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► **To cite this version:**

Xinyong Wang, Ying Tang, Christophe Fiter, Laurentiu Hetel. Sampled-data Distributed Control for Homo-directional Linear Hyperbolic System with Spatially Sampled State Measurements. *Automatica*, 2022, 10.1016/j.automatica.2022.110183 . hal-03822880

**HAL Id: hal-03822880**

**<https://hal.science/hal-03822880>**

Submitted on 20 Oct 2022

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# Sampled-data Distributed Control for Homo-directional Linear Hyperbolic System with Spatially Sampled State Measurements <sup>★</sup>

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## Abstract

This paper addresses the stability analysis problem for a class of linear hyperbolic systems with distributed controllers sampled in space and time. First, the considered system is recast in an equivalent form with a continuous time control loop and operators representing the discretization errors (spatio-temporal sampling errors). Then with the help of the Lyapunov-Razumikhin approach, the  $R\varepsilon$ -stability of the proposed linear hyperbolic systems of balance laws is verified via sufficient conditions. At last, the proposed method is illustrated numerically.

*Key words:* Linear hyperbolic system, sampled-data control, aperiodic sampling, Lyapunov-Razumikhin function.

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## 1 Introduction

In recent decades, with the widespread use of digital platforms, sampled-data control has become omnipresent [1, 6, 22]. A large number of results have been obtained concerning the stability and control design of sampled-data finite dimensional systems [23, 26]. For the moment, the analysis and control of infinite-dimensional sampled-data systems are very attractive topics. The mathematical problems are very challenging and very few results have been obtained [15, 25]. In this paper we address the distributed sampled-data control problem for 1-D hyperbolic systems of balance laws. This class of systems is motivated by various practical applications, such as road traffic control [4, 10], chemicals reactions and navigation channels [2]. In particular in [4], the traffic control of Adaptive Cruise Control-equipped (ACC-equipped) vehicles leads to the study of hyperbolic systems with in-domain control.

Roughly speaking, in the literature, there are four basic methods to deal with the sampled-data control of finite-

dimensional systems (see the survey paper [14]): discrete time [18, 34], time-delay [12, 24], Input/Output [13, 28], hybrid approach [27, 32]. For infinite-dimensional systems described by partial differential equations (PDEs), the time-delay method was used for parabolic PDEs [11, 17, 33]. Sampled-data boundary control of 1-D parabolic systems was considered in [20]. Concerning the sampled-data hyperbolic PDEs, few results exist in the literature. Event-triggered boundary control has been developed in [3, 8]. In [7, 9, 19], boundary sampled-data control via backstepping approach have been considered. In [35], a sampled-data observer was created for a hyperbolic system controlled by a one-dimensional semi-linear wave equation. In [36], a network-based distributed controller was designed for the damped semi-linear beam equation.

In summary, the analysis of sampled-data hyperbolic PDEs is still a wide open problem. In the present paper, we aim at studying the stability properties of sampled-data controlled linear hyperbolic systems with discrete-space measurements. More precisely, we study the case of first-order hyperbolic system which is different from [35, 36] where higher-order systems are considered. We consider that the state-space is divided into several sub-domains, where sensors provide point state measurements to the controller. By generalizing the Input/Output approach [29] used for finite dimensional systems, an equivalent system with two sampling errors is deduced. We derive sufficient LMI conditions for the  $R\varepsilon$ -stability by utilizing appropriate Lyapunov

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<sup>★</sup> A preliminary version of this work was presented at the 21st IFAC World Congress, Berlin, Germany, 2020. Corresponding author X. Wang. Tel. +33 772772595.

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Razumikhin technique [21]. This paper is a continuation of our preliminary results in [37, 38], in which we used a simplified version of the current method for the case of hyperbolic systems where the controller is discretized only in time. Here we extend the approach for controllers with both time and space discretizations.

The structure of this paper is given as follows. Section 2 presents the system under study and the problem formulation. In Section 3, we give our main results. The system is first reformulated as a nominal continuous control loop with perturbations induced by time sampling and space discretization. Then the main stability theorems are given. A numerical example is shown in Section 4. Section 5 concludes the paper. Several useful lemmas as well as the proof of Theorem 1 and some steps of the proof of Theorem 2 are given in the appendix.

**Notations:**  $\mathbb{N}$  is the set of nonnegative integers from 0 to infinity,  $\mathbb{R}_+$  is the set of nonnegative reals,  $\mathbb{R}^n$  is used to denote the set of  $n$ -dimensional Euclidean space with the norm  $|\cdot|$ .  $L^2(0, L)$  stands for the Hilbert space of square integrable scalar functions on  $(0, L)$  with the corresponding norm  $\|\cdot\|_{L^2(0,L)}$ , defined by  $\|\tau\|_{L^2(0,L)} = \sqrt{\int_0^L |\tau(x)|^2 dx}$ . The norm to Sobolev space  $H^1(0, L)$  is defined as  $\|\tau\|_{H^1(0,L)} = \sqrt{\int_0^L (|\tau(x)|^2 + |\tau'(x)|^2) dx}$ .

A functional  $V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$  is given such that  $\mathfrak{L}_{V \leq C} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq C\}$ . The notation  $W \preceq 0$  denotes that  $W$  is a symmetric and negative semidefinite matrix. The symmetric elements are denoted by  $*$  in the symmetric matrix. The identity matrix is denoted by  $I$  and  $\lambda_{\min}(\Theta)$  and  $\lambda_{\max}(\Theta)$  are the minimum and maximum eigenvalues of the matrix  $\Theta$ .  $\mathcal{C}^0$  is the space of continuous functions, whereas  $\mathcal{C}^1$  is the space of continuously differentiable functions.  $\lceil \cdot \rceil$  is the ceiling function.

## 2 System description and problem formulation

### 2.1 System Description

We consider the linear hyperbolic system (1) given below

$$\begin{cases} \partial_t z(t, x) + \Lambda \partial_x z(t, x) + \Gamma z(t, x) \\ + F \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i) = 0, & (1a) \\ z(t, 0) = Gz(t, L), \forall t \geq 0, & (1b) \\ z(0, x) = z_0(x), \forall x \in [0, L], & (1c) \end{cases}$$

where  $z : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$ ,  $t \in \mathbb{R}_+$ ,  $x \in [0, L]$ ,  $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  with  $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ ,  $G$ ,  $\Gamma$  and  $F$  are real  $n \times n$  matrices.

Following [11], we assume that  $N$  sensors are uniformly distributed over the interval  $[0, L]$ . The location of the

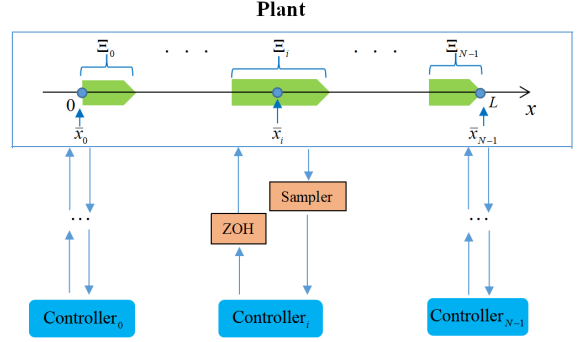


Fig. 1. The system control setup.

sensor is denoted by  $\bar{x}_i, i \in \{0, \dots, N-1\}$ , such that  $\bar{x}_0 = 0, \bar{x}_i = \bar{x}_{i-1} + \bar{b}, i \in \{1, \dots, N-1\}$  where  $\bar{b} = L/(N-1)$ . Each sensor is in charge of an interval  $\Xi_i = [x_i, x_{i+1}], i \in \{0, \dots, N-1\}$  where

$$\begin{cases} x_i = \frac{\bar{x}_{i-1} + \bar{x}_i}{2}, i \in \{1, \dots, N-1\}, & (2a) \\ x_0 = 0, x_N = L. & (2b) \end{cases}$$

We consider the sampling time instants

$$0 = t_0 < t_1 < \dots < t_k \dots, \quad \lim_{k \rightarrow \infty} t_k = \infty,$$

the sampling sequence is defined as  $v = \{t_k\}_{k \in \mathbb{N}}$ . The sampling intervals in time is bounded  $t_{k+1} - t_k \in [\underline{h}, \bar{h}]$ , and  $\bar{h} \geq \underline{h} > 0$  are the corresponding bounds. The control setup is schematically presented in Fig. 1. The plant is a linear hyperbolic system. Each sub-domain  $\Xi_i$  provides discrete time point measurements of the state. Then the sampling state  $z(t_k, \bar{x}_i)$  is transferred to the controller, and the resulting feedback with some constant gains is further implemented to the hyperbolic system through a zero-order hold (ZOH). We consider that a ZOH control is applied using the shape function

$$\begin{cases} d_i(x) = 1, & x \in \Xi_i, \\ d_i(x) = 0, & \text{otherwise}, \end{cases} \quad i \in \{0, \dots, N-1\}. \quad (3)$$

The shape function is used to obtain a linear combination of controllers, each controller is responsible for the control of a region.

