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► **To cite this version:**

Ahmed Gaber. Intersections of balancing, Lucas-balancing and sums of two Jacobsthal numbers. 2022. ⟨hal-03823108⟩

HAL Id: hal-03823108

<https://hal.science/hal-03823108v1>

Preprint submitted on 20 Oct 2022

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Intersections of balancing, Lucas-balancing and sums of two Jacobsthal numbers

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Abstract

We completely solve the two Diophantine equation $B_k = J_n + J_m$ and $C_k = J_n + J_m$ where $\{B_k\}_{k \geq 0}$, $\{C_k\}_{k \geq 0}$ and $\{J_k\}_{k \geq 0}$ are the sequences of balancing, Lucas-balancing and Jacobsthal numbers, respectively. We use Matveev's theorem on linear forms in logarithms and a related reduction procedure of Dujella and Pethő.

Keywords: Balancing numbers, Lucas-balancing numbers, Jacobsthal numbers, Linear forms in logarithms.

2020 Mathematics Subject Classification: 11B39, 11D72, 11J70.

1 Introduction

Balancing numbers are defined by $B_0 = 0$, $B_1 = 1$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$. The first few terms of this sequence are

$$0, 1, 6, 35, 204, 1189, 6930, \dots$$

Lucas-Balancing numbers are strongly related to Balancing numbers and defined by $C_0 = 1$, $C_1 = 3$ and $C_{n+1} = 6C_n - C_{n-1}$ for $n \geq 1$. The first few terms of this sequence are

$$1, 3, 17, 99, 591, 3543, 21255, \dots$$

Jacobsthal sequence is defined by $J_0 = 0$, $J_1 = 1$, and $J_n + 1 = J_n + 2J_{n-1}$ for $n \geq 1$. Its first few terms are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \dots$$

Many Diophantine equations related to balancing numbers has been recently investigated. For instance, see [3], [5], [10] and [12]. The aim of this paper is to study the two Diophantine equations

$$B_k = J_n + J_m, \tag{1.1}$$

and

$$C_k = J_n + J_m. \tag{1.2}$$

The complete sets of solutions are provided in the following theorems.

Theorem 1.1. *The only non-negative solutions (k, n, m) which satisfy Eq.(1.1) and $n \geq m$ are*

$$(1, 1, 0), (2, 3, 3), (2, 4, 1), (2, 4, 2).$$

Theorem 1.2. *The only non-negative solutions (k, n, m) which satisfy Eq.(1.2) and $n \geq m$ are*

$$(0, 1, 0), (0, 2, 0), (1, 3, 0).$$

2 Preliminary results

2.1 Balancing and Lucas-balancing sequences

The characteristic equation of the sequences of balancing and Lucas-balancing numbers is

$$\Theta(x) := x^2 - 6x + 1 = 0.$$

Let $\rho = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$ be the roots of the characteristic equation. The Binet formulas of the balancing and Lucas-balancing numbers are

$$B_n = \frac{\rho^n - \delta^n}{2\sqrt{8}} \quad \text{for all } n \geq 0. \quad (2.1)$$

and

$$C_n = \frac{\rho^n + \delta^n}{2} \quad \text{for all } n \geq 0. \quad (2.2)$$

It is easy to prove that

$$\rho^{n-1} \leq B_n < \rho^n \quad \text{holds for all } n \geq 1. \quad (2.3)$$

and

$$\frac{\rho^n}{2} \leq C_n < \frac{\rho^{n+1}}{2} \quad \text{holds for all } k \geq 2. \quad (2.4)$$

For more details concerning balancing and Lucas-balancing numbers, see [1] and [11].

2.2 Jacobsthal sequence

The characteristic equation of the Jacobsthal sequence is

$$P(x) = x^2 - x - 2.$$

Its Binet formula is

$$J_n = \frac{2^n - (-1)^n}{3}. \quad (2.5)$$

It is known that

$$2^{n-2} \leq J_n \leq 2^{n-1} \quad \text{for all } n \geq 1. \quad (2.6)$$

Basic properties of Jacobsthal numbers can be found in [6] and [7].

