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On Geodesically Reversible Finsler Manifolds

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Abstract. A Finsler manifold is said to be geodesically reversible if the reversed curve of any geodesic remains a geometrical geodesic. Well-known examples of geodesically reversible Finsler metrics are Randers metrics with closed 1-forms. Another family of well-known examples are projectively flat Finsler metrics on the 2-sphere that have constant positive curvature. In this paper, we prove some geometrical and dynamical characterizations of geodesically reversible Finsler metrics, and we prove several rigidity results about a family of so-called Randers-type Finsler metrics. One of our results is as follows: let \sqrt{g} be a Riemannian Finsler metric on a closed surface Σ , and p be a small antisymmetric potential on Σ that is a natural generalization of 1-form (see Section 1). If the Randers-type Finsler metric $\sqrt{g} - p$ is geodesically reversible, and the geodesic flow of \sqrt{g} is topologically transitive, then we prove that p must be a closed 1-form. We also prove that this rigidity result is not true for the family of projectively flat Finsler metrics on the 2-sphere of constant positive curvature.

Keywords: Finsler metric; flip map; geodesic flows; Anosov flows

AMS Mathematics subject classification: 37A35; 34D20; 37D35; 37D40

1 Introduction

Let M be a C^∞ manifold, $\pi : TM \rightarrow M$ the tangent bundle and $TM_0 = TM - \{0\}$. A function $F : TM \rightarrow \mathbb{R}^+$ is called a *Finsler metric* on M if it

has the following properties :

- (a) $F(tv) = tF(v)$ for all $v \in TM$ and $t \geq 0$;
- (b) F^2 is strictly positive and C^∞ on TM_0 ;
- (c) in standard local coordinates (x_i, y_i) on TM , the matrix of the partial derivatives $\frac{\partial^2 F^2}{\partial y_i \partial y_j}$ is positive-definite.

The pair (M, F) is called a *Finsler manifold*. We say that (M, F) is *reversible* if $F(-u) = F(u)$ for all $u \in TM$. A Finsler manifold (M, F) is said to be *Riemannian* if for any $x \in M$ there exists an inner product g_x on $T_x M$ such that $F_x(v) = \sqrt{g_x(v, v)}$ for all $v \in T_x M$. Another important class of Finsler metrics are the so-called *Randers metrics*, which are constructed as follows: let \sqrt{g} be a Riemannian Finsler metric and θ be a C^∞ 1-form on M . We define for any $v \in TM$, $F(v) = \sqrt{g(v, v)} - \theta(v)$. It is well-known that F is a Finsler metric if and only if $\sup_{x \in M} \|\theta_x\| < 1$, where $\|\theta_x\|$ denotes the norm of θ_x with respect to g_x (see [11]). We note that Riemannian Finsler metrics are reversible, while Randers metrics are irreversible if θ is not zero.

Given a Finsler manifold (M, F) , we can define a function $\hat{F} : TM \rightarrow \mathbb{R}^+$ such that for all $u \in TM$,

$$\hat{F}(u) = \frac{1}{2}(F(-u) + F(u)).$$

It is proved in [9] that \hat{F} is also a Finsler metric on M , which is moreover reversible. We will call \hat{F} the *associated reversible Finsler metric* of F .

A function $p : TM \rightarrow \mathbb{R}$ is called an *antisymmetric potential* if it has the following properties :

- (a) $p(tu) = tp(u)$ for all $u \in TM$ and $t > 0$;
- (b) p is C^∞ on TM_0 ;
- (c) $p(-u) = -p(u)$ for any $u \in TM$.

We note that important examples of antisymmetric potentials are 1-forms. More generally, given any Finsler manifold (M, F) , we can define $p = \hat{F} - F$, which implies that for all $u \in TM$

$$p(u) = \frac{1}{2}(F(-u) - F(u)).$$

It is easy to see that p is an antisymmetric potential, which will be called the *associated antisymmetric potential* of F . Since $F = \hat{F} - p$, we see that any Finsler metric can be written as the difference between a reversible Finsler metric and an antisymmetric potential.

Definition 1.1. Let (M, F) be a Finsler manifold. We say that (M, F) is *geodesically reversible* if for any geodesic $c : [a, b] \rightarrow M$, its reversed curve

$\bar{c} : [-b, -a] \rightarrow M$ defined as $\bar{c}(t) = c(-t)$ is a geometrical geodesic, i.e. it can be reparametrized to be a geodesic.

Obviously, reversible Finsler manifolds are geodesically reversible. Moreover, we can see that if the associated antisymmetric potential of a Finsler manifold (M, F) is given by a closed 1-form, then (M, F) is geodesically reversible (see Section 4 and [1]). Therefore, one may wonder whether any geodesically reversible Finsler metric F can be written as $F = \hat{F} - \theta$, where θ is a closed 1-form.

Surprisingly, it is not the case for certain positively curved Finsler metrics. In fact, Robert L. Bryant proved in [2] that on the 2-sphere \mathbb{S}^2 , modulo diffeomorphism, there exists a 2-parameter family of Finsler metrics whose Finsler-Gauss curvature is identically 1 and whose geodesic path geometry is projectively flat, i.e. locally equivalent to that of straight lines in the plane (see Section 2). These interesting Finsler metrics are geodesically reversible since they are projectively flat and the reversed curve of a straight line is still geometrically a straight line. Moreover, according to Theorem 10 in [2], any Finsler metric F in this family can be rewritten in the form of $\hat{F} - \theta$, where θ is a 1-form on \mathbb{S}^2 . However, we will prove in Section 2 that θ is rarely closed.

Therefore, the following rigidity question seems natural and interesting : if (M, F) is geodesically reversible, in which case can we prove that its associated antisymmetric potential p must be a closed 1-form ? We note that according to [2], except the Riemannian metric of constant curvature, the above mentioned Finsler metrics on \mathbb{S}^2 are not Randers metrics, which means that their associated reversible Finsler metrics are not Riemannian. Therefore, with respect to our question, it seems appropriate to consider the following family of Finsler metrics.

Definition 1.2. Let (M, F) be a Finsler manifold. We say that (M, F) is of *Randers-type* if its associated reversible Finsler metric is Riemannian.

We note that the Finsler metrics whose associated antisymmetric potentials are 1-forms have been extensively studied (see for example [2] and [1]), while those with general antisymmetric potentials are much less studied. But as we will show in this paper, they are equally interesting and worthy of study. We will firstly prove the following result for Randers metrics.

Theorem 1.3. *Let $F = \sqrt{g} - \theta$ be a Randers Finsler metric on a closed manifold M , such that the 1-form θ is small in the C^2 topology. Then (M, F) is geodesically reversible if and only if θ is closed.*

For Randers-type Finsler metrics on closed surfaces, we will prove the following result.