The last item in (1a) can be represented as the following control law

$$u(t, x) = F \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i), t \in [t_k, t_{k+1}). \quad (4)$$

*Remark 1.* To analyze the stability of the closed-loop system (1)-(4), the compatibility condition:  $z_0(0) = Gz_0(L)$  is guaranteed by (1b) and (1c).

*Remark 2.* Let us discuss the notion of solution used in the present paper. The system (1)-(4) can be rewritten as a first order system

$$\begin{cases} \frac{dz(t, \cdot)}{dt} = \Upsilon z(t, \cdot) + f(z(t, \cdot)), t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ z(0, \cdot) = z_0(\cdot), \end{cases}$$

where  $f(z(t, \cdot)) = -F \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i)$ , and the operator  $\Upsilon$  is defined by  $\Upsilon z = -\Lambda \partial_x z - \Gamma z$ , with domain

$$\mathcal{D}(\Upsilon) = \left\{ z \in H^1([0, L]; \mathbb{R}^n) \mid z(t, 0) = Gz(t, L). \right\}$$

The operator  $\Upsilon$  generates a stable  $C_0$  semigroup (see the proof of theorem A.1. in [2]). Moreover, we note that  $f : H^1(0, L) \rightarrow H^1(0, L)$  is continuously differentiable for  $t \in [t_k, t_{k+1})$ . If  $z_0 \in \mathcal{D}(\Upsilon)$ , then according to Theorem 6.1.5 of [30], there exists a classical solution for each  $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ . Therefore, we can construct a solution by choosing the last value of the previous sampling interval as the initial condition of the following sampling interval such that it is continuous at each sampling instant.

## 2.2 Problem Formulation

In our paper, we prove  $R\varepsilon$ -stability defined below.

*Definition 1.*  $R\varepsilon$ -stability [31]: Consider positive scalars  $R$  and  $\varepsilon$ , such that  $0 < \varepsilon < R$ , and a candidate Lyapunov function  $V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$ . If for all solutions of system (1)-(4) with  $z_0(x) \in \mathfrak{L}_{V < R}$ , the trajectory of the state  $z(t, x)$  converges to  $\mathfrak{L}_{V \leq \varepsilon}$  as  $t$  goes to infinity, then, system (1)-(4) is  $R\varepsilon$ -stable from  $\mathfrak{L}_{V < R}$  to  $\mathfrak{L}_{V \leq \varepsilon}$ .

*Remark 3.* Intuitively, Definition 1 means that for a given controller (4) and an arbitrary initial condition satisfying  $z_0(0) = Gz_0(L)$  in the domain where  $V < R$ , the solution of the system (1)-(4) converges from the attraction domain ( $R$ -neighborhood) to a steady motion domain ( $\varepsilon$ -neighborhood), and will never go out.

In this paper, our goal is to provide numerical tools for analysis of the  $R\varepsilon$ -stability of the system (1)-(4), while ensuring some performances in terms of the convergence.

## 3 Main Result

This section is divided into two parts. First, we represent the sampled-data system as a continuous time hyperbolic PDE. In the equivalent system, spatio-temporal sampling errors appear in the input as the disturbances. Secondly, based on the provided model, constructive  $R\varepsilon$ -stability criteria are provided.

### 3.1 System Remodelling

Before going into technical details, we first define two parameters  $\varpi$  and  $\vartheta$ , where  $\varpi$  is the time sampling error  $\varpi(t, x) = z(t_k, x) - z(t, x)$ , and  $\vartheta$  is the space discretization error  $\vartheta(t_k, x) = z(t_k, x) - \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i)$ .

Due to (4), we can rewrite  $u(t, x), \forall t \in [t_k, t_{k+1})$  as

$$\begin{aligned} u(t, x) &= Fz(t, x) + F(z(t_k, x) - z(t, x)) \\ &\quad - F \left( z(t_k, x) - \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i) \right) \\ &= Fz(t, x) + F\varpi(t, x) - F\vartheta(t_k, x). \end{aligned} \quad (5)$$

According to (5), the closed-loop system (1)-(4) can be equivalently re-expressed as

$$\begin{cases} \partial_t z(t, x) + \Lambda \partial_x z(t, x) + (\Gamma + F)z(t, x) \\ \quad + F\varpi(t, x) - F\vartheta(t_k, x) = 0, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (6a) \\ z(t, 0) = Gz(t, L), \forall t \geq 0, & (6b) \\ z(0, x) = z_0(x), \forall x \in [0, L]. & (6c) \end{cases}$$

In (6) the parameter  $\varpi$ , as a perturbation input, is the time sampling induced error which can be expressed by a sampling  $\nu$ -dependent operator  $J_\nu$ :

$$J_\nu : \begin{cases} \varpi(t, x) = (J_\nu \varphi)(t, x) = - \int_{t_k}^t \varphi(\theta, x) d\theta, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L], \end{cases} \quad (7)$$

with  $\varphi$  as an auxiliary output for system (6)

$$\varphi(t, x) = \frac{\partial z(t, x)}{\partial t}, \forall t \in (t_k, t_{k+1}), x \in [0, L]. \quad (8)$$

The parameter  $\vartheta$  in (6) is another disturbance input, which is an error caused by space discretization and can be handled by the operator  $E$ :

$$E : \begin{cases} \vartheta(t_k, x) = (E\phi)(t_k, x) \\ \quad = \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x \phi(t_k, \varsigma) d\varsigma, \\ \forall x \in \Xi_i, i = 0, \dots, N-1. \end{cases} \quad (9)$$

with the function  $\phi$  as another auxiliary output for system (6),  $\forall t \geq 0, x \in (x_i, x_{i+1}), i = 0, \dots, N-1$ ,

$$\phi(t, x) = \frac{\partial z(t, x)}{\partial x}. \quad (10)$$

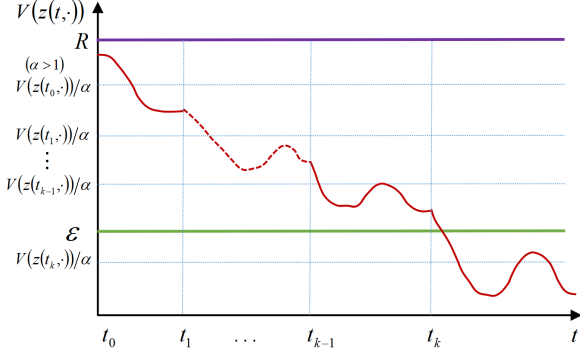


Fig. 2. Schematic diagram of  $R\varepsilon$ -stability under Lyapunov-Razumikhin method.

### 3.2 Stability Analysis

As follows, our main results and  $R\varepsilon$ -stability conditions are given.

**Theorem 1** Consider systems (6)-(10) with (2)-(3) and a candidate Lyapunov function  $V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$  which is differentiable w.r.t. its argument and  $V(0) = 0$ ,  $V(\varrho) > 0, \forall \varrho \in \mathcal{O} \setminus \{0\}$ , where  $\mathcal{O}$  is the neighborhood of  $\varrho = 0$ . Suppose that along the trajectories of the system (6)-(10), the corresponding solution  $z(t, \cdot)$  satisfies  $\dot{V}(z) + 2\delta V(z) \leq 0$ , for some  $\delta > 0$ , whenever

(i)  $R > V(z(t, \cdot)) \geq \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}$ , with some  $\alpha > 1$ ,

(ii)  $z(t_k, \cdot) \in \mathfrak{L}_{V < R}$ .

Then the system is  $R\varepsilon$ -stable from  $\mathfrak{L}_{V < R}$  to  $\mathfrak{L}_{V \leq \varepsilon}$ .

