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Abstract. In this paper, we consider an incompressible viscous fluid in an infinitely deep ocean, being bounded above by a free moving boundary. The governing equations are the gravity-driven incompressible Navier-Stokes equations with variable density and no surface tension is taken into account on the free surface. After using the Lagrangian transformation, we write the main equations in a perturbed form in a fixed domain. In the first part, we describe a spectral analysis of the linearized equations around a hydrostatic equilibrium $(\rho_0(x_3), 0, P_0(x_3))$ for a smooth increasing density profile ρ_0. Precisely, we prove that there exist infinitely many normal modes to the linearized equations by following the operator method initiated by Lafitte and Nguyễn [20]. In the second part, we study the nonlinear Rayleigh-Taylor instability around the above profile by constructing a wide class of initial data for the nonlinear perturbation problem departing from the equilibrium, based on the finding of infinitely many normal modes. Our nonlinear result extends the previous framework of Guo and Strauss [6] and also of Grenier [10].

Contents

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1. Introduction

The Rayleigh–Taylor (RT) instability, studied first by Lord Rayleigh in [25] and then Taylor [26] is well known as a gravity-driven instability in two semi-infinite inviscid and incompressible fluids when the heavy one is on top of the light one. It has attracted much attention due to both its physical and mathematical importance. Two applications worth mentioning are implosion of inertial confinement fusion capsules [22] and core-collapse of supernovae [24]. For a detailed physical comprehension of the linear RT instability, we refer to three survey papers [18, 30, 31]. Mathematically speaking, the nonlinear study of classical RT instability is proven by Desjardins and Grenier [4]. For the inviscid and incompressible fluid with a smooth density profile, the classical RT instability was investigated by Lafitte [19], by Guo and Hwang [5] and by Helffer and Lafitte [14].

Concerning the viscous linear RT instability, one of the first studies can be seen in the book of Chandrasekhar [3, Chapter X]. He considers two uniform viscous fluid separated by a horizontal boundary and generalize the classical result of Rayleigh and Taylor. We refer the readers to mathematical viscous linear/nonlinear RT studies for two (in-)compressible channel flows in [7], [29] and [12]. For the incompressible fluid in the whole space $\mathbf{R}^3$, with a smooth density profile, we mention the results of Jiang et. al [13] and of Lafitte and Nguyễn [20], respectively.

Let $\mathbb{T} = \mathbf{R}/\mathbb{Z}$ be the usual 1D torus and $\mathbf{T}^2 = 2\pi L_1\mathbb{T} \times 2\pi L_2\mathbb{T}$ $(L_1, L_2 > 0)$. The aim of this paper is to study nonlinear RT instability of the viscous surface wave in an infinitely deep ocean. The domain of the fluid is

$$\Omega(t) = \{x = (x_1, x_2, x_3) \in \mathbf{T}^2 \times \mathbf{R}, x_3 < \eta(t, x_1, x_2)\} \tag{1.1}$$

and $\Omega(t)$ is bounded above by the free surface $\Gamma(t) = \{x_3 = \eta(t, x_1, x_2)\}$, where η is an unknown of the problem. In Eulerian coordinate, the fluid dynamics are governed by the incompressible Navier–Stokes equations (2.1) in the presence of a gravity force field $-ge_3(g > 0)$, and the effect of surface tension is neglected on

the free surface. The movement of $\Omega(t)$ and $\Gamma(t)$ raises numerous mathematical difficulties. Let $\mathbf{R}_- = (-\infty, 0)$, we use the unknown function η and the Lagrangian coordinate transformation to transform the free boundary problem into the equivalent problem, (2.9) in a fixed domain $\Omega = \mathbf{T}^2 \times \mathbf{R}_-$, of which the fixed upper boundary is $\Gamma = \mathbf{T}^2 \times \{x_3 = 0\}$.

Let $' = d/dx_3$, we then consider two smooth functions ρ_0 and P_0 depending only on x_3. Hence $(\rho_0(x_3), 0, P_0(x_3))$ is a steady state solution of Eq. (2.9), provided that $P_0' = -g\rho_0$. Since we are interested in the RT instability, we assume that

$$C_0^\infty(\mathbf{R}_-) \ni \rho_0' \geqslant 0 \quad \text{such that } \mathrm{supp}\rho_0' = [-a, 0], \quad \text{with } a > 0. \tag{1.2}$$

We denote by

$$0 < \rho_- = \rho_0(x_3) \text{ for all } x_3 \leqslant -a, \quad \rho_0(0) = \rho_+. \tag{1.3}$$

This means that a layer of finite depth models the heavier fluid before the perturbation. We write Eq. (2.9) in the perturbed form, Eq. (2.11) and obtain the linearized equations (2.13). We study the linear instability by seeking normal mode solutions $e^{\lambda t}V(x)$ of Eq. (2.13). It turns out that the investigation of normal modes relies on the finding of bounded solutions of a fourth-order ordinary differential equation on $\mathbf{R}_-$, which is

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi, \tag{1.4}$$

which wave number is k and $\lambda \in \mathbb{C}$ ($\mathrm{Re}\lambda > 0$) is called a characteristic value of the linearized problem (see [3, Chapter X]). The boundary conditions at $x_3 = 0$ are

$$\begin{cases} \mu(k^2\phi(0) + \phi''(0)) = 0, \\ -\lambda\mu\phi'''(0) + (3\lambda\mu k^2 + \lambda^2\rho_+)\phi'(0) + gk^2\rho_+\phi(0) = 0 \end{cases} \tag{1.5}$$

and we physically have that ϕ decays at $-\infty$, i.e.

$$\lim_{x_3 \to -\infty} \phi(x_3) = 0. \tag{1.6}$$

Our first result is to show that there exists an infinite sequence of normal modes to the linearized equations (2.13) by following the operator method introduced by Lafitte and Nguyễn [20]. The key ingredients are the followings:

(1) seek the exact solutions on $(-\infty, -a)$,
(2) deduce the boundary conditions for (1.4) on $(-a, 0)$ from the outer solutions and (1.5),
(3) write a variational formulation on $H^2((-a, 0))$ for (1.4) on $(-a, 0)$,
(4) use the spectral theory for a self-adjoint and compact operator for a Sturm-Liouville problem on $H^2((-a, 0))$.

The spectral analysis allows us to study fully nonlinear perturbation equations (2.11). To prove the nonlinear instability, we follow the procedure:

Step 1. establish some *a priori* energy estimates to the nonlinear equations,
Step 2. formulate a linear combination of normal modes to the linearized equations (2.13) to set its value at initial time $t = 0$ of size $0 < \delta \ll 1$ as an initial datum to the nonlinear perturbation equations,
Step 3. obtain the difference between the local exact solution and the approximate solution in Step 2 and exploit some energy estimates for the difference,
Step 4. deduce the bound in time of the difference functions and prove the nonlinear instability.

Our nonlinear study is inspired by the abstract frameworks of Guo and Strauss [6] and of Grenier [10]. In the spirit of these above frameworks, only the maximal normal mode was used in Step 2 to set its value at $t = 0$ of size $0 < \delta \ll 1$ to be an initial datum to derive a solution of the nonlinear equations. Let us emphasize that, our nonlinear results show that a wide class of initial data (related to a linear combination of normal modes) to the nonlinear problem departing from the equilibrium is formulated in Step 2 and it gives rise to the nonlinear instability. The nonlinear result in this work is in the same spirit as in [23], where the author studies the nonlinear viscous RT instability in a slab domain $2\pi L\mathbb{T} \times (-1, 1)$ $(L > 0)$, with Navier-slip boundary conditions.

We organize this paper as follows. In Section 2, from the formulation in Eulerian coordinates of the governing equations (2.1), we derive the formulation in Lagrangian coordinates, see (2.9). We introduce our two theorems, Theorem 2.1 describing the spectral analysis of the linearized equations and Theorem 2.2 proving the nonlinear instability. Section 3 is devoted to the proof of Theorem 2.1. In Section 4, we construct the *a priori* energy estimates to the nonlinear equations. In the last part, Section 5, we prove Theorem 2.2.

We employ the Einstein convention of summing over repeated indices. Throughout the paper $C > 0$ will denote universal constants that depend on the physical parameters of the problem, μ, g, k, a and $\rho_\pm$. Such constants are allowed to change from line to line. We will employ the notation $a \lesssim b$ $(a \gtrsim b)$ to mean that $a \leqslant Cb$ $(a \geqslant Cb)$ for a universal constant $C > 0$.

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2. The governing equations and main results

2.1. **Formulation in Eulerian coordinates.** We are concerned with the viscous RT of the nonhomogeneous incompressible Navier-Stokes equations without any effects of surface tension, that read as

$$\begin{cases} \partial_t\tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{u}) = 0 & \text{in } \Omega(t), \\ \partial_t(\tilde{\rho}\tilde{u}) + \mathrm{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + \nabla\tilde{p} = \mu\Delta\tilde{u} - g\tilde{\rho}e_3 & \text{in } \Omega(t), \\ \mathrm{div}\tilde{u} = 0 & \text{in } \Omega(t), \\ (\tilde{p}\mathrm{Id} - \mu\mathbb{S}\tilde{u})n = p_{atm}n & \text{on } \Gamma(t), \\ \partial_t\eta = \tilde{u}_3 - \tilde{u}_1\partial_1\eta - \tilde{u}_2\partial_2\eta & \text{on } \Gamma(t). \end{cases} \tag{2.1}$$

The unknowns $\tilde{\rho} := \tilde{\rho}(t, x)$, $\tilde{u} := \tilde{u}(t, x)$ and $\tilde{p} := \tilde{p}(t, x)$ denote the density, the velocity and the pressure of the fluid, respectively, while $\mu > 0$ is the viscosity coefficient and $g > 0$ is the gravitational constant. $\mathbb{S}\tilde{u} = \nabla\tilde{u} + \nabla\tilde{u}^T$ is the stress tensor. The outward normal vector to the boundary $\Gamma(t)$, n is given by

$$n = \frac{(-\partial_1\eta, -\partial_2\eta, 1)^T}{\sqrt{1 + |\partial_1\eta|^2 + |\partial_2\eta|^2}}. \tag{2.2}$$

The given constant p_{atm} is the atmospheric pressure. For a more physical description of the equations (2.1) and the boundary conditions in (2.1), we refer to [21, Sect. 1.8].

To complete the statement of the problem, we must specify the initial conditions. We suppose that the initial surface $\Gamma(0)$ is given by the graph of the function $\eta(0) = \eta_0$, which yields the open set $\Omega(0)$ on which we specify the initial data for the velocity, $u(0) = u_0 : \Omega(0) \to \mathbf{R}^3$. We assume that the initial surface function satisfies the "zero-average" condition

$$\int_{\mathbf{T}^2} \eta_0 = 0 \tag{2.3}$$

and $\eta(0), u(0)$ satisfy certain compatibility conditions, which we will present later (see Proposition 4.1). Note that, for sufficiently regular solutions to the problem, the condition (2.3) holds for all $t \geqslant 0$, that is

$$\int_{\mathbf{T}^2} \eta(t) = 0 \quad \text{for all } t \geqslant 0. \tag{2.4}$$

It is a consequence of

$$\frac{d}{dt}\int_{\mathbf{T}^2} \eta = \int_{\mathbf{T}^2} \partial_t \eta = \int_{\Gamma(t)} \tilde{u} \cdot n = \int_{\Omega(t)} \mathrm{div}\tilde{u} = 0.$$

2.2. **Formulation in Lagrangian coordinates.** The movement of the free boundary $\Gamma(t)$ and the domain $\Omega(t)$ raises numerous mathematical difficulties. To handle that, following Beale [2], we use the unknown function η. So that we can transform the free boundary problem (2.1) into a new problem in a fixed domain. We define (see Appendix A)

$$\theta := \text{Poisson extension of } \eta \text{ into } \mathbf{T}^2 \times \{x_3 \leqslant 0\} \tag{2.5}$$

and the following coordinate transformation:

$$\Omega \ni x = (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + \theta(t, x_1, x_2, x_3)) =: \Theta(t, x) = (y_1, y_2, y_3) \in \Omega(t). \tag{2.6}$$

If η is sufficiently small (in an appropriate Sobolev norm), then the mapping Θ is a diffeomorphism. From the definition of Θ (2.6), we first compute

$$\nabla\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1\theta & \partial_2\theta & 1 + \partial_3\theta \end{pmatrix}.$$

Following [2] again, we denote by

$$A = \partial_1\theta, \quad B = \partial_2\theta, \quad J = 1 + \partial_3\theta, \quad K = J^{-1} \tag{2.7}$$

and

$$\mathcal{A} := ((\nabla\Theta)^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix}. \tag{2.8}$$

We write the differential operators $\nabla_{\mathcal{A}}, \mathrm{div}_{\mathcal{A}}, \Delta_{\mathcal{A}}$ with their actions given by

$$(\nabla_{\mathcal{A}} f)_i := \sum_{j=1}^{3} \mathcal{A}_{ij}\partial_j f, \quad \mathrm{div}_{\mathcal{A}} X := \sum_{1 \leqslant i,j \leqslant 3} \mathcal{A}_{ij}\partial_j X_i, \quad \Delta_{\mathcal{A}} f = \mathrm{div}_{\mathcal{A}}\nabla_{\mathcal{A}} f.$$

We write

$$\mathcal{N} := (-\partial_1\eta, -\partial_2\eta, 1)^T$$

for the non-unit normal vector to $\Gamma(t)$, and we also write the stress tensor $\mathbb{S}_{\mathcal{A}}u$ as

$$(\mathbb{S}_{\mathcal{A}}u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i.$$

We now define the density ρ, the velocity u and the pressure p on the domain Ω by the composition

$$(\rho, u, p)(t, x) = (\tilde{\rho}, \tilde{u}, \tilde{p})(t, \Theta(t, x)).$$

We transform (2.1) into the following system in the new coordinates

$$\begin{cases} \partial_t \rho - K\partial_t\theta\partial_3\rho + \mathrm{div}_{\mathcal{A}}(\rho u) = 0 & \text{in } \Omega, \\ \rho(\partial_t u - K\partial_t\theta\partial_3 u + u\cdot\nabla_{\mathcal{A}}u) + \nabla_{\mathcal{A}}p - \mu\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}u = -g\rho e_3 & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}}u = 0 & \text{in } \Omega, \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Gamma, \\ (p\mathrm{Id} - \mu\mathbb{S}_{\mathcal{A}}u)\mathcal{N} = p_{atm}\mathcal{N} & \text{on } \Gamma. \end{cases} \tag{2.9}$$

2.3. **Equilibrium state and and main results.** We now rewrite (2.9) in a perturbed formulation around the steady-state solution

$$(\rho(t,x), u(t,x), p(t,x), \eta(t,x_h)) = (\rho_0(x_3), 0, P_0(x_3), 0),$$

satisfying that $P_0' = -g\rho_0$ and adding the condition $P_0(0) = p_{atm}$. We define a special density and pressure perturbation by

$$\zeta = \rho - \rho_0 - \rho_0'\theta, \quad q = p - P_0 + g\rho_0\theta. \tag{2.10}$$

We still call the perturbations of the velocity and of the characterization of surface as (u, η) respectively. The equations for the perturbation $U = (\zeta, u, q, \eta)$ write

$$\begin{cases} \partial_t\zeta + \rho_0' u_3 = \mathcal{Q}^1(U) & \text{in } \Omega, \\ \rho_0\partial_t u + \nabla q - \mu\Delta u + g\zeta e_3 = \mathcal{Q}^2(U) & \text{in } \Omega, \\ \mathrm{div}u = \mathcal{Q}^3(U) & \text{in } \Omega, \\ \partial_t\eta - u_3 = \mathcal{Q}^4(U) & \text{on } \Gamma, \\ ((q - g\rho_+\eta)\mathrm{Id} - \mu\mathbb{S}u)e_3 = \mathcal{Q}^5(U) & \text{on } \Gamma. \end{cases} \tag{2.11}$$

The nonlinear terms $\mathcal{Q}^i(U)$ $(1 \leqslant i \leqslant 5)$ (for short $\mathcal{Q}^i$) are given by

$$\begin{aligned} \mathcal{Q}^1 &= \rho_0' u_3 - \rho_0'\partial_t\theta + K\partial_t\theta(\partial_3\zeta + \rho_0' + \rho_0''\theta + \rho_0'\partial_3\theta), \\ &\quad - \mathrm{div}_{\mathcal{A}}((\rho_0 + \rho_0'\theta + \zeta)u) \\ \mathcal{Q}^2 &= -(\zeta + \rho_0'\theta)\partial_t u - (\zeta + \rho_0 + \rho_0'\theta)K\partial_t\theta\partial_3 u - (\nabla_{\mathcal{A}}p - \nabla q - g\zeta e_3) \\ &\quad - (\zeta + \rho_0 + \rho_0'\theta)u\cdot\nabla_{\mathcal{A}}u - \mu(\Delta u - \mathrm{div}_{\mathcal{A}}(\mathbb{S}_{\mathcal{A}}u)), \\ \mathcal{Q}^3 &= \mathrm{div}u - \mathrm{div}_{\mathcal{A}}u, \\ \mathcal{Q}^4 &= -u_1\partial_1\eta - u_2\partial_2\eta, \\ \mathcal{Q}^5 &= (q - g\rho_+\eta)\mathrm{Id}\cdot(e_3 - \mathcal{N}) - \mu\mathbb{S}ue_3 + \mu(\mathbb{S}_{\mathcal{A}}u)\mathcal{N}. \end{aligned} \tag{2.12}$$

We refer to Appendix B for the precise expression of $\mathcal{Q}^i (1 \leqslant i \leqslant 5)$ as a polynomial of U, A, B, J, K.

The linearized equations are

$$\begin{cases} \partial_t\zeta + \rho_0' u_3 = 0 & \text{in } \Omega, \\ \rho_0\partial_t u + \nabla q - \mu\Delta u + g\zeta e_3 = 0, & \text{in } \Omega, \\ \mathrm{div}u = 0 & \text{in } \Omega, \\ \partial_t\eta = u_3 & \text{on } \Gamma, \\ ((q - g\rho_+\eta)\mathrm{Id} - \mu\mathbb{S}u)e_3 = 0 & \text{on } \Gamma. \end{cases} \tag{2.13}$$

As in [3, Chapter XI], we seek normal modes $U(t,x) = e^{\lambda t}V(x)$ of (2.13), which are

$$(\zeta, u, q)(t,x) = e^{\lambda t}(\omega, v, r)(x), \quad \eta(t, x_h) = e^{\lambda t}\nu(x_h). \tag{2.14}$$

We deduce the following system on (ω, v, r, ν),

$$\begin{cases} \lambda\omega + \rho_0' v_3 = 0 & \text{in } \Omega, \\ \lambda\rho_0 v + \nabla r - \mu\Delta v + g\omega e_3 = 0 & \text{in } \Omega, \\ \mathrm{div} v = 0 & \text{in } \Omega, \\ \lambda\nu = v_3 & \text{on } \Gamma, \\ ((r - g\rho_+\nu)\mathrm{Id} - \mu(\nabla v + \nabla v^T))e_3 = 0 & \text{on } \Gamma. \end{cases} \tag{2.15}$$

That implies

$$\omega = -\frac{1}{\lambda}\rho_0' v_3, \quad \nu = \frac{1}{\lambda} v_3|_\Gamma \tag{2.16}$$

and

$$\begin{cases} \lambda^2\rho_0 v + \lambda\nabla r - \lambda\mu\Delta v - g\rho_0' v_3 e_3 = 0 & \text{in } \Omega, \\ \mathrm{div} v = 0 & \text{in } \Omega, \\ ((\lambda r - g\rho_+ v_3)\mathrm{Id} - \lambda\mu(\nabla v + \nabla v^T))e_3 = 0 & \text{on } \Gamma. \end{cases} \tag{2.17}$$

Let $\mathbf{k} = (k_1, k_2) \in L_1^{-1}\mathbb{Z} \times L_2^{-1}\mathbb{Z}\backslash\{0\}$, we further assume that

$$\begin{cases} v_1(x) = \sin(k_1 x_1 + k_2 x_2)\psi(\mathbf{k}, x_3), \\ v_2(x) = \sin(k_1 x_1 + k_2 x_2)\varphi(\mathbf{k}, x_3), \\ v_3(x) = \cos(k_1 x_1 + k_2 x_2)\phi(\mathbf{k}, x_3), \\ r(x) = \cos(k_1 x_1 + k_2 x_2)\pi(\mathbf{k}, x_3). \end{cases} \tag{2.18}$$

Denote by $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, we deduce from (2.17) that

$$\begin{cases} \lambda^2\rho_0\psi - \lambda k_1\pi + \lambda\mu(k^2\psi - \psi'') = 0 & \text{in } \mathbf{R}_-, \\ \lambda^2\rho_0\varphi - \lambda k_2\pi + \lambda\mu(k^2\varphi - \varphi'') = 0 & \text{in } \mathbf{R}_-, \\ \lambda^2\rho_0\phi + \lambda\pi' + \lambda\mu(k^2\phi - \phi'') = g\rho_0'\phi & \text{in } \mathbf{R}_-, \\ k_1\psi + k_2\varphi + \phi' = 0 & \text{in } \mathbf{R}_-, \end{cases} \tag{2.19}$$

At $x_3 = 0$, we have the boundary conditions

$$\begin{cases} \mu(k_1\phi(0) - \psi'(0)) = 0, \\ \mu(k_2\phi(0) - \varphi'(0)) = 0, \\ \lambda\pi(0) - g\rho_+\phi(0) - 2\lambda\mu\phi'(0) = 0. \end{cases} \tag{2.20}$$

We also need the decaying condition at $-\infty$,

$$\lim_{x_3 \to -\infty} (\psi, \varphi, \phi, \pi)(x_3) = 0. \tag{2.21}$$

Note that, due to $(2.19)_{1,2,4}$

$$\pi = -\frac{1}{k^2}(\lambda\rho_0\phi' + \mu(k^2\phi' - \phi''')) \quad \text{in } \mathbf{R}_-. \tag{2.22}$$

Hence, from (2.22) and $(2.19)_3$, we get a fourth-order ODE for ϕ, (1.4), i.e.

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi.$$

The boundary conditions at $x_3 = 0$ deduced from $(2.19)_4$, (2.20) and (2.22) are (1.5), i.e.

$$\mu(k^2\phi(0) + \phi''(0)) = 0, \quad -\lambda\mu\phi'''(0) + (3\lambda\mu k^2 + \lambda^2\rho_+)\phi'(0) + gk^2\rho_+\phi(0) = 0$$

and from (2.21), we have that ϕ decays at $-\infty$, (1.6), i.e.

$$\lim_{x_3\to-\infty}\phi(x_3)=0.$$

The finding of normal modes of the form (2.14) to Eq. (2.13) relies on the investigation of the characteristic values $\lambda\in\mathbb{C}$ ($\mathrm{Re}\lambda>0$) such that (1.4)-(1.5)-(1.6) has a nontrivial solution ϕ living at least in $H^4(\mathbf{R}_-)$.

We first show that all characteristic values λ are real. Since our goal is to study the instability, we only consider positive λ in what follows. Let L_0 be the characteristic length such that $L_0^{-1}=\|\frac{\rho_0'}{\rho_0}\|_{L^\infty(\mathbf{R}_-)}$, we further obtain the uniform upper bound $\sqrt{\frac{g}{L_0}}$ of λ.

Lemma 2.1. *For any $k>0$,*

- *all characteristic values λ are always real,*
- *all characteristic values λ satisfy that $\lambda\leqslant\sqrt{\frac{g}{L_0}}$.*

Proof of Lemma 2.1 is given in the beginning of Section 3. In view of Lemma 2.1, we look for functions ϕ being real and we only consider the vector spaces of real-valued functions in what follows in the linear analysis.

We state our first theorem solving the ODE (1.4)–(1.5)–(1.6) .

Theorem 2.1. *Let ρ_0 satisfy (1.2)–(1.3), there exists an infinite sequence $(\lambda_n,\phi_n)_{n\geqslant1}$ with $\lambda_n\in(0\sqrt{\frac{g}{L_0}})$ and $\phi_n\in H^\infty(\mathbf{R}_-)$ satisfying (1.4)-(1.5)-(1.6).*

Let us discuss about the strategy for proving Theorem 2.1, which is in the same spirit as [20]. We first look for a solution $\phi\in H^4(\mathbf{R}_-)$.

On $(-\infty,-a)$, the ODE (1.4) is an ODE with constant coefficients, for which we can find explicit solutions in Proposition 3.1 decaying to 0 at $-\infty$. Hence, we transform the problem for the normal modes on $\mathbf{R}_-$ into an ODE problem stated on a compact interval $(-a,0)$ with appropriate boundary conditions deduced from the outer solutions. They are described by (1.5) stated above and (3.7) (to be seen in Lemma 3.1).

In order to solve a fourth-order ODE (1.4) with the boundary conditions (3.7) and (1.5), the crucial step is to construct a continuous and coercive bilinear form $\mathscr{B}_{a,k,\lambda}$ on $H^2((-a,0))$ (see (3.9) in Proposition 3.2), such that the finding of a solution $\phi\in H^4((-a,0))$ of Eq. (1.4)-(1.5)-(3.7) is equivalent to finding a weak solution $\phi\in H^2((-a,0))$ to the variational problem

$$\lambda\mathscr{B}_{a,k,\lambda}(\phi,\omega)=gk^2\int_{-a}^0\rho_0'\phi\omega dx_3\quad\text{for all }\omega\in H^2((-a,0)) \tag{2.23}$$

and thus improving the regularity of that weak solution ϕ.

As $\mathscr{B}_{a,k,\lambda}$ is a coercive form on $H^2((-a,0))$, we have that $\sqrt{\mathscr{B}_{a,k,\lambda}(\cdot,\cdot)}$ is a norm on $H^2((-a,0))$. Let $(H^2((-a,0)))'$ be the dual space of the functional space $H^2((-a,0))$, associated with the norm $\sqrt{\mathscr{B}_{a,k,\lambda}(\cdot,\cdot)}$. In view of Riesz's representation theorem, we obtain an abstract operator $Y_{a,k,\lambda}$ from $H^2((-a,0))$ to $(H^2((-a,0)))'$, such that

$$\mathscr{B}_{a,k,\lambda}(\vartheta,\varrho)=\langle Y_{a,k,\lambda}\vartheta,\varrho\rangle\quad\text{for all }\vartheta,\varrho\in H^2((-a,0)). \tag{2.24}$$

Hence, from (2.23) and (2.24), we have that the existence of a solution $\phi\in H^4((-a,0))$ of Eq. (1.4)-(1.5)-(3.7) is reduced to the finding of a weak solution $\phi\in H^2((-a,0))$

of

$$\lambda Y_{a,k,\lambda}\phi = gk^2\rho_0'\phi \quad \text{ in } (H^2((-a,0)))'. \tag{2.25}$$

Restricting to $\varrho \in C_0^\infty((-a,0))$ in (2.24), we find the precise expression of $Y_{a,k,\lambda}$ (see Proposition 3.3(1)), i.e. for all $\vartheta \in H^2((-a,0))$,

$$Y_{a,k,\lambda}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) \quad \text{in } \mathcal{D}'((-a,0)).$$

Furthermore, a classical bootstrap argument (see Proposition 3.3(2)) shows that we are able to define the inverse operator $Y_{a,k,\lambda}^{-1}$ of $Y_{a,k,\lambda}$, from $L^2((-a,0))$ to a subspace of $H^4((-a,0))$ requiring all elements satisfy (3.7)-(1.5). Note that, because ϕ belongs to $H^4((-a,0))$, these boundary conditions (involving the derivatives ϕ'', ϕ''' of ϕ at $x_3 = -a$ and at $x_3 = 0$) are well defined. Composing the above operator $Y_{a,k,\lambda}^{-1}$ with the continuous injection from $H^4((-a,0))$ to $L^2((-a,0))$ (see Proposition 3.4), we obtain that $Y_{a,k,\lambda}^{-1}$ is a compact and self-adjoint operator from $L^2((-a,0))$ to itself.

We introduce $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((-a,0))$. Note from (2.25) that, we thus find v satisfying

$$\frac{\lambda}{gk^2}v = \mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}v. \tag{2.26}$$

Indeed, if v satisfies (2.26), $Y_{a,k,\lambda}^{-1}\mathcal{M}v$ will satisfy (2.25).

We show that the operator $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ is compact and self-adjoint from $L^2((-a,0))$ to itself (see Proposition 3.5), which enables to use the spectral theory of self-adjoint and compact operators to obtain that

the discrete spectrum of the operator $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ is an infinite sequence of eigenvalues (denoted by $\{\gamma_n(\lambda,k)\}_{n\geqslant 1}$).

Let $v_{n,k,\lambda}$ be an eigenfunction of $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ associated with the eigenvalue $\gamma_n(\lambda,k)$ and let $\phi_{n,k,\lambda} = Y_{a,k,\lambda}^{-1}\mathcal{M}v_{n,k,\lambda} \in H^4((-a,0))$, we obtain

$$\gamma_n(\lambda,k)Y_{a,k,\lambda}\phi_{n,k,\lambda} = \mathcal{M}^2\phi_{n,k,\lambda_n} = \rho_0'\phi_{n,k,\lambda_n}. \tag{2.27}$$

From (2.25) and (2.27), we see that the problem of finding characteristic values of (1.4) amounts to solving all the equations

$$\gamma_n(\lambda,k) = \frac{\lambda}{gk^2}. \tag{2.28}$$

In Proposition 3.6, for each n, we will show the existence and uniqueness of a solution λ_n to (2.28) owing first to the differentiability in λ of $\gamma_n(\lambda,k)$ (see Lemma 3.3), which is an extension of Kato's perturbation theory of the spectrum of operators (see [15]), and to the fact that $\lambda \to \gamma_n(\lambda,k)$ is decreasing in λ (see Lemma 3.4), through the derivative $\frac{d}{d\lambda}(\frac{1}{\gamma_n(\lambda,k)})$ which exists also thanks to a similar argument (see [15]). Furthermore, we have that $\{\lambda_n\}_{n\geqslant 1}$ is a decreasing sequence towards 0.

For each λ_n, we have that $\phi_{n,k,\lambda_n} = Y_{a,k,\lambda_n}^{-1}\mathcal{M}v_{n,k,\lambda_n} \in H^4((-a,0))$ satisfies Eq. (1.4) with the boundary conditions (3.7)-(1.5) thanks to Proposition 3.3(2) again. Hence, ϕ_{n,k,λ_n} is glued with the decaying solutions of (1.4) in the outer region $(-\infty,-a)$ by the boundary conditions at $x_3 = -a$ to become a solution of (1.4)-(1.5)-(1.6) in $H^4(\mathbf{R}_-)$ associated with $\lambda = \lambda_n$. Theorem 2.1 is proven.

Once Eq. (1.4)-(1.5)-(1.6) is solved, we go back to the linearized equations (2.13). For a fixed $\mathbf{k} \in L_1^{-1}\mathbb{Z} \times L_2^{-1}\mathbb{Z}\backslash\{0\}$, we obtain a sequence of real solutions to the

linearized equations (2.13) (see Proposition 3.7), which are

$$e^{\lambda_j(\mathbf{k})t}V_j(\mathbf{k},x)=(e^{\lambda_j(\mathbf{k})t}(\zeta_j(\mathbf{k},x),u_j(\mathbf{k},x),q_j(\mathbf{k},x),\eta_j(\mathbf{k},x_h))^T.$$

Note from Lemma 2.1 that

$$0<\Lambda:=\sup_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}}\lambda_1(\mathbf{k})\leqslant\sqrt{\frac{g}{L_0}},\tag{2.29}$$

we show that Λ is the maximal growth rate of the linearized equations, see Proposition 3.9.

We move to solve the nonlinear instability.

The local well-posedness of (2.11) in our functional framework (see Proposition 4.1) can be established similarly as in [8, Theorem 6.3] for the incompressible viscous surface wave problem, which is used in [29] for the incompressible viscous surface-internal wave problem and [28] for the incompressible viscous fluid with magnetic field. We refer to [8, 29, 28] for the construction of local solutions to (2.11) with some specific compatibility conditions.

We derive the *a priori* energy estimate (4.6) to the nonlinear equations in Proposition 4.2.

Thanks to (2.29), we define the non-empty set

$$S_\Lambda:=\Big\{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}:\lambda_1(\mathbf{k})>\frac{2\Lambda}{3}\Big\}.$$

We further fix a $\mathbf{k}_0\in S_\Lambda$. Hence, there is a unique $N=N(\mathbf{k}_0)\in\mathbb{N}^\star$ such that

$$\Lambda\geqslant\lambda_1(\mathbf{k}_0)>\lambda_2(\mathbf{k}_0)>\cdots>\lambda_N(\mathbf{k}_0)>\frac{2\Lambda}{3}>\lambda_{N+1}(\mathbf{k}_0)>\ldots.\tag{2.30}$$

Let $M\in\mathbb{N}^\star$ be arbitrary. In view of getting infinitely many characteristic values of the linearized problem, we consider a linear combination of normal modes

$$U^M(t,x)=\sum_{j=1}^M\mathsf{c}_je^{\lambda_j(\mathbf{k}_0)t}V_j(\mathbf{k}_0,x)\tag{2.31}$$

to be an approximate solution to the nonlinear equations (2.11), with constants c_j being chosen such that

$$\text{at least one of }\mathsf{c}_j\ (1\leqslant j\leqslant N)\text{ is non-zero}\tag{2.32}$$

and let $j_m:=\min\{j:1\leqslant j\leqslant N,\mathsf{c}_j\neq0\}$,

$$|\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)}>\sum_{j\geqslant j_m+1}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}.\tag{2.33}$$

In order to prove the nonlinear instability, we would like to use $U^M(0,x)$ as the initial data for the nonlinear equations (2.11). However, to ensure the local existence of solutions, the initial data for the nonlinear equations (2.11) must satisfy the compatibility conditions (4.4) (see again Proposition 4.1), which are not satisfied by the normal modes $V_j(\mathbf{k},x)$ $(j\geqslant1)$. Thanks to an abstract argument from [11, Section 5C], which was also used in [29, 28], we modify $U^M(0,x)$ in Proposition 5.1 as follows: there exist $\delta_0>0$ such that the family of initial data

$$U^{\delta,M}(x)=\delta U^M(0,x)+\delta^2U_\star^{\delta,M}(x)\tag{2.34}$$

for $\delta\in(0,\delta_0)$ satisfies the compatibility conditions (4.4). Eq. (2.11) with the initial data $U^{\delta,M}(x)$ (2.34) has a unique local strong solution $U^{\delta,M}(t,x)$ on $[0,T_{\max})$.

Hence, we observe that $U^d(t) = U^{\delta,M}(t) - \delta U^M(t)$ solves (2.11) with the initial data $U^d(0) = \delta^2 U_\star^{\delta,M}$ and the same nonlinear terms $\mathcal{Q}^i(1 \leqslant i \leqslant 5)$ (see Eq. (5.2)). For t small enough, we deduce the following bound in time (see Proposition 5.2),

$$\begin{aligned}&\|(\zeta^d, u^d)(t)\|^2_{L^2(\Omega)} + \|\eta^d(t)\|^2_{L^2(\Gamma)}\\ &\qquad\lesssim \delta^3\big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \max_{N+1\leqslant j\leqslant M} |\mathsf{c}_j| e^{\frac{2}{3}\Lambda t}\big)^3,\end{aligned}$$

That relies on some energy estimates of Eq. (5.2) and the bound in time of a suitable Sobolev norm of $U^{\delta,M}(t)$ (see Proposition 5.3), which we obtain thanks to the *a priori* energy estimate established in Proposition 4.2. Combining those estimates, we obtain the following nonlinear result.

Theorem 2.2. *Assume that ρ_0 satisfies (1.2)-(1.3). Let $M \in \mathbb{N}^\star$ be arbitrary, there exist positive constants δ_0, ε_0 sufficiently small and another constant $m_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the nonlinear equations (2.11) with the initial data (2.34), i.e.*

$$\delta \sum_{j=1}^{M} \mathsf{c}_j V_j(x) + \delta^2 U_\star^{\delta,M}(x),$$

satisfying (2.32)-(2.33) admits a unique local strong solution $U^{\delta,M}$ such that

$$\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)} \geqslant m_0 \epsilon_0, \tag{2.35}$$

where $T^\delta \in (0, T_{\max})$ is given by $\delta \sum_{j=j_m}^{M} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \epsilon_0$.

3. The linear analysis

We begin with the proof of Lemma 2.1.

Proof of Lemma 2.1. Multiplying by $\overline{\phi}$ on both sides of (1.4) and then integrating by parts, we obtain that

$$\begin{aligned}&\lambda^2\Big(\int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 - \rho_0\phi'\overline{\phi}\Big|_{-\infty}^{0}\Big)\\ &\qquad + \lambda\mu\Big(\int_{\mathbf{R}_-} (|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_3 + (\phi'''\overline{\phi} - \phi''\overline{\phi}' - 2k^2\phi'\overline{\phi})\Big|_{-\infty}^{0}\Big)\\ &= gk^2\int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3.\end{aligned}$$

Using (1.5) and (1.6), we get

$$\begin{aligned}&\lambda^2\Big(\int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 - \rho_+\phi(0)\overline{\phi}'(0)\Big) + \lambda\mu\int_{\mathbf{R}_-} (|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_3\\ &\qquad + (3\lambda\mu k^2 + \lambda^2\rho_+)\phi'(0)\overline{\phi}(0) + gk^2\rho_+|\phi(0)|^2 + \lambda\mu k^2\overline{\phi}'(0)\phi(0) - 2\lambda\mu k^2\phi'(0)\overline{\phi}(0)\\ &= gk^2\int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3.\end{aligned}$$

This yields

$$\begin{aligned}&\lambda^2\int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 + \lambda\mu\int_{\mathbf{R}_-} (|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2)dx_3\\ &\qquad + \lambda\mu k^2(\phi'(0)\overline{\phi}(0) + \overline{\phi}'(0)\phi(0)) + gk^2\rho_+|\phi(0)|^2 = gk^2\int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3.\end{aligned}$$

Using the integration by parts and (1.5) again, we have

$$\lambda^2 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 + \lambda\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 \\ = -gk^2\rho_+|\phi(0)|^2 + gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3. \tag{3.1}$$

Suppose that $\lambda = \lambda_1 + i\lambda_2$, then one deduces from (3.1) that

$$(\lambda_1^2 - \lambda_2^2) \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 + \lambda_1\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 \\ = -gk^2\rho_+|\phi(0)|^2 + gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3 \tag{3.2}$$

and that

$$-2\lambda_1\lambda_2 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 = \lambda_2\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3. \tag{3.3}$$

If $\lambda_2 \neq 0$, (3.3) leads us to

$$-2\lambda_1 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 = \mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 < 0,$$

that contradiction yields $\lambda_2 = 0$, i.e. λ is real. Using (3.1) again, we further get that

$$\lambda^2 \int_{\mathbf{R}_-} \rho_0(k^2|\phi|^2 + |\phi'|^2)dx_3 \leqslant gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 2.1. □

Note again that, thanks to Lemma 2.1, in what follows in this section, we only use real-valued functions.