Theorem 1.4. *Let $F = \sqrt{g} - p$ be a Randers-type Finsler metric on a closed surface Σ , such that p is small in the C^2 topology. If (Σ, F) is geodesically reversible, and the Riemannian geodesic flow of \sqrt{g} is topologically transitive, then p must be a closed 1-form on Σ .*

In any dimension, we will prove the following result about certain deformations.

Theorem 1.5. *Let M be a closed manifold and \sqrt{g} be a hyperbolic Riemannian Finsler metric on M , i.e. of constant sectional curvature -1 . Let F be a Randers-type Finsler metric on M , which is close to \sqrt{g} in the C^4 topology. If (M, F) is geodesically reversible, then the associated antisymmetric potential of F must be a closed 1-form on M .*

This article is organized as follows : in Section 2, we prove some general geometrical and dynamical characterizations of geodesically reversible Finsler metrics. Moreover, we show that the associated antisymmetric potentials of the geodesically reversible Finsler metrics on \mathbb{S}^2 constructed in [2] are rarely closed 1-forms. In Section 3, we define and study the so-called generalized Tonelli Lagrangians and their Euler-Lagrange flows. We prove that the Euler-Lagrange flows are smoothly orbit equivalent to the Finsler geodesic flows, which generalizes a result in [10] about Randers metrics. In Section 4, we determine the tangent vector fields of the Euler-Lagrange flows associated to Randers-type Finsler metrics. Finally in Section 5, we give the proofs of the three theorems above.

2 Preliminaries

2.1 Topology on the space of positively homogeneous functions

In this article, we frequently speak that one Finsler metric is close to another Finsler metric in the C^k topology. So it seems necessary to recall the meaning of this. Let M be a closed C^∞ manifold. Let $\mathcal{H}(TM)$ be the space of functions on TM which have the following properties :

- (a) $f(tv) = tf(v)$ for all $v \in TM$ and $t > 0$;
- (b) f is continuous on TM and C^∞ on TM_0 .

We choose an arbitrary Finsler metric F on M , a real number $a \in (0, 1)$ and a finite atlas of M , $\{(U_i, \varphi_i)\}_{i \in I}$. Let $T^{a < F < \frac{1}{a}} M = \{v \in TM \mid a < F(v) < \frac{1}{a}\}$. For any $k \in \mathbb{N}$, any $f \in \mathcal{H}(TM)$ and any $\epsilon > 0$, we define

$$B_k(f, \epsilon) = \{g \in \mathcal{H}(TM) \mid |\partial^\alpha(g - f)| < \epsilon \text{ on } T^{a < F < \frac{1}{a}} M, \forall |\alpha| \leq k\},$$

where α is a multiple index and the partial derivatives are calculated via the chosen finite atlas.

The family $\{B_k(f, \epsilon) \mid f \in \mathcal{H}(TM), \epsilon > 0\}$ constitutes the basis of a topology on the space $\mathcal{H}(TM)$, which is called the C^k topology. It is easy to see that this topology is independent of the choice of F , of a and of the finite atlas of M . Let $f, h \in \mathcal{H}(TM)$. We say that h is close to f in the C^k topology, if there exists $\epsilon > 0$ such that $h \in B_k(f, \epsilon)$.

2.2 Finsler geodesic flows and geodesical reversibility

Given a closed C^∞ Finsler manifold (M, F) , its geodesic flow φ^F on TM_0 is defined as follows : for any $u \in TM_0$ and $t \in \mathbb{R}$, let $\gamma(t)$ be the unique maximal geodesic such that $\gamma'(t) = u$. We define the geodesic flow $\varphi^F : TM_0 \rightarrow TM_0$ such that $\varphi_t^F(u) = \gamma'(t)$. The sphere bundle $S_F M = \{v \in TM \mid F(v) = 1\}$ is preserved by φ and the restriction of φ to $S_F M$ is still called the geodesic flow of (M, F) .

The Hilbert form of F , denoted by α^F , is defined as follows : for any $v \in TM_0$ and $\xi \in T_v(TM_0)$,

$$\alpha_v^F(\xi) = \frac{d}{dt} \left[\frac{1}{2} F^2(v + tD\pi(\xi)) \right] \Big|_{t=0}.$$

Let $\omega^F = -d\alpha^F$. It is well-known that ω^F is a symplectic form on TM_0 . Let X^F be the tangent vector field of φ on TM_0 . Then the Finsler geodesic flow φ^F on TM_0 is the Hamiltonian flow of $\frac{1}{2}F^2$ with respect to ω^F , i.e.

$$i_{X^F} \omega^F = d\left(\frac{1}{2}F^2\right).$$

Moreover, the Finsler geodesic flow φ^F on $S_F M$ is the Reeb flow of the contact form α^F , i.e.

$$\alpha^F(X^F) = 1 \quad \text{and} \quad i_{X^F} d\alpha^F = 0.$$

Let $i : TM \rightarrow TM$ be the involution map defined as $i(v) = -v$ for all $v \in TM$. We have the following proposition.

Proposition 1. *Given a closed C^∞ Finsler manifold (M, F) , let $\varphi^F : S_F M \rightarrow S_F M$ be its geodesic flow and X^F be the tangent field of φ^F . Then (M, F) is geodesically reversible if and only if the flow $i \circ \varphi^F \circ i$ is a time change of the flow generated by $-X^F$.*

Proof. We suppose that (M, F) is geodesically reversible. For any $v \in S_F M$, let $c : [0, +\infty) \rightarrow M$ be a geodesic such that $c'(0) = v$. Thus for any $t > 0$, $\varphi_t^F(v) = c'(t)$. The reversed curve $\bar{c} : [-t, 0] \rightarrow M$ is a geometrical geodesic, i.e. its reparametrization by arc-length is a geodesic. Thus there exists a C^∞ positive function $\tau(v, t)$ such that $\varphi_{\tau(v, t)}^F(-c'(t)) = -v$. We deduce that $i \circ \varphi^F \circ i = \varphi_\tau^F$, i.e. $i \circ \varphi^F \circ i$ is a time change of the flow generated by $-X^F$. Inversely, if the flow $i \circ \varphi^F \circ i$ on $S_F M$ is a time change of the flow generated by $-X^F$, then by projecting their orbits to M , we see that the reversed curve of any geodesic is also a geometrical geodesic. \square