The proof of Theorem 1 can be found in the appendix.

*Remark 4.* In the Lyapunov Razumikhin approach, the main idea is that it is not necessary to ensure the negative definiteness of  $\dot{V}(z(t, \cdot))$  along all the trajectories of the system. In fact, it is sufficient to guarantee its negative definiteness only for the solutions that tend to escape the neighborhood of  $V(z(t, \cdot)) \leq V(z(t_k, \cdot))/\alpha$  of the equilibrium (see Fig. 2). Furthermore, the Razumikhin method is also adapted here to local, practical stability from one large open set, to a smaller one, with guaranteed exponential decay. Theorem 1 is a generic result concerning the  $R\varepsilon$ -stability properties of Lyapunov Razumikhin functionals. It represents a stepping stone for Theorem 2 which is our main theoretical result.

**Theorem 2** Consider systems (6)-(10) with (2)-(3) and an initial condition satisfying  $z_0(0) = Gz_0(L)$ :

(i) Let  $\underline{\lambda} = \min_{i \in \{1, \dots, n\}} \lambda_i$ . Assume that there exist constants  $\mu, \gamma, \kappa > 0$ ,  $\alpha > 1$  and symmetric positive definite

matrices  $\Theta_1 \in \mathbb{R}^{n \times n}$ ,  $\Theta_2 \in \mathbb{R}^{n \times n}$  satisfying the commutativity conditions:  $\Lambda\Theta_1 = \Theta_1\Lambda$ ,  $\Lambda\Theta_2 = \Theta_2\Lambda$  and

$$-\Lambda e^{-2\mu L}\Theta_1 + G^\top \Lambda \Theta_1 G \preceq 0, \quad (11)$$

$$P := U^\top \Lambda \Theta_2 U - \Xi \preceq 0 \quad (12)$$

with

$$U = \begin{bmatrix} \Lambda^{-1}G\Lambda & \Lambda^{-1}G\Gamma - \Lambda^{-1}\Gamma G & \Lambda^{-1}GF - \Lambda^{-1}FG \end{bmatrix},$$

$$\Xi = \begin{bmatrix} \Lambda e^{-2\mu L}\Theta_2 & \mathbf{0}_{n \times 2n} \\ * & \mathbf{0}_{2n \times 2n} \end{bmatrix},$$

and

$$M(0) \succeq 0, \quad M(L) \succeq 0, \quad (13)$$

with  $M(x)$  defined for all  $x \in [0, L]$  as

$$M(x) = \begin{bmatrix} Q(x) & S(x) \\ * & I \end{bmatrix}, \quad (14)$$

where

$$Q(x) = \begin{bmatrix} \Omega_1(x) & e^{-2\mu x}\Theta_1 F + \kappa e^{-2\mu x}\Theta_1 & \mathbf{0} & \mathbf{0} \\ * & \gamma I + \kappa e^{-2\mu x}\Theta_1 & \mathbf{0} & \mathbf{0} \\ * & * & \Omega_2(x) & \mathbf{0} \\ * & * & * & \Omega_3(x) \end{bmatrix}, \quad (15)$$

$$S(x) = \begin{bmatrix} e^{-2\mu x}\Theta_1^\top \\ \mathbf{0}_{3n \times n} \end{bmatrix}, \quad (16)$$

and

$$\Omega_1(x) = e^{-2\mu x} \left[ (F + \Gamma)^\top \Theta_1 + \Theta_1 (F + \Gamma) - \kappa(\alpha - 1)\Theta_1 \right],$$

$$\Omega_2(x) = e^{-2\mu x} \left[ \Gamma^\top \Theta_2 + \Theta_2 \Gamma + \beta\Theta_2 - \kappa\alpha\Theta_2 \right],$$

$$\Omega_3(x) = -\frac{\bar{b}^2}{\pi^2} F^\top F + \kappa e^{-2\mu x}\Theta_2. \quad (17)$$

(ii) Assume that there exist  $\varepsilon, R \in \mathbb{R}_+$  s.t.  $0 < \varepsilon < R$  and

$$\gamma 3\bar{h} \left( (|\Lambda|^2 + 2|F|^2 L^2) \Phi_1 + (|\Gamma|^2 + 2|F|^2 L|G|^2) \Phi_2 \right) \leq (2\sigma - \beta)\varepsilon - 2\delta R, \quad (18)$$

with  $\Phi_1 = \frac{R}{\lambda_{\min}(\Theta_2)e^{-2\mu L}}$ ,  $\Phi_2 = \frac{R}{\lambda_{\min}(\Theta_1)e^{-2\mu L}}$ , where  $\sigma = \mu\underline{\lambda}$ ,  $0 < \beta < 2\sigma$ ,  $\delta > 0$ .

Then the considered system is  $R\varepsilon$ -stable from  $\mathfrak{L}_{V < R}$  to  $\mathfrak{L}_{V \leq \varepsilon}$  for any sampling sequence satisfying  $t_{k+1} - t_k \in$

$[\underline{h}, \bar{h}]$ , with the Lyapunov function defined by

$$V(z) = V_1(z) + V_2(z), \quad (19)$$

with

$$V_1(z) = \int_0^L z^\top e^{-2\mu x} \Theta_1 z dx, \quad (20)$$

$$V_2(z) = \int_0^L z_x^\top e^{-2\mu x} \Theta_2 z_x dx. \quad (21)$$

**PROOF.** Consider the Lyapunov function (19)-(21). It can be bounded as follows:

$$\Phi \|z(t, \cdot)\|_{H^1([0,L]; \mathbb{R}^n)}^2 \leq V(z(t, \cdot)) \leq \Psi \|z(t, \cdot)\|_{H^1([0,L]; \mathbb{R}^n)}^2, \quad (22)$$

where  $\Phi = \min\{\lambda_{\min}(\Theta_1), \lambda_{\min}(\Theta_2)\}e^{-2\mu L}, \Psi = \max\{\lambda_{\max}(\Theta_1), \lambda_{\max}(\Theta_2)\}$ .

*Step 1:* In this step we first show that  $V$  defined in (19) is continuous, so we can proceed to the following steps. The proof is in the appendix.

*Remark 5.* Here  $V_1$  is used in order to bound  $z$ , and  $V_2$  is used to deal with the term  $z_x$  that appears in the derivative of  $V_1$ .

*Step 2:* In this step we study the time derivative of the function of  $V(z)$  defined in (19), and its upper bound is defined by the following inequality:

$$\begin{aligned} \dot{V}(z) &\leq -2\sigma V_1(z) - (2\sigma - \beta)V_2(z) \\ &\quad + \int_0^L \eta^\top W(x) \eta dx + \gamma \|\varpi(s, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \end{aligned} \quad (23)$$

with  $\sigma = \mu\lambda$ ,  $\eta = [z^\top \varpi^\top (\partial_x z)^\top (\partial_x z(t_k, \cdot))^\top]^\top$ ,  $\mathcal{A}_1(x) = e^{-2\mu x} [(F + \Gamma)^\top \Theta_1 + \Theta_1(F + \Gamma)]$ ,  $\mathcal{A}_2(x) = e^{-2\mu x} [\Gamma^\top \Theta_2 + \Theta_2 \Gamma + \beta \Theta_2]$ ,  $\mathcal{A}_3(x) = -\mathcal{A}_1(x) + e^{-4\mu x} \Theta_1^\top \Theta_1$ , and

$$W(x) = \begin{bmatrix} \mathcal{A}_3(x) & -e^{-2\mu x} \Theta_1 F & \mathbf{0} & \mathbf{0} \\ * & -\gamma I & \mathbf{0} & \mathbf{0} \\ * & * & -\mathcal{A}_2(x) & \mathbf{0} \\ * & * & * & \frac{\bar{b}^2}{\pi^2} F^\top F \end{bmatrix}.$$

A detailed proof of (23) is given in the Appendix.