3.1. Solutions on the outer region $(-\infty, -a)$ and reduction to an ODE on the finite interval $(-a, 0)$.

Proposition 3.1. *Let $\tau_- = \sqrt{k^2 + \lambda\rho_-/\mu}$. There are two linearly independent solutions of (1.4) decaying to 0 at $-\infty$ as $x_3 \in (-\infty, -a]$, i.e.*

$$\phi_1^-(x_3) = e^{kx_3} \quad \textit{and} \quad \phi_2^-(x_3) = e^{\tau_- x_3}. \tag{3.4}$$

The space of solutions decaying to 0 at $-\infty$ are spanned by (ϕ_1^-, ϕ_2^-).

Proof. On the interval $(-\infty, -a)$, Eq. (1.4) is an ODE with constant coefficients,

$$-\lambda\rho_-(k^2\phi - \phi'') = \mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi). \tag{3.5}$$

We seek ϕ as $\phi(x_3) = e^{rx_3}$. Hence,

$$-\lambda\rho_-(k^2 - r^2) = \mu(r^4 - 2k^2r^2 + k^4),$$

which yields $r = \pm k$ or $r = \pm(k^2 + \lambda\rho_-/\mu)^{1/2}$. Since ϕ tends to 0 at $-\infty$, we get two independent solutions of (3.5),

$$\phi_1^-(x_3) = e^{kx_3} \quad \text{and} \quad \phi_2^-(x_3) = e^{(k^2+\lambda\rho_-/\mu)^{1/2}x_3}.$$

Hence, all bounded solutions of (3.5) are of the form

$$\phi(x_3) = A_1 e^{k(x_3+a)} + A_2 e^{\tau_-(x_3+a)}. \tag{3.6}$$

Proposition 3.1 is proven. □

Once it is proven that $\phi(x_3)$ outside $(-a,0)$ is known precisely, we look for ϕ on $(-a,0)$. That solution has to match with (3.6) well, i.e. there is a condition on $(\phi,\phi',\phi'',\phi''')$ at $x_3=-a$. We will show that in the following lemma.

Lemma 3.1. *The boundary conditions of (1.4) at $x_3=-a$, for $\phi\in H^4(\mathbf{R}_-)$, are*

$$\begin{cases} k\tau_-\phi(-a)-(k+\tau_-)\phi'(-a)+\phi''(-a)=0,\\ k\tau_-(k+\tau_-)\phi(-a)-(k^2+k\tau_-+\tau_-^2)\phi'(-a)+\phi'''(-a)=0. \end{cases} \tag{3.7}$$

and at $x_3=0$, are (1.5).

Proof. For a solution ϕ of Eq. (1.4) on $(-a,0)$, the boundary conditions at $x_3=-a$ are equivalent to the fact that ϕ belongs to the space of decaying solutions at ∞. On the one hand, it can be seen from (3.6) that

$$\begin{pmatrix}\phi(x_3)\\ \phi'(x_3)\\ \phi''(x_3)\\ \phi'''(x_3)\end{pmatrix} = A_1^- e^{k(x_3+a)}\begin{pmatrix}1\\ k\\ k^2\\ k^3\end{pmatrix} + A_2^- e^{\tau_-(x_3+a)}\begin{pmatrix}1\\ \tau_-\\ \tau_-^2\\ \tau_-^3\end{pmatrix} \quad \text{for } x_3\leqslant -a.$$

On the other hand, direct computations show that the orthogonal complement of the subspace of $\mathbf{R}^4$ spanned by two vectors $(1,k,k^2,k^3)^T$ and $(1,\tau_-,\tau_-^2,\tau_-^3)^T$ is spanned by

$$(k\tau_-,-(k+\tau_-),1,0)^T \quad \text{and} \quad (k\tau_-(k+\tau_-),-(k^2+k\tau_-+\tau_-^2),0,1)^T.$$

The above arguments allow us to set (3.7) as boundary conditions of Eq. (1.4) at $x_3=-a$. □

3.2. **A bilinear form and a self-adjoint invertible operator.**

Proposition 3.2. *Let us denote by*

$$\begin{aligned} BV_{-a,k,\lambda}(\vartheta,\varrho) &:= \mu\begin{pmatrix} k\tau_-(k+\tau_-)\vartheta(-a)\varrho(-a)-k\tau_-\vartheta'(-a)\varrho(-a)\\ \qquad -k\tau_-\vartheta(-a)\varrho'(-a)+(k+\tau_-)\vartheta'(-a)\varrho'(-a)\end{pmatrix},\\ BV_{0,k,\lambda}(\vartheta,\varrho) &:= \mu k^2(\vartheta'(0)\varrho(0)+\vartheta(0)\varrho'(0))+\frac{gk^2\rho_+}{\lambda}\vartheta(0)\varrho(0), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \mathscr{B}_{a,k,\lambda}(\vartheta,\varrho) := BV_{0,k,\lambda}(\vartheta,\varrho)+BV_{-a,k,\lambda}(\vartheta,\varrho)+\lambda\int_{-a}^0\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_3\\ +\mu\int_{-a}^0(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_3. \end{aligned} \tag{3.9}$$

The bilinear form $\mathscr{B}_{a,k,\lambda}$ is a continuous and coercive on $H^2((-a,0))$.

Furthermore, let $(H^2((-a,0)))'$ be the dual space of $H^2((-a,0))$ associated with the norm $\sqrt{\mathscr{B}_{a,k,\lambda}(\cdot,\cdot)}$, there exists a unique operator

$$Y_{a,k,\lambda}\in\mathcal{L}(H^2((-a,0)),(H^2((-a,0)))'),$$

that is also bijective, such that

$$\mathscr{B}_{a,k,\lambda}(\vartheta,\varrho)=\langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.10}$$

for all $\vartheta,\varrho\in H^2((-a,0))$.

Before proving Proposition 3.2, we state our key lemma, whose proof is postponed to Appendix D. This yields the coercivity of $\mathscr{B}_{a,k,\lambda}$ as it will appear in the proof of Proposition 3.2.

Lemma 3.2. *We have*

$$\min_{\vartheta\in H^2((-a,0))}\begin{pmatrix}2k^2(\vartheta'(0)\vartheta(0)-\vartheta'(-a)\vartheta(-a)),\vartheta \text{ satisfies the constraint}\\ \int_{-a}^{0}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_3=1.\end{pmatrix}$$
$$=-\frac{\sinh(ka)+ka}{3\sinh(ka)-ka},$$

and

$$\max_{\vartheta\in H^2((-a,0))}\begin{pmatrix}2k^2(\vartheta'(0)\vartheta(0)-\vartheta'(-a)\vartheta(-a)),\vartheta \text{ satisfies the constraint}\\ \int_{-a}^{0}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_3=1.\end{pmatrix}=1.$$

Proof of Proposition 3.2. Clearly, $\mathscr{B}_{a,k,\lambda}$ is a bilinear form on $H^2((-a,0))$ since the terms $BV_{a,k,\lambda}(\vartheta,\varrho)$ and $BV_{0,k,\lambda}(\vartheta,\varrho)$ are well defined. We then establish the boundedness of $\mathscr{B}_{a,k,\lambda}$. The integral terms of $\mathscr{B}_{a,k,\lambda}$ are clearly $\lesssim \|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}$. About the two boundary value terms, it follows from the general Sobolev inequality that

$$\max(\vartheta^2(0),\vartheta^2(-a))\lesssim \|\vartheta\|^2_{H^1((-a,0))}$$

and that

$$\max((\vartheta'(0))^2,(\vartheta'(-a))^2)\lesssim \|\vartheta'\|^2_{H^1((-a,0))}.$$

Consequently, we get

$$\begin{aligned}|BV_{-a,k,\lambda}(\vartheta,\varrho)|&\lesssim(|\vartheta(-a)|+|\vartheta'(-a)|)(|\varrho(-a)|+|\varrho'(-a)|)\\&\lesssim\|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}.\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}|BV_{0,k,\lambda}(\vartheta,\varrho)|&\lesssim\frac{\lambda+1}{\lambda}(|\vartheta(0)|+|\vartheta'(0)|)(|\varrho(0)|+|\varrho'(0)|)\\&\lesssim\frac{\lambda+1}{\lambda}\|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}.\end{aligned}\tag{3.12}$$

In view of (3.11) and (3.12), we find that

$$|\mathscr{B}_{a,k,\lambda}(\vartheta,\varrho)|\lesssim\frac{\lambda+1}{\lambda}\|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))},\tag{3.13}$$

i.e. $\mathscr{B}_{a,k,\lambda}$ is bounded.

We move to show the coercivity of $\mathscr{B}_{a,k,\lambda}$. We have that

$$\begin{aligned}\mathscr{B}_{a,k,\lambda}(\vartheta,\vartheta)=BV_{0,k,\lambda}(\vartheta,\vartheta)+BV_{-a,k,\lambda}(\vartheta,\vartheta)+\lambda\int_{-a}^{0}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx_3\\+\mu\int_{-a}^{0}((\vartheta'')^2+2k^2(\vartheta')^2+k^4(\vartheta)^2)dx_3.\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{\mu}BV_{-a,k,\lambda}(\vartheta,\vartheta) &= k\tau_-(k+\tau_-)(\vartheta(-a))^2 - 2k\tau_-\vartheta(-a)\vartheta'(-a) + (k+\tau_-)(\vartheta'(-a))^2\\ &= (k+\tau_-)\Big(\vartheta'(-a) + \frac{k(k-\tau_-)}{k+\tau_-}\vartheta(-a)\Big)^2\\ &\quad + \frac{k(\tau_-(k+\tau_-)^2 - k(k-\tau_-)^2)}{k+\tau_-}(\vartheta(-a))^2 - 2k^2\vartheta(-a)\vartheta'(-a)\\ &\geqslant -2k^2\vartheta(-a)\vartheta'(-a).\end{aligned}$$

Therefore, we deduce that

$$\frac{1}{\mu}\mathscr{B}_{a,k,\lambda}(\vartheta,\vartheta) \geqslant 2k^2(\vartheta(0)\vartheta'(0) - \vartheta(-a)\vartheta'(-a)) + \int_{-a}^{0}((\vartheta'')^2 + 2k^2(\vartheta')^2 + k^4\vartheta^2)dx_3.$$

Notice from Lemma 3.2 that

$$\frac{1}{\mu}\mathscr{B}_{a,k,\lambda}(\vartheta,\vartheta) \geqslant \frac{2(\sinh(ka)-ka)}{3\sinh(ka)-ka}\int_{-a}^{0}((\vartheta'')^2 + 2k^2(\vartheta')^2 + k^4\vartheta^2)dx_3. \tag{3.14}$$

The inequalities (3.13) and (3.14) tell us that $\mathscr{B}_{a,k,\lambda}$ is a continuous and coercive bilinear form on $H^2((-a,0))$. It follows from Riesz's representation theorem that there is a unique operator $Y_{a,k,\lambda} \in \mathcal{L}(H^2((-a,0)),(H^2((-a,0)))')$, that is also bijective, satisfying (3.10) for all $\vartheta,\varrho \in H^2((-a,0))$. Proof of Proposition 3.2 is complete. □

The next proposition is to devoted to studying the properties of $Y_{a,k,\lambda}$.

Proposition 3.3. *We have the following results.*

(1) *For all $\vartheta \in H^2((-a,0))$,*

$$Y_{a,k,\lambda}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \lambda\mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta)$$

in $\mathcal{D}'((-a,0))$.

(2) *Let $f \in L^2((-a,0))$ be given, there exists a unique solution $\vartheta \in H^2((-a,0))$ of*

$$Y_{a,k,\lambda}\vartheta = f \text{ in } (H^2((-a,0)))'. \tag{3.15}$$

Moreover, $\vartheta \in H^4((-a,0))$ and satisfies the boundary conditions (3.7)–(1.5).

Proof. It follows from Proposition 3.2 that there is a unique $\vartheta \in H^2((-a,0))$ such that

$$\lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 + \mu\int_{-a}^{0}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3 = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.16}$$

for all $\varrho \in C_0^\infty((-a,0))$. We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'' which is in $L^2((-a,0))$. Hence, Eq. (3.16) is equivalent to

$$\lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 + \mu\langle(\vartheta'')'',\varrho\rangle + \mu\int_{-a}^{0}(2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3 = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.17}$$

for all $\varrho \in C_0^\infty((-a,0))$. Eq. (3.17) implies that

$$\mu((\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta) + \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') = Y_{a,k,\lambda}\vartheta \quad \text{in } \mathcal{D}'((-a,0)). \tag{3.18}$$

The first assertion holds.

Under the assumption $f \in L^2((-a,0))$, we improve the regularity of the weak solution $\vartheta \in H^2((-a,0))$ of (3.18). Indeed, we rewrite (3.18) as

$$\mu\langle(\vartheta'')'', \varrho\rangle = \int_{-a}^{0} (Y_{a,k,\lambda}\vartheta + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')\varrho dx_3$$

for all $\varrho \in C_0^\infty((-a,0))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-a,0))$, it follows from (3.17) that $(\vartheta'')'' \in L^2((-a,0))$.

Let $\chi_1 \in C_0^\infty((-a,0))$ satisfy $\int_{-a}^{0}\chi_1(y)dy = 1$. Using the distribution theory, we define $\Sigma \in \mathcal{D}'((-a,0))$ such that

$$\langle\Sigma, \varrho\rangle = \langle(\vartheta'')'', \zeta_\varrho\rangle \tag{3.19}$$

for all $\varrho \in C_0^\infty((-a,0))$, where

$$\zeta_\varrho(x_3) = \int_{-a}^{x_3}\Big(\varrho(y) - \chi_1(y)\int_{-a}^{0}\varrho(s)ds\Big)dy$$

for all $-a < x_3 < 0$. We obtain

$$\langle\Sigma', \varrho\rangle = -\langle\Sigma, \varrho'\rangle = -\langle(\varrho'')'', \zeta_{\varrho'}\rangle.$$

Note that

$$\langle(\vartheta'')'', \zeta_{\varrho'}\rangle = \langle(\vartheta'')'', \varrho(x_3) - \int_{-a}^{x_3}\chi_1(y)\int_{-a}^{0}\varrho'(s)dsdy\rangle = \langle(\vartheta'')'', \varrho\rangle,$$

this yields $\langle\Sigma', \varrho\rangle = -\langle(\vartheta'')'', \varrho\rangle$. Hence, we have that $(\vartheta'')' + \Sigma \equiv$ constant. In view of $(\vartheta'')'' \in L^2((-a,0))$ and (3.19), we know that $(\vartheta'')' \in L^2((-a,0))$. Since $\vartheta \in H^2((-a,0))$ and $(\vartheta'')', (\vartheta'')'' \in L^2((-a,0))$, it tells us that ϑ belongs to $H^4((-a,0))$ and we can take their traces up to order 3.

By exploiting (3.16), we then show that ϑ satisfies (3.7)-(1.5). Indeed, for all $\varrho \in H^2((-a,0))$, we use the integration by parts to obtain from (3.16) that

$$\begin{aligned}&\lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 + \mu\int_{-a}^{0}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3\\ &\quad - \lambda\rho_0\vartheta'\varrho\Big|_{-a}^{0} + \mu\Big(\vartheta'''\varrho\Big|_{-a}^{0} - \vartheta''\varrho'\Big|_{-a}^{0} - 2k^2\vartheta'\varrho\Big|_{-a}^{0}\Big) = \int_{-a}^{0}(Y_{a,k,\lambda}\vartheta)\varrho dx_3.\end{aligned}$$

It then follows from the definition of the bilinear form $\mathscr{B}_{a,k,\lambda}$ that

$$-\lambda\rho_0\vartheta'\varrho\Big|_{-a}^{0} + \mu\Big(\vartheta'''\varrho\Big|_{-a}^{0} - \vartheta''\varrho'\Big|_{-a}^{0} - 2k^2\vartheta'\varrho\Big|_{-a}^{0}\Big) = BV_0(\vartheta,\varrho) + BV_{-a}(\vartheta,\varrho), \tag{3.20}$$

for all $\varrho \in H^2((-a,0))$. By collecting all terms corresponding to $\varrho(-a)$ in (3.20), we deduce that

$$\lambda\rho_-\vartheta'(-a) - \mu\vartheta'''(-a) + 2\mu k^2\vartheta'(-a) = \mu k\tau_-(k+\tau_-)\vartheta(-a) - \mu k\tau_-\vartheta'(-a).$$

This yields,

$$\begin{aligned}\vartheta'''(-a) &= (\tau_-^2 - k^2)\vartheta'(-a) + 2k^2\vartheta'(-a) + k\tau_-\vartheta'(-a) - k\tau_-(k+\tau_-)\vartheta(-a)\\ &= (k^2 + k\tau_- + \tau_-^2)\vartheta'(-a) - k\tau_-(k+\tau_-)\vartheta(-a).\end{aligned}$$

We just proved that ϑ satisfies $(3.7)_2$. Similarly, ϑ also fulfils $(3.7)_1$ and (1.5). This ends the proof of Proposition 3.2. □

We have the following proposition on $Y_{a,k,\lambda}^{-1}$.

Proposition 3.4. *The operator $Y_{a,k,\lambda}^{-1} : L^2((-a,0)) \to L^2((-a,0))$ is compact and self-adjoint.*

Proof. It follows from Proposition 3.3 that $Y_{a,k,\lambda}$ admits an inverse operator $Y^{-1}_{a,k,\lambda}$ from $L^2((-a,0))$ to a subspace of $H^4((-a,0))$ requiring all elements satisfy (1.5)-(3.7), which is symmetric due to Proposition 3.2. We compose $Y^{-1}_{a,k,\lambda}$ with the continuous injection from $H^4((-a,0))$ to $L^2((-a,0))$. Notice that the embedding $H^p((-a,0)) \hookrightarrow H^q((-a,0))$ for $p > q \geqslant 0$ is compact. Therefore, the operator $Y^{-1}_{a,k,\lambda}$ is compact and self-adjoint from $L^2((-a,0))$ to $L^2((-a,0))$. Proposition 3.4 is shown. □

3.3. **A sequence of characteristic values.** We continue considering $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$ and we study the operator $S_{a,k,\lambda} := \mathcal{M} Y^{-1}_{a,k,\lambda} \mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$.

Proposition 3.5. *The operator $S_{a,k,\lambda} : L^2((-a,0)) \to L^2((-a,0))$ is compact and self-adjoint.*

Proof. Due to the assumptions on ρ_0, the operator $S_{a,k,\lambda}$ is well-defined from $L^2((-a,a))$ to itself. The operator $Y^{-1}_{a,k,\lambda}$ is compact, so is $S_{a,k,\lambda}$. Moreover, because both the inverse $Y^{-1}_{a,k,\lambda}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{a,k,\lambda}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{a,k,\lambda}$ is discrete, i.e. is a sequence $\{\gamma_n(\lambda,k)\}_{n\geqslant 1}$ of eigenvalues of $S_{a,k,\lambda}$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-a,0))$. That means

$$\gamma_n(\lambda,k)\varpi_n = S_{a,k,\lambda}\varpi_n = \mathcal{M} Y^{-1}_{a,k,\lambda}\mathcal{M}\varpi_n.$$

So that with $\phi_n = Y^{-1}_{a,k,\lambda}\mathcal{M}\varpi_n \in H^4((-a,0))$, one has

$$\gamma_n(\lambda,k) Y_{a,k,\lambda}\phi_n = \rho_0'\phi_n \tag{3.21}$$

and ϕ_n satisfies (3.7)-(1.5). Eq. (3.21) also tells us that $\gamma_n(\lambda,k) > 0$ for all n. Indeed, we obtain

$$\gamma_n(\lambda,k)\int_{-a}^{0}(Y_{a,k,\lambda}\phi_n)\phi_n dx_3 = \int_{-a}^{0}\rho_0'\phi_n^2 dx_3.$$

That implies

$$\gamma_n(\lambda,k)\mathscr{B}_{a,k,\lambda}(\phi_n,\phi_n) = \int_{-a}^{0}\rho_0'\phi_n^2 dx_3. \tag{3.22}$$

Since $\mathscr{B}_{a,k,\lambda}(\phi_n,\phi_n) > 0$ and $\rho_0' > 0$ on $(-a,0)$, we know that $\gamma_n(\lambda,k)$ is positive for all n. Hence, by reordering and using the spectral theory of compact and self-adjoint operators again, we obtain that $\{\gamma_n(\lambda,k)\}_{n\geqslant 1}$ is a positive sequence decreasing towards 0 as $n \to \infty$.

For each n, in order to verify that ϕ_n is a solution of (1.4)-(3.7)-(1.5), we are left to look for real values of λ_n such that

$$\gamma_n(\lambda_n,k) = \frac{\lambda_n}{gk^2}. \tag{3.23}$$

To solve (3.23), we need the two following lemmas.

Lemma 3.3. *For each n, $\gamma_n(\lambda,k)$ and ϕ_n are differentiable in λ.*

Proof. The proof of Lemma 3.3 is the same as [20, Lemma 3.2], we omit the details here. □

Lemma 3.4. *For each n, $\gamma_n(\lambda,k)$ is strictly decreasing in λ.*

Proof. Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (3.21) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{a,k,\lambda}z_n = \frac{1}{\gamma_n(\lambda,k)}\rho_0'z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\rho_0'\phi_n \tag{3.24}$$

on $(-a,0)$. At $x_3=-a$, we have

$$\begin{cases} z_n''(-a) - (k+\tau_-)z_n'(-a) + k\tau_- z_n(-a) = \frac{\rho_-}{2\mu\tau_-}\phi_n'(-a) - \frac{k\rho_-}{2\mu\tau_-}\phi_n(-a), \\ z_n'''(-a) - (k^2 + k\tau_- + \tau_-^2)z_n'(-a) + k\tau_-(k+\tau_-)z_n(-a) \\ \qquad = \Big(\frac{k\rho_-}{2\mu\tau_-} + \frac{\rho_-}{\mu}\Big)\phi_n'(-a) - \Big(\frac{k^2\rho_-}{2\mu\tau_-} + \frac{k\rho_-}{\mu}\Big)\phi_n(-a), \end{cases} \tag{3.25}$$

and at $x_3=0$, we also have

$$\begin{cases} z_n''(0) + k^2 z_n(0) = 0, \\ z_n'''(0) - (3k^2 + \frac{\lambda\rho_+}{\mu})z_n'(0) + \frac{gk^2\rho_+}{\lambda\mu}z_n(0) = \frac{\rho_+}{\mu}\phi_n'(0) + \frac{gk^2\rho_+}{\lambda^2\mu}\phi_n(0). \end{cases} \tag{3.26}$$

Multiplying by ϕ_n on both sides of (3.24), we obtain that

$$\begin{aligned} &\int_{-a}^0 (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_3 + \int_{-a}^0 (Y_{a,k,\lambda}z_n)\phi_n dx_3 \\ &\qquad = \frac{1}{\gamma_n(\lambda,k)}\int_{-a}^0 \rho_0' z_n\phi_n dx_3 + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^0 \rho_0'\phi_n^2 dx_3. \end{aligned} \tag{3.27}$$

Thanks to the integration by parts, we have

$$\int_{-a}^0 (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_3 = \int_{-a}^0 \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 - (\rho_0\phi_n'\phi_n)\Big|_{-a}^0 \tag{3.28}$$

and

$$\begin{aligned} \int_{-a}^0 (Y_{a,k,\lambda}z_n)\phi_n dx_3 = \int_{-a}^0 (Y_{a,k,\lambda}\phi_n)z_n dx_3 + \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^0 \\ - \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n'z_n) - \lambda\rho_0\phi_n'z_n\Big)\Big|_{-a}^0. \end{aligned} \tag{3.29}$$

Owing to (3.28), (3.29) and (3.21), Eq. (3.27) becomes

$$\begin{aligned} &\int_{-a}^0 \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 + \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^0 \\ &\qquad - \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n'z_n) - \lambda\rho_0\phi_n'z_n\Big)\Big|_{-a}^0 - (\rho_0\phi_n'\phi_n)\Big|_{-a}^0 \\ &= \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^0 \rho_0'\phi_n^2 dx_3. \end{aligned} \tag{3.30}$$

Using (3.25), we obtain

$$\begin{aligned} &- \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(-a) \\ &\quad + \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n'z_n) - \lambda\rho_0\phi_n'z_n\Big)(-a) + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= \Big(\frac{k^2\rho_-}{2\tau_-} + k\rho_-\Big)(\phi_n(-a))^2 - \Big(\frac{k\rho_-}{2\tau_-} + \rho_-\Big)\phi_n'(-a)\phi_n(-a) \\ &\quad - \frac{k\rho_-}{2\tau_-}\phi_n(-a)\phi_n'(-a) + \frac{\rho_-}{2\tau_-}(\phi_n'(-a))^2 + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= k\rho_-(\phi_n(-a))^2 + \frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2. \end{aligned} \tag{3.31}$$

Using (3.26) and (1.5), we also have

$$\begin{aligned}&-\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(0) + \rho_+\phi_n'(0)\phi_n(0)\\ &\quad+\Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(0) = \frac{gk^2\rho_+}{\lambda^2}(\phi_n(0))^2.\end{aligned} \tag{3.32}$$

Combining (3.30), (3.31) and (3.32), we deduce that

$$\begin{aligned}\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^0 \rho_0'\phi_n^2 dx_3 = &\int_{-a}^0 \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 + k\rho_-(\phi_n(-a))^2\\ &+\frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2 + \frac{gk^2\rho_+}{\lambda^2}(\phi_n(0))^2.\end{aligned} \tag{3.33}$$

This yields that $\frac{1}{\gamma_n(\lambda,k)}$ is strictly increasing in λ, i.e. $\gamma_n(\lambda,k)$ is strictly decreasing in λ. This ends the proof of Lemma 3.4. □

Now we are in position to solve (3.23).

Proposition 3.6. *For each $n \geqslant 1$, there exists a unique $\lambda_n > 0$ solving (2.28). In addition, λ_n decreases towards 0 as n goes to ∞.*

Proof. Using (3.22), we know that

$$\frac{1}{\gamma_n(\lambda,k)}\int_{-a}^0 \rho_0'\phi_n^2 dx_3 = \int_{-a}^0 (Y_{a,k,\lambda}\phi_n)\phi_n dx_3 = \mathscr{B}_{a,k,\lambda}(\phi_n,\phi_n),$$

Keep in mind (3.9), we deduce that

$$\frac{1}{\gamma_n(\lambda,k)}\int_{-a}^0 \rho_0'\phi_n^2 dx_3 \geqslant \lambda\int_{-a}^0 k^2\rho_0\phi_n^2 dx_3 + \mu\int_{-a}^0 k^4\phi_n^2 dx_3,$$

that implies

$$\frac{1}{L_0\gamma_n(\lambda,k)} \geqslant \lambda k^2 + \frac{\mu k^4}{\rho_+}.$$

Since $\gamma_n(\lambda,k)$ is a decreasing function, we have that $\frac{\lambda}{\gamma_n(\lambda,k)}$ is an increasing function. Hence

$$\lim_{\lambda\to\sqrt{\frac{g}{L_0}}}\frac{\lambda}{\gamma_n(\lambda,k)} > gk^2. \tag{3.34}$$

Meanwhile, we also have $\frac{1}{\gamma_n(\lambda,k)} \leqslant \frac{1}{\gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}},k)}$ for all $\lambda \leqslant \frac{1}{2}\sqrt{\frac{g}{L_0}}$. It yields,

$$\lim_{\lambda\to 0}\frac{\lambda}{\gamma_n(\lambda,k)} = 0. \tag{3.35}$$

Combining (3.34), (3.35) and the fact that γ_n is decreasing in λ, we obtain a unique λ_n solving (3.23).

We prove that the sequence $(\lambda_n)_{n\geqslant 1}$ is decreasing. Indeed, if $\lambda_m < \lambda_{m+1}$ for some $m \geqslant 1$, we have

$$\gamma_m(\lambda_m,k) > \gamma_m(\lambda_{m+1},k).$$

Meanwhile, we also have

$$\gamma_m(\lambda_{m+1},k) > \gamma_{m+1}(\lambda_{m+1},k).$$

That implies

$$\frac{\lambda_m}{gk^2} = \gamma_m(\lambda_m,k) > \gamma_{m+1}(\lambda_{m+1},k) = \frac{\lambda_{m+1}}{gk^2}.$$

That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence.

To conclude Proposition 3.6, we prove that $\lim_{n\to\infty}\lambda_n = 0$. Indeed, suppose that $\lim_{n\to\infty}\lambda_n = c_0 > 0$, one has that $\lambda_n \geqslant c_0$ for all $n \geqslant 1$. This yields

$$\gamma_n(c_0, k) \geqslant \gamma_n(\lambda_n, k) = \frac{\lambda_n}{gk^2} \geqslant \frac{c_0}{gk^2}.$$

Letting $n \to \infty$, we obtain that $0 \geqslant \frac{c_0}{gk^2}$, which is a contradiction. Hence, $\lim_{n\to\infty}\lambda_n = 0$. Proposition 3.6 is proven. □

3.4. **Proof of Theorem 2.1 and normal modes to the linearized equations.** We are in position to prove Theorem 2.1.

Proof of Theorem 2.1. For each $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ being found from Proposition 3.6, let $\phi_n(x_3) = Y^{-1}_{a,k,\lambda_n}\mathcal{M}\varpi_n(x_3) \in H^4((-a, 0))$, which is a solution of (1.4) on $(-a, 0)$ with the boundary conditions (3.7)-(1.5).

Keep in mind our computations in Subsection 3.1, we extend ϕ_n to $\mathbf{R}_-$ by requiring ϕ_n satisfies (3.6) for some constants $A_{n,1}, A_{n,2}$ as $\lambda = \lambda_n$. Those constants $A_{n,1}, A_{n,2}$ are defined by

$$\begin{cases} \phi_n(-a) = A_{n,1} + A_{n,2}, \\ \phi_n'(-a) = kA_{n,1} + A_{n,2}\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}}. \end{cases} \tag{3.36}$$

Solving (3.36), we get that

$$A_{n,1} = \frac{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}}\phi_n(-a) - \phi_n'(-a)}{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}} - k}, \quad A_{n,2} = \frac{\phi_n'(-a) - k\phi_n(-a)}{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}} - k}. \tag{3.37}$$

Therefore, the function $\phi_n \in H^4(\mathbf{R}_-)$ is a solution of (1.4) satisfying (1.5) and (1.6) as $\lambda = \lambda_n$ for each $n \geqslant 1$. Using a bootstrap argument, we have $\phi_n \in H^\infty(\mathbf{R}_-)$. Proof of Theorem 2.1 is complete. □

Once we have solutions of (1.4)-(1.5)-(1.6), we go back to the linearized equations (2.13).

Proposition 3.7. *For each* $\mathbf{k} = (k_1, k_2) \in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}$, *there exists an infinite sequence of normal modes*

$$e^{\lambda_n(\mathbf{k})t}V_n(\mathbf{k}, x) = e^{\lambda_n(\mathbf{k})t}(\zeta_n(\mathbf{k}, x), u_n(\mathbf{k}, x), q_n(\mathbf{k}, x), \eta_n(\mathbf{k}, x_h)) \tag{3.38}$$

to the linearized equations (2.13), *such that*

$$\zeta_n \in H^\infty(\Omega), u_n \in (H^\infty(\Omega))^3 \text{ and } q_n \in H^\infty(\Omega). \tag{3.39}$$

Proof. For each solution $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ of (2.28), we have a solution ϕ_n in $H^4(\mathbf{R}_-)$ of (1.4) as $\lambda = \lambda_n$, being found in Theorem 2.1. Furthermore, $\phi_n \in H^\infty(\mathbf{R}_-)$. We find uniquely $\pi_n \in H^\infty(\mathbf{R}_-)$ from (2.22) such that

$$\pi_n(\mathbf{k}, x_3) = -\frac{1}{k^2}(\lambda_n\rho_0\phi_n' + \mu(k^2\phi_n' - \phi_n'''))(\mathbf{k}, x_3).$$

To look for ψ_n, we rewrite (2.19) as a second order ODE,

$$-\mu\psi_n'' + (\lambda_n\rho_0\psi_n + \mu k^2\psi_n - k_1\pi_n) = 0.$$

Note from (2.20) and (2.21) that ψ_n satisfies that $\psi_n'(0) = k_1\phi_n(0)$ and that $\lim_{x_3\to-\infty}\psi_n(x_3) = 0$. By the ODE theory on a bounded interval and the domain expansion technique, we obtain a unique solution $\psi_n \in H^\infty(\mathbf{R}_-)$, where the

solution ψ_n depends on the known functions ϕ_n and π_n. We get φ_n in a similar way. Hence, $(\psi_n, \varphi_n, \phi_n, \pi_n) \in (H^\infty(\mathbf{R}_-))^4$ is a solution of (2.19)-(2.20).

Following (2.18), we then construct the functions

$$\begin{aligned} v_{1,n}(\mathbf{k}, x) &= \sin(k_1 x_1 + k_2 x_2)\psi_n(\mathbf{k}, x_3), \\ v_{2,n}(\mathbf{k}, x) &= \sin(k_1 x_1 + k_2 x_2)\varphi_n(\mathbf{k}, x_3), \\ v_{3,n}(\mathbf{k}, x) &= \cos(k_1 x_1 + k_2 x_2)\phi_n(\mathbf{k}, x_3), \\ r_n(\mathbf{k}, x_3) &= \cos(k_1 x_1 + k_2 x_2)\pi_n(\mathbf{k}, x_3). \end{aligned}$$

Keep in mind (2.16), let us define also

$$\omega_n(\mathbf{k}, x) = -\frac{1}{\lambda_n(\mathbf{k})}\rho_0'(x_3)v_{3,n}(\mathbf{k}, x_3) \quad \text{and} \quad \nu_n(\mathbf{k}, x_h) = \frac{1}{\lambda_n(\mathbf{k})}v_{3,n}(\mathbf{k}, x_h, 0).$$

Hence

$$(\zeta_n(t, \mathbf{k}, x), u_n(t, \mathbf{k}, x), q_n(t, \mathbf{k}, x), \eta_n(t, \mathbf{k}, x_h)) = e^{\lambda_n(\mathbf{k})t}(\omega_n, v_n, r_n, \nu_n)(\mathbf{k}, x)$$

is a real-valued solution of the linearized equations (2.13). We claim (3.39) by virtue of $(\psi_n, \varphi_n, \phi_n, \pi_n) \in (H^\infty(\mathbf{R}_-))^4$. □

3.5. **Maximal growth rate.** We derive the following proposition on the largest characteristic value λ_1 found in Theorem 2.1.

Proposition 3.8. *Let us recall the bilinear form $\mathscr{B}_{a,k,\lambda}$ on $H^2((-a, 0))$ (3.9) and (λ_1, ϕ_1) from Theorem 2.1. We have that*

$$\frac{1}{gk^2} = \max_{\phi \in H^2((-a,0))} \frac{\int_{-a}^0 \rho_0' \phi^2 dx_3}{\lambda_1 \mathscr{B}_{a,k,\lambda_1}(\phi, \phi)}, \tag{3.40}$$

and the extremal problem (3.40) is attained by ϕ_1 restricted on $(-a, 0)$ up to a constant.

Furthermore, let us define the following bilinear form on $H^2(\mathbf{R}_-)$,

$$\begin{aligned} \mathbf{B}_{k,\lambda}(\phi, \theta) := \lambda \int_{\mathbf{R}_-} \rho_0(k^2\phi\theta + \phi'\theta')dx_3 + \mu \int_{\mathbf{R}_-} ((\phi'' + k^2\phi)(\theta'' + k^2\theta) + 4k^2\phi'\theta')dx_3 \\ + \frac{gk^2\rho_+}{\lambda}\phi(0)\theta(0). \end{aligned}$$

Hence, we have

$$\frac{1}{gk^2} = \max_{\phi \in H^2(\mathbf{R}_-)} \frac{\int_{\mathbf{R}_-} \rho_0' \phi^2 dx_3}{\lambda_1 \mathbf{B}_{k,\lambda_1}(\phi, \phi)}. \tag{3.41}$$

The extremal problem (3.41) is attained by ϕ_1 up to a constant.

Proof of Proposition 3.8. We divide the proof into two parts, proving (3.40) and (3.41), respectively.

Part 1. We show that (3.40) holds. For all $\lambda > 0$, we solve the variational problem

$$\alpha_1(\lambda, k) = \max\Big(\int_{-a}^0 \rho_0' \phi^2 dx_3 \Big| \phi \in H^2((-a, 0)), \quad \lambda \mathscr{B}_{a,k,\lambda}(\phi, \phi) = 1 \Big). \tag{3.42}$$

Let us define the Lagrangian functional

$$\mathscr{L}_{\mathscr{B}}(\nu, \phi) = \int_{-a}^0 \rho_0' \phi^2 dx_3 - \nu(\lambda \mathscr{B}_{a,k,\lambda}(\phi, \phi) - 1). \tag{3.43}$$

It follows from the Lagrange multiplier theorem that the extrema of the quotient

$$\frac{\int_{-a}^{0}\rho_0'\phi^2dx_3}{\lambda\mathscr{B}_{a,k,\lambda}(\phi,\phi)}$$

are necessarily obtained at the stationary points $(\nu_\star,\phi_\star)$ of $\mathscr{L}_{\mathscr{B}}$, which satisfy

$$\lambda\mathscr{B}_{a,k,\lambda}(\phi_\star,\phi_\star)=1 \tag{3.44}$$

and

$$\int_{-a}^{0}\rho_0'\phi_\star\theta dx_3-\lambda\nu_\star\mathscr{B}_{a,k,\lambda}(\phi_\star,\theta)=0, \tag{3.45}$$

for all $\theta\in H^2((-a,0))$. Restricting $\theta\in C_0^\infty((-a,0))$ and following the line of the proof of Proposition 3.3, one deduces from (3.45) that $\phi_\star$ has to satisfy

$$\lambda\nu_\star Y_{a,k,\lambda}\phi_\star=\rho_0'\phi_\star \tag{3.46}$$

in a weak sense. We further get that $\phi_\star\in H^4((-a,0))$ and satisfies (3.44) and the boundary conditions (3.7)-(1.5). Hence, all stationary points $(\nu_\star,\phi_\star)$ of $\mathcal{L}_{\mathscr{B}}$ satisfy that, $\lambda\nu_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda}=\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ from $L^2((-a,0))$ to itself, with

$$\mathcal{M}^{-1}Y_{a,k,\lambda}\phi_\star=\frac{1}{\lambda\nu_\star}\mathcal{M}\phi_\star\in L^2((-a,0))$$

being an associated eigenfunction. That implies

$$\alpha_1(\lambda,k)\leqslant\lambda^{-1}\gamma_1(\lambda,k). \tag{3.47}$$

Meanwhile, since the operator $S_{a,k,\lambda}$ is self-adjoint and positive, we thus obtain that

$$\gamma_1(\lambda,k)=\sup_{\phi\in L^2((-a,0))}\frac{\langle S_{a,k,\lambda}\phi,\phi\rangle}{\|\phi\|^2_{L^2((-a,0))}}.$$

Hence, for all $\phi\in L^2((-a,0))$ and for $\psi=Y_{a,k,\lambda}^{-1}\mathcal{M}\phi\in H^4((-a,0))$, we have

$$\langle Y_{a,k,\lambda}\psi,\psi\rangle=\langle S_{a,k,\lambda}\phi,\phi\rangle,$$

which yields

$$\gamma_1(\lambda,k)\langle Y_{a,k,\lambda}\psi,\psi\rangle\leqslant\frac{\langle S_{a,k,\lambda}\phi,\phi\rangle^2}{\|\phi\|^2_{L^2((-a,0)}}\leqslant\|S_{a,k,\lambda}\phi\|^2_{L^2((-a,0))}.$$

This yields

$$\gamma_1(\lambda,k)\leqslant\sup\Big\{\frac{\|\mathcal{M}\psi\|^2_{L^2((-a,0))}}{\langle Y_{a,k,\lambda}\psi,\psi\rangle}|\psi\in H^4((-a,0))\text{ and }\mathcal{M}^{-1}Y_{a,k,\lambda}\psi\in L^2((-a,0))\Big\}.$$

Owing to (3.10), we have that

$$\gamma_1(\lambda,k)\leqslant\sup\Big\{\frac{\int_{-a}^{0}\rho_0'\psi^2dx_3}{\mathscr{B}_{a,k,\lambda}(\psi,\psi)}|\psi\in H^4((-a,0))\text{ and }\mathcal{M}^{-1}Y_{a,k,\lambda}\psi\in L^2((-a,0))\Big\}.$$

We thus obtain

$$\lambda^{-1}\gamma_1(\lambda,k)\leqslant\alpha_1(\lambda,k) \tag{3.48}$$

The two inequalities (3.47) and (3.48) tell us that $\alpha_1(k,\lambda)=\lambda^{-1}\gamma_1(k,\lambda)$ for all $\lambda>0$, from which we deduce $\alpha_1(\lambda_1,k)=\frac{1}{gk^2}$ and the extremal problem (3.40) is attained by the function $\phi_1|_{(-a,0)}$ up to a constant.

