



**HAL**  
open science

## Pascal's formulas and vector fields

Philippe Chassaing, Jules Flin, Alexis Zevio

► **To cite this version:**

Philippe Chassaing, Jules Flin, Alexis Zevio. Pascal's formulas and vector fields. 2022. hal-03821769v2

**HAL Id: hal-03821769**

**<https://hal.science/hal-03821769v2>**

Preprint submitted on 20 Oct 2022 (v2), last revised 14 Sep 2023 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Pascal's formulas and vector fields

Philippe Chassaing <sup>\*</sup>, Jules Flin <sup>†</sup>, Alexis Zevio <sup>‡</sup>

October 20, 2022

## Abstract

We consider four examples of combinatorial triangles  $(T(n, k))_{0 \leq k \leq n}$  (Pascal, Stirling of both types, Euler) : their *Pascal's formulas* define four vector fields, together with their field lines that turn out to be the limit of sample paths of four well known Markov chains. Using saddle-point asymptotics, we prove fluid approximation results in some of the four cases.

## 1 Introduction

### 1.1 Pascal's formulas

Set  $S = \{(n, k) \in \mathbb{N}^2, 0 \leq k \leq n\}$ ,  $S^\circ = \{(n, k) \in \mathbb{N}^2, 0 < k < n\}$ , and let  $S^* = S \setminus \{(0, 0)\}$ . Besides Pascal's triangle, other triangular arrays  $(T(n, k))_{(n, k) \in S}$  of interest satisfy a recursion formula similar to Pascal's formula, i.e. of the following form, for  $(n, k) \in S^*$  :

$$T(n, k) = a(n, k)T(n-1, k-1) + b(n, k)T(n-1, k), \quad (1)$$

with the convention that either  $(n, k) \in S$  or  $T(n, k) = 0$ . For instance, relation (1) holds true for the following triangular arrays :

- for Pascal's triangle, if  $(a, b)(n, k) = (1, 1)$  ;
- for Stirling numbers of the second kind, if  $(a, b)(n, k) = (1, k)$  ;
- for Stirling numbers of the first kind, if  $(a, b)(n, k) = (1, n-1)$  ;
- for Euler's triangle, if  $(a, b)(n, k) = (n-k, k+1)$ .

---

<sup>\*</sup>Institut Élie Cartan, Université de Lorraine Email: chassaingph@gmail.com

<sup>†</sup>Institut Élie Cartan, Université de Lorraine Email: jules.flin8@etu.univ-lorraine.fr

<sup>‡</sup>Institut Élie Cartan, Université de Lorraine Email: alexis.zevio1@etu.univ-lorraine.fr

## 1.2 Transition probabilities

In view of (1), for  $(n, k) \in S^*$ , consider

$$(p_0(n, k), p_1(n, k)) = \left( \frac{b(n, k)T(n-1, k)}{T(n, k)}, \frac{a(n, k)T(n-1, k-1)}{T(n, k)} \right) \quad (2)$$

as some transition probabilities from  $(n, k)$  to  $(n-1, k)$ , resp. to  $(n-1, k-1)$ . For each of these four triangular arrays, the transition probabilities

$$(p_\varepsilon(n, k))_{(\varepsilon, (n, k)) \in \{0,1\} \times S^*},$$

together with the initial state  $(m, \ell)$ , define a Markov chain  $W = (W_k)_{0 \leq k \leq n}$  with terminal state  $(0, 0)$ . These four Markov chains are closely related to the simple random walk, the coupon collector problem, the chinese restaurant process and the one-dimensional internal DLA, respectively : they are the time-reversed versions of these processes, once these processes are conditioned to be at level  $\ell$  at time  $m$ , as explained in the next section.

## 1.3 Random walk, coupon collector, chinese restaurant, and internal DLA

Consider a random process defined by  $X_0 = 0$ , and, for  $n \geq 0$ ,  $X_{n+1} = X_n + Y_{n+1}$ , in which the  $Y_i$ 's are Bernoulli random variables. Set

$$W_n = (m - n, X_{m-n}), \quad 0 \leq n \leq m.$$

### 1.3.1 Simple random walk

Assume that  $(Y_i)_{i \geq 1}$  is a Bernoulli process, i.e. a sequence of i.i.d. Bernoulli random variables with parameter  $p \in (0, 1)$ . Then

**Proposition 1.** *The stochastic process  $W = (W_n)_{0 \leq n \leq m}$ , conditioned to  $W_0 = (m, \ell)$ , or equivalently to  $X_m = \ell$ , is the Markov chain with transition probabilities  $(p_\varepsilon(n, k))_{(\varepsilon, (n, k)) \in \{0,1\} \times S^*}$  related to Pascal's triangle. Its distribution does not depend on  $p$ .*

This result goes back at least to Kennedy [Ken75]. We recall its proof at Subsection 2. In the next cases, the Bernoulli random variables  $Y_i$  are not i.i.d. .

### 1.3.2 Coupon collector's problem

Consider the coupon collector's problem with  $N$  different items. Let  $X_n$  denote the number of different items in the collection after the  $n$ th step. Again :

**Proposition 2.** *The stochastic process  $W = (W_n)_{0 \leq n \leq m}$ , conditioned on  $W_0 = (m, \ell)$ , or equivalently on the number of different items in the collection after the  $m$ th step,  $X_m$ , to be equal to  $\ell$ , is the Markov chain with transition probabilities  $(p_\varepsilon(n, k))_{\varepsilon, n, k}$  related to Stirling numbers of the second kind. Its distribution does not depend on  $N$ .*

### 1.3.3 Chinese restaurant process

In the Chinese restaurant process with  $(0, \theta)$  seating plan, defined at Section 2 (see also, e.g., [Pit06, Ch. 3]), let  $X_n$  denote the number of occupied tables after the arrival of the  $n$ th customer.

**Proposition 3.** *The stochastic process  $W = (W_n)_{0 \leq n \leq m}$ , conditioned to  $W_0 = (m, \ell)$ , or equivalently to  $X_m = \ell$ , is the Markov chain with transition probabilities  $(p_\varepsilon(n, k))_{\varepsilon, n, k}$  related to Stirling numbers of the first kind. Its distribution does not depend on  $\theta$ .*

**Remark 1.** As a consequence, in the three previous cases, given the data  $(X_n)_{0 \leq n \leq m}$ ,  $X_m$  (or  $W_0$ ) are sufficient statistics for the parameters  $p$ ,  $N$  or  $\theta$ , respectively.

### 1.3.4 One-dimensional Internal Diffusion Limited Aggregation

Finally, in the one-dimensional Internal Diffusion Limited Aggregation process (iDLA), let  $X_n$  denote the number of particles settled to the right of the origin after the release of the  $n$ th particle. Then

**Proposition 4.** *The stochastic process  $W = (W_n)_{0 \leq n \leq m}$ , conditioned to  $W_0 = (m, \ell)$ , is the Markov chain with transition probabilities  $(p_\varepsilon(n, k))_{\varepsilon, n, k}$  related to Euler's triangle.*

More precise definitions, and proofs, are to be found at Section 2.