2.3 Anosov flows and geometric flip

In the case that the Finsler manifold (M, F) is of negative flag curvature (see [11]), it is proved in [5] that the geodesic flow $(\varphi, S_F M)$ is a so-called Anosov flow. This means that there exists a $D\varphi$ -invariant splitting of the tangent bundle

$$T(S_F M) = E^{ss} \oplus \mathbb{R}X \oplus E^{su},$$

where X denotes the tangent vector field of φ , a Riemannian metric on $S_F M$ and two positive numbers a and b , such that for any $t \geq 0$, $u \in E^{ss}$ and $v \in E^{su}$,

$$\|D\varphi_t(u)\| \leq a \cdot e^{-bt} \|u\|, \quad \|D\varphi_{-t}(v)\| \leq a \cdot e^{-bt} \|v\|.$$

The vector bundles E^{ss} and E^{su} are called respectively the *strong stable* and *strong unstable* distributions of φ , which are generally just Hölder continuous (see [7]). For any $k \geq 1$ we say that ψ has C^k *Anosov splitting* if E^{ss} and E^{su} are both C^k subbundles of $T(S_F M)$. It is well-known that if M is of dimension two or φ is $\frac{1}{2}$ -pinched (see [8]), then its geodesic flow φ has C^1 Anosov splitting (see [8]).

Now we suppose that (M, F) is of negative flag curvature. A positive (resp. negative) geodesic ray is a unit-speed geodesic defined on $[0, +\infty)$ (resp. $(-\infty, 0]$) in a Finsler universal covering space $(\widetilde{M}, \widetilde{F})$. Let $S_{\widetilde{F}}(\widetilde{M}) = \{u \in T\widetilde{M} \mid \widetilde{F}(u) = 1\}$. According to [4], the *lifted geometric flip* $\sigma_{\widetilde{F}} : S_{\widetilde{F}}(\widetilde{M}) \rightarrow S_{\widetilde{F}}(\widetilde{M})$ is defined as follows : for any $u \in S_{\widetilde{F}}(\widetilde{M})$ such that $\pi(u) = x$, let c be the unique negative geodesic ray such that $c'(0) = u$.

By Proposition 3.14 in [4], there exists a unique positive geodesic ray γ such that $\gamma(0) = x$ and the Hausdorff distance $\tilde{d}_H(c((-\infty, 0]), \gamma([0, +\infty)))$ is finite. Thus we define $\sigma_{\tilde{F}}(u) = \gamma'(0)$.

It is proved in [4] that $\sigma_{\tilde{F}}$ is a homeomorphism, which is obviously invariant under the natural action of the fundamental group $\pi_1(M)$ over \tilde{M} . Therefore the quotient map $\sigma_F : S_F M \rightarrow S_F M$ is well-defined and extends naturally to a map defined on TM which preserves F viewed as a function on TM . This map will still be denoted by $\sigma_F : TM \rightarrow TM$ and is called the *geometric flip* of (M, F) . For any $u \in TM$, we can consider $\sigma_F(u)$ as the opposite vector of u from the point of view of the large-scale geometry of (\tilde{M}, \tilde{F}) .

There exists another natural flip which is of infinitesimal nature, $\sigma_F^a : TM \rightarrow TM$ such that $\sigma_F^a(u) = -\frac{F(u)}{F(-u)}u$. We call this map the *algebraic flip* of (M, F) . In the case that (M, F) is reversible, the algebraic flip is simply the involution map i . We have the following proposition.

Proposition 2. *Let (M, F) be a closed Finsler manifold of negative flag curvature. Then (M, F) is geodesically reversible if and only if the geometric flip and the algebraic flip of (M, F) are the same.*

Proof. If (M, F) is geodesically reversible, then given any geodesic $c : [a, b] \rightarrow M$, its reversed curve \bar{c} is, up to a reparametrization, a geodesic. We deduce that the reversed curve of any negative geodesic ray in \tilde{M} is, up to a reparametrization, a positive geodesic ray. Therefore, $\sigma_F = \sigma_F^a$.

Inversely, we suppose that $\sigma_F = \sigma_F^a$. Thus on the universal covering space (\tilde{M}, \tilde{F}) we have $\sigma_{\tilde{F}} = \sigma_{\tilde{F}}^a$. We suppose on the contrary that (M, F) is not geodesically reversible. Then there exists a geodesic line γ , i.e. a unit-speed geodesic defined on \mathbb{R} in \tilde{M} , such that $\bar{\gamma}$ does not locally minimize the length. Let γ_1 be the geodesic line such that $\gamma_1'(0) = \sigma_{\tilde{F}}(\gamma'(0))$. Since $\sigma_{\tilde{F}} = \sigma_{\tilde{F}}^a$, then by the definition of the geometric flip, there exists a constant C such that

$$\tilde{d}_H(\gamma((-\infty, 0]), \gamma_1([0, +\infty))) \leq C, \quad \tilde{d}_H(\gamma([0, +\infty)), \gamma_1((-\infty, 0])) \leq C.$$

Since $\bar{\gamma}$ is not, up to a reparametrization, a geodesic, there exists $s \in \mathbb{R}$ such that $\gamma(s) \notin \gamma_1$. Let γ_2 be the geodesic line such that $\gamma_2'(0) = \sigma_{\tilde{F}}(\gamma'(s))$. Again by $\sigma_{\tilde{F}} = \sigma_{\tilde{F}}^a$ we obtain for the same C as above,

$$\tilde{d}_H(\gamma((-\infty, 0]), \gamma_2([0, +\infty))) \leq C, \quad \tilde{d}_H(\gamma([0, +\infty)), \gamma_2((-\infty, 0])) \leq C.$$

We deduce that

$$\tilde{d}_H(\gamma_1([0, +\infty)), \gamma_2([0, +\infty))) \leq C, \quad \tilde{d}_H(\gamma_1((-\infty, 0]), \gamma_2((-\infty, 0])) \leq C.$$

Therefore, γ_1 and γ_2 are two geodesic lines that are distinct and a bounded distance apart, which is impossible by Proposition 4.4 in [4]. \square

2.4 Projectively flat Finsler metrics on the 2-sphere of constant curvature

Let $\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ be the 2-sphere. With respect to its canonical Riemannian metric, all its geodesics are the great circles and its Gauss curvature is identically 1. Surprisingly, it is proved in [2] that modulo diffeomorphism, there exists a 2-parameter family of Finsler metrics on \mathbb{S}^2 that are still projectively flat and have constant Finsler-Gauss curvature.