Part 2. We prove that (3.41) holds. We set

$$\alpha_2(\lambda,k)=\max_{\phi\in H^2(\mathbf{R}_-)}\Big(\int_{\mathbf{R}_-}\rho_0'\phi^2dx_3\Big|\lambda\mathbf{B}_{k,\lambda}(\phi,\phi)=1\Big).$$

and consider the Lagrangian functional

$$\mathbf{L_B}(\omega,\phi)=\int_{\mathbf{R}_-}\rho_0'\phi^2dx_3-\omega(\mathbf{B}_{k,\lambda}(\phi,\phi)-1).$$

Thanks to Lagrange multiplier theorem again, the extrema of the quotient

$$\frac{\int_{\mathbf{R}_-}\rho_0'\phi^2dx_3}{\lambda\mathbf{B}_{k,\lambda}(\phi,\phi)}$$

are necessarily obtained at the stationary points $(\omega_\star,\Phi_\star)\in\mathbf{R}_+\times H^2(\mathbf{R}_-)$ of $\mathbf{L_B}$, which satisfy

$$\lambda\mathbf{B}_{k,\lambda}(\Phi_\star,\Phi_\star)=1 \tag{3.49}$$

and

$$\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3-\lambda\omega_\star\mathbf{B}_{k,\lambda}(\Phi_\star,\theta)=0 \tag{3.50}$$

for all $\theta\in H^2(\mathbf{R}_-)$.

We now improve the regularity of $\Phi_\star$. We respectively define $(\Phi_\star'')'$ and $(\Phi_\star'')''$ in the distributional sense as the first and second derivative of $\Phi_\star''$ which is in $L^2(\mathbf{R}_-)$. Hence, (3.50) will imply that

$$\begin{aligned}\lambda\int_{\mathbf{R}_-}\rho_0(k^2\Phi_\star\theta+\Phi_\star'\theta')dx_3+\mu\langle(\Phi_\star'')'',\theta\rangle+\mu\int_{\mathbf{R}_-}(2k^2\Phi_\star''\theta+4k^2\Phi_\star'\theta'+k^4\Phi_\star\theta)dx_3\\=\frac{1}{\lambda\omega_\star}\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3\end{aligned} \tag{3.51}$$

for all $\theta\in C_0^\infty(\mathbf{R}_-)$. We deduce from (3.51) that

$$\mu((\Phi_\star'')''-2k^2\Phi_\star''+k^4\Phi_\star)+\lambda(k^2\rho_0\Phi_\star-(\rho_0\Phi_\star')')=\frac{1}{\lambda\omega_\star}\rho_0'\Phi_\star\quad\text{in }\mathcal{D}'(\mathbf{R}_-). \tag{3.52}$$

Thanks to (3.51) again, we obtain $(\Phi_\star'')''\in L^2(\mathbf{R}_-)$. Let $b>0$ be fixed and arbitrary, we have that $\Phi\in H^2((-b,0))$. Let $\chi_2\in C_0^\infty((-b,0))$ satisfy $\int_{-b}^0\chi_2(y)dy=1$. Using the distribution theory, we define $\Sigma_b\in\mathcal{D}'((-b,0))$ such that

$$\langle\Sigma_b,\theta\rangle=\langle(\Phi_\star'')'',\zeta_{\theta,b}\rangle \tag{3.53}$$

for all $\theta\in C_0^\infty((-b,0))$, where

$$\zeta_{\theta,b}(x_3)=\int_{-b}^{x_3}\Big(\theta(y)-\chi_2(y)\int_{-b}^0\theta(s)ds\Big)dy$$

for all $-b<x_3<0$. We obtain

$$\langle\Sigma_b',\theta\rangle=-\langle\Sigma_b,\theta'\rangle=-\langle(\Phi_\star'')'',\zeta_{\theta',b}\rangle.$$

Note that

$$\langle(\Phi_\star'')'',\zeta_{\theta',b}\rangle=\langle(\Phi_\star'')'',\theta(x_3)-\int_{-b}^{x_3}\chi_2(y)\int_{-b}^0\theta'(s)dsdy\rangle=\langle(\Phi_\star'')'',\theta\rangle,$$

this yields $\langle\Sigma_b',\theta\rangle=-\langle(\Phi_\star'')'',\theta\rangle$. Hence, we have that $(\Phi_\star'')'+\Sigma_b\equiv$ constant. In view of $(\Phi_\star'')''\in L^2((-b,0))$ and (3.53), we know that $(\Phi_\star'')'\in L^2((-b,0)))$. Since $\Phi_\star\in H^2(\mathbf{R}_-)$ and $(\Phi_\star'')',(\Phi_\star'')''\in L^2((-b,0))$, it tells us that Φ belongs to $H^4((-b,0))$.

Next, let us take $\theta\in C_0^\infty((-\infty,-b))$ with $b>a$. Due to (1.2) and (1.3), one has

$$\mu((\Phi_\star'')''-2k^2\Phi_\star''+k^4\Phi_\star)+\lambda\rho_+(k^2\Phi_\star-\Phi_\star'')=0\quad\text{in }\mathcal{D}'((-\infty,-b)).$$

As Φ is bounded at $-\infty$, we have

$$\Phi_\star(x_3) = a_1 e^{kx_3} + a_2 e^{(k^2+\lambda\rho_-/\mu)^{1/2}x_3}.$$

Since $\Phi_\star$ is explicit, we see that $\Phi_\star \in H^4((-\infty,-b))$. Consequently, $\Phi_\star \in H^4(\mathbf{R}_-)$ and $\Phi_\star$ decays to 0 at infinity.

By exploiting (3.50), we show that $\Phi_\star$ satisfies (1.5). Indeed, for all $\theta \in H^2(\mathbf{R}_-)$, we use the integration by parts to obtain from (3.50) that

$$\begin{aligned}
&\lambda\int_{\mathbf{R}_-}(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')')\theta dx_3 + \lambda\rho_0\Phi_\star'\theta\Big|_{-\infty}^0 \\
&\quad + \mu\int_{\mathbf{R}_-}((\Phi_\star''+k^2\Phi_\star)(\theta_\star''+k^2\theta_\star) + 4k^2\Phi_\star'\theta')dx_3 \\
&\quad + \mu\Big(\Phi_\star''\theta'\Big|_{-\infty}^0 - \Phi_\star'''\theta\Big|_{-\infty}^0 + k^2\Phi_\star\theta'\Big|_{-\infty}^0 + 3k^2\Phi_\star'\theta\Big|_{-\infty}^0\Big) + \frac{gk^2\rho_+}{\lambda}\Phi_\star(0)\theta(0) \\
&= \frac{1}{\lambda\omega_\star}\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3.
\end{aligned}$$

By collecting all terms corresponding to $\theta'(0)$ and $\theta'(0)$ respectively, we obtain that $\Phi_\star''(0) + k^2\Phi_\star(0) = 0$ and that

$$\lambda\rho_+\Phi_\star'(0) - \mu\Phi_\star'''(0) + 3k^2\Phi_\star'(0) + \frac{gk^2\rho_+}{\lambda}\Phi_\star(0) = 0.$$

This yields that $\Phi_\star$ satisfies (1.5).

We have just shown that $\Phi_\star \in H^4(\mathbf{R}_-)$ is a solution to

$$\mu(\Phi_\star^{(4)} - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')') = \frac{1}{\lambda\omega_\star}\rho_0'\Phi_\star \quad \text{on } \mathbf{R}_- \tag{3.54}$$

satisfying (1.5)-(1.6). Since $\mathrm{supp}\rho_0' = [-a,0]$, we see that $\Phi_\star$ is a solution of

$$\mu(\Phi_\star^{(4)} - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')') = 0 \quad \text{on } (-\infty,-a).$$

Then, $\Phi_\star$ on $(-\infty,-a)$ is of the form (3.6). Mimicking the computations in the proof of Lemma 3.1, we deduce $\Phi_\star$ on $(-a,0)$ is a solution of

$$\lambda\omega_\star Y_{a,k,\lambda}(\Phi_\star|_{(-a,0)}) = \rho_0'\Phi_\star|_{(-a,0)} = \mathcal{M}^2\Phi_\star|_{(-a,0)}$$

with the boundary conditions (3.7)-(1.5). Set

$$\tilde{\Phi} = \mathcal{M}^{-1}Y_{a,k,\lambda}(\Phi_\star|_{(-a,0)}) = \frac{1}{\lambda\omega_\star}\mathcal{M}\Phi_\star|_{(-a,0)} \in L^2((-a,0)), \tag{3.55}$$

it yields

$$\lambda\omega_\star\tilde{\Phi} = \mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}\tilde{\Phi} = S_{a,k,\lambda}\tilde{\Phi}.$$

That means $\lambda\omega_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda}$ from $L^2((-a,0))$ to itself, with $\tilde{\Phi} \in L^2((-a,0))$ (defined as in (3.55)) being an associated eigenfunction. Hence, we get

$$\lambda\alpha_2(\lambda,k) \leqslant \gamma_1(\lambda,k). \tag{3.56}$$

Let us recall the function ϕ_1 from Theorem 2.1. One thus has

$$\alpha_2(\lambda,k) \geqslant \frac{\int_{\mathbf{R}_-}\rho_0'\phi_1^2 dx_3}{\lambda\mathbf{B}_{k,\lambda}(\phi_1,\phi_1)}. \tag{3.57}$$

Note that from the proof of Theorem 2.1,

$$\phi_1(x_3) = A_1 e^{k(x_3+a)} + A_2 e^{\sqrt{k^2+\frac{\lambda_1\rho_-}{\mu}}(x_3+a)} \quad \text{as } -\infty < x_3 < -a.$$

Let us write $\phi_1|_{(-a,0)}$ as the function ϕ_1 being restricted on $(-a,0)$. Hence, the direct computations show that

$$\mathbf{B}_{k,\lambda}(\phi_1,\phi_1)=\mathscr{B}_{a,k,\lambda}(\phi_1|_{(-a,0)},\phi_1|_{(-a,0)}), \tag{3.58}$$

and we keep in mind the assumption $\mathrm{supp}\rho_0'=[-a,0]$. Then, from (3.57) and (3.58), we have

$$\alpha_2(\lambda,k)\geqslant\frac{\int_{-a}^0\rho_0'\phi_1^2dx_3}{\lambda\mathscr{B}_{a,k,\lambda}(\phi_1|_{(-a,0)},\phi_1|_{(-a,0)})}.$$

It then follows

$$\alpha_2(\lambda_1,k)\geqslant\frac{\int_{-a}^0\rho_0'\phi_1^2dx_3}{\lambda_1\mathscr{B}_{a,k,\lambda_1}(\phi_1|_{(-a,0)},\phi_1|_{(-a,0)})}=\frac{1}{gk^2}. \tag{3.59}$$

Combining (3.56) and (3.59) gives us that $\alpha_2(\lambda_1,k)=\frac{1}{gk^2}$ and the extremal problem (3.41) is attained by ϕ_1 up to a constant. We finish the proof of Proposition 3.8. □

Recall the definition of Λ from (2.29), we prove that Λ is the maximal growth rate of the linearized equations (2.13) in the following sense:

Proposition 3.9. *For all $t\geqslant 0$, the following inequalities hold*

$$\begin{aligned}&\|\zeta(t)\|_{H^1(\Omega)}^2+\|u(t)\|_{H^1(\Omega)}^2+\|\partial_t u(t)\|_{L^2(\Omega)}^2+\int_0^t\|u(s)\|_{H^1(\Omega)}^2ds\\&\lesssim(\|\zeta(0)\|_{H^1(\Omega)}^2+\|\partial_t u(0)\|_{L^2(\Omega)}^2+\|u(0)\|_{H^1(\Omega)}^2)e^{2\Lambda t},\end{aligned} \tag{3.60}$$

and

$$\begin{aligned}&\|\eta(t)\|_{H^{1/2}(\Gamma)}^2+\|\partial_t\eta(t)\|_{H^{1/2}(\Gamma)}^2+\int_0^t\|\partial_t\eta(s)\|_{H^{1/2}(\Gamma)}^2\\&\lesssim(\|\eta(0)\|_{H^{1/2}(\Gamma)}^2+\|\partial_t u(0)\|_{L^2(\Omega)}^2+\|u(0)\|_{H^1(\Omega)}^2)e^{2\Lambda t}.\end{aligned} \tag{3.61}$$

The proof of Proposition 3.9 relies on the two lemmas below.

Lemma 3.5. *There holds*

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega\rho_0|\partial_t u|^2-\int_\Omega g\rho_0'|u_3|^2+\int_\Gamma g\rho_+|u_3|^2\Big)+\frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u|^2=0. \tag{3.62}$$

Proof. We differentiate $(2.13)_2$ in time, multiply the resulting equation by $\partial_t u$ and then use $(2.13)_1$ to obtain

$$\int_\Omega\rho_0\partial_t^2u\cdot\partial_t u+\int_\Omega\nabla\partial_t q\cdot\partial_t u-\mu\int_\Omega\Delta\partial_t u\cdot\partial_t u-\int_\Omega g\rho_0'u_3\partial_t u_3=0.$$

That is equivalent to

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega\rho_0|\partial_t u|^2-\int_\Omega g\rho_0'|u_3|^2\Big)+\int_\Omega\nabla\partial_t q\cdot\partial_t u-\mu\int_\Omega\Delta\partial_t u\cdot\partial_t u=0. \tag{3.63}$$

We use the integration by parts over Ω to have

$$\begin{aligned}\int_\Omega\nabla\partial_t q\cdot\partial_t u-\mu\int_\Omega\Delta\partial_t u\cdot\partial_t u=&\int_\Gamma(\partial_t q\mathrm{Id}-\mu\mathbb{S}\partial_t u)e_3\cdot\partial_t u-\int_\Omega\partial_t q\mathrm{div}\partial_t u\\&+\frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u|^2\end{aligned}$$

Thanks to $(2.13)_{3,4,5}$, we obtain

$$\begin{aligned}\int_\Omega \nabla\partial_t q\cdot\partial_t u-\mu\int_\Omega \Delta\partial_t u\cdot\partial_t u&=\int_\Gamma g\rho_+\partial_t\eta\partial_t u_3+\frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u|^2\\&=\int_\Gamma g\rho_+u_3\partial_t u_3+\frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u|^2.\end{aligned}\tag{3.64}$$

Substituting (3.64) into (3.63), we conclude (3.62). □

Lemma 3.6. *There holds*

$$\int_\Omega g\rho_0'|u_3|^2\leqslant\int_\Gamma g\rho_+|u_3|^2+\Lambda^2\int_\Omega\rho_0|u|^2+\frac{1}{2}\Lambda\int_\Omega\mu|\mathbb{S}u|^2.\tag{3.65}$$

Proof. Let $\mathbf{k}=(k_1,k_2)\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}$ be fixed and $\hat f$ be the horizontal Fourier transform of f, i.e.

$$\hat f(\mathbf{k},x_3)=\int_{\mathbf{T}}f(x_h,x_3)e^{-i\mathbf{k}\cdot x_h}dx_h.$$

We write

$$\hat u_1(\mathbf{k},x)=-i\psi(\mathbf{k},x_3),\quad \hat u_2(\mathbf{k},x)=-i\varphi(\mathbf{k},x_3),\quad \hat u_3(\mathbf{k},x)=\phi(\mathbf{k},x_3).$$

Notice that for $\mathbf{k}=0$,

$$\phi(0,0)=\int_\Gamma u_3=\int_\Omega \mathrm{div}u=0.$$

Hence, together with Parseval's theorem, we have

$$\int_\Gamma g\rho_+|u_3|^2=\frac{1}{4\pi^2L_1L_2}\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}}g\rho_+|\phi(\mathbf{k},0)|^2.\tag{3.66}$$

We may reduce to estimate (3.66) when ψ,φ and ϕ are real-valued and then continue the estimate to the real and imaginary parts of ψ,φ and ϕ

For each $\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}$, we deduce from Proposition 3.8 that

$$\begin{aligned}\int_{\mathbf{R}_-}g\rho_0'\phi^2(\mathbf{k},x_3)dx_3\leqslant g\rho_+(\phi(k,0))^2+\lambda_1^2\int_{\mathbf{R}_-}\rho_0\Big(\phi^2+\frac{(\phi')^2}{k^2}\Big)(\mathbf{k},x_3)dx_3\\+\lambda_1\mu\int_{\mathbf{R}_-}\Big(\Big(\frac{\phi''}{k}+k\phi\Big)^2+4(\phi')^2\Big)(\mathbf{k},x_3)dx_3.\end{aligned}$$

It thus follows from the definition of Λ (2.29) that

$$\begin{aligned}\int_{\mathbf{R}_-}g\rho_0'\phi^2(\mathbf{k},x_3)dx_3\leqslant g\rho_+(\phi(\mathbf{k},0))^2+\Lambda^2\int_{\mathbf{R}_-}\rho_0\Big(\phi^2+\frac{(\phi')^2}{k^2}\Big)(\mathbf{k},x_3)dx_3\\+\Lambda\mu\int_{\mathbf{R}_-}\Big(\Big(\frac{\phi''}{k}+k\phi\Big)^2+4(\phi')^2\Big)(\mathbf{k},x_3)dx_3\end{aligned}\tag{3.67}$$

for all $\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\backslash\{0\}$.

Meanwhile, for $\mathbf{k}\neq0$, notice that $k_1\psi+k_2\varphi+\phi'=0$. One thus has

$$(\phi')^2\leqslant(k_1\psi+k_2\varphi)^2+(k_1\varphi-k_2\psi)^2=k^2(\psi^2+\varphi^2),\tag{3.68}$$

and

$$2(\phi')^2=2k_1^2\psi^2+2k_2^2\varphi^2+4k_1k_2\psi\varphi\leqslant2k_1^2\psi^2+2k_2^2\varphi^2+(k_1\varphi+k_2\psi)^2.\tag{3.69}$$

Furthermore, we obtain that

$$(\phi'')^2\leqslant(k_1\psi'+k_2\varphi')^2+(k_1\varphi'-k_2\psi')^2=k^2((\psi')^2+(\varphi')^2).$$

This yields

$$\Big(\frac{1}{k}\phi''+k\phi\Big)^2=\frac{1}{k^2}(\phi'')^2+2\phi\phi''+k^2\phi^2\leqslant(\psi')^2+(\varphi')^2-2\phi(k_1\psi'+k_2\varphi')+k^2\phi^2,$$

so that

$$\Big(\frac{1}{k}\phi''+k\phi\Big)^2\leqslant(k_1\phi-\psi')^2+(k_2\phi-\varphi')^2. \tag{3.70}$$

Then, in view of Fubini's and Parseval's theorem again, we find that due to (3.68),

$$\begin{aligned}\int_\Omega\rho_0|u|^2&=\frac{1}{4\pi^2L_1L_2}\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}\int_{\mathbf{R}_-}\rho_0(\psi^2+\varphi^2+\phi^2)(\mathbf{k},x_3)dx_3\\&\geqslant\frac{1}{4\pi^2L_1L_2}\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\setminus\{0\}}\int_{\mathbf{R}_-}\rho_0\Big(\phi^2+\frac{(\phi')^2}{k^2}\Big)(\mathbf{k},x_3)dx_3\end{aligned} \tag{3.71}$$

and that due to (3.69) and (3.70),

$$\begin{aligned}&\frac{1}{2}\int_\Omega\mu|\mathbb{S}u|^2\\&=\frac{\mu}{4\pi^2L_1L_2}\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}\int_{\mathbf{R}_-}\begin{pmatrix}2(\phi')^2+2k_1^2\psi^2+2k_2^2\varphi^2+(k_1\varphi+k_2\psi)^2\\+(k_1\phi-\psi')^2+(k_2\phi-\varphi')^2\end{pmatrix}dx_3\\&\geqslant\frac{\mu}{4\pi^2L_1L_2}\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}\setminus\{0\}}\int_{\mathbf{R}_-}\Big(\Big(\frac{\phi''}{k}+k\phi\Big)^2+4(\phi')^2\Big)dx_3.\end{aligned} \tag{3.72}$$

Combining (3.66), (3.67), (3.71) and (3.72), the inequality (3.65) follows, we end the proof here. □

We are in position to prove Proposition 3.9.

Proof of Proposition 3.9. Owing to (3.62) and (3.65), we have that

$$\begin{aligned}\int_\Omega\rho_0|\partial_tu(t)|^2+\int_0^t\int_\Omega\mu|\mathbb{S}\partial_tu(s)|^2ds&=y_1+\int_\Omega g\rho_0'|u_3(t)|^2-\int_\Gamma g\rho_+|u_3(t)|^2\\&\leqslant y_1+\Lambda^2\int_\Omega\rho_0|u(t)|^2+\frac{1}{2}\Lambda\int_\Omega\mu|\mathbb{S}u(t)|^2,\end{aligned} \tag{3.73}$$

where

$$y_1=\int_\Omega\rho_0|\partial_tu(0)|^2-\int_\Omega g\rho_0'|u_3(0)|^2+\int_\Gamma g\rho_+|u_3(0)|^2.$$

Using Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}\int_\Omega\mu|\mathbb{S}u(t)|^2&=\int_\Omega\mu|\mathbb{S}u(0)|^2+2\int_0^t\int_\Omega\mu\mathbb{S}u(s):\mathbb{S}\partial_tu(s)ds\\&\leqslant\int_\Omega\mu|\mathbb{S}u(0)|^2+\frac{1}{\Lambda}\int_0^t\int_\Omega\mu|\mathbb{S}\partial_tu(s)|^2ds+\Lambda\int_0^t\int_\Omega\mu|\mathbb{S}u(s)|^2ds\end{aligned} \tag{3.74}$$

and that

$$\frac{d}{dt}\int_\Omega\rho_0|u|^2\leqslant\frac{1}{\Lambda}\int_\Omega\rho_0|\partial_tu|^2+\Lambda\int_\Omega\rho_0|u|^2. \tag{3.75}$$

The three inequalities (3.73), (3.74) and (3.75) imply that

$$\frac{d}{dt}\int_\Omega\rho_0|u(t)|^2+\frac{1}{2}\int_\Omega\mu|\mathbb{S}u(t)|^2\leqslant y_2+2\Lambda\int_\Omega\rho_0|u(t)|^2+\Lambda\int_0^t\int_\Omega\mu|\mathbb{S}u(s)|^2ds. \tag{3.76}$$

where

$$y_2 = \frac{y_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u(0)|^2.$$

In view of Gronwall's inequality, we obtain from (3.76) that

$$\int_\Omega \rho_0|u(t)|^2 + \frac{1}{2}\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds \leqslant e^{2\Lambda t}\int_\Omega \rho_0|u(0)|^2 + \frac{y_2}{2\Lambda}(e^{2\Lambda t}-1). \tag{3.77}$$

Hence,

$$\begin{aligned}\frac{1}{\Lambda}\int_\Omega \rho_0|\partial_t u(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u(t)|^2 &\leqslant y_2 + \Lambda\int_\Omega \rho_0|u(t)|^2 + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds\\ &\leqslant \Big(y_2 + 2\Lambda\int_\Omega \rho_0|u(0)|^2\Big)e^{2\Lambda t}\end{aligned} \tag{3.78}$$

Using the trace theorem, we have

$$y_1 + y_2 \lesssim \|u(0)\|^2_{H^1(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)}. \tag{3.79}$$

Because of (3.77), (3.78) and (3.79), we observe

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|\mathbb{S}u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\mathbb{S}u(s)\|^2_{L^2(\Omega)} ds\\ &\quad\lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.80}$$

In view of Korn's inequality (see (C.5)), that implies

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds\\ &\quad\lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.81}$$

Using $(2.13)_1$ and (3.81) also, we get

$$\begin{aligned}\|\zeta(t)\|^2_{H^1(\Omega)} &\lesssim \|\zeta(0)\|^2_{H^1(\Omega)} + \int_0^t \|u(s)\|^2_{H^1(\Omega)} ds\\ &\lesssim (\|\zeta(0)\|^2_{H^1(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.82}$$

The inequality (3.60) follows from (3.81) and (3.82).

To prove (3.61), we use the trace theorem to obtain that

$$\begin{aligned}\|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)} ds &= \|u_3(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|u_3(s)\|^2_{H^{1/2}(\Gamma)} ds\\ &\leqslant \|u_3(t)\|^2_{H^1(\Omega)} + \int_0^t \|u_3(s)\|^2_{H^1(\Omega)} ds.\end{aligned}$$

Together with (3.77), (3.79) and (3.81), we deduce that

$$\|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)} ds \lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}. \tag{3.83}$$

The resulting inequality tells us that

$$\begin{aligned}\|\eta(t)\|^2_{H^{1/2}(\Gamma)} &\leqslant \|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)} ds\\ &\lesssim (\|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.84}$$

The inequality (3.61) follows from (3.83) and (3.84). Proposition 3.9 is proven. □

4. A priori energy estimates

Let us recall the perturbation terms $U = (\zeta, u, q, \eta)$ from (2.10). We define the full energy functional $\mathcal{E}_f(U(t)) > 0$ such that

$$\begin{aligned}\mathcal{E}_f^2(U(t)) := \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \sum_{l=0}^{2} \|\partial_t^l \eta\|^2_{H^{4-2l}(\Gamma)} + \sum_{l=0}^{2} \|\partial_t^l(\zeta, u)(t)\|^2_{H^{4-2l}(\Omega)} \\ + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\end{aligned} \tag{4.1}$$

and its corresponding dissipation $\mathcal{D}_f((u,q)(t)) > 0$,

$$\mathcal{D}_f^2((u,q)(t)) := \sum_{l=0}^{2} \|\partial_t^l u(t)\|^2_{H^{5-2l}(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} + \|q\|^2_{H^4(\Omega)}. \tag{4.2}$$

For notational convenience, we only write $\mathcal{E}_f(t)$ and $\mathcal{D}_f(t)$ in this section.

The local existence of regular solution to (2.11) then follows from [8, Theorem 6.3], that we restate below.

Let us recall the definition of K from (2.7) and of $\mathcal{A}$ from (2.8) and define $\mathbb{M} = K\mathcal{A}$, $R = \partial_t \mathbb{M}\mathbb{M}^{-1}$ and $D_t u = \partial_t u - Ru$. We also define an orthogonal projection onto the tangent space of the surface $\{x_3 = \eta_0(x_1, x_2)\}$ according to

$$\Pi_0 v = v - \frac{v \cdot \mathcal{N}_0}{|\mathcal{N}_0|^2}\mathcal{N}_0 \quad \text{for } \mathcal{N}_0 = (-\partial_1\eta_0, \partial_2\eta_0, 1)^T. \tag{4.3}$$

Let us write

$$\begin{aligned} G^{2,0} &= g\rho_+\eta\mathcal{N} \text{ on } \Gamma, \\ G^{2,1} &= D_t G^{2,0} + \mu\mathbb{S}_{\mathcal{A}}(Ru)\mathcal{N} + (\mu\mathbb{S}_{\mathcal{A}}u - q\mathrm{Id})\partial_t\mathcal{N} + \mu\mathbb{S}_{\partial_t\mathcal{A}}u\mathcal{N} \text{ on } \Gamma. \end{aligned}$$

Proposition 4.1. *Suppose that there is a sufficiently small constant $\nu_1 \in (0,1)$ such that $(\zeta_0, u_0, q_0, \eta_0)$ satisfying*

$$\|\zeta_0\|^2_{H^4(\Omega)} + \|u_0\|^2_{H^4(\Omega)} + \|q_0\|^2_{H^3(\Omega)} + \|\eta_0\|^2_{H^{9/2}(\Gamma)} \leqslant \nu_1.$$

Suppose also that the following compatibility conditions hold for $j = 0$ and 1,

$$\begin{cases} div_{\mathcal{A}_0} D_t^j u_0 = 0 & \text{in } \Omega, \\ \Pi_0(G^{2,j}(0) + \mu\mathbb{S}_{\mathcal{A}_0} D_t^j u_0 \mathcal{N}_0) = 0 & \text{on } \Gamma. \end{cases} \tag{4.4}$$

Then, there exist $\nu_2 > 0$ and $T_{\max} > 0$ such that if $\mathcal{E}_f(0) \leqslant \nu_2$, Eq. (2.11) with the initial data $(\zeta_0, u_0, q_0, \eta_0)$ satisfying the compatibility conditions (4.4) has a unique solution (ζ, u, q, η) on the time interval $[0, T_{\max})$. Moreover, we have

$$\mathcal{E}_f(t) \lesssim (1 + T_{\max})\mathcal{E}_f(0),$$

and η is such that the mapping $\Theta(\cdot, t)$ defined by (2.6) is a C^2-diffeomorphism for each $t \in [0, T_{\max})$.

With that regular solution (ζ, u, q, η) of (2.11) on a finite time interval $[0, T_{\max})$, we aim at showing the *a priori* energy estimates for the nonlinear equations (2.11).

Proposition 4.2. *Let C_{emb} be the optimal constant of the Sobolev embedding*

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega).$$

Let δ_0 be sufficiently small such that

$$0 < \delta_0 \leqslant \frac{\rho_-}{2C_{emb}\max(1, \max_{\mathbf{R}_-}\rho_0'(x_3))}, \tag{4.5}$$

and (4.173) *holds later. Hence, there are* $\varepsilon > 0$ *sufficiently small and another* $C_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$ *if* $\sup_{0\leqslant s\leqslant t} \mathcal{E}_f(s) \leqslant \delta$, *we have*

$$\begin{aligned}
&\mathcal{E}_f^2(t) + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_0\Big(\varepsilon^{-5}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-5}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds + \varepsilon^{-5}\mathcal{E}_f^3(t)\Big) \\
&\quad + C_0\varepsilon^{-59}\int_0^t (\|(\zeta, u)(s)\|_{L^2(\Omega)}^2 + \|\eta(s)\|_{L^2(\Gamma)}^2)ds.
\end{aligned} \tag{4.6}$$

Strategy of the proof. Respectively, we derive the *a priori* energy estimates for the space-time derivatives of η in Propositions 4.3, 4.4, for the temporal derivatives of u in Proposition 4.5, for the horizontal space-time derivatives of u in Proposition 4.6 and for the space-time derivatives of ζ in Proposition 4.7. Then, we derive some estimates thanks to the elliptic regularity theory (see Propositions 4.9, 4.10). In view of these above estimates, we obtain (4.6) and complete the proof of Proposition 4.2.

In what follows, the constants C_i $(i \geqslant 1)$ are to indicate some constants, which are referred later.

4.1. **Energy estimates of the perturbation transport.** We first derive the *a priori* energy estimates for η.

Proposition 4.3. *The following inequalities hold*

$$\begin{aligned}
\|\eta(t)\|_{H^4(\Gamma)}^2 &\leqslant C_1\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\eta(s)\|_{H^4(\Gamma)}^2 + \varepsilon^{-1}\|\nabla u(s)\|_{H^4(\Omega)}^2)ds\Big) \\
&\quad + C_1\int_0^t \mathcal{E}_f^3(s)ds,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\|\partial_t\eta(t)\|_{H^2(\Gamma)}^2 &\leqslant C_2\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\partial_t\eta(s)\|_{H^2(\Gamma)}^2 + \varepsilon^{-1}\|\nabla\partial_t u(s)\|_{H^2(\Omega)}^2)ds\Big) \\
&\quad + C_2\int_0^t \mathcal{E}_f^3(s)ds,
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2 &\leqslant C_3\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\partial_t^2\eta(s)\|_{L^2(\Gamma)}^2 + \varepsilon^{-1}\|\nabla\partial_t^2 u(s)\|_{L^2(\Omega)}^2)ds\Big) \\
&\quad + C_3\int_0^t \mathcal{E}_f^3(s)ds.
\end{aligned} \tag{4.9}$$

Proof. Let us prove (4.7). For any $\alpha \in \mathbb{N}^2, |\alpha| \leqslant 4$, we have by $(2.11)_4$,

$$\partial_t\partial^\alpha\eta = \partial^\alpha u_3 - (u_1\partial^\alpha\partial_1\eta + u_2\partial^\alpha\partial_2\eta) - \underbrace{\sum_{0\neq\beta\leqslant\alpha} (\partial^\beta u_1\partial^{\alpha-\beta}\partial_1\eta + \partial^\beta u_2\partial^{\alpha-\beta}\partial_2\eta)}_{=:R_1^\alpha}.$$

Using the integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\eta\|_{L^2(\Gamma)}^2 = \frac{1}{2}\int_\Gamma (\partial_1 u_1 + \partial_2 u_2)|\partial^\alpha\eta|^2 + \int_\Gamma (\partial^\alpha u_3 - R_1^\alpha)\partial^\alpha\eta.$$

So that, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\eta\|^2_{L^2(\Gamma)} \lesssim (\|\partial_1 u_1\|_{L^\infty(\Gamma)} + \|\partial_2 u_2\|_{L^\infty(\Gamma)})\|\partial^\alpha\eta\|^2_{L^2(\Gamma)} \\ + (\|\partial^\alpha u_3\|_{L^2(\Gamma)} + \|R_1^\alpha\|_{L^2(\Gamma)})\|\partial^\alpha\eta\|_{L^2(\Gamma)}.\end{aligned} \tag{4.10}$$

We make use of the trace theorem to obtain that

$$\|\partial_j u_j\|_{L^\infty(\Gamma)} \lesssim \|u\|_{H^3(\Gamma)} \lesssim \|u\|_{H^4(\Omega)}, \tag{4.11}$$

that

$$\|\partial^\alpha u_3\|_{L^2(\Gamma)} \lesssim \|\partial^\alpha u\|_{H^1(\Omega)} \tag{4.12}$$

and that

$$\|R_1^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\neq\beta\leqslant\alpha} \|\partial^\beta u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\eta\|_{H^1(\Gamma)} \lesssim \|\partial^\alpha u\|_{H^1(\Omega)}\|\eta\|_{H^{|\alpha|}(\Gamma)}. \tag{4.13}$$

By summing over $\alpha\in\mathbb{N}^2, |\alpha|\leqslant 4$, it follows from (4.10), (4.11), (4.12) and (4.13) that

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^4(\Gamma)} &\lesssim \|\nabla u\|_{H^4(\Omega)}\|\eta\|_{H^4(\Gamma)} + \|u\|_{H^4(\Omega)}\|\eta\|^2_{H^4(\Gamma)} \\ &\lesssim \|\nabla u\|_{H^4(\Omega)}\|\eta\|_{H^4(\Gamma)} + \mathcal{E}_f^3.\end{aligned}$$

Using Cauchy-Schwarz's inequality and then integrating the resulting inequality from 0 to t, we obtain (4.7).

We show (4.8). Let $\alpha\in\mathbb{N}^2, |\alpha|\leqslant 2$, we get

$$\begin{aligned}\partial_t^2\partial^\alpha\eta = \partial^\alpha\partial_t u_3 - (u_1\partial^\alpha\partial_1\partial_t\eta + u_2\partial^\alpha\partial_2\partial_t\eta) \\ \underbrace{- \sum_{0\neq\beta\leqslant\alpha} (\partial^\beta u_1\partial^{\alpha-\beta}\partial_1\partial_t\eta + \partial^\beta u_2\partial^{\alpha-\beta}\partial_2\partial_t\eta)}_{=:R_2^\alpha} \\ \underbrace{- \sum_{0\leqslant\beta\leqslant\alpha} (\partial^\beta\partial_t u_1\partial^{\alpha-\beta}\partial_1\eta + \partial^\beta\partial_t u_2\partial^{\alpha-\beta}\partial_2\eta)}_{=:R_3^\alpha}.\end{aligned}$$

Via the integration by parts, this yields

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\partial_t\eta\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma(\partial_1 u_1 + \partial_2 u_2)|\partial^\alpha\partial_t\eta|^2 + \int_\Gamma(\partial^\alpha\partial_t u_3 - R_2^\alpha - R_3^\alpha)\partial^\alpha\partial_t\eta.$$

Using the trace theorem again, we have

$$\begin{aligned}\int_\Gamma(\partial_t\partial_1 u_1 + \partial_t\partial_2 u_2)\partial^\alpha\eta\partial^\alpha\partial_t\eta \lesssim \|\partial_t u\|_{H^1(\Gamma)}\|\partial^\alpha\eta\|_{H^2(\Gamma)}\|\partial_t\partial^\alpha\eta\|_{H^2(\Gamma)} \\ \lesssim \|\partial_t u\|_{H^2(\Omega)}\|\eta\|_{H^2(\Gamma)}\|\partial_t\eta\|_{H^2(\Gamma)},\end{aligned} \tag{4.14}$$

and

$$\|\partial^\alpha\partial_t u_3\|_{L^2(\Gamma)} \lesssim \|\partial_t u\|_{H^{|\alpha|+1}(\Omega)} \tag{4.15}$$

We follow (4.13) to get that

$$\|R_2^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\neq\beta\leqslant\alpha} \|\partial^\beta u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\partial_t\eta\|_{H^1(\Gamma)} \lesssim \|u\|_{H^3(\Omega)}\|\partial_t\eta\|_{H^2(\Gamma)} \tag{4.16}$$

and that

$$\|R_3^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\leqslant\beta\leqslant\alpha} \|\partial^\beta\partial_t u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\eta\|_{H^1(\Gamma)} \lesssim \|\nabla\partial_t u\|_{H^2(\Omega)}\|\eta\|_{H^3(\Gamma)}. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17), we deduce that

$$\frac{d}{dt}\|\partial_t\eta\|^2_{H^2(\Gamma)} \lesssim \|\nabla\partial_t u\|_{H^2(\Omega)}\|\partial_t\eta\|_{H^2(\Gamma)} + \mathcal{E}_f^3.$$

Using Cauchy-Schwarz's inequality and then integrating from 0 to t, we obtain (4.8).

