition 10.2.2]. We use the same terminology as previously, and to lighten the notations we will write A for the structure generated by the set A . As previously we work in a monster model of sT_∞^K .

In this context we will say that a finite tuple $(x_i)_{i < n}$ is generic over K if x_i is generic over $K(x_j)_{j \neq i}$ for all $i < n$. We remind that in a simple field the notion of genericity for the additive group and the multiplicative group coincide. These notions also coincide with genericity for affine transformations which is defined by : if x is generic over $K \subseteq \mathbb{K}$, $g \in \mathbb{K}^*$ and $h \in \mathbb{K}$ such that $x \downarrow_K^f g, h$ then $g \cdot x + h \downarrow_0^f K, g, h$.

- Remark 5.1.**
1. *Extension* : If a tuple $(x_i)_{i < n}$ is generic over K and if $(x_i)_{i < n} \downarrow_{K'}^f K'$ for $K \subseteq K'$ then $(x_i)_{i < n}$ is generic over K' .
 2. *'Genericity'* : If a tuple $(x_i)_{i < n}$ is generic over K , $M \in GL_n(K)$ is an invertible matrix and $(k_i)_{i < n} \in K$ then the tuple $M \times (x_i)_{i < n} + (k_i)_{i < n}$ is also generic over K .
 3. *Transitivity* : If \bar{x} and \bar{y} are finite tuples, \bar{x} is generic over K and \bar{y} is generic over $K(\bar{x})$ if and only if $\bar{x}\bar{y}$ is generic over K .

Proof. The first and last points are just transitivity of \downarrow^f and the fact that in a simple group if g is generic over A , $A \subseteq B$ and $g \downarrow_A^f B$ then g is generic over B . For the second point what we need to know is that in a simple field genericity for the additive group and the multiplicative group are the same. This tells us that if $(\alpha_i) \in K^*$ and $(k_i)_{i < n} \in K$ the tuple $(\alpha_i \cdot x_i + k_i)_{i < n}$ is generic over K . Now with the same reasoning as before we can show that the tuple $(x_1 + x_2, x_2, \dots, x_{n-1})_{i < n}$ is generic over K , and it is clear that these two types of transformations and the fact that we can change the order of the tuple we can generate the transformations $(x_i)_{i < n} \rightarrow M \times (x_i)_{i < n} + (k_i)_{i < n}$ described in the statement. \square

Definition 5.2. Let $d \in \mathbb{M}$ and $A \subseteq B \subseteq M$. We will say that d is Gamma-independent of B over A , written a $\downarrow_A^\Gamma B$, if $K(Ad) \downarrow_{K(A)}^f K(B)$ and one of the following conditions holds :

1. $d \in \mathbb{K}$,
2. $d \in \langle A \rangle$,
3. $d \notin \langle B \rangle$ and for all $b_1, \dots, b_n \in V(B) \setminus \langle A \rangle$ linearly independent the tuple $([d, b_i])_{i \leq n}$ is generic over $K(Ad)K(B)$.

We inductively extend this definition to finitary types by setting that $\bar{d}d' \perp_A^\Gamma B$ iff $\bar{d} \perp_A^\Gamma B$ and $d' \perp_{A\bar{d}}^\Gamma B\bar{d}$.

Remark 5.3. Remark 5.1 tells us in particular that when B has finite dimension over A it is enough to check 3. for a single tuple b_1, \dots, b_n such that $\langle Ab_1, \dots, b_n \rangle = \langle B \rangle$.

Proposition 5.4. The relation of Gamma-forking is invariant, transitive, symmetric, satisfies existence, monotonicity, finite character and extension.

