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Abstract: We study an interconnection between a slow switched affine system and a fast LTI system. The lower dimensional slow and fast subsystems are computed via the singular perturbation approach. We consider the classical stabilization method based on the existence of a stable convex combination (while ignoring the fast dynamics) and we provide an LMI condition for checking ultimate boundedness when the fast dynamics is taken into account. Numerical example illustrates the main result.

Keywords: switched affine system, multiple time scales, Lyapunov method, singular perturbation

1. INTRODUCTION

The interest in study of switched affine systems comes from their wide use in various fields in physics and engineering. For example, in DC-DC power converters, the switched circuit topology indicates different modes of operation that have no common equilibrium, can be considered as such systems [Beccuti et al. (2005); Buisson et al. (2005); Deaecto et al. (2010)]. Different approaches have been considered for the stability and the stabilization of such systems: Hurwitz convex combinations [Bolzern and Spinelli (2004); Beccuti et al. (2005)], Lyapunov techniques [Hetel and Fridman (2013); Kader (2017)], Adaptive based method [Beneux et al. (2017)], Hybrid approach [Sanchez et al. (2019)], etc. Besides dealing with switching among various equilibria in such electronic devices one also has to handle the slow and fast variables (for example, in such systems we usually have fast variables such as currents interconnected with slow variables such as voltages).

Motivated by such control problems we study in this paper the interconnection between a slow switched affine system and a fast linear time invariant (LTI) system. The powerful singular perturbation techniques [Kokotović et al. (1986); Khalil (1996)] existing in the literature are not applicable to the case under study. We deal with systems with state-dependent switching, which results in complex phenomena such as sliding modes and non unique Filippov solutions. The main contribution of this work is to provide a method of stability analysis for slow switched affine systems interconnected to fast LTI dynamics. We consider the classical stabilization method based on the existence of a stable convex combination (while ignoring the fast dynamics) and we provide an LMI condition for checking ultimate boundedness when the fast dynamics is taken into account. To the best of our knowledge, this setting is new in the literature. However, some results exist in a different context with time-dependent switching [Rejeb et al. (2018); Yang et al. (2020); Malloci et al. (2010); Hachemi et al. (2012)].

The paper is organized as follows. Section 2 is dedicated to preliminary results concerning the Filippov solution of switched affine systems. Section 3 presents the system under consideration and the decomposition of both fast and slow subsystems via the singular perturbation method. Section 4 states the stability properties of such system under a state-dependent switching law. In Section 5, the main result is illustrated by numerical examples. Finally, concluding remarks end the paper.

Notation. Given a matrix M, M^{-1} and $M^\top$ represent the inverse and the transpose matrix of M respectively. $M \succeq 0$ ($M \preceq 0$) means that the matrix M is positive (negative) semi-definite. $M \succ 0$ ($M \prec 0$) means that the matrix M is positive (negative) definite. For a symmetric matrix, $\star$ denotes the symmetric part. $\|\ \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. For a positive integer N, $\mathcal{I}_N$ denotes the set $\{1,\dots,N\}$. The n dimensional open ball in $\mathbb{R}^n$ centred on a with radius r is denoted by $\mathcal{B}(a,r) := \{b \in \mathbb{R}^n : |a-b| < r\}$. μ denotes the Lebesgue measure. The convex closure of the set S is denoted by $\mathrm{Conv}\{S\}$, and Δ_N stands for

$$\Delta_N = \left\{ v = [v_1,\dots,v_N]^\top \in \mathbb{R}^N : \sum_{i=1}^{N} v_i = 1, v_i \geqslant 0, \forall i \in \mathcal{I}_N \right\}.$$

2. PRELIMINARIES

We present some preliminary results about the solution of switched affine systems which will be used in the sequel. Let us consider the following switched affine systems

$$\dot{x} = A_{\sigma(x)}x + b_{\sigma(x)} =: g(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $\sigma(x)$ is the switching law and $\sigma \in \mathcal{I}_N = \{1,2,\dots,N\}$. A_i for $i \in \{1,2,\dots,N\}$ are the evolution matrices of the subsystems, $b_i \in \mathbb{R}^n$ presents the affine term.

The system (1) does not admit classical solutions since the right-hand side is discontinuous. While we can consider Filippov solution [Filippov (1988)] of (1) if the map g :

$\mathbb{R}^n \to \mathbb{R}^n$ is locally bounded and discontinuous. In this sense, the differential equation (1) is associated to a set-valued map taking into account all the possible values of the derivative. Then we may consider the convex hull of the possible trajectories over the discontinuities, and differential inclusions are used to construct the solutions.