We have (4.9) by following the same strategy as for proving (4.8). The proof of Proposition 4.3 is complete. □

Proposition 4.4. *There holds*

$$\begin{aligned}\|\eta(t)\|^2_{H^{9/2}(\Gamma)} \leqslant C_4\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\eta(s)\|^2_{H^{9/2}(\Gamma)} + \varepsilon^{-1}\|u(s)\|^2_{H^5(\Omega)})ds\Big)\\ + C_4\int_0^t \mathcal{E}_f^3(s)ds.\end{aligned} \tag{4.18}$$

Proof. To prove (4.18), we borrow the idea from [29, Lemma 3.9]. Let $\mathcal{J} = \sqrt{1-\partial_1^2-\partial_2^2}$. We apply $\mathcal{J}^{9/2}$ to $(2.11)_4$ and then multiply the resulting equation by $\mathcal{J}^{9/2}\eta$. Hence, we find that

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^{9/2}(\Gamma)} &= -\frac{1}{2}\int_\Gamma (u_1\partial_1|\mathcal{J}^{9/2}\eta|^2 + u_2\partial_2|\mathcal{J}^{9/2}\eta|^2)\\ &\quad + \int_\Gamma (\mathcal{J}^{9/2}u_3 - [\mathcal{J}^{9/2},u_1]\partial_1\eta - [\mathcal{J}^{9/2},u_2]\partial_2\eta)\mathcal{J}^{9/2}\eta\\ &= \frac{1}{2}\int_\Gamma (\partial_1u_1+\partial_2u_2)|\mathcal{J}^{9/2}\eta|^2\\ &\quad + \int_\Gamma (\mathcal{J}^{9/2}u_3 - [\mathcal{J}^{9/2},u_1]\partial_1\eta - [\mathcal{J}^{9/2},u_2]\partial_2\eta)\mathcal{J}^{9/2}\eta.\end{aligned}$$

Thanks to (C.6), we have the following estimates,

$$\int_\Gamma \partial_j u_j|\mathcal{J}^{9/2}\eta|^2 \lesssim \|\partial_j u_j\|_{L^\infty(\Gamma)}\|\mathcal{J}^{9/2}\eta\|^2_{L^2(\Gamma)} \lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)}, \tag{4.19}$$

$$\int_\Gamma \mathcal{J}^{9/2}u_3\mathcal{J}^{9/2}\eta \lesssim \|\mathcal{J}^{9/2}u_3\|_{L^2(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)} \lesssim \|\mathcal{J}^4u\|_{H^1(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)}, \tag{4.20}$$

and

$$\begin{aligned}\int_\Gamma [\mathcal{J}^{9/2},u_j]\partial_j\eta\mathcal{J}^{9/2}\eta &\lesssim \|\partial_j u_j\|_{L^\infty(\Gamma)}\|\mathcal{J}^{7/2}\eta\|_{L^2(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)}\\ &\quad + \|\mathcal{J}^{9/2}u\|_{L^2(\Gamma)}\|\partial_j\eta\|_{L^\infty(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)}\\ &\lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)} + \|\mathcal{J}^4u\|_{H^1(\Omega)}\|\eta\|_{H^3(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}.\end{aligned} \tag{4.21}$$

In view of (4.19), (4.20) and (4.21), we get

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^{9/2}(\Gamma)} &\lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)} + \|\mathcal{J}^4u\|_{H^1(\Omega)}(1+\|\eta\|_{H^3(\Gamma)})\|\eta\|_{H^{9/2}(\Gamma)}\\ &\lesssim \|u\|_{H^5(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)} + \mathcal{E}_f^3.\end{aligned}$$

Using Cauchy-Schwarz's inequality and then integrating from 0 to t, we obtain (4.18). □

We provide some additional estimates on η, which will be used later.

Lemma 4.1. *We have*

$$\|\partial_t\eta\|_{H^{7/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2, \tag{4.22}$$

$$\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2, \tag{4.23}$$

and

$$\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2u\|_{H^1(\Omega)}(1+\mathcal{E}_f) + \mathcal{E}_f^2. \tag{4.24}$$

Proof. By $(2.11)_4$, we have that

$$\|\partial_t\eta\|_{H^{7/2}(\Gamma)} \lesssim \|u_3\|_{H^{7/2}(\Gamma)} + \|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)} \lesssim \|u_3\|_{H^4(\Omega)} + \|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}. \tag{4.25}$$

We use (C.1) and the trace theorem to estimate $\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}$ (see $\mathcal{Q}^4$ in (2.12)) as

$$\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)} \lesssim \|u\|_{H^{7/2}(\Gamma)}\|(\partial_1\eta, \partial_2\eta)\|_{H^{7/2}(\Gamma)} \lesssim \|u\|_{H^4(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)},$$

Substituting the resulting inequality into (4.25), we have (4.22).

Using (C.1) again, we have

$$\begin{aligned}\|\partial_t\mathcal{Q}^4\|_{H^{3/2}(\Gamma)} &\lesssim \|\partial_t\partial_h\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)} + \|\partial_t u\|_{H^{3/2}(\Gamma)}\|\partial_h\eta\|_{H^{7/2}(\Gamma)}\\ &\lesssim \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^4(\Omega)} + \|\eta\|_{H^{9/2}(\Gamma)}\|\partial_t u\|_{H^2(\Omega)}.\end{aligned}$$

Together with (4.22), that implies

$$\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)} \lesssim \|\partial_t u_3\|_{H^2(\Omega)} + \|\partial_t\mathcal{Q}^4\|_{H^{3/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2.$$

One thus has (4.23).

We continue using $(2.11)_4$ to have that

$$\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u_3\|_{H^{1/2}(\Gamma)} + \|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u_3\|_{H^1(\Omega)} + \|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)}. \tag{4.26}$$

As a consequence of the product estimate (C.1) and Sobolev embedding, we obtain

$$\begin{aligned}\|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} &\lesssim \|\partial_t^2 u\|_{H^{1/2}(\Gamma)}\|(\partial_1\eta, \partial_2\eta)\|_{H^{5/2}(\Gamma)} + \|\partial_t u\|_{H^{1/2}(\Gamma)}\|\partial_t(\partial_1\eta, \partial_2\eta)\|_{H^{5/2}(\Gamma)}\\ &\quad + \|u\|_{H^{5/2}(\Gamma)}\|\partial_t^2(\partial_1\eta, \partial_2\eta)\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|\partial_t^2 u\|_{H^1(\Omega)}\|\eta\|_{H^{7/2}(\Gamma)} + \|\partial_t u\|_{H^1(\Omega)}\|\partial_t\eta\|_{H^{7/2}(\Gamma)}\\ &\quad + \|u\|_{H^3(\Omega)}\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)}.\end{aligned}$$

We continue using (4.22) and (4.23) to observe

$$\|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u\|_{H^1(\Omega)}\mathcal{E}_f + \mathcal{E}_f^2. \tag{4.27}$$

The inequality (4.24) follows from (4.26) and (4.27). Lemma 4.1 is proven. □

4.2. **Temporal estimates for the perturbation velocity.** If we use the nonlinear equations in the perturbed form (2.11), there will be no control of the highest temporal derivative of q appearing in the nonlinear term $\mathcal{Q}^2$. Instead, we switch our original nonlinear equations (2.9) to a new formulation using a geometric transformation of the domain. The equations are

$$\begin{cases}\partial_t\zeta + \mathrm{div}_{\mathcal{A}}(\rho_0 u) = F^1 & \text{in } \Omega,\\ (\rho_0 + \rho_0'\theta + \zeta)\partial_t u + \nabla_{\mathcal{A}} q - \mu\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}u + g\zeta e_3 = F^2 & \text{in } \Omega,\\ \mathrm{div}_{\mathcal{A}}u = 0 & \text{in } \Omega,\\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Gamma,\\ (qId - \mu\mathbb{S}_{\mathcal{A}}u)\mathcal{N} = g\rho_+\eta\mathcal{N}, & \text{on } \Gamma.\end{cases} \tag{4.28}$$

Here,

$$\begin{aligned}F^1 &= K\partial_t\theta(\rho_0''\theta + \partial_3\zeta) - \mathrm{div}_{\mathcal{A}}((\rho_0'\theta + \zeta)u),\\ F^2 &= -(\rho_0 + \rho_0'\theta + \zeta)(-K\partial_t\theta\partial_3 u + u\cdot\nabla_{\mathcal{A}}u) - g\rho_0'(AK\theta, BK\theta, (1-K)\theta)^T.\end{aligned} \tag{4.29}$$

Applying the temporal differential operator ∂_t^l ($l \geqslant 1$) to (4.28), the resulting equations are

$$\begin{cases} \partial_t(\partial_t^l\zeta) + \mathrm{div}_{\mathcal{A}}(\rho_0\partial_t^l u) = F^{1,l} & \text{in } \Omega, \\ (\rho_0 + \rho_0'\theta + \zeta)\partial_t(\partial_t^l u) + \nabla_{\mathcal{A}}\partial_t^l q - \mu\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u + g\partial_t^l\zeta e_3 = F^{2,l} & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}}\partial_t^l u = F^{3,l} & \text{in } \Omega, \\ \partial_t(\partial_t^l\eta) = \partial_t^l u\cdot\mathcal{N} + F^{4,l} & \text{on } \Gamma, \\ (\partial_t^l q Id - \mu\mathbb{S}_{\mathcal{A}}\partial_t^l u)\mathcal{N} = g\rho_+\partial_t^l\eta\mathcal{N} + F^{5,l} & \text{on } \Gamma. \end{cases} \tag{4.30}$$

The terms $F^{j,l}$ with $l \geqslant 1$ and $1 \leqslant j \leqslant 5$ are given in Appendix B (see (B.7), (B.8), (B.9) and (B.10)). We use the convention that $F^{1,0} = F^1, F^{2,0} = F^2$ and $F^{j,0} = 0$ for $3 \leqslant j \leqslant 5$. We now derive the following proposition.

Proposition 4.5. *For $l = 0$ and 1, we have*

$$\begin{aligned} &\|\partial_t^l u(t)\|_{L^2(\Omega)}^2 + \|\partial_t^l\eta(t)\|_{L^2(\Gamma)}^2 + \int_0^t \|\nabla\partial_t^l u(s)\|_{L^2(\Omega)}^2 ds \\ &\qquad \leqslant C_5\Big(\mathcal{E}_f^2(0) + \int_0^t \|(u,\zeta)(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \mathcal{E}_f^3(s)ds\Big). \end{aligned} \tag{4.31}$$

We also have

$$\begin{aligned} &\|\partial_t^2 u(t)\|_{L^2(\Omega)}^2 + \|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2 + \int_0^t \|\nabla\partial_t^2 u(s)\|_{L^2(\Omega)}^2 ds \\ &\qquad \leqslant C_6\Big(\mathcal{E}_f^2(0) + \int_0^t \|(u,\zeta)(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds + \mathcal{E}_f^3(t)\Big). \end{aligned} \tag{4.32}$$

The proof of Proposition 4.5 relies on Lemmas 4.2, 4.3 and 4.4 below.

Lemma 4.2. *Let J be defined as in (2.7). For any scalar function $\vartheta \in \mathbf{R}$ and any vector function $\varrho \in \mathbf{R}^3$, there holds*

$$\int_\Omega (\nabla_{\mathcal{A}}\vartheta)\cdot J\varrho = \int_\Gamma \vartheta(\mathcal{N}\cdot\varrho) - \int_\Omega J\vartheta\, div_{\mathcal{A}}\varrho. \tag{4.33}$$

Proof. We have from the integration by parts that

$$\int_\Omega (\nabla_{\mathcal{A}}\vartheta)\cdot J\varrho = \int_\Omega J\mathcal{A}_{ij}\partial_j\vartheta\varrho_i = \int_\Gamma \vartheta(J\mathcal{A}_{i3}\varrho_i) - \int_\Omega \vartheta\partial_j(J\mathcal{A}_{ij}\varrho_i) \tag{4.34}$$

Note that $J\mathcal{A}_{i3}\varrho_i = \mathcal{N}\cdot\varrho$, hence

$$\int_\Gamma \vartheta(J\mathcal{A}_{i3}\varrho_i) = \int_\Gamma \vartheta(\mathcal{N}\cdot\varrho) \tag{4.35}$$

Note also that

$$\partial_j(J\mathcal{A}_{ij}) = 0 \quad \text{for all } 1 \leqslant i \leqslant 3,$$

this implies

$$\int_\Omega \vartheta\partial_j(J\mathcal{A}_{ij}\varrho_i) = \int_\Omega \vartheta J\mathcal{A}_{ij}\partial_j\varrho_i = \int_\Omega J\vartheta\mathrm{div}_{\mathcal{A}}\varrho. \tag{4.36}$$

Substituting (4.35), (4.36) into (4.34), we obtain (4.33), i.e. Lemma 4.2. □

Lemma 4.3. *There holds for all $l \geqslant 0$,*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega (\rho_0 + \rho_0'\theta + \zeta)J|\partial_t^l u|^2 + \int_\Gamma g\rho_+|\partial_t^l\eta|^2\Big) + \frac{1}{2}\mu\int_\Omega J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2 \\ &= \frac{1}{2}\int_\Omega \partial_t((\rho_0 + \rho_0'\theta + \zeta)J)|\partial_t^l u|^2 + \int_\Omega J(F^{2,l}\cdot\partial_t^l u - g\partial_t^l\zeta\partial_t^l u_3 + F^{3,l}\partial_t^l q) \\ &\quad - \int_\Gamma (g\rho_+\partial_t^l\eta F^{4,l} + F^{5,l}\cdot\partial_t^l u). \end{aligned} \tag{4.37}$$

If $l \geqslant 1$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2+\int_\Gamma g\rho_+|\partial_t^l\eta|^2-\int_\Omega g\rho_0'|\partial_t^{l-1}u_3|^2\Big)+\frac{1}{2}\mu\int_\Omega J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\\
&=\frac{1}{2}\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2+\int_\Omega J(F^{2,l}\cdot\partial_t^l u+F^{3,l}\partial_t^l q)\\
&\quad-\int_\Gamma(g\rho_+\partial_t^l\eta F^{4,l}+F^{5,l}\cdot\partial_t^l u)+\int_\Omega g\rho_0 JF^{3,l-1}\partial_t^l u_3\\
&\quad-\int_\Omega g\rho_0'(A\partial_t^{l-1}\partial_3 u_1+B\partial_t^{l-1}\partial_3 u_2)\partial_t^l u_3-\int_\Omega gJF^{1,l-1}\partial_t^l u_3.
\end{aligned}\tag{4.38}$$

Proof. We multiply by $J\partial_t^l u$ on both sides of $(4.30)_2$ to have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2\Big)\\
&=\frac{1}{2}\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2-\int_\Omega\nabla_{\mathcal{A}}\partial_t^l q\cdot J\partial_t^l u+\int_\Omega\mu(\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u)\cdot J\partial_t^l u\\
&\quad-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3+\int_\Omega JF^{2,l}\cdot\partial_t^l u.
\end{aligned}\tag{4.39}$$

Thanks to Lemma 4.2, one deduces

$$\begin{aligned}
&-\int_\Omega\nabla_{\mathcal{A}}\partial_t^l q\cdot J\partial_t^l u+\int_\Omega\mu(\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u)\cdot J\partial_t^l u\\
&=\int_\Gamma(\mu\mathbb{S}_{\mathcal{A}}\partial_t^l u-\partial_t^l q Id)\mathcal{N}\cdot\partial_t^l u+\int_\Omega J(\mathrm{div}_{\mathcal{A}}\partial_t^l u)\partial_t^l q-\frac{1}{2}\int_\Omega\mu J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2
\end{aligned}\tag{4.40}$$

Substituting (4.40) into (4.39), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2\Big)+\frac{1}{2}\int_\Omega\mu J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\\
&=\frac{1}{2}\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2+\int_\Omega J(\mathrm{div}_{\mathcal{A}}\partial_t^l u)\partial_t^l q\\
&\quad+\int_\Gamma(\mu\mathbb{S}_{\mathcal{A}}\partial_t^l u-\partial_t^l q Id)\mathcal{N}\cdot\partial_t^l u-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3+\int_\Omega JF^{2,l}\cdot\partial_t^l u.
\end{aligned}\tag{4.41}$$

Using $(4.30)_{3,4,5}$, we obtain (4.37) from (4.41).

To prove (4.38), we use $(4.30)_1$ at order $l-1$ to get that

$$\begin{aligned}
-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3&=\int_\Omega gJ\mathrm{div}_{\mathcal{A}}(\rho_0\partial_t^{l-1}u)\partial_t^l u_3-\int_\Omega gJF^{1,l-1}\partial_t^l u_3\\
&=\int_\Omega g\rho_0'\partial_t^{l-1}u_3\partial_t^l u_3+\int_\Omega g\rho_0 JF^{3,l-1}\partial_t^l u_3\\
&\quad-\int_\Omega g\rho_0'(A\partial_t^{l-1}\partial_3 u_1+B\partial_t^{l-1}\partial_3 u_2)\partial_t^l u_3-\int_\Omega gJF^{1,l-1}\partial_t^l u_3.
\end{aligned}\tag{4.42}$$

Combining (4.42) and (4.37), we obtain (4.38). Lemma 4.3 is proven. □

Lemma 4.4. *The following inequalities hold*

$$\sum_{l=0}^{1}\Big(\|(F^{1,l},F^{2,l},F^{3,l})\|_{L^2(\Omega)}+\|(F^{4,l},F^{5,l})\|_{L^2(\Gamma)}\Big)\lesssim\mathcal{E}_f^2,\tag{4.43}$$

$$\|(F^{1,2},F^{2,2})\|_{L^2(\Omega)}+\|(F^{4,2},F^{5,2})\|_{L^2(\Gamma)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}),\tag{4.44}$$

and

$$\|(F^{3,2},JF^{3,2})\|_{L^2(\Omega)}\lesssim\mathcal{E}_f^2,\quad \|\partial_t(JF^{3,2})\|_{L^2(\Omega)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).\tag{4.45}$$

Proof. For $\Sigma=\Omega$ or Γ, all quadratic terms $\|X_1X_2\|_{L^2(\Sigma)}$ or cubic ones $\|X_1X_2X_3\|_{L^2(\Sigma)}$ appearing in $F^{j,l}$ with $1\leqslant j\leqslant 5$ will be bounded by using Sobolev embedding, Lemma A.1 and other inequalities in Appendix C. Precisely, we have

$$\|X_1X_2\|_{L^2(\Sigma)}\lesssim\|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^2(\Sigma)}\lesssim\|X_1\|_{H^2(\Sigma)}\|X_2\|_{L^2(\Sigma)}$$

and

$$\begin{aligned}\|X_1X_2X_3\|_{L^2(\Sigma)}&\lesssim\|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^\infty(\Sigma)}\|X_3\|_{L^2(\Sigma)}\\&\lesssim\|X_1\|_{H^2(\Sigma)}\|X_2\|_{H^2(\Sigma)}\|X_3\|_{L^2(\Sigma)}.\end{aligned}$$

We only show the estimates of the term $F^{2,l}(0\leqslant l\leqslant 2)$ (see (4.29) and (B.8)), the estimates of others terms are proven in the same way.

For F^2 (see (4.29)), we have

$$(\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u=(\rho_0+\rho_0'\theta+\zeta)(K-1)\partial_t\theta\partial_3 u+(\rho_0+\rho_0'\theta+\zeta)\partial_t\theta\partial_3 u.$$

Thanks to Lemma A.1 and (C.10), we obtain

$$\begin{aligned}&\|(\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u\|_{L^2(\Omega)}\\&\quad\lesssim(1+\|(\theta,\zeta)\|_{H^2(\Omega)})(1+\|K-1\|_{H^2(\Omega)})\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\&\quad\lesssim(1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^1(\Omega)}\\&\quad\lesssim\mathcal{E}_f^2.\end{aligned}\tag{4.46}$$

Note that

$$u\cdot\nabla_{\mathcal{A}}u=u\cdot\nabla_{\mathcal{A}-\mathrm{Id}}u+u\cdot\nabla u,$$

we use Lemma A.1 and (C.12) to get that

$$\begin{aligned}&\|(\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u\|_{L^2(\Omega)}\\&\quad\lesssim\|(\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}-\mathrm{Id}}u\|_{L^2(\Omega)}+\|(\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla u\|_{L^2(\Omega)}\\&\quad\lesssim(1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(1+\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)})\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\&\quad\lesssim(1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)})\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\&\quad\lesssim\mathcal{E}_f^2.\end{aligned}\tag{4.47}$$

Due to Lemma A.1 again and (C.10), (C.11), we have

$$\begin{aligned}\|(AK\theta,BK\theta,(1-K)\theta)\|_{L^2(\Omega)}&\lesssim\|(AK,BK,K-1)\|_{H^2(\Omega)}\|\theta\|_{L^2(\Omega)}\\&\lesssim\|\eta\|_{H^{5/2}(\Gamma)}\|\eta\|_{L^2(\Gamma)}\\&\lesssim\mathcal{E}_f^2.\end{aligned}\tag{4.48}$$

It follows from (4.46), (4.47) and (4.48) that $\|F^2\|_{L^2(\Omega)}\lesssim\mathcal{E}_f^2$.

For $F^{2,1}$ (see (B.8)), we obtain

$$\begin{aligned}\|F^{2,1}\|_{L^2(\Omega)}\lesssim{}&\|\partial_t F^2\|_{L^2(\Omega)}+(1+\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)})\|\partial_t\mathcal{A}\|_{H^1(\Omega)}\|\nabla u\|_{H^2(\Omega)}\\&+(1+\|\mathcal{A}-\mathrm{Id}\|_{H^3(\Omega)})\|\nabla^2 u\|_{H^2(\Omega)}\|\partial_t\mathcal{A}\|_{H^1(\Omega)}+\|\zeta\|_{H^3(\Omega)}\|\partial_t\mathcal{A}\|_{L^2(\Omega)}\\&+(\|\partial_t\zeta\|_{H^2(\Omega)}+\|\partial_t\theta\|_{H^2(\Omega)})\|\partial_t u\|_{L^2(\Omega)}.\end{aligned}\tag{4.49}$$

According to Lemma A.1 and (C.12), it follows from (4.49) that

$$\begin{aligned}\|F^{2,1}\|_{L^2(\Omega)} &\lesssim \|\partial_t F^2\|_{L^2(\Omega)} + (1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^4(\Omega)}\\ &\quad + \|\zeta\|_{H^3(\Omega)}\|\partial_t\eta\|_{H^{1/2}(\Gamma)} + (\|\partial_t\zeta\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)})\|\partial_t u\|_{L^2(\Omega)}\\ &\lesssim \|\partial_t F^2\|_{L^2(\Omega)} + \mathcal{E}_f^2.\end{aligned} \tag{4.50}$$

We calculate each term of $\partial_t F^2$,

$$\begin{aligned}\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u) &= (\rho_0+\rho_0'\theta+\zeta)(\partial_t K\partial_t\theta\partial_3 u + K\partial_t^2\theta\partial_3 u + K\partial_t\theta\partial_t\partial_3 u)\\ &\quad + (\rho_0'\partial_t\theta+\partial_t\zeta)K\partial_t\theta\partial_3 u,\end{aligned}$$

which will be bounded as follows

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)}\\ &\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\ &\quad + (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|K-1\|_{H^2(\Omega)}+1)\\ &\qquad \times (\|\partial_3 u\|_{H^2(\Omega)}\|\partial_t^2\theta\|_{L^2(\Omega)} + \|\partial_t\theta\|_{H^2(\Omega)}\|(u,\partial_t u)\|_{H^1(\Omega)})\\ &\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|K-1\|_{H^2(\Omega)}+1)\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}.\end{aligned}$$

Using Lemma A.1 and (C.10), we deduce that

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)}\\ &\lesssim (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})\|\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^1(\Omega)}\\ &\quad + (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)})\\ &\qquad \times (\|u\|_{H^3(\Omega)}\|\partial_t^2\eta\|_{L^2(\Gamma)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|(u,\partial_t u)\|_{H^1(\Omega)})\\ &\quad + (\|\partial_t\zeta\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)})\|u\|_{H^1(\Omega)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{4.51}$$

Next, we compute

$$\begin{aligned}&\partial_t((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\\ &= (\rho_0+\rho_0'\theta+\zeta)(\partial_t u_i\mathcal{A}_{ij}\partial_j u_k + u_i\partial_t\mathcal{A}_{ij}\partial_j u_k + u_i\mathcal{A}_{ij}\partial_t\partial_j u_k)\\ &\quad + (\rho_0'\partial_t\theta+\partial_t\zeta)u_i\mathcal{A}_{ij}\partial_j u_k.\end{aligned}$$

Hence, it follows from Lemma A.1 and (C.12) that

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\ &\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|_{H^3(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\\ &\quad + (1+\|(\theta,\zeta)\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}^2\|\partial_t\mathcal{A}\|_{L^2(\Omega)}\\ &\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\ &\lesssim (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|_{H^3(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\\ &\quad + (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})\|u\|_{H^3(\Omega)}^2\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\\ &\quad + (\|\partial_t\zeta\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{4.52}$$

Using again Lemma A.1 and (C.10), (C.11), one has

$$\begin{aligned}&\|\partial_t(AK\theta, BK\theta, (1-K)\theta)\|_{L^2(\Omega)}\\ &\lesssim \|(AK,BK,K-1)\|_{H^2(\Omega)}\|\partial_t\theta\|_{L^2(\Omega)} + \|\partial_t(AK,BK,K-1)\|_{L^2(\Omega)}\|\theta\|_{H^2(\Omega)}\\ &\lesssim \|\eta\|_{H^{5/2}(\Gamma)}\|\partial_t\eta\|_{L^2(\Gamma)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\eta\|_{H^{3/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{4.53}$$

We deduce $\|\partial_t F^2\|_{L^2(\Omega)} \lesssim \mathcal{E}_f^2$ from (4.50), (4.51), (4.52) and (4.53). So that, $\|F^{2,1}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f^2$.

For $F^{2,2}$ (see again (4.29)), we use the product estimate (C.1) and Sobolev embedding to obtain that

$$\begin{aligned}\|F^{2,2}\|_{L^2(\Omega)} &\lesssim \|\partial_t^2 F^2\|_{L^2(\Omega)} + \|\mathcal{A}\|_{H^2(\Omega)}(\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t u\|_{H^2(\Omega)} + \|\partial_t^2\mathcal{A}\|_{H^1(\Omega)}\|\nabla u\|_{H^3(\Omega)})\\ &\quad + \|\partial_t\mathcal{A}\|_{H^2(\Omega)}(\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla u\|_{H^2(\Omega)} + \|\mathcal{A}\|_{H^3(\Omega)}\|\nabla\partial_t u\|_{H^1(\Omega)})\\ &\quad + \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\mathcal{A}\|_{H^3(\Omega)}\|\nabla u\|_{H^3(\Omega)} + \|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\partial_t\zeta\|_{H^1(\Omega)}\\ &\quad + \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla\zeta\|_{H^2(\Omega)} + \|(\partial_t\zeta,\partial_t\theta)\|_{H^2(\Omega)}\|\partial_t^2 u\|_{L^2(\Omega)}\\ &\quad + \|(\partial_t^2\zeta,\partial_t^2\theta)\|_{L^2(\Omega)}\|\partial_t u\|_{H^2(\Omega)}.\end{aligned}$$

We make use of Lemma A.1 and (C.12) to further get that

$$\begin{aligned}\|F^{2,2}\|_{L^2(\Omega)} &\lesssim \|\partial_t^2 F^2\|_{L^2(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\nabla\partial_t u\|_{H^2(\Omega)}\\ &\quad + \|\eta\|_{H^{3/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}\|u\|_{H^4(\Omega)} + \|\partial_t\eta\|^2_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)}\\ &\quad + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t u\|_{H^2(\Omega)} + \|\partial_t^2\eta\|_{L^2(\Gamma)}\|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}\\ &\quad + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\zeta\|_{H^1(\Omega)} + \|\partial_t^2\eta\|_{L^2(\Gamma)}\|\zeta\|_{H^3(\Omega)}\\ &\quad + (\|\partial_t\zeta\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)})\|\partial_t^2 u\|_{L^2(\Omega)}\\ &\quad + (\|\partial_t^2\zeta\|_{L^2(\Omega)} + \|\partial_t^2\eta\|_{L^2(\Gamma)})\|\partial_t u\|_{H^2(\Omega)}\\ &\lesssim \|\partial_t^2 F^2\|_{L^2(\Omega)} + \mathcal{E}_f(\mathcal{E}_f + \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\nabla\partial_t u\|_{H^2(\Omega)}).\end{aligned}$$

Together with (4.23), we deduce from the resulting inequality that

$$\|F^{2,2}\|_{L^2(\Omega)} \lesssim \|\partial_t^2 F^2\|_{L^2(\Omega)} + \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)}).$$

Hence, in order to show that

$$\|F^{2,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}),$$

we will prove that

$$\|\partial_t^2 F^2\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}).$$

We now estimate each term of $\partial_t^2 F^2$. Due to Sobolev embedding, one has that

$$\begin{aligned}&\|\partial_t^2((\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)}\\ &\lesssim (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})(\|K-1\|_{H^2(\Omega)} + 1)\\ &\qquad \times \begin{pmatrix}\|\partial_3 u\|_{H^2(\Omega)}\|\partial_t^3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3 u\|_{L^2(\Omega)}\|\partial_t^2\theta\|_{H^2(\Omega)}\\ + \|\partial_t^2\partial_3 u\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\end{pmatrix}\\ &\quad + (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t K\|_{H^2(\Omega)}\\ &\qquad \times \Big(\|\partial_t\theta\|_{H^2(\Omega)}\|\partial_t u\|_{H^1(\Omega)} + \|\partial_t^2\theta\|_{L^2(\Omega)}\|u\|_{H^3(\Omega)}\Big)\\ &\quad + (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t^2 K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)}\\ &\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|K-1\|_{H^2(\Omega)} + 1)\\ &\qquad \times \Big(\|\partial_t^2\theta\|_{L^2(\Omega)}\|u\|_{H^3(\Omega)} + \|\partial_t\theta\|_{H^2(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\Big)\\ &\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}\|\partial_t K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)}\\ &\quad + \|(\partial_t^2\theta,\partial_t^2\zeta)\|_{L^2(\Omega)}(\|K-1\|_{H^2(\Omega)} + 1)\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)}.\end{aligned}$$

Thanks to Lemma A.1 and (C.10), this yields

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\\
&\qquad\times\begin{pmatrix}\|u\|_{H^3(\Omega)}\|\partial_t^3\eta\|_{L^2(\Gamma)}+\|\partial_t u\|_{H^1(\Omega)}\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)}\\ +\|\nabla\partial_t^2 u\|_{L^2(\Omega)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\end{pmatrix}\\
&\quad+(1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\\
&\qquad\times\Big(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}+\|\partial_t^2\eta\|_{L^2(\Gamma)}\|u\|_{H^3(\Omega)}\Big)\\
&\quad+(1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})(\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|^2_{H^{1/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\\
&\qquad\times\Big(\|\partial_t^2\eta\|_{L^2(\Gamma)}\|u\|_{H^3(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}\Big)\\
&\quad+(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t^2\eta\|_{L^2(\Gamma)}+\|\partial_t^2\zeta\|_{L^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)}.
\end{aligned}\tag{4.54}$$

Using (4.23) and (4.24), we thus have from (4.54) that

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)}\\
&\quad\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}+\|\partial_t^3\eta\|_{L^2(\Gamma)}+\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)})\\
&\quad\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).
\end{aligned}\tag{4.55}$$

In a same way, we have

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)(\|\partial_t^2 u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}+\|\partial_t u\|^2_{H^2(\Omega)})\\
&\quad+(1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\partial_t\mathcal{A}\|_{L^2(\Omega)}\|\partial_t u\|_{H^3(\Omega)}+\|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|u\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}\\
&\quad+\|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|\partial_t u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}\\
&\quad+\|(\partial_t^2\theta,\partial_t^2\zeta)\|_{L^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|^2_{H^3(\Omega)}.
\end{aligned}$$

Thanks to Lemma A.1 and (C.12), we further get that

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)(\|\partial_t^2 u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}+\|\partial_t u\|^2_{H^2(\Omega)})\\
&\quad+(1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})\\
&\qquad\times(\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t u\|_{H^3(\Omega)}+(\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|^2_{H^{5/2}(\Gamma)})\|u\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|\partial_t u\|_{H^3(\Omega)}\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t^2\eta\|_{L^2(\Gamma)}+\|\partial_t^2\zeta\|_{L^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|^2_{H^3(\Omega)}\\
&\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).
\end{aligned}$$

Due to (4.23), we obtain

$$\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).\tag{4.56}$$

Furthermore, thanks to (4.23) again and Lemma A.1, (C.10), (C.11), one has

$$\begin{aligned}
&\|\partial_t^2(AK\theta, BK\theta, (1-K)\theta)\|_{L^2(\Omega)}\\
&\quad\lesssim \|\partial_t^2(AK, BK, K-1)\|_{L^2(\Omega)}\|\theta\|_{H^2(\Omega)} + \|\partial_t(AK, BK, K-1)\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\\
&\qquad + \|(AK, BK, K-1)\|_{H^2(\Omega)}\|\partial_t^2\theta\|_{L^2(\Omega)}\\
&\quad\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|\eta\|_{H^{3/2}(\Gamma)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\\
&\qquad + \|\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{L^2(\Gamma)}\\
&\quad\lesssim \mathcal{E}_f(\mathcal{E}_f + \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\\
&\quad\lesssim \mathcal{E}_f^2.
\end{aligned} \tag{4.57}$$

Consequently, there holds

$$\|\partial_t^2 F^2\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t^2 u\|_{L^2(\Omega)})$$

thanks to (4.55), (4.56) and (4.57).

We are left to prove (4.45). From the formula of $F^{3,2}$ (see (B.9)), we use Sobolev embedding and (C.12) to get

$$\begin{aligned}
\|F^{3,2}\|_{L^2(\Omega)} &\lesssim \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla u\|_{H^2(\Omega)} + \|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t u\|_{L^2(\Omega)}\\
&\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|u\|_{H^3(\Omega)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}.
\end{aligned}$$

Owing to (4.22) and (4.23), we deduce that

$$\begin{aligned}
\|F^{3,2}\|_{L^2(\Omega)} &\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\mathcal{E}_f + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\mathcal{E}_f\\
&\lesssim \mathcal{E}_f^2.
\end{aligned} \tag{4.58}$$

Together with (C.8), this yields

$$\|JF^{3,2}\|_{L^2(\Omega)} \lesssim (1 + \|J-1\|_{L^\infty(\Omega)})\|F^{3,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f^2. \tag{4.59}$$

We continue using (C.8), Lemma A.1 and (4.58) to get

$$\begin{aligned}
\|\partial_t(JF^{3,2})\|_{L^2(\Omega)} &\lesssim \|\partial_t J\|_{L^\infty(\Omega)}\|F^{3,2}\|_{L^2(\Omega)} + (1 + \|J-1\|_{L^\infty(\Omega)})\|\partial_t F^{3,2}\|_{L^2(\Omega)}\\
&\lesssim \|\partial_t\partial_3\theta\|_{H^2(\Omega)}\mathcal{E}_f^2 + \|\partial_t F^{3,2}\|_{L^2(\Omega)}\\
&\lesssim \|\partial_t\eta\|_{H^{5/2}(\Omega)}\mathcal{E}_f^2 + \|\partial_t F^{3,2}\|_{L^2(\Omega)}.
\end{aligned} \tag{4.60}$$

By Sobolev embedding and (C.12), let us estimate that

$$\begin{aligned}
\|\partial_t F^{3,2}\|_{L^2(\Omega)} &\lesssim \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla\partial_t u\|_{H^2(\Omega)} + \|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t^2 u\|_{L^2(\Omega)}\\
&\quad + \|\partial_t^3\mathcal{A}\|_{L^2(\Omega)}\|\nabla u\|_{H^2(\Omega)}\\
&\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|\nabla\partial_t u\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\nabla\partial_t^2 u\|_{L^2(\Omega)}\\
&\quad + (\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}^3)\|u\|_{H^3(\Omega)}.
\end{aligned} \tag{4.61}$$

Combining the resulting inequality (4.59) with all inequalities in Lemma 4.1, we obtain

$$\|\partial_t F^{3,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}). \tag{4.62}$$

Combining (4.22), (4.60) and (4.62) gives us that

$$\|\partial_t(JF^{3,2})\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}). \tag{4.63}$$

The inequality (4.45) follows from (4.58), (4.59) and (4.63). □

We are in position to prove Proposition 4.5.

Proof of Proposition 4.5. In view of (4.37) at order $l = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|u(t)|^2+\int_\Gamma g\rho_+|\eta(t)|^2\Big)+\frac{1}{2}\mu\int_0^t\int_\Omega J|\mathbb{S}_{\mathcal{A}}u(s)|^2ds\\
&=\frac{1}{2}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|u(t)|^2+\int_\Gamma g\rho_+|\eta(t)|^2\Big)\Big|_{t=0}\\
&+\frac{1}{2}\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|u|^2(s)ds+\int_0^t\int_\Omega J(F^2\cdot u-g\zeta u_3)ds
\end{aligned}\tag{4.64}$$

We first estimate the l.h.s of (4.31). Notice that

$$J\|\mathbb{S}_{\mathcal{A}}u\|^2_{L^2(\Omega)}=\|\mathbb{S}u\|^2_{L^2(\Omega)}+\int_\Omega(J-1)|\mathbb{S}u|^2+\int_\Omega J(\mathbb{S}_{\mathcal{A}}u+\mathbb{S}u):(\mathbb{S}_{\mathcal{A}}u-\mathbb{S}u).$$

Since

$$\mathbb{S}_{\mathcal{A}}u\pm\mathbb{S}u=(\mathcal{A}_{ik}\pm\delta_{ik})\partial_k u_j+(\mathcal{A}_{jk}\pm\delta_{jk})\partial_k u_j,$$

we use (C.9) to obtain

$$\begin{aligned}
\int_\Omega J(\mathbb{S}_{\mathcal{A}}u+\mathbb{S}u):(\mathbb{S}_{\mathcal{A}}u-\mathbb{S}u)&=4\int_\Omega(A^2(\partial_1u_2+\partial_2u_1)^2+B^2(\partial_1u_3+\partial_3u_1)^2)\\
&\lesssim\|(A,B)\|^2_{H^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\
&\lesssim\mathcal{E}_f^4.
\end{aligned}$$

Note also that $\|J-1\|_{L^\infty(\Omega)}\lesssim 1$ (see (C.8)), we use Korn's inequality (C.5) to have

$$J\|\mathbb{S}_{\mathcal{A}}u\|^2_{L^2(\Omega)}\gtrsim\|\nabla u\|^2_{L^2(\Omega)}-\mathcal{E}_f^3.\tag{4.65}$$

Due to the assumption on δ_0 (4.5) and Sobolev embedding, we then have

$$\inf_\Omega(\rho_0+\rho_0'\theta+\zeta)\geqslant\rho_--C_{\rm emb}\max(1,\max_{\mathbf{R}_-}\rho_0'(x_3))\|(\theta,\zeta)\|_{H^2(\Omega)}\geqslant\frac{1}{2}\rho_-.\tag{4.66}$$

The l.h.s of (4.64) will be estimated as

$$\begin{aligned}
&\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|u(t)|^2+\int_\Gamma g\rho_+|\eta(t)|^2+\mu\int_0^t\int_\Omega J|\mathbb{S}_{\mathcal{A}}u(s)|^2ds\\
&\gtrsim\|u(t)\|^2_{L^2(\Omega)}+\|\eta(t)\|^2_{L^2(\Gamma)}+\int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}ds-\int_0^t\mathcal{E}_f^3(s)ds.
\end{aligned}\tag{4.67}$$

We now estimate the r.h.s of (4.64). By Gagliardo-Nirenberg's inequality (see (C.2)) and Sobolev embedding, one has

$$\begin{aligned}
&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)J)\|_{L^\infty(\Omega)}\\
&\quad\lesssim\|(\rho_0+\rho_0'\theta+\zeta)\partial_tJ\|_{H^2(\Omega)}+\|(\rho_0'\partial_t\theta+\partial_t\zeta)J\|_{L^\infty(\Omega)}\\
&\quad\lesssim(1+\|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t\theta\|_{H^3(\Omega)}+\|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(1+\|J-1\|_{L^\infty(\Omega)}).
\end{aligned}$$

Together with Lemma A.1, (4.22) and (C.8), we observe

$$\begin{aligned}
&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)J)\|_{L^\infty(\Omega)}\\
&\lesssim(1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\zeta\|_{H^2(\Omega)}\\
&\lesssim\mathcal{E}_f,
\end{aligned}\tag{4.68}$$

which yields

$$\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|u|^2(s)ds\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\tag{4.69}$$

Furthermore, thanks to (C.8) and (4.43), we get

$$\begin{aligned}\int_\Omega J(F^2\cdot u-g\zeta u_3)&\lesssim(\|J-1\|_{L^\infty(\Omega)}+1)(\|F^2\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+\|\zeta\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)})\\&\lesssim\mathcal{E}_f^3+\|\zeta\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)}.\end{aligned}\tag{4.70}$$

Substituting (4.65), (4.67), (4.69) and (4.70) into (4.64), we deduce $(4.31)_{l=0}$.

