STOCHASTIC REDUCED ORDER MODELS FOR BAYESIAN ESTIMATION PROBLEMS IN FLUID MECHANICS

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G. Le Pape, M. Ladvig, A. M. Picard,
V. Resseguier, D. Heitz & G. Stabile

CONTENT

PART I

CONTEXT :

OBSERVER FOR WIND TURBINE APPLICATIONS

CONTEXT
Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow
using strongly-limited computational resources & few sensors

Wind turbine blade

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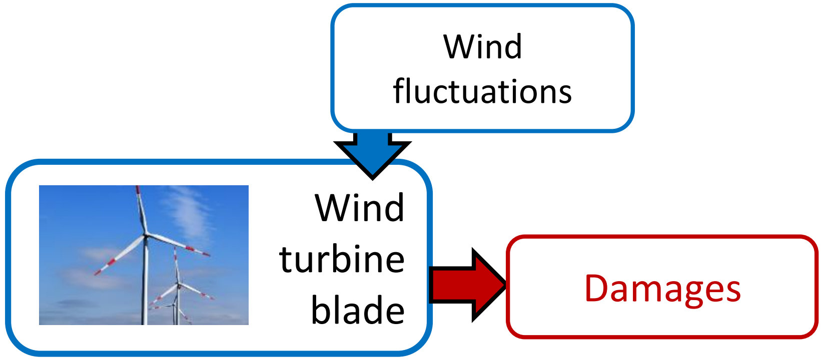

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Estimation and prediction:
- Air flow
- Lift, drag, inflow
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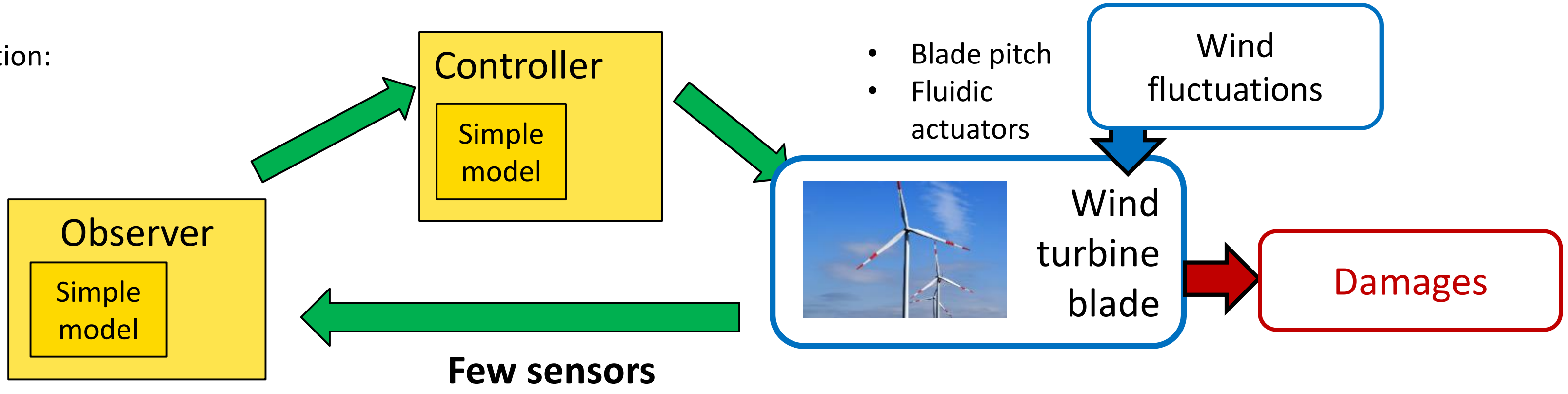

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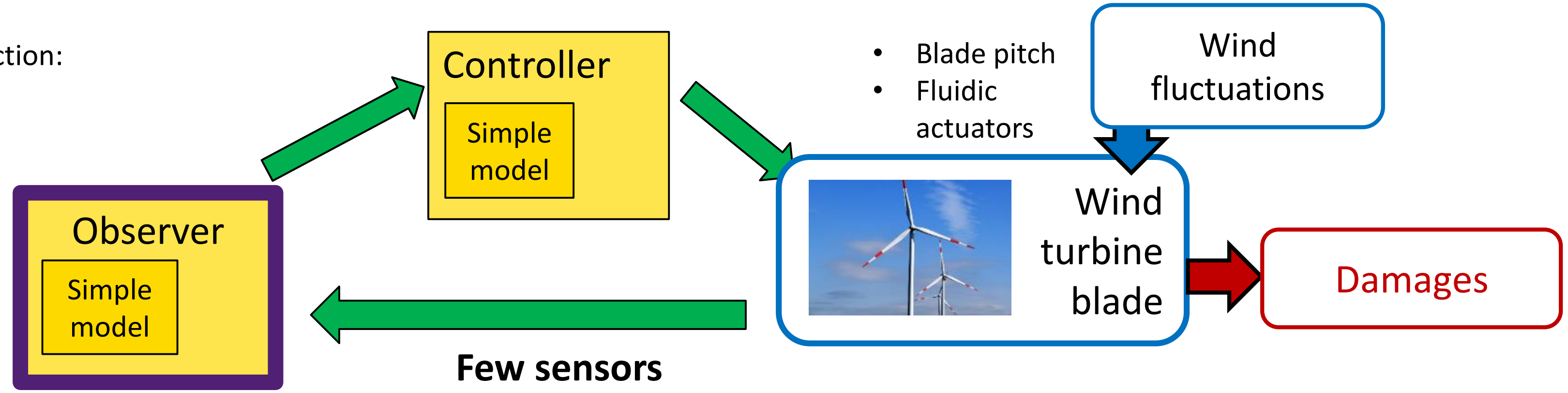

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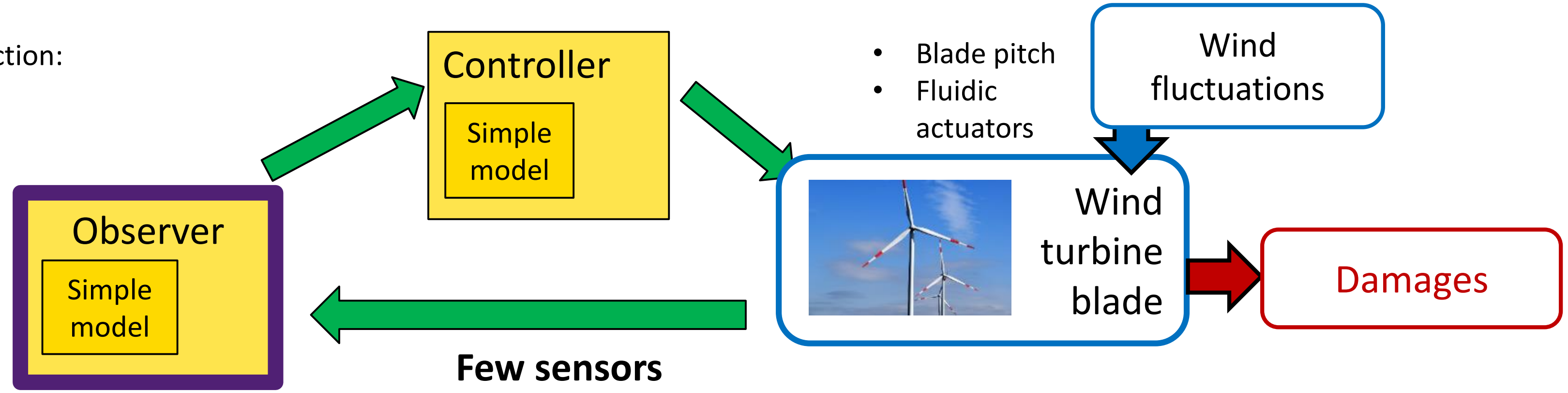

Which simple model?   How to combine model & measurements?

SCALIAN

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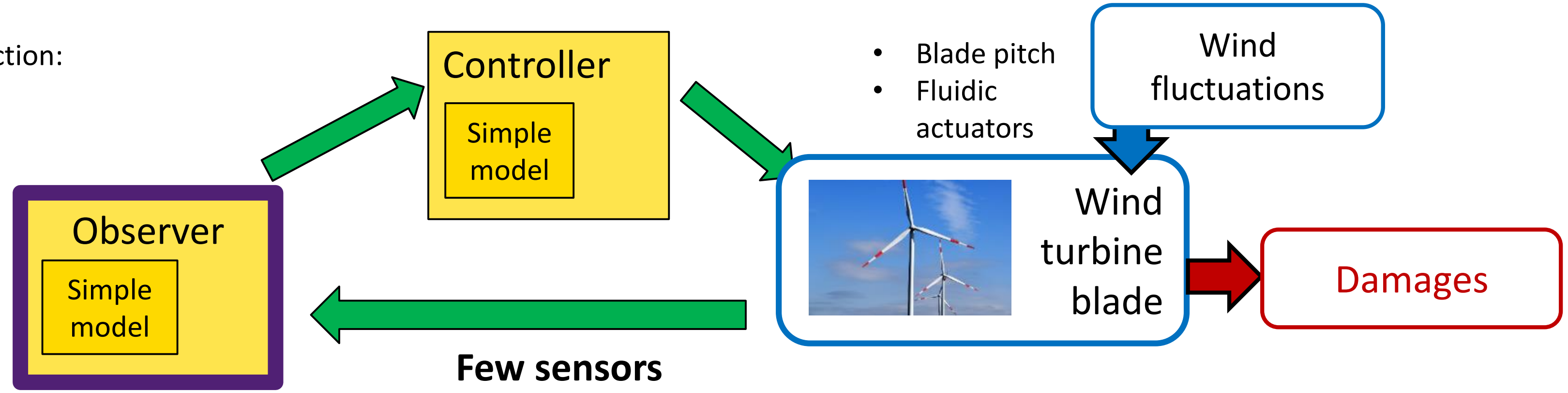