2.3 Linear forms in logarithms.

Let α be an algebraic number of degree d with minimal polynomial (over \mathbb{Z}) given by

$$a_0x^d + a_1x^{d-1} + \dots + a_d = a_0 \prod_{i=1}^d (x - \alpha^{(i)}),$$

where the leading coefficient a_0 is positive and the $\alpha^{(i)}$'s are the conjugates of α . Then the logarithmic Weil height (over algebraic real field) of α is given by

$$h(\alpha) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log (\max \{ |\alpha^{(i)}|, 1 \}) \right)$$

The following are some basic of the properties of the logarithmic height function $h(\cdot)$:

$$\begin{aligned} h(\alpha_1 \pm \alpha_2) &\leq h(\alpha_1) + h(\alpha_2) + \log 2; \\ h(\alpha_1 \alpha_2^{\pm 1}) &\leq h(\alpha_1) + h(\alpha_2); \\ h(\alpha^s) &= |s| h(\alpha) \quad (s \in \mathbb{Z}). \end{aligned}$$

The proof of the following result can be found in [9].

Theorem 2.1. (Matveev) *Let $\alpha_1, \dots, \alpha_l$ be positive real algebraic numbers in a real algebraic number field $\mathbb{L} \subset \mathbb{R}$ of degree D , b_1, \dots, b_l be a non zero integers, and assume that*

$$\Lambda_1 := \alpha_1^{b_1} \alpha_2^{b_2} \dots \alpha_l^{b_l} - 1 \neq 0.$$

Then,

$$\log |\Lambda| > -1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) A_1 \dots A_l, \quad (2.7)$$

where

$$B \geq \max\{|b_1|, \dots, |b_l|\},$$

and

$$A_i \geq \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}, \quad \text{for all } i = 1, \dots, l.$$

2.4 Dujella and Pethö reduction lemma

Let X be a real number. Set $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$. In [4], Dujella and Pethö proved the following significant result .

Lemma 2.2. *Let M be a positive integer. Let $\tau, \mu, A > 0, B > 1$ be given real numbers. Assume $\frac{p}{q}$ is a convergent of τ such that $q > 6M$ and $\epsilon := \|\mu q\| - M\|\tau q\| > 0$. If (n, m, ω) is a positive solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^\omega}$$

with $n \leq M$, then

$$\omega < \frac{\log \left(\frac{Aq}{\epsilon} \right)}{\log B}.$$

2.5 Legendre theorem

The following theorem is due to Legendre and appeared in his book [8]. We will use this theorem in some cases of our investigation of Balancing numbers that can be written as the sum of two Jacobsthal numbers. The interested reader can see [2] for more details.

Theorem 2.3. *Let x be a real number, let $p, q \in \mathbb{Z}$ and let $x = [a_0, a_1, \dots]$. If*

$$\left| \frac{p}{q} - x \right| < \frac{1}{2q^2}$$

then $\frac{p}{q}$ is a convergent of the continued fraction of x . Furthermore, if M and n are non negative integers such that $q_n > M$, then

$$\frac{1}{(b+2)q^2} < \left| \frac{p}{q} - x \right|$$

where $b = \max\{a_i : i = 0, 1, 2, \dots, n\}$.

3 Proof of Theorem 1.1

3.1 Bounding m , n and k .

Applying the inequalities (2.6) and (2.3), we get

$$\rho^{k-1} \leq B_k \leq 2^n \quad \text{and} \quad 2^{n-2} \leq B_k \leq \rho^{k'}. \quad (3.1)$$

These implies that,

$$(n-2) \frac{\log 2}{\log \rho} \leq k \leq n \frac{\log 2}{\log \rho} + 1. \quad (3.2)$$

The value of $\frac{\log 2}{\log \rho}$ is approximately 0.39, so we can take $k < 2n$. Using the Binet formulas of the Balancing and Jacobsthal sequences in (1.1). We get

$$\frac{\rho^k - \delta^k}{2\sqrt{8}} = \frac{2^n - (-1)^n}{3} + \frac{2^m - (-1)^m}{3}. \quad (3.3)$$