For $l=1$, we make use of (4.38) at order $l=1$ to have that

$$\begin{aligned}&\frac12\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2(t)+\int_\Gamma g\rho_+|\partial_t\eta(t)|^2-\int_\Omega g\rho_0'|u_3(t)|^2\Big)\\&\qquad+\frac{\mu}{2}\int_0^t\|\mathbb{S}_{\mathcal{A}}\partial_t u(s)\|_{L^2(\Omega)}^2ds\\&=\frac12\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2+\int_\Gamma g\rho_+|\partial_t\eta|^2-\int_\Omega g\rho_0'|u_3|^2\Big)\Big|_{t=0}\\&\quad+\frac12\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t u|(s)^2ds+\int_0^t\int_\Omega J(F^{2,1}\cdot\partial_t u+F^{3,1}\partial_t q)(s)ds\\&\quad-\int_0^t\int_\Gamma(g\rho_+\partial_t\eta F^{4,1}+F^{5,1}\cdot\partial_t u)(s)ds-\int_0^t\int_\Omega g\rho_0'(A\partial_3u_1+B\partial_3u_2)\partial_t u_3(s)ds\\&\quad-\int_0^t\int_\Omega gJF^1\partial_t u_3(s)ds.\end{aligned}\tag{4.71}$$

By a similar argument as the proof of (4.67), we estimate the l.h.s of (4.71) as

$$\begin{aligned}&\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2(t)+\int_\Gamma g\rho_+|\partial_t\eta(t)|^2-\int_\Omega g\rho_0'|u_3(t)|^2\\&\qquad+\mu\int_0^t\|\mathbb{S}_{\mathcal{A}}\partial_t u(s)\|_{L^2(\Omega)}^2ds\\&\gtrsim\|\partial_t u(t)\|_{L^2(\Omega)}^2+\|\partial_t\eta(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla\partial_t u(s)\|_{L^2(\Omega)}^2-\|u_3(t)\|_{L^2(\Omega)}^2-\int_0^t\mathcal{E}_f^3(s)ds.\end{aligned}\tag{4.72}$$

For the r.h.s of (4.71), we use (4.68) to obtain that

$$\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t u|^2(s)ds\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\tag{4.73}$$

Next, thanks to (C.9), we see that

$$\int_0^t\int_\Omega g\rho_0'(A\partial_3u_1+B\partial_3u_2)\partial_t u_3(s)ds\lesssim\int_0^t\mathcal{E}_f^2\|(A,B)\|_{H^2(\Omega)}(s)ds\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\tag{4.74}$$

Let us use (C.8) and (4.43) to estimate that

$$\begin{aligned}&\int_0^t\int_\Omega J(F^{2,1}\cdot\partial_t u+F^{3,1}\partial_t q-gF^1\partial_3u)(s)ds-\int_0^t\int_\Gamma(g\rho_+\partial_t\eta F^{4,1}+F^{5,1}\cdot\partial_t u)(s)ds\\&\lesssim\int_0^t((\|J-1\|_{L^\infty(\Omega)}+1)\|(F^1,F^{2,1},F^{3,1})(s)\|_{L^2(\Omega)}+\|(F^{4,1},F^{5,1})(s)\|_{L^2(\Gamma)})\mathcal{E}_f(s)ds\\&\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\end{aligned}\tag{4.75}$$

Combining (4.72), (4.73) and (4.75), we obtain

$$\begin{aligned}&\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t \eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla \partial_t u(s)\|^2_{L^2(\Omega)} ds\\ &\lesssim \mathcal{E}_f^2(0) + \|u_3(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{E}_f^3(s) ds.\end{aligned} \tag{4.76}$$

As a consequence of (4.76) and $(4.31)_{l=0}$, the inequality $(4.31)_{l=1}$ follows.

For $l = 2$, we use (4.38) at order $l = 2$ to have that

$$\begin{aligned}&\frac{1}{2}\Big(\int_\Omega (\rho_0 + \rho_0'\theta + \zeta) J |\partial_t^2 u|^2(t) + \int_\Gamma g\rho_+ |\partial_t^2 \eta(t)|^2 - \int_\Omega g\rho_0' |\partial_t u_3(t)|^2 \Big)\\ &\qquad + \frac{1}{2}\mu \int_0^t \int_\Omega J |\mathbb{S}_{\mathcal{A}} \partial_t^2 u(s)|^2 ds\\ &= \frac{1}{2}\Big(\int_\Omega (\rho_0 + \rho_0'\theta + \zeta) J |\partial_t^2 u|^2 + \int_\Gamma g\rho_+ |\partial_t^2 \eta|^2 - \int_\Omega g\rho_0' |\partial_t u_3|^2 \Big)\Big|_{t=0}\\ &\quad + \frac{1}{2}\int_0^t \int_\Omega \partial_t((\rho_0 + \rho_0'\theta + \zeta)J) |\partial_t^2 u(s)|^2 ds + \int_0^t \int_\Omega J(F^{2,2} \cdot \partial_t^2 u + F^{3,2} \partial_t^2 q)(s) ds\\ &\quad - \int_0^t \int_\Gamma (g\rho_+ \partial_t^2 \eta F^{4,2} + F^{5,2} \cdot \partial_t^2 u)(s) ds + \int_0^t \int_\Omega g\rho_0 J F^{3,1} \partial_t^2 u_3(s) ds\\ &\quad - \int_0^t \int_\Omega g\rho_0' (A\partial_t \partial_3 u_1 + B\partial_t \partial_3 u_2) \partial_t^2 u_3(s) ds - \int_0^t \int_\Omega g J F^{1,1} \partial_t^2 u_3(s) ds.\end{aligned} \tag{4.77}$$

We follow the previous arguments to observe that

$$\begin{aligned}&\|\partial_t^2 u(t)\|^2_{L^2(\Omega)} + \|\partial_t^2 \eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla \partial_t^2 u(s)\|^2_{L^2(\Omega)} ds\\ &\lesssim \mathcal{E}_f^2(0) + \|\partial_t u_3(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{E}_f^2(s) \|(A, B)(s)\|_{H^2(\Omega)} ds\\ &\qquad + \int_0^t (\|J - 1\|_{L^\infty(\Omega)} + 1) \|(F^{1,1}, F^{2,2}, F^{3,1})(s)\|_{L^2(\Omega)} ds\\ &\qquad + \int_0^t \|(F^{4,2}, F^{5,2})(s)\|_{L^2(\Gamma)} \mathcal{E}_f(s) ds + \int_0^t \int_\Omega (J F^{3,2} \partial_t^2 q)(s) ds.\end{aligned} \tag{4.78}$$

Since $\partial_t^2 q$ does not appear in $\mathcal{E}_f$ or $\mathcal{D}_f$, we use the integration in time to have

$$\begin{aligned}\int_0^t \int_\Omega (J F^{3,2} \partial_t^2 q)(s) ds = &\int_\Omega (\partial_t q J F^{3,2})(t) - \int_\Omega (\partial_t q J F^{3,2})(0)\\ &- \int_0^t \int_\Omega \partial_t q(s) \partial_t (J F^{3,2})(s) ds.\end{aligned}$$

Thanks to (4.45), we observe

$$\int_0^t \int_\Omega (J F^{3,2} \partial_t^2 q)(s) ds \lesssim \mathcal{E}_f^2(0) + \mathcal{E}_f^3(t) + \int_0^t \mathcal{E}_f^2 (\mathcal{E}_f + \|\nabla \partial_t u\|_{H^2(\Omega)} + \|\nabla \partial_t^2 u\|_{L^2(\Omega)})(s) ds. \tag{4.79}$$

It follows from (C.8), (4.43) and (4.44) that

$$\begin{aligned}&\int_0^t ((\|J - 1\|_{L^\infty(\Omega)} + 1) \|(F^{2,2}, F^{3,1}, F^{1,1})(s)\|_{L^2(\Omega)} + \|(F^{4,2}, F^{5,2})(s)\|_{L^2(\Gamma)}) \mathcal{E}_f(s) ds\\ &\lesssim \int_0^t \mathcal{E}_f^2 (\mathcal{E}_f + \|\nabla \partial_t u\|_{H^2(\Omega)} + \|\nabla \partial_t^2 u\|_{L^2(\Omega)})(s) ds.\end{aligned} \tag{4.80}$$

Using (4.78), (4.79), (4.80) and (C.9), we deduce that

$$\begin{aligned}&\|\partial_t^2 u(t)\|_{L^2(\Omega)}^2+\|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla\partial_t^2 u(s)\|_{L^2(\Omega)}^2ds\\&\lesssim \mathcal{E}_f^2(0)+\|\partial_t u_3(t)\|_{L^2(\Omega)}^2+\mathcal{E}_f^3(t)+\int_0^t\mathcal{E}_f^2(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)})(s)ds.\end{aligned}$$

We obtain (4.32) thanks to the resulting inequality and $(4.31)_{l=1}$. □

4.3. **Horizontal estimates of the perturbation velocity.** We continue deriving the mixed horizontal space-time derivatives of u. Note that ρ_0 only depends on x_3. Let $\beta=(\beta_1,\beta_2)\in\mathbb{N}^2$ and let us apply the horizontal derivative $\partial_h^\beta=\partial_1^{\beta_1}\partial_2^{\beta_2}$ to (2.11), we obtain the following equations.

$$\begin{cases}\partial_t\partial_h^\beta\zeta+\rho_0'\partial_h^\beta u_3=\partial_h^\beta\mathcal{Q}^1 & \text{in }\Omega,\\ \rho_0\partial_t\partial_h^\beta u+\nabla\partial_h^\beta q-\mu\Delta\partial_h^\beta u+g\partial_h^\beta\zeta e_3=\partial_h^\beta\mathcal{Q}^2 & \text{in }\Omega,\\ \mathrm{div}\partial_h^\beta u=\partial_h^\beta\mathcal{Q}^3 & \text{in }\Omega,\\ \partial_t\partial_h^\beta\eta-\partial_h^\beta u_3=\partial_h^\beta\mathcal{Q}^4 & \text{on }\Gamma,\\ (\partial_h^\beta q\mathrm{Id}-\mu\mathbb{S}\partial_h^\beta u)e_3=g\rho_+\partial_h^\beta\eta e_3+\partial_h^\beta\mathcal{Q}^5 & \text{on }\Gamma.\end{cases}\tag{4.81}$$

Proposition 4.6. *The following inequalities hold*

$$\begin{aligned}&\sum_{\beta\in\mathbb{N}^2,1\leqslant|\beta|\leqslant4}\Big(\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla\partial_h^\beta u(s)\|_{L^2(\Omega)}^2ds\Big)\\&\leqslant C_7\Big(\mathcal{E}_f^2(0)+\varepsilon^3\int_0^t(\mathcal{E}_f^2(s)+\|\nabla u_3(s)\|_{H^4(\Omega)}^2)ds+\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\&\quad+C_7\varepsilon^{-27}\int_0^t(\|u_3(s)\|_{L^2(\Omega)}^2+\|\eta(s)\|_{L^2(\Omega)}^2)ds,\end{aligned}\tag{4.82}$$

and

$$\begin{aligned}&\sum_{\beta\in\mathbb{N}^2,1\leqslant|\beta|\leqslant2}\Big(\|\partial_h^\beta\partial_t u(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla\partial_h^\beta\partial_t u(s)\|_{L^2(\Omega)}^2ds\Big)\\&\leqslant C_8\Big(\mathcal{E}_f^2(0)+\varepsilon^3\int_0^t(\mathcal{E}_f^2(s)+\|\nabla u_3(s)\|_{H^4(\Omega)}^2)ds+\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\&\quad+C_8\varepsilon^{-27}\int_0^t(\|u_3(s)\|_{L^2(\Omega)}^2+\|\eta(s)\|_{L^2(\Omega)}^2)ds.\end{aligned}\tag{4.83}$$

To prove Proposition 4.6, we need the following lemma.

Lemma 4.5. *The following inequalities hold*

$$\begin{aligned}&\|\mathcal{Q}^1\|_{H^2(\Omega)}+\|\partial_t\mathcal{Q}^1\|_{L^2(\Omega)}+\|\partial_t\mathcal{Q}^2\|_{L^2(\Omega)}+\|\mathcal{Q}^2\|_{H^2(\Omega)}+\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}\\&\qquad+\|\mathcal{Q}^3\|_{H^3(\Omega)}+\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}+\|\mathcal{Q}^5\|_{H^{5/2}(\Gamma)}+\|\partial_t\mathcal{Q}^5\|_{H^{1/2}(\Gamma)}\lesssim\mathcal{E}_f^2,\end{aligned}\tag{4.84}$$

and

$$\begin{aligned}&\|\mathcal{Q}^2\|_{H^3(\Omega)}+\|\partial_t\mathcal{Q}^2\|_{H^1(\Omega)}+\|\mathcal{Q}^3\|_{H^4(\Omega)}+\|\partial_t\mathcal{Q}^3\|_{H^2(\Omega)}\\&\qquad+\|\mathcal{Q}^5\|_{H^{7/2}(\Gamma)}+\|\partial_t\mathcal{Q}^5\|_{H^{3/2}(\Gamma)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).\end{aligned}\tag{4.85}$$

Proof. For (4.84), we only present estimates for some terms of the l.h.s, precisely,

$$\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}+\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}+\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)}\lesssim\mathcal{E}_f^2,$$

the estimates of the other terms in the l.h.s of (4.84) follow the same way. To get $\|\partial_t \mathcal{Q}^3\|_{H^1(\Omega)} \lesssim \mathcal{E}_f^2$, we use (C.1) and (C.10), (C.11) to bound each term of $\mathcal{Q}^3$ (B.5). Indeed, we have

$$\begin{aligned}\|\partial_t((1-K)\partial_3 u_3)\|_{H^1(\Omega)} &\lesssim \|\partial_t K\|_{H^1(\Omega)}\|\partial_3 u_3\|_{H^3(\Omega)} + \|K-1\|_{H^3(\Omega)}\|\partial_t\partial_3 u_3\|_{H^1(\Omega)}\\ &\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u_3\|_{H^4(\Omega)} + \|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t u_3\|_{H^2(\Omega)}\\ &\lesssim \mathcal{E}_f^2,\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}&\|\partial_t(AK\partial_3 u_1 + BK\partial_3 u_2)\|_{H^1(\Omega)}\\ &\lesssim \|\partial_t(AK, BK)\|_{H^1(\Omega)}\|\partial_3 u\|_{H^3(\Omega)} + \|(AK, BK)\|_{H^3(\Omega)}\|\partial_t\partial_3 u\|_{H^1(\Omega)}\\ &\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^4(\Omega)} + \|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t u\|_{H^2(\Omega)}\\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{4.87}$$

Hence, $\|\partial_t \mathcal{Q}^3\|_{H^1(\Omega)} \lesssim \mathcal{E}_f^2$ follows from (4.86) and (4.87). We apply the product estimate (C.1) and the trace theorem to have that

$$\begin{aligned}\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)} &\lesssim \|u_1\|_{H^{7/2}(\Gamma)}\|\partial_1\eta\|_{H^{7/2}(\Gamma)} + \|u_2\|_{H^{7/2}(\Gamma)}\|\partial_2\eta\|_{H^{7/2}(\Gamma)}\\ &\lesssim \|u\|_{H^4(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f^2.\end{aligned}$$

Moreover, using (C.1), (C.10), (C.11) again and the trace theorem, we show

$$\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)} \lesssim \mathcal{E}_f^2.$$

From the expression of $\mathcal{Q}_1^5$ (B.6), we have that

$$\begin{aligned}&\|\partial_1\eta(q - g\rho_+\eta - 2\mu(\partial_1 u_1 - AK\partial_3 u_1))\|_{H^{5/2}(\Gamma)}\\ &\quad\lesssim \|\partial_1\eta\|_{H^{5/2}(\Gamma)}(\|(q, \eta, \partial_1 u_1)\|_{H^{5/2}(\Gamma)} + \|AK\|_{H^{5/2}(\Gamma)}\|\partial_3 u_1\|_{H^{5/2}(\Gamma)})\\ &\quad\lesssim \|\eta\|_{H^{7/2}(\Gamma)}(\|q\|_{H^3(\Omega)} + \|u_1\|_{H^4(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)} + \|AK\|_{H^3(\Omega)}\|u_1\|_{H^4(\Omega)})\\ &\quad\lesssim \|\eta\|_{H^{7/2}(\Gamma)}(\|q\|_{H^3(\Omega)} + \|u_1\|_{H^4(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)} + \|\eta\|_{H^{7/2}(\Gamma)}\|u_1\|_{H^4(\Omega)}),\end{aligned} \tag{4.88}$$

that

$$\begin{aligned}&\|\partial_2\eta(\partial_1 u_2 + \partial_2 u_1 - AK\partial_3 u_2 - BK\partial_3 u_1)\|_{H^{5/2}(\Gamma)}\\ &\quad\lesssim \|\partial_2\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)}(1 + \|(AK, BK)\|_{H^{5/2}(\Gamma)})\\ &\quad\lesssim \|\partial_2\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)}(1 + \|(AK, BK)\|_{H^3(\Omega)})\\ &\quad\lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}(1 + \|\eta\|_{H^{7/2}(\Gamma)}),\end{aligned} \tag{4.89}$$

and that

$$\begin{aligned}\|(1-K)\partial_3 u_1 + AK\partial_3 u_3\|_{H^{5/2}(\Gamma)} &\lesssim \|(K-1, AK)\|_{H^{5/2}(\Gamma)}\|\partial_3 u\|_{H^{5/2}(\Gamma)}\\ &\lesssim \|(K-1, AK)\|_{H^3(\Omega)}\|u\|_{H^4(\Omega)}\\ &\lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}\end{aligned} \tag{4.90}$$

Hence, the inequality $\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)} \lesssim \mathcal{E}_f^2$ follows from the three above estimates (4.88), (4.89) and (4.90).

Similarly, for (4.85), we show only

$$\|\partial_t \mathcal{Q}_1^2\|_{H^1(\Omega)} + \|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \mathcal{D}_f).$$

The inequality $\|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f\mathcal{D}_f$ (see $\mathcal{Q}^3$ in (B.5)) is proven by using (C.1) and (C.10), (C.11),

$$\|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \|(AK, BK, K-1)\|_{H^4(\Omega)}\|\partial_3 u\|_{H^4(\Omega)} \lesssim \|\eta\|_{H^{9/2}(\Gamma)}\|\nabla u\|_{H^4(\Omega)}.$$

Let us prove $\|\partial_t \mathcal{Q}_1^2\|_{H^1(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \mathcal{D}_f)$ (see $\mathcal{Q}_1^2$ in (B.2)). In view of (C.1) and Lemma A.1, we obtain that

$$\begin{aligned}\|\partial_t((\zeta+\rho_0'\theta)\partial_t u_1)\|_{H^1(\Omega)} &\lesssim \|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}\|\partial_t u_1\|_{H^3(\Omega)} + \|(\zeta,\theta)\|_{H^3(\Omega)}\|\partial_t^2 u_1\|_{H^1(\Omega)}\\ &\lesssim (\|\partial_t\zeta\|_{H^1(\Omega)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)})\|\partial_t u_1\|_{H^3(\Omega)}\\ &\quad + (\|\zeta\|_{H^3(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t^2 u_1\|_{H^1(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u_1\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u_1\|_{L^2(\Omega)}).\end{aligned} \tag{4.91}$$

We further use (C.1) to have

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)Ku_3\partial_3 u_1)\|_{H^1(\Omega)}\\ &\lesssim (1+\|(\theta,\zeta)\|_{H^3(\Omega)})\|\partial_t(Ku_3\partial_3 u_1)\|_{H^1(\Omega)} + \|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}\|Ku_3\partial_3 u_1\|_{H^3(\Omega)}\\ &\lesssim (1+\|(\theta,\zeta)\|_{H^3(\Omega)})\|\partial_t K\|_{H^1(\Omega)}\|u_3\|_{H^3(\Omega)}\|\partial_3 u_1\|_{H^3(\Omega)}\\ &\quad + (1+\|(\theta,\zeta)\|_{H^3(\Omega)})(\|K-1\|_{H^3(\Omega)}+1)\\ &\qquad \times (\|\partial_t u_3\|_{H^1(\Omega)}\|\partial_3 u_1\|_{H^3(\Omega)} + \|\partial_t\partial_3 u_1\|_{H^1(\Omega)}\|u_3\|_{H^3(\Omega)})\\ &\quad + \|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}(\|K-1\|_{H^3(\Omega)}+1)\|u_3\|_{H^3(\Omega)}\|\partial_3 u_1\|_{H^3(\Omega)}.\end{aligned}$$

Thanks to Lemma A.1 and (C.10), we deduce

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)Ku_3\partial_3 u_1)\|_{H^1(\Omega)}\\ &\lesssim (1+\|\zeta\|_{H^3(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^4(\Omega)}^2\\ &\quad + (1+\|\zeta\|_{H^3(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)})(\|\eta\|_{H^{7/2}(\Gamma)}+1)\|\partial_t u\|_{H^2(\Omega)}\|u\|_{H^4(\Omega)}\\ &\quad + (\|\partial_t\zeta\|_{H^1(\Omega)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)})(\|\eta\|_{H^{7/2}(\Gamma)}+1)\|u\|_{H^4(\Omega)}^2\\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{4.92}$$

Since $K^2-1 = -J^{-2}(2\partial_3\theta + (\partial_3\theta)^2)$, let us use (C.1) to obtain

$$\begin{aligned}&\|\partial_t((K^2+A^2+B^2-1)\partial_{33}^2 u_1 - 2AK\partial_{13}^2 u_1 - 2BK\partial_{23}^2 u_1)\|_{H^1(\Omega)}\\ &\lesssim (\|(A^2,B^2,AK,BK)\|_{H^3(\Omega)} + \|K^2-1\|_{H^3(\Omega)})\|\partial_t u_1\|_{H^3(\Omega)}\\ &\quad + \|\partial_t(A^2,B^2,K^2-1,AK,BK)\|_{H^1(\Omega)}\|\nabla^2 u_1\|_{H^3(\Omega)}\\ &\lesssim (\|(A,B)\|_{H^3(\Omega)}^2 + \|(AK,BK)\|_{H^3(\Omega)} + \|\partial_3\theta\|_{H^3(\Omega)}(1+\|\partial_3\theta\|_{H^3(\Omega)}))\|\partial_t u_1\|_{H^3(\Omega)}\\ &\quad + (\|(A,B)\|_{H^3(\Omega)}\|(\partial_t A,\partial_t B)\|_{H^1(\Omega)} + \|\partial_t\partial_3\theta\|_{H^1(\Omega)}(1+\|\partial_3\theta\|_{H^3(\Omega)}))\|\nabla^2 u_1\|_{H^3(\Omega)}\\ &\quad + \|\partial_t(AK,BK)\|_{H^1(\Omega)}\|\nabla^2 u_1\|_{H^3(\Omega)}.\end{aligned}$$

Owing to (C.9) and (C.11), we deduce

$$\begin{aligned}&\|\partial_t((K^2+A^2+B^2-1)\partial_{33}^2 u_1 - 2AK\partial_{13}^2 u_1 - 2BK\partial_{23}^2 u_1)\|_{H^1(\Omega)}\\ &\lesssim \|\eta\|_{H^{7/2}(\Gamma)}(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_t u_1\|_{H^3(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\nabla^2 u_1\|_{H^3(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u_1\|_{H^2(\Omega)} + \|\nabla u_1\|_{H^4(\Omega)}).\end{aligned} \tag{4.93}$$

We continue using (C.1), Lemma A.1 and (C.11) to get

$$\begin{aligned}&\|\partial_t(AK(\partial_3 q - g\rho_0'\theta))\|_{H^1(\Omega)}\\ &\lesssim \|\partial_t(AK)\|_{H^1(\Omega)}\|(q,\theta)\|_{H^3(\Omega)} + \|AK\|_{H^3(\Omega)}(\|\partial_t\partial_3 q\|_{H^1(\Omega)} + \|\partial_t\theta\|_{H^1(\Omega)})\\ &\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}(\|q\|_{H^3(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)}) + \|\eta\|_{H^{7/2}(\Gamma)}(\|\partial_t q\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)})\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f + \|\partial_t q\|_{H^2(\Omega)}).\end{aligned} \tag{4.94}$$

From the product estimate (C.1), we obtain also

$$\begin{aligned}
&\|\partial_t((K\partial_3K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1K-B\partial_2K)\partial_3u_1)\|_{H^1(\Omega)}\\
&\lesssim (\|K\partial_3K(A^2+B^2+1)\|_{H^3(\Omega)}+\|\nabla(AK,BK)\|_{H^3(\Omega)})\|\partial_t\partial_3u_1\|_{H^1(\Omega)}\\
&\quad+(\|A\partial_1K\|_{H^3(\Omega)}+\|B\partial_2K\|_{H^3(\Omega)})\|\partial_t\partial_3u_1\|_{H^1(\Omega)}\\
&\quad+(\|\partial_t(K\partial_3K)(A^2+B^2+1)\|_{H^1(\Omega)}+\|\nabla\partial_t(AK,BK)\|_{H^1(\Omega)})\|\partial_3u_1\|_{H^3(\Omega)}\\
&\quad+(\|(\partial_tA\partial_1K,\partial_tB\partial_2K)\|_{H^1(\Omega)}+\|(A\partial_t\partial_1K,B\partial_t\partial_2K)\|_{H^1(\Omega)})\|\partial_3u_1\|_{H^3(\Omega)}.
\end{aligned}\tag{4.95}$$

We will bound each term in the r.h.s of (4.99). Thanks to the product estimate (C.1) again and (C.9), (C.10), we have

$$\begin{aligned}
\|K\partial_3K(A^2+B^2+1)\|_{H^3(\Omega)}&\lesssim\|K\partial_3K\|_{H^3(\Omega)}(1+\|(A,B)\|^2_{H^3(\Omega)})\\
&\lesssim(\|K-1\|_{H^3(\Omega)}+1)\|\partial_3K\|_{H^3(\Omega)}(1+\|(A,B)\|^2_{H^3(\Omega)})\\
&\lesssim(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\eta\|_{H^{9/2}(\Gamma)}(1+\|\eta\|^2_{H^{7/2}(\Gamma)})\\
&\lesssim\mathcal{E}_f.
\end{aligned}\tag{4.96}$$

In a same way, we have

$$\begin{aligned}
\|A\partial_1K\|_{H^3(\Omega)}+\|B\partial_2K\|_{H^3(\Omega)}&\lesssim\|(A,B)\|_{H^3(\Omega)}\|\nabla K\|_{H^3(\Omega)}\\
&\lesssim\|\eta\|_{H^{7/2}(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}\\
&\lesssim\mathcal{E}_f^2,
\end{aligned}\tag{4.97}$$

and

$$\begin{aligned}
\|(\partial_tA\partial_1K,\partial_tB\partial_2K)\|_{H^1(\Omega)}&\lesssim\|(\partial_tA,\partial_tB)\|_{H^1(\Omega)}\|\nabla K\|_{H^3(\Omega)}\\
&\lesssim\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}\\
&\lesssim\mathcal{E}_f^2.
\end{aligned}\tag{4.98}$$

Using also (4.22), we deduce

$$\begin{aligned}
\|(A\partial_t\partial_1K,B\partial_t\partial_2K)\|_{H^1(\Omega)}&\lesssim\|(A,B)\|_{H^3(\Omega)}\|\partial_t\nabla K\|_{H^1(\Omega)}\\
&\lesssim\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\\
&\lesssim\mathcal{E}_f^2,
\end{aligned}\tag{4.99}$$

and

$$\begin{aligned}
\|\nabla(AK,BK)\|_{H^3(\Omega)}+\|\nabla\partial_t(AK,BK)\|_{H^1(\Omega)}&\lesssim\|\eta\|_{H^{9/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\\
&\lesssim\mathcal{E}_f.
\end{aligned}\tag{4.100}$$

Thanks to (C.1), (C.10) and (4.22) again, let us estimate the term $\|\partial_t(K\partial_3K)\|_{H^1(\Omega)}$ as follows

$$\begin{aligned}
\|\partial_t(K\partial_3K)\|_{H^1(\Omega)}&\lesssim\|\partial_tK\|_{H^1(\Omega)}\|\partial_3K\|_{H^3(\Omega)}+(1+\|K-1\|_{H^3(\Omega)})\|\partial_t\partial_3K\|_{H^1(\Omega)}\\
&\lesssim\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}+(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\\
&\lesssim\mathcal{E}_f.
\end{aligned}$$

Owing to (C.9), this yields

$$\begin{aligned}
\|\partial_t(K\partial_3K)(A^2+B^2+1)\|_{H^1(\Omega)}&\lesssim\|\partial_t(K\partial_3K)\|_{H^1(\Omega)}(1+\|(A,B)\|^2_{H^3(\Omega)})\\
&\lesssim\mathcal{E}_f(1+\|\eta\|^2_{H^{7/2}(\Gamma)})\\
&\lesssim\mathcal{E}_f.
\end{aligned}\tag{4.101}$$

The five inequalities above (4.96), (4.97), (4.98), (4.100) and (4.101) help us to obtain from (4.99) that

$$\begin{aligned}&\|\partial_t((K\partial_3 K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1 K-B\partial_2 K)\partial_3 u_1)\|_{H^1(\Omega)}\\&\lesssim \mathcal{E}_f(\|\partial_t u_1\|_{H^2(\Omega)}+\|u_1\|_{H^4(\Omega)})\\&\lesssim \mathcal{E}_f^2.\end{aligned}\tag{4.102}$$

Combining (4.91), (4.92), (4.93), (4.94) and (4.102), we conclude

$$\begin{aligned}\|\partial_t \mathcal{Q}_1^2\|_{H^1(\Omega)}&\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u_1\|_{L^2(\Omega)}+\|\partial_t q\|_{H^2(\Omega)}+\|\nabla\partial_t u_1\|_{H^2(\Omega)})\\&\lesssim \mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).\end{aligned}$$

□

We are in position to show Proposition 4.6.

Proof of Proposition 4.6. For any $\beta\in\mathbb{N}^2$ such that $1\leqslant|\beta|\leqslant 4$, multiplying by $\partial_h^\beta u$ on both sides of $(4.81)_2$ and integrating over Ω, one has the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta u|^2+\int_\Omega(\nabla\partial_h^\beta q-\mu\Delta\partial_h^\beta u)\cdot\partial_h^\beta u+\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3=\int_\Omega\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2.$$

Using the integration by parts and $(4.81)_{3,5}$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho_0|\partial_h^\beta u|^2+\frac{1}{2}\int_\Omega\mu|\mathbb{S}\partial_h^\beta u|^2\\&=-\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3+\int_\Omega\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2-\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3\\&\quad+\int_\Omega\partial_h^\beta q\partial_h^\beta\mathcal{Q}^3-\int_\Gamma\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^5.\end{aligned}\tag{4.103}$$

We estimate each integral in the r.h.s of (4.103). For the first integral, we use Young's inequality and (C.7) to get that

$$\begin{aligned}\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3&\lesssim\|\zeta\|_{H^4(\Omega)}\|u_3\|_{H^4(\Omega)}\\&\lesssim\varepsilon^3\|\zeta\|_{H^4(\Omega)}^2+\varepsilon^{-3}\|u_3\|_{H^4(\Omega)}^2\\&\lesssim\varepsilon^3(\|\zeta\|_{H^4(\Omega)}^2+\|u_3\|_{H^5(\Omega)}^2)+\varepsilon^{-27}\|u_3\|_{L^2(\Omega)}^2.\end{aligned}\tag{4.104}$$

For the third integral, it follows from the trace theorem and Young's inequality that

$$\begin{aligned}\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3&\lesssim\|\partial_h^\beta\eta\|_{H^{-1/2}(\Gamma)}\|\partial_h^\beta u_3\|_{H^{1/2}(\Gamma)}\\&\lesssim\|\eta\|_{H^{|\beta|-1/2}(\Gamma)}\|u_3\|_{H^{|\beta|+1/2}(\Gamma)}\\&\lesssim\|\eta\|_{H^{7/2}(\Gamma)}\|u_3\|_{H^5(\Omega)}\\&\lesssim\varepsilon^3\|u_3\|_{H^5(\Omega)}^2+\varepsilon^{-3}\|\eta\|_{H^{7/2}(\Gamma)}^2.\end{aligned}$$

Thanks to (C.7) again, we have

$$\|\eta\|_{H^{7/2}(\Gamma)}^2\lesssim\varepsilon^6\|\eta\|_{H^{9/2}(\Gamma)}^2+\varepsilon^{-21}\|\eta\|_{L^2(\Gamma)}^2.$$

Hence,

$$\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3\lesssim\varepsilon^3(\|\eta\|_{H^{9/2}(\Gamma)}^2+\|u_3\|_{H^5(\Omega)}^2)+\varepsilon^{-24}\|\eta\|_{L^2(\Gamma)}^2.\tag{4.105}$$

For the fourth integral, we use Cauchy-Schwarz's inequality and (4.85) to have

$$\int_\Omega \partial_h^\beta q \partial_h^\beta \mathcal{Q}^3 \lesssim \|q\|_{H^4(\Omega)} \|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f \mathcal{D}_f (\mathcal{E}_f + \mathcal{D}_f). \tag{4.106}$$

For the second and fifth integral, we split into two cases. For $\beta \in \mathbb{N}^2$ such that $1 \leqslant |\beta| \leqslant 3$, we use trace theorem and (4.85) to bound

$$\begin{aligned} \int_\Omega \partial_h^\beta u \cdot \partial_h^\beta \mathcal{Q}^2 - \int_\Gamma \partial_h^\beta u \cdot \partial_h^\beta \mathcal{Q}^5 &\lesssim \|u\|_{H^3(\Omega)} \|\mathcal{Q}^2\|_{H^3(\Omega)} + \|u\|_{H^3(\Gamma)} \|\mathcal{Q}^5\|_{H^3(\Gamma)} \\ &\lesssim \|u\|_{H^3(\Omega)} \|\mathcal{Q}^2\|_{H^3(\Omega)} + \|u\|_{H^4(\Omega)} \|\mathcal{Q}^5\|_{H^3(\Gamma)} \\ &\lesssim \mathcal{E}_f^2 (\mathcal{E}_f + \mathcal{D}_f). \end{aligned} \tag{4.107}$$

For $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ such that $|\beta| = 4$, we assume $\beta_1 \geqslant 1$ and write

$$\beta_- = (\beta_1 - 1, \beta_2) \quad \text{and} \quad \beta_+ = (\beta_1 + 1, \beta_2). \tag{4.108}$$

Thanks to (4.85) again, we estimate that

$$\begin{aligned} \Big| \int_\Omega \partial_h^\beta u \cdot \partial_h^\beta \mathcal{Q}^2 \Big| = \Big| \int_\Omega \partial_h^{\beta_+} u \cdot \partial_h^{\beta_-} \mathcal{Q}^2 \Big| &\lesssim \|\partial_h^{\beta_+} u\|_{L^2(\Omega)} \|\partial_h^{\beta_-} \mathcal{Q}^2\|_{L^2(\Omega)} \\ &\lesssim \|u\|_{H^5(\Omega)} \|\mathcal{Q}^2\|_{H^3(\Omega)} \\ &\lesssim \mathcal{E}_f (\mathcal{E}_f^2 + \mathcal{D}_f^2). \end{aligned} \tag{4.109}$$

Using the trace theorem also, we have

$$\begin{aligned} \Big| \int_\Gamma \partial_h^\beta u \cdot \partial_h^\beta \mathcal{Q}^5 \Big| = \Big| \int_\Gamma \partial_h^{\beta_+} u \cdot \partial_h^{\beta_-} \mathcal{Q}^5 \Big| &\lesssim \|\partial_h^{\beta_+} u\|_{H^{-1/2}(\Gamma)} \|\partial_h^{\beta_-} \mathcal{Q}^5\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|u\|_{H^{|\beta_+|-1/2}(\Gamma)} \|\mathcal{Q}^5\|_{H^{|\beta_-|+1/2}(\Gamma)} \\ &\lesssim \|u\|_{H^5(\Omega)} \|\mathcal{Q}^5\|_{H^{7/2}(\Gamma)} \\ &\lesssim \mathcal{E}_f (\mathcal{E}_f^2 + \mathcal{D}_f^2). \end{aligned} \tag{4.110}$$

Substituting (4.104), (4.105), (4.106), (4.107), (4.109), (4.110) into (4.103), we obtain

$$\begin{aligned} \frac{d}{dt} \int_\Omega \rho_0 |\partial_h^\beta u|^2 + \int_\Omega \mu |\mathbb{S} \partial_h^\beta u|^2 \lesssim \varepsilon^3 (\mathcal{E}_f^2 + \|\nabla u_3\|_{H^4(\Omega)}^2) + \mathcal{E}_f (\mathcal{E}_f^2 + \mathcal{D}_f^2) \\ + \varepsilon^{-27} (\|u_3\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Gamma)}^2). \end{aligned} \tag{4.111}$$

By Korn's inequality (C.5), one has

$$\int_\Omega \mu |\mathbb{S} \partial_h^\beta u|^2 \gtrsim \|\nabla \partial_h^\beta u\|_{L^2(\Omega)}^2.$$

Hence, we deduce from (4.111) that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \rho_0 |\partial_h^\beta u|^2 + \|\nabla \partial_h^\beta u\|_{L^2(\Omega)}^2 \lesssim \varepsilon^3 (\mathcal{E}_f^2 + \|\nabla u_3\|_{H^4(\Omega)}^2) + \varepsilon^{-27} (\|u_3\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Gamma)}^2) \\ + \mathcal{E}_f (\mathcal{E}_f^2 + \mathcal{D}_f^2), \end{aligned}$$

Integrating the resulting inequality in time, we obtain (4.82).