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Scientific problem :

Simulation & data assimilation under severe dimensional reduction

typically, $10^7 \rightarrow O(10)$ degrees of freedom

SCALIAN

PART II

STATE OF THE ART

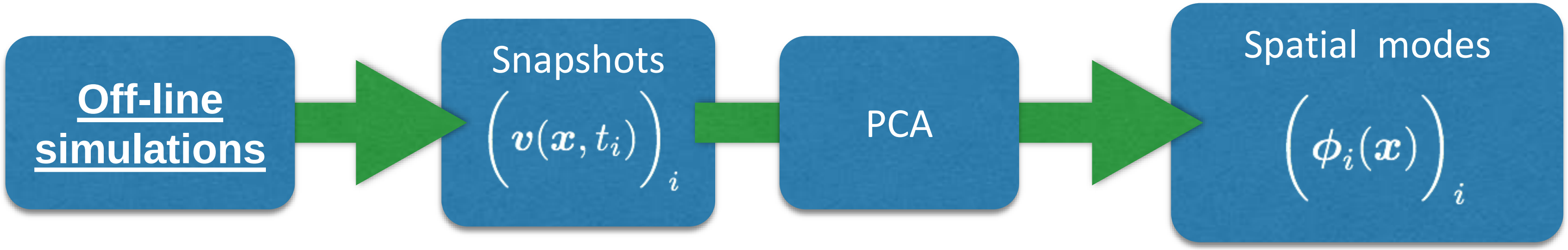
a. Intrusive reduced order model (ROM)

b. Data assimilation

INTRUSIVE REDUCED ORDER MODEL (ROM)

Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:



- Approximation:

$$v(x,t) \approx \sum_{i=0}^{n} b_i(t)\ \phi_i(x)$$

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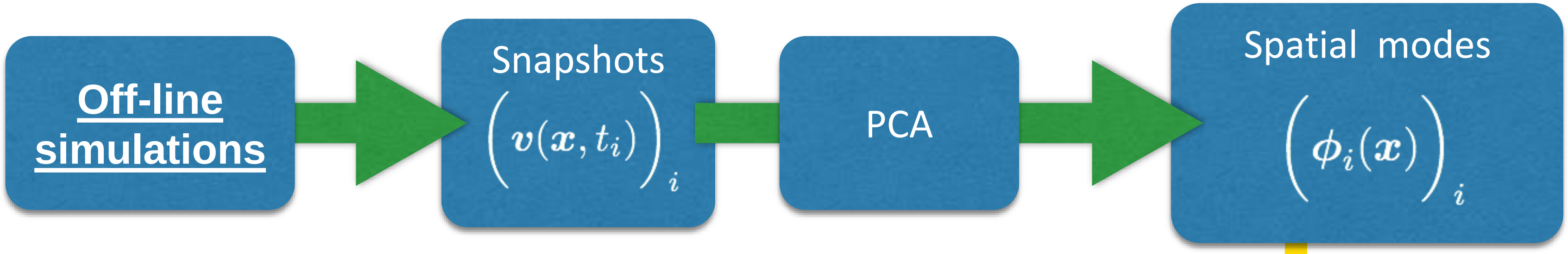

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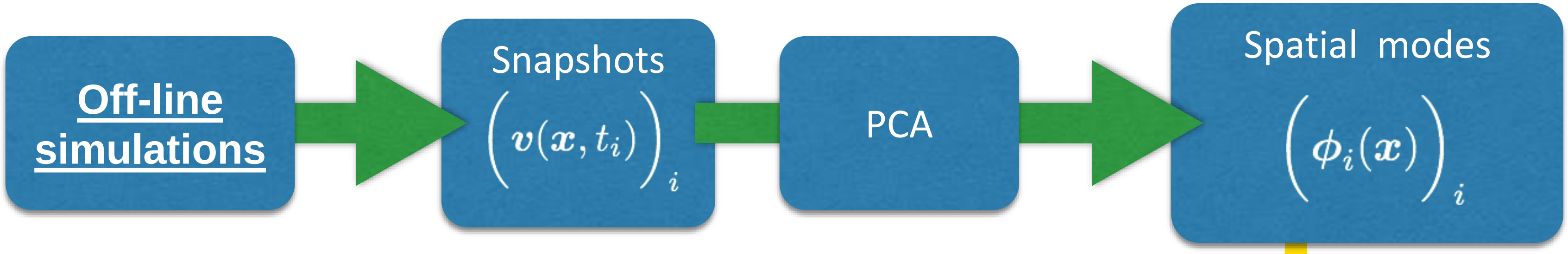

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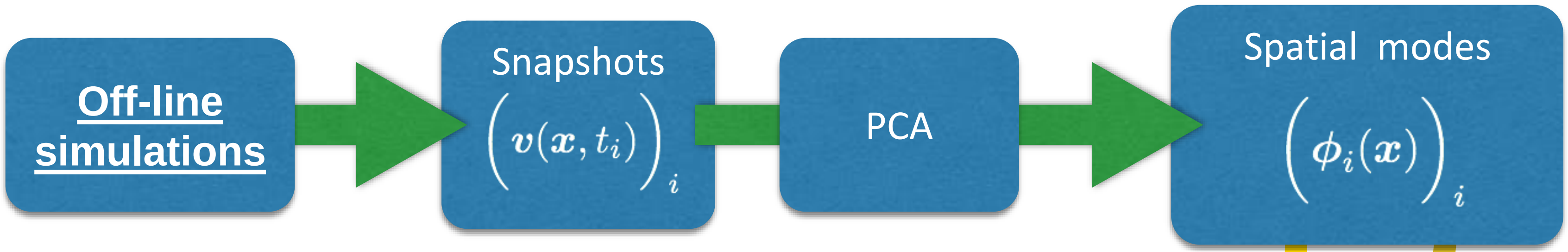

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$$\int_{\Omega} dx\, \phi_i(x) \cdot \ (\textit{Physical equation} \text{ (e.g. Navier-Stokes)})$$

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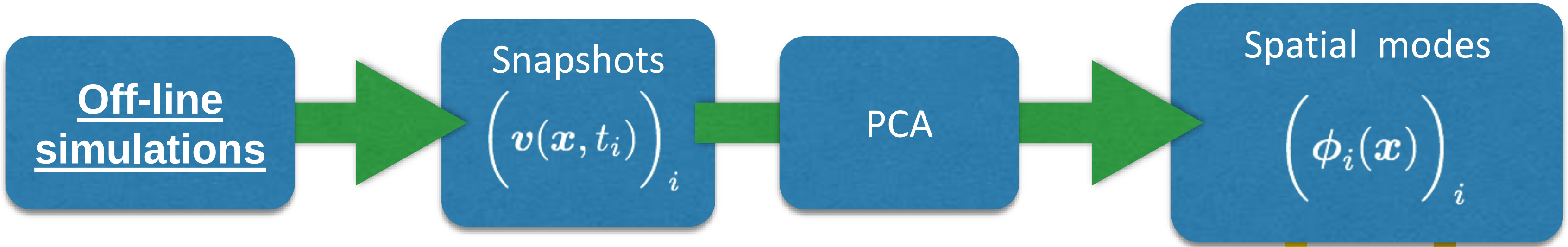

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SPOILER

$\int_\Omega dx\, \phi_i(x) \cdot$ (Randomized Navier-Stokes)

→ **ROM for very fast simulation of temporal modes**

DATA ASSIMILATION
= Coupling simulations and measurements y

Numerical Simulation (ROM)

→ erroneous

On-line measurements

→ incomplete
→ possibly noisy

Velocity

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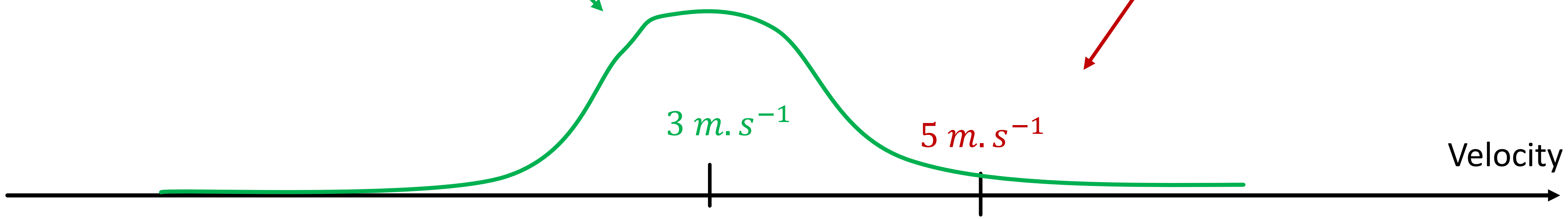