Then,

$$\left| \frac{\rho^k}{2\sqrt{8}} - \frac{2^n}{3} \right| = \left| \frac{2^m}{3} - \frac{((-1)^n + (-1)^m)}{3} + \frac{\delta^k}{2\sqrt{8}} \right|. \quad (3.4)$$

This implies that

$$\left| \frac{\rho^k}{2\sqrt{8}} - \frac{2^n}{3} \right| < \frac{4 \cdot 2^m}{3}. \quad (3.5)$$

Thus,

$$\left| \frac{3\rho^k 2^{-n}}{2\sqrt{8}} - 1 \right| < \frac{4}{2^{n-m}}. \quad (3.6)$$

Let

$$\Lambda_1 = \frac{3\rho^k 2^{-n}}{2\sqrt{2}} - 1, \quad l = 3, \quad \alpha_1 = \frac{3}{2\sqrt{8}}, \quad \alpha_2 = \rho, \quad \alpha_3 = 2, \quad b_1 = 1, \quad b_2 = k, \quad b_3 = -n.$$

If $\Lambda_1 = 0$, then $3\rho^k = 2^n \cdot 2\sqrt{8}$. Consider the automorphism σ such that $\sigma(\rho) = \delta$. Then $|3\delta^k| = 2^n \cdot 2\sqrt{8}$. But $|3\delta^k| < 3$, then $2^n \cdot 2\sqrt{8} < 3$ which is a contradiction. So, $\Lambda_1 \neq 0$.

Take $\mathbb{L} = \mathbb{Q}(\rho)$. Then $D = 2$. The logarithmic heights are

$$h(\alpha_1) \leq h(3) + h(2\sqrt{8}) \leq \log 3 + \frac{5}{2} \log 2;$$

$$h(\alpha_2) = \frac{1}{2} \log \rho;$$

$$h(\alpha_3) = \log 2.$$

Taking,

$$A_1 = 2 \log 3 + 5 \log 2, \quad A_2 = \log \rho, \quad \text{and} \quad A_3 = 2 \log 2, \quad B = 2n$$

and applying Matveev's Theorem(2.1), we obtain

$$\log |\Lambda_1| > -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(1 + \log 2n)(2 \log 3 + 5 \log 2)(2 \log 2 \log \rho),$$

then,

$$\log |\Lambda_1| > -2 \times 10^{13}(1 + \log 2n). \quad (3.7)$$

Also, from (3.6) we have,

$$\log |\Lambda_1| < \log 4 + (m - n) \log 2. \quad (3.8)$$

Thus by comparing inequalities in (3.7) and (3.8), we get

$$(n - m) \log 2 - \log 4 < 2 \times 10^{13}(1 + \log 2n). \quad (3.9)$$

Hence,

$$m \log 2 > n \log 2 - 2 \times 10^{13}(1 + \log 2n) - \log 4. \quad (3.10)$$

Eq.(3.3) can be written as,

$$\frac{\rho^k}{2\sqrt{8}} - \frac{2^n(1 + 2^{m-n})}{3} = \frac{\delta^k}{2\sqrt{8}} - \frac{(-1)^n - (-1)^m}{3}, \quad (3.11)$$

Therefore,

$$\left| \frac{3\rho^k 2^{-n}}{2\sqrt{8}(1 + 2^{m-n})} - 1 \right| = \left| \frac{3 \cdot 2^{-n}}{2\sqrt{8}(1 + 2^{m-n})} \left(\frac{\delta^k}{2\sqrt{8}} - \frac{(-1)^n - (-1)^m}{3} \right) \right|. \quad (3.12)$$

Hence,

$$\left| \frac{3\rho^k 2^{-n}}{2\sqrt{8}(1 + 2^{m-n})} - 1 \right| < \frac{5}{2^m}. \quad (3.13)$$