## 1.4 Simulations

In the case of Pascal's triangle, the behaviour of this time-reversed Markov chain is well understood since forever. Quite recently, [AC19] gave a rather precise analysis of the same time-reversed Markov chain in case of Stirling numbers of the second kind, with combinatorial analysis of finite automata as a motivation.

In this section, in order to surmise the behaviour of these time-reversed Markov chains, we present the result of some simulations. For each case, the figures below show sample paths starting at  $(m, mt)$  with  $t \in \{0.05, \dots, 0.95\}$  and  $m = 500$  :

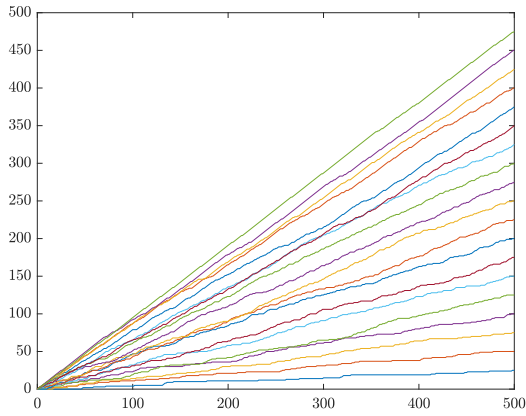


Figure 1: Pascal's triangle.

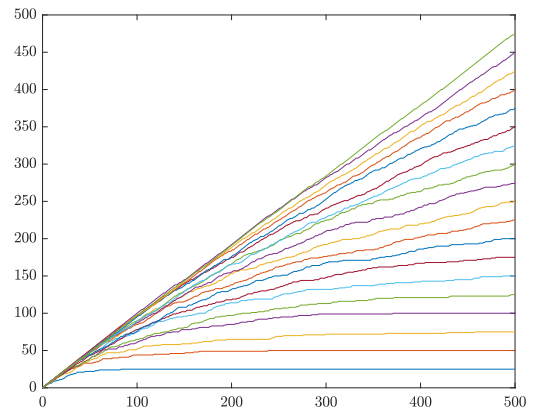


Figure 2: Stirling numbers of the second kind.

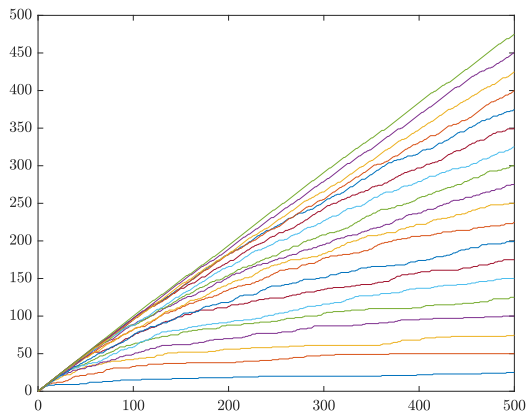


Figure 3: Stirling numbers of the first kind.

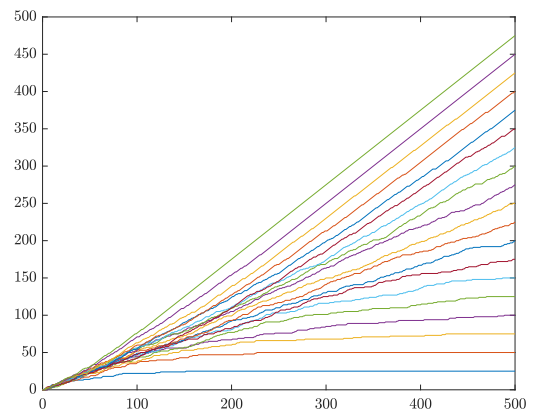


Figure 4: Eulerian numbers.

Now, in order to compare the four combinatorial triangles, we show the average of 100 sample paths for each triangle, for  $m = 1000$  and  $t \in \{0.05, \dots, 0.95\}$  :

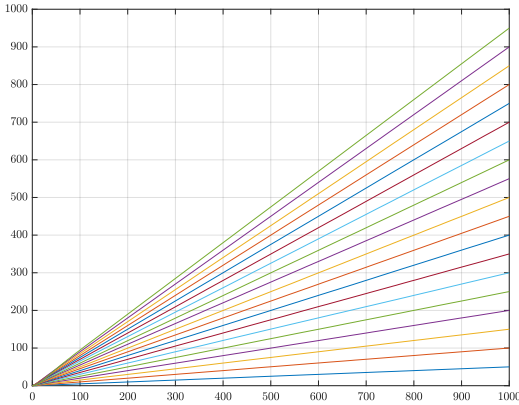


Figure 5: Pascal's triangle.

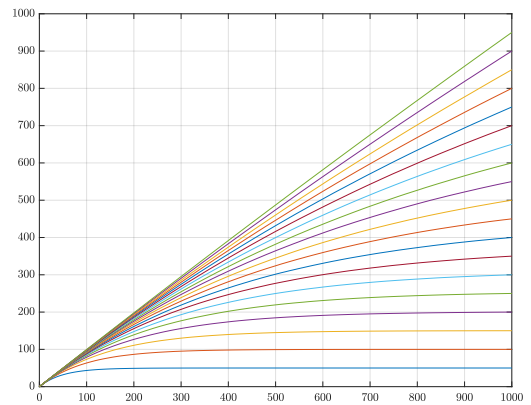


Figure 6: Stirling numbers of the second kind.

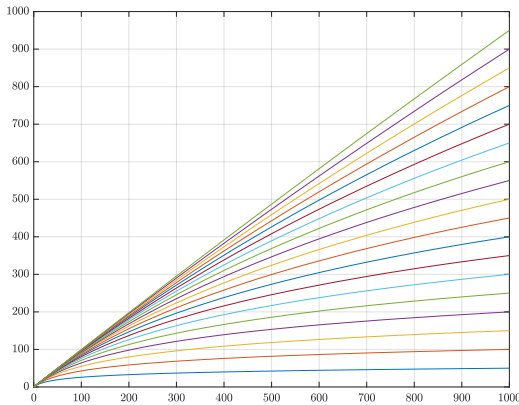


Figure 7: Stirling numbers of the first kind.

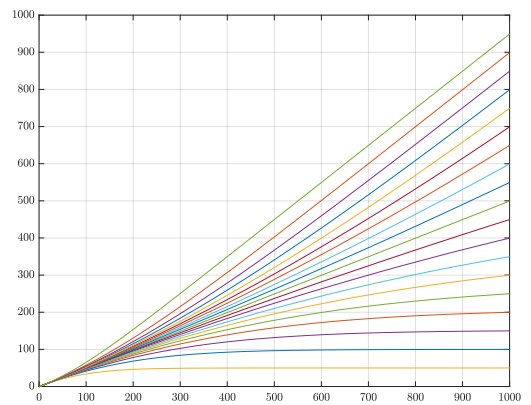


Figure 8: Eulerian numbers.

In the first two cases, the smooth nature of these averaged paths is not unexpected due to old, and more recent, fluid approximation results, see [AC19] or the next sections. This paper aims at a global explanation of the asymptotic behaviour of the four Markov chains exhibited by these simulations.