Let us firstly recall the construction of these Finsler metrics (see Section 4 of [2]). Let V be a real vector space of dimension 3 and \mathbb{S} be the space of oriented lines through the origin in V . It is clear that \mathbb{S} is diffeomorphic to the 2-sphere \mathbb{S}^2 . For any $v \in V \setminus \{0\}$, we denote by $[v]$ the oriented line determined by v . For any pair of vectors in V , (v, w) with $v \neq 0$, a curve γ can be defined in \mathbb{S} by

$$\gamma(t) = [v + tw].$$

The tangent vector at $t = 0$ of this curve will be denoted by $[v, w]$, which is an element in the tangent space $T_{[v]}\mathbb{S}$. According to Subsection 4.2.2 in [2], a Finsler metric on \mathbb{S} determines and is determined by a function F on $(V \setminus \{0\}) \times V$ that satisfies

$$F(v, w) = F(av, aw + bv), \quad \forall a > 0, b \in \mathbb{R}$$

and

$$F(v, cw) = cF(v, w), \quad \forall c > 0.$$

In fact, the corresponding Finsler metric on \mathbb{S} will be defined as $F([v, w]) = F(v, w)$ for all $(v, w) \in (V \setminus \{0\}) \times V$, which will still be denoted F . According to Proposition 4 and Theorem 10 in [2], let p and q be real numbers satisfying $|q| \leq p < \frac{\pi}{2}$ and fix a basis of V^* such that V is identified with \mathbb{R}^3 (Only in this subsection, p does not denote an antisymmetric potential, but a real number). We consider the following quadratic form Q on $V \otimes \mathbb{C}$ defined as

$$Q(v) = e^{ip}x^2 + e^{iq}y^2 + e^{-ip}z^2, \quad \forall v = (x, y, z) \in V.$$

Let the inner product of two vector v and w with respect to Q be denoted $v \cdot w$. We define

$$F_{p,q}(v, w) = \Re \left[\frac{\sqrt{(w \cdot w)(v \cdot v) - (w \cdot v)^2} - i(v \cdot w)}{v \cdot v} \right].$$

It is proved in ([2], Theorem 10) that the Finsler metric corresponding to $F_{p,q}$ on \mathbb{S} is projectively flat, whose Finsler-Gauss curvature is identically 1. By the remark following Theorem 10 in [2],

$$F_{p,q}(v, w) - F_{p,q}(v, -w) = 2\mathfrak{I}\left[\frac{v \cdot w}{v \cdot v}\right].$$

Let θ be the associated antisymmetric potential of $F_{p,q}$ defined on $T\mathbb{S}$. Then we have

$$\theta(v, w) = -\mathfrak{I}\left[\frac{v \cdot w}{v \cdot v}\right].$$

More precisely, for all $v = (x, y, z)$ and $w = (a, b, c)$,

$$\theta(v, w) = -\mathfrak{I}\left[\frac{e^{ip}ax + e^{iq}by + e^{-ip}cz}{e^{ip}x^2 + e^{iq}y^2 + e^{-ip}z^2}\right].$$

Therefore, θ is a C^∞ 1-form on \mathbb{S} .

Proposition 3. *Under the notations above, θ is closed if and only if $p = |q|$.*

Proof. Recall that as mentioned above, we have $|q| \leq p < \frac{\pi}{2}$. Let $U \subset \mathbb{S}$ be the open subset defined as

$$U = \{(x, y, z) \mid z > 0\}.$$

We consider the chart $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^2$ defined by

$$\phi([x, y, z]) = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}\right).$$

Therefore, $\phi(U) = \{(x, y) \mid x^2 + y^2 < 1\}$ and for all $(x, y) \in \phi(U)$,

$$\phi^{-1}(x, y) = [x, y, \sqrt{1 - x^2 - y^2}].$$

Let $\alpha = 2 \sin p \cdot x^2 - \sin p + (\sin q + \sin p)y^2$. Then by a straightforward calculation, we get

$$\phi_*\theta = -\frac{2 \sin p \sin q \cdot x}{\alpha^2 + \cos^2 q} dx - \frac{(\sin q + \sin p) \cos q \cdot y - \alpha(\cos q - \cos p)y}{\alpha^2 + \cos^2 q} dy.$$

Therefore,

$$d(\phi_*\theta) = -\frac{4(\cos q - \cos p) \sin p(\alpha^2 - \cos^2 q)xy}{(\alpha^2 + \cos^2 q)^2} dx \wedge dy.$$

Since $|q| \leq p < \frac{\pi}{2}$, then $d(\phi_*\theta) = 0$ on $\phi(U)$ if and only if $p = |q|$. Moreover, by similar calculations, we can see that if $p = |q|$, then θ is closed on the whole \mathbb{S} . \square

For $p = q = 0$, it is easy to see that $F_{0,0}$ is the unique reversible Finsler metric in this family, i.e. its antisymmetric potential is 0. By the proposition above, for any (p, q) such that $|q| = p < \frac{\pi}{2}$, the antisymmetric potential of $F_{p,q}$ is a closed 1-form; and for any (p, q) such that $|q| < p < \frac{\pi}{2}$, the antisymmetric potential of $F_{p,q}$ is a 1-form, but not closed.

Moreover, according to the remark following Theorem 10 in [2], except the case where $p = q = 0$, $F_{p,q}$ is not a Randers metric. Since we have seen that the antisymmetric potential of $F_{p,q}$ is a 1-form, then $F_{p,q}$ is neither of Randers-type.

3 Associated Euler-Lagrange flow

3.1 Generalized Tonelli Lagrangian

Definition 3.1. Let (M, F) be a closed C^∞ Finsler manifold. For any $a > 0$, let $T^{a < F}M = \{v \in TM \mid F(v) > a\}$, which projects naturally onto M . A function $L : TM \rightarrow \mathbb{R}$ is said to be a *generalized (F, a) -Tonelli Lagrangian* if it has the following properties :

- (a) L is continuous on TM and C^∞ on TM_0 ;
- (b) for any $x \in M$, the restriction of L to $T_x^{a < F}M$ has everywhere positive-definite Hessian;
- (c) $\lim_{F(v) \rightarrow +\infty} \frac{L(v)}{F(v)} = +\infty$ uniformly on $x \in M$.

The energy function of L is defined as $E(v) = \frac{\partial L}{\partial v} \cdot v - L(v)$ for all $v \in TM_0$. For any $x \in M$, because of the absence of differentiability of L on $M \subseteq TM$, it is possible that L does not have everywhere positive-definite Hessian on T_xM_0 , but only on some $T_x^{a < F}M$.

Now let N be a reversible Finsler metric and p be an antisymmetric potential on M . We define $L : TM \rightarrow \mathbb{R}$ such that for all $v \in TM$,

$$L(v) = \frac{1}{2}(N(v))^2 - p(v).$$

For any $x \in M$, the Hessian of $\frac{1}{2}N^2$ on T_xM_0 is positive-definite and positively homogeneous of degree 0. On the other side, the Hessian of p on T_xM_0 is positively homogeneous of degree -1 . Therefore, there exists $a > 0$ such that L is a generalized (N, a) -Tonelli Lagrangian. Moreover, if p is small enough in the C^2 topology, we can take $a \in (0, 1)$.