Proof. It is clear that \perp^Γ satisfies **invariance**. For **symmetry**, we will first prove it when a and b are single elements : assume that $d \perp_A^\Gamma Ab$, we want to show that $b \perp_A^\Gamma Ad$. Then condition on the field sort is symmetric, so if $b \in \mathbb{K}$ or $b \in \langle A \rangle$ it holds. Now assume that $b \notin \langle A \rangle$. If $d \in \langle A \rangle$ then $\langle Ad \rangle = \langle A \rangle$ and $b \notin \langle A \rangle$ and the condition 3. is trivially satisfied. If $d \notin \langle A \rangle$ we have that $d \notin \langle Ab \rangle$ and $[d, b]$ is generic over $K(Ad)K(Ab)$. So $b \notin \langle Ad \rangle$ and 3. is satisfied.

Let us now prove **transitivity** : We begin by showing by induction on $len(\bar{b})$ that for d, c single elements and $\bar{b} : a \perp_A^\Gamma A\bar{b}c$ iff $d \perp_A^\Gamma A\bar{b}$ and $d \perp_{A\bar{b}}^\Gamma A\bar{b}c$. (†)

The case $len(b) = 0$ is trivial, now suppose the claim holds for all \bar{b} such that $len(\bar{b}) \leq n$. Let \bar{b} be any tuple such that $len(\bar{b}) \leq n$ and b', c, d be any single elements and A any set. We prove $d \perp_A^\Gamma A\bar{b}b'c$ iff $d \perp_A^\Gamma A\bar{b}b'$ and $d \perp_{A\bar{b}b'}^\Gamma A\bar{b}b'c$.

- If $d \in K$, the equivalence holds since $K(A\bar{b}b'c) = K(A\bar{b}b')(d)$ and since forking independence in the sort \mathbb{K} satisfies transitivity. The case of $d \in \langle A \rangle$ is similar since $K(A\bar{b}b'c) = K(A\bar{b}b')K(Ad)$ and $K(Ad)$ is just $K(A)$ where we add coefficients for d .

- If $d \in \langle A\bar{b}b'c \rangle \setminus \langle A \rangle$ then the equivalence holds since neither of the sides can be satisfied.

- If $d \notin \langle A\bar{b}b'c \rangle$: Assume that $d \perp_A^\Gamma A\bar{b}b'c$. Clearly $d \perp_A^\Gamma A\bar{b}b'$. Now if $c \notin \langle A\bar{b}b' \rangle$ let b'_1, \dots, b'_m be a base of $\langle A\bar{b}b' \rangle$ in $A\bar{b}b'$. Then by hypothesis $[d, c], ([d, b'_i])_{i \leq m}$ is generic over $K(Ad)K(A\bar{b}b'c)$, $K(A\bar{b}b'c) = K(Ad)K(A\bar{b}b')([d, b'_i])_{i \leq m}$, so by Remark 5.1 $[d, c]$ is generic over $K(A\bar{b}b'c)K(A\bar{b}b'd)$. So $[d, c] \perp_{\emptyset}^f K(A\bar{b}b'c)K(A\bar{b}b'd)$, and using $([d, b'_i])_{i \leq m} \perp_{\emptyset}^f K(Ad)K(A\bar{b}b'c)$ we get that $K(A\bar{b}b'd) \perp_{K(Ad)K(A\bar{b}b'c)}^f K(A\bar{b}b'c)$, now with $K(Ad) \perp_{K(A)} K(A\bar{b}b'c)$ we finally get that $K(A\bar{b}b'd) \perp_{K(A\bar{b}b'c)}^f K(A\bar{b}b'c)$. The case where $c \in \langle A\bar{b}b' \rangle$ is easier, the condition 3. is trivially satisfied and with the same computations as we did we get that $K(A\bar{b}b'd) \perp_{K(A\bar{b}b'c)}^f K(A\bar{b}b'c)$, so $d \perp_{A\bar{b}b'}^\Gamma A\bar{b}b'c$.