## 1.5 Asymptotics of sample paths, and field lines of vector fields

Combinatorial analysis, see [Goo61] or [Ben73], yields that,

**Theorem 1.** *In each of the four cases, there exists a function  $\varphi : (0, +\infty) \rightarrow [0, 1]$  such that, for any positive number  $\lambda_\infty$ , when  $(m, \ell) \rightarrow +\infty$  and  $\lim m/\ell = 1 + \lambda_\infty$ ,*

$$\lim p_1(m, \ell) = \varphi(\lambda_\infty).$$

At the end of this section, the function  $\varphi$  is described for each of the four cases.

We set, for  $(m, \ell) \in \mathring{S}$ ,

$$\lambda = \lambda(m, \ell) = \frac{m - \ell}{\ell}.$$

As a direct consequence of Theorem 1, one expects a fluid approximation of the previous Markov chains by a special family of curves : let  $\mathbb{P}_{(m, \ell)}$  denote the probability distribution of the Markov chain  $W$  starting from  $(m, \ell)$ , and let  $(x, \gamma_\lambda(x))_{0 \leq x \leq n}$  be the field line going through the point  $(1, \ell/m) = (1, 1/(1 + \lambda))$  for the vector field  $(1, \varphi(-1 + x/y))$ , or, equivalently, let  $\gamma_\lambda$  denote the solution of the ODE

$$y' = \varphi\left(\frac{x - y}{y}\right) \quad (3)$$

that satisfies  $y(1) = 1/(1 + \lambda)$ . So far we have a complete proof of this approximation only in the first 2 cases :

**Theorem 2.** *In the first 2 cases, there exists  $\eta > 0$  such for any  $\varepsilon > 0$  and for any  $\lambda_\infty > 0$ , when  $(m, \ell) \rightarrow +\infty$  and  $\lim \lambda(m, \ell) = \lambda_\infty$ ,*

$$\lim \mathbb{P}_{(m, \ell)} \left( \sup_{\varepsilon \leq t \leq 1} \left( \left\| \frac{1}{m} W_{\lfloor m(1-t) \rfloor} - (t, \gamma_{\lambda_\infty}(t)) \right\| \right) \geq m^{-\eta} \right) = 0.$$

This kind of statement seems to hold true for eulerian numbers, according to our simulations (see Section 1.4), but remains an open question. For Stirling numbers of the first kind, our proof is still incomplete at the moment, but should be completed in the next few weeks. For Stirling numbers of the second kind, the proofs rely mainly on uniform bounds for

$$m |p_1(m, \ell) - \varphi(\lambda(m, \ell))|,$$

on domains that approach  $\mathring{S}$  as well as possible. These bounds follow from a careful asymptotic analysis of  $T(m, \ell)$ .

Our choice of combinatorial triangles may seem arbitrary, and we confess it is : for instance, Bell's triangle or Delannoy's triangle have also Pascal's formulas, but of a slightly different form. It remains to see if the approach of this paper still produces results for Bell's triangle or Delannoy's triangle, in spite of these slight differences.

### 1.5.1 Description of $\varphi$

- *Pascal's triangle.* It is well known that for all  $(m, \ell) \in S^*$ ,

$$p_1(m, \ell) = \frac{\ell}{m},$$

so that

$$\varphi_1(\lambda) = \frac{1}{1 + \lambda}.$$

Relation (3) reduces to  $y' = y/x$ , with the linear functions as solutions, as expected.

- *Stirling numbers of the second kind.* For  $\lambda > 0$ , let  $\varphi_2$  be defined, through  $\zeta_2(\lambda)$ , the unique positive solution of

$$\frac{\zeta_2}{1 - e^{-\zeta_2}} = 1 + \lambda, \quad \text{by } \varphi_2(\lambda) = e^{-\zeta_2}.$$

- *Stirling numbers of the first kind.* For  $\lambda > 0$ , let  $\varphi_3$  be defined, through  $\zeta_3(\lambda)$ , the unique solution, in  $(0, 1)$ , of

$$\frac{\zeta_3}{(\zeta_3 - 1) \ln(1 - \zeta_3)} = 1 + \lambda, \quad \text{by } \varphi_3(\lambda) = 1 - \zeta_3.$$

- *Eulerian numbers.* For  $\lambda > 0$ , let  $\varphi_4$  be defined, through  $\zeta_4(\lambda)$ , the unique solution, in  $\mathbb{R}$ , of

$$\frac{1}{1 + \lambda} = \frac{e^{\zeta_4}}{e^{\zeta_4} - 1} - \frac{1}{\zeta_4}, \quad \text{by } \varphi_4(\lambda) = 1 - \frac{\zeta_4}{(1 + \lambda)(e^{\zeta_4} - 1)}.$$

## 2 Time reversal and Markov property for $W$

In this section, we prove Propositions 1, 2, 3 and 4. The notations  $X_n, Y_n, W_n$  are defined at Section 1.3.

### 2.1 Time reversal

As already known since the beginning of the theory, a time-reversed Markov process is still a Markov process (cf. [CW05, Chap. 10]), but it is an inhomogeneous one. Let us recall the basic facts that we need here : if  $h_k$  denotes the probability distribution of  $X_k$  and if  $X = (X_k)_{k \geq 0}$  is an inhomogeneous Markov chain with kernels  $(Q_k)_{k \geq 0}$ , i.e.

$$Q_{k,i,j} = \mathbb{P}(X_{k+1} = j \mid X_k = i),$$

then

**Proposition 5.**  $W = (W_n)_{0 \leq n \leq m}$  is a Markov chain with kernel  $P$  defined by

$$P_{(n,i),(n-1,j)} = \frac{h_{n-1}(j)Q_{n-1,j,i}}{h_n(i)}.$$

*Proof.* First,  $(P_n)$  is a sequence of probability kernels due the Chapman-Kolmogorov equations for  $(h_n)$  and  $(Q_n)$ . Then,

$$\mathbb{P}((X_k)_{0 \leq k \leq m} = (x_k)_{0 \leq k \leq m}) = h_0(x_0) \prod_{k=0}^{m-1} Q_{k,x_k,x_{k+1}},$$

thus, provided that  $x_m = \ell$ ,

$$\begin{aligned} \mathbb{P}((X_k)_{0 \leq k \leq m} = (x_k)_{0 \leq k \leq m} \mid X_m = \ell) &= h_0(x_0) \prod_{k=0}^{m-1} Q_{k,x_k,x_{k+1}} / h_m(\ell), \\ &= \prod_{k=0}^{m-1} P_{(k+1,x_{k+1}),(k,x_k)}. \end{aligned}$$



That is,

$$\mathbb{P}((W_k)_{0 \leq k \leq m} = (m - k, x_{m-k})_{0 \leq k \leq m} \mid W_0 = (m, \ell)) = \prod_{k=0}^{m-1} P_{(k+1, x_{k+1}), (k, x_k)},$$

as expected.  $\square$

In the first three examples,  $h_n(k) = T(n, k)\theta^k/T_n(\theta)$ , or  $h_n(k) = T(n, k)\theta^{k\downarrow}/T_n(\theta)$ , in which  $T_n(\theta)$  is a normalizing constant :

$$T_n(\theta) = \sum_{k=0}^n T(n, k)\theta^k, \quad \text{or} \quad T_n(\theta) = \sum_{k=0}^n T(n, k)\theta^{k\downarrow}. \quad (4)$$

For eulerian numbers,  $h_n(k) = T(n, k)/T_n(1) = T(n, k)/n!$ . Also,  $Q_n$  results from a natural growing mechanism with independent steps, that is, a Markovian growth process obtained as follows :

- by addition of an  $n + 1$ th letter, either **a** or **b**, at the end of a random word of  $\{\mathbf{a}, \mathbf{b}\}^n$ , in order to form an  $n + 1$ -letters long word, for Pascal's triangle,
- by addition of the image of  $n + 1$  to a random mapping from  $\llbracket n \rrbracket$  to  $\llbracket N \rrbracket$ , in order to form a random mapping from  $\llbracket n + 1 \rrbracket$  to  $\llbracket N \rrbracket$ , for the the second Stirling triangle,
- by random insertion of  $n + 1$  in order to form a permutation on  $\llbracket n + 1 \rrbracket$ , starting from a permutation on  $\llbracket n \rrbracket$ , for the 2 other examples.