Since N and p are both positively homogeneous of degree one, a simple calculation shows that $E(v) = \frac{1}{2}(N(v))^2$ for all $v \in TM_0$. Let $\gamma : [a, b] \rightarrow M$ be an extremal of the functional defined for any C^1 curve c as $\mathcal{L}(c) =$

$\int_a^b L(c(t), c'(t))dt$. Then the energy function E is constant along the curve $(\gamma(t), \gamma'(t))$, which means that γ has constant N -speed. Therefore, the extremals of \mathcal{L} give rise to a complete flow $\psi : T^{a<N}M \rightarrow T^{a<N}M$, which is called the *associated Euler-Lagrange flow* of the pair (N, p) . Its restriction to $S_N M$ is still said to be the *associated Euler-Lagrange flow*.

Recall that antisymmetric potentials are natural generalizations of 1-forms. Moreover, given any antisymmetric potential $p : TM \rightarrow \mathbb{R}$, we can naturally define a C^∞ associated 1-form θ^p on TM_0 : for any $u \in TM_0$ and $\xi \in T_u(TM_0)$,

$$\theta^p(\xi) = \frac{d}{dt}p(u + tD\pi(\xi)) \Big|_{t=0},$$

where $\pi : TM_0 \rightarrow M$ denotes the canonical projection. In the case that p is a C^∞ 1-form θ on M , it is easy to see that $\theta^p = \pi^*\theta$.

Let α^N be the Hilbert form of N . We define the *Hilbert form* of L as $\alpha = \alpha^N - \theta^p$. It is easy to see that the C^∞ 2-form $\omega = -d\alpha$ is a symplectic structure on $T^{a<N}M$, and the Euler-Lagrange flow ψ on $T^{a<N}M$ is the Hamiltonian flow of E with respect to ω .

3.2 Smooth orbit equivalence

Given a closed Finsler manifold (M, F) , let \hat{F} be its associated reversible Finsler metric and p be its associated antisymmetric potential. We define

$$\Psi : S_{\hat{F}}M \rightarrow S_F M, \quad \Psi(v) = \frac{v}{1 - p(v)}$$

and

$$\Phi : S_F M \rightarrow S_{\hat{F}}M, \quad \Phi(v) = \frac{v}{1 + p(v)}.$$

Since for all $v \in S_N M$, $F(v) = \hat{F}(v) - p(v) = 1 - p(v) > 0$, then Ψ is well-defined. Similarly, Φ is also well-defined and is the inverse application of Ψ . Therefore, Ψ and Φ are both C^∞ diffeomorphisms. We have the following proposition, which generalizes a result of G. Paternain in [10].

Proposition 4. *Under the notations above, if there exists $a \in (0, 1)$ such that L is a generalized (\hat{F}, a) -Tonelli Lagrangian, then the associated Euler-Lagrange flow ψ on $S_{\hat{F}}M$ and the geodesic flow φ^F on $S_F M$ are C^∞ orbit equivalent. More precisely, we have $D\Psi(\frac{X}{1-p}) = X^F$ and $D\Phi(\frac{X^F}{1+p}) = X$, where X and X^F are respectively the tangent fields of ψ and φ^F .*

Proof. We consider the pull-back cotangent bundle $\pi^*T^*M = \{(u, p) \mid u \in TM_0, p \in T^*M, \pi(u) = \pi(p)\}$. We can define two C^∞ sections of this bundle as follows : for any $v \in TM_0$ and $u \in TM$ such that $\pi(u) = \pi(v)$, let

$$\bar{\alpha}_v(u) = \frac{d}{dt}[L(v + tu)]|_{t=0} \quad \text{and} \quad \bar{\alpha}_v^F(u) = \frac{d}{dt}[\frac{1}{2}F^2(v + tu)]|_{t=0} .$$

It is easy to see that $\alpha = \bar{\alpha}_{\pi(\cdot)}(D\pi(\cdot))$ and $\alpha^F = \bar{\alpha}_{\pi(\cdot)}^F(D\pi(\cdot))$, where $\pi : T(TM_0) \rightarrow TM_0$ denotes the natural projection.

Take a standard local coordinate system (x^i, y^i) in TM_0 . For any $(x, y) \in S_{\hat{F}}M$, since $\hat{F}(x, y) = 1$, then

$$\bar{\alpha} = \hat{F} \frac{\partial \hat{F}}{\partial y^i} dx^i - \frac{\partial p}{\partial y^i} dx^i = \left(\frac{\partial \hat{F}}{\partial y^i} - \frac{\partial p}{\partial y^i} \right) dx^i .$$

Similarly for any $(x, y) \in S_F M$, since $F(x, y) = 1$, then

$$\bar{\alpha}^F = F \frac{\partial F}{\partial y^i} dx^i = \left(\frac{\partial F}{\partial y^i} - \frac{\partial p}{\partial y^i} \right) dx^i .$$

Since \hat{F} and p are both positively homogeneous of degree one, we deduce from the above local formulas of $\bar{\alpha}$ and $\bar{\alpha}^F$ that $\bar{\alpha}_v = \bar{\alpha}_{\Psi(v)}^F$ for all $v \in S_{\hat{F}}M$.

Now take $v \in S_{\hat{F}}M$ and $\xi \in T_v(S_{\hat{F}}M)$. Since $\pi \circ \Psi = \pi$, we have

$$\begin{aligned} (\Psi^* \alpha^F)_v(\xi) &= \alpha_{\Psi(v)}^F(D\Psi(\xi)) = \bar{\alpha}_{\Psi(v)}^F(D\pi(D\Psi(\xi))) \\ &= \bar{\alpha}_{\Psi(v)}^F(D\pi(\xi)) = \bar{\alpha}_v(D\pi(\xi)) = \alpha_v(\xi) \end{aligned}$$

Therefore, $\Psi^* \alpha^F = \alpha$.