Now assume that $d \perp_A^\Gamma A\bar{b}b'$ and $d \perp_{A\bar{b}b'}^\Gamma A\bar{b}b'c$. By transitivity of \perp^f we get that $K(Ad) \perp_{K(A)}^f K(A\bar{b}b'c)$. Now let b'_1, \dots, b'_m, c' be a l.i. tuple over $\langle A \rangle$ inside of $A\bar{b}b'c$ with $b'_1, \dots, b'_m \in \langle A\bar{b}b' \rangle$ and $c' \in \langle A\bar{b}b'c \rangle$ (or if $c \in \mathbb{K}$ or $c \in \langle A\bar{b}b' \rangle$ then no c'), by adding elements we can assume that $\langle Ab'_1 \dots b'_m c' \rangle = A\bar{b}b'c$. We have that $b'_i \in Lin_{K(A\bar{b}b'c)}(A\bar{b}b')$ for all $i \leq m$, so since $[d, c']$ is generic over $K(A\bar{b}b'd)K(A\bar{b}b'c)$ we have $[d, c']$ generic over $K(Ad)K(A\bar{b}b'c)([d, b'_i])_{i \leq m}$. Now we have to check that $([d, b'_i])_{i \leq m}$ is generic over $K(Ad)K(A\bar{b}b'c)[d, c']$. Let $(b_i)_{i \leq m}$ be a linearly independent tuple over $\langle A \rangle$ and inside of $A\bar{b}b'$ such that $\langle A\bar{b}b' \rangle = \langle Ab_1 \dots b_m \rangle$, by hypothesis $([d, b_i])_{i \leq m}$ is generic over $K(Ad)K(A\bar{b}b')$ and $K(A\bar{b}b'd) \perp_{K(A\bar{b}b'c)}^f K(A\bar{b}b'c)[d, c']$ (using transitivity) so $([d, b_i])_{i \leq m}$ is generic over $K(Ad)K(A\bar{b}b'c)[d, c']$, writing $(b'_i)_{i \leq m}$ gives us an invertible matrix M with coefficients in $K(A\bar{b}b'c)$, so by Remark 5.1 we have that the tuple $([d, b'_i])_{i \leq m} = M \times ([d, b_i])_{i \leq m}$ is generic over $K(Ad)K(A\bar{b}b'c)[d, c']$, and now also by Remark 5.1 $([d, b'_i])_{i \leq m}[d, c']$ is generic over $K(Ad)K(A\bar{b}b'c)$, so $d \perp_A^\Gamma A\bar{b}b'c$.

Now that we have shown (†) it is clear that for any tuples \bar{b}, \bar{c} if d is a single element $d \downarrow_A^\Gamma A\bar{b}\bar{c}$ iff $d \downarrow_A^\Gamma A\bar{b}$ and $d \downarrow_{A\bar{b}}^\Gamma A\bar{b}\bar{c}$. We will now prove **full transitivity** by induction on $len(\bar{d})$. Assume it holds for all tuples \bar{d} with $len(\bar{d}) \leq n$. Let \bar{d} be such that $len(\bar{d}) \leq n$ and d' an single element. Assume that $\bar{d}d' \downarrow_A^\Gamma C$ with $A \subseteq B \subseteq C$. By the definition of Γ independence we have that $\bar{d} \downarrow_A^\Gamma C$ and $d' \downarrow_{A\bar{d}}^\Gamma C\bar{d}$. By inductive hypothesis we have : $\bar{d} \downarrow_A^\Gamma B$, $\bar{d} \downarrow_B^\Gamma C$ and $d' \downarrow_{A\bar{d}}^\Gamma B\bar{d}$, $d' \downarrow_{B\bar{d}}^\Gamma C\bar{d}$. So $\bar{d}d' \downarrow_A^\Gamma B$ and $\bar{d}d' \downarrow_B^\Gamma C$. The argument for the converse is similar, and we have proven full transitivity.