The differential inclusion associated to (1) is given by

$$\dot{x}(t) \in \mathcal{F}[g](x), \tag{2}$$

with the set-valued map

$$\mathcal{F}[g](x) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \text{Conv}\{g(\mathcal{B}(x,\delta) \setminus \mathcal{S})\}, \quad \forall x \in \mathbb{R}^n. \tag{3}$$

The solutions of (1) in the sense of Filippov are defined as follows.

Definition 1. Consider the system (1) and its associated differential inclusion (2). A Filippov solution [Filippov (1988)] of the system on the interval $I = [t_1, t_2] \in [0, +\infty)$ is an absolutely continuous map $\Xi : [t_1, t_2] \to \mathbb{R}^n$ such that $\dot{\Xi}(t) \in \mathcal{F}[g](\Xi(t))$ is satisfied for almost every $t \in [t_1, t_2]$, with $\mathcal{F}[g](x)$ given by (3). □

The existence of at least one solution of the differential inclusion (2) for some initial conditions is guaranteed if $\mathcal{F}$ is locally bounded and takes nonempty compact and convex values.

3. PROBLEM STATEMENT

Consider the following interconnection

$$\begin{cases} \dot{x}_s = A^{11}_{\sigma(x_s)} x_s + A^{12}_{\sigma(x_s)} x_f + b_{\sigma(x_s)}, & \text{(4a)} \\ \varepsilon \dot{x}_f = A^{21} x_s + A^{22} x_f, & \text{(4b)} \end{cases}$$

where the state variables are $x_s \in \mathbb{R}^n$, $x_f \in \mathbb{R}^m$ and the switching law is $\sigma \in \mathcal{I}_N = \{1, 2, \ldots, N\}$. The small perturbation parameter ε satisfies $0 < \varepsilon \ll 1$. The affine term is $b_i \in \mathbb{R}^n$ for $i \in \{1, 2, \ldots, N\}$, and the matrices A^{11}_i, A^{12}_i, A^{21}, A^{22} are of appropriate dimensions.

Due to the small parameter ε, the dynamics of the system (4) evolve according to two time scales. We follow the standard singular perturbation approach [Kokotović et al. (1986)] to decompose (4) into the slow and fast subsystems. Assuming A^{22} invertible and setting $\varepsilon = 0$ in (4b), we have

$$x_f = -{A^{22}}^{-1} A^{21} x_s. \tag{5}$$

Replacing x_f in (4a) by the right-hand side of (5), we obtain the slow subsystem which is a switched affine system

$$\dot{x}_s = A^s_{\sigma(x_s)} x_s + b_{\sigma(x_s)}, \tag{6}$$

where

$$A^s_{\sigma(x_s)} = A^{11}_{\sigma(x_s)} - A^{12}_{\sigma(x_s)} {A^{22}}^{-1} A^{21}. \tag{7}$$

To compute the fast subsystem, we first perform the following change of variable

$$y = x_f + {A^{22}}^{-1} A^{21} x_s,$$

we then write

$$\varepsilon \dot{y} = A^{22} y + \varepsilon (G^1_{\sigma(x_s)} x_s + G^2_{\sigma(x_s)} y + {A^{22}}^{-1} A^{21} b_{\sigma(x_s)}),$$

where

$G^1_{\sigma(x_s)} = {A^{22}}^{-1} A^{21} A^s_{\sigma(x_s)}$, $G^2_{\sigma(x_s)} = {A^{22}}^{-1} A^{21} A^{12}_{\sigma(x_s)}$.

Using a new time scale $\tau = \frac{t}{\varepsilon}$ and $\varepsilon \to 0$, the fast LTI subsystem is computed as

$$\frac{dy}{d\tau} = A^{22} y. \tag{8}$$

The standing assumptions using in this work are given as follows.