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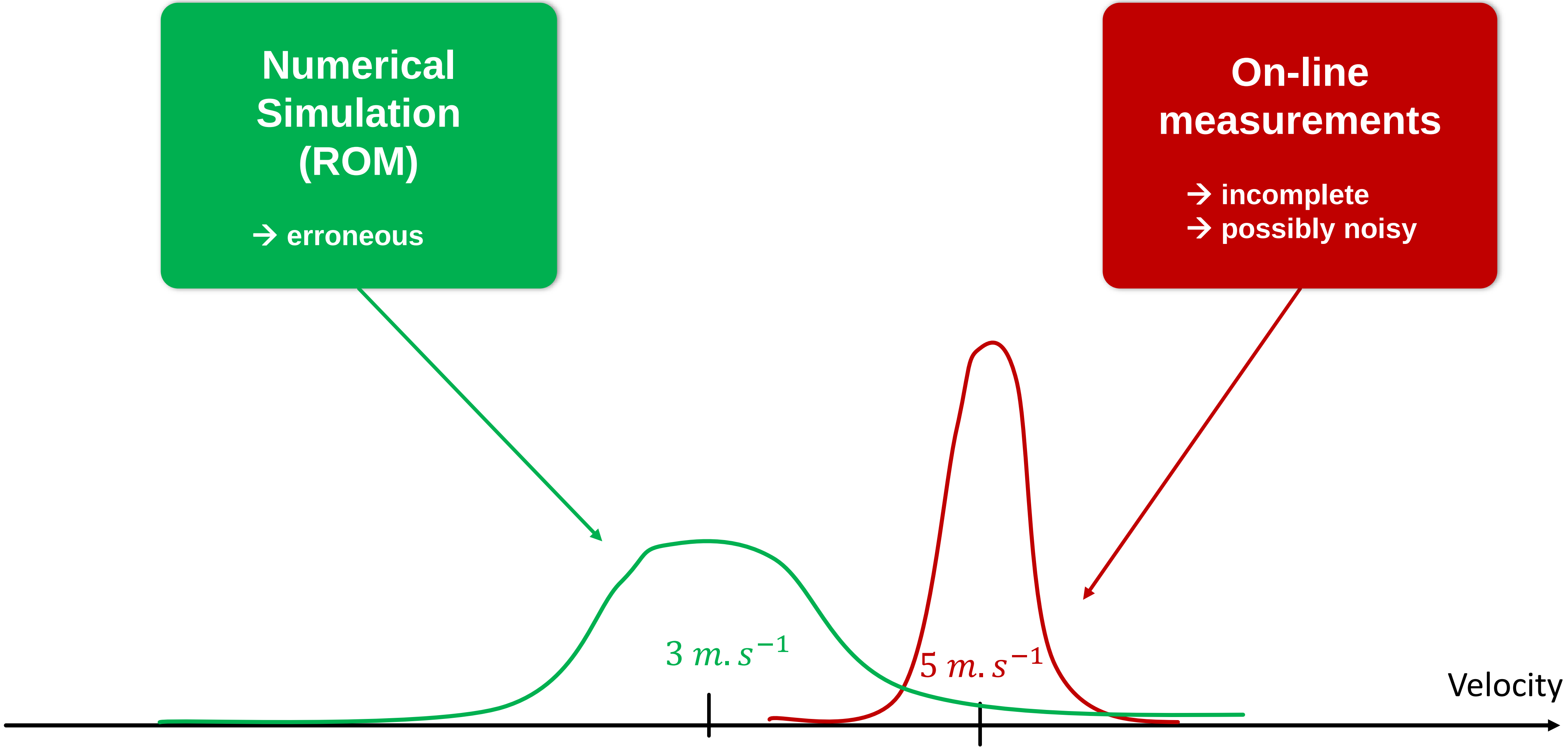

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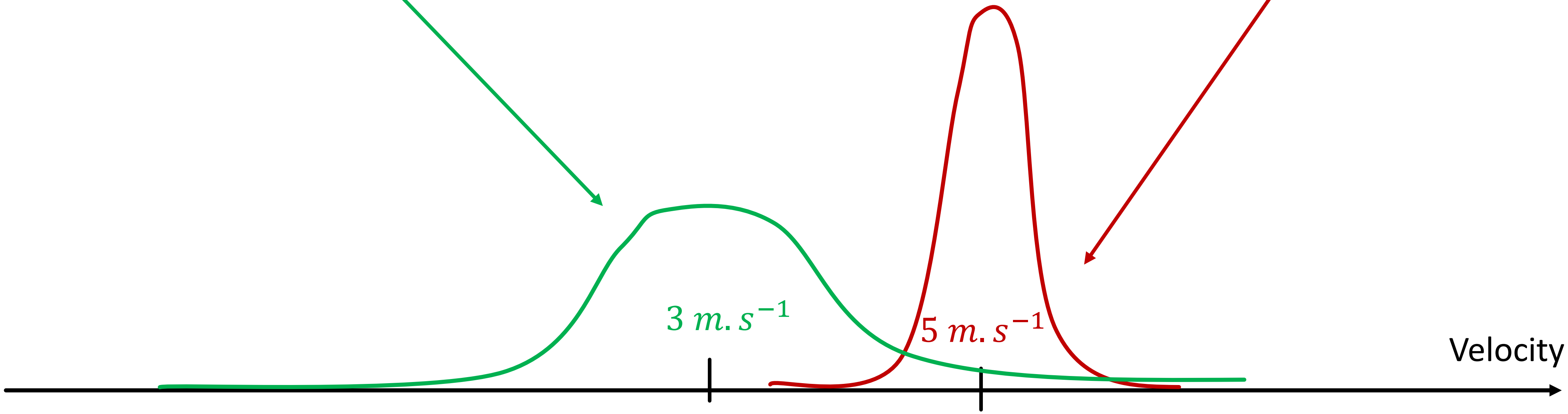

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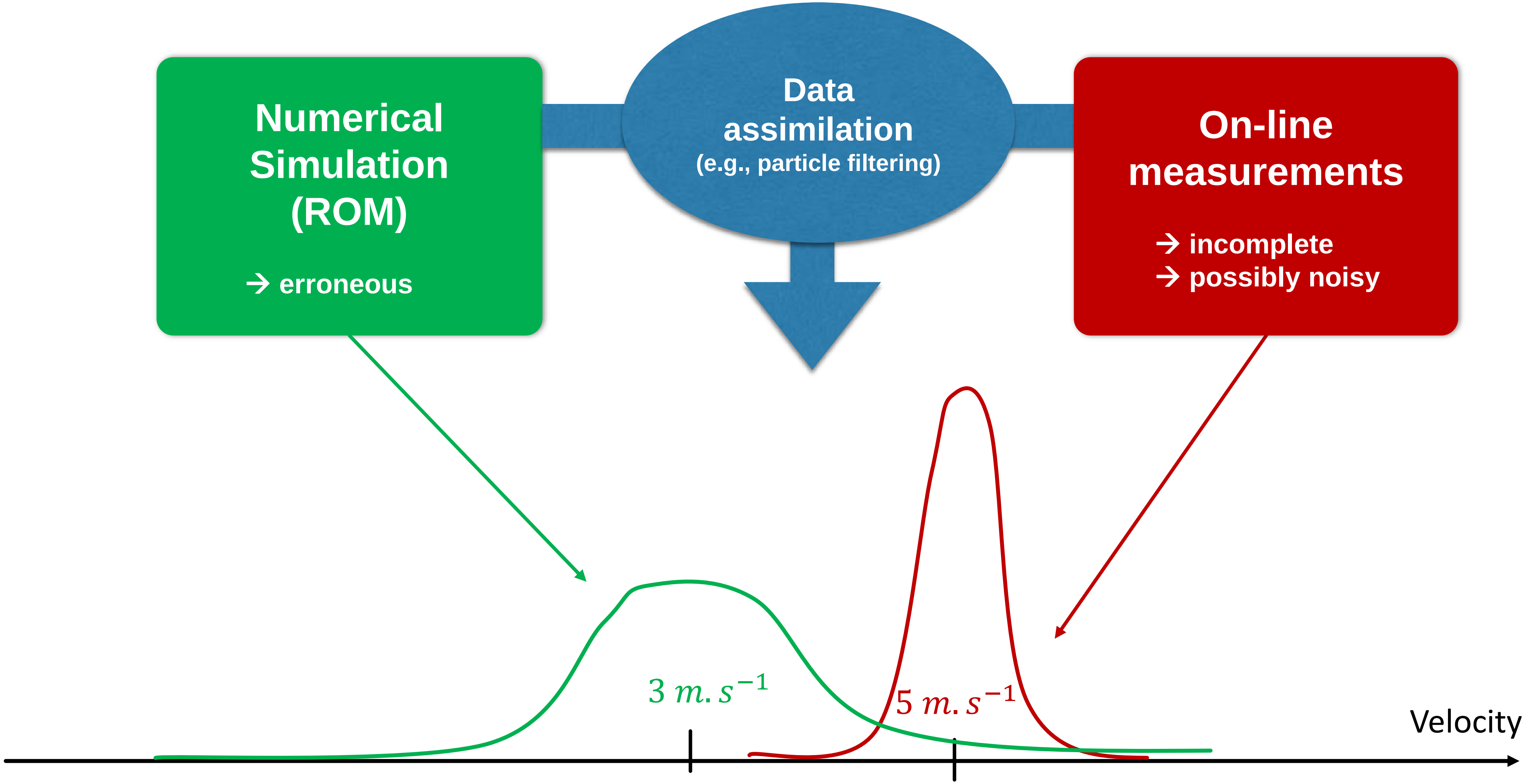