In each case, the added letter, or integer, is chosen independently of the history of the growth process, hence the Markovian character of these growth processes. For the sake of brevity, in this paper, we call this last growth process the *random permutation process*. For these three nonhomogeneous Markov growth processes, there exist well studied functionals that retain the Markov property, and whose one-dimensional distributions are given by the rows of the corresponding combinatorial triangle :

- the sequence of counts of letter **a**, in the sequence of words defined previously, forms one of the most studied Markov chain : the simple random walk, whose one-dimensional distributions  $h_n$  are binomial distributions, famously related to Pascal's triangle ;
- the sequence of sizes of images, derived from the sequence of random mappings, is a famous inhomogeneous Markov chain, related to the coupon collector problem : it is the sequence of successive sizes of the collection. Its one-dimensional distributions  $h_n$  have a simple expression in terms of the Stirling numbers of the second kind ;
- the sequence of number of cycles, derived from the random permutation process, is an inhomogeneous Markov chain, related to the chinese restaurant process. Its one-dimensional distributions  $h_n$  have a simple expression in terms of the Stirling numbers of the first kind ;

- the sequence of the number of descents, also derived from the random permutation process, is an inhomogeneous Markov chain, related to the internal diffusion limited aggregation process. Its one-dimensional distributions  $h_n$  have a simple expression in terms of eulerian numbers.

The Chapman-Kolmogorov equations for these Markov chains are derived from the Pascal's formulas for the corresponding triangles through renormalization : in our settings,  $Q_n$  is defined by  $(a, b)$  as follows

$$Q_{n,x,y} = c_n(\theta) \left( b(n+1, y) \mathbb{1}_{y=x \in \llbracket n \rrbracket} + a(n+1, y) \theta \mathbb{1}_{y=x+1 \in \llbracket n+1 \rrbracket} \right), \quad (5)$$

in which  $c_n(\theta)$  denotes a normalizing factor  $T_n(\theta)/T_{n+1}(\theta)$ , and  $\theta = \theta^{x+1}/\theta^x$  should be replaced, in (5), with  $\theta - x = \theta^{x+1\downarrow}/\theta^{x\downarrow}$  in the case of Stirling numbers of the second kind. For eulerian numbers,  $\theta = 1$ . Then, Pascal's formulas appear as special cases of the Chapman-Kolmogorov equation  $h_{n-1}Q_{n-1} = h_n$ , and relation (2) is just a special case of Proposition 5.

Here,  $Q_{n,x_n,x_{n+1}} \neq 0$  only if  $\varepsilon_{n+1} = x_{n+1} - x_n$  belongs to  $\{0, 1\}$ , thus  $P_{(n,x),(n-1,y)} \neq 0$  only if  $\varepsilon = x - y$  belongs to  $\{0, 1\}$  : in this paper,  $P_{(n,x),(n-1,x-\varepsilon)}$  is abridged to  $p_\varepsilon(n, x)$ .

## 2.2 Simple random walk

*Proof of Proposition 1.* Here

$$\begin{aligned} T(n, k) &= \binom{n}{k}, \quad \theta = \frac{p}{1-p}, \quad T_n(\theta) = (1+\theta)^n \\ h_n(k) &= \binom{n}{k} p^k (1-p)^{n-k}, \quad c_n(\theta) = \frac{1}{1+\theta} = 1-p. \end{aligned}$$

Then, for instance,

$$\begin{aligned} P_{(n,i),(n-1,i-1)} &= \frac{h_{n-1}(i-1)Q_{n-1,i-1,i}}{h_n(i)} \\ &= \frac{\binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \times p}{\binom{n}{i} p^i (1-p)^{n-i}} = \frac{\binom{n-1}{i-1}}{\binom{n}{i}} = p_1(n, i) \end{aligned}$$

as expected. □

## 2.3 Coupon collector's problem

Let us recall the famous problem studied by Gauss and Laplace, among others : a collector wants to complete a collection of  $N$  different items (denoted  $1, \dots, N$ ). At each step, he receives a coupon chosen uniformly from  $\llbracket 1, N \rrbracket$ . The average time to complete the collection is known to be  $NH_N$ , where

$$H_N = \sum_{k=1}^N \frac{1}{k}$$

is the  $N$ th harmonic number. If  $X_n$  denotes the number of different items in the collection after the  $n$ th step, then we call the graph of  $t \mapsto X_{[t]}$  the *completion curve*.

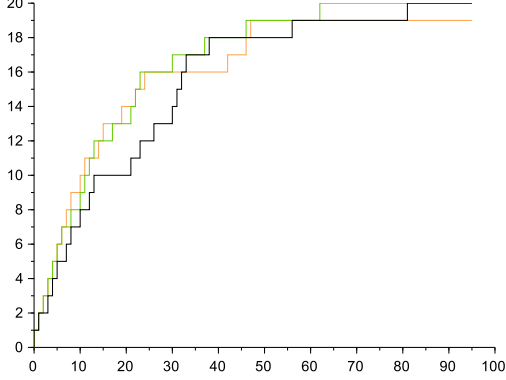


Figure 9: Three completion curves for a  $n = 20$  items collection.

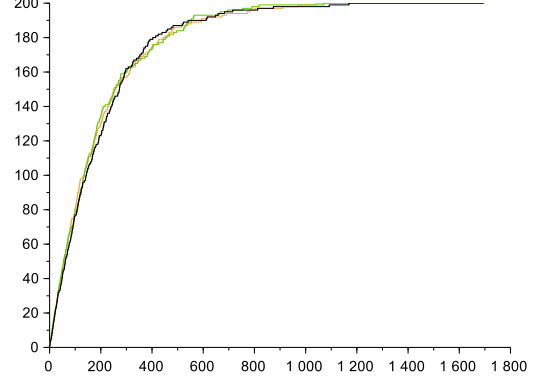


Figure 10: Three completion curves for a  $n = 200$  items collection.