To prove (4.83), we compute from (4.81) that

$$\begin{cases} \rho_0 \partial_t^2 \partial_h^\beta u + \nabla \partial_t \partial_h^\beta q - \mu \Delta \partial_t \partial_h^\beta u - g \rho_0' \partial_h^\beta u_3 e_3 = -g \partial_h^\beta \mathcal{Q}^1 e_3 + \partial_t \partial_h^\beta \mathcal{Q}^2 & \text{in } \Omega, \\ \operatorname{div} \partial_t \partial_h^\beta u = \partial_t \partial_h^\beta \mathcal{Q}^3 & \text{in } \Omega, \\ (\partial_t \partial_h^\beta q \mathrm{Id} - \mu \mathbb{S} \partial_t \partial_h^\beta u) e_3 = g \rho_+ \partial_h^\beta u_3 e_3 + g \rho_+ \partial_h^\beta \mathcal{Q}^4 e_3 + \partial_h^\beta \mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{4.112}$$

For any $\beta \in \mathbb{N}^2$ with $|\beta| = 1$ or 2, multiplying by $\partial_t \partial_h^\beta u$ on both sides of $(4.112)_1$ and integrating over Ω, one has the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_0|\partial_t\partial_h^\beta u|^2 - g\rho_0'|\partial_h^\beta u_3|^2) + \int_\Omega (\nabla \partial_t \partial_h^\beta q - \mu \Delta \partial_t \partial_h^\beta u)\cdot \partial_t \partial_h^\beta u\\ &= \int_\Omega (-g\partial_h^\beta \mathcal{Q}^1 \partial_t \partial_h^\beta u_3 + \partial_t \partial_h^\beta \mathcal{Q}^2 \cdot \partial_t \partial_h^\beta u).\end{aligned}$$

Using the integration by parts, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t\partial_h^\beta u|^2 - \int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big) + \int_\Gamma (\partial_t\partial_h^\beta q \mathrm{Id} - \mu \mathbb{S}\partial_t\partial_h^\beta u)e_3 \cdot \partial_t\partial_h^\beta u\\ &= \int_\Omega \partial_t\partial_h^\beta q \mathrm{div}\partial_t\partial_h^\beta u - \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t\partial_h^\beta u|^2 + \int_\Omega (-g\partial_h^\beta \mathcal{Q}^1\partial_t\partial_h^\beta u_3 + \partial_t\partial_h^\beta \mathcal{Q}^2\cdot\partial_t\partial_h^\beta u)\end{aligned}$$

By $(4.112)_{2,3}$, we observe

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t\partial_h^\beta u|^2 + \int_\Gamma g\rho_+|\partial_h^\beta u_3|^2 - \int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}\partial_t\partial_h^\beta u|^2\\ &= \int_\Omega (-g\partial_h^\beta \mathcal{Q}^1\partial_t\partial_h^\beta u_3 + \partial_t\partial_h^\beta \mathcal{Q}^2\cdot\partial_t\partial_h^\beta u) + \int_\Omega \partial_t\partial_h^\beta q\partial_t\partial_h^\beta \mathcal{Q}^3\\ &\quad - \int_\Gamma (\partial_t\partial_h^\beta \mathcal{Q}^5 + g\rho_+\partial_h^\beta \mathcal{Q}^4 e_3)\cdot\partial_t\partial_h^\beta u.\end{aligned} \tag{4.113}$$

We now estimate each integral in the r.h.s of (4.113). For the first and third integral, we use Cauchy-Schwarz's inequality and (4.84), (4.85) to have

$$\begin{aligned}\int_\Omega \partial_h^\beta \mathcal{Q}^1\partial_t\partial_h^\beta u_3 + \int_\Omega \partial_t\partial_h^\beta q\partial_t\partial_h^\beta \mathcal{Q}^3 &\lesssim \|\partial_t u_3\|_{H^2(\Omega)}\|\mathcal{Q}^1\|_{H^2(\Omega)} + \|\partial_t q\|_{H^2(\Omega)}\|\partial_t\mathcal{Q}^3\|_{H^2(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2).\end{aligned} \tag{4.114}$$

With the same notations $\beta_\pm$ (4.108), we use Cauchy-Schwarz's inequality again and (4.85) to bound the second integral as

$$\begin{aligned}\Big|\int_\Omega \partial_t\partial_h^\beta u\cdot\partial_t\partial_h^\beta \mathcal{Q}^2\Big| = \Big|\int_\Omega \partial_t\partial_h^{\beta_+} u\cdot\partial_t\partial_h^{\beta_-}\mathcal{Q}^2\Big| &\lesssim \|\partial_t u\|_{H^{|\beta_+|}(\Gamma)}\|\partial_t\mathcal{Q}^2\|_{H^{|\beta_-|}(\Omega)}\\ &\lesssim \|\partial_t u\|_{H^3(\Omega)}\|\partial_t\mathcal{Q}^2\|_{H^1(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2),\end{aligned} \tag{4.115}$$

and to bound the fourth integral as

$$\begin{aligned}\Big|\int_\Gamma \partial_t\partial_h^\beta u\cdot\partial_t\partial_h^\beta \mathcal{Q}^5\Big| = \Big|\int_\Gamma \partial_t\partial_h^{\beta_+} u\cdot\partial_t\partial_h^{\beta_-}\mathcal{Q}^5\Big| &\lesssim \|\partial_t\partial_h^{\beta_+}u\|_{H^{-1/2}(\Gamma)}\|\partial_t\partial_h^{\beta_-}\mathcal{Q}^5\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|\partial_t u\|_{H^{|\beta_+|-1/2}(\Gamma)}\|\partial_t\mathcal{Q}^5\|_{H^{|\beta_-|+1/2}(\Gamma)}\\ &\lesssim \|\partial_t u\|_{H^3(\Omega)}\|\partial_t\mathcal{Q}^5\|_{H^{3/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2).\end{aligned} \tag{4.116}$$

Thanks to the trace theorem and (4.84), we bound the fifth integral as

$$\begin{aligned}\int_\Gamma \partial_t\partial_h^\beta u_3\partial_h^\beta \mathcal{Q}^4 &\lesssim \|\partial_t\partial^\beta u_3\|_{H^{-1/2}(\Gamma)}\|\partial^\beta\mathcal{Q}^4\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|\partial_t u_3\|_{H^{|\beta|-1/2}(\Omega)}\|\mathcal{Q}^4\|_{H^{|\beta|+1/2}(\Gamma)}\\ &\lesssim \|\partial_t u_3\|_{H^2(\Omega)}\|\mathcal{Q}^4\|_{H^{5/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f^3.\end{aligned} \tag{4.117}$$

In view of (4.114), (4.115), (4.116) and (4.117), we get

$$\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t\partial_h^\beta u|^2+\int_\Gamma g\rho_+|\partial_h^\beta u_3|^2-\int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big)+\int_\Omega|\mathbb{S}\partial_t\partial_h^\beta u|^2\lesssim\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).$$

Integrating in time and using Korn's inequality (C.5), we obtain

$$\begin{aligned}&\|\partial_t\partial_h^\beta u(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla\partial_t\partial_h^\beta u(s)\|_{L^2(\Omega)}^2ds\\&\lesssim\mathcal{E}_f^2(0)+\|\partial_h^\beta u_3(t)\|_{L^2(\Omega)}^2+\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds.\end{aligned}$$

Combining the resulting inequality and (4.82), the inequality (4.83) follows. Proof of Proposition 4.6 is complete. □

4.4. **Estimates of the perturbation density.** We continue deriving the energy evolution of the space-time derivatives of ζ. We rewrite $(2.9)_1$ as

$$\partial_t\zeta=K\partial_t\theta\partial_3\zeta-u_j\mathcal{A}_{jk}\partial_k\zeta-\rho_0'u_3-\rho_0\mathcal{Q}^3+\tilde{\mathcal{Q}}^1,\tag{4.118}$$

where

$$\tilde{\mathcal{Q}}^1=\rho_0''K\theta\partial_t\theta-q\mathcal{A}_{lk}\partial_ku_l-\mathcal{A}_{lk}\partial_k(\rho_0'\theta u_l)-(\mathcal{A}_{lk}-\delta_{lk})\partial_k(\rho_0u_l).\tag{4.119}$$

We first present the estimate of $\tilde{\mathcal{Q}}^1$.

Lemma 4.6. *There holds*

$$\|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).\tag{4.120}$$

Proof. We use (C.1), (C.10) and Lemma A.1 to have that

$$\begin{aligned}\|\rho_0''K\theta\partial_t\theta\|_{H^4(\Omega)}&\lesssim(1+\|K-1\|_{H^4(\Omega)})\|\theta\|_{H^4(\Omega)}\|\partial_t\theta\|_{H^4(\Omega)}\\&\lesssim(1+\|\eta\|_{H^{9/2}(\Gamma)})\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t\eta\|_{H^{7/2}(\Gamma)}.\end{aligned}$$

Combining (4.22) and the resulting inequality, we have

$$\|\rho_0''K\theta\partial_t\theta\|_{H^4(\Omega)}\lesssim\mathcal{E}_f^2.\tag{4.121}$$

Using Lemma A.1 again and (C.1), (C.12), one has

$$\begin{aligned}\|\mathcal{A}_{lk}\partial_k(\rho_0'\theta u_l)\|_{H^4(\Omega)}&\lesssim(1+\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)})\|\theta\|_{H^5(\Omega)}\|u\|_{H^5(\Omega)}\\&\lesssim(1+\|\eta\|_{H^{9/2}(\Gamma)})\|\eta\|_{H^{9/2}(\Gamma)}\|u\|_{H^5(\Omega)}\\&\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla u\|_{H^4(\Omega)})\end{aligned}\tag{4.122}$$

and

$$\|(\mathcal{A}_{lk}-\delta_{lk})\partial_k(\rho_0u_l)\|_{H^4(\Omega)}\lesssim\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)}\|u\|_{H^5(\Omega)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla u\|_{H^4(\Omega)}).\tag{4.123}$$

Thanks to Gagliardo-Nireberg's inequality also and (C.12), we obtain

$$\begin{aligned}\|q\mathcal{A}_{lk}\partial_ku_l\|_{H^4(\Omega)}&\lesssim(1+\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)})(\|q\|_{H^2(\Omega)}\|\nabla u\|_{H^4(\Omega)}+\|q\|_{H^4(\Omega)}\|\nabla u\|_{H^2(\Omega)})\\&\lesssim\mathcal{E}_f(\mathcal{E}_f+\|q\|_{H^4(\Omega)}+\|\nabla u\|_{H^4(\Omega)}),\end{aligned}\tag{4.124}$$

Those above estimates, (4.121), (4.122), (4.123) and (4.124) imply

$$\|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|q\|_{H^4(\Omega)}+\|\nabla u\|_{H^4(\Omega)})\lesssim\mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).$$

Lemma 4.6 is proven. □

We derive the following proposition.

Proposition 4.7. *The following inequality holds*

$$\begin{aligned}\|\zeta(t)\|^2_{H^4(\Omega)} \leqslant C_9\Big(\mathcal{E}_f^2(0) + \varepsilon^3\int_0^t (\|\zeta(s)\|^2_{H^4(\Omega)} + \|u_3(s)\|^2_{H^5(\Omega)})ds\Big)\\ + C_9\Big(\varepsilon^{-27}\int_0^t \|(u_3,\zeta)(s)\|^2_{L^2(\Omega)}ds + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big).\end{aligned} \tag{4.125}$$

Proof. It can be seen from $(2.11)_1$ that

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} = -\int_\Omega \rho_0' u_3\zeta + \int_\Omega \mathcal{Q}^1\zeta \lesssim (\|u_3\|_{L^2(\Omega)} + \|\mathcal{Q}^1\|_{L^2(\Omega)})\|\zeta\|_{L^2(\Omega)}.$$

Due to (4.84), we thus have

$$\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} \lesssim \|u_3\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)} + \mathcal{E}_f^3.$$

This yields

$$\begin{aligned}\|\zeta(t)\|^2_{L^2(\Omega)} &\lesssim \mathcal{E}_f^2(0) + \int_0^t \|(u_3,\zeta)(s)\|^2_{L^2(\Omega)}ds + \int_0^t \mathcal{E}_f^3(s)ds\\ &\lesssim \mathcal{E}_f^2(0) + \int_0^t (\varepsilon^3\|u_3(s)\|^2_{L^2(\Omega)} + \varepsilon^{-3}\|\zeta(s)\|^2_{L^2(\Omega)})ds + \int_0^t \mathcal{E}_f^3(s)ds\end{aligned} \tag{4.126}$$

For $\alpha\in\mathbb{N}^3, 1\leqslant|\alpha|\leqslant 4$, we have from (4.118) that

$$\begin{aligned}\partial_t\partial^\alpha\zeta &= K\partial^\alpha(\partial_t\theta\partial_3\zeta) + \sum_{0\neq\beta\leqslant\alpha}\partial^\beta K\partial^{\alpha-\beta}(\partial_t\theta\partial_3\zeta) - \partial^\alpha(u_j\mathcal{A}_{jk}\partial_k\zeta)\\ &\quad + \partial^\alpha(-\rho_0'u_3 - \rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1)\\ &= (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta) + \sum_{0\neq\beta\leqslant\alpha} K\partial^\beta\partial_t\theta\partial^{\alpha-\beta}\partial_3\zeta\\ &\quad + \sum_{0\neq\beta\leqslant\alpha}\partial^\beta K\partial^{\alpha-\beta}(\partial_t\theta\partial_3\zeta) - \sum_{0\neq\beta\leqslant\alpha}\partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta\\ &\quad + \partial^\alpha(-\rho_0'u_3 - \rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1).\end{aligned} \tag{4.127}$$

We deduce from (4.127) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} &= \int_\Omega (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta + \sum_{0\neq\beta\leqslant\alpha}\int_\Omega K\partial^\beta\partial_t\theta\partial^{\alpha-\beta}\partial_3\zeta\partial^\alpha\zeta\\ &\quad + \sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta K\partial^{\alpha-\beta}(\partial_t\theta\partial_3\zeta)\partial^\alpha\zeta - \sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta\partial^\alpha\zeta\\ &\quad + \int_\Omega \partial^\alpha(-\rho_0'u_3 - \rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1)\partial^\alpha\zeta.\end{aligned} \tag{4.128}$$

We bound each integral in the r.h.s of (4.128). For the first integral, using the integration by parts, one has

$$\begin{aligned}2\int_\Omega (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta &= \int_\Omega (K\partial_t\theta\partial_3|\partial^\alpha\zeta|^2 - u_j\mathcal{A}_{jk}\partial_k|\partial^\alpha\zeta|^2)\\ &= \int_\Gamma (K\partial_t\theta - u_j\mathcal{A}_{j3})|\partial^\alpha\zeta|^2\\ &\quad - \int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2.\end{aligned}$$

On Γ, we have $K\partial_t\theta - u_j\mathcal{A}_{j3} = 0$ by the definition of $\mathcal{A}$ (2.8) and by $(2.9)_4$. This yields

$$2\int_\Omega (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta = -\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2.$$

Due to Sobolev embedding and the product estimate (C.1), it can be seen that

$$\begin{aligned}
&\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2\\
&\quad\lesssim \|\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk})\|_{H^2(\Omega)}\|\zeta\|^2_{H^4(\Omega)}\\
&\quad\lesssim ((\|K-1\|_{H^3(\Omega)}+1)\|\theta\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^3(\Omega)}+1))\|\zeta\|_{H^4(\Omega)}.
\end{aligned}$$

Owing to Lemma A.1 and (C.10), (C.12), we have

$$\begin{aligned}
\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2 &\lesssim (1+\|\eta\|_{H^{7/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)} + \|u\|_{H^3(\Omega)})\|\zeta\|^2_{H^4(\Omega)}\\
&\lesssim \mathcal{E}_f^3.
\end{aligned} \tag{4.129}$$

For the second integral in the r.h.s of (4.128), we use Cauchy-Schwarz's inequality, (C.8) and Lemma A.1 to obtain

$$\begin{aligned}
\sum_{0\neq\beta\leqslant\alpha}\int_\Omega K\partial^\beta\partial_t\theta\partial^{\alpha-\beta}\partial_3\zeta\partial^\alpha\zeta &\lesssim (1+\|K-1\|_{L^\infty(\Omega)})\|\partial_t\theta\|_{H^3(\Omega)}\|\partial_3\zeta\|_{H^3(\Omega)}\|\zeta\|_{H^4(\Omega)}\\
&\lesssim \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\zeta\|^2_{H^4(\Omega)}.
\end{aligned}$$

In view of (4.22), one has

$$\sum_{0\neq\beta\leqslant\alpha}\int_\Omega K\partial^\beta\partial_t\theta\partial^{\alpha-\beta}\partial_3\zeta\partial^\alpha\zeta \lesssim \mathcal{E}_f^3. \tag{4.130}$$

Next, for the third integral, we use Cauchy-Schwarz's inequality and the product estimate (C.1) to have

$$\begin{aligned}
\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta K\partial^{\alpha-\beta}(\partial_t\theta\partial_3\zeta)\partial^\alpha\zeta &\lesssim \|\nabla K\|_{H^2(\Omega)}\|\partial_t\theta\partial_3\zeta\|_{H^3(\Omega)}\|\zeta\|_{H^4(\Omega)}\\
&\lesssim \|\nabla K\|_{H^2(\Omega)}\|\partial_t\theta\|_{H^3(\Omega)}\|\zeta\|^2_{H^4(\Omega)}.
\end{aligned}$$

Owing to (C.10), Lemma A.1, and (4.22) again, we deduce

$$\begin{aligned}
\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta K\partial^{\alpha-\beta}(\partial_t\theta\partial_3\zeta)\partial^\alpha\zeta &\lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\zeta\|^2_{H^4(\Omega)}\\
&\lesssim \mathcal{E}_f^4.
\end{aligned} \tag{4.131}$$

For the fourth integral, we continue using Cauchy-Schwarz's inequality and (C.12) also to observe

$$\begin{aligned}
\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta\partial^\alpha\zeta &\lesssim (\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)}+1)\|u\|_{H^4(\Omega)}\|\zeta\|^2_{H^4(\Omega)}\\
&\lesssim (1+\|\eta\|_{H^{9/2}(\Gamma)})\|u\|_{H^4(\Omega)}\|\zeta\|^2_{H^4(\Omega)}\\
&\lesssim \mathcal{E}_f^3.
\end{aligned} \tag{4.132}$$

Let us bound the fifth integral. Thanks to Young's inequality, we have

$$\int_\Omega \partial^\alpha(\rho_0' u_3)\partial^\alpha\zeta \lesssim \|\partial^\alpha\zeta\|_{L^2(\Omega)}\|u_3\|_{H^{|\alpha|}(\Omega)} \lesssim \varepsilon^3\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \varepsilon^{-3}\|u_3\|^2_{H^4(\Omega)}$$

By Young's inequality again and (C.7), this yields

$$\begin{aligned}\int_\Omega \partial^\alpha(\rho_0' u_3)\partial^\alpha\zeta &\lesssim \varepsilon^3\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \varepsilon^{-3}(\varepsilon^6\|u_3\|^2_{H^5(\Omega)} + \varepsilon^{-24}\|u_3\|^2_{L^2(\Omega)})\\ &\lesssim \varepsilon^3(\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \|u_3\|^2_{H^5(\Omega)}) + \varepsilon^{-27}\|u_3\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.133}$$

Thanks to (4.85) and (4.120), we have

$$\begin{aligned}\int_\Omega \partial^\alpha(-\rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1)\partial^\alpha\zeta &\lesssim (\|\mathcal{Q}^3\|_{H^4(\Omega)} + \|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)})\|\zeta\|_{H^4(\Omega)}\\ &\lesssim \mathcal{E}_f^2(\mathcal{E}_f + \mathcal{D}_f).\end{aligned} \tag{4.134}$$

We substitute (4.129), (4.130) (4.131), (4.132), (4.133) and (4.134) into (4.128) to have

$$\frac{d}{dt}\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} \lesssim \varepsilon^3(\|\zeta\|^2_{H^4(\Omega)} + \|u_3\|^2_{H^5(\Omega)}) + \varepsilon^{-27}\|u_3\|^2_{L^2(\Omega)} + \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2).$$

Integrating the resulting inequality from 0 to t, together with (4.126), one has (4.125). Proof of Proposition 4.7 is complete. □

In addition, we have the following estimate.

Proposition 4.8. *There holds*

$$\|\partial_t\zeta\|^2_{H^2(\Omega)} + \|\partial_t^2\zeta\|^2_{L^2(\Omega)} \leqslant C_{10}(\|u_3\|^2_{H^2(\Omega)} + \|\partial_t u_3\|^2_{L^2(\Omega)} + \mathcal{E}_f^4). \tag{4.135}$$

Proof. It follows directly from $(2.11)_1$ and (4.84) that

$$\|\partial_t\zeta\|^2_{H^2(\Omega)} \lesssim \|u_3\|^2_{H^2(\Omega)} + \|\mathcal{Q}^1\|^2_{H^2(\Omega)} \lesssim \|u_3\|^2_{H^2(\Omega)} + \mathcal{E}_f^4 \tag{4.136}$$

and

$$\|\partial_t^2\zeta\|^2_{L^2(\Omega)} \lesssim \|\partial_t u_3\|^2_{L^2(\Omega)} + \|\partial_t\mathcal{Q}^1\|^2_{L^2(\Omega)} \lesssim \|\partial_t u_3\|^2_{L^2(\Omega)} + \mathcal{E}_f^4. \tag{4.137}$$

Hence, we obtain (4.135) by combining (4.136) and (4.137). □

4.5. **Elliptic estimates.** We use the elliptic estimate (C.3) to derive some inequalities.

Proposition 4.9. *There holds*

$$\begin{aligned}&\|u\|^2_{H^4(\Omega)} + \|q\|^2_{H^3(\Omega)} + \|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)}\\ &\quad\leqslant C_{11}\Big(\|\partial_t^2 u\|^2_{L^2(\Omega)} + \|u_3\|^2_{L^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)} + \|\eta\|^2_{H^{5/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{1/2}(\Gamma)} + \mathcal{E}_f^4\Big).\end{aligned} \tag{4.138}$$

Proof. We derive from (2.11) that

$$\begin{cases} -\mu\Delta\partial_t u + \nabla\partial_t q = -\rho_0\partial_t^2 u - g\partial_t\zeta e_3 + \partial_t\mathcal{Q}^2 & \text{in } \Omega,\\ \operatorname{div}\partial_t u = \partial_t\mathcal{Q}^3 & \text{in } \Omega,\\ (\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3 = g\rho_+\partial_t\eta e_3 + \partial_t\mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{4.139}$$

Applying the elliptic estimate (C.3) to (4.139), it tells us that

$$\begin{aligned}\|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} &\lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|\partial_t\zeta\|^2_{L^2(\Omega)} + \|\partial_t\eta\|^2_{H^{1/2}(\Gamma)}\\ &\quad + \|\partial_t\mathcal{Q}^2\|^2_{L^2(\Omega)} + \|\partial_t\mathcal{Q}^3\|^2_{H^1(\Omega)} + \|\partial_t\mathcal{Q}^5\|^2_{H^{1/2}(\Gamma)}.\end{aligned}$$

Note that from $(2.11)_1$ that ,

$$\|\partial_t\zeta\|^2_{L^2(\Omega)} \lesssim \|u_3\|^2_{L^2(\Omega)} + \|\mathcal{Q}^1\|^2_{L^2(\Omega)}.$$

Hence, we have

$$\|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} \lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|\partial_t \eta\|^2_{H^{1/2}(\Gamma)} + \|u_3\|^2_{L^2(\Omega)} + \|\mathcal{Q}^1\|^2_{L^2(\Omega)} + \|\partial_t \mathcal{Q}^2\|^2_{L^2(\Omega)} + \|\partial_t \mathcal{Q}^3\|^2_{H^1(\Omega)} + \|\partial_t \mathcal{Q}^5\|^2_{H^{1/2}(\Gamma)}.$$

Due to (4.84), this yields

$$\|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} \lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|u_3\|^2_{L^2(\Omega)} + \|\partial_t \eta\|^2_{H^{1/2}(\Gamma)} + \mathcal{E}_f^4. \tag{4.140}$$

Meanwhile, we obtain from (2.11) that

$$\begin{cases} -\Delta u + \nabla q = -\rho_0 \partial_t u - g\zeta e_3 + \mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}^3 & \text{in } \Omega, \\ (q\mathrm{Id} - \mu \mathbb{S} u)e_3 = g\rho_+ \eta e_3 + \mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{4.141}$$

Owing to (4.84) and by applying the elliptic estimate (C.3) again to (4.141), we observe that

$$\begin{aligned} \|u\|^2_{H^4(\Omega)} + \|q\|^2_{H^3(\Omega)} &\lesssim \|\partial_t u\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)} + \|\mathcal{Q}^2\|^2_{H^2(\Omega)} + \|\mathcal{Q}^3\|^2_{H^3(\Omega)} \\ &\quad + \|\eta\|^2_{H^{5/2}(\Gamma)} + \|\mathcal{Q}^5\|^2_{H^{5/2}(\Gamma)} \\ &\lesssim \|\partial_t u\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)} + \|\eta\|^2_{H^{5/2}(\Gamma)} + \mathcal{E}_f^4. \end{aligned} \tag{4.142}$$

Combining (4.140) and (4.142), one has (4.138). Proof of Proposition 4.9 is complete. □

Let us define the "horizontal" dissipation $\mathcal{D}_h > 0$ as follows,

$$\mathcal{D}_h^2 := \sum_{\beta \in \mathbb{N}^2, |\beta| \leqslant 4} \|\nabla \partial_h^\beta u\|^2_{L^2(\Omega)} + \sum_{\beta \in \mathbb{N}^2, |\beta| \leqslant 2} \|\nabla \partial_h^\beta \partial_t u\|^2_{L^2(\Omega)} + \|\nabla \partial_t^2 u\|^2_{L^2(\Omega)}. \tag{4.143}$$

The next proposition is to compare $\mathcal{D}_f$ (4.2) and $\mathcal{D}_h$ (4.143).

Proposition 4.10. *Assuming δ_0 sufficiently small (see (4.173)), there holds*

$$\mathcal{D}_f^2 \leqslant C_{12}\Big(\mathcal{D}_h^2 + \varepsilon^3 \mathcal{E}_f^2 + \varepsilon^{-9}(\|(\zeta, u)\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}) + \mathcal{E}_f^3\Big). \tag{4.144}$$

To prove Proposition 4.10, we use the two lemmas below.

Lemma 4.7. *For any $s \geqslant 0$, there holds*

$$\|f\|^2_{H^{s+1/2}(\Gamma)} \lesssim \|f\|^2_{H^{1/2}(\Gamma)} + \sum_{\beta \in \mathbb{N}^2, |\beta| = s} \|\partial_h^\beta f\|^2_{H^{1/2}(\Gamma)}. \tag{4.145}$$

Proof. Since $\Gamma = 2\pi L_1 \mathbb{Z} \times 2\pi L_2 \mathbb{Z} \times \{0\}$, we exploit the definition of the Sobolev norm on Γ to have that

$$\|f\|^2_{H^{s+1/2}(\Gamma)} = \sum_{n \in \mathbb{Z}^2} (1 + |n|^2)^{s+1/2} |\hat{f}(n)|^2,$$

where $\hat{f}$ is the Fourier series of f. By Cauchy-Schwarz's inequality, one has

$$\|f\|^2_{H^{s+1/2}(\Gamma)} \lesssim \sum_{n \in \mathbb{Z}^2} (1 + |n|^2)^{1/2} |\hat{f}(n)|^2 + \sum_{\beta \in \mathbb{N}^2, |\beta| = s} \sum_{n \in \mathbb{Z}^2} (1 + |n|^2)^{1/2} |n^\beta \hat{f}(n)|^2,$$

which immediately yields (4.145). □

Lemma 4.8. *Let us write*

$$\mathscr{W} := \sqrt{\|\zeta\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \|\eta\|^2_{L^2(\Gamma)} + \mathcal{E}_f^3}.$$

The following inequalities holds

$$\|\partial_t u\|_{L^2(\Omega)} \lesssim \|\nabla \partial_t u\|_{L^2(\Omega)} + \mathscr{W}, \tag{4.146}$$

and

$$\|\partial_t^2 u\|_{L^2(\Omega)} \lesssim \|\nabla \partial_t u\|_{L^2(\Omega)} + \|\nabla \partial_t^2 u\|_{L^2(\Omega)} + \mathscr{W}. \tag{4.147}$$

Proof. Let us show (4.146) first. Multiplying by $\partial_t u$ on both sides of $(2.11)_2$, we obtain

$$\begin{aligned}\int_\Omega \rho_0 |\partial_t u|^2 &= -\int_\Omega \nabla q \cdot \partial_t u + \mu \int_\Omega \Delta u \cdot \partial_t u - \int_\Omega g\zeta \partial_t u_3 + \int_\Omega \mathcal{Q}^2 \cdot \partial_t u \\ &= -\int_\Gamma (q\mathrm{Id} - \mu \mathbb{S}u) e_3 \cdot \partial_t u + \int_\Omega q \mathrm{div} \partial_t u - \frac{\mu}{2} \int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u \\ &\quad - \int_\Omega g\zeta \partial_t u_3 + \int_\Omega \mathcal{Q}^2 \cdot \partial_t u,\end{aligned}$$

after using the integration by parts. Using $(2.11)_{3,5}$, this yields

$$\begin{aligned}\int_\Omega \rho_0 |\partial_t u|^2 &= -\int_\Gamma g\rho_+ \eta \partial_t u_3 - \int_\Gamma \mathcal{Q}^5 \cdot \partial_t u + \int_\Omega q \partial_t \mathcal{Q}^3 + \mu \int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u \\ &\quad - \int_\Omega g\zeta \partial_t u_3 + \int_\Omega \mathcal{Q}^2 \cdot \partial_t u.\end{aligned} \tag{4.148}$$

By Cauchy-Schwarz's inequality, we have

$$\frac{\mu}{2} \int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u - \int_\Omega g\zeta \partial_t u_3 \lesssim \|u\|_{H^1(\Omega)} \|\partial_t u\|_{H^1(\Omega)} + \|\zeta\|_{L^2(\Omega)} \|\partial_t u_3\|_{L^2(\Omega)}. \tag{4.149}$$

Using also the trace theorem, we have

$$\int_\Gamma g\rho_+ \eta \partial_t u_3 \lesssim \|\eta\|_{L^2(\Gamma)} \|\partial_t u\|_{H^1(\Omega)}. \tag{4.150}$$

Because of (4.84) and the trace theorem again, one has

$$\begin{aligned}\int_\Gamma \mathcal{Q}^5 \cdot \partial_t u + \int_\Omega q \partial_t \mathcal{Q}^3 + \int_\Omega \mathcal{Q}^2 \cdot \partial_t u &\lesssim (\|\mathcal{Q}^5\|_{L^2(\Gamma)} + \|\mathcal{Q}^2\|_{L^2(\Omega)}) \|\partial_t u\|_{H^1(\Omega)} \\ &\quad + \|\partial_t \mathcal{Q}^3\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)} \\ &\lesssim \mathcal{E}_f^3.\end{aligned} \tag{4.151}$$

Combining (4.149), (4.150) and (4.151), we obtain from (4.148) that

$$\begin{aligned}\|\partial_t u\|_{L^2(\Omega)}^2 &\lesssim \|\partial_t u\|_{L^2(\Omega)} (\|\eta\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)} + \|\zeta\|_{L^2(\Omega)}) \\ &\quad + \|\nabla \partial_t u\|_{L^2(\Omega)} (\|\eta\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)}) + \mathcal{E}_f^3.\end{aligned}$$

Using Young's inequality, we get further that for any $\nu > 0$,

$$\|\partial_t u\|_{L^2(\Omega)}^2 \lesssim \nu \|\partial_t u\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u\|_{L^2(\Omega)}^2 + (1 + \nu^{-1}) \mathscr{W}. \tag{4.152}$$

Let $\nu > 0$ be sufficiently small, the inequality (4.146) follows from (4.152).

To prove (4.147), we differentiate $(2.11)_{2,5}$ with respect to t and then eliminate the terms $\partial_t \zeta, \partial_t \eta$ by using $(2.11)_{1,4}$ to deduce that

$$\begin{cases} \rho_0 \partial_t^2 u + \nabla \partial_t q - \mu \Delta \partial_t u - g\rho_0' u_3 e_3 = \partial_t \mathcal{Q}^2 - g\mathcal{Q}^1 e_3 & \text{in } \Omega, \\ \mathrm{div} \partial_t u = \partial_t \mathcal{Q}^3 & \text{in } \Omega, \\ (\partial_t q \mathrm{Id} - \mu \mathbb{S}\partial_t u) e_3 = g\rho_+ u e_3 + g\rho_+ \mathcal{Q}^4 e_3 + \partial_t \mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{4.153}$$

Multiplying both sides of $(4.153)_1$ by $\partial_t^2 u$, we obtain that

$$\int_\Omega \rho_0 |\partial_t^2 u|^2 + \int_\Omega (\nabla \partial_t q - \mu \Delta \partial_t u) \cdot \partial_t^2 u - \int_\Omega g\rho_0' u_3 \partial_t^2 u_3 = \int_\Omega (\partial_t \mathcal{Q}^2 - g\mathcal{Q}^1 e_3) \cdot \partial_t^2 u. \tag{4.154}$$

Using the integration by parts, we have that

$$\int_\Omega \rho_0|\partial_t^2 u|^2 = -\int_\Gamma (\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3\cdot\partial_t u + \int_\Omega \partial_t q\mathrm{div}\partial_t u - \frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u \\ - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 + \int_\Omega (\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u.$$

Substituting $(4.153)_{2,3}$ into the resulting equality yields

$$\int_\Omega \rho_0|\partial_t^2 u|^2 = -\int_\Gamma g\rho_+ u_3\partial_t^2 u_3 - \int_\Gamma (g\rho_+\mathcal{Q}^4 e_3 + \partial_t\mathcal{Q}^5)\cdot\partial_t^2 u + \int_\Omega \partial_t q\partial_t\mathcal{Q}^3 \\ - \frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 + \int_\Omega (\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u. \tag{4.155}$$

We estimate each integral in the r.h.s of (4.155). For the first integral, we use Cauchy-Schwarz's inequality and the trace theorem to have

$$-\int_\Gamma g\rho_+ u_3\partial_t^2 u_3 \lesssim \|u_3\|_{L^2(\Gamma)}\|\partial_t^2 u_3\|_{L^2(\Gamma)} \lesssim \|u_3\|_{H^1(\Omega)}\|\partial_t^2 u_3\|_{H^1(\Omega)}. \tag{4.156}$$

For the fourth and fifth integral, we bound as

$$-\frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 \lesssim \|\partial_t u\|_{H^1(\Omega)}\|\partial_t^2 u\|_{H^1(\Omega)} + \|u_3\|_{L^2(\Omega)}\|\partial_t^2 u_3\|_{L^2(\Omega)}. \tag{4.157}$$

For the other integrals, we use (4.84) to obtain

$$\begin{aligned}&\int_\Omega (\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u + \int_\Omega \partial_t q\partial_t\mathcal{Q}^3 \\ &\lesssim (\|\partial_t\mathcal{Q}^2\|_{L^2(\Omega)} + \|\mathcal{Q}^1\|_{L^2(\Omega)})\|\partial_t^2 u\|_{L^2(\Omega)} + \|\partial_t q\|_{L^2(\Omega)}\|\partial_t\mathcal{Q}^3\|_{L^2(\Omega)} \\ &\lesssim \mathcal{E}_f^3,\end{aligned} \tag{4.158}$$

and use the trace theorem also to obtain

$$\begin{aligned}\int_\Gamma (g\rho_+\mathcal{Q}^4 e_3 + \partial_t\mathcal{Q}^5)\cdot\partial_t^2 u &\lesssim (\|\mathcal{Q}^4\|_{L^2(\Gamma)} + \|\partial_t\mathcal{Q}^5\|_{L^2(\Gamma)})\|\partial_t^2 u\|_{H^1(\Omega)} \\ &\lesssim \mathcal{E}_f^2\|\partial_t^2 u\|_{H^1(\Omega)}.\end{aligned} \tag{4.159}$$

Substituting (4.156), (4.157), (4.158) and (4.159) into (4.155) yields

$$\begin{aligned}\|\partial_t^2 u\|_{L^2(\Omega)}^2 &\lesssim (\|u\|_{H^1(\Omega)} + \|\partial_t u\|_{H^1(\Omega)} + \mathcal{E}_f^2)\|\partial_t^2 u\|_{H^1(\Omega)} + \mathcal{E}_f^3 \\ &\lesssim (\|u\|_{H^1(\Omega)} + \|\partial_t u\|_{H^1(\Omega)} + \mathcal{E}_f^2)(\|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \mathcal{E}_f).\end{aligned} \tag{4.160}$$

Combining (4.160) and (4.146) gives us that

$$\|\partial_t^2 u\|_{L^2(\Omega)}^2 \lesssim (\|\nabla\partial_t u\|_{L^2(\Omega)} + \mathscr{W})(\|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \mathscr{W}). \tag{4.161}$$

Thanks to Cauchy-Schwarz's inequality, the resulting inequality (4.161) implies (4.147). Lemma 4.8 is shown. □

We are able to show Proposition 4.10.