For any $v = (x, y) \in S_{\hat{F}}M$, we $D\pi(X(v)) = v$. Thus

$$\alpha(X(v)) = \bar{\alpha}(D\pi(X(v))) = \frac{\partial \hat{F}}{\partial y^i} y^i - \frac{\partial p}{\partial y^i} y^i = 1 - p(v).$$

So $\alpha(X) = 1 - p$. We deduce that $\alpha(\frac{X}{1-p}) = 1$ on $S_{\hat{F}}M$. Moreover, we have $i_X d\alpha = 0$ on $S_{\hat{F}}M$. Therefore, the vector field $\frac{X}{1-p}$ generates the Reeb flow of the contact form α on $S_{\hat{F}}M$. On the other side, we know that X^F generates the Reeb flow of the contact form α^F on $S_F M$. Since we showed above that $\Psi^* \alpha^F = \alpha$, then we must have $D\Psi(\frac{X}{1-p}) = X^F$, i.e. Ψ is a C^∞ flow equivalence carrying the flow of $\frac{X}{1-p}$ to φ . Similar arguments show that $D\Phi(\frac{X^F}{1+p}) = X$. \square

The following corollary of the above proposition is a special case of Lemma 11 in [1].

Corollary 1. *Under the same hypothesis as in the above proposition, if p is given by a closed C^∞ 1-form θ , then the Finsler manifold (M, F) is geodesically reversible.*

Proof. We have $\theta^p = \pi^*\theta$, thus $d\theta^p = d(\pi^*\theta) = 0$. Therefore, $\omega = -d\alpha = -d\alpha^{\hat{F}}$ and the tangent vector field X of ψ is determined by $i_X(-d\alpha^{\hat{F}}) = d(\frac{1}{2}\hat{F}^2)$. Therefore, ψ is exactly the geodesic flow of the reversible Finsler metric \hat{F} . Moreover, since by the proposition above, the geodesic flow of F is C^∞ orbit equivalent to ψ , we deduce that for any unit-speed F -geodesic γ , its reversed curve is also a geometric F -geodesic, which implies that (M, F) is geodesically reversible. \square

4 Tangent field of the Euler-Lagrange flow associated to Randers-type metrics

Let $F = \sqrt{g} - p$ be a Randers-type Finsler metric on a closed manifold M , where g is a Riemannian metric. Let $SM = \{v \in TM \mid g(v, v) = 1\}$. We suppose that there exists $a \in (0, 1)$ such that L is a generalized (\sqrt{g}, a) -Tonelli Lagrangian. We will determine the tangent vector field X of the associated Euler-Lagrange flow $(\psi, T^{a < \sqrt{g}}M)$. Let us firstly recall some elementary facts in Riemannian geometry.

Let (M, g) be a Riemannian manifold and $\pi : TM \rightarrow M$ be the natural projection. Let $V(TM) = \text{Ker}D\pi$, which is called the *vertical subbundle* of $T(TM)$. The *connection map* $K : T(TM) \rightarrow TM$ is defined as follows: for any $v \in TM$ and $\xi \in T_v(TM)$, let $U(t)$ be a C^∞ curve in TM such that $U(0) = v$ and $U'(0) = \xi$. Let $c(t) = \pi(U(t))$. We define $K_v : T_v(TM) \rightarrow T_{\pi(v)}M$ by $K_v(\xi) = \nabla_{c'}U(0)$, where ∇ denotes the Levi-Civita connection of (M, g) . The *horizontal subbundle* is defined as $H(TM) = \text{Ker}K$. It is well-known that

$$T(TM) = V(TM) \oplus H(TM),$$

and $(D\pi, K) : T(TM) \rightarrow TM \oplus TM$ is a vector bundle isomorphism. The Hilbert form of \sqrt{g} can be written as

$$\alpha_v^{\sqrt{g}}(\xi) = g(D\pi(\xi), v),$$

for all $v \in TM$ and $\xi \in T_v(TM)$. The symplectic structure $\omega^{\sqrt{g}} = -d\alpha^{\sqrt{g}}$ can be written as

$$\omega^{\sqrt{g}}(\xi, \eta) = g(D\pi(\xi), K(\eta)) - g(K(\xi), D\pi(\eta)).$$

Now let us return to the tangent vector field X of the Euler-Lagrange flow $(\psi, T^{a < \sqrt{g}}M)$. For any $v \in T^{a < \sqrt{g}}M$ and $\xi \in T_v(T^{a < \sqrt{g}}M)$, we define $\xi_1 = d\pi(\xi)$ and $\xi_2 = K(\xi)$. Thus in particular, we have the decomposition $X = (X_1, X_2)$.

It is easy to see that $dE(\xi) = g(\xi_2, v)$, where $E = \frac{1}{2}g$ is the energy function preserved by ψ . Combined with the equation $i_X\omega = dE$, we obtain the following equation (1) :

$$(1) \quad g(\xi_2, v) = g(X_1, \xi_2) - g(X_2, \xi_1) + d\theta^p(X, \xi).$$

Let $D\pi^{-1} : TM \rightarrow H(TM)$ and $K^{-1} : TM \rightarrow V(TM)$ be the inverse bundle isomorphisms of the restrictions $D\pi : H(TM) \rightarrow TM$ and $K : V(TM) \rightarrow TM$. For any $v \in TM_0$, we define the linear endomorphism $Y_v : T_{\pi(v)}M \rightarrow T_{\pi(v)}M$ such that for all $u_1, u_2 \in T_v(TM)$,

$$d\theta_v^p(D\pi^{-1}(u_1), D\pi^{-1}(u_2)) = g(Y_v(u_1), u_2).$$

Similarly we define $Y_v^{VH} : T_{\pi(v)}M \rightarrow T_{\pi(v)}M$ such that

$$d\theta_v^p(K^{-1}(u_1), D\pi^{-1}(u_2)) = g(Y_v^{VH}(u_1), u_2),$$

and $Y_v^{HV} : T_{\pi(v)}M \rightarrow T_{\pi(v)}M$ such that

$$d\theta^p(D\pi^{-1}(u_1), K^{-1}(u_2)) = g(Y_v^{HV}(u_1), u_2).$$

It is clear that $Y_v^{VH} = -Y_v^{HV}$. Since $d\theta^p|_{V(TM_0)} = 0$, we obtain

$$d\theta^p(X, \xi) = g(Y_v(X_1), \xi_1) + g(Y_v^{VH}(X_2), \xi_1) + g(Y_v^{HV}(X_1), \xi_2).$$

Combined with the equation (1) above, we get a system of two equations:

$$v = X_1(v) + Y_v^{HV}(X_1(v))$$

and

$$X_2(v) - Y_v^{VH}(X_2(v)) = Y_v(X_1(v)).$$

Since X defines a second-order ordinary differential equation, we have necessarily $X_1(v) = v$. Therefore, we obtain for any $v \in T^{a < \sqrt{g}}M$,

$$X_1(v) = v, \quad (Id - Y_v^{VH})(X_2(v)) = Y_v(v).$$

Moreover, since ω is a symplectic form on $T^{a<\sqrt{g}}M$, the vector field (X_1, X_2) defined by $i_X\omega = dE$ exists uniquely. Therefore, for any $v \in T^{a<\sqrt{g}}M$, the solution of the linear system $(Id - Y_v^{VH})(X_2(v)) = Y_v(v)$ exists uniquely. We deduce that the linear endomorphism $Id - Y_v^{VH}$ is injective, thus invertible. Therefore, we obtain the following proposition.