*Step 3:* In this step, we show that under the conditions of Theorem 2, for each sampling interval,  $V$  is decaying with a decay rate  $2\delta$  whenever it is greater than a target level set  $V(z(t_k, \cdot))/\alpha$  and a positive invariant level set

$\varepsilon$ . This can be more intuitively observed through Fig. 2 and can be expressed as  $\dot{V}(z) + 2\delta V(z) \leq 0$ , whenever

$$R > V(z(t, \cdot)) \geq \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}, \quad (24a)$$

$$z(\theta, \cdot) \in \mathfrak{L}_{V < R}, \forall \theta \in [t_k, t], k \in \mathbb{N}. \quad (24b)$$

Let us assume that conditions (24) holds. Since condition (13) holds, by convexity, we have  $M(x) \succeq 0$ , for  $x \in [0, L]$ . Moreover, by using Lemma 2 in the appendix, we have  $W(x) + \kappa N(x) \preceq 0$ , for  $x \in [0, L]$ . Therefore, we get

$$\int_0^L \eta^\top (W(x) + \kappa N(x)) \eta dx \leq 0, \quad (25)$$

with  $N(x)$  given in (A.7).

Now, consider  $t \in [t_k, t_{k+1})$  and a trajectory  $z$  satisfying (24). Since condition (24a) is satisfied, we have  $V(z(t, \cdot)) \geq V(z(t_k, \cdot))/\alpha$  with some  $\alpha > 1$ , which can be rewritten as  $\alpha V(z(t, \cdot)) - V(z(t_k, \cdot)) \geq 0$ . According to (19) and (21), by simple manipulation we get

$$\int_0^L \eta^\top N(x) \eta dx \geq 0. \quad (26)$$

According to (25) and (26), we get

$$\int_0^L \eta^\top W(x) \eta dx \leq 0. \quad (27)$$

And because condition (24b) is satisfied, we have

$$\begin{aligned} \int_0^L z^\top e^{-2\mu x} \Theta_1 z dx &< R, \forall z(\theta, \cdot) \in \mathfrak{L}_{V < R}, \\ \int_0^L \partial_x z^\top e^{-2\mu x} \Theta_2 \partial_x z dx &< R, \forall \partial_x z(\theta, \cdot) \in \mathfrak{L}_{V < R}. \end{aligned}$$

The following inequalities are further derived

$$\begin{aligned} \|z(\theta, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 &< \frac{R}{\lambda_{\min}(\Theta_1)e^{-2\mu L}}, \forall \theta \in [t_k, t], \\ \|\partial_x z(\theta, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 &< \frac{R}{\lambda_{\min}(\Theta_2)e^{-2\mu L}}, \forall \theta \in [t_k, t]. \end{aligned} \quad (28)$$

Using (7), (8) and (28), we can compute the upper bound

$$\begin{aligned} &\|\varpi(s, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \\ &= \int_0^L |\varpi(s, x)|^2 dx = \int_0^L \left| \int_{t_k}^t \frac{\partial z(\theta, x)}{\partial \theta} d\theta \right|^2 dx \\ &= \int_0^L \left| \int_{t_k}^t (\Lambda \partial_x z(\theta, x) + \Gamma z(\theta, x) + F \epsilon) d\theta \right|^2 dx \\ &\leq 3 \int_0^L \int_{t_k}^t (|\Lambda|^2 |\partial_x z(\theta, x)|^2 + |\Gamma|^2 |z(\theta, x)|^2) \end{aligned}$$

$$+|F|^2 \left| \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i) \right|^2 \Big) d\theta dx, \quad (29)$$

where  $\epsilon = \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i)$ . According to (6b),

$$\begin{aligned} z(t_k, \bar{x}_i) &= z(t_k, \bar{x}_i) - z(t_k, 0) + z(t_k, 0) \\ &= \int_0^{\bar{x}_i} \partial_x z(t_k, x) dx + Gz(t_k, L) \\ &\leq \int_0^L |\partial_x z(t_k, x)| dx + Gz(t_k, L). \end{aligned} \quad (30)$$

This inequality corresponds to an upper bound on the control error due to space discretization. Cauchy-Schwarz inequality yields

$$\begin{aligned} &\left| \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i) \right|^2 \\ &\leq L \left| \int_0^L |\partial_x z(t_k, x)| dx + Gz(t_k, L) \right|^2 \\ &\leq 2L \left( \|\partial_x z(t_k, x)\|^2 + |G|^2 |z(t_k, L)|^2 \right). \end{aligned} \quad (31)$$

Then by substituting (31) into (29) we have

$$\begin{aligned} &\|\varpi(s, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\ &\leq 3 \int_{t_k}^t \left( |\Lambda|^2 \|\partial_x z(\theta, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \right. \\ &\quad \left. + |\Gamma|^2 \|z(\theta, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \right. \\ &\quad \left. + |F|^2 \int_0^L 2L \|\partial_x z(t_k, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 dx \right. \\ &\quad \left. + 2L|F|^2 |G|^2 \|z(t_k, L)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \right) d\theta \\ &\leq 3\bar{h} \left( (|\Lambda|^2 + 2|F|^2 L^2) \frac{R}{\lambda_{\min}(\Theta_2) e^{-2\mu L}} \right. \\ &\quad \left. + (|\Gamma|^2 + 2L|F|^2 |G|^2) \frac{R}{\lambda_{\min}(\Theta_1) e^{-2\mu L}} \right) = \omega. \end{aligned} \quad (32)$$

This upper bound corresponds to an estimate of the maximum time sampling error. In addition, since condition (24a) is satisfied, we have

$$\begin{aligned} &-2\sigma V_1(z) - (2\sigma - \beta)V_2(z) \\ &\leq -(2\sigma - \beta)(V_1(z) + V_2(z)) < -(2\sigma - \beta)\epsilon. \end{aligned} \quad (33)$$

The inequality (33) corresponds to an estimate of the domain of attraction. Therefore, instituting (27), (32) and (33) into (23), we have for all  $t \in [t_k, t_{k+1})$ ,  $\dot{V}(z) < -(2\sigma - \beta)\epsilon + \gamma\omega$ .

Since (18) holds, we have  $\dot{V}(z) < -2\delta R \leq -2\delta V(z)$ . Therefore, we have shown that  $\dot{V}(z) + 2\delta V(z) \leq 0$ , whenever conditions (24) are satisfied.

*Step 4:* In this step, we show that if  $z(t_k, \cdot) \in \mathfrak{L}_{V < R}$ , then  $z(t, \cdot) \in \mathfrak{L}_{V < R}, \forall t \in [t_k, t_{k+1})$ . Consider  $z$  such that  $z(t_k, \cdot) \in \mathfrak{L}_{V < R}$ , assume that  $\exists t^\circ \in (t_k, t_{k+1})$  s.t.  $V(z(t^\circ, \cdot)) \geq R$ . Let us then call  $T^\circ$  the minimum of such  $t^\circ$ , then  $\forall t \in [t_k, T^\circ), V(z(t, \cdot)) < R$ . Therefore conditions (24) are going to be satisfied for any  $t \in [t_k, T^\circ)$ . From step 3, we know that  $V$  is going to decrease during that time interval, either continuously, or until  $V$  reaches below  $\max\{\epsilon, V(z(t_k, \cdot))/\alpha\}$  and when it reaches that region, it never gets back out. Therefore, we have  $V(z(T^\circ, \cdot)) < V(z(t_k, \cdot)) < R$ , which contradicts the assumption that there exists  $t^\circ \in (t_k, t_{k+1})$  such that  $V(z(t^\circ, \cdot)) \geq R$ .

*Summary:* From step 3 and step 4, it is clear that  $\dot{V}(z) + 2\delta V(z) \leq 0$  wherever

$$\left\{ \begin{aligned} R > V(z(t, \cdot)) &\geq \max\{\epsilon, V(z(t_k, \cdot))/\alpha\}, \\ z(t_k, \cdot) &\in \mathfrak{L}_{V < R}, \end{aligned} \right. \quad (34a)$$

$$\left\{ \begin{aligned} R > V(z(t, \cdot)) &\geq \max\{\epsilon, V(z(t_k, \cdot))/\alpha\}, \\ z(t_k, \cdot) &\in \mathfrak{L}_{V < R}, \end{aligned} \right. \quad (34b)$$

and therefore, the conditions of Theorem 1 are satisfied, which concludes the proof of  $R\epsilon$ -stability.  $\blacksquare$

Theorem 2 provides constructive conditions for the analysis of the sampled-data hyperbolic system (6)-(10).