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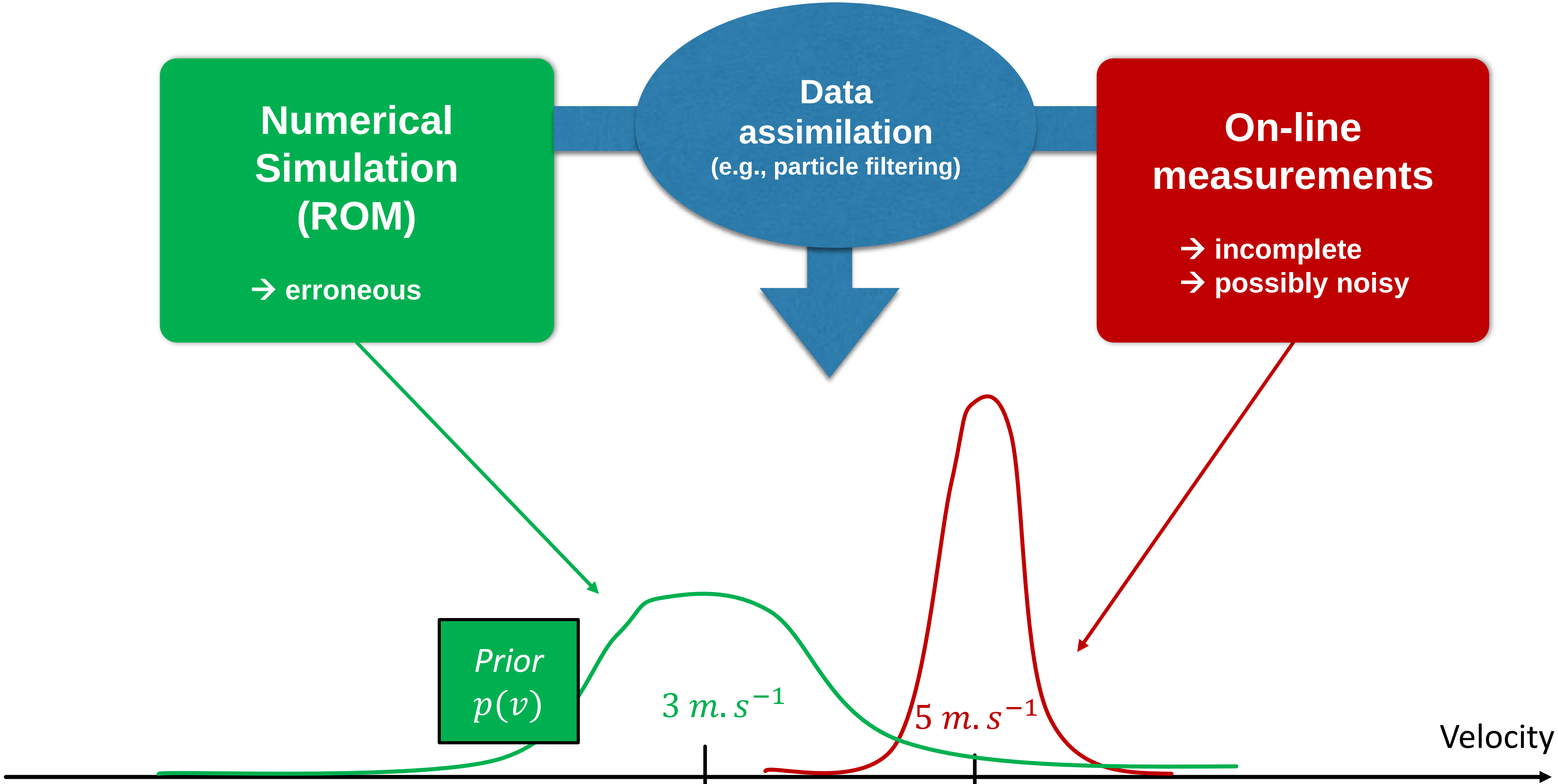

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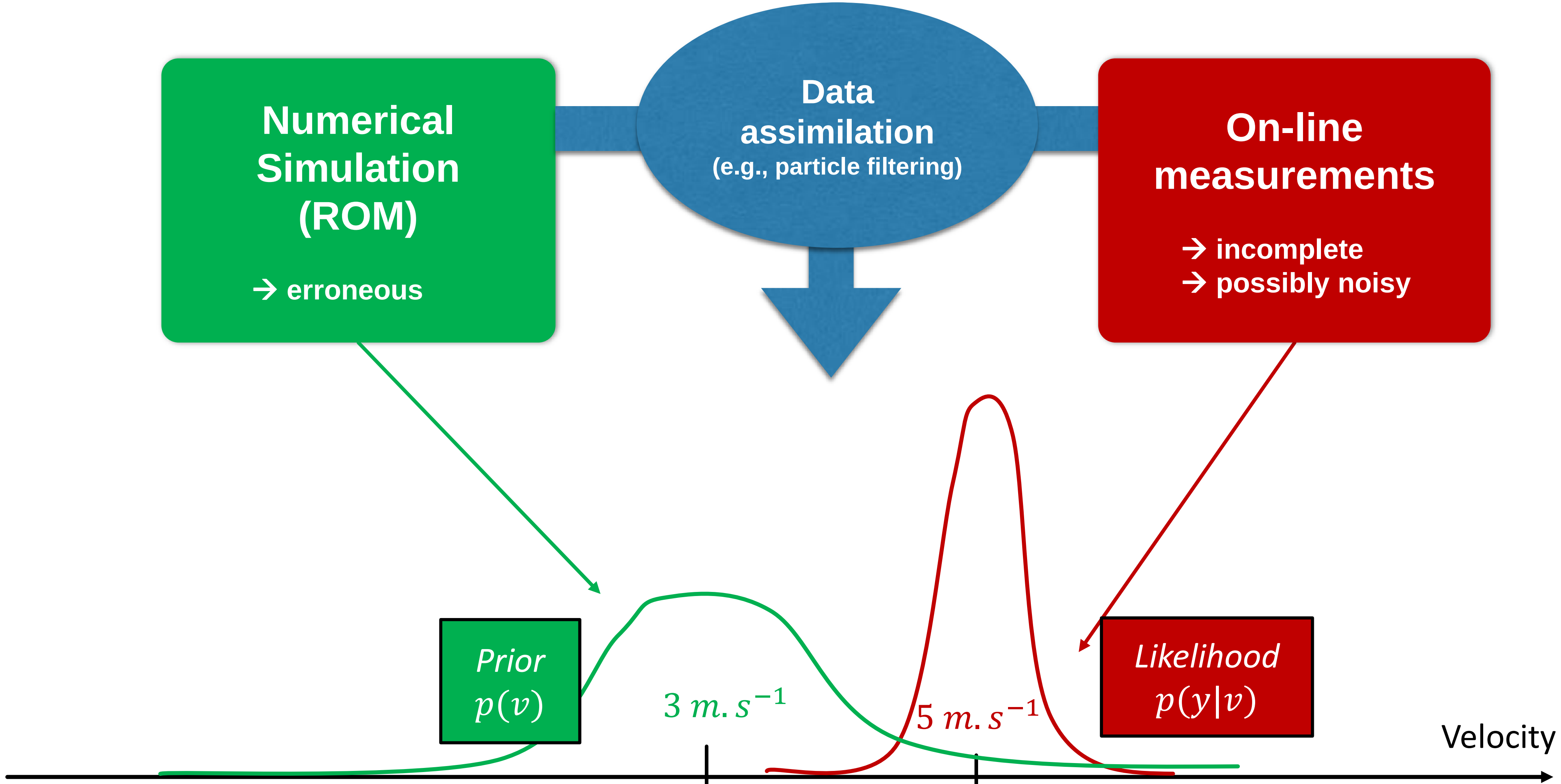