Let $\Lambda_2 = \frac{3\rho^k 2^{-n}}{2\sqrt{8}(1 + 2^{m-n})} - 1$. Then,

$$\log |\Lambda_2| < \log 5 - m \log 2. \quad (3.14)$$

Let

$$\alpha_1 = \frac{3}{2\sqrt{8}(1 + 2^{m-n})}, \quad \alpha_2 = \rho, \quad \alpha_3 = 2, \quad l = 3, \quad b_1 = 1, \quad b_2 = k, \quad b_3 = -n, \quad B = 2n.$$

First we show that $\Lambda_2 \neq 0$. If $\Lambda_2 = 0$, then $3\rho^k = 2\sqrt{8}(2^n + 2^m)$. Consider the automorphism σ such that $\sigma(\rho) = \delta$. Then $|3\delta^k| = 2\sqrt{8}(2^n + 2^m)$. But $|3\delta^k| < 3$, which is a contradiction. Then we take $\mathbb{L} = \mathbb{Q}(\rho)$, for which $D = 2$. We compute the logarithmic heights as follows:

$$h(\alpha_1) \leq h(3) + h(2) + h(2\sqrt{8}) + h(1 + 2^{m-n}) \leq \log 3 + \frac{7}{2} \log 2 + (n - m) \log 2;$$

$$h(\alpha_2) = \frac{1}{2} \log \rho;$$

$$h(\alpha_3) = \log 2.$$

We take,

$$A_1 = 2 \log 3 + 2(n - m) \log 2 + 7 \log 2, \quad A_2 = \log \rho, \quad \text{and} \quad A_3 = 2 \log 2.$$

By Matveev's Theorem, we get

$$\log |\Lambda_2| > -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(1 + \log 2n)(2 \log 3 + 2(n - m) \log 2 + 7 \log 2)(2 \log 2 \log \rho),$$

Using Eqs.(3.9),(3.10),(3.14) and direct computations, we find

$$n \log 2 < 6 \times 10^{13}(1 + \log 2n) + 12 \times 10^{25}(1 + \log 2n)^2 + 3. \quad (3.15)$$

We deduce that

$$n < 3 \times 10^{29}. \quad (3.16)$$

3.2 Reducing bound on n

Now, we use the reduction lemma to reduce the upper bound on n .

$$\Gamma_1 = \log\left(\frac{3}{2\sqrt{8}}\right) + k \log \rho - n \log 2.$$

Eq.(3.6) gives, for $m - n \geq 5$,

$$\Lambda_1 = e^{\Gamma_1} - 1 < \frac{4}{2^{m-n}} < \frac{1}{4}, \quad (3.17)$$

which implies that

$$|\Gamma_1| < \frac{1}{2}. \quad (3.18)$$

Then $|\Gamma_1| < 2|e^{\Gamma_1} - 1|$. Therefore we get

$$|\Gamma_1| < \frac{8}{2^{n-m}}. \quad (3.19)$$

We observe that $\Gamma_1 \neq 0$ since $\Lambda_1 \neq 0$. Then

$$0 < \left| \frac{\log\left(\frac{3}{2\sqrt{8}}\right)}{\log 2} - n + k \left(\frac{\log \rho}{\log 2} \right) \right| < \frac{8}{2^{n-m} \log 2} < \frac{12}{2^{n-m}}. \quad (3.20)$$

We apply lemma (2.2) with $M = 6 \times 10^{29}$ ($M > 2n > k$), $\tau = \frac{\log \rho}{\log 2}$, $\mu = \frac{\log(\frac{3}{2\sqrt{8}})}{\log 2}$, $A = 12$, $B = 2$. Write τ as a continued fraction $[a_0, a_1, \dots]$ we get $q_{61} = 6332847229674209482244367144203 > 6M$. We compute

$$\epsilon = ||\mu q_{61}|| - M ||\tau q_{61}|| > 0.4.$$

Thus by lemma (2.2), we get $n - m < 108$. Now we put

$$\Gamma_2 = \log \left(\frac{3}{2\sqrt{8}(1 + 2^{m-n})} \right) + k \log \rho - n \log 2.$$