*Remark 6.* We explain as follows the selection of parameters in the previous theorem. For  $R\epsilon$ -stability,  $R$  is the domain of attraction for a given Lyapunov function,  $\epsilon$  specifies the positive invariant level set of  $V$ . They satisfy  $0 < \epsilon < R$ . In this paper, we can fix  $R$  then compute  $\epsilon$  or vice versa.  $\alpha$  is a parameter introduced in the Lyapunov-Razumikhin method to define level sets in which the time derivative of  $V(z(t, \cdot))$  should be negative between two sampling times. We choose it greater than 1 but as small as possible to reduce the conservativeness of conditions.  $\mu$  is related to the decay rate of  $V_1, V_2$ , and  $2\delta$  is related to the decay rate of  $V$ .  $\gamma, \kappa, \bar{h}$  and  $\bar{b}$  are found by line search to satisfy the conditions given in Theorem 1.  $\Theta_1, \Theta_2$  can be found by solving the LMIs in (13) and (14). Due to (18), we adjust  $\gamma, \beta$  to be the smallest possible and  $\mu$  to be the largest possible.

## 4 Numerical example

Consider system (1)-(4) with

$$\Lambda = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}, \Gamma = \begin{bmatrix} 2 & 0.5 \\ -0.5 & 1 \end{bmatrix}, F = \begin{bmatrix} 9 & -0.1 \\ -0.5 & 8 \end{bmatrix},$$

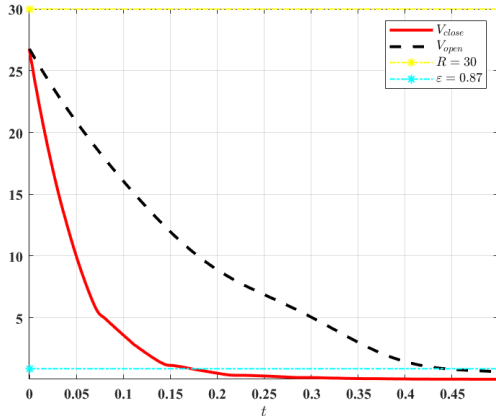


Fig. 3. Time-evolutions of  $V$  in open-loop (black dashed line) and closed-loop (red real line) systems for  $\bar{h} = 0.0001$  and  $\bar{b} = 0.1$ .

$$L = 1, G = \begin{bmatrix} 0.1 & 0.02 \\ -0.1 & -0.01 \end{bmatrix}.$$

The system is open-loop stable. The controller is added in order to improve locally the system performance in terms of reaching time. We apply Theorem 2 in order to verify the  $R\varepsilon$ -stability of the closed-loop system with a sampled-data control for several values of the maximum sampling interval  $\bar{h}$  with a fixed decay rate  $2\delta = 0.002$  and a fixed space discretization step  $\bar{b} = 0.1$ . Table 1 summarizes the results obtained based on the same Lyapunov-Razumikhin functional  $V$  defined in (19)-(21) with a fixed  $R = 30$  and

$$\Theta_1 = 10^{-3} \times \begin{bmatrix} 4.7 & 0.24 \\ * & 4.8 \end{bmatrix}, \Theta_2 = \begin{bmatrix} 1.24 & 0.74 \\ * & 0.91 \end{bmatrix}.$$

The parameters  $\beta = 0.01$ ,  $\mu = 0.7$ ,  $\kappa = 1$ ,  $\gamma = 0.02$ ,  $\alpha = 1.3$  are selected according to Remark 6. First, we solve the LMI conditions (11)-(13) in order to compute the matrices  $\Theta_1$ ,  $\Theta_2$ . Then we use the linear search algorithm in order to find the maximum sampling bound  $\bar{h}$  satisfying (18). The system can be shown to be  $R\varepsilon$ -stable up to  $\bar{h} = 0.0035$ . It can be seen that  $\varepsilon$  linearly depends on  $\bar{h}$  which is consistent with condition (ii) of Theorem 2. Fig. 3 illustrates the time evolution of  $V$  in open-loop and closed-loop for  $\bar{h} = 0.0001$  respectively with

$$z_0(x) = \begin{bmatrix} (1 - \cos 2\pi x) \sin 2\pi x & (\cos 2\pi x - 1) \sin 2\pi x \end{bmatrix}^\top.$$

The closed-loop dynamics ( $V_{close}$  in Fig. 3) has a reaching time of  $t = 0.17s$  (convergence time to the level set  $\varepsilon = 0.87$ ), while the reaching time in open-loop is  $t = 0.43s$ .

Table 1

Evaluation of  $\varepsilon$  for different values of  $\bar{h}$  when  $R = 30$ .

| $\bar{h} _{\bar{b}=0.1}$ | 0.0001 | 0.0005 | 0.0015 | 0.0025 | 0.0035 |
|--------------------------|--------|--------|--------|--------|--------|
| $\varepsilon$            | 0.87   | 4.23   | 12.64  | 21.04  | 29.45  |

## 5 Conclusion

This paper provided methods for the analysis of distributed sampled-data control of linear hyperbolic balance laws under space discretization. By using the Input/Output approach to sampled-data control and the Lyapunov-Razumikhin method, sufficient numerical conditions for the  $R\varepsilon$ -stability were derived. In the future we will work on methods for the global exponential stability analysis of such systems.

## References

- [1] K. J. Åström and B. Wittenmark. *Computer-controlled systems: theory and design*. Courier Corporation, 2013.
- [2] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*. Birkhäuser, 2016.
- [3] L. Baudouin, S. Marx, and S. Tarbouriech. Event-triggered damping of a linear wave equation. *IFAC CPDE*, Oaxaca, 2019.
- [4] N. Bekiaris-Liberis and A. Delis. Feedback control of freeway traffic flow via time-gap manipulation of ACC-equipped vehicles: A PDE-based approach. *IFAC-PapersOnLine*, 52(6):1–6, 2019.
- [5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*. SIAM journal on applied mathematics, 1994.
- [6] T. Chen and B. A. Francis. *Optimal sampled-data control systems*. Springer Science & Business Media, 2012.
- [7] M. A. Davó, D. Bresch-Pietri, C. Prieur, and F. Di Meglio. Stability analysis of a  $2 \times 2$  linear hyperbolic system with a sampled-data controller via backstepping method and looped-functionals. *IEEE Transactions on Automatic Control*, 64(4):1718–1725, 2018.
- [8] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based control of linear hyperbolic systems of conservation laws. *Automatica*, 70:275–287, 2016.
- [9] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based boundary control of a linear  $2 \times 2$  hyperbolic system via backstepping approach. *IEEE Transactions on Automatic Control*, 63(8):2686–2693, 2017.
- [10] N. Espitia, H. Yu, and M. Krstic. Event-triggered varying speed limit control of stop-and-go traffic. *IFAC-PapersOnLine*, 53(2):7509–7514, 2020.
- [11] E. Fridman and A. Blichovsky. Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, 48(5):826–836, 2012.