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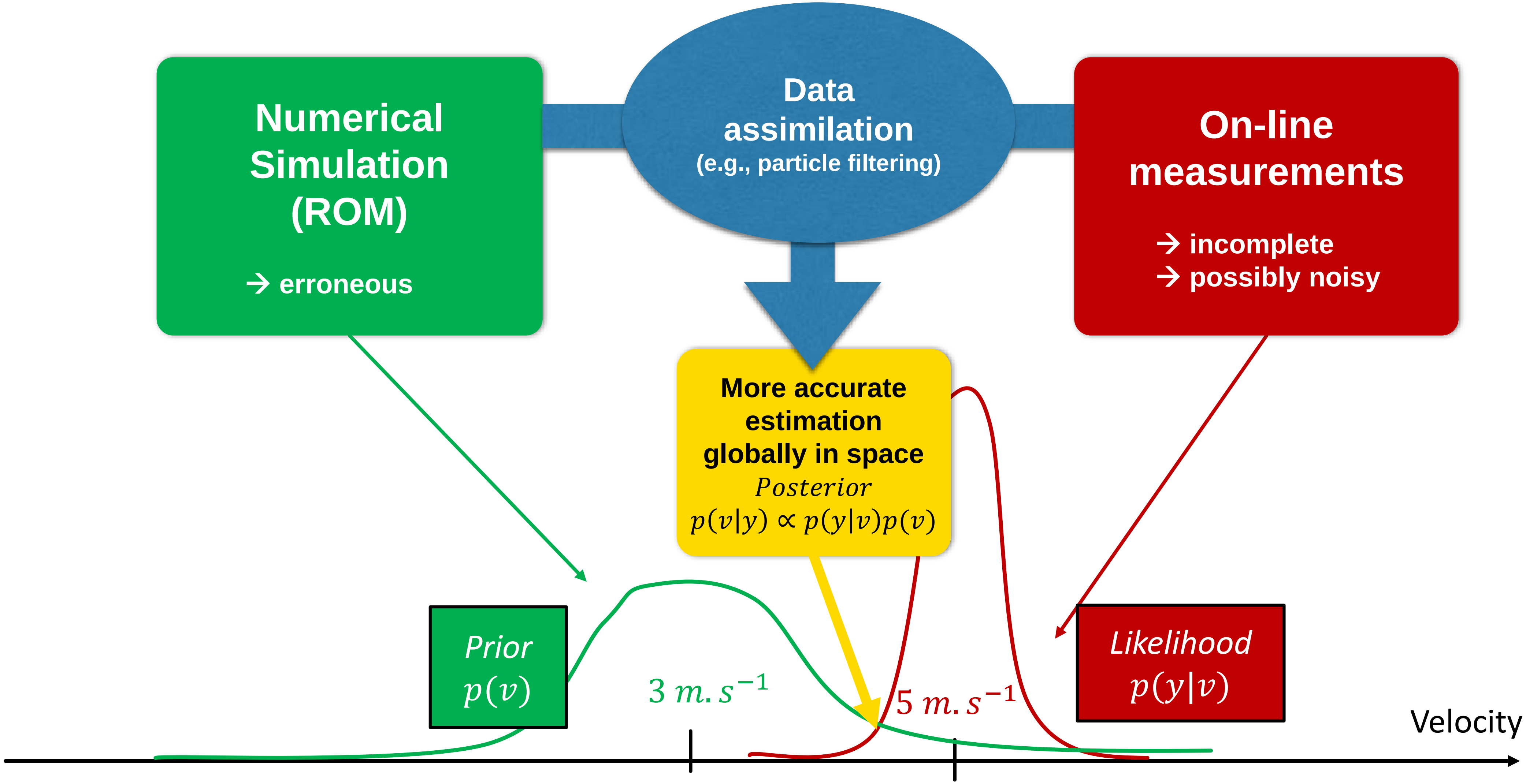

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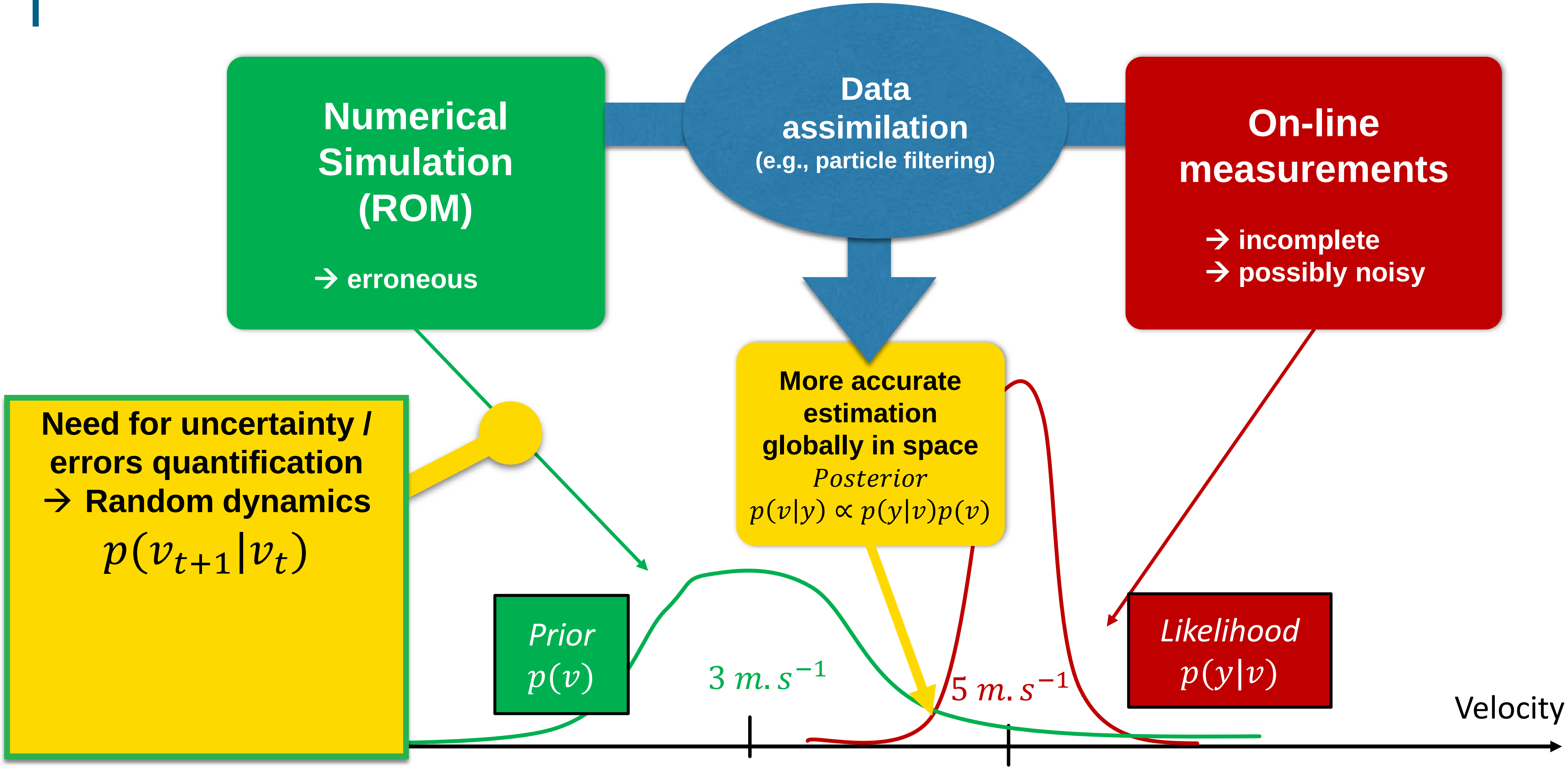

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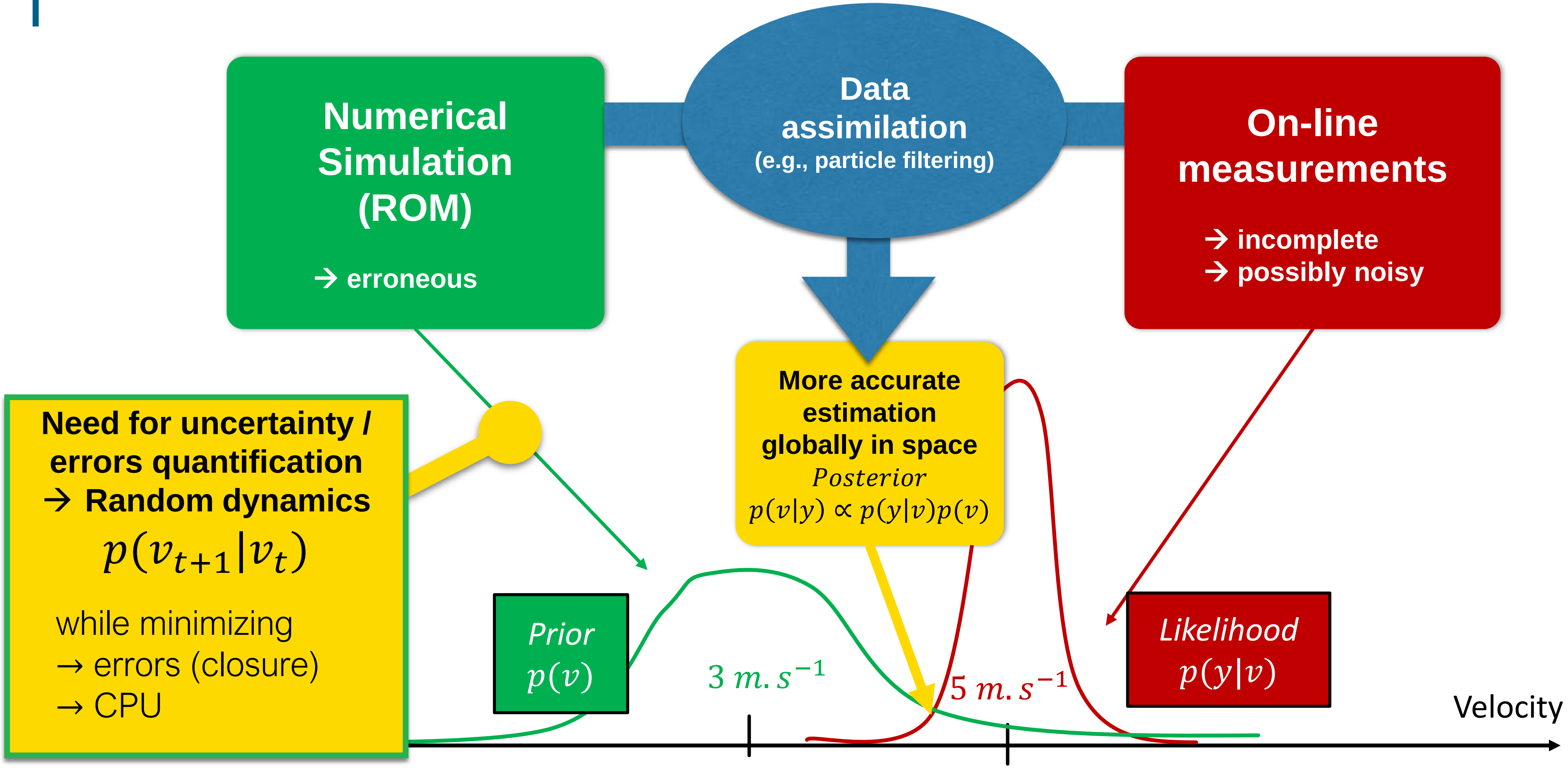