Assumption 1. Assume that there exists $\alpha \in \Delta_N$ such that the convex combinations $b(\alpha) = \sum_{i=1}^{N} \alpha_i b_i = 0$, and $A^s(\alpha) = \sum_{i=1}^{N} \alpha_i A^s_i$ is Hurwitz, that is

$$A^s(\alpha)^\top P + P A^s(\alpha) \preceq -c_s P, \tag{9}$$

with a matrix $P = P^\top \succ 0$, and a scalar $c_s > 0$. □

Remark 1. In Assumption 1, let us remark that the condition $b(\alpha) = \sum_{i=1}^{N} \alpha_i b_i = 0$ is a necessary condition [Hetel and Bernuau (2015)] for the existence of the switching law $\sigma(x_s)$ such that 0 is an equilibrium of the differential inclusion associated with system (6). Moreover, it is worth to emphasize that condition (9) does not require the stability of each mode of the slow subsystem (6). □

Assumption 2. A^{22} is Hurwitz, that is

$${A^{22}}^\top Q + Q A^{22} \preceq -c_f Q, \tag{10}$$

with a matrix $Q = Q^\top \succ 0$, and a scalar $c_f > 0$. □

Remark 2. Condition (10) is a standard stability condition of LTI systems. Thus the stability of the fast subsystem is ensured. □

In the following, we rewrite the whole system (4) in the coordinate (x_s, y) as

$$\begin{pmatrix} \dot{x}_s \\ \dot{y} \end{pmatrix} = H(x_s, y) = \begin{cases} h_1(x_s, y, \sigma(x_s)) \\ h_{2\varepsilon}(x_s, y, \sigma(x_s)) \end{cases}$$

$$:= \begin{cases} A^s_{\sigma(x_s)} x_s + A^{12}_{\sigma(x_s)} y + b_{\sigma(x_s)}, \\ \frac{1}{\varepsilon} A^{22} y + \left(G^1_{\sigma(x_s)} x_s + G^2_{\sigma(x_s)} y + {A^{22}}^{-1} A^{21} b_{\sigma(x_s)} \right). \end{cases} \tag{11}$$

Considering system (11), we propose a state-dependent switching law

$$\sigma(x_s) \in \arg\min_{i \in \mathcal{I}_N} \{ x_s^\top ({A^s_i}^\top P + P A^s_i) x_s + 2 x_s^\top P b_i \}. \tag{12}$$

In order to define the Filippov solution of the closed-loop system (11) and (12), we introduce the following differential inclusion associated to the closed-loop system

$$\begin{pmatrix} \dot{x}_s(t) \\ \dot{y}(t) \end{pmatrix} \in \mathcal{F}[H](x_s, y), \tag{13}$$

with the set-valued map

$$\mathcal{F}[H](x_s, y) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \text{Conv}\{H(\mathcal{B}((x_s, y), \delta) \setminus \mathcal{S})\}. \tag{14}$$

Definition 2. Consider system (11) with the switching law (12) and its associated differential inclusion (13). A Filippov solution [Filippov (1988)] of the system on the interval $I = [t_1, t_2] \in [0, +\infty)$ is an absolutely continuous map $\Xi : [t_1, t_2] \to \mathbb{R}^{n+m}$ such that $\dot{\Xi}(t) \in \mathcal{F}[H](\Xi(t))$

is satisfied for almost every $t \in [t_1, t_2]$, with $\mathcal{F}[H](x_s, y)$ given by (14). □

The objective of this work is to study the stability properties of the system (11) under the state-dependent switching law (12). The stability notion adopted in the work is given as follows.

Definition 3. Let $\mathcal{S}$ be a neighborhood of the origin. The closed-loop system (11) and (12) is said to be *Uniformly Ultimately Bounded* in $\mathcal{S}$ if for all $a > 0$ there exists $T(a) > 0$ such that, for any $\left\| \begin{matrix} x_s(0) \\ y(0) \end{matrix} \right\| \leq a$, the solutions $\begin{pmatrix} x_s(t) \\ y(t) \end{pmatrix} \in \mathcal{S}$, for all $t \geq T(a)$. □

4. STABILITY ANALYSIS

In this section, we aim at showing the stability properties of the system (11) under the switching law (12).

The main result is stated as follows.

Theorem 1. If symmetric matrices P, Q satisfy Assumptions 1-2 for some positive scalars c_s, c_f and if there exist constants $c > 0, \theta > 0$, $0 < \varepsilon^* \ll 1$ such that for $i \in \mathcal{I}_N$

$$\begin{pmatrix} -c_s P + \theta P & 0 & \sqrt{\varepsilon^*}(PA_i^{12} + {G_i^1}^\top Q) \\ \star & -\theta c I & \sqrt{\varepsilon^*}(QA^{22^{-1}}A^{21}b_i)^\top \\ \star & \star & -c_f Q + \varepsilon^*({G_i^2}^\top Q + QG_i^2) + \theta Q \end{pmatrix} \prec 0, \tag{15}$$

then, for $\varepsilon \in (0, \varepsilon^*]$, system (11) with the state-dependent switching law (12) is uniformly ultimately bounded with an estimation of the ultimate bounded set

$$\mathcal{D}(c^*) = \left\{ \begin{pmatrix} x_s \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \ : \begin{pmatrix} x_s \\ y \end{pmatrix}^\top \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} x_s \\ y \end{pmatrix} \leq c^* \right\},$$

where $c^* = \max\{c\}$ satisfies (15).