Proposition 5. *For any $v \in T^{a<\sqrt{g}}M$,*

$$X(v) = (v, (Id - \lambda Y_v^{VH})^{-1}(Y_v(v))).$$

We deduce that a C^∞ curve C in $T^{a<\sqrt{g}}M$ is an orbit of ψ if and only if $C(t)$ is of the form $(c(t), c'(t))$ such that

$$\nabla_{c'}c' = (Id - Y_{c'}^{VH})^{-1}(Y_{c'}(c')).$$

where ∇ denotes the Levi-Civita connection of (M, g) . The Euler-Lagrange flow (ψ, SM) is generated by the restriction of X to SM .

5 Proofs of our Theorems

5.1 A vanishing result

In this section, we suppose that $F = \sqrt{g} - p$ is a Randers-type Finsler metric on a closed manifold M , and that there exists $a \in (0, 1)$ such that $L = \frac{1}{2}g - p$ is a generalized (\sqrt{g}, a) -Tonelli Lagrangian. We denote by SM the sphere bundle of the Riemannian metric g .

Lemma 5.1. *Under the notations above, for any $v \in SM$, $Y_{-v} = Y_v$ and $Y_{-v}^{VH} = -Y_v^{VH}$.*

Proof. Let $i : TM \rightarrow TM$ be the involution map defined as $i(v) = -v$. Since the antisymmetric potential p is antisymmetric (see Section 1), we have $i^*\theta = \theta$, which implies that $i^*(d\theta^p) = d\theta^p$. We have obviously $\pi \circ i = \pi$, which implies that $D\pi \circ Di = D\pi$. Therefore, according to the definition of Y , we obtain for all $v \in SM$, $Y_{i(v)} = Y_v$, i.e. $Y_{-v} = Y_v$. Let K be the connection map of g . It is easy to see that $K \circ Di = -K$. Therefore, $Y_{-v}^{VH} = -Y_v^{VH}$. \square

Proposition 6. *Under the notations above, if (M, F) is geodesically reversible, then for any $v \in SM$, $Y_v(v) = 0$.*

Proof. For any $u \in S_FM$, let c be a F -geodesic such that $c'(0) = u$. Since (M, F) is geodesically reversible, there exists a reparametrization ϕ , which is an orientation-preserving diffeomorphism such that $c(-\phi(t))$ is a unit-speed F -geodesic.

Since by Proposition 3, the diffeomorphism $\Phi : S_FM \rightarrow SM$ verifies $D\Phi(\frac{X^F}{1+p}) = X$ and $1+p = \sqrt{g}$ on S_FM , then $\Phi(c, c')$ is tangent to the vector field $\sqrt{g} \circ \Phi^{-1} \cdot X$ on SM . Therefore, by Proposition 4, we obtain

$$\begin{aligned} \nabla_{c'} \frac{c'}{\sqrt{g}(c')} &= \sqrt{g}(c') \cdot (Id - Y_{\frac{c'}{\sqrt{g}(c')}}^{VY})^{-1} Y_{\frac{c'}{\sqrt{g}(c')}} \left(\frac{c'}{\sqrt{g}(c')} \right) \\ &= (Id - Y_{\frac{c'}{\sqrt{g}(c')}}^{VY})^{-1} Y_{\frac{c'}{\sqrt{g}(c')}} (c'). \end{aligned}$$

Thus a C^∞ curve c is a F -geodesic if and only if

$$\nabla_{c'} \frac{c'}{\sqrt{g}(c')} = (Id - Y_{\frac{c'}{\sqrt{g}(c')}}^{VY})^{-1} Y_{\frac{c'}{\sqrt{g}(c')}} (c').$$

Since $c(-\phi(t))$ is also a F -geodesic and $(c(-\phi(t)))' = -c'(-\phi(t)) \cdot \phi'(t)$ with $\phi'(t) > 0$, then we obtain

$$\begin{aligned} &\nabla_{-c'(-\phi(t)) \cdot \phi'(t)} \frac{-c'(-\phi(t)) \cdot \phi'(t)}{\sqrt{g}(-c'(-\phi(t)) \cdot \phi'(t))} \\ &= (Id - Y_{\frac{-c'(-\phi(t)) \cdot \phi'(t)}{\sqrt{g}(-c'(-\phi(t)) \cdot \phi'(t))}}^{VY})^{-1} Y_{\frac{-c'(-\phi(t)) \cdot \phi'(t)}{\sqrt{g}(-c'(-\phi(t)) \cdot \phi'(t))}} (-c'(-\phi(t)) \cdot \phi'(t)). \end{aligned}$$

Since \sqrt{g} is reversible, we deduce from Lemma 5.1 that

$$\begin{aligned} \nabla_{c'(-\phi(t))} \frac{c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))} &= (Id - Y_{\frac{-c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}}^{VY})^{-1} Y_{\frac{-c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}} (-c'(-\phi(t))). \\ &= -(Id + Y_{\frac{c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}}^{VY})^{-1} Y_{\frac{c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}} (c'(-\phi(t))). \end{aligned}$$

Let $A = Y_{\frac{c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}}^{VY}$ and $U = Y_{\frac{c'(-\phi(t))}{\sqrt{g}(c'(-\phi(t)))}} (c'(-\phi(t)))$. Thus we obtain

$$(Id - A)^{-1}U = -(Id + A)^{-1}U.$$

Multiplied by $(Id - A)(Id + A)$ on both sides, we get $(Id + A)U = -(Id - A)U$, i.e. $U + AU = -U + AU$, thus $U = 0$. We deduce that for any $v \in SM$, $Y_v(v) = 0$. \square

5.2 Transverse rigidity

Definition 5.2. Let M be a closed manifold and α be a contact form on M . Let ϕ be the Reeb flow of α and X be the tangent vector field of ϕ . We say that ϕ is *transversely rigid* if it has the following property: for any C^∞ 1-form β on M , if $i_X d\beta = 0$, then there exists a constant $a \in \mathbb{R}$ such that $d\beta = a \cdot d\alpha$.