*Proof of Proposition 2.* See [AC19, Proposition 1], in which the proof is given for  $m = N$ . It fits with the frame given at Section 2.1 as follows : set

$$T(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, \quad \theta = N,$$

but consider a variant of  $T_n$ . Here :

$$T_n(\theta) = \sum_k T(n, k) \theta^{k\downarrow} = N^n$$

$$h_n(k) = \binom{N}{k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! \frac{1}{N^n} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{N^{k\downarrow}}{N^n}, \quad c_n(\theta) = \frac{1}{N}.$$

Then, for instance,

$$Q_{n,k,k+1} = a(n+1, k) \frac{\theta^{k+1\downarrow}}{\theta^{k\downarrow}} c_n(\theta) = \frac{N-k}{N}.$$

and

$$p_1(n, k) = P_{(n,k),(n-1,k-1)} = \frac{h_{n-1}(k-1) Q_{n-1,k-1,k}}{h_n(k)}$$

$$= \frac{\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \frac{N^{k-1\downarrow}}{N^{n-1}} \times \frac{N-k+1}{N}}{\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{N^{k\downarrow}}{N^n}} = \frac{\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}}{\left\{ \begin{matrix} n \\ k \end{matrix} \right\}},$$

as expected. □

## 2.4 Chinese restaurant process

Set  $\theta \in (0, +\infty)$ . The chinese restaurant process, introduced in 1974 by Antoniak in [Ant74], is defined as follows : when entering a metaphoric chinese restaurant, the

first customer seats at the first table. For  $n > 1$ , the  $n$ th customer seats at the  $k$ th (non-empty) table with probability  $\frac{c_{n,k}}{n-1+\theta}$  (where  $c_{n,k}$  is the number of customers seated at this table), or at an empty table with probability  $\frac{\theta}{n-1+\theta}$ . Let  $X_n$  denote the number of non-empty tables after the arrival of the  $n$ th customer. For exemple, let us sample the first 50 steps of the process, for  $\theta = 1$  :

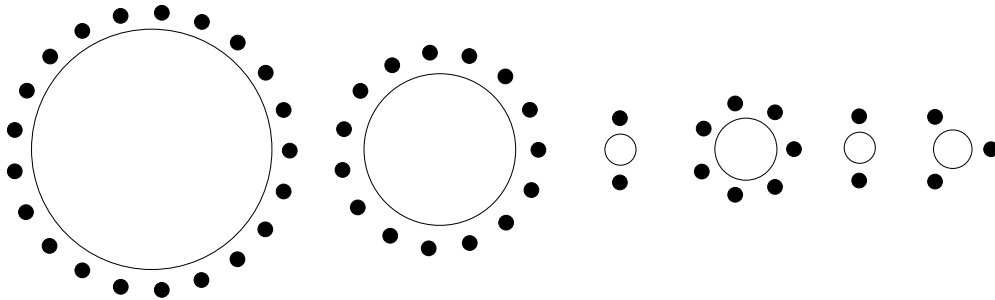


Figure 11: A realization of the chinese restaurant process (here  $X_{50} = 6$ ).

*Proof of Proposition 3.* In this example,  $(Y_i)_{i \geq 1}$  is a family of independent Bernoulli random variables with respective parameters  $\mu_i = \theta/(i - 1 + \theta)$ . We have :

$$T(n, k) = \binom{n}{k}, \quad T_n(\theta) = \sum_k T(n, k) \theta^k = (\theta)^{\uparrow n}, \quad c_n(\theta) = \frac{1}{\theta + n}.$$

Thus the probability distribution of  $X_n$  is given, for  $n \geq 1$ , by:

$$h_n(\ell) = \mathbb{P}(X_n = \ell) = \frac{\theta^\ell}{(\theta)^{\uparrow n}} \binom{n}{\ell} \mathbb{1}_{1 \leq \ell \leq n},$$

see [Pit06, Section 3.1.3]. For instance,

$$Q_{n,k,k+1} = \frac{\theta}{n + \theta} = c_n(\theta) a(n + 1, k) \theta.$$

and

$$\begin{aligned} p_1(n, k) &= P_{(n,k), (n-1,k-1)} = \frac{h_{n-1}(k-1) Q_{n-1,k-1,k}}{h_n(k)} \\ &= \frac{\frac{\theta^{k-1}}{(\theta)^{n-1 \uparrow}} \binom{n-1}{k-1} \times \frac{\theta}{n-1+\theta}}{\binom{n}{k} \frac{\theta^k}{(\theta)^{n \uparrow}}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}, \end{aligned}$$

as expected. □

## 2.5 One-dimensional Internal Diffusion Limited Aggregation process

Diaconis and Fulton [DF91] introduced the internal Diffusion Limited Aggregation process (iDLA). Lawler, Bramson and Griffeath [LBG92] coined the terminology *iDLA*, and obtained an asymptotic shape behaviour. In the iDLA process, an aggregate of particles on  $\mathbb{Z}^d$  is built as follows:

- i) the first particle settles at the origin;
- ii) the next particles perform a symmetric random walk on  $\mathbb{Z}^d$ , starting from the origin, and settle at the first empty site they encounter.

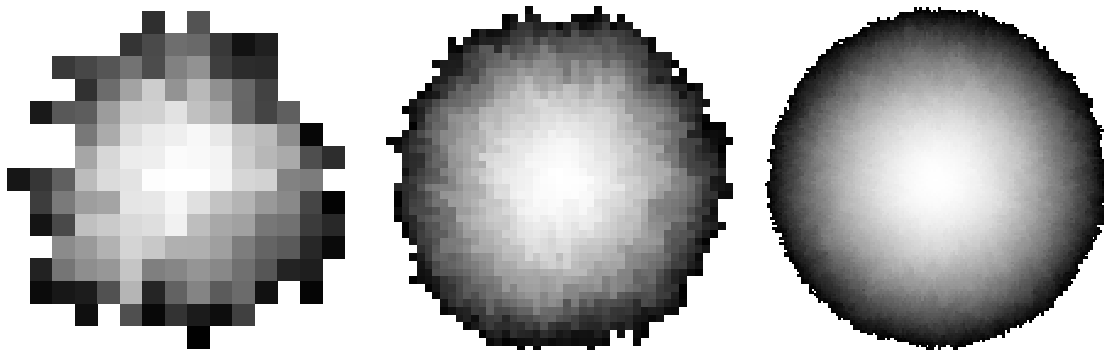


Figure 12: Normalized iDLA aggregates with 150, 1,500 and 15,000 particles on  $\mathbb{Z}^2$ .

When  $d = 1$ , let  $X_n$  denote the number of particles settled to the right of the origin after the  $n$ th step. Then, according to [Mit20], the process  $(X_n)_n$  is an inhomogeneous Markov chain with the same distribution as the sequence of number of descents of the sequence of random permutations defined previously. Both processes have the one-dimensional distribution below

$$\mathbb{P}(X_n = k) = h_n(k) = \frac{\langle n \atop k \rangle}{n!} \mathbb{1}_{(n,k) \in S}.$$

In the case of the one-dimensional iDLA we can stack successive aggregates upon one another to form a space-time diagram. As with Figure 12, the longer it took to visit a cell, the darker we color it.