Proof. The proof is based on the Lyapunov analysis. First, for $x_s \in \mathbb{R}^n$ we define the set of minimizers in which the switching law choose values as

$$\begin{aligned} \mathcal{I}^*(x_s) = \{ & i \in \mathcal{I}_N : x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i \leq \\ & x_s^\top ({A_j^s}^\top P + PA_j^s)x_s + 2x_s^\top P b_j, \forall j \in \mathcal{I}_N \}. \end{aligned} \tag{16}$$

For the system (11) with the switching law (12), we define the set-valued map $\mathcal{F}^*[H](x_s, y)$ for $x_s \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $i \in \mathcal{I}^*(x_s)$

$$\mathcal{F}^*[H](x_s, y) = \underset{i \in \mathcal{I}^*(x_s)}{\text{Conv}} \left\{ \begin{pmatrix} A_i^s x_s + A_i^{12} y + b_i \\ \dfrac{A^{22}}{\varepsilon} y + (G_i^1 x_s + G_i^2 y + A^{22^{-1}} A^{21} b_i) \end{pmatrix} \right\}. \tag{17}$$

We can remark that the set-valued map $\mathcal{F}[H](x_s, y)$ given in (14) satisfies

$$\mathcal{F}[H](x_s, y) \subseteq \mathcal{F}^*[H](x_s, y). \tag{18}$$

More precisely, if there is only one mode satisfies (16) (i.e. $\mathcal{I}^*(x_s) = \{i\}$) then $\mathcal{F}[H](x_s, y) = \mathcal{F}^*[H](x_s, y)$. Otherwise, we have $\mathcal{F}[H](x_s, y) \subset \mathcal{F}^*[H](x_s, y)$ if there are several modes verify (16). The set-valued map $\mathcal{F}^*[H](x_s, y)$ is locally bounded, has a nonempty, compact and convex values. Then the differential inclusion $\begin{pmatrix} \dot{x}_s(t) \\ \dot{y}(t) \end{pmatrix} \in \mathcal{F}^*[H](x_s, y)$ admits at least one solution for some initial conditions. Due to the relation given in (18), this also holds for the differential inclusion $\begin{pmatrix} \dot{x}_s(t) \\ \dot{y}(t) \end{pmatrix} \in \mathcal{F}[H](x_s, y)$.

Then, we consider the following candidate Lyapunov function

$$V(x_s, y) = x_s^\top P x_s + y^\top Q y, \tag{19}$$

with P and Q satisfying (9) and (10) respectively. For all $i \in \mathcal{I}^*(x_s)$, we have

$$\begin{aligned} & \frac{\partial V}{\partial x_s} h_1(x_s, y, \sigma(x_s) = i) \\ & = x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i + 2x_s^\top PA_i^{12} y. \end{aligned} \tag{20}$$

From (16), the switching law gives that

$$\begin{aligned} & x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i \\ & \leq x_s^\top ({A_j^s}^\top P + PA_j^s)x_s + 2x_s^\top P b_j, \forall j \in \mathcal{I}_N. \end{aligned} \tag{21}$$

Let $\alpha \in \Delta_N$. We deduce from equation (21)

$$\begin{aligned} & \sum_{j=1}^N \alpha_j (x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i) \\ & \leq \sum_{j=1}^N \alpha_j (x_s^\top ({A_j^s}^\top P + PA_j^s)x_s + 2x_s^\top P b_j). \end{aligned}$$

Then using $\sum_{j=1}^N \alpha_j = 1$, we have

$$\begin{aligned} & x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i \\ & \leq x_s^\top (A^s(\alpha)^\top P + PA(\alpha))x_s + 2x_s^\top P b(\alpha). \end{aligned}$$

From Assumption 1 and recalling $h_1(x_s, y, \sigma(x_s))$ defined in (11), we deduce, for $i \in \mathcal{I}^*(x_s)$

$$\begin{aligned} & \frac{\partial V}{\partial x_s} h_1(x_s, y, \sigma(x_s) = i) \\ & = x_s^\top ({A_i^s}^\top P + PA_i^s)x_s + 2x_s^\top P b_i + 2x_s^\top PA_i^{12} y \\ & \leq x_s^\top (-c_s P)x_s + 2x_s^\top PA_i^{12} y. \end{aligned} \tag{22}$$