Proof of Proposition 4.10. We apply the elliptic estimate (C.4) to

$$\begin{cases} -\mu\Delta\partial_t u + \nabla\partial_t q = -\rho_0\partial_t^2 u + g(\rho_0' u_3 - \mathcal{Q}^1) + \partial_t\mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div}\partial_t u = \partial_t\mathcal{Q}^3 & \text{in } \Omega, \\ \partial_t u = \partial_t u & \text{on } \Gamma, \end{cases}$$

to have that

$$\|\partial_t u\|^2_{H^3(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} \lesssim \|\partial_t^2 u\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^1(\Omega)} + \|(\mathcal{Q}^1, \partial_t \mathcal{Q}^2)\|^2_{H^1(\Omega)} + \|\partial_t \mathcal{Q}^3\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{H^{5/2}(\Gamma)}.$$

This yields

$$\|\partial_t u\|^2_{H^3(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} \lesssim \|\partial_t^2 u\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^1(\Omega)} + \|\partial_t u\|^2_{H^{5/2}(\Gamma)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2), \tag{4.162}$$

due to (4.85) also. It follows from (4.145) and the trace theorem that

$$\begin{aligned} \|\partial_t u\|^2_{H^{5/2}(\Gamma)} &\lesssim \|\partial_t u\|^2_{H^{1/2}(\Gamma)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|^2_{H^{1/2}(\Gamma)} \\ &\lesssim \|\partial_t u\|^2_{H^1(\Omega)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|^2_{H^1(\Omega)}. \end{aligned} \tag{4.163}$$

Combining (4.162) and (4.163) gives us that

$$\|\partial_t u\|^2_{H^3(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} \lesssim \|(u, \partial_t u, \partial_t^2 u)\|^2_{H^1(\Omega)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|^2_{H^1(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{4.164}$$

Thanks to the interpolation inequality (C.7), we get that, for $\nu > 0$,

$$\|\partial_h^\beta \partial_t u\|^2_{L^2(\Omega)} \lesssim \|\partial_t u\|^2_{H^2(\Omega)} \lesssim \nu \|\partial_t u\|^2_{H^3(\Omega)} + \nu^{-2} \|\partial_t u\|^2_{L^2(\Omega)}.$$

Hence, it follows from (4.164) that

$$\|\partial_t u\|^2_{H^3(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} \lesssim \|(u, \partial_t u, \partial_t^2 u)\|^2_{H^1(\Omega)} + \nu \|\partial_t u\|^2_{H^3(\Omega)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|^2_{L^2(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2).$$

Let $\nu > 0$ be sufficiently small, one has

$$\|\partial_t u\|^2_{H^3(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} \lesssim \|(u, \partial_t u, \partial_t^2 u)\|^2_{H^1(\Omega)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|^2_{L^2(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{4.165}$$

Meanwhile, applying the elliptic estimate (C.4) to

$$\begin{cases} -\mu \Delta u + \nabla q = -\rho_0 \partial_t u - g\zeta e_3 + \mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}^3 & \text{in } \Omega, \\ u = u & \text{on } \Gamma, \end{cases}$$

we have

$$\|u\|^2_{H^5(\Omega)} + \|q\|^2_{H^4(\Omega)} \lesssim \|\partial_t u\|^2_{H^3(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \|\mathcal{Q}^2\|^2_{H^3(\Omega)} + \|\mathcal{Q}^3\|^2_{H^4(\Omega)} + \|u\|^2_{H^{9/2}(\Gamma)}.$$

Using (4.165) and (4.85), we further obtain

$$\|u\|^2_{H^5(\Omega)} + \|q\|^2_{H^4(\Omega)} \lesssim \|(u, \partial_t u, \partial_t^2 u)\|^2_{H^1(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \|u\|^2_{H^{9/2}(\Gamma)} + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|^2_{L^2(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{4.166}$$

Using (4.145) again and the trace theorem, we obtain that

$$\begin{aligned}\|u\|^2_{H^{9/2}(\Gamma)} &\lesssim \|u\|^2_{H^{1/2}(\Gamma)} + \sum_{\beta\in\mathbb{N}^2,|\beta|=4} \|\partial_h^\beta u\|^2_{H^{1/2}(\Gamma)}\\ &\lesssim \|u\|^2_{H^1(\Omega)} + \sum_{\beta\in\mathbb{N}^2,|\beta|=4} \|\partial_h^\beta u\|^2_{H^1(\Omega)}.\end{aligned} \tag{4.167}$$

Notice from (C.7) again that, for $\nu > 0$,

$$\|\partial_h^\beta u\|^2_{L^2(\Omega)} \lesssim \|u\|^2_{H^4(\Omega)} \lesssim \nu\|u\|^2_{H^5(\Omega)} + \nu^{-4}\|u\|^2_{L^2(\Omega)}. \tag{4.168}$$

In view of (4.167) and (4.168), we deduce from (4.166) that

$$\begin{aligned}\|u\|^2_{H^5(\Omega)} + \|q\|^2_{H^4(\Omega)} \lesssim \|(u,\partial_t u,\partial_t^2 u)\|^2_{H^1(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \nu\|u\|^2_{H^5(\Omega)}\\ + \sum_{\beta\in\mathbb{N}^2,|\beta|=4} \|\nabla\partial_h^\beta u\|^2_{L^2(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2).\end{aligned} \tag{4.169}$$

Let $\nu > 0$ be sufficiently small, the inequality (4.169) implies that

$$\begin{aligned}\|u\|^2_{H^5(\Omega)} + \|q\|^2_{H^4(\Omega)} \lesssim \|(u,\partial_t u,\partial_t^2 u)\|^2_{H^1(\Omega)} + \|\zeta\|^2_{H^3(\Omega)}\\ + \sum_{\beta\in\mathbb{N}^2,|\beta|=4} \|\nabla\partial_h^\beta u\|^2_{L^2(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2).\end{aligned} \tag{4.170}$$

Keep in mind the definition of $\mathcal{D}_h$ (4.143), we obtain from (4.165) and (4.170) that

$$\begin{aligned}&\|u\|^2_{H^5(\Omega)} + \|\partial_t u\|^2_{H^3(\Omega)} + \|q\|^2_{H^4(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)}\\ &\quad\lesssim \mathcal{D}_h^2 + \|(u,\partial_t u,\partial_t^2 u)\|^2_{H^1(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2).\end{aligned}$$

Thanks to the definition of $\mathcal{D}_f$ (4.2), that implies

$$\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \|(u,\partial_t u,\partial_t^2 u)\|^2_{L^2(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{4.171}$$

Thanks to Lemma 4.8, we deduce from (4.171) that

$$\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \|u\|^2_{H^1(\Omega)} + \|\zeta\|^2_{H^3(\Omega)} + \|\eta\|^2_{L^2(\Gamma)} + \mathcal{E}_f^3 + \mathcal{E}_f^2\mathcal{D}_f^2.$$

We continue using (C.7) to further have

$$\begin{aligned}\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \varepsilon^3(\|u\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^4(\Omega)}) + \varepsilon^{-3}\|u\|^2_{L^2(\Omega)} + \varepsilon^{-9}\|\zeta\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}\\ + \mathcal{E}_f^3 + \mathcal{E}_f^2\mathcal{D}_f^2.\end{aligned}$$

That means

$$\mathcal{D}_f^2 \leqslant C_{13}\Big(\mathcal{D}_h^2 + \varepsilon^3\mathcal{E}_f^2 + \varepsilon^{-9}(\|(\zeta,u)\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}) + \mathcal{E}_f^3 + \mathcal{E}_f^2\mathcal{D}_f^2\Big). \tag{4.172}$$

Restricting further

$$C_{13}\delta_0^2 \leqslant \frac{1}{2}, \tag{4.173}$$

we obtain (4.144) from (4.172). Proposition 4.10 is proven. □

4.6. **Proof of Proposition 4.2.** Let us denote

$$\begin{cases} C_{1,\varepsilon} = (C_1 + C_2 + C_3 + C_4)\varepsilon^3 + C_5 + C_6 + C_7 + C_8 + C_9,\\ C_{14} = \sum_{i=1}^9 C_i,\\ C_{2,\varepsilon} = (C_1 + C_2 + C_3 + C_4)\varepsilon + (C_7 + C_8 + C_9)\varepsilon^3,\\ C_{3,\varepsilon} = C_5 + C_6 + (C_7 + C_8 + C_9)\varepsilon^{-27}.\end{cases}$$

We obtain from Propositions 4.3, 4.4, 4.5, 4.6, 4.7 that

$$\begin{aligned}
&\varepsilon^2\Big(\|\eta(t)\|^2_{H^4(\Gamma)}+\|\partial_t\eta(t)\|^2_{H^2(\Gamma)}+\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}+\|\eta(t)\|^2_{H^{9/2}(\Gamma)}\Big)\\
&\quad+\|\zeta(t)\|^2_{H^4(\Omega)}+\|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla(u,\partial_t u,\partial_t^2 u)(s)\|^2_{L^2(\Omega)}ds\\
&\quad+\sum_{\beta\in\mathbb{N}^2,1\leqslant|\beta|\leqslant 4}\int_0^t\|\nabla\partial_h^\beta u(s)\|^2_{L^2(\Omega)}ds+\sum_{\beta\in\mathbb{N}^2,1\leqslant|\beta|\leqslant 2}\int_0^t\|\nabla\partial_t\partial_h^\beta u(s)\|^2_{L^2(\Omega)}ds\\
&\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0)+C_6\mathcal{E}_f^3(t)+C_{14}\varepsilon^3\int_0^t\mathcal{E}_f^2(s)ds+C_{2,\varepsilon}\int_0^t\mathcal{D}_f^2(s)ds\\
&\quad+C_{3,\varepsilon}\int_0^t(\|(u,\zeta)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds+C_{1,\varepsilon}\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds.
\end{aligned}\tag{4.174}$$

Keep in mind the definition of $\mathcal{D}_h$ (4.143). It follows from (4.174) that

$$\begin{aligned}
&\varepsilon^2\Big(\|\eta(t)\|^2_{H^4(\Gamma)}+\|\partial_t\eta(t)\|^2_{H^2(\Gamma)}+\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}+\|\eta(t)\|^2_{H^{9/2}(\Gamma)}\Big)\\
&\quad+\|\zeta(t)\|^2_{H^4(\Omega)}+\|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)}+\int_0^t\mathcal{D}_h^2(s)ds\\
&\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0)+C_6\mathcal{E}_f^3(t)+C_{14}\varepsilon^3\int_0^t\mathcal{E}_f^2(s)ds+C_{2,\varepsilon}\int_0^t\mathcal{D}_f^2(s)ds\\
&\quad+C_{3,\varepsilon}\int_0^t(\|(u,\zeta)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds+C_{1,\varepsilon}\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds.
\end{aligned}\tag{4.175}$$

Chaining (4.175) with (4.144) in Proposition 4.10, we get that

$$\begin{aligned}
&\varepsilon^2\Big(\|\eta(t)\|^2_{H^4(\Gamma)}+\|\partial_t\eta(t)\|^2_{H^2(\Gamma)}+\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}+\|\eta(t)\|^2_{H^{9/2}(\Gamma)}+\|\zeta(t)\|^2_{H^4(\Omega)}\Big)\\
&\quad+\|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)}+\frac{1}{C_{12}}\int_0^t\mathcal{D}_f^2(s)ds\\
&\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0)+C_6\mathcal{E}_f^3(t)+(C_{14}+1)\varepsilon^3\int_0^t\mathcal{E}_f^2(s)ds+C_{2,\varepsilon}\int_0^t\mathcal{D}_f^2(s)ds\\
&\quad+C_{4,\varepsilon}\int_0^t(\|(u,\zeta)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds+(C_{1,\varepsilon}+1)\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds,
\end{aligned}\tag{4.176}$$

where $C_{4,\varepsilon}=C_{3,\varepsilon}+\varepsilon^{-9}$. Let $0<\varepsilon\leqslant 1$ be sufficiently small such that

$$C_{2,\varepsilon}\leqslant\frac{1}{2C_{12}}.$$

So that, the inequality (4.176) implies

$$\begin{aligned}
&\varepsilon^2\Big(\|\eta(t)\|^2_{H^4(\Gamma)}+\|\partial_t\eta(t)\|^2_{H^2(\Gamma)}+\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}+\|\eta(t)\|^2_{H^{9/2}(\Gamma)}+\|\zeta(t)\|^2_{H^4(\Omega)}\Big)\\
&\quad+\|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)}+\frac{1}{2C_{12}}\int_0^t\mathcal{D}_f^2(s)ds\\
&\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0)+C_6\mathcal{E}_f^3(t)+(C_{14}+1)\varepsilon^3\int_0^t\mathcal{E}_f^2(s)ds+(C_{1,\varepsilon}+1)\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\\
&\quad+C_{4,\varepsilon}\int_0^t(\|(u,\zeta)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds.
\end{aligned}\tag{4.177}$$

By dividing both sides of (4.177) by ε^2, we have

$$\begin{aligned}
&\|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\eta(t)\|^2_{H^4(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} \\
&\qquad + \|\zeta(t)\|^2_{H^4(\Omega)} + \|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big) \\
&\qquad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t).
\end{aligned} \tag{4.178}$$

Combining (4.138) and (4.178), one has

$$\begin{aligned}
&\varepsilon^{1/4}\Big(\|u(t)\|^2_{H^4(\Omega)} + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t u(t)\|^2_{H^2(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\Big) \\
&\quad + \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\eta(t)\|^2_{H^4(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} \\
&\quad + \|\zeta(t)\|^2_{H^4(\Omega)} + \|u(t)\|^2_{L^2(\Omega)} + \|\partial_t^2 u(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_{11}\varepsilon^{1/4}\Big(\|(\partial_t^2 u, u_3)(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{H^2(\Omega)} + \|\eta(t)\|^2_{H^{5/2}(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)}\Big) \\
&\quad + C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big) \\
&\quad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t) + C_{11}\varepsilon^{1/4}\mathcal{E}_f^4(t).
\end{aligned} \tag{4.179}$$

Let us refine ε so that

$$C_{11}\varepsilon^{1/4} \leqslant \frac{1}{2},$$

it follows from (4.179) that

$$\begin{aligned}
&\varepsilon^{1/4}\Big(\|u(t)\|^2_{H^4(\Omega)} + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t u(t)\|^2_{H^2(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\Big) \\
&\quad + \frac{1}{2}\Big(\|(\partial_t^2 u, u)(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)}\Big) \\
&\quad + \|\eta\|^2_{H^4(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big) \\
&\quad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t) + C_{11}\varepsilon^{1/4}\mathcal{E}_f^4(t).
\end{aligned} \tag{4.180}$$

Dividing both sides of (4.180) by $\varepsilon^{1/4}$, one has

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)} + \|\partial_t^j\eta(t)\|^2_{H^{4-2j}(\Gamma)}) + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta\|^2_{H^{9/2}(\Gamma)} \\
&\qquad + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_{16}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0) + \varepsilon^{3/4}\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-9/4}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big) \\
&\qquad + C_{16}\varepsilon^{-117/4}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{16}\varepsilon^{-9/4}\mathcal{E}_f^3(t).
\end{aligned} \tag{4.181}$$

Combining (4.181) and (4.135) in Proposition 4.8, we obtain

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)}+\|\partial_t^j\eta(t)\|^2_{H^{4-2j}(\Gamma)})+\|\zeta(t)\|^2_{H^4(\Omega)}+\|\eta\|^2_{H^{9/2}(\Gamma)}\\
&\quad+\|q(t)\|^2_{H^3(\Omega)}+\|\partial_t q(t)\|^2_{H^1(\Omega)}+\int_0^t\mathcal{D}_f^2(s)ds+\varepsilon^{1/4}(\|\partial_t\zeta\|^2_{H^2(\Omega)}+\|\partial_t^2\zeta\|^2_{L^2(\Omega)})\\
&\leqslant C_{16}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0)+\varepsilon^{3/4}\int_0^t\mathcal{E}_f^2(s)ds+\varepsilon^{-9/4}\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\
&\quad+C_{16}\varepsilon^{-117/4}\int_0^t(\|(\zeta,u)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds\\
&\quad+C_{16}\varepsilon^{-9/4}\mathcal{E}_f^3(t)+C_{10}\varepsilon^{1/4}(\|u_3\|^2_{H^2(\Omega)}+\|\partial_t u_3\|^2_{L^2(\Omega)}+\mathcal{E}_f^4(t)).
\end{aligned}\tag{4.182}$$

We continue refining ε so that

$$C_{10}\varepsilon^{1/4}\leqslant\frac{1}{2}.$$

It follows from (4.182) that

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)}+\|\partial_t^j\eta(t)\|^2_{H^{4-2j}(\Gamma)})+\|\zeta(t)\|^2_{H^4(\Omega)}+\|\eta\|^2_{H^{9/2}(\Gamma)}\\
&\quad+\|q(t)\|^2_{H^3(\Omega)}+\|\partial_t q(t)\|^2_{H^1(\Omega)}+\int_0^t\mathcal{D}_f^2(s)ds+\varepsilon^{1/4}(\|\partial_t\zeta\|^2_{H^2(\Omega)}+\|\partial_t^2\zeta\|^2_{L^2(\Omega)})\\
&\leqslant C_{17}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0)+\varepsilon^{3/4}\int_0^t\mathcal{E}_f^2(s)ds+\varepsilon^{-9/4}\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\
&\quad+C_{17}\varepsilon^{-117/4}\int_0^t(\|(\zeta,u)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds+C_{17}(\varepsilon^{-9/4}\mathcal{E}_f^3(t)+\varepsilon^{1/4}\mathcal{E}_f^4(t)).
\end{aligned}\tag{4.183}$$

Let us recall the definition of $\mathcal{E}_f$ (4.1) and divide both sides of (4.183) by $\varepsilon^{1/4}$ to deduce

$$\begin{aligned}
&\mathcal{E}_f^2(t)+\int_0^t\mathcal{D}_f^2(s)ds\\
&\leqslant C_{18}\Big(\varepsilon^{-5/2}\mathcal{E}_f^2(0)+\varepsilon^{1/2}\int_0^t\mathcal{E}_f^2(s)ds+\varepsilon^{-5/2}\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\
&\quad+C_{18}\varepsilon^{-59/2}\int_0^t(\|(\zeta,u)(s)\|^2_{L^2(\Omega)}+\|\eta(s)\|^2_{L^2(\Gamma)})ds+C_{18}\varepsilon^{-5/2}\mathcal{E}_f^3(t).
\end{aligned}\tag{4.184}$$

Switching $\varepsilon^{1/2}$ by ε in (4.181), one has (4.6). Proof of Proposition 4.2 is finished.

5. Nonlinear instability

Note again that, we compactly write $U=(\zeta,u,q,\eta)$ throughout this paper.

Thanks to Proposition 3.7, we will consider a sequence of approximate solutions $e^{\lambda_n(\mathbf{k})}V_n(\mathbf{k},x)$ to the nonlinear equations (2.11), that are solutions to the linearized ones (2.13). Let us fix a $\mathbf{k}=\mathbf{k}_0\in S_\Lambda$ such that (2.30) holds. We recall (2.31),

$$U^M(t,x):=\sum_{j=1}^{M}\mathsf{c}_j e^{\lambda_j(\mathbf{k}_0)t}V_j(\mathbf{k}_0,x)$$

and require that the coefficients c_j satisfying (2.32)-(2.33). Due to the compatibility conditions (4.4), we cannot set $U^M(0,x)$ as the initial data for the nonlinear

equations (2.11). With the help of an abstract argument in [11, Section 5C], we obtain the modified initial data $U_0^{\delta,M}(x)$.

Proposition 5.1. *There exist a number $\delta_0 > 0$ and a family of initial data*

$$U_0^{\delta,M}(x) = \delta U^M(0,x) + \delta^2 U_\star^{\delta,M}(x) \tag{5.1}$$

for $\delta \in (0,\delta_0)$ such that

(1) *$U_0^{\delta,M}$ satisfies the compatibility conditions (4.4) and $\mathcal{E}_f(U_\star^{\delta,M}) \leqslant C_M^\star < \infty$ with $C_M^\star$ being independent of δ,*
(2) *the nonlinear equations (2.11) with the above initial data $U_0^{\delta,M}$ has a unique solution $U^{\delta,M}$ satisfying that $\sup_{0\leqslant t<T^{\max}} \mathcal{E}_f(U^{\delta,M}(t)) < \infty$.*

5.1. **The difference functions.** Set

$$U^d(t,x) = U^{\delta,M}(t,x) - \delta U^M(t,x).$$

Since $U^{\delta,M}$ solves the nonlinear equations (2.11) and U^M solves the linearized equations (2.13), we obtain that U^d satisfies

$$\begin{cases} \partial_t \zeta^d + \rho_0' u_3^d = \mathcal{Q}^1(U^{\delta,M}) & \text{in } \Omega, \\ \rho_0 \partial_t u^d - \mu\Delta u^d + \nabla q^d + g\zeta^d e_3 = \mathcal{Q}^2(U^{\delta,M}) & \text{in } \Omega, \\ \mathrm{div} u^d = \mathcal{Q}^3(U^{\delta,M}) & \text{in } \Omega, \\ \partial_t \eta^d = u_3^d + \mathcal{Q}^4(U^{\delta,M}) & \text{on } \Gamma, \\ ((q^d - g\rho_+\eta^d)\mathrm{Id} - \mu\mathbb{S}u^d)e_3 = \mathcal{Q}^5(U^{\delta,M}) & \text{on } \Gamma. \end{cases} \tag{5.2}$$

The initial condition for (5.2) is

$$U^d(0) = (\zeta^d, u^d, \eta^d, q^d)(0) = \delta^2 U_\star^{\delta,M}. \tag{5.3}$$

Let $\|U\|_{\mathcal{E}_f} := \mathcal{E}_f(U)$, which is defined as in (4.1). Let $F_M(t) = \sum_{j=j_m}^M |\mathsf{c}_j| e^{\lambda_j t}$ and $0 < \epsilon_0 \ll 1$ be fixed later (5.39). There exists a unique T^δ such that $\delta F_M(T^\delta) = \epsilon_0$. Let

$$C_{19} = \|U^M(0)\|_{\mathcal{E}_f}, \quad C_{20} = \sqrt{\|(\zeta^M, u^M)(0)\|_{L^2(\Omega)}^2 + \|\eta^M(0)\|_{L^2(\Gamma)}^2}. \tag{5.4}$$

We define

$$\begin{aligned} T^\star &:= \sup\Big\{ t \in (0,T^{\max}) | \|U^{\delta,M}(t)\|_{\mathcal{E}_f} \leqslant 2C_{19}\delta_0 \}, \\ T^{\star\star} &:= \sup\{ t \in (0,T^{\max}) | \|(\zeta^{\delta,M}, u^{\delta,M})(t)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(t)\|_{L^2(\Gamma)} \leqslant 2C_{20}\delta F_M(t) \Big\}. \end{aligned} \tag{5.5}$$

Note that

$$\|U^{\delta,M}(0)\|_{\mathcal{E}_f} \leqslant \delta \|U^M(0)\|_{\mathcal{E}_f} + \|U^d(0)\|_{\mathcal{E}_f} \leqslant C_{19}\delta + C_M^\star \delta^2 < 2C_{19}\delta_0,$$

we then have $T^\star > 0$. Similarly, we have $T^{\star\star} > 0$.

The aim of this part is to derive the bound in time of $\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)} + \|\eta^d(t)\|_{L^2(\Gamma)}$ in the following proposition.

Proposition 5.2. *For all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$, there holds*

$$\begin{aligned} &\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)}^2 + \|\eta^d(t)\|_{L^2(\Gamma)}^2 \\ &\quad \leqslant C_{21}\delta^3 \Big(\sum_{j=j_m}^N |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \max_{N+1\leqslant j\leqslant M} |\mathsf{c}_j| e^{\frac{2}{3}\Lambda t} \Big)^3. \end{aligned} \tag{5.6}$$

In order to prove Proposition 5.2, we need the following bound in time of $\|U^{\delta,M}(t)\|_{\mathcal{E}_f}$.

Proposition 5.3. *For all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, there holds*

$$\|U^{\delta,M}(t)\|_{\mathcal{E}_f} \leqslant C_{22}\delta F_M(t) \quad \textit{for all } t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}. \tag{5.7}$$

Proof. We fix a sufficiently small constant ε such that

$$C_0\varepsilon \leqslant \frac{\lambda_M}{4} \tag{5.8}$$

and Proposition 4.2 holds. Hence, it follows from (4.6) that

$$\begin{aligned}\mathcal{E}_f^2(t) + \int_0^t \mathcal{D}_f^2(s)ds \leqslant \frac{\lambda_M}{4}\int_0^t \mathcal{E}_f^2(s)ds + C_{\lambda_M}\Big(\mathcal{E}_f^2(0) + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds + \mathcal{E}_f^3(t)\Big)\\ + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds.\end{aligned} \tag{5.9}$$

Refining also δ_0, we get

$$C_{\lambda_M}\delta_0 \leqslant \frac{1}{2} \quad \text{and} \quad C_{\lambda_M}\delta_0 \leqslant \frac{\lambda_M}{4}, \tag{5.10}$$

one thus has

$$\begin{aligned}\frac{1}{2}\mathcal{E}_f^2(t) + \frac{1}{2}\int_0^t \mathcal{D}_f^2(s)ds &\leqslant C_{\lambda_M}\mathcal{E}_f^2(0) + \Big(\frac{\lambda_M}{4} + \delta C_{\lambda_M}\Big)\int_0^t \mathcal{E}_f^2(s)ds\\ &\quad + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds\\ &\leqslant C_{\lambda_M}\mathcal{E}_f^2(0) + \frac{\lambda_M}{2}\int_0^t \mathcal{E}_f^2(s)ds\\ &\quad + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds.\end{aligned} \tag{5.11}$$

Consequently, for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$,

$$\begin{aligned}\|U^{\delta,M}(t)\|^2_{\mathcal{E}_f} &\leqslant 2C_{\lambda_M}\|U^{\delta,M}(0)\|^2_{\mathcal{E}_f} + \lambda_M\int_0^t \|U^{\delta,M}(s)\|^2_{\mathcal{E}_f}ds\\ &\quad + 2C_{\lambda_M}\int_0^t (\|(\zeta^{\delta,M},u^{\delta,M})(s)\|^2_{L^2(\Omega)} + \|\eta^{\delta,M}(s)\|^2_{L^2(\Gamma)})ds\\ &\leqslant \lambda_M\int_0^t \|U^{\delta,M}(s)\|^2_{\mathcal{E}_f}ds + C_{23}\delta^2F_M^2(t).\end{aligned}$$

Applying Gronwall's inequality, the resulting inequality tells us that

$$\|U^{\delta,M}(t)\|^2_{\mathcal{E}_f} \leqslant C_{23}\Big(\delta^2F_M^2(t) + \delta^2\int_0^t e^{\lambda_M(t-s)}F_M^2(s)ds\Big). \tag{5.12}$$

Note that $\lambda_M < \lambda_j$ for all $1 \leqslant j \leqslant M-1$, we have

$$\begin{aligned}\int_0^t e^{\lambda_M(t-s)}F_M^2(s)ds &\leqslant M^2\sum_{j=j_m}^M \int_0^t e^{\lambda_M(t-s)}|\mathsf{c}_j|^2e^{2\lambda_j s}ds\\ &\leqslant M^2e^{\lambda_M t}\sum_{j=j_m}^M |\mathsf{c}_j|^2\frac{e^{(2\lambda_j-\lambda_M)t}}{2\lambda_j-\lambda_M}.\end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.12), this yields (5.7). We deduce Proposition 5.3. □

We now prove Proposition 5.2.

Proof of Proposition 5.2. Differentiating $(5.2)_{2,5}$ with respect to t and then eliminating the terms $\partial_t\zeta^d, \partial_t\eta^d$ by using $(5.2)_{1,4}$, we deduce from (5.2) that

$$\begin{cases} \rho_0\partial_t^2 u^d + \nabla\partial_t q^d - \mu\Delta\partial_t u^d - g\rho_0' u_3^d e_3 = \partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3 & \text{in } \Omega,\\ \mathrm{div}\partial_t u^d = \partial_t\mathcal{Q}^3(U^{\delta,M}) & \text{in } \Omega,\\ (\partial_t q^d \mathrm{Id} - \mu\mathbb{S}\partial_t u^d)e_3 = g\rho_+ u^d e_3 + g\rho_+\mathcal{Q}^4(U^{\delta,M})e_3 + \partial_t\mathcal{Q}^5(U^{\delta,M}) & \text{on } \Gamma. \end{cases} \tag{5.14}$$

Multiplying both sides of $(5.14)_1$ by $\partial_t u^d$, we obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_t u^d|^2 + \int_\Omega(\nabla\partial_t q^d - \mu\Delta\partial_t u^d)\cdot\partial_t u^d - \int_\Omega g\rho_0' u_3^d\partial_t u_3^d\\ &= \int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d. \end{aligned} \tag{5.15}$$

Using the integration by parts, we deduce from the resulting equality that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u^d|^2 - \int_\Omega g\rho_0'|u_3^d|^2\Big) + \int_\Gamma(\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u^d)e_3\cdot\partial_t u^d\\ &= \int_\Omega \partial_t q^d \mathrm{div}\partial_t u^d - \frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u^d|^2 + \int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d. \end{aligned}$$

Substituting $(5.14)_{2,3}$ into (5.15), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u^d|^2 - \int_\Omega g\rho_0'|u_3^d|^2\Big) + \int_\Gamma g\rho_+\partial_t u_3^d\partial_t u_3^d\\ &\qquad + \int_\Gamma(\partial_t\mathcal{Q}^5(U^{\delta,M}) + g\rho_+\mathcal{Q}^4(U^{\delta,M})e_3)\cdot\partial_t u^d\\ &= \int_\Omega \partial_t q^d\partial_t\mathcal{Q}^3(U^{\delta,M}) - \frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u^d|^2 + \int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d. \end{aligned}$$

This yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u^d|^2 - \int_\Omega g\rho_0'|u_3^d|^2 + \int_\Gamma g\rho_+|u_3^d|^2\Big) + \frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t u^d|^2\\ &= \int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d + \int_\Omega \partial_t q^d\partial_t\mathcal{Q}^3(U^{\delta,M})\\ &\qquad - \int_\Gamma(\partial_t\mathcal{Q}^5(U^{\delta,M}) + g\rho_+\mathcal{Q}^4(U^{\delta,M})e_3)\cdot\partial_t u^d, \end{aligned} \tag{5.16}$$

Note that

$$\begin{aligned} &\int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d\\ &\quad\lesssim (\|\partial_t\mathcal{Q}^2(U^{\delta,M})\|_{L^2(\Omega)} + \|\mathcal{Q}^1(U^{\delta,M})\|_{L^2(\Omega)})(\|\partial_t u^{\delta,M}\|_{L^2(\Omega)} + \delta\|\partial_t u^M\|_{L^2(\Omega)}). \end{aligned}$$

In view of (4.84), (5.3) and the definition of U^M, we have

$$\begin{aligned} \int_\Omega(\partial_t\mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot\partial_t u^d &\lesssim \|U^{\delta,M}\|_{\mathcal{E}_f}^2(\|U^{\delta,M}\|_{\mathcal{E}_f} + \delta\|\partial_t u^M\|_{L^2(\Omega)})\\ &\lesssim \delta^3 F_M^3(t). \end{aligned} \tag{5.17}$$

Similarly, we observe

$$\begin{aligned}\int_\Omega \partial_t q^d \partial_t \mathcal{Q}^3(U^{\delta,M}) &\lesssim \|\partial_t \mathcal{Q}^3(U^{\delta,M})\|_{L^2(\Omega)}(\|\partial_t q^{\delta,M}\|_{L^2(\Omega)} + \delta\|\partial_t q^M\|_{L^2(\Omega)}) \\ &\lesssim \delta^3 F_M^3(t).\end{aligned} \tag{5.18}$$

We continue applying (4.84), (5.3) and use the trace theorem to get

$$\begin{aligned}&\int_\Gamma (\partial_t \mathcal{Q}^5(U^{\delta,M}) + g\rho_+ \mathcal{Q}^4(U^{\delta,M})e_3)\cdot \partial_t u^d \\ &\lesssim (\|\partial_t \mathcal{Q}^5(U^{\delta,M})\|_{L^2(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)})\|\partial_t u^d\|_{L^2(\Gamma)} \\ &\lesssim (\|\partial_t \mathcal{Q}^5(U^{\delta,M})\|_{H^{1/2}(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{H^{1/2}(\Gamma)})\|\partial_t u^d\|_{H^1(\Omega)} \\ &\lesssim \delta^3 F_M^3(t).\end{aligned} \tag{5.19}$$

Substituting (5.17), (5.18) and (5.19) into (5.16), we obtain that

$$\begin{aligned}&\int_\Omega \rho_0 |\partial_t u^d(t)|^2 + \int_0^t \int_\Omega \mu |\mathbb{S}\partial_t u^d(s)|^2 ds \\ &\qquad \leqslant z_1 + \int_\Omega g\rho_0' |u_3^d(t)|^2 - \int_\Gamma g\rho_+ |u_3(t)|^2 + C_{24}\delta^3 F_M^3(t),\end{aligned} \tag{5.20}$$

where

$$z_1 = \int_\Omega \rho_0 |\partial_t u^d(0)|^2 - \int_\Omega g\rho_0' |u_3^d(0)|^2 + \int_\Gamma g\rho_+ |u_3^d(0)|^2.$$

Thanks to Lemma 3.6, we deduce from (5.20) that

$$\begin{aligned}&\int_\Omega \rho_0 |\partial_t u^d(t)|^2 + \int_0^t \int_\Omega \mu |\mathbb{S}\partial_t u^d(s)|^2 ds \\ &\qquad \leqslant z_1 + \Lambda^2 \int_\Omega \rho_0 |u^d(t)|^2 + \frac{1}{2}\Lambda \int_\Omega \mu |\mathbb{S}u^d(t)|^2 + C_{23}\delta^3 F_M^3(t).\end{aligned} \tag{5.21}$$

Using Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}\int_\Omega \mu |\mathbb{S}u^d(t)|^2 &= \int_\Omega \mu |\mathbb{S}u^d(0)|^2 + 2\int_0^t \int_\Omega \mu \mathbb{S}u^d(s) : \mathbb{S}\partial_t u^d(s) ds \\ &\leqslant \int_\Omega \mu |\mathbb{S}u^d(0)|^2 + \frac{1}{\Lambda}\int_0^t \int_\Omega \mu |\mathbb{S}\partial_t u^d(s)|^2 ds + \Lambda \int_0^t \int_\Omega \mu |\mathbb{S}u^d(s)|^2 ds\end{aligned} \tag{5.22}$$

and that

$$\frac{d}{dt}\int_\Omega \rho_0 |u^d|^2 \leqslant \frac{1}{\Lambda}\int_\Omega \rho_0 |\partial_t u^d|^2 + \Lambda \int_\Omega \rho_0 |u^d|^2. \tag{5.23}$$

Three above inequalities (5.21), (5.22) and (5.23) imply that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_0 |u^d(t)|^2 + \frac{1}{2}\int_\Omega \mu |\mathbb{S}u^d(t)|^2 &\leqslant \frac{z_1}{\Lambda} + \int_\Omega \mu |\mathbb{S}u^d(0)|^2 + 2\Lambda \int_\Omega \rho_0 |u^d(t)|^2 \\ &\quad + \Lambda \int_0^t \int_\Omega \mu |\mathbb{S}u^d(s)|^2 ds + C_{24}\delta^3 F_M^3(t),\end{aligned} \tag{5.24}$$

It follows from $U^d(0) = \delta^2 U_\star^{\delta,M}$ that $z_1 \lesssim \delta^4$, this yields

$$\frac{z_1}{\Lambda} + \int_\Omega \mu |\mathbb{S}u^d(0)|^2 \lesssim \delta^4.$$

Hence, the inequality (5.24) implies

$$\frac{d}{dt}\int_\Omega \rho_0|u^d(t)|^2+\frac{1}{2}\int_\Omega \mu|\mathbb{S}u^d(t)|^2 \leqslant 2\Lambda\int_\Omega \rho_0|u^d(t)|^2+\Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2ds + C_{25}\delta^3F_M^3(t), \tag{5.25}$$

In view of Gronwall's inequality, we obtain from (5.25) that

$$\begin{aligned}\int_\Omega \rho_0|u^d(t)|^2+\frac{1}{2}\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2ds &\leqslant C_{25}\delta^3\int_0^t e^{2\Lambda(t-s)}F_M^3(s)ds\\ &\leqslant C_{26}\delta^3\int_0^t e^{2\Lambda(t-s)}F_M(3s)ds.\end{aligned} \tag{5.26}$$

Due to (2.30), we obtain for $1\leqslant j\leqslant N$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds=\frac{1}{3\lambda_j-2\Lambda}(e^{(3\lambda_j-2\Lambda)t}-1)\leqslant\frac{1}{3\lambda_j-2\Lambda}e^{(3\lambda_j-2\Lambda)t} \tag{5.27}$$

and for $j\geqslant N+1$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds=\frac{1}{3\lambda_j-2\Lambda}(e^{(3\lambda_j-2\Lambda)t}-1)\leqslant\frac{1}{2\Lambda-3\lambda_j}. \tag{5.28}$$

Substituting (5.27) and (5.28) into (5.26), we observe that if $M\leqslant N$,

$$\|u^d(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla u^d(s)\|_{L^2(\Omega)}^2ds\leqslant C_{26}\delta^3\Big(\sum_{j=j_m}^M\frac{|\mathsf{c}_j|}{3\lambda_j-2\Lambda}e^{3\lambda_jt}\Big) \tag{5.29}$$

and if $M\geqslant N+1$,

$$\begin{aligned}&\|u^d(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla u^d(s)\|_{L^2(\Omega)}^2ds\\ &\qquad\leqslant C_{26}\delta^3\Big(\sum_{j=j_m}^M\frac{|\mathsf{c}_j|}{3\lambda_j-2\Lambda}e^{3\lambda_jt}+\sum_{j=N+1}^M\frac{|\mathsf{c}_j|}{2\Lambda-3\lambda_j}e^{2\Lambda t}\Big).\end{aligned} \tag{5.30}$$

We then estimate $\|\zeta^d(t)\|_{L^2(\Omega)}$. Due to $(5.2)_1$, we obtain

$$\|\zeta^d(t)\|_{L^2(\Omega)}^2\lesssim\|\zeta^d(0)\|_{L^2(\Omega)}^2+\int_0^t(\|u_3^d(s)\|_{L^2(\Omega)}^2+\|\mathcal{Q}^1(U^{\delta,M})(s)\|_{L^2(\Omega)}^2)ds. \tag{5.31}$$

Note that $\zeta^d(0)=\delta^2\zeta_\star^{\delta,M}$ and thanks to (4.84) also, the inequality (5.31) implies

$$\begin{aligned}\|\zeta^d(t)\|_{L^2(\Omega)}^2&\lesssim\delta^4+\int_0^t(\|u_3^d(s)\|_{L^2(\Omega)}^2+\|U^{\delta,M}(s)\|_{\mathcal{E}_f}^4)ds\\ &\lesssim\delta^4+\int_0^t(\|u_3^d(s)\|_{L^2(\Omega)}^2+\delta^4F_M^4(s))ds.\end{aligned} \tag{5.32}$$

Note that $\delta F_M(t)\leqslant\varepsilon_0\leqslant 1$ for any $t\leqslant T^\delta$. Hence, we have

$$\delta^4\int_0^t F_M^4(s)ds\lesssim\delta^3\int_0^t F_M^3(s)ds\lesssim\delta^3\int_0^t F_M(3s)ds,$$

this yields

$$\delta^4\int_0^t F_M^4(s)ds\lesssim\delta^3\sum_{j=j_m}^M|\mathsf{c}_j|e^{3\lambda_jt}. \tag{5.33}$$

Combining (5.29), (5.30) and (5.33), we deduce from (5.32) that, if $M \leqslant N$,

$$\|\zeta^d(t)\|^2_{L^2(\Omega)} \leqslant C_{27}\delta^3 \sum_{j=j_m}^{M} |\mathsf{c}_j| e^{3\lambda_j t} \tag{5.34}$$

and if $M \geqslant N+1$,

$$\|\zeta^d(t)\|^2_{L^2(\Omega)} \leqslant C_{27}\delta^3 \Big(\sum_{j=j_m}^{M} |\mathsf{c}_j| e^{3\lambda_j t} + \sum_{j=N+1}^{M} |\mathsf{c}_j| e^{2\Lambda t} \Big). \tag{5.35}$$

To estimate $\|\eta^d(t)\|_{L^2(\Gamma)}$, we use $(5.2)_4$ and the trace theorem to obtain

$$\begin{aligned} \frac{d}{dt}\|\eta^d\|^2_{L^2(\Gamma)} &\leqslant \|\eta^d\|_{L^2(\Gamma)}(\|u_3^d\|_{L^2(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}) \\ &\lesssim \|\eta^d\|_{L^2(\Gamma)}(\|u_3^d\|_{H^1(\Omega)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}). \end{aligned}$$

This yields

$$\frac{d}{dt}\|\eta^d\|_{L^2(\Gamma)} \lesssim \|u_3^d\|_{H^1(\Omega)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}.$$

Thanks to (4.84) and (5.3), we further get

$$\begin{aligned} \|\eta^d(t)\|^2_{L^2(\Gamma)} &\lesssim \|\eta^d(0)\|^2_{L^2(\Gamma)} + \int_0^t (\|u_3^d(s)\|^2_{H^1(\Omega)} + \|U^{\delta,M}\|^2_{\mathcal{E}_f})ds \\ &\lesssim \delta^4 + \int_0^t (\|u_3^d(s)\|^2_{H^1(\Omega)} + \delta^4 F_M^4(s))ds. \end{aligned}$$

Using (5.29), (5.30) and (5.33) again, we have that $\|\eta^d(t)\|^2_{L^2(\Gamma)}$ is bounded above like (5.34) or (5.35). Together with (5.29), (5.30), (5.34) and (5.35), Proposition 5.2 is proven. □

5.2. **Proof of Theorem 2.2.** Since $j_m = \min\{j : 1 \leqslant j \leqslant N, \mathsf{c}_j \neq 0\}$, we have

$$\|u^M(t)\|^2_{L^2(\Omega)} = \sum_{i=j_m}^{M} \mathsf{c}_i^2 e^{2\lambda_i t}\|u_i\|^2_{L^2(\Omega)} + 2\sum_{j_m \leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i \cdot u_j. \tag{5.36}$$

It can be seen that

$$\begin{aligned} \|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} &+ 2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i \cdot u_j \\ &- |\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M} |\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}. \end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} &2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i \cdot u_j \\ &\quad\geqslant -\sum_{j_m\leqslant i<j\leqslant M} |\mathsf{c}_i||\mathsf{c}_j| e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_i\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} \\ &\quad\geqslant -e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M} |\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2. \end{aligned}$$

This implies

$$\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2 - |\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big).$$

Due to the assumption (2.33), we deduce that

$$\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} - \mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_{j_m}\|^2_{L^2(\Omega)} - \mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|^2_{L^2(\Omega)}.$$

This yields

$$\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \mathsf{c}_{j_m}^2\Big(e^{2\lambda_{j_m}t} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} - e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\Big)\|u_{j_m}\|^2_{L^2(\Omega)} + \sum_{j=j_m+1}^{M}\mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}.$$

Notice that for all $t \geqslant 0$,

$$e^{2\lambda_{j_m}t} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} - e^{(\lambda_{j_m}+\lambda_{j_m+1})t} \geqslant \frac{1}{2}e^{2\lambda_{j_m}t}.$$

Hence, we have

$$\|u^M(t)\|_{L^2(\Omega)} \geqslant C_{28}F_M(t), \tag{5.37}$$

for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$.