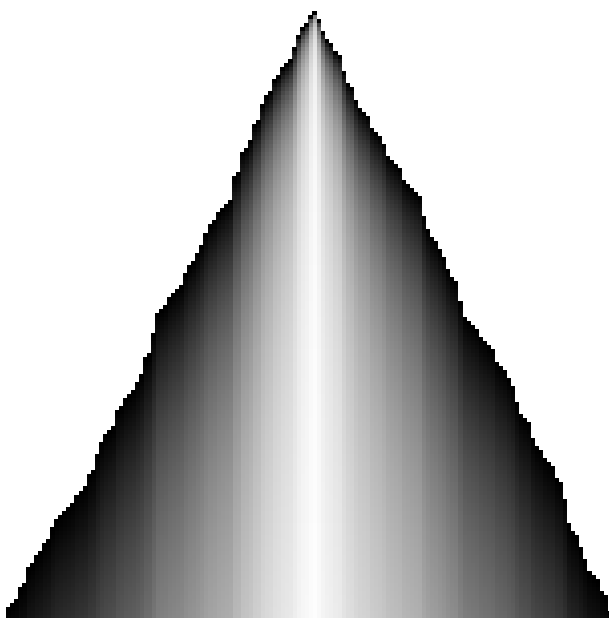


Figure 13: Space-time diagram of a one-dimensional iDLA.

*Proof of Proposition 4.* In this example, we have :

$$T(n, k) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle, \quad T_n(1) = \sum_k T(n, k) = n!, \quad c_n(1) = \frac{1}{n}.$$

Thus the probability distribution of  $X_n$  is given, for  $n \geq 1$ , by:

$$h_n(\ell) = \mathbb{P}(X_n = \ell) = \frac{1}{n!} \left\langle \begin{matrix} n \\ \ell \end{matrix} \right\rangle \mathbb{1}_{1 \leq \ell \leq n},$$

see [Pit06, Section 3.1.3]. For instance,

$$Q_{n,k,k+1} = \frac{n-k}{n+1} = c_n(1) a(n+1, k).$$

and

$$\begin{aligned} p_1(n, k) &= P_{(n,k), (n-1,k-1)} = \frac{h_{n-1}(k-1) Q_{n-1,k-1,k}}{h_n(k)} \\ &= \frac{\frac{1}{(n-1)!} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle \times \frac{n-k}{n}}{\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \frac{1}{n!}} = \frac{\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle (n-k)}{\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle}, \end{aligned}$$

as expected. □

### 3 The limit vector field

#### 3.1 Proof of Theorem 1

One can see the set of average jumps from  $(n, k) \in S$ ,

$$\begin{aligned} v(n, k) &= p_0(n, k) \times (-1, 0) + p_1(n, k) \times (-1, -1) \\ &= (-1, -p_1(n, k)), \end{aligned}$$

as a kind of discrete vector field on  $S$ , with slope  $p_1(n, k)$  at point  $(n, k)$ . As a consequence, the convergence of the sample paths of the time-reversed Markov chains of Section 1.3 (see Theorem 2) requires a precise asymptotic analysis of

$$p_1(n, k) = \frac{a(n, k)T(n-1, k-1)}{T(n, k)},$$

and thus, of  $T(n, k)$ . Consider the generating functions  $V_k$  and  $H_n$  defined by

$$V_k(z) = \sum_{n=k}^{+\infty} \frac{1}{\varepsilon_n} T(n, k) z^n, \quad H_n(w) = \sum_{k=0}^n \frac{1}{\varepsilon_n} T(n, k) w^k,$$

respectively. In the first three cases,  $V_k$  exhibits a factorisation  $A \times B^k$  suitable for the saddle-point method, while, for eulerian numbers,  $H_n$  is approximately of the form  $B^n$ , allowing the use of large deviations methods. Due to these factorisations, the limit vector field depends only on the slope  $y/x$ , and the function  $\varphi$  depends on  $B$  alone, in the first 3 cases through the saddle-point equation

$$\frac{B'(\zeta)}{B(\zeta)} = \frac{1 + \lambda}{\zeta}, \tag{6}$$

obtained by optimisation of the function  $x \rightarrow \frac{B(x)}{x^{1+\lambda}}$  on  $(0, +\infty)$ , and, for eulerian numbers, through the Legendre transformation of  $\ln B$ , leading to the equation :

$$\frac{B'(\zeta)}{B(\zeta)} = \frac{1}{1 + \lambda}. \tag{7}$$

*Proof of Theorem 1.* But for eulerian numbers, let  $\zeta(\lambda)$  be defined implicitly by (6), i.e. let  $\zeta(\cdot)$  be the inverse function of :

$$x \longrightarrow \frac{x B'(x)}{B(x)} - 1.$$

The eulerian case is similar, but uses large deviations rather than saddle-point methods, and will be handled separately. In the remaining 3 cases, recall that  $a(n, k) = 1$ , and set :

$$1 + \lambda = \frac{n}{k}, \quad \zeta = \zeta(\lambda), \quad 1 + \tilde{\lambda} = \frac{n-1}{k-1}, \quad \tilde{\zeta} = \zeta(\tilde{\lambda}).$$

For these 3 cases, the saddle-point method, see [FS09, Part B, Chap. VIII], leads to

$$T(n, k) \sim \frac{\varepsilon_n}{\varepsilon_k} \left( \frac{B(\zeta)}{\zeta^{1+\lambda}} \right)^k f(n, k),$$

in which  $f(., .)$  is some factor such that  $f(n, k) \sim f(n - 1, k - 1)$ . The factor  $\varepsilon_n/\varepsilon_k$  matters only for the enumeration of sets of labelled structures (subsets or cycles, for the 2 Stirling's triangles, cf. [FS09, Part A]) and thus does not appear for Pascal's triangle : in the case of labelled structures, the factor  $n!$  is due to the use of EGFs, and since we consider *sets*, not sequences, of  $k$  objects, i.e. *unordered collections*, the generating function  $V_k$  has a factor  $1/k!$ . As a consequence, for the two Stirling triangles, we have

$$\begin{aligned} p_1(n, k) &\sim \frac{a(n, k)}{n} k \frac{\tilde{\zeta}^{1+\tilde{\lambda}}}{B(\tilde{\zeta})} \left( \frac{B(\tilde{\zeta})}{\tilde{\zeta}^{1+\tilde{\lambda}}} \frac{\zeta^{1+\lambda}}{B(\zeta)} \right)^k \\ &\sim \frac{1}{1+\lambda} \frac{\zeta^{1+\lambda}}{B(\zeta)} \left( \frac{\tilde{\zeta}^{1+\lambda}}{\tilde{\zeta}^{1+\tilde{\lambda}}} \right)^k \left( \frac{B(\tilde{\zeta})}{\tilde{\zeta}^{1+\tilde{\lambda}}} \frac{\zeta^{1+\lambda}}{B(\zeta)} \right)^k \\ &\sim \frac{1}{1+\lambda} \frac{\zeta^{1+\lambda}}{B(\zeta)} \zeta^{(\lambda-\tilde{\lambda})k}, \end{aligned}$$

the last step due to

$$\lim_k k \ln \left( \frac{B(\tilde{\zeta})}{\tilde{\zeta}^{1+\tilde{\lambda}}} \frac{\zeta^{1+\lambda}}{B(\zeta)} \right) = 0. \quad (8)$$