Moreover, by recalling $h_{2\varepsilon}(x_s, y, \sigma(x_s))$ defined in (11), we have

$$\begin{aligned} & \frac{\partial V}{\partial y} h_{2\varepsilon}(x_s, y, \sigma(x_s) = i) \\ & = \frac{1}{\varepsilon} y^\top \left({A^{22}}^\top Q + QA^{22} + \varepsilon({G_i^2}^\top Q + QG_i^2) \right) y \\ & \quad + 2x_s^\top {G_i^1}^\top Q y + 2y^\top Q A^{22^{-1}} A^{21} b_i. \end{aligned} \tag{23}$$

Under Assumption 2, inequality (23) becomes

$$\begin{aligned}
&\frac{\partial V}{\partial y}h_{2\varepsilon}(x_s,y,\sigma(x_s)=i)\\
&\leq \frac{1}{\varepsilon}y^\top\left(-c_fQ+\varepsilon({G_i^2}^\top Q+QG_i^2)\right)y\\
&\quad +2x_s^\top {G_i^1}^\top Qy+2y^\top QA^{22^{-1}}A^{21}b_i. \qquad (24)
\end{aligned}$$

In view of (22) and (24), we have

$$\begin{aligned}
&\frac{\partial V}{\partial x_s}h_1(x_s,y)+\frac{\partial V}{\partial y}h_{2\varepsilon}(x_s,y)\\
&\leq x_s^\top(-c_sP)x_s+\frac{1}{\varepsilon}y^\top\left(-c_fQ+\varepsilon({G_i^2}^\top Q+QG_i^2)\right)y\\
&+2x_s^\top\left(PA_i^{12}+{G_i^1}^\top Q\right)y+2y^\top QA^{22^{-1}}A^{21}b_i\\
&=\begin{pmatrix}x_s\\1\\y\end{pmatrix}^\top\begin{pmatrix}-c_sP & 0 & PA_i^{12}+{G_i^1}^\top Q\\ \star & 0 & (QA^{22^{-1}}A^{21}b_i)^\top\\ \star & \star & \dfrac{-c_fQ}{\varepsilon}+{G_i^2}^\top Q+QG_i^2\end{pmatrix}\begin{pmatrix}x_s\\1\\y\end{pmatrix}\\
&=: f_i(x_s,y),\ i\in\mathcal{I}^*(x_s). \qquad (25)
\end{aligned}$$

In the case where the switching law allows to choose several modes $i\in\mathcal{I}^*(x_s)$, we can define the set $\Delta^*_{N(x_s)}$ such that

$$\Delta^*_{N(x_s)}=\{\beta\in\Delta_N:\beta_i=0, i\notin\mathcal{I}^*(x_s)\}. \qquad (26)$$

In this case (17) becomes

$$\begin{aligned}
&\mathcal{F}^*[H](x_s,y)\\
&=\left\{\sum_{i=1}^N\beta_i\begin{pmatrix}A_i^sx_s+A_i^{12}y+b_i\\ \dfrac{A^{22}}{\varepsilon}y+(G_i^1x_s+G_i^2y+A^{22^{-1}}A^{21}b_i)\end{pmatrix}:\right.\\
&\qquad\left.\beta_i\in\Delta^*_{N(x_s)}\right\}. \qquad (27)
\end{aligned}$$

The relation (18) also holds for $\mathcal{F}^*[H](x_s,y)$ given in (27). Consequently, due to the compactness of the set $\Delta^*_{N(x_s)}$ and in view of (27) and (18), we have