We now prove **full symmetry**, assume that for all tuples \bar{b}, \bar{c} with $len(\bar{b}), len(\bar{c}) \leq n$ and for all set A $\bar{b} \downarrow_A^\Gamma \bar{c}$ iff $\bar{c} \downarrow_A^\Gamma \bar{b}$. Let \bar{b}, \bar{c} be tuples of length $\leq n$ and b' be a single element. Assume that $\bar{c} \downarrow_A^\Gamma \bar{b}b'$. By transitivity this is equivalent to $\bar{c} \downarrow_A^\Gamma \bar{b}$ and $\bar{c} \downarrow_{A\bar{b}}^\Gamma b'$. By inductive hypothesis this is equivalent to $\bar{b} \downarrow_A^\Gamma \bar{c}$ and $b' \downarrow_{A\bar{b}}^\Gamma \bar{c}$, which by definition means that $\bar{b}b' \downarrow_A^\Gamma \bar{c}$.

Finite character is a direct consequence of the strong local character of \downarrow^f and **monotonicity** is clear. We now prove **extension**. Let $A \subseteq B \subseteq C$ and d a finite tuple such that $d \downarrow_A^\Gamma B$. We want to find $d' \equiv_B d$ such that $d' \downarrow_A^\Gamma C$. Using transitivity and choosing the element of the tuple d' one by one we can assume that d is a single element. For this let us choose some bases as in the previous section $A = (Lin_{K(A)}((a_i)_{i < \alpha}), K(A))$, $Ad = (Lin_{K(A)}((a_i)_{i < \alpha}, (d_i)_{i < n}), K(Ad))$ and $B = (Lin_{K(B)}((a_i)_{i < \alpha}, (b_i)_{i < \beta}), K(B))$. By hypothesis we have that $\langle Ad \rangle \cap \langle B \rangle = \langle A \rangle$, So using extension for \downarrow^f , Remark 5.1, Q.E. and Lemma 4.1 we can find the element d' . \square

Proposition 5.5. *For a structure A , $\bar{d} = (d_i)_{i < n}$ and $\bar{b} = (b_j)_{j < m}$ finite tuples of vectors each linearly disjoint over $\langle A \rangle$, F and L two fields containing $K(A\bar{d})$ and $K(A\bar{b})$ respectively we have that $(Lin_F(A\bar{d}), F) \downarrow_A^\Gamma (Lin_L(A\bar{b}), L)$ if and only if $F \downarrow_{K(A)}^f L$, $\langle A\bar{d} \rangle \cap \langle A\bar{b} \rangle = \langle A \rangle$ and the tuple $([d_i, b_j])_{i < n, j < m}$ is generic over FL .*

Proof. We show the equivalence by induction on n . Assume that it holds for any set A , any finite tuple \bar{b} , any tuple \bar{d} of length $\leq n$ and any fields K, K' as in the statement. Let \bar{d} be a tuple of length $\leq n$ and d' be a single vector not in $\langle A\bar{d} \rangle$.

Assume that $(Lin_F(A\bar{d}, d), F) \downarrow_A^\Gamma (Lin_L(A\bar{b}), L)$. Then $F \downarrow_{K(A)}^f L$ (using monotonicity and transitivity in the field sort). By transitivity $Ad \downarrow_{A\bar{d}}^\Gamma A\bar{b}\bar{d}$ and $A\bar{d} \downarrow_A^\Gamma A\bar{b}$, so $\langle A\bar{d}d' \rangle \cap \langle A\bar{b}\bar{d} \rangle = \langle A\bar{d} \rangle$ and $\langle A\bar{d} \rangle \cap \langle A\bar{b} \rangle = \langle A \rangle$ by induction hypothesis, so $\langle A\bar{d}d' \rangle \cap \langle A\bar{b} \rangle = \langle A \rangle$. Now by induction hypothesis we have that the tuple $([d_i, b_j])_{i < n, j < m}$ is generic over FL , and since :

$$(Lin_F(Ad), F) \downarrow_{(Lin_F(A\bar{d}), F)}^\Gamma (Lin_{FL}([d_i, b_j])_{i < n, j < m}(A\bar{b}\bar{d}), FL([d_i, b_j])_{i < n, j < m}),$$

the tuple $([d', b_i])_{j < m}$ is generic over $FL([d_i, b_j])_{i < n, j < m}$. By Remark 5.1 we have that the tuple $([d_i, b_j])_{i < n, j < m}([d', b_i])_{j < m}$ is generic over FL . The converse is proven the same way using transitivity of \downarrow^Γ and Remark 5.1. \square