- [12] E. Fridman, A. Seuret, and J.-P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441–1446, 2004.
- [13] H. Fujioka. Stability analysis of systems with aperiodic sample-and-hold devices. *Automatica*, 45(3):771–775, 2009.
- [14] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.-P. Richard, and S. I. Niculescu. Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76:309–335, 2017.
- [15] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic pdes. *IEEE Transactions on Automatic Control*, 61(11):3301–3314, 2015.
- [16] R. C. James. Advanced calculus belmont. *Calif.: Wadsworth*, 1966.
- [17] W. Kang and E. Fridman. Distributed sampled-data control of Kuramoto–Sivashinsky equation. *Automatica*, 95:514–524, 2018.
- [18] C.-Y. Kao and H. Fujioka. On stability of systems with aperiodic sampling devices. *IEEE Transactions on Automatic Control*, 58(8):2085–2090, 2013.
- [19] I. Karafyllis and M. Krstic. Sampled-data boundary feedback control of 1-D linear transport PDEs with non-local terms. *Systems & Control Letters*, 107:68–75, 2017.
- [20] I. Karafyllis and M. Krstic. Sampled-data boundary feedback control of 1-D parabolic PDEs. *Automatica*, 87:226–237, 2018.
- [21] V. Kolmanovskii and A. Myshkis. *Introduction to the theory and applications of functional differential equations*. Springer Science & Business Media, 2013.
- [22] B. C. Kuo. *Discrete-data control systems*. Prentice-Hall Englewood Cliffs, NJ, 1970.
- [23] D. S. Laila, D. Nešić, and A. Astolfi. 3 sampled-data control of nonlinear systems. In *Advanced topics in control systems theory*, pages 91–137. Springer, 2006.
- [24] K. Liu, V. Suplin, and E. Fridman. Stability of linear systems with general sawtooth delay. *IMA Journal of Mathematical Control and Information*, 27(4):419–436, 2010.
- [25] H. Logemann, R. Rebarber, and S. Townley. Generalized sampled-data stabilization of well-posed linear infinite-dimensional systems. *SIAM journal on control and optimization*, 44(4):1345–1369, 2005.
- [26] S. Monaco and D. Normand-Cyrot. Issues on nonlinear digital control. *European Journal of Control*, 7(2-3):160–177, 2001.
- [27] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel. Exponential stability of impulsive systems with application to uncertain sampled-data systems. *Systems & Control Letters*, 57(5):378–385, 2008.
- [28] D. Nesić and A. R. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.
- [29] H. Omran, L. Hetel, J.-P. Richard, and F. Lamnabhi-Lagarrigue. Stability analysis of bilinear systems under aperiodic sampled-data control. *Automatica*, 50(4):1288–1295, 2014.
- [30] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer Science & Business Media, 2012.
- [31] A. Polyakov. Practical stabilization via relay delayed control. In *47th IEEE Conference on Decision and Control*, pages 5306–5311. Cancun, Mexico, 2008.
- [32] R. Postoyan and D. Nesić. A framework for the observer design for networked control systems. *IEEE Transactions on Automatic Control*, 57(5):1309–1314, 2011.
- [33] A. Selivanov and E. Fridman. Sampled-data relay control of diffusion PDEs. *Automatica*, 82:59–68, 2017.
- [34] Y.S. Suh. Stability and stabilization of nonuniform sampling systems. *Automatica*, 44(12):3222–3226, 2008.
- [35] M. Terushkin and E. Fridman. Sampled-data observers for semilinear damped wave equations under spatially sampled state measurements. *Automatica*, 106:150–160, 2019.
- [36] M. Terushkin and E. Fridman. Network-based control of a semilinear damped beam equation under point and pointlike measurements. *Systems & Control Letters*, 136:104617, 2020.
- [37] X. Wang, C. Fiter, Y. Tang, and L. Hetel. Sampled-data control for a class of linear hyperbolic system via the Lyapunov-Razumikhin technique. <https://hal.archives-ouvertes.fr/hal-02566290>, 2020.
- [38] X. Wang, Y. Tang, C. Fiter, and L. Hetel. Stability analysis for a class of linear hyperbolic system of balance laws with sampled-data control. In *IFAC World Congress*. Berlin, Germany, 2020.

## A Some useful Lemmas

*Lemma 1.* Consider system (1)-(3) with an initial condition  $z_0$  satisfying  $z_0(0) = Gz_0(L)$ . Then  $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$ ,  $\partial_x z(t, 0) = \Lambda^{-1}G\Lambda\partial_x z(t, L) + (\Lambda^{-1}G\Gamma - \Lambda^{-1}\Gamma G)z(t, L) + (\Lambda^{-1}GF - \Lambda^{-1}FG)z(t_k, L)$ .

**PROOF.** From (1), the time derivative of the boundary condition is

$$\partial_t z(t, 0) = G\partial_t z(t, L), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (\text{A.1})$$

Combining (1a), (A.1) with (2b), we obtain

$$\begin{aligned}
& \partial_x z(t, 0) = \Lambda^{-1}(-\Gamma z(t, 0) - Fz(t_k, 0) - \partial_t z(t, 0)) \\
& = \Lambda^{-1}(-\Gamma Gz(t, L) - FGz(t_k, L) - G(-\Lambda \partial_x z(t, L) \\
& \quad - \Gamma z(t, L) - Fz(t_k, L))) \\
& = \Lambda^{-1}G\Lambda \partial_x z(t, L) + (\Lambda^{-1}G\Gamma - \Lambda^{-1}\Gamma G)z(t, L) \\
& \quad + (\Lambda^{-1}GF - \Lambda^{-1}FG)z(t_k, L) \tag{A.2}
\end{aligned}$$

■

*Lemma 2.* Consider the condition (13) is satisfied. Then

$$W(x) + \kappa N(x) \preceq 0, \forall x \in [0, L]. \tag{A.3}$$

**PROOF.** According to [5], (14) is equivalent to

$$Q(x) - S_{4n \times n}(x)I_n^{-1}S_{4n \times n}^\top(x) \succeq 0 \tag{A.4}$$

where  $Q(x)$  and  $S(x)$  are defined as (15) and (16). If (A.4) holds, we have

$$Q(x) - \bar{S}_{4n \times 4n}(x)I_{4n \times 4n}^{-1}\bar{S}_{4n \times 4n}^\top(x) \succeq 0 \tag{A.5}$$

with

$$\bar{S}(x) = \begin{bmatrix} S_{4n \times n}(x) & \mathbf{0}_{4n \times 3n} \end{bmatrix}.$$

Then (A.5) can be re-expressed for all  $x \in [0, L]$  as

$$W(x) + \kappa N(x) \preceq 0, \tag{A.6}$$

with  $W(x)$  defined in step 2 and

$$N(x) = e^{-2\mu x} \begin{bmatrix} (\alpha - 1)\Theta_1 & -\Theta_1 & \mathbf{0} & \mathbf{0} \\ * & -\Theta_1 & \mathbf{0} & \mathbf{0} \\ * & * & \alpha\Theta_2 & \mathbf{0} \\ * & * & * & -\Theta_2 \end{bmatrix}. \tag{A.7}$$

■

## B Proof of Theorem 1

**PROOF.** During a sampling interval  $[t_k, t_{k+1})$  with initial state  $z(t_k, \cdot)$ :

- (1) If  $V(z(t_k, \cdot)) \leq \varepsilon$ ,  $V(z(t, \cdot))$  will remain in  $\varepsilon$  during  $[t_k, t_{k+1})$ . Otherwise, we will have  $\dot{V}(z) > 0 > -2\delta V(z)$  at some point when  $V(z(t, \cdot)) \geq \varepsilon \geq V(z(t_k, \cdot))/\alpha$ , which would contradict the proposition in Theorem 1.

- (2) If  $R > V(z(t_k, \cdot)) \geq \varepsilon$ :  
(a) We have  $V(z(t, \cdot)) \leq V(z(t_k, \cdot))$  during  $[t_k, t_{k+1})$ . Otherwise, we will have  $\dot{V}(z) > 0 > -2\delta V(z)$  at some point when  $V(z(t, \cdot)) \geq V(z(t_k, \cdot)) \geq V(z(t_k, \cdot))/\alpha$ , which would contradict the proposition in Theorem 1.  
(b) We can further show that

$$V(z(t, \cdot)) \leq \max\{\varepsilon, \frac{V(z(t_k, \cdot))}{\alpha}, e^{-2\delta(t-t_k)}V(z(t_k, \cdot))\} \tag{B.1}$$

during  $[t_k, t_{k+1})$ . In the following, we will discuss two possibilities in case (b):

(b1) If there exists  $t' \in [t_k, t_{k+1})$  such that  $V(z(t', \cdot)) = \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}$ . If  $t \in [t_k, t')$ ,  $\dot{V}(z) + 2\delta V(z) \leq 0$  holds, and we have  $V(z(t, \cdot)) \leq e^{-2\delta(t-t_k)}V(z(t_k, \cdot)), \forall t \in [t_k, t')$ . Otherwise, we will have  $\dot{V}(z) > 0 > -2\delta V(z)$  at some point when  $V(z(t, \cdot)) \geq V(z(t_k, \cdot))/\alpha$ , which would contradict the proposition in Theorem 1. If  $t \in [t', t_{k+1})$ ,  $V(z(t, \cdot))$  cannot go back above  $\max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}$ , otherwise, according to the same principle, it would contradict the proposition in Theorem 1, then we have  $V(z(t, \cdot)) \leq \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}, \forall t \in [t', t_{k+1})$ . So, over the whole sampling interval  $t \in [t_k, t_{k+1})$ , we can get inequality (B.1).

(b2) When  $V(z(t, \cdot)) > \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}, \forall t \in [t_k, t_{k+1})$ , since  $\dot{V}(z) + 2\delta V(z) \leq 0$ , we have  $V(z(t, \cdot)) \leq e^{-2\delta(t-t_k)}V(z(t_k, \cdot))$ . Then it is not hard to get (B.1).