We note that if $i_X d\beta = 0$, then $d\beta$ must be ϕ -invariant since

$$\mathcal{L}_X d\beta = d(i_X(d\beta)) + i_X(d(d\beta)) = 0.$$

Proposition 7. *Let $F = \sqrt{g} - p$ be a Randers-type Finsler metric on a closed manifold M . We suppose that there exists $a \in (0, 1)$ such that $L = \frac{1}{2}g - p$ is a generalized (\sqrt{g}, a) -Tonelli Lagrangian. If (M, F) is geodesically reversible, and the Riemannian geodesic flow of \sqrt{g} on SM is transversely rigid, then the antisymmetric potential p of F must be a closed 1-form.*

Proof. By Proposition 5, we have $Y_v(v) = 0$ on SM . Then by Proposition 4, a \sqrt{g} -unit speed curve c is a L -extremal if and only if $\nabla_{c'} c' = 0$, where ∇ denotes the Levi-Civita connection of (M, g) . Thus in M , the L -extremals are just \sqrt{g} -geodesics. We deduce that X is just the tangent vector field of the Riemannian geodesic flow, say $\varphi^{\sqrt{g}}$, of g on $S_N M$, which implies that $i_X d\theta^p = 0$ on SM . Since $\varphi^{\sqrt{g}}$ is transversely rigid, there exists $\delta \in \mathbb{R}$ such that

$$d\theta^p = \delta \cdot d\alpha^{\sqrt{g}},$$

where $\alpha^{\sqrt{g}}$ denotes the Hilbert form of \sqrt{g} . Let $i : SM \rightarrow SM$ be the involution map. Since $i^* \alpha^{\sqrt{g}} = -\alpha^{\sqrt{g}}$ and $i^* \theta^p = \theta^p$, then $d\theta^p = i^* d\theta^p = \delta \cdot i^* d\alpha^{\sqrt{g}} = -\delta \cdot d\alpha^{\sqrt{g}}$. Therefore $\delta = 0$, thus $d\theta^p = 0$ on SM .

Take a standard local coordinate system (x^i, y^i) in TM_0 . We have locally

$$d\theta^p = [p]_{y^i x^j} dx^j \wedge dx^i + [p]_{y^i y^j} dy^j \wedge dx^i.$$

Let $Z = y^i \frac{\partial}{\partial y^i}$ which is locally transverse to SM . Since p is positively homogeneous of degree one then

$$d\theta^p(Z, \cdot) = [p]_{y^i y^j} y^j dx^i = 0.$$

Since in addition, it is shown above that $d\theta^p = 0$ considered as a differential form on SM , then for any $(x, y) \in SM$,

$$[p]_{y^i x^j}(x, y) dx^j \wedge dx^i + [p]_{y^i y^j}(x, y) dy^j \wedge dx^i = 0.$$

In particular, $[p]_{y^i y^j}(x, y) = 0$ for any $(x, y) \in SM$. Since p is positively homogeneous of degree one then $[p]_{y^i y^j}$ is positively homogeneous of degree (-1) . Therefore, $[p]_{y^i y^j}(x, y) = 0$ for any $(x, y) \in TM_0$, which implies that p is given by a C^∞ 1-form on M , say θ . Thus we have $\theta^p = \pi^*\theta$. Therefore, the equation $i_X d\theta^p = 0$ on SM implies that

$$0 = \pi^* d\theta(X(v), \xi) = d\theta(D\pi(X(v)), D\pi(\xi)) = d\theta(v, D\pi(\xi))$$

for all $v \in SM$ and $\xi \in H_v(TM_0)$. We deduce that $d\theta = 0$. \square

We note that in the proof of the proposition above, the transverse rigidity is only used to prove that the antisymmetric potential is given by a 1-form, i.e. to prove that F is in fact a Randers metric. Therefore, Proposition 6 proves Theorem 1.3.

In order to prove Theorems 1.4 and 1.5, we need only to prove the transverse rigidity of the Riemannian geodesic flow $\varphi^{\sqrt{g}}$. If M is of dimension 2, the following fact is well-known, which combined with Proposition 6 proves Theorem 1.4. We give its proof only for the convenience of the reader.

Proposition 8. *Under the same hypothesis as in the above proposition, if M is a closed surface, and $\varphi^{\sqrt{g}}$ is topologically transitive, i.e. admitting a dense orbit, then $\varphi^{\sqrt{g}}$ is transversely rigid.*

Proof. Let β be a C^∞ 1-form on $S\Sigma$ such that $i_{X^{\sqrt{g}}} d\beta = 0$, where $X^{\sqrt{g}}$ denotes the tangent vector field of $\varphi^{\sqrt{g}}$. We have seen above that $d\beta$ is necessarily $\varphi^{\sqrt{g}}$ -invariant. Let $\alpha^{\sqrt{g}}$ be the Hilbert form of $\varphi^{\sqrt{g}}$, which is a $\varphi^{\sqrt{g}}$ -invariant contact form on the sphere bundle $S\Sigma$. Thus there exists a continuous function f on $S\Sigma$ such that $\alpha^{\sqrt{g}} \wedge d\beta = f \cdot \alpha^{\sqrt{g}} \wedge d\alpha^{\sqrt{g}}$. Since $\alpha^{\sqrt{g}} \wedge d\beta$ and $\alpha^{\sqrt{g}} \wedge d\alpha^{\sqrt{g}}$ are both $\varphi^{\sqrt{g}}$ -invariant, f is also $\varphi^{\sqrt{g}}$ -invariant. Therefore, because $\varphi^{\sqrt{g}}$ is supposed to be topologically transitive, f must be a constant. Moreover, since $i_{X^{\sqrt{g}}} d\beta = 0$ and $X^{\sqrt{g}}$ generates the Reeb flow of the contact form $\alpha^{\sqrt{g}}$, we get $d\beta = f \cdot d\alpha^{\sqrt{g}}$. \square

Proof of Theorem 1.5. Let $F = \sqrt{\bar{g}} - p$ be a Randers-type Finsler metric and $\sqrt{\bar{g}}$ be a hyperbolic Riemannian Finsler metric on a closed manifold M . We suppose that F is close to $\sqrt{\bar{g}}$ in the C^4 topology, which implies that the sectional curvature of (M, \bar{g}) is close to -1 . Then it is well-known that the Riemannian geodesic flow $\varphi^{\sqrt{\bar{g}}}$ on SM is a $\frac{1}{2}$ -pinched Anosov flow (see [7]). Therefore, by [8], $\varphi^{\sqrt{\bar{g}}}$ has C^1 Anosov splitting. Thus it is proved in [6], Theorem A, (3), that $\varphi^{\sqrt{\bar{g}}}$ is transversely rigid (see also [3], Proposition 9 and [4], Theorem 4.9 for some Finsler generalizations of this result).

The proof of Theorem 1.5 is complete.

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