Actually, since  $\zeta$  is solution of the saddle-point equation, the derivative of

$$x \rightarrow \ln \left( \frac{B(x)}{x^{1+\lambda}} \right)$$

vanishes at  $\zeta$ , thus

$$\ln \left( \frac{B(\tilde{\zeta})}{\tilde{\zeta}^{1+\tilde{\lambda}}} \frac{\zeta^{1+\lambda}}{B(\zeta)} \right) = o(\tilde{\zeta} - \zeta) = o(\lambda - \tilde{\lambda}),$$

but

$$\lambda - \tilde{\lambda} = \frac{n}{k} - \frac{n-1}{k-1} \sim \frac{-\lambda}{k},$$

entailing (8). Thus

$$p_1(n, k) \sim \frac{1}{1+\lambda} \frac{\zeta}{B(\zeta)}.$$

Finally, for Stirling's triangles, the saddle-point equation (6) gives

$$p_1(n, k) \sim \frac{1}{B'(\zeta)},$$

For Pascal's triangle,  $\binom{n}{k}$  enumerates words with  $n$  letters,  $k$  among them being a's and the  $n - k$  others being b's, thus Pascal's triangle enumerates *sequences* (not



sets) of *unlabelled* objects<sup>1</sup>, for which one usually uses OGFs. As a consequence, compared with the previous computation, we are rid of the factor  $n!/k!$  in  $T(n, k)$ , and of the factor  $k/n = 1/(1 + \lambda)$  in  $p_1(n, k)$ , thus we obtain

$$p_1(n, k) \sim \frac{\zeta}{B(\zeta)}.$$

Before we turn to the case of eulerian numbers, let us derive  $\varphi$  for each of the 3 first cases :

- *Pascal's triangle* :

$$\begin{aligned} V_{k,1}(z) &= \sum_{n \geq k} \binom{n}{k} z^n = \frac{1}{1-z} \left( \frac{z}{1-z} \right)^k, \\ B_1(z) &= \frac{z}{1-z}, \\ p_1(n, k) &\sim \frac{\zeta}{B_1(\zeta)} = 1 - \zeta. \end{aligned}$$

Here (6) can be written

$$\frac{1}{1-\zeta} = 1 + \lambda,$$

thus  $\varphi_1(k/n) = \frac{1}{1+\lambda} = k/n$ , that is :

$$p_1(n, k) \sim \frac{k}{n},$$

which is not a surprise, since it is well known that, actually,  $p_1(n, k) = \frac{k}{n}$ .

- *Stirling numbers of the second kind*

Here :

$$\begin{aligned} V_{k,2}(z) &= \sum_{n \geq k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{z^n}{n!} = \frac{1}{k!} (e^z - 1)^k, \\ B_2(z) &= e^z - 1, \\ p_1(n, k) &\sim \frac{1}{B_2'(\zeta)} = e^{-\zeta}, \end{aligned}$$

and (6) can be written

$$\frac{\zeta}{1 - e^{-\zeta}} = 1 + \lambda, \tag{9}$$

see [AC19]. According to Good and others,  $\zeta$  is a smooth concave function of  $\lambda > 0$ , with positive values. Thus

$$\varphi_2(k/n) = e^{-\zeta(\frac{n}{k}-1)}.$$

Note that (9) is the equation to be solved when one wants to tune the parameter  $\zeta$  of a Poisson random variable *conditioned to be positive* in order to obtain the expectation  $1 + \lambda$ .

---

<sup>1</sup>Such a word is a sequence of  $k$  words of the form  $\mathbf{b}^m \mathbf{a}$  followed by a word of the form  $\mathbf{b}^m$ .

- *Stirling numbers of the first kind (unsigned)*

$$V_{k,3}(z) = \sum_{n \geq k} \binom{n}{k} \frac{z^n}{n!} = \frac{1}{k!} (-\ln(1-z))^k,$$

$$B_3(z) = -\ln(1-z),$$

$$p_1(n, k) \sim \frac{1}{B_3'(\zeta)} = 1 - \zeta.$$

Here (6) can be written

$$\frac{\zeta}{(\zeta - 1) \ln(1 - \zeta)} = 1 + \lambda, \quad (10)$$

which defines  $\zeta$  as smooth concave function of  $\lambda > 0$ , with values in  $(0, 1)$ . Thus

$$\varphi_3(k/n) = 1 - \zeta\left(\frac{n}{k} - 1\right). \quad (11)$$

Note that (10) is the equation to be solved when one wants to tune the parameter  $\zeta$  of a logarithmic probability distribution in order to obtain the expectation  $1 + \lambda$ .

- *Eulerian numbers*

Though the computation of  $\varphi_4$  has a similar flavour, it presents some notable differences. In order to sum up the asymptotic analysis of eulerian numbers, set, as done in [Ben73] :

$$t = \frac{k}{n} = \frac{1}{1 + \lambda}.$$

In [Ben73], the main tool is the approximation of  $H_n(e^s)$ , the Laplace transform of  $h_n$ , by

$$B(s)^{n+1} = r(s)^{-n-1} = \left(\frac{e^s - 1}{s}\right)^{n+1}.$$

In other terms, the key point in [Ben73] is that  $h_n$  is approximately the distribution of the sum of  $n + 1$  i.i.d. uniform random variables, with Laplace transform  $B(s)$ . This is reminiscent of Tanny's representation of eulerian numbers (cf. [Tan73]) :

$$h_n(k) = \frac{\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle}{n!} = \mathbb{P}([U_1 + U_2 + \dots + U_n] = k). \quad (12)$$

Bender obtains the following asymptotic formula for  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$  when  $(n, k)$  goes to infinity

$$\frac{\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle}{n!} \sim (B(\zeta)e^{-\zeta t})^n f(n, k)$$

in which  $f(.,.)$  is some factor such that  $f(n, k) \sim f(n-1, k-1)$ , and in which  $\zeta$  is the only real number such that

$$\begin{aligned} \frac{1}{1+\lambda} = t &= \frac{\partial}{\partial \zeta} \ln \left( \frac{e^\zeta - 1}{\zeta} \right) \\ &= \frac{e^\zeta}{e^\zeta - 1} - \frac{1}{\zeta}. \end{aligned} \quad (13)$$

One recognizes in  $\zeta(\cdot)$  the derivative of the Legendre-Fenchel transformation of the cumulant-generating function of the uniform distribution, i.e. the unique solution of

$$\frac{\partial}{\partial \zeta} \ln (B(\zeta)e^{-\zeta t}) = \frac{B'(\zeta)}{B(\zeta)} - t = 0. \quad (14)$$

As a consequence, for eulerian numbers, we have

$$\begin{aligned} p_1(n, k) &\sim \frac{a(n, k)}{n} \left( \frac{B(\tilde{\zeta})^{n-1} e^{-\tilde{\zeta}(k-1)}}{B(\zeta)^n e^{-\zeta k}} \right) \\ &\sim \frac{a(n, k)}{n} \frac{e^{\tilde{\zeta}}}{B(\tilde{\zeta})} \left( \frac{B(\tilde{\zeta})^n e^{-\tilde{\zeta} k}}{B(\zeta)^n e^{-\zeta k}} \right) \\ &\sim \frac{(1-t)e^\zeta}{B(\zeta)} \left( \frac{B(\tilde{\zeta})e^{-\tilde{\zeta} t}}{B(\zeta)e^{-\zeta t}} \right)^n \\ &\sim \frac{(1-t)e^\zeta}{B(\zeta)} \end{aligned}$$

the last step due to

$$\lim_n n \ln \left( \frac{B(\tilde{\zeta})e^{-\tilde{\zeta} t}}{B(\zeta)e^{-\zeta t}} \right) = 0. \quad (15)$$