PART III

REDUCED LOCATION UNCERTAINTY MODELS

a. Location uncertainty models (LUM)

b. Reduced LUM (Red LUM)

LOCATION UNCERTAINTY MODELS (LUM)

$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
$v' = \sigma \dot{B}$

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

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Navier-Stokes model

- Good closure
- Good model error quantification

for data assimilation

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Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LUM

Memin, 2014
Resseguier et al. 2017 a, b, c, d
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019
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SALT

Holm, 2015
Holm and Tyranowski, 2016
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LOCATION UNCERTAINTY MODELS (LUM),
Randomized Navier-Stokes

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Resolved fluid velocity:
w

Unresolved fluid velocity:
$v' = \frac{\sigma dB_t}{dt}$ (Gaussian, white wrt t)

(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Momentum conservation

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(w(t, X_t)\right) = F \text{ (Forces)}$$

Positions of fluid parcels X_t :

$$\frac{d}{dt} X_t = w(t, X_t) + \underbrace{\sigma(t, X_t) \frac{dB_t}{dt}}_{\text{Gaussian process white in time}}$$

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

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Variance tensor:
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From Ito-Wentzell formula (Kunita 1990) with Ito notations

$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

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Advection — "Turbulent" diffusion — Forces

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Skew-symmetric multiplicative random forcing

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Advection | “Turbulent” diffusion | Forces

$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

Usual terms

Skew-symmetric multiplicative random forcing

Balanced energy fluxes

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

LOCATION UNCERTAINTY MODELS (LUM),
Randomized Navier-Stokes

SCALIAN

$v = w + v'$

Resolved fluid velocity:
w

Unresolved fluid velocity:
$v' = \sigma \dot{B}$ (Gaussian, white wrt t)

Symmetric negative

From Ito-Wentzell formula (Kunita 1990) with Ito notations

Advection | Turbulent Diffusion | Forces

$$\partial_t w + C(w,w) + C(\sigma \dot{B}, w) + F(w) = F$$

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REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

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Reduced order : $n \sim 10$

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2nd order polynomial

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PCA modes

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Multiplicative skew-symmetric noise

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(n+1) x (n+1)

n x 1

M x 1

$$\frac{d}{dt} b(t) = \boxed{H\big(b(t)\big)} + \boxed{K\left(\boxed{\sigma\dot{B}}\right)}\, b(t)$$

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Multiplicative skew-symmetric noise

2nd order polynomial

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}} \frac{\Delta b_i}{\Delta t}\, v'}\right]$$

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2nd order polynomial

Multiplicative skew-symmetric noise

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REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

$v = w + v'$

Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \frac{\sigma dB_t}{dt}$ (Gaussian, white wrt t)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

$$\int_\Omega \phi_i(x) \cdot \left(\partial_t w + C(w,w) + F(w) + C(\sigma\dot{B}, w) = F\right) dx$$

(n+1) x (n+1)   n x 1   M x 1

$$\frac{d}{dt} b(t) = \boxed{H\big(b(t)\big)} + \boxed{K\big(\sigma\dot{B}\big)}\, b(t)$$

Multiplicative skew-symmetric noise

2nd order polynomial

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\, \frac{\Delta b_i}{\Delta t}\, v'}\right]$$

Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t\, \overline{v'\,(v')^T}$

Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$

$\overline{f} = \frac{1}{T}\int_0^T f$

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Randomized Navier-Stokes

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$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$

$\overline{f} = \frac{1}{T}\int_0^T f$

New estimator

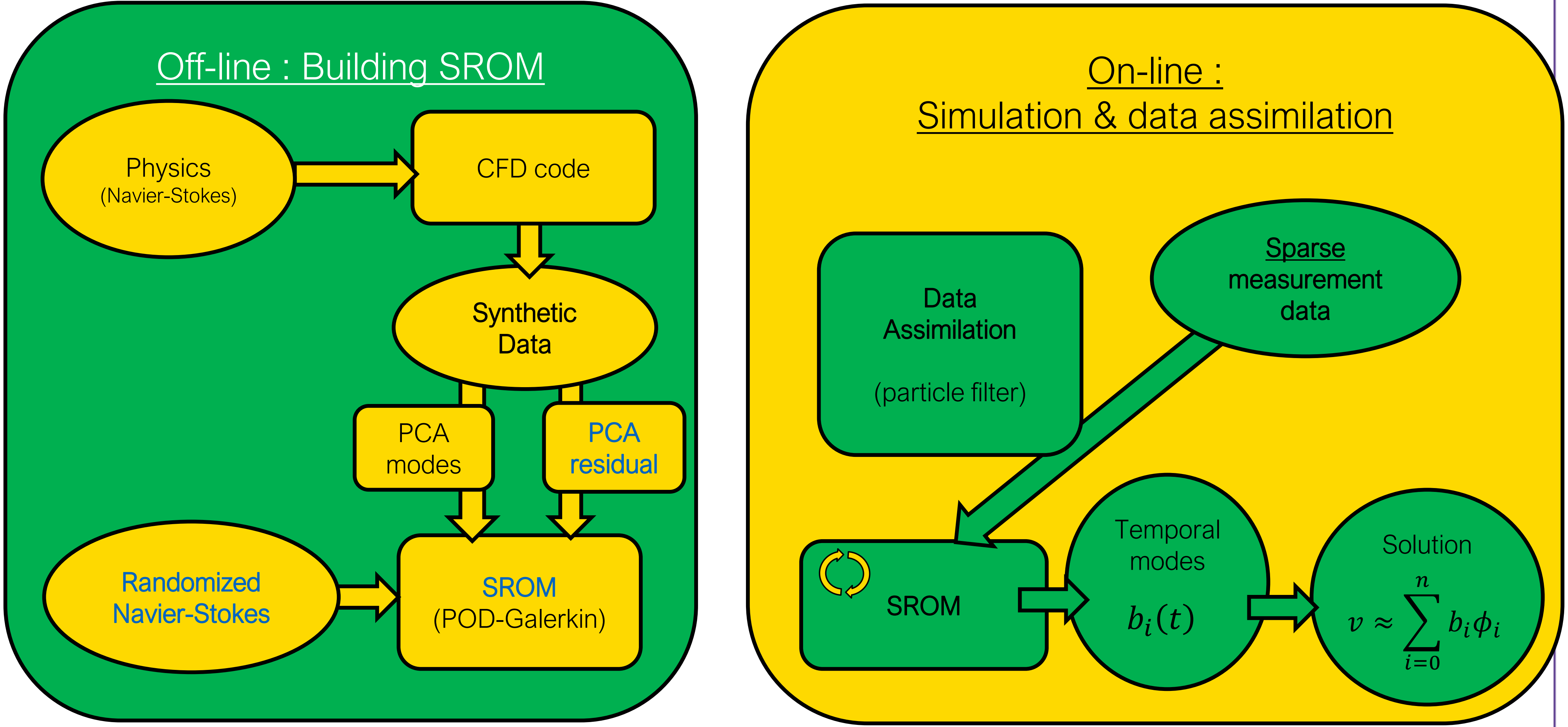
- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based