Proposition 5.6. *Let A be a structure such that \downarrow^f in T_K satisfies the independence theorem over $K(A)$. If D^0, D^1 and D^0, D^1 are such that $B^0 \downarrow_A^\Gamma B^1$, $B^0 \downarrow_A^\infty B^1$ and $D^j \downarrow_A^\Gamma B^j$ then there is D' such that $D' \equiv_{AB^j} D^j$ for $j = 0, 1$ and $D' \downarrow_A^\Gamma B^0 B^1$ In particular the relation of Gamma-forking satisfies type amalgamation over models.*

Proof. By type definability and local character of \downarrow^Γ it is enough to show this for finite sets. Let $A = (Lin_{K(A)}((e_i)_{i < \alpha}), K(A)) \models sT_\infty^K$, $D^j = (Lin_{K(D^j)}((e_i)_{i < \alpha}, d_1^j, \dots, d_n^j), K(D^j))$ for $j = 0, 1$ and $B^j = (Lin_{K(B^j)}((e_i)_{i < \alpha}, b_1^j, \dots, b_{n_j}^j), K(B^j))$ for $j = 0, 1$ be such that $B^0 \downarrow_A^\Gamma B^1$, $D^0 \equiv_A D^1$ and $D^j \downarrow_A^\Gamma B^j$ for $j = 0, 1$. Let k_D^0 enumerate $K(D^j)$ and k_D^1 be the corresponding enumeration of $K(D^1)$ by the equivalence $D^0 \equiv_A D^1$.

Then $\text{tp}(k_D^0/K(A)) = \text{tp}(k_D^1/K(A))$, $k_D^j \downarrow_{K(A)}^f K(B^j)$ for $j = 0, 1$ and $K(B^0) \downarrow^f K(B^1)$, so by type amalgamation in T_K there is $k' \models \text{tp}(k_D^0/K(B^0)) = \text{tp}(k_D^1/K(B^1))$ such that $k' \downarrow_{K(A)}^f K(B^0)K(B^1)$. By extension we can assume that $k' \downarrow_{K(A)}^f K(B^0B^1)$. Now we need to find values for $[d'_i, b'_j]$. By assumption the tuple $([d_i^0, b_l^0])_{i \leq n, j \leq n^0}$ is generic over $K(D^0)K(B^0)$, by extension there is $(\delta_{i,j}^0)_{i \leq n, j \leq n^0}$ such that $k'K(B^0)(\delta_{i,j}^0)_{i \leq n, j \leq n^0} \equiv_{K(A)} K(D^0)K(B^0)([d_i^0, b_l^0])_{i \leq n, j \leq n^0}$ and $(\delta_{i,j}^0)_{i \leq n, j \leq n^0}$ is generic over $k'K(B^0B^1)$. By the same argument there is $(\delta_{i,j}^1)_{i \leq n, j \leq n^1}$ such that $k'K(B^1)(\delta_{i,j}^1)_{i \leq n, j \leq n^1} \equiv_{K(A)} K(D^1)K(B^1)([d_i^1, b_l^1])_{i \leq n, j \leq n^1}$ and $(\delta_{i,j}^1)_{i \leq n, j \leq n^1}$ is generic over $k'K(B^0B^1)(\delta_{i,j}^0)_{i \leq n, j \leq n^0}$. Then by Remark 5.1 the tuple $(\delta_{i,j}^0)_{i \leq n, j \leq n^0}(\delta_{i,j}^1)_{i \leq n, j \leq n^1}$ is generic over $k'K(B^0B^1)$. We can now define a structure D' as previously since we have found our field sort and the values of the bilinear form by using Lemma 4.1 and using Q.E. to conclude that we have the right type. \square