Let

$$\tilde{\mathsf{c}}(M) = \max_{N+1\leqslant j\leqslant M}\frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|} \geqslant 0.$$

Now, we show that

$$T^\delta \leqslant \min\{T^\star, T^{\star\star}\} \tag{5.38}$$

by choosing

$$\varepsilon_0 < \min\Big(\frac{2C_{19}\delta_0}{C_{22}}, \frac{C_{20}^2}{C_{21}(1+M\tilde{\mathsf{c}}(M))^3}, \frac{C_{28}^2}{4C_{21}(1+M\tilde{\mathsf{c}}(M))^2}\Big). \tag{5.39}$$

Indeed, if $T^\star < T^\delta$, we have from (5.7) that

$$\|U^{\delta,M}(T^\star)\|_{\mathcal{E}_f} \leqslant C_{22}\delta F_M(T^\star) < C_{22}\delta F_M(T^\delta) = C_{22}\varepsilon_0 < 2C_{19}\delta_0,$$

which contradicts the definition of $T^\star$ in (5.5). If $T^{\star\star} < T^\delta$, we obtain from the definition of C_{20} (5.4) and the inequality (5.6) that

$$\begin{aligned}
&\|(\zeta^{\delta,M}, u^{\delta,M})(T^\delta)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)}\\
&\leqslant \|(\zeta^d, u^d)(T^\delta)\|_{L^2(\Omega)} + \|\eta^d(T^\delta)\|_{L^2(\Omega)} + \delta(\|(\zeta^M, u^M)(T^\delta)\|_{L^2(\Omega)} + \|\eta^M(T^\delta)\|_{L^2(\Gamma)})\\
&\leqslant \sqrt{C_{21}}\delta^{\frac{3}{2}}\Big(\sum_{j=j_m}^{N}|\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{2\Lambda T^\delta/3}\Big)^{3/2}\\
&\qquad + C_{20}\delta F_M(T^\delta).
\end{aligned} \tag{5.40}$$

Notice from (2.30) that for $N+1 \leqslant j \leqslant M$,

$$|\mathsf{c}_j|\delta e^{\frac{2}{3}\Lambda T^\delta} < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}(\delta|\mathsf{c}_{j_m}|e^{\lambda_{j_m}T^\delta}) < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\delta F_M(T^\delta) = \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\epsilon_0.$$

Then, it follows from (5.40) that

$$\begin{aligned}&\|(\zeta^{\delta,M},u^{\delta,M})(T^\delta)\|_{L^2(\Omega)}+\|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)}\\&\quad\leqslant C_{20}\delta F_M(T^\delta)+\sqrt{C_{21}}\delta^{3/2}(1+M\tilde{\mathsf{c}}(M))^{3/2}F_M^{3/2}(T^\delta)\\&\quad\leqslant C_{20}\epsilon_0+\sqrt{C_{21}}(1+M\tilde{\mathsf{c}}(M))^{3/2}\epsilon_0^{3/2}.\end{aligned}$$

Using (5.39) again, we deduce

$$\|(\zeta^{\delta,M},u^{\delta,M})(T^\delta)\|_{L^2(\Omega)}+\|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)}<2C_{20}\epsilon_0=2C_{20}\delta F_M(T^\delta),$$

which also contradicts the definition of $T^{\star\star}$ in (5.5). So, (5.38) holds.

Once we have (5.38), it follows from (5.6) and (5.37) that

$$\begin{aligned}&\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)}\\&\geqslant\delta\|u^M(T^\delta)\|_{L^2(\Omega)}-\|u^d(T^\delta)\|_{L^2(\Omega)}\\&\geqslant C_{28}\delta F_M(T^\delta)\\&\quad-\sqrt{C_{21}}\delta^{3/2}\Big(\sum_{j=j_m}^{N}|\mathsf{c}_j|e^{\lambda_jT^\delta}+\max(0,M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{2\Lambda T^\delta/3}\Big)^{3/2}.\end{aligned}$$

Thanks to (5.39) again, we conclude that

$$\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)}\geqslant C_{28}\epsilon_0-\sqrt{C_{21}}(1+M\tilde{\mathsf{c}}(M))^{3/2}\epsilon_0^{3/2}\geqslant\frac{1}{2}C_{28}\epsilon_0>0.$$

Theorem 2.2 follows by taking δ_0 satisfying Propositions 4.2, 5.1 and the inequality (5.10), ε_0 satisfying (5.39) and $m_0=\frac{1}{2}C_{28}$.

Appendix A. Poisson extension

We define the appropriate Poisson sum that allows us to extend η defined on Γ to a function θ defined on Ω. For any $\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}$, we write

$$\hat{f}(\mathbf{k})=\int_{\mathbf{T}^2}f(x_h)\frac{e^{-i\mathbf{k}\cdot x_h}}{2\pi\sqrt{L_1L_2}}dx_h\tag{A.1}$$

and define the Poisson sum on Ω by

$$(\mathsf{p}f)(x_h,x_3)=\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}\frac{e^{i\mathbf{k}\cdot x_h}}{2\pi\sqrt{L_1L_2}}e^{|\mathbf{k}|x_3}\hat{f}(\mathbf{k})\tag{A.2}$$

We then have $\mathsf{p}:H^s(\Gamma)\to H^{s+1/2}(\Omega)$ is a bounded linear operator for $s>0$.

Lemma A.1. *For $q\in\mathbb{N}$, let H_h^q be the usual homogeneous Sobolev space of order q and $\mathsf{p}f$ be the Poisson sum of a function f in $H_h^{q-1/2}(\Gamma)$. There holds*

$$\|\nabla^q\mathsf{p}f\|_{L^2(\Omega)}^2\lesssim\|f\|_{H_h^{q-1/2}(\Gamma)}^2.\tag{A.3}$$

Proof. Thanks to Fubini's theorem and Parseval's formula, we obtain

$$\begin{aligned}\|\nabla^q\mathsf{p}f\|_{L^2(\Omega)}^2&\lesssim\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}\int_{-\infty}^0|\mathbf{k}|^{2q}|\hat{f}(\mathbf{k})|^2e^{2|\mathbf{k}|x_3}dx_3\\&\lesssim\sum_{\mathbf{k}\in L_1^{-1}\mathbb{Z}\times L_2^{-1}\mathbb{Z}}|\mathbf{k}|^{2q-1}|\hat{f}(\mathbf{k})|^2.\end{aligned}$$

The inequality (A.3) then follows. □

We extend η defined on Γ to be a function defined on Ω,

$$\theta(t,x) := (\mathsf{p}\eta)(t,x_h,x_3) \tag{A.4}$$

for all $x_h \in \mathbf{T}^2, x_3 \leqslant 0$. Lemma A.1 implies in particular that if $\eta \in H^{q-1/2}(\Gamma)$, then $\theta \in H^q(\Omega)$ for $q \geqslant 0$.

Appendix B. Nonlinear terms

The nonlinear terms $\mathcal{Q}^i (1 \leqslant i \leqslant 5)$ in (2.12) are presented by that

$$\begin{aligned}\mathcal{Q}^1 = &-K\rho_0''\theta u_3 + K\partial_t\theta(\partial_3\zeta + \rho_0''\theta) - K\rho_0'\theta(Au_1 + Bu_2)\\ &- u_1\partial_1\zeta - u_2\partial_2\zeta - Ku_3\partial_3\zeta + K\partial_3\zeta(Au_1 + Bu_2)\end{aligned} \tag{B.1}$$

that

$$\begin{aligned}\mathcal{Q}_1^2 = &-(\zeta + \rho_0'\theta)\partial_t u_1 - (\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u_1 + AK(\partial_3 q - g\rho_0'\theta)\\ &- (\zeta + \rho_0 + \rho_0'\theta)\Big(u_1(\partial_1 u_1 - AK\partial_3 u_1) + u_2(\partial_2 u_1 - BK\partial_3 u_1) + Ku_3\partial_3 u_1\Big)\\ &+ \mu\begin{pmatrix}(K^2 + A^2 + B^2 - 1)\partial_{33}^2 u_1 - 2AK\partial_{13}^2 u_1 - 2BK\partial_{23}^2 u_1\\ (K\partial_3 K(A^2 + B^2 + 1) - \partial_1(AK) - \partial_2(BK) - A\partial_1 K - B\partial_2 K)\partial_3 u_1\end{pmatrix},\end{aligned} \tag{B.2}$$

$$\begin{aligned}\mathcal{Q}_2^2 = &-(\zeta + \rho_0'\theta)\partial_t u_2 - (\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u_1 + BK(\partial_3 q - g\rho_0'\theta)\\ &- (\zeta + \rho_0 + \rho_0'\theta)\Big(u_1(\partial_1 u_2 - AK\partial_3 u_2) + u_2(\partial_2 u_2 - BK\partial_3 u_2) + Ku_3\partial_3 u_2\Big)\\ &+ \mu\begin{pmatrix}(K^2 + A^2 + B^2 - 1)\partial_{33}^2 u_2 - 2AK\partial_{13}^2 u_2 - 2BK\partial_{23}^2 u_2\\ (K\partial_3 K(A^2 + B^2 + 1) - \partial_1(AK) - \partial_2(BK) - A\partial_1 K - B\partial_2 K)\partial_3 u_2\end{pmatrix},\end{aligned} \tag{B.3}$$

$$\begin{aligned}\mathcal{Q}_3^2 = &-(\zeta + \rho_0'\theta)\partial_t u_3 - (\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u_3 + (1 - K)(\partial_3 q - g\rho_0'\theta)\\ &- (\zeta + \rho_0 + \rho_0'\theta)\Big(u_1(\partial_1 u_3 - AK\partial_3 u_3) + u_2(\partial_2 u_3 - BK\partial_3 u_3) + Ku_3\partial_3 u_3\Big)\\ &+ (K - 1)(\partial_{13}^2 u_1 + \partial_{23}^2 u_2) + \partial_1 K\partial_3 u_1 + \partial_2 K\partial_3 u_2\\ &- \partial_1(AK\partial_3 u_3) - AK\partial_3(K\partial_3 u_1 + \partial_1 u_3 - AK\partial_3 u_3)\\ &- \partial_2(BK\partial_3 u_3) - BK\partial_3(K\partial_3 u_2 + \partial_2 u_3 - BK\partial_3 u_3)\\ &+ 2(K^2 - 1)\partial_{33}^2 u_3 + 2K\partial_3 K\partial_3 u_3,\end{aligned} \tag{B.4}$$

that

$$\begin{aligned}\mathcal{Q}^3 &= (1 - K)\partial_3 u_3 + AK\partial_3 u_1 + BK\partial_3 u_2,\\ \mathcal{Q}^4 &= -u_1\partial_1\eta - u_2\partial_2\eta,\end{aligned} \tag{B.5}$$

and that

$$\begin{aligned}\mathcal{Q}^5 = \partial_1\eta\begin{pmatrix} q-g\rho_+\eta-2\mu(\partial_1u_1-AK\partial_3u_1)\\ -\mu(\partial_1u_2+\partial_2u_1-AK\partial_3u_2-BK\partial_3u_1)\\ -\mu(\partial_1u_3-AK\partial_3u_3+K\partial_3u_1)\end{pmatrix}\\ +\partial_2\eta\begin{pmatrix} -\mu(\partial_1u_2+\partial_2u_1-AK\partial_3u_2-BK\partial_3u_1)\\ q-g\rho_+\eta-2\mu(\partial_2u_2-BK\partial_3u_2)\\ -\mu(\partial_2u_3-AK\partial_3u_3+K\partial_3u_2)\end{pmatrix}\\ -\mu\begin{pmatrix}(1-K)\partial_3u_1+AK\partial_3u_3\\ (1-K)\partial_3u_2+BK\partial_3u_3\\ 2(1-K)\partial_3u_3\end{pmatrix}.\end{aligned}\tag{B.6}$$

The terms $F^{j,l}(1\leqslant j\leqslant 5)$ in (4.30) are given by

$$F^{1,l}=\partial_t^lF^1-\sum_{0<j\leqslant l}C_l^j\partial_t^j\mathcal{A}_{jk}\partial_k(\rho_0\partial_t^{l-j}u_j),\tag{B.7}$$

$$\begin{aligned}F_i^{2,l}=\partial_t^lF^2+\sum_{0<j\leqslant l}C_l^j\mu(\mathcal{A}_{jk}\partial_k(\partial_t^j\mathcal{A}_{jm}\partial_t^{l-j}\partial_mu_i)+\partial_t^j\mathcal{A}_{jk}\partial_t^{l-j}\partial_k(\mathcal{A}_{jm}\partial_mu_i))\\ -\sum_{0<j\leqslant l}C_l^j(\rho_0\partial_t^j\mathcal{A}_{ik}\partial_k\partial_t^{l-j}\zeta+\partial_t^j(\zeta+\rho_0'\theta)\partial_t(\partial_t^{l-j}u_i)),\end{aligned}\tag{B.8}$$

$$\begin{aligned}F^{3,l}&=-\sum_{0<j\leqslant l}C_l^j\partial_t^j\mathcal{A}_{ik}\partial_k(\partial_t^{l-j}u_i),\\ F^{4,l}&=\sum_{0<j\leqslant l}C_l^j\partial_t^j\mathcal{N}\cdot\partial_t^{l-j}u,\end{aligned}\tag{B.9}$$

$$\begin{aligned}F_i^{5,l}=\mu\sum_{0<j\leqslant l}C_l^j(\partial_t^j(\mathcal{A}_{ik}\mathcal{N}_m)\partial_k\partial_t^{l-j}u_m+\partial_t^j(\mathcal{A}_{mk}\mathcal{N}_m)\partial_k\partial_t^{l-j}u_i)\\ +\sum_{0<j\leqslant l}C_l^j\partial_t^j\mathcal{N}_i\partial_t^{l-j}(g\rho_+\eta-q).\end{aligned}\tag{B.10}$$

Appendix C. Some useful estimates

Product estimate. Suppose that $\Sigma=\Omega$ or Γ, let $f\in H^{s_1}(\Sigma),g\in H^{s_2}(\Sigma)$,

(1) if $0\leqslant r\leqslant s_1\leqslant s_2$ and $s_2>r+3/2$, then $fg\in H^r(\Sigma)$,
(2) if $0\leqslant r\leqslant s_1\leqslant s_2$ and $s_1>3/2$, then $fg\in H^r(\Sigma)$.

In both cases, we have

$$\|fg\|_{H^r(\Sigma)}\lesssim\|f\|_{H^{s_1}(\Sigma)}\|g\|_{H^{s_2}(\Sigma)},\tag{C.1}$$

We refer to [8, Lemma 10.1] for the proof of (C.1).

Gagliardo-Nirenberg's inequality. Let $s\geqslant 0$, $\Sigma=\Omega$ or Γ and $f,g\in H^s(\Sigma)\cap L^\infty(\Sigma)$, we have

$$\|fg\|_{H^s(\Sigma)}\lesssim\|f\|_{H^s(\Sigma)}\|g\|_{L^\infty(\Sigma)}+\|f\|_{L^\infty(\Sigma)}\|g\|_{H^s(\Sigma)}.\tag{C.2}$$

Elliptic estimates. Let $r \geqslant 2$ and $\phi \in H^{r-2}(\Omega), \psi \in H^{r-1}(\Omega)$ and $\alpha \in H^{r-1/2}(\Gamma)$. There exist unique $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = \phi & \text{in } \Omega, \\ \mathrm{div} u = \psi & \text{in } \Omega, \\ (q\mathrm{Id} - \mu \mathbb{S}u)e_3 = \alpha & \text{on } \Sigma. \end{cases}$$

Moreover, we have

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|\phi\|^2_{H^{r-2}(\Omega)} + \|\psi\|^2_{H^{r-1}(\Omega)} + \|\alpha\|^2_{H^{r-3/2}(\Omega)}. \tag{C.3}$$

thanks to [8, Lemma A.15] for example.

We also recall the classical regularity theory for the Stokes problem with Dirichlet boundary conditions (see [27, Theorem 2.4] after using the domain expansion technique). Let $r \geqslant 2$ and $f \in H^{r-2}(\Omega), g \in H^{r-1}(\Omega)$ and $h \in H^{r-1/2}(\Gamma)$ such that

$$\int_\Omega g = \int_\Gamma h \cdot \nu, \text{ where } \nu \text{ is the outward unit normal vector to the boundary.}$$

There exist uniquely $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = f & \text{in } \Omega, \\ \mathrm{div} u = g & \text{in } \Omega, \\ u = h & \text{on } \Gamma. \end{cases}$$

There also holds

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|f\|^2_{H^{r-2}(\Omega)} + \|g\|^2_{H^{r-1}(\Omega)} + \|h\|^2_{H^{r-1/2}(\Gamma)}. \tag{C.4}$$

Korn's inequality. The following Korn's inequality is proven in [17, Theorem 5.12],

$$\|\nabla u\|^2_{L^2(\Omega)} \lesssim \|\mathbb{S}u\|^2_{L^2(\Omega)}. \tag{C.5}$$

Commutator estimates.

Let $\mathcal{J} = \sqrt{1 - \partial_1^2 - \partial_2^2}$ and let us define the commutator

$$[\mathcal{J}^s, f]g = \mathcal{J}^s(fg) - f\mathcal{J}^s g.$$

We have

$$\|[\mathcal{J}^s, f]g\|_{L^2(\Gamma)} \lesssim \|\nabla f\|_{L^\infty(\Gamma)} \|\mathcal{J}^{s-1} g\|_{L^2(\Gamma)} + \|\mathcal{J}^s f\|_{L^2(\Gamma)} \|g\|_{L^\infty(\Gamma)}. \tag{C.6}$$

The proof of (C.6) is similar to that one of [16, Lemma X1].

Interpolation inequality. It can be found in [1, Chapter 5] that

$$\|u\|_{H^j(\Omega)} \lesssim \|u\|^{1/(j+1)}_{L^2(\Omega)} \|u\|^{j/(j+1)}_{H^{j+1}(\Omega)}.$$

That implies for $\varepsilon > 0$, there is a universal constant $C(j)$ such that

$$\|u\|_{H^j(\Omega)} \leqslant \varepsilon \|u\|_{H^{j+1}(\Omega)} + C(j)\varepsilon^{-j} \|u\|_{L^2(\Omega)}. \tag{C.7}$$

Coefficient estimates. If $\|\eta\|_{H^{5/2}(\Gamma)} \lesssim 1$, we have

$$\|J - 1\|_{L^\infty(\Omega)} + \|\mathcal{N} - 1\|_{L^\infty(\Gamma)} + \|K - 1\|_{L^\infty(\Gamma)} \lesssim \|\eta\|_{H^{5/2}(\Gamma)}. \tag{C.8}$$

Also, the map Θ defined by (2.6) is a diffeomorphism. We refer to [9, Lemma 2.4] for the proof of (C.8). In the following lemma, we provide some additional estimates.

Lemma C.1. *Under the assumption* $\|\eta\|_{H^{9/2}(\Gamma)} \lesssim 1$*, the following inequalities hold*

$$\|\partial_t^l(A,B)\|_{H^s(\Omega)} \lesssim \|\partial_t^l\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for any } 0 \leqslant l \leqslant 2 \text{ and } 0 \leqslant s \leqslant 4, \tag{C.9}$$

and

$$\begin{cases} \|K-1\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t K\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2 K\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3 K\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}, \end{cases} \tag{C.10}$$

and

$$\begin{cases} \|(AK,BK)\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t(AK,BK)\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2(AK,BK)\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3(AK,BK)\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}, \end{cases} \tag{C.11}$$

and

$$\begin{cases} \|\mathcal{A}-Id\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t\mathcal{A}\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3\mathcal{A}\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}. \end{cases} \tag{C.12}$$

Proof. To prove (C.9), we use Lemma A.1 to obtain

$$\|\partial_t^l(A,B)\|_{H^s(\Omega)} = \|\partial_t^l(\partial_1\theta,\partial_2\theta)\|_{H^s(\Omega)} \lesssim \|\partial_t^l\theta\|_{H^{s+1}(\Omega)} \lesssim \|\partial_t^l\eta\|_{H^{s+1/2}(\Gamma)}.$$

We then claim (C.10). Since $K-1 = J^{-1}(1-J) = -J^{-1}\partial_3\theta$, we have

$$\|K-1\|_{H^s(\Omega)} \lesssim \|J^{-1}\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\theta\|_{H^{s+1}(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)}.$$

Note that $K = J^{-1}$, we have $\partial_t K = -J^{-2}\partial_t J$. Owing to the product estimate (C.1), Lemma A.1 and the fact that $\|J-1\|_{L^\infty(\Omega)} \lesssim 1$ (C.8), we get

$$\|\partial_t K\|_{H^s(\Omega)} \lesssim \|J^{-2}\partial_t\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\partial_t\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)}.$$

Since $\partial_t^2 K = -J^{-2}\partial_t^2 J + 2J^{-3}(\partial_t J)^2$, we continue applying Sobolev embedding, Lemma A.1 and (C.8) to obtain

$$\begin{aligned} \|\partial_t^2 K\|_{L^2(\Omega)} &\lesssim \|J^{-2}\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|J^{-3}(\partial_t\partial_3\theta)^2\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2\partial_3\theta\|_{L^2(\Omega)}(1+\|\partial_3\theta\|_{H^2(\Omega)}) + \|\partial_t\partial_3\theta\|_{L^2(\Omega)}\|\partial_t\partial_3\theta\|_{H^2(\Omega)} \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}(1+\|\eta\|_{H^{5/2}(\Gamma)}) + \|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{5/2}(\Gamma)} \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}. \end{aligned}$$

Similarly, we have

$$\partial_t^3 K = -J^{-2}\partial_t^3 J + 6J^{-3}\partial_t J\partial_t^2 J - 6J^{-4}(\partial_t J)^3.$$

This yields

$$\begin{aligned} \|\partial_t^3 K\|_{L^2(\Omega)} &\lesssim \|J^{-2}\partial_t^3\partial_3\theta\|_{L^2(\Omega)} + \|J^{-3}\partial_t\partial_3\theta\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|J^{-4}(\partial_t\partial_3\theta)^3\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\partial_3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3\theta\|_{H^2(\Omega)}\|\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3\theta\|^2_{H^2(\Omega)}\|\partial_t\partial_3\theta\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}. \end{aligned}$$

Hence, (C.10) is proven.

We combine (C.9) and (C.10) to prove (C.11). Note that $XK = X(K-1) + X$ for $X = A$ or B, we use Sobolev embedding and (C.10) to obtain that

$$\|XK\|_{L^2(\Omega)} \lesssim \|X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) \lesssim \|\eta\|_{H^{1/2}(\Gamma)}.$$

We make use (C.1) and (C.9), (C.10) to obtain

$$\|XK\|_{H^1(\Omega)} \lesssim \|X\|_{H^1(\Omega)}(1 + \|K-1\|_{H^3(\Omega)}) \lesssim \|\eta\|_{H^{3/2}(\Gamma)}$$

and if $s = 2, 3$ or 4, we use also Gagliardo-Nirenberg's inequality to have

$$\begin{aligned}\|XK\|_{H^s(\Omega)} &\lesssim \|X\|_{H^s(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}(1 + \|K-1\|_{H^s(\Omega)}) \\ &\lesssim \|\eta\|_{H^{s+1/2}(\Gamma)}.\end{aligned}$$

We further obtain

$$\|\partial_t(XK)\|_{H^s(\Omega)} \lesssim \|\partial_t X\|_{H^s(\Omega)} + \|\partial_t X(K-1)\|_{H^s(\Omega)} + \|X\partial_t(K-1)\|_{H^s(\Omega)}.$$

If $s = 0$, we use Sobolev embedding and (C.9), (C.10) again to have

$$\begin{aligned}\|\partial_t(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}\|\partial_t K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t\eta\|_{H^{1/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}).\end{aligned}$$

If $s = 1$ or 2, we use (C.1) and also (C.9), (C.10) to obtain

$$\begin{aligned}\|\partial_t(XK)\|_{H^1(\Omega)} &\lesssim \|\partial_t X\|_{H^1(\Omega)}(1 + \|K-1\|_{H^3(\Omega)}) + \|X\|_{H^3(\Omega)}\|\partial_t K\|_{H^1(\Omega)} \\ &\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}(1 + \|\eta\|_{H^{7/2}(\Gamma)}).\end{aligned}$$

or

$$\begin{aligned}\|\partial_t(XK)\|_{H^2(\Omega)} &\lesssim \|\partial_t X\|_{H^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\lesssim \|\partial_t\eta\|_{H^{5/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}).\end{aligned}$$

Similarly, it can be seen that

$$\begin{aligned}\|\partial_t^2(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t^2 X\|_{L^2(\Omega)} + \|\partial_t^2 X(K-1)\|_{L^2(\Omega)} + \|\partial_t X\partial_t(K-1)\|_{L^2(\Omega)} \\ &\quad + \|X\partial_t^2(K-1)\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2 X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|\partial_t X\|_{L^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\quad + \|X\|_{H^2(\Omega)}\|\partial_t^2 K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}) + \|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{5/2}(\Gamma)} \\ &\quad + \|\eta\|_{H^{5/2}(\Gamma)}(\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}) \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}.\end{aligned}$$

In a same way, we have

$$\begin{aligned}\|\partial_t^3(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t^3 X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|\partial_t^2 X\|_{L^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\quad + \|\partial_t X\|_{H^2(\Omega)}\|\partial_t^2 K\|_{L^2(\Omega)} + \|X\|_{H^2(\Omega)}\|\partial_t^3 K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}.\end{aligned}$$

Thus, the proof of (C.11) is complete.

Note that

$$\|\partial_t^l(\mathcal{A} - \mathrm{Id})\|_{H^s(\Omega)} \leqslant \|\partial_t^l(K-1)\|_{H^s(\Omega)} + \|\partial_t^l(AK)\|_{H^s(\Omega)} + \|\partial_t^l(BK)\|_{H^s(\Omega)}.$$

Hence, (C.12) follows from (C.9), (C.10) and (C.11). □

Appendix D. Proof of Lemma 3.2

Note that the quotient

$$\frac{2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a))}{\int_{-a}^0((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3} \tag{D.1}$$

is bounded because of the embedding $H^2((-a,0))\hookrightarrow C^1((-a,0))$. To prove Lemma 3.2, let us consider the Lagrangian functional

$$\mathscr{L}_k(\phi,\beta)=\beta\Big(\int_{-a}^0((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3-1\Big)-2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a)),$$

for any $\phi\in H^2((-a,0))$. Using Lagrange multiplier theorem again, we find that the extrema of the quotient (D.1) are necessarily the stationary points (ϕ_k,β_k) of $\mathscr{L}_k$, which satisfy

$$\int_{-a}^0((\phi_k'')^2+2k^2(\phi_k')^2+k^4\phi_k^2)dx_3=1 \tag{D.2}$$

and

$$\begin{aligned}&\beta_k\int_{-a}^0(\phi_k''\theta''+2k^2\phi_k'\theta'+k^4\phi_k\theta)dx_3\\&\quad=k^2(\phi_k'(0)\theta(0)+\phi_k(0)\theta'(0)-\phi_k'(-a)\theta(-a)-\phi_k(-a)\theta'(-a)).\end{aligned} \tag{D.3}$$

for all $\theta\in H^2((-a,0))$.

Taking the integration by parts, we obtain that

$$\begin{aligned}&\beta_k\int_{-a}^0(\phi_k^{(4)}-2k^2\phi_k''+k^4\phi_k)\theta dx_3+\beta_k(\phi_k''\theta'-\phi_k'''\theta+2k^2\phi_k'\theta)\Big|_{-a}^0\\&\quad=k^2(\phi_k'(0)\theta(0)+\phi_k(0)\theta'(0)-\phi_k'(-a)\theta(-a)-\phi_k(-a)\theta'(-a)).\end{aligned} \tag{D.4}$$

Restricting $\theta\in C_0^\infty((-a,0))$, the resulting equality yields

$$\phi_k^{(4)}-2k^2\phi_k''+k^4\phi_k=0\quad\text{on }(-a,0). \tag{D.5}$$

Hence, (D.4) tells us that

$$\begin{cases}\beta_k\phi_k''(0)=k^2\phi_k(0),\\ \beta_k(-\phi_k'''(0)+2k^2\phi_k'(0))=k^2\phi_k'(0),\\ \beta_k\phi_k''(-a)=k^2\phi_k(-a),\\ \beta_k(-\phi_k'''(-a)+2k^2\phi_k'(-a))=k^2\phi_k'(-a).\end{cases} \tag{D.6}$$

Any solution ϕ_k of (D.5) is of the form

$$\phi_k(x_3)=(Ax_3+B)\sinh(kx_3)+(Cx_3+D)\cosh(kx_3), \tag{D.7}$$

with A,B,C,D are four constants such that $A^2+B^2+C^2+D^2>0$. Let us compute from (D.7) that

$$\phi_k'(x_3)=(A+kD+kCx_3)\sinh(kx_3)+(C+kB+kAx_3)\cosh(kx_3),$$

$$\phi_k''(x_3)=(2kC+k^2B+k^2Ax_3)\sinh(kx_3)+(2kA+k^2D+k^2Cx_3)\cosh(kx_3).$$

and

$$\phi_k'''(x_3)=(3k^2A+k^3D+k^3C)\sinh(kx_3)+(3k^2C+k^3B+k^3Ax_3)\cosh(kx_3).$$

Substituting these formulas into (D.6), we obtain

$$\begin{cases}\beta_k(2kA+k^2D)=k^2D,\\ \beta_k(-k^2C+k^3B)=k^2(C+kB),\end{cases} \tag{D.8}$$

and

$$\begin{cases} \beta_k\Big(-(2kC + k^2(B - Aa))\sinh(ka) + (2kA + k^2(D - Ca))\cosh(ka)\Big) \\ \qquad = k^2(-(B - Aa)\sinh(ka) + (D - Ca)\cosh(ka)), \\ \beta_k\Big(-(3k^2A + k^3(D - Ca))\sinh(ka) + (3k^2C + k^3(B - Aa))\cosh(ka)\Big) \\ \qquad = k^2(2\beta_k - 1)\Big(-(A + k(D - Ca))\sinh(ka) + (C + k(B - Aa))\cosh(ka)\Big). \end{cases} \tag{D.9}$$

System (D.8) is equivalent to

$$\begin{cases} k(\beta_k - 1)B = (\beta_k + 1)C, \\ k(\beta_k - 1)D = -2\beta_k A. \end{cases} \tag{D.10}$$

We also obtain that (D.9) is equivalent to

$$\begin{cases} \Big((-\beta_k(ka\sinh(ka) + 2\cosh(ka)) + ka\sinh(ka)\Big)A + (\beta_k - 1)k\sinh(ka)B \\ \quad +\Big((2\sinh(ka) + ka\cosh(ka))\beta_k - ka\cosh(ka)\Big)C + (-\beta_k + 1)k\cosh(ka)D = 0, \\ \Big(-(\beta_k + 1)\sinh(ka) + (\beta_k - 1)ka\cosh(ka)\Big)A + (-\beta_k + 1)k\cosh(ka)B \\ \quad +\Big((-\beta_k + 1)ka\sinh(ka) + (\beta_k + 1)\cosh(ka)\Big)C + (\beta_k - 1)k\sinh(ka)D = 0. \end{cases} \tag{D.11}$$

Substituting (D.10) into (D.11), we deduce

$$\begin{cases} ka\tanh(ka)(-\beta_k + 1)A + ((3\beta_k + 1)\tanh(ka) + ka(\beta_k - 1))C = 0, \\ (-(3\beta_k + 1)\tanh(ka) + ka(\beta_k - 1))A + (-\beta_k + 1)ka\tanh(ka)C = 0. \end{cases}$$

Hence, system (D.6) is equivalent to

$$\begin{cases} (\beta_k + 1)C - k(\beta_k - 1)B = 0, \\ 2\beta_k A + k(\beta_k - 1)D = 0, \\ ka\tanh(ka)(-\beta_k + 1)A + (\tanh(ka)(3\beta_k + 1) + ka(\beta_k - 1))C = 0, \\ (-\tanh(ka)(3\beta_k + 1) + ka(\beta_k - 1))A + ka\tanh(ka)(-\beta_k + 1)C = 0. \end{cases} \tag{D.12}$$

System (D.12) admits a nontrivial solution (A, C, B, D) if and only if the determinant of the corresponding matrix is equal to zero. This yields

$$k^2(\beta_k - 1)^2\Big((ka)^2\tanh^2(ka)(\beta_k - 1)^2 - \Big((ka)^2(\beta_k - 1)^2 - \tanh^2(ka)(3\beta_k + 1)^2\Big)\Big) = 0.$$

Equivalently,

$$k^2(\beta_k - 1)^2((ka)^2(\beta_k - 1)^2 - \sinh^2(ka)(3\beta_k + 1)^2) = 0. \tag{D.13}$$

We have three possible values of β_k, which are solutions of (D.13) and ordered as

$$1 \text{ (multiplicity 2)} > -\frac{\sinh(ka) - ka}{3\sinh(ka) + ka} > -\frac{\sinh(ka) + ka}{3\sinh(ka) - ka}.$$

Let us take the maximal value $\beta_k = 1$. Clearly, we obtain $A = C = 0$ from (D.10) and

$$\phi_k(x_3) = B\sinh(kx_3) + D\cosh(kx_3).$$

Substituting the above ϕ_k into (D.2), we have

$$\int_{-a}^{0}(B\sinh(kx_3) + D\cosh(kx_3))^2dx_3 + \int_{-a}^{0}(D\sinh(kx_3) + B\cosh(kx_3))^2dx_3 = \frac{1}{2k^4}.$$

Equivalently,

$$(B^2+D^2)\int_{-a}^{0}\cosh(2kx_3)dx_3+2BD\int_{-a}^{0}\sinh(2kx_3)dx_3=\frac{1}{2k^4}.$$

We directly have

$$\frac{1}{2}\sinh(2ka)(B^2+D^2)-2\sinh^2(ka)BD=\frac{1}{2k^3}.$$

This yields

$$\begin{cases} D \text{ is arbitrary and} \\ B=\frac{2\sinh^2(ka)\pm\sqrt{\sinh^2(ka)(2\cosh^2(ka)+\cosh(2ka))D^2+\frac{1}{k^3}\sinh(2ka)}}{2\sinh(2ka)}. \end{cases} \tag{D.14}$$

Let us consider the minimal value $\beta_k=-\frac{\sinh(ka)+ka}{3\sinh(ka)-ka}$. It can be seen from (D.12) that

$$D=-\frac{\sinh(ka)+ka}{2k\sinh(ka)}A,\quad C=\frac{\cosh(ka)-1}{\sinh(ka)}A, \tag{D.15}$$

and

$$B=-\frac{(\sinh(ka)-ka)(\cosh(ka)-1)}{2k\sinh^2(ka)}A. \tag{D.16}$$

Hence, $\phi_k(x_3)=Az_k(x_3)$, where

$$\begin{aligned} z_k(x_3)=&\Big(x_3-\frac{(\sinh(ka)-ka)(\cosh(ka)-1)}{2k\sinh^2(ka)}\Big)\sinh(kx_3)\\ &+\Big(\frac{\cosh(ka)-1}{\sinh(ka)}x_3-\frac{\sinh(ka)+ka}{2k\sinh(ka)}\Big)\cosh(kx_3). \end{aligned}$$

To find A, we trace back to (D.2). That means

$$A^2\int_{-a}^{0}((z_k'')^2+2k^2(z_k')^2+k^4z_k^2)dx_3=1. \tag{D.17}$$

From the above cases, we conclude that

- $$\max_{\phi\in H^2((-a,0))}\frac{2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a))}{\int_{-a}^{0}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3}=1.$$
  That variational problem is attained by functions
  $$\phi(x_3)=B\sinh(kx_3)+D\cosh(kx_3),$$
  where B,D satisfy (D.14).
- $$\min_{\phi\in H^2((-a,0))}\frac{2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a))}{\int_{-a}^{0}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3}=-\frac{\sinh(ka)+ka}{3\sinh(ka)-ka}.$$
  That variational problem is attained by functions
  $$\phi(x_3)=(Ax_3+B)\sinh(kx_3)+(Cx_3+D)\cosh(kx_3),$$
  where A,B,C,D satisfy (D.17), (D.16) and (D.15).

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