Now, let us consider  $z(t_k, \cdot) \in \mathfrak{L}_{V < R}, t \in [t_k, t_{k+1}), k \in \mathbb{N}$ . We have

$$\begin{aligned}
& V(z(t, \cdot)) \\
& \leq \max\{\varepsilon, V(z(t_k, \cdot))/\alpha, e^{-2\delta(t-t_k)}V(z(t_k, \cdot))\} \\
& = \max\{\varepsilon, \xi V(z(t_k, \cdot))\}, \tag{B.2}
\end{aligned}$$

with  $\xi = \max\{1/\alpha, e^{-2\delta(t-t_k)}\} \leq 1$ , then

$$V(z(t_k, \cdot)) \leq \max\{\varepsilon, \zeta V(z(t_{k-1}, \cdot))\}, \tag{B.3}$$

with  $\zeta = \max\{1/\alpha, e^{-2\delta h}\} < 1, \forall k \in \mathbb{N} \setminus \{0\}$ , where  $h$  is the lower bound of the sampling interval.

By recursion, if  $z(t_0, \cdot) \in \mathfrak{L}_{V < R}, \forall k \in \mathbb{N}$  we have

$$\begin{aligned}
V(z(t_k, \cdot)) & \leq \max\{\varepsilon, \zeta \max\{\varepsilon, \zeta V(z(t_{k-2}, \cdot))\}\} \\
& \leq \max\{\varepsilon, \zeta \varepsilon, \zeta^2 V(z(t_{k-2}, \cdot))\} \\
& \leq \max\{\varepsilon, \zeta^2 V(z(t_{k-2}, \cdot))\} \\
& \leq \dots \\
& \leq \max\{\varepsilon, \zeta^k V(z(t_0, \cdot))\}. \tag{B.4}
\end{aligned}$$

Then combining (B.2) and (B.4), we get that

$$\begin{aligned} V(z(t, \cdot)) &\leq \max\{\varepsilon, \xi \max\{\varepsilon, \zeta^k V(z(t_0, \cdot))\}\} \\ &= \max\{\varepsilon, \xi \varepsilon, \xi \zeta^k V(z(t_0, \cdot))\} \\ &= \max\{\varepsilon, \zeta^k V(z(t_0, \cdot))\} = \varepsilon \end{aligned} \quad (\text{B.5})$$

when  $k$  is large enough. Therefore,  $\exists t_k \geq 0, z(t, \cdot) \in \mathfrak{L}_{V \leq \varepsilon}, \forall t \geq t_k$  ( $k = \lceil \log_{\zeta} \varepsilon / V(z(t_0, \cdot)) \rceil$ ), which define the proof of  $R\varepsilon$ -stability.  $\blacksquare$

## C Proof of Step 1 and Step 2 in Theorem 2

### Proof of Step 1:

- (1) Since  $z(t, x)$  is continuous with respect to  $t$  for all  $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ , and continuous at sampling instants by construction (see Remark 2), then  $V_1$  is continuous for all  $t \geq 0$ .
- (2) From system (1) and (4), we can get

$$\begin{aligned} z_x(t, x) &= \Lambda^{-1}(-z_t(t, x) - \Gamma z(t, x) \\ &\quad - F \sum_{i=0}^{N-1} d_i(x) z(t_k, \bar{x}_i)), \end{aligned} \quad (\text{C.1})$$

for all  $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ . Since all the terms on the right of the equation (C.1) are continuous in  $t$  on  $(t_k, t_{k+1}), \forall k \in \mathbb{N}$ , then  $z_x(t, x)$  and thus  $V_2$  are also continuous in  $t$  for all  $(t_k, t_{k+1}), k \in \mathbb{N}$ .

Now we consider the time interval  $[t_k, t_{k+1})$ , for some  $k \in \mathbb{N}$  and an initial condition  $z_k(x)$ . The solution of (1) is defined as  $z(t, x)$  on the time interval  $[t_k, t_{k+1})$ , and is such that  $z$  and  $z_x$  are both  $\mathcal{C}^0$  in  $t \in [t_k, t_{k+1})$ .

Next, we prolong the solution to  $\mathcal{C}^1$  in  $t$  on  $[t_k, t_{k+1}]$ . We denote  $y(t, x)$  the solution on  $[t_k, t_{k+1}]$  with initial condition  $z_k(x)$ .  $y(t, x)$  and  $y_x(t, x)$  are  $\mathcal{C}^0$  in  $t$  on  $[t_k, t_{k+1}]$ . We get on  $[t_k, t_{k+1})$

$$\begin{cases} z(t, x) = y(t, x), \\ z_x(t, x) = y_x(t, x). \end{cases} \quad (\text{C.2})$$

Then the left limit can be calculated as

$$\lim_{t \rightarrow t_{k+1}^-} z_x(t, x) = \lim_{t \rightarrow t_{k+1}^-} y_x(t, x) = y_x(t_{k+1}, x).$$

For the next time interval  $[t_{k+1}, t_{k+2})$ , we set the initial condition  $z_{k+1}(x) = y(t_{k+1}, x)$ . Then the solution  $z(t, x)$  of system (1) on  $[t_{k+1}, t_{k+2})$  has  $z_x(t, x)$  is  $\mathcal{C}^0$ . Therefore,

$$\lim_{t \rightarrow t_{k+1}^+} z_x(t, x) = z_x(t_{k+1}, x) = y_x(t_{k+1}, x).$$

In conclusion, we can see that by construction,  $z_x(t, x)$  is continuous in  $t$  at time instant  $t_{k+1}$ . Similarly, we can show that the function  $z_x(t, x)$  is continuous at all sampling instants, which shows both the continuity of  $z_x(t, x)$  with respect to time for all  $t \geq 0$  and the continuity of  $V_2$ .

**Proof of Step 2:** Thanks to commutativity condition:  $\Lambda \Theta_1 = \Theta_1 \Lambda$ , we first compute the time derivative of  $V_1(z)$  along the solutions to (6)-(10),  $\forall t \in [t_k, t_{k+1})$ ,

$$\begin{aligned} \dot{V}_1(z) &= \int_0^L \left( \partial_t z^\top e^{-2\mu x} \Theta_1 z + z^\top e^{-2\mu x} \Theta_1 \partial_t z \right) dx \\ &= \int_0^L \left( (-\Lambda \partial_x z - (F + \Gamma)z - F\varpi)^\top e^{-2\mu x} \Theta_1 z \right. \\ &\quad \left. + (F\vartheta)^\top e^{-2\mu x} \Theta_1 z + z^\top e^{-2\mu x} \Theta_1 F\vartheta \right. \\ &\quad \left. + z^\top e^{-2\mu x} \Theta_1 (-\Lambda \partial_x z - (F + \Gamma)z - F\varpi) \right) dx \\ &= \int_0^L -\partial_x [z^\top \Lambda e^{-2\mu x} \Theta_1 z] dx \\ &\quad + \int_0^L \left( -z^\top (F + \Gamma)^\top e^{-2\mu x} \Theta_1 z \right. \\ &\quad \left. + (F\vartheta)^\top e^{-2\mu x} \Theta_1 z + z^\top e^{-2\mu x} \Theta_1 F\vartheta \right. \\ &\quad \left. - z^\top e^{-2\mu x} \Theta_1 (F + \Gamma)z - z^\top e^{-2\mu x} \Theta_1 F\varpi \right. \\ &\quad \left. - 2\mu z^\top \Lambda e^{-2\mu x} \Theta_1 z - \varpi^\top F^\top e^{-2\mu x} \Theta_1 z \right) dx \\ &= z^\top(\cdot, L) (-\Lambda e^{-2\mu L} \Theta_1 + G^\top \Lambda \Theta_1 G) z(\cdot, L) \\ &\quad + \int_0^L \left( -z^\top \left( (F + \Gamma)^\top e^{-2\mu x} \Theta_1 \right. \right. \\ &\quad \left. \left. + e^{-2\mu x} \Theta_1 (F + \Gamma) \right) z \right. \\ &\quad \left. + (F\vartheta)^\top e^{-2\mu x} \Theta_1 z + z^\top e^{-2\mu x} \Theta_1 F\vartheta \right. \\ &\quad \left. - \varpi^\top F^\top e^{-2\mu x} \Theta_1 z - z^\top e^{-2\mu x} \Theta_1 F\varpi \right) dx \\ &\quad - 2\mu \int_0^L z^\top \Lambda e^{-2\mu x} \Theta_1 z dx. \end{aligned} \quad (\text{C.3})$$