Then we have from equation(3.13) that, for $m \geq 5$,

$$\Lambda_2 = e^{\Gamma_2} - 1 < \frac{5}{2^m} < \frac{1}{4}, \quad (3.21)$$

which implies that

$$|\Gamma_2| < \frac{1}{2}. \quad (3.22)$$

Then $|\Gamma_2| < 2 |e^{\Gamma_2} - 1|$. Therefore we get

$$|\Gamma_2| < \frac{10}{2^m}. \quad (3.23)$$

Then

$$0 < \left| \frac{\log \left(\frac{3}{2\sqrt{8}(1+2^{m-n})} \right)}{\log 2} - n + k \left(\frac{\log \rho}{\log 2} \right) \right| < \frac{15}{2^m}. \quad (3.24)$$

We apply lemma (2.2) with $M = 6 \times 10^{29}$ ($M > 2n > k$), $\tau = \frac{\log \rho}{\log 2}$, $\mu = \frac{\log(\frac{3}{2\sqrt{8}(1+2^{m-n})})}{\log 2}$, $A = 15$ and $B = 2$. It can be seen that $q_{65} > 6M$. . We consider the values of ϵ in two cases

Case I: if $n - m < 108$ and $n - m \neq 1$

$$\epsilon = ||\mu q_{65}|| - M ||\tau q_{65}|| > 0.0008.$$

Thus by lemma (2.2), we get $m < 124$ so $n < 232$ and $k < 464$.

Case II: $n - m = 1$ we get ϵ always negative. So we solve equation(1.1) if $n - m = 1$. In this case equation(1.1) can be written as $B_k = J_m + J_{m+1}$ and can be reduced as

$$B_k = 2^m. \quad (3.25)$$

Then $k < 2m$ and from Eq.(4.16) we get $m < 3 \times 10^{29}$. As before, we can prove that

$$\gamma^k 2^{-(m+3)} - 1 < \frac{1}{2^m},$$

This gives, for $m \geq 3$, that

$$\left| k \frac{\log \gamma}{\log 2} - (m + 3) \right| < \frac{4}{2^m} < \frac{1}{4}.$$

Employing that $16m < 2^m$ for $m \geq 7$, we deduce that $\frac{4}{2^m} < \frac{1}{2k^2}$. Then $\left| \frac{\log \rho}{\log 2} - \frac{m+3}{k} \right| < \frac{1}{2k^2}$. So, by Legendre's theorem, $\frac{m+3}{k}$ is a convergent of $\frac{\log \rho}{\log 2}$. Using $k < M$ and some computations we find that

$$q_{58} < M < q_{59} \quad \text{and} \quad b := \max\{a_i : i = 0, 1, 2, \dots, 59\} = 200.$$

Consequently,

$$\frac{1}{(200+2)k} < \frac{4}{2^m}.$$

Thus

$$2^m < 5 \cdot 10^{32}.$$

Then $m \leq 108$. Solution of Eq.(4.25) for $m < 108$ gives no solutions and solution of Eq.(4.25) for $m < 124$, $n < 232$ and $k < 464$ gives the triples in Theorem (1.1).

4 Proof of Theorem 1.2

By symmetry of Eq.(1.2), we assume that $n \geq m$.

4.1 bounding m , n and k .

By (2.4) and (2.6), we have

$$\frac{\rho^k}{2} \leq C_k \leq 2^n \quad \text{and} \quad 2^{n-2} \leq C_k \leq \frac{\rho^{k+1}}{2}. \quad (4.1)$$

These imply that,

$$(n-1) \frac{\log 2}{\log \rho} - 1 \leq k \leq (n+1) \frac{\log 2}{\log \rho}. \quad (4.2)$$

We take $k < 2n$. Replacing the Lucas-Balancing and Jacobsthal sequences in (1.2) by their Binet formulas, we get

$$\frac{\rho^k + \delta^k}{2} = \frac{2^n - (-1)^n}{3} + \frac{2^m - (-1)^m}{3}. \quad (4.3)$$

Then,

$$\left| \frac{\rho^k}{2} - \frac{2^n}{3} \right| = \left| \frac{2^m}{3} - \frac{((-1)^n + (-1)^m)}{3} - \delta^k \right|. \quad (4.4)$$