$$\begin{aligned}
&\sup_{(h_1,h_{2\varepsilon})\in\mathcal{F}[H](x_s,y)}\left\{\frac{\partial V}{\partial x_s}h_1(x_s,y)+\frac{\partial V}{\partial y}h_{2\varepsilon}(x_s,y)\right\}\\
&\leq\sup_{(h_1,h_{2\varepsilon})\in\mathcal{F}^*[H](x_s,y)}\left\{\frac{\partial V}{\partial x_s}h_1(x_s,y)+\frac{\partial V}{\partial y}h_{2\varepsilon}(x_s,y)\right\}\\
&=\sup_{\beta\in\Delta^*_{N(x_s)}}\left\{\sum_{i=1}^N\beta_i\left(x_s^\top({A_i^s}^\top P+PA_i^s)x_s+2x_s^\top Pb_i\right.\right.\\
&+\frac{1}{\varepsilon}y^\top\left(A^{22^\top}Q+QA^{22}+\varepsilon({G_i^2}^\top Q+QG_i^2)\right)y\\
&\left.\left.+2x_s^\top\left(PA_i^{12}+{G_i^1}^\top Q\right)y+2y^\top QA^{22^{-1}}A^{21}b_i\right)\right\}\\
&=\max_{\beta\in\Delta^*_{N(x_s)}}\left\{\sum_{i=1}^N\beta_i\left(x_s^\top({A_i^s}^\top P+PA_i^s)x_s+2x_s^\top Pb_i\right.\right.\\
&+\frac{1}{\varepsilon}y^\top\left(A^{22^\top}Q+QA^{22}+\varepsilon({G_i^2}^\top Q+QG_i^2)\right)y\\
&\left.\left.+2x_s^\top\left(PA_i^{12}+{G_i^1}^\top Q\right)y+2y^\top QA^{22^{-1}}A^{21}b_i\right)\right\}. \qquad (28)
\end{aligned}$$

Since $\beta_i=0$ for all $i\in\mathcal{I}_N\setminus\mathcal{I}^*(x_s)$ and $\sum_{i=1}^N\beta_i=1$, for $i\in\mathcal{I}^*(x_s)$, in view of (22), the following holds

$$\begin{aligned}
&\sum_{i=1}^N\beta_i\left(x_s^\top({A_i^s}^\top P+PA_i^s)x_s+2x_s^\top Pb_i\right)\\
&=\sum_{i\in\mathcal{I}^*(x_s)}\beta_i\left(x_s^\top({A_i^s}^\top P+PA_i^s)x_s+2x_s^\top Pb_i\right)\\
&\leq x_s^\top(-c_sP)x_s. \qquad (29)
\end{aligned}$$

Substituting (10) and (29) into (28) and reorganizing it, the following holds

$$\begin{aligned}
&\sup_{(h_1,h_{2\varepsilon})\in\mathcal{F}[H](x_s,y)}\left\{\frac{\partial V}{\partial x_s}h_1(x_s,y)+\frac{\partial V}{\partial y}h_{2\varepsilon}(x_s,y)\right\}\\
&\leq\max_{\beta\in\Delta^*_{N(x_s)}}\left\{\sum_{i=1}^N\beta_if_i(x_s,y)\right\}, \qquad (30)
\end{aligned}$$

with $f_i(x_s,y)$ defined in (25).

In the following we prove that $f_i(x_s,y)<0$, for $i\in\mathcal{I}^*(x_s)$. By using the Schur complement, condition (15) is equivalent to

$$\begin{pmatrix}-c_sP+\theta P & 0\\ \star & -\theta cI\end{pmatrix}\prec 0 \qquad (31)$$

and

$$\begin{aligned}
&-c_fQ+\varepsilon^*({G_i^2}^\top Q+QG_i^2)+\theta Q\\
&-\varepsilon^*\begin{pmatrix}PA_i^{12}+{G_i^1}^\top Q\\ (QA^{22^{-1}}A^{21}b_i)^\top\end{pmatrix}^\top\\
&\begin{pmatrix}-c_sP+\theta P & 0\\ \star & -\theta cI\end{pmatrix}^{-1}\begin{pmatrix}PA_i^{12}+{G_i^1}^\top Q\\ (QA^{22^{-1}}A^{21}b_i)^\top\end{pmatrix}\prec 0. \qquad (32)
\end{aligned}$$

The above inequality (32) can be rewritten as follows

$$\varepsilon^* \left\{ \left(-\frac{c_f Q}{\varepsilon^*} + {G_i^2}^\top Q + Q G_i^2 \right) - \begin{pmatrix} P A_i^{12} + {G_i^1}^\top Q \\ (Q A^{22^{-1}} A^{21} b_i)^\top \end{pmatrix}^\top \begin{pmatrix} -c_s P + \theta P & 0 \\ \star & -\theta c I \end{pmatrix}^{-1} \begin{pmatrix} P A_i^{12} + {G_i^1}^\top Q \\ (Q A^{22^{-1}} A^{21} b_i)^\top \end{pmatrix} \right\} + \theta Q \prec 0. \tag{33}$$