Actually, since  $\zeta$  is solution of (14), the derivative of

$$x \rightarrow \ln (B(x)e^{-xt})$$

vanishes at  $\zeta$ , thus

$$\ln \left( \frac{B(\tilde{\zeta})e^{-\tilde{\zeta} t}}{B(\zeta)e^{-\zeta t}} \right) = o(\tilde{\zeta} - \zeta) = o(t - \tilde{t}),$$

but

$$t - \tilde{t} = \frac{k}{n} - \frac{k-1}{n-1} \sim \frac{1-t}{n},$$

entailing (15). Thus

$$p_1(n, k) \sim \frac{(1-t)\zeta e^\zeta}{e^\zeta - 1} = \frac{\lambda \zeta}{(1+\lambda)(1-e^{-\zeta})} = \phi_4(\lambda).$$

Note that :

$$\zeta(1-t) = -\zeta(t), \quad \varphi_4(1-t) = 1 - \varphi_4(t) = \frac{t\zeta}{e^\zeta - 1},$$

as expected. □

## 4 Sample path convergence

For the sake of completeness, this section is devoted to the well known proof of Theorem 2, in the case of Pascal's triangle. In the case of Stirling numbers of the second kind, Theorem 2 was obtained in [AC19] at the price of a more tedious proof. For Stirling numbers of the first kind, and for Euler's triangle as well, we think that the same property holds true, but the proofs are still works in progress.

*Proof of Theorem 2.* (Pascal's triangle case) Set  $p = \ell/m$  and let  $(Y_i)_i$  be a sequence of i.i.d. Bernoulli random variables with parameter  $p$ . Consider  $\hat{W} = (\hat{W}_k)_{0 \leq k \leq m}$  defined by  $\hat{W}_k = (m - k, X_{m-k})$  where  $X_n = Y_1 + \dots + Y_n$ . We have seen in Proposition 1 that the conditional distribution of  $\hat{W}$  given  $\hat{W}_0 = (m, \ell)$  and the distribution of  $W$  are the same. By Hoeffding's inequality, for all  $t > 0$ , and all  $n \in \llbracket 0, m \rrbracket$ ,

$$\begin{aligned} \mathbb{P}(|X_n - np| \geq t) &\leq 2 \exp\left(-\frac{2t^2}{n}\right) \\ &\leq 2 \exp\left(-\frac{2t^2}{m}\right). \end{aligned}$$

In particular, for all  $\varepsilon \in (0, 1/2)$  and for  $t = m^{1/2+\varepsilon}$ ,

$$\mathbb{P}(|X_n - np| \geq m^{1/2+\varepsilon}) \leq 2 \exp(-2m^{2\varepsilon})$$

Thus

$$\begin{aligned} \mathbb{P}(\exists n \in \llbracket 0, m \rrbracket \text{ s.t. } |X_n - np| \geq m^{1/2+\varepsilon}) &\leq \sum_{n=1}^m \mathbb{P}(|X_n - np| \geq m^{1/2+\varepsilon}) \\ &\leq 2m \exp(-2m^{2\varepsilon}) \end{aligned}$$

Set

$$\begin{aligned} A_m &= \left\{ \exists n \in \llbracket 0, m \rrbracket \text{ s.t. } \|W_n - (m - n, (m - n)p)\|_1 \geq m^{1/2+\varepsilon} \right\}, \\ \hat{A}_m &= \left\{ \exists n \in \llbracket 0, m \rrbracket \text{ s.t. } \|\hat{W}_n - (m - n, (m - n)p)\|_1 \geq m^{1/2+\varepsilon} \right\}. \end{aligned}$$

Then  $\mathbb{P}(\hat{A}_m) \leq 2m \exp(-2m^{2\varepsilon})$  and

$$\mathbb{P}(A_m) = \frac{\mathbb{P}(\hat{A}_m \cap \{X_m = \ell\})}{\mathbb{P}(X_m = \ell)} \leq \frac{\mathbb{P}(\hat{A}_m)}{\mathbb{P}(X_m = \ell)}$$

But

$$\begin{aligned} \mathbb{P}(X_m = \ell) &= \binom{m}{\ell} p^\ell (1-p)^{m-\ell} \\ &\sim \frac{1}{\sqrt{2\pi p(1-p)}} \cdot \frac{1}{\sqrt{m}}. \end{aligned}$$

Finally,  $\mathbb{P}(A_m) = O(m^{3/2} e^{-2m^{2\varepsilon}})$  and for  $\eta = 1/2 - r \in (0, 1/2)$  and  $\varepsilon \in (0, 1)$ ,

$$\lim_m \mathbb{P}_{(m,\ell)} \left( \sup_{\varepsilon \leq t \leq 1} \left( \left\| \frac{1}{m} W_{\lfloor m(1-t) \rfloor} - (t, \gamma_{m,\ell}(t)) \right\|_1 \right) \geq m^{-\eta} \right) = 0.$$

□

## References

- [AC19] Anis Amri and Philippe Chassaing, *The impatient collector*, working paper or preprint, June 2019.
- [Ant74] Charles E. Antoniak, *Mixtures of Dirichlet processes with applications to Bayesian nonparametric problems*, *Ann. Statist.* **2** (1974), 1152–1174. MR 365969
- [Ben73] Edward A. Bender, *Central and local limit theorems applied to asymptotic enumeration*, *J. Combinatorial Theory Ser. A* **15** (1973), 91–111. MR 375433
- [CW05] Kai Lai Chung and John B. Walsh, *Markov processes, Brownian motion, and time symmetry.*, 2nd ed. ed., Grundlehren Math. Wiss., vol. 249, New York, NY: Springer, 2005 (English).
- [DF91] P. Diaconis and W. Fulton, *A growth model, a game, an algebra, Lagrange inversion, and characteristic classes*, vol. 49, 1991, *Commutative algebra and algebraic geometry, II (Italian) (Turin, 1990)*, pp. 95–119 (1993). MR 1218674
- [FS09] Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009. MR 2483235
- [Goo61] I. J. Good, *An asymptotic formula for the differences of the powers at zero*, *Ann. Math. Statist.* **32** (1961), 249–256. MR 0120204
- [Ken75] Douglas P. Kennedy, *The Galton-Watson process conditioned on the total progeny*, *J. Appl. Probab.* **12** (1975), 800–806 (English).
- [LBG92] Gregory F. Lawler, Maury Bramson, and David Griffeath, *Internal diffusion limited aggregation*, *Ann. Probab.* **20** (1992), no. 4, 2117–2140 (English).
- [Mit20] Kiana Mittelstaedt, *A stochastic approach to Eulerian numbers*, *Am. Math. Mon.* **127** (2020), no. 7, 618–628 (English).
- [Pit06] J. Pitman, *Combinatorial stochastic processes*, *Lecture Notes in Mathematics*, vol. 1875, Springer-Verlag, Berlin, 2006, *Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002*, With a foreword by Jean Picard. MR 2245368
- [Tan73] S. Tanny, *A probabilistic interpretation of Eulerian numbers*, *Duke Math. J.* **40** (1973), 717–722 (English).