Therefore,

$$\left| \frac{\rho^k}{2} - \frac{2^n}{3} \right| \leq \frac{2^m}{3} + \frac{2}{3} + |\delta^k|$$

Then

$$\left| \frac{\rho^k}{2} - \frac{2^n}{3} \right| < \frac{4 \cdot 2^m}{3} \quad (4.5)$$

Thus,

$$\left| \frac{3\rho^k}{2^{n+1}} - 1 \right| < \frac{4}{2^{n-m}}. \quad (4.6)$$

Consider the following:

$$\Lambda_3 = 3\rho^k 2^{-(n+1)} - 1, \quad l = 3, \quad \alpha_1 = 3, \quad \alpha_2 = \rho, \quad \alpha_3 = 2, \quad b_1 = 1, \quad b_2 = k, \quad b_3 = -(n+1).$$

As before, we can prove that $\Lambda_3 \neq 0$. Take $\mathbb{L} = \mathbb{Q}(\rho)$, for which $D = 2$. Then,

$$h(\alpha_1) = \log 3;$$

$$h(\alpha_2) = \frac{1}{2} \log \rho;$$

$$h(\alpha_3) = \log 2.$$

We take,

$$A_1 = 2 \log 3, \quad A_2 = \log \rho, \quad \text{and} \quad A_3 = 2 \log 2.$$

Let $B = 2n$. Then Theorem(2.1) shows that

$$\log |\Lambda_3| > -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(1 + \log 2n)(2 \log 3)(2 \log 2 \log \rho).$$

Consequently,

$$\log |\Lambda_3| > -3 \times 10^{12}(1 + \log 2n). \quad (4.7)$$

Then also from (4.6) we have,

$$\log |\Lambda_3| < \log 4 + (m - n) \log 2. \quad (4.8)$$

Thus by comparing inequalities in (4.7) and (4.8) we get,

$$(n - m) \log 2 - \log 6 < 3 \times 10^{12}(1 + \log 2n). \quad (4.9)$$

Hence,

$$m \log 2 > n \log 2 - 3 \times 10^{12}(1 + \log 2n) - \log 4. \quad (4.10)$$

Eq.(4.3) is equivalent to

$$\frac{\rho^k}{2} - \frac{2^n(1 + 2^{m-n})}{3} = -\frac{\delta^k}{2} - \frac{(-1)^n - (-1)^m}{3}. \quad (4.11)$$

So,

$$\left| \frac{3\rho^k 2^{-(n+1)}}{1 + 2^{m-n}} - 1 \right| = \left| \frac{3 \cdot 2^{-n}}{1 + 2^{m-n}} \left(-\frac{\delta^k}{2} - \frac{(-1)^n - (-1)^m}{3} \right) \right|. \quad (4.12)$$

Then,

$$\left| \frac{3\rho^k 2^{-(n+1)}}{1 + 2^{m-n}} - 1 \right| < \frac{5}{2^m}. \quad (4.13)$$

Let $\Lambda_4 = \frac{3}{1+2^{m-n}}\rho^k 2^{-(n+1)} - 1$. Hence,

$$\log |\Lambda_4| < \log 5 - m \log 2. \quad (4.14)$$

Set

$$\alpha_1 = \frac{3}{1 + 2^{m-n}}, \quad \alpha_2 = \rho, \quad \alpha_3 = 2, \quad l = 3, \quad b_1 = 1, \quad b_2 = k, \quad b_3 = -(n+1).$$

Again, we have that $\Lambda_4 \neq 0$. Let $\mathbb{L} = \mathbb{Q}(\rho)$. Then,

$$h(\alpha_1) \leq \log 3 + (n - m) \log 2 + \log 2;$$

$$h(\alpha_2) = \frac{1}{2} \log \rho;$$

$$h(\alpha_3) = \log 2.$$

We take,

$$A_1 = 2 \log 3 + 2(n - m) \log 2 + 2 \log 2, \quad A_2 = \log \rho, \quad A_3 = 2 \log 2, \text{ and } B = 2n.$$