→ Robustness in extrapolation

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

SUMMARY

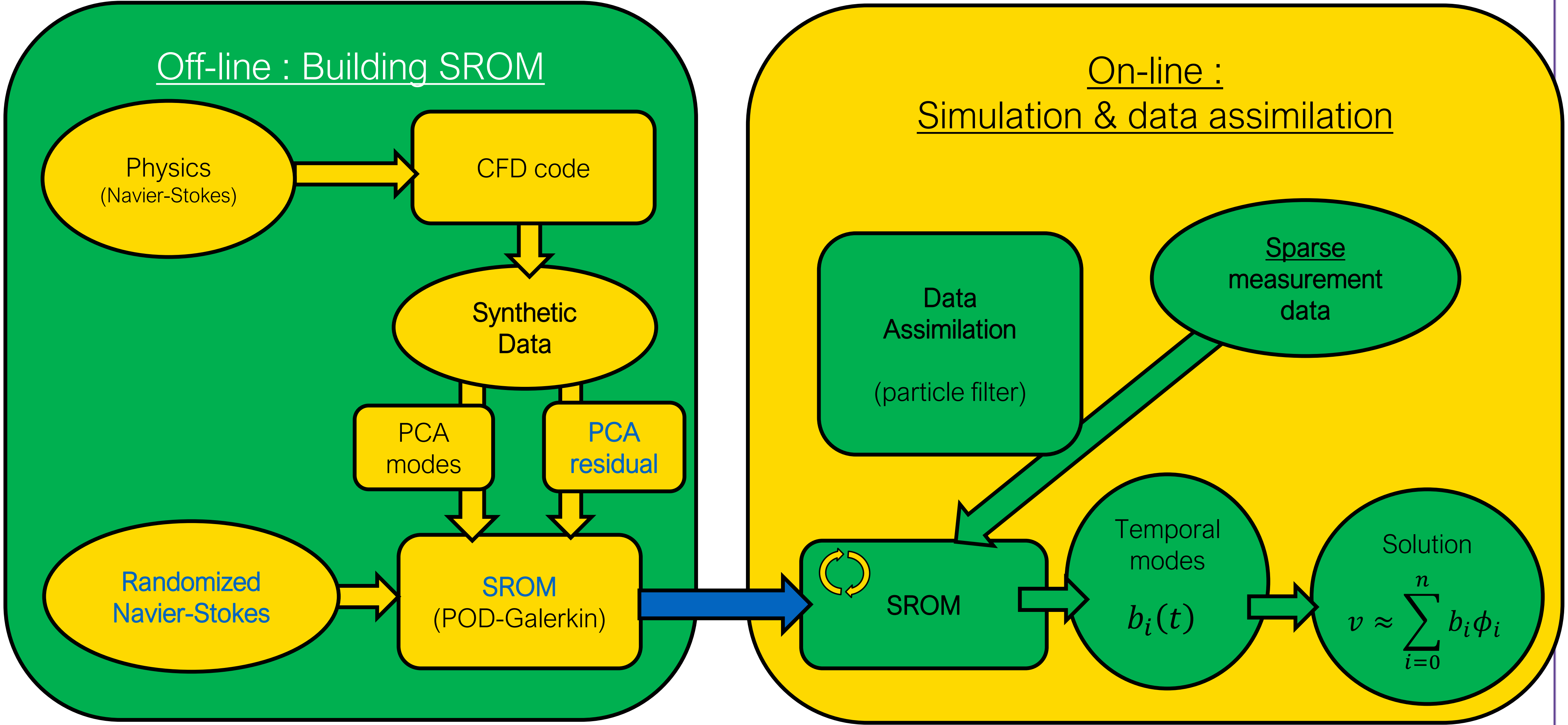
Stochastic ROM + Data assimilation



Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

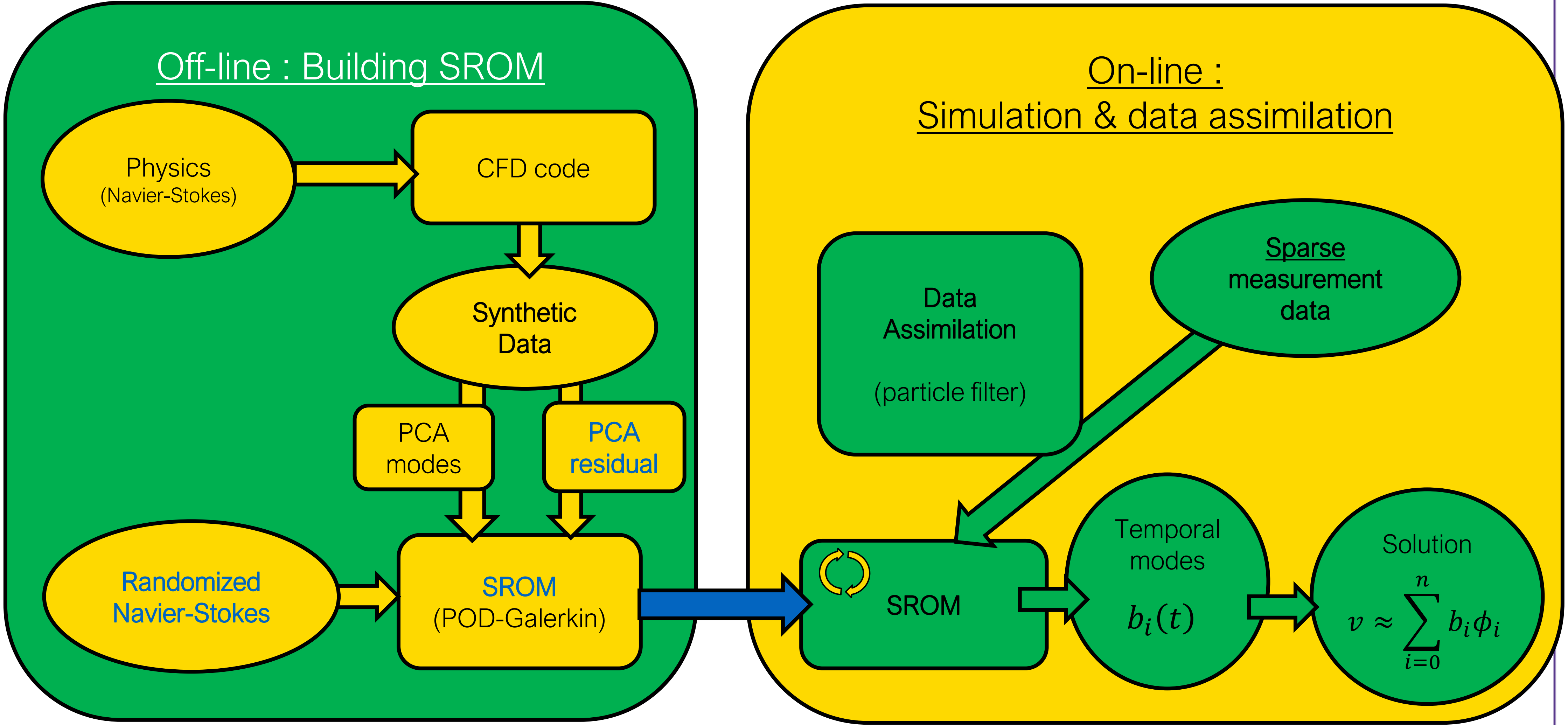
SUMMARY

Stochastic ROM + Data assimilation



Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

SUMMARY
Stochastic ROM + Data assimilation



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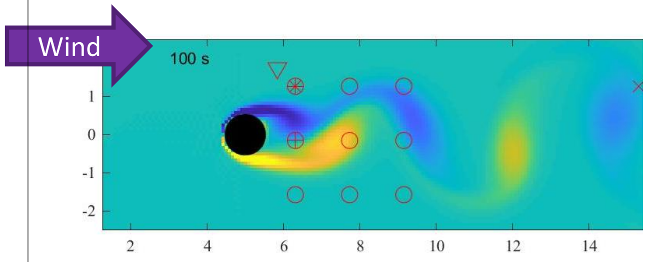
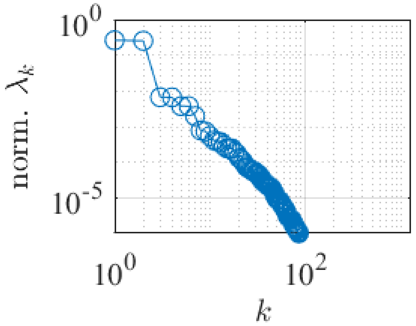
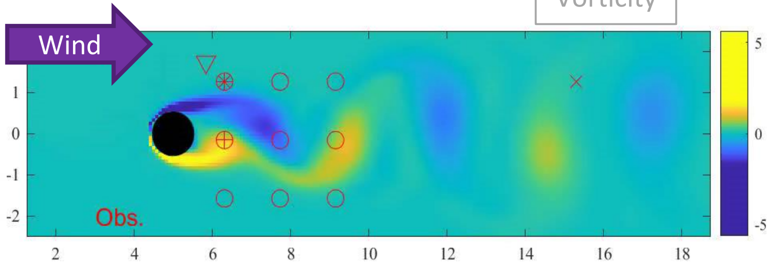
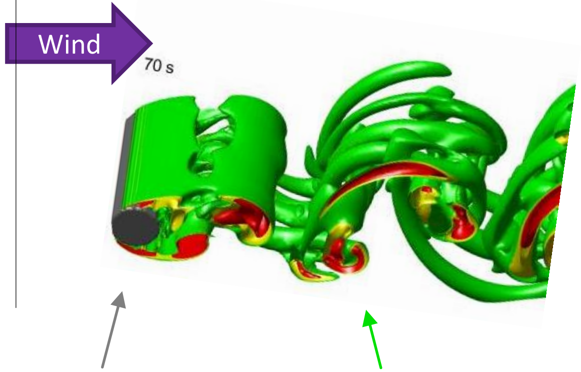
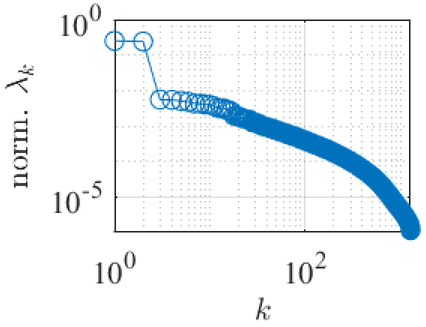
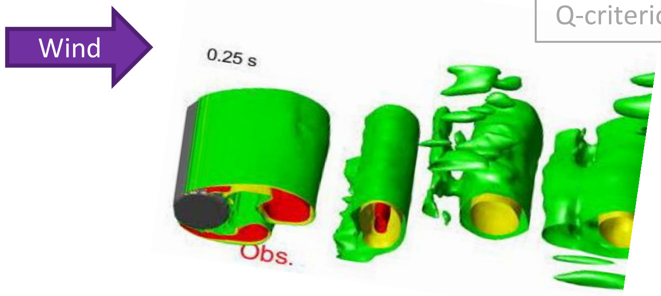
PART IV