Proposition 5.7. *If $A \subseteq D, B$ are small structures $D \downarrow_A^\Gamma B$ implies $D \downarrow_A^f B$. In particular the theory sT_∞^K satisfies existence.*

Proof. By finite character it is enough to prove this for $D = d$ and $B = b$ finite tuples. Since \downarrow^Γ satisfies transitivity and \downarrow^f satisfies left transitivity it is enough to show this when d is a single element. Now since \downarrow^Γ also satisfies extension it is enough to show that if d is a single element, \bar{b} is a finite tuple and A an arbitrary structure then $d \downarrow_A^\Gamma \bar{b}$ implies $d \downarrow_A^f \bar{b}$.

Let $A = (\text{Lin}_{K(A)}((e_i)_{i < \alpha}), K(A))$, $B_0 = (\text{Lin}_{K(B_0)}((e_i)_{i < \alpha}, b_0^0, \dots, b_{n-1}^0), K(B_0))$ and

$$B_i = (\text{Lin}_{K(B_i)}((e_i)_{i < \alpha}, b_0^i, \dots, b_{n-1}^i), K(B_0))$$

such that $(B_i)_{i < \omega}$ is an indiscernible sequence over A . We want to find a d' such that $d'B_i \equiv_A dB_0$ for all $i < \omega$. We find a negative part and extend the sequence to $(B_i)_{i \in \mathbb{Z}}$. By extension for \downarrow^Γ we can assume that $a \downarrow_A^\Gamma B_\mathbb{Z}$, where $B_\mathbb{Z}$ is the structure generated by $\{B_i : i \in \mathbb{Z}\}$. We will write $B_{[i,j]}$ for the structure generated by $B_i \dots B_j$. By base monotonicity of Gamma-forking $a \downarrow_{AB_{<0}}^\Gamma B_\mathbb{Z}$. Let us define $A' = AB_{<0}$, by indiscernibility the B_i for $i \in \omega$ are linearly disjoint over A' and form an A' indiscernible sequence. In fact if $b_k^i \in \langle B_{<i} \rangle$ then by indiscernibility $b_k^0 \in \langle B_{<0} \rangle$ and so $b_k^i \in \langle B_{<0} \rangle$.

Thanks to this we can assume without loss of generality (eventually replacing A by A') that the sequence $(B_i)_{i < \omega}$ is linearly disjoint over A . What happens here is that the elements of $(B_i)_{i < \omega}$ might overlap outside of $\langle A \rangle$ but since the property of not being linearly disjoint has local character by indiscernibility all the overlapping that might happen is already happening inside of the negative part of the sequence.

Let k be a tuple enumerating $K(Ad)$ and $p(x, y, z) = \text{tp}(k, ([d, b_j^0])_{j < n}, K(B_0)/A)$. By assumption $k([d, b_j^0])_{j < n} \downarrow_{K(A)}^f K(B_0)$, so there is $k', (\delta_j)_{j < n}$ such that $k', (\delta_j)_{j < n} \models \bigcup_{i < \omega} p(x, y, K(B_i))$. Now let us choose an element d' in the vector sort such that $d' \notin \langle B_\omega \rangle$, $[d', e_i] = \delta_i$, where δ_i is the element of k' that correspond to $[d, e_i]$ in k for all $i < \alpha$, and $[d', b_j^i] = \delta_j$ for all $j < n$ and $i < \omega$. This is possible with Lemma 4.1 since the family $(e_i)_{i < \alpha}(b_j^i)_{j < n, i < \omega}$ is free. By Q.E. we get that $d'B_i \equiv_A dB_0$ for all $i < \omega$, which concludes our proof. \square

Remark 5.8. *The use we make of genericity in this section is completely contained in the properties of Remark 5.1. Being able to find a notion of elements that can satisfy these tree properties inside some theory of fields could eventually allow a similar construction of \downarrow^Γ and a similar proof of existence.*

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