Since $0 < \varepsilon^* \ll 1$, $\theta > 0$, and $Q \succ 0$, inequality (33) implies

$$\left(-\frac{c_f Q}{\varepsilon^*} + {G_i^2}^\top Q + Q G_i^2 \right) - \begin{pmatrix} P A_i^{12} + {G_i^1}^\top Q \\ (Q A^{22^{-1}} A^{21} b_i)^\top \end{pmatrix}^\top \begin{pmatrix} -c_s P + \theta P & 0 \\ \star & -\theta c I \end{pmatrix}^{-1} \begin{pmatrix} P A_i^{12} + {G_i^1}^\top Q \\ (Q A^{22^{-1}} A^{21} b_i)^\top \end{pmatrix} + \theta Q \prec 0. \tag{34}$$

In view of (31) and (34), using again the Schur complement, we have

$$\begin{pmatrix} -c_s P + \theta P & 0 & P A_i^{12} + {G_i^1}^\top Q \\ \star & -\theta c I & (Q A^{22^{-1}} A^{21} b_i)^\top \\ \star & \star & \frac{-c_f Q}{\varepsilon^*} + {G_i^2}^\top Q + Q G_i^2 + \theta Q \end{pmatrix} \prec 0. \tag{35}$$

From (35), we get, for $\varepsilon \in (0, \varepsilon^*]$

$$f_i(x_s, y) + \begin{pmatrix} x_s \\ 1 \\ y \end{pmatrix}^\top \begin{pmatrix} \theta P & 0 & 0 \\ 0 & -\theta c I & 0 \\ 0 & 0 & \theta Q \end{pmatrix} \begin{pmatrix} x_s \\ 1 \\ y \end{pmatrix} < 0 \tag{36}$$

with $f_i(x_s, y)$ defined in (25).
Whenever $V(x_s, y) \geqslant c$, it holds

$$\begin{pmatrix} x_s \\ 1 \\ y \end{pmatrix}^\top \begin{pmatrix} P & 0 & 0 \\ 0 & -cI & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} x_s \\ 1 \\ y \end{pmatrix} \geq 0. \tag{37}$$

Since $\theta > 0$ and in view of (36) and (37), it holds $f_i(x_s, y) < 0$ when $i \in \mathcal{I}^*(x_s)$. Thus,

$$\frac{\partial V}{\partial x_s} h_1(x_s, y) + \frac{\partial V}{\partial y} h_{2\varepsilon}(x_s, y) < 0, \tag{38}$$

when $V(x_s, y) \geqslant c$. Then, system (11) with the state-dependent switching law (12) is uniformly ultimately bounded. ■

Remark 3. Theorem 1 provides a simple condition for checking ultimate boundedness of system (11) with the state-dependent switching law (12). In the following, let us summarize the tuning parameters in detail. The matrices P and Q are the corresponding Lyapunov matrices, which can be found by solving the LMIs (9) and (10) respectively. The constant c relates to an estimation of the ultimate bound. It can be obtained by solving (15). For a given perturbation parameter ε^* sufficiently small, a line search can be used to find θ. □

5. NUMERICAL EXAMPLE

In this section we use an academic numerical example to illustrate the main result. We consider the system (4) with $N = 3$ modes, and matrices defined as

$$A_1^{11} = \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}, \; A_2^{11} = \begin{pmatrix} -2 & 3 \\ -2 & 1 \end{pmatrix}, \; A_3^{11} = \begin{pmatrix} -3 & -3 \\ 2 & -1 \end{pmatrix},$$

$$A_1^{12} = \begin{pmatrix} 2 & -3 \\ -1 & -2 \end{pmatrix}, \; A_2^{12} = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}, \; A_3^{12} = \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \; b_2 = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \; b_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$A^{21} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \; A^{22} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

The small perturbation parameter $\varepsilon = 1.5 \times 10^{-3}$. Considering $\alpha = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^\top$, we have $b(\alpha) = \sum_{i=1}^{N} \alpha_i b_i = 0$. From the definition of the matrices A_i^s ($i \in \mathcal{I}_N$) in (7), we get that $A^s(\alpha) = \sum_{i=1}^{N} \alpha_i A_i^s = \begin{pmatrix} -5.67 & 8 \\ -5 & 4 \end{pmatrix}$ is Hurwitz, which means that Assumption 1 is satisfied. Furthermore, we can see that A^{22} is also Hurwitz, which means that Assumption 2 is also satisfied.