In order to get the time derivative of  $z_x$  in  $V_2$ , we refer to the original system (1). Since  $z : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$  has consecutive partial derivatives in  $[0, +\infty) \times [0, L]$ , according to Schwartz's theorem ([16]) we can obtain

$$\begin{aligned} \partial_{xt} z(t, x) &= \partial_{tx} z(t, x) = -\Lambda \partial_{xx} z(t, x) \\ &\quad - \Gamma \partial_x z(t, x), \forall t \in (t_k, t_{k+1}). \end{aligned} \quad (\text{C.4})$$

Using Lemma 1 in the appendix and commutativity condition:  $\Lambda \Theta_2 = \Theta_2 \Lambda$ , the time derivative of  $V_2(z)$  along the solutions to (A.2) and (C.4),  $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$  is shown as follows

$$\begin{aligned} \dot{V}_2(z) &= Z^\top P Z - 2\mu \int_0^L \partial_x z^\top \Lambda e^{-2\mu x} \Theta_2 \partial_x z dx \\ &\quad + \int_0^L \left( -\partial_x z^\top (\Gamma^\top e^{-2\mu x} \Theta_2 + e^{-2\mu x} \Theta_2 \Gamma) \partial_x z \right) dx. \end{aligned} \quad (\text{C.5})$$

where  $Z = [(\partial_x z(\cdot, L))^\top z^\top(\cdot, L) z^\top(t_k, L)]^\top$ ,  $P$  is defined in (12). Adding and subtracting  $\gamma \|\varpi(s, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$  to (C.3) and  $\beta \int_0^L z_x^\top e^{-2\mu x} \Theta_2 z_x dx$  to (C.5) for some  $\gamma > 0, \beta > 0$ , and using conditions (11) and (12) we have

$$\begin{aligned} \dot{V}(z) &= \dot{V}_1(z) + \dot{V}_2(z) \\ &\leq \int_0^L (-z^\top \mathcal{A}_1(x) z - \varpi^\top F^\top e^{-2\mu x} \Theta_1 z \\ &\quad + (F\vartheta)^\top e^{-2\mu x} \Theta_1 z + z^\top e^{-2\mu x} \Theta_1 F\vartheta \\ &\quad - z^\top e^{-2\mu x} \Theta_1 F\varpi - \gamma \|\varpi\|_{L^2([0, L]; \mathbb{R}^n)}^2) dx \\ &\quad - 2\sigma V_1(z) + \gamma \|\varpi(s, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\ &\quad + \int_0^L (-\partial_x z^\top \mathcal{A}_2(x) \partial_x z) dx - 2\sigma V_2(z) \\ &\quad + \beta \int_0^L z_x^\top e^{-2\mu x} \Theta_2 z_x dx. \end{aligned} \quad (\text{C.6})$$

with  $\mathcal{A}_1(x) = e^{-2\mu x} [(F + \Gamma)^\top \Theta_1 + \Theta_1 (F + \Gamma)]$ ,  $\mathcal{A}_2(x) = e^{-2\mu x} [\Gamma^\top \Theta_2 + \Theta_2 \Gamma + \beta \Theta_2]$ . Then by using Young's inequality to  $\int_0^L (F\vartheta)^\top e^{-2\mu x} \Theta_1 z dx$ , we get

$$\begin{aligned} &\int_0^L (F\vartheta)^\top e^{-2\mu x} \Theta_1 z dx \\ &\leq \frac{1}{2} \int_0^L (F\vartheta)^\top (F\vartheta) dx \\ &\quad + \frac{1}{2} \int_0^L (e^{-2\mu x} \Theta_1 z)^\top (e^{-2\mu x} \Theta_1 z) dx \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (F\vartheta)^\top (F\vartheta) dx \\ &\quad + \frac{1}{2} \int_0^L (e^{-2\mu x} \Theta_1 z)^\top (e^{-2\mu x} \Theta_1 z) dx \end{aligned} \quad (\text{C.7})$$

Let us recall the space discretization error (9) with (10)

$$\begin{aligned} \vartheta(t_k, x) &= \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x z_\zeta(t_k, \zeta) d\zeta, \\ \forall x \in [x_i, x_{i+1}), i &= 0, \dots, N-1. \end{aligned}$$

and  $\vartheta(t_k, \bar{x}_i) = 0$ , for  $\bar{x}_0 = 0, \bar{x}_{i+1} = \bar{x}_i + \bar{b}, \bar{b} = L/(N-1), i = 0, \dots, N-2$ .

We rewrite the first term in (C.7) for  $x \in \Xi_i, i = 0, \dots, N-1$  as

$$\frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (F\vartheta)^\top (F\vartheta) dx$$

$$\begin{aligned} &= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{\bar{x}_i} \left( F \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x z_\zeta(t_k, \zeta) d\zeta \right)^\top \\ &\quad \left( F \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x z_\zeta(t_k, \zeta) d\zeta \right) dx \\ &\quad + \frac{1}{2} \sum_{i=0}^{N-1} \int_{\bar{x}_i}^{x_{i+1}} \left( F \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x z_\zeta(t_k, \zeta) d\zeta \right)^\top \\ &\quad \left( F \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x z_\zeta(t_k, \zeta) d\zeta \right) dx \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{\bar{x}_i} (z(t_k, x) - z(t_k, \bar{x}_i))^\top F^\top \\ &\quad F (z(t_k, x) - z(t_k, \bar{x}_i)) dx \\ &\quad + \frac{1}{2} \sum_{i=0}^{N-1} \int_{\bar{x}_i}^{x_{i+1}} (z(t_k, x) - z(t_k, \bar{x}_i))^\top F^\top \\ &\quad F (z(t_k, x) - z(t_k, \bar{x}_i)) dx \end{aligned} \quad (\text{C.8})$$

Using Wirtinger's inequality [24] with  $d - c \leq \bar{b}/2$  on each integral term, the above (C.8) becomes

$$\begin{aligned} &\frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (F\vartheta)^\top (F\vartheta) dx \\ &\leq \frac{1}{2} \frac{\bar{b}^2}{\pi^2} \sum_{i=0}^{N-1} \left[ \int_{x_i}^{\bar{x}_i} z_x^\top(t_k, x) F^\top F z_x(t_k, x) \right. \\ &\quad \left. + \int_{\bar{x}_i}^{x_{i+1}} z_x^\top(t_k, x) F^\top F z_x(t_k, x) dx \right] \\ &= \frac{\bar{b}^2}{2\pi^2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} z_x^\top(t_k, x) F^\top F z_x(t_k, x) dx \\ &= \frac{\bar{b}^2}{2\pi^2} \int_0^L z_x^\top(t_k, x) F^\top F z_x(t_k, x) dx \end{aligned} \quad (\text{C.9})$$

The above inequality (C.9) involves the upper bound of the space discretization error. Combining (C.7) and (C.9), we get

$$\begin{aligned} &\int_0^L (F\vartheta)^\top e^{-2\mu x} \Theta_1 z dx \\ &\leq \frac{\bar{b}^2}{2\pi^2} \int_0^L z_x^\top(t_k, x) F^\top F z_x(t_k, x) dx \\ &\quad + \frac{1}{2} \int_0^L e^{-4\mu x} z^\top \Theta_1^\top \Theta_1 z dx \end{aligned} \quad (\text{C.10})$$

Then substituting (C.10) into (C.6), we get

$$\dot{V}(z) \leq -2\sigma V_1(z) - 2\sigma V_2(z) + \int_0^L \eta^\top W(x) \eta dx$$

$$\begin{aligned}
& + \gamma \|\varpi(s, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
& + \beta \int_0^L z_x^\top e^{-2\mu x} \Theta_2 z_x dx \\
= & -2\sigma V_1(z) - (2\sigma - \beta)V_2(z) \\
& + \int_0^L \eta^\top W(x) \eta dx + \gamma \|\varpi(s, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2
\end{aligned} \tag{C.11}$$

with  $\sigma, \eta, W(x)$  defined below (23).

The proof of step 2 is complete. ■