NUMERICAL RESULTS

a. Test cases
b. Data assimilation

TEST CASES

From 10^7 to 8 degrees of freedom

	Full-order reference	POD eigenvalues	Reduced-order reference PCA-projection of the full-order simulation (Optimal from 8-dof linear decomposition)
Easy case Reynolds number (Re) = 100 2D (10^4 dof)			
Difficult case Reynolds number (Re) = 300 3D (10^7 dof)			

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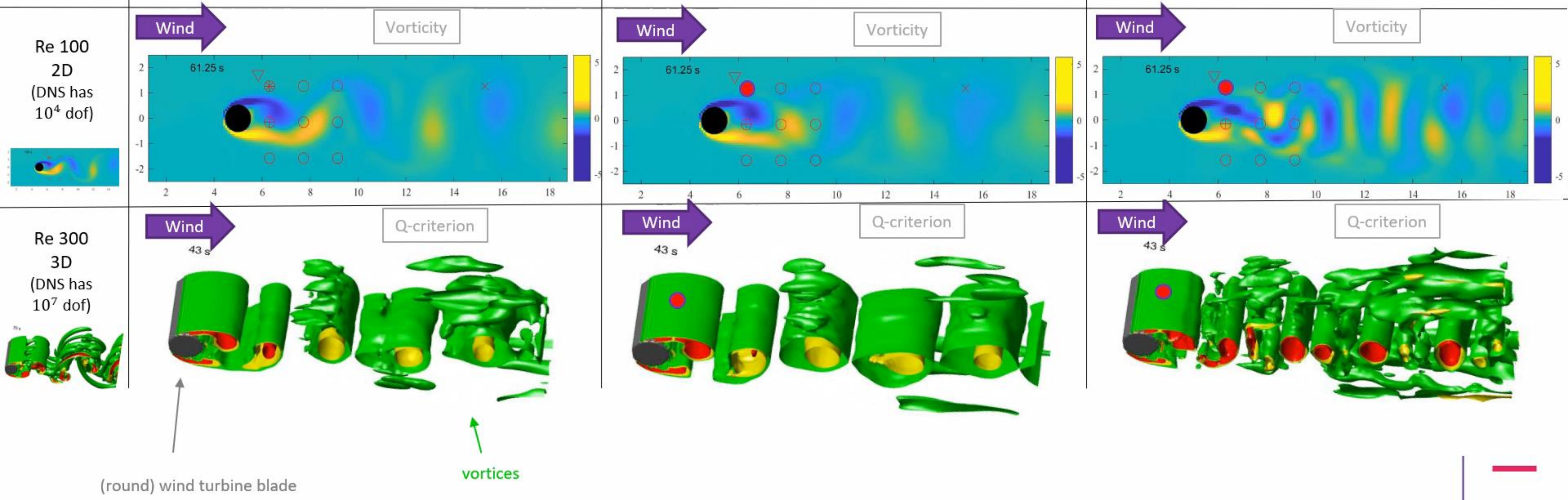
DATA ASSIMILATION
On-line estimation of the solution

From 10^7 to 8 degrees of freedom
Single measurement point (blurred & noisy velocity)

	Reference	**Our method**	**State-of-the-art**
	PCA-projection of the DNS (Optimal from 8-dof linear decomposition)	POD-Galerkin with Navier-Stokes under location uncertainty (LUM)	POD-Galerkin with Navier-Stokes + optimally tuned eddy viscosity & additive noise



Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

DATA ASSIMILATION

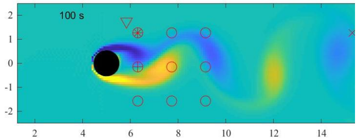
Error on the solution estimation

$$v = w + v'$$

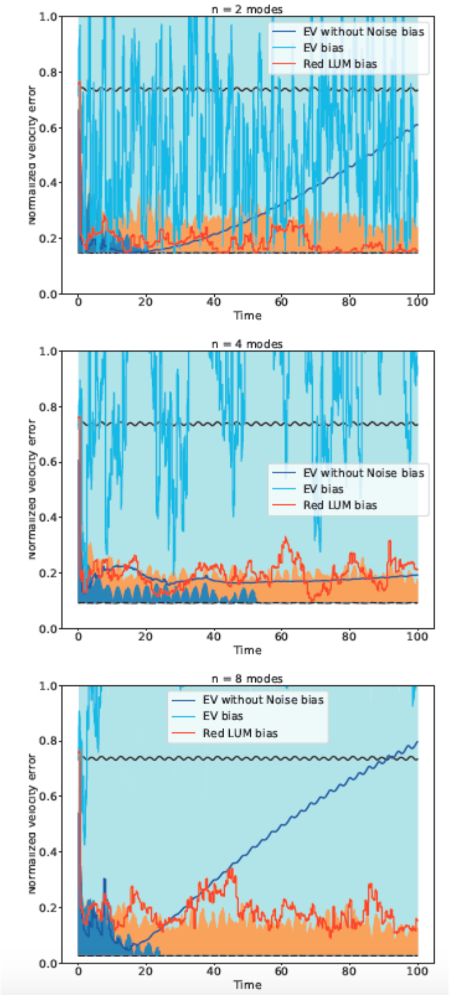
Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

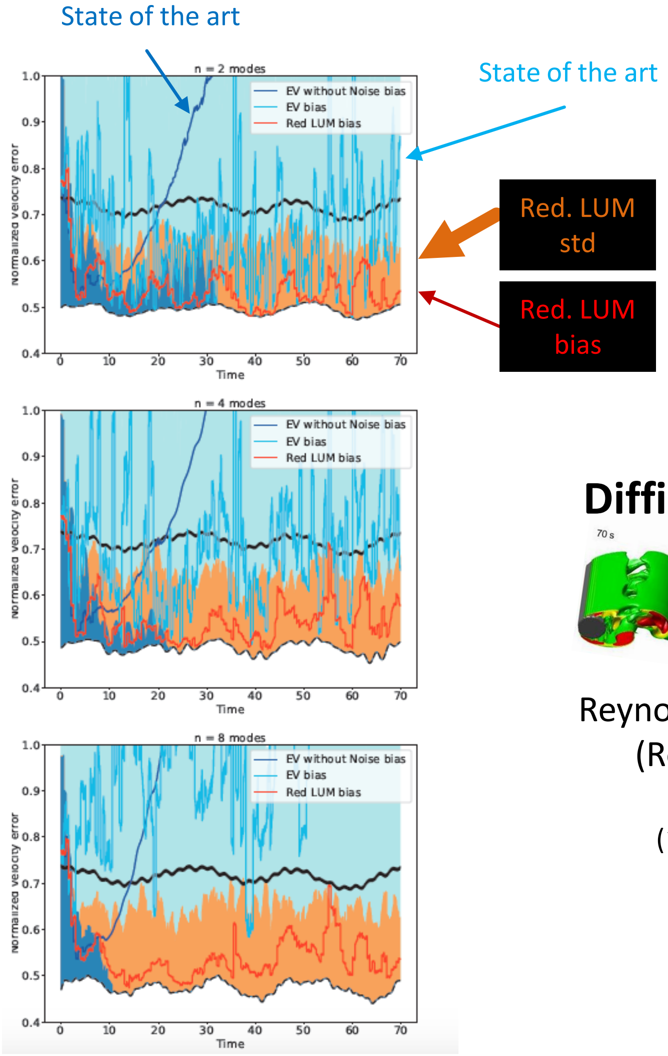
Unresolved fluid velocity:
v'

Easy case

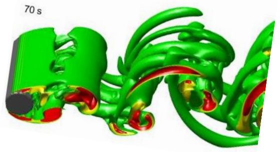


Reynolds number
(Re) = 100
2D
(10^4 dof)





Difficult case



Reynolds number
(Re) = 300
3D
(10^7 dof)

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CONCLUSION

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CONCLUSION

- Intrusive ROM : for very fast and robust CFD $(10^7 \rightarrow 8$ degrees of freedom.)
  - Closure problem handled by LUM
  - Efficient estimator for the multiplicative noise
  - Efficient generation of prior / Model error quantification
  - Now implemented in ITHACA-FV
- Data assimilation (Bayesian inverse problem) :
  to correct the fast simulation on-line by incomplete/noisy measurements
- First results
  - Optimal unsteady flow estimation/prediction in the whole spatial domain (large-scale structures)
  - Robust far outside the training set

NEXT STEPS

- Real measurements
- Parametric ROM (unknown inflow)
- Increasing Reynolds
  (ROM of (non-polynomial) turbulence models)

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