It is important to note that in this example, none of the A_i^s matrices ($i \in \mathcal{I}_N$) in (11) is Hurwitz, which means that even if the fast subsystem (with state y) converges, the slow subsystem (with state x_s) will converge only by using of a switching law (*i.e.* staying only in one mode will lead to the slow subsystem's unstability). Using the state-dependent switching law proposed in (12), with $P = \begin{pmatrix} 2.8660 & -2.7788 \\ \star & 4.7849 \end{pmatrix}$ obtained after solving the LMI (9) with $c_s = 1$, we manage to stabilize the system and guarantee its stability thanks to Theorem 1, with $Q = \begin{pmatrix} 1.1970 & -0.9892 \\ \star & 0.9982 \end{pmatrix}$ obtained after solving the LMI (10) with $c_f = 1$, and the scalar $\theta = 0.1$. The estimation of the ultimate bound $c = 4.2243$. Some simulations obtained with initial conditions $x_s(0) = (2.9020 \; 0.0986)^\top$ and $x_f(0) = (0.4127 \; 1.0804)^\top$ are shown in Figures 1-5.

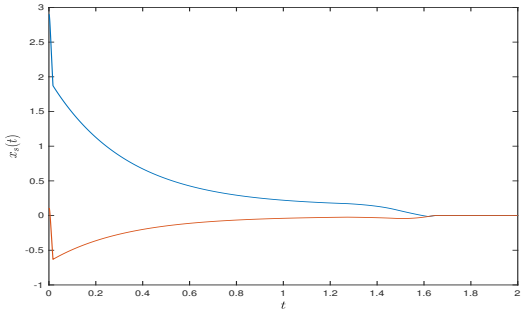

Fig. 1: System's evolution (state x_s)

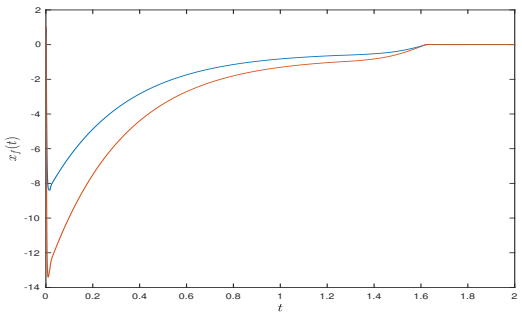

Fig. 2: System's evolution (state x_f)

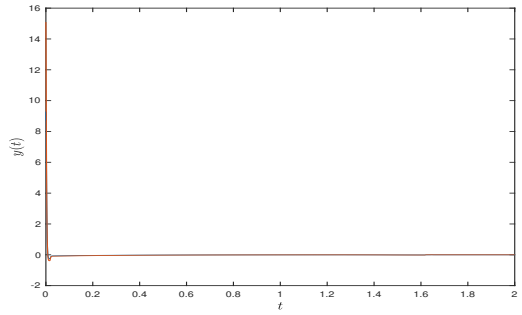

Fig. 3: System's evolution (state y)

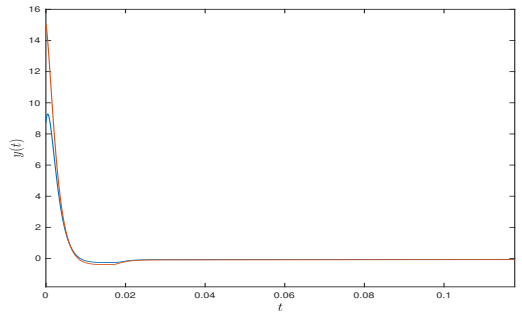

Fig. 4: Zoom of system's evolution (state y)

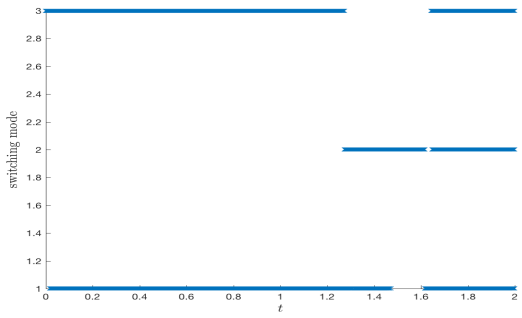

Fig. 5: System's evolution (switching mode $\sigma(x_s)$)

6. CONCLUSION

In this work, we studied the interconnection between a slow switched affine system and a fast LTI system. We considered the classical stabilization method based on the existence of a stable convex combination (while ignoring the fast dynamics) and we provide an LMI condition for checking ultimate boundedness when the fast dynamics has been taken into account.

Further extensions will include a generalization of the switched fast and slow dynamics as well as applying such result to physical systems (for example, DC-DC power converters).

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