Then

$$\log |\Lambda_4| > -1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(1 + \log 2n)(2 \log 3 + 2 \log 2 + 2(n - m) \log 2)(2 \log 2 \log \rho),$$

Then using equations (4.9), (4.10) and (4.14) with some computation we get,

$$n \log 2 < 11 \times 10^{12}(1 + \log 2n) + 12 \times 10^{24}(1 + \log 2n)^2 + 3. \quad (4.15)$$

Hence,

$$n < 3 \times 10^{28}. \quad (4.16)$$

4.2 Reducing bound on n .

Assume that $n - m \geq 5$ and let

$$\Gamma_3 = \log(3) + k \log \rho - (n + 1) \log 2.$$

By Eq.(4.6), we have

$$\Lambda_3 = e^{\Gamma_3} - 1 < \frac{4}{2^{m-n}} < \frac{1}{4}, \quad (4.17)$$

So,

$$|\Gamma_3| < \frac{1}{2}. \quad (4.18)$$

Then, $|\Gamma_3| < 2 |e^{\Gamma_3} - 1|$. Therefore, we have

$$|\Gamma_3| < \frac{8}{2^{n-m}}. \quad (4.19)$$

Since $\Gamma_3 \neq 0$, then

$$0 < \left| \frac{\log 3}{\log 2} - (n + 1) + k \left(\frac{\log \rho}{\log 2} \right) \right| < \frac{8}{2^{n-m} \log 2} < \frac{12}{2^{n-m}}. \quad (4.20)$$

We apply lemma (2.2) with $M = 6 \times 10^{28}$ ($M > 2n > k$), $\tau = \frac{\log \rho}{\log 2}$, $\mu = \frac{\log 3}{\log 2}$, $A = 12$, and $B = 2$. Considering the continued fraction of τ , we find that $q_{65} > 6M$. We compute

$$\epsilon = ||\mu q_{65}|| - M ||\tau q_{65}|| > 0.3.$$

Thus, by lemma (2.2), we get $n - m < 117$. Set

$$\Gamma_4 = \log \left(\frac{3}{1 + 2^{m-n}} \right) + k \log \rho - (n + 1) \log 2.$$

and let $m > 5$. Then we have from equation(4.13) that

$$\Lambda_4 = e^{\Gamma_4} - 1 < \frac{5}{2^m} < \frac{1}{4}. \quad (4.21)$$

We conclude that

$$|\Gamma_4| < \frac{1}{2}. \quad (4.22)$$

Thus $|\Gamma_4| < 2|e^{\Gamma_4} - 1|$. Therefore we get

$$|\Gamma_4| < \frac{10}{2^m}. \quad (4.23)$$

We observe that $\Gamma_4 \neq 0$. So,

$$0 < \left| \frac{\log \left(\frac{3}{1+2^{m-n}} \right)}{\log 2} - (n + 1) + k \left(\frac{\log \rho}{\log 2} \right) \right| < \frac{15}{2^m}. \quad (4.24)$$

We apply lemma (2.2) again with $M = 6 \times 10^{28}$ ($M > 2n > k$), $\tau = \frac{\log \rho}{\log 2}$, $\mu = \frac{\log \left(\frac{3}{1+2^{m-n}} \right)}{\log 2}$, $A = 15$, and $B = 2$. We have $q_{65} > 6M$. We consider the values of ϵ in two cases

Case I: if $n - m < 117$ and $n - m \neq 1$

$$\epsilon = ||\mu q_{65}|| - M ||\tau q_{65}|| > 0.01.$$

Thus, by lemma (2.2), we get $m < 122$. So, $n < 239$ and $k < 478$.

Case II: If $n - m = 1$, we get ϵ always negative. So we solve equation(1.2) separately if $n - m = 1$. In this case equation(1.2) can be reduced to

$$C_k = 2^m. \quad (4.25)$$

By induction, we can prove that all the Lucas-Balancing numbers are odd. Therefore, we have no solutions in this case. Solutions of Eq.(1.2) for $m < 122$, $n < 239$ and $k < 478$ yields the solutions which appear in Theorem (1.2).

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