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DIFFUSIVE LIMIT OF THE VLASOV-POISSON-FOKKER-PLANCK MODEL: QUANTITATIVE AND STRONG CONVERGENCE RESULTS

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ABSTRACT. This work tackles the diffusive limit for the Vlasov-Poisson-Fokker-Planck model. We derive *a priori* estimates which hold without restriction on the phase-space dimension and propose a strong convergence result in a L^2 space. Furthermore, we strengthen previous results by obtaining an explicit convergence rate arbitrarily close to the (formal) optimal rate, provided that the initial data lies in some L^p space with p large enough. Our result holds on bounded time intervals whose size grow to infinity in the asymptotic limit with explicit lower bound. The analysis relies on identifying the right set of phase-space coordinates to study the regime of interest. In this set of coordinates the limiting model arises explicitly.

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1. INTRODUCTION

1.1. Physical model and motivation. In this article, we study a plasma composed of moving electrons and fixed ions. We denote by $f^\varepsilon(t, \mathbf{x}, \mathbf{v}) \geq 0$ the density of electron at time $t \in \mathbb{R}^+$, position $\mathbf{x} \in \mathbb{K}^d$ with $\mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}$ and velocity $\mathbf{v} \in \mathbb{R}^d$ whereas the density of ions is given by $\rho_i(\mathbf{x}) \geq 0$ with $\rho_i \in L^1(\mathbb{K}^d)$. We focus on the Vlasov-Poisson-Fokker-Planck (VPFP) model, which describes a situation where the particles interact through electrostatic effects and where electrons are subjected to collisions with the ion background. Considering the regime in which the electron/ion mass ratio

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and the mean free path of electrons have the same magnitude, the VPFP model reads

$$(1.1) \quad \begin{cases} \partial_t f^\varepsilon + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \frac{1}{\varepsilon} \mathbf{E}^\varepsilon \cdot \nabla_{\mathbf{v}} f^\varepsilon = \frac{1}{\varepsilon^2} \nabla_{\mathbf{v}} \cdot [\mathbf{v} f^\varepsilon + \nabla_{\mathbf{v}} f^\varepsilon], \\ \mathbf{E}^\varepsilon = -\nabla_{\mathbf{x}} \phi^\varepsilon, \quad -\Delta_{\mathbf{x}} \phi^\varepsilon = \rho^\varepsilon - \rho_i, \quad \rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon d\mathbf{v}, \\ f^\varepsilon(0, \mathbf{x}, \mathbf{v}) = f_0^\varepsilon(\mathbf{x}, \mathbf{v}), \end{cases}$$

where the self consistent electric field \mathbf{E}^ε is induced by Coulombian interactions between charges whereas the Fokker-Planck operator on the right-hand side of the first line in (1.1) accounts for collisions with the ion background. A detailed description of the re-scaling process in order to derive (1.1) may be found in [9, 19]. Since mass is conserved along the trajectories of (1.1), we normalize f^ε as follows

$$\int_{\mathbb{K}^d \times \mathbb{R}^d} f_0^\varepsilon d\mathbf{x} d\mathbf{v} = 1.$$

When $\mathbb{K} = \mathbb{T}$, we also impose the compatibility assumption

$$\int_{\mathbb{K}^d \times \mathbb{R}^d} f_0^\varepsilon d\mathbf{x} d\mathbf{v} = \int_{\mathbb{K}^d} \rho_i d\mathbf{x},$$

which is then satisfied for all positive time $t \geq 0$. In this article, we focus on the asymptotic analysis of (1.1) in the diffusive regime corresponding to the limit $\varepsilon \ll 1$.

1.2. Formal derivation. In this section, we carry out a formal analysis of the diffusive regime. Our purpose is to derive the limit of f^ε as $\varepsilon \rightarrow 0$ and to explain the key idea of this article in order to rigorously justify this limit. Considering the leading order term in equation (1.1), we deduce that as ε vanishes, f^ε converges to the following local equilibrium of the Fokker-Planck operator

$$f^\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \rho^\varepsilon \otimes \mathcal{M},$$

where \mathcal{M} stands for the standard Maxwellian distribution over \mathbb{R}^d

$$(1.2) \quad \mathcal{M}(\mathbf{v}) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2} |\mathbf{v}|^2\right),$$

and where the spatial distribution ρ^ε given in (1.1) solves the following equation, obtained after integrating the first line in (1.1) with respect to \mathbf{v}

$$\partial_t \rho^\varepsilon + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \mathbf{v} f^\varepsilon d\mathbf{v} = 0.$$

It is unfortunately trickier to compute the limit of ρ^ε . The difficulty stems from the transport operator in (1.1) which induces a stiff dependence with respect to f^ε in the equation on ρ^ε . To overcome the latter difficulty, the key idea is to consider the following re-scaled version g^ε of f^ε

$$(\tau_{\varepsilon \mathbf{v}} g^\varepsilon)(t, \mathbf{x}, \mathbf{v}) = f^\varepsilon(t, \mathbf{x}, \mathbf{v}),$$

where $\tau_{\mathbf{x}_0} g$ denotes the function obtained by translating any function g in the direction $\mathbf{x}_0 \in \mathbb{R}^d$, that is

$$\tau_{\mathbf{x}_0} g(t, \mathbf{x}, \mathbf{v}) := g(t, \mathbf{x} + \mathbf{x}_0, \mathbf{v}), \quad \forall (t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}^+ \times \mathbb{K}^d \times \mathbb{R}^d.$$

Indeed, operating the change of variable $\mathbf{x} \rightarrow \mathbf{x} + \varepsilon \mathbf{v}$ in equation (1.1), we deduce that the marginal π^ε of g^ε given by

$$\pi^\varepsilon(t, \mathbf{x}) = \int_{\mathbb{R}^d} g^\varepsilon(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} = \int_{\mathbb{R}^d} \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$

solves the following equation, obtained after operating the change of variable $\mathbf{x} \rightarrow \mathbf{x} + \varepsilon \mathbf{v}$ in equation (1.1) and integrating with respect to \mathbf{v}

$$(1.3) \quad \partial_t \pi^\varepsilon + \nabla_{\mathbf{x}} \cdot \left[\int_{\mathbb{R}^d} \tau_{\varepsilon \mathbf{v}}^{-1} (\mathbf{E}^\varepsilon f^\varepsilon)(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} \right] - \Delta_{\mathbf{x}} \pi^\varepsilon = 0.$$

In the limit $\varepsilon \rightarrow 0$, we have $\rho^\varepsilon \sim \pi^\varepsilon$ and thus $\mathbf{E}^\varepsilon \sim \mathbf{I}^\varepsilon$, where

$$(1.4) \quad \mathbf{I}^\varepsilon = -\nabla_{\mathbf{x}} \psi^\varepsilon, \quad -\Delta_{\mathbf{x}} \psi^\varepsilon = \pi^\varepsilon - \rho_i.$$

Furthermore, the translation operator $\tau_{\varepsilon \mathbf{v}}^{-1}$ in (1.3) cancels as $\varepsilon \rightarrow 0$. Therefore, the dependence with respect to f^ε is removed from the non-linear term in (1.3), that is

$$\int_{\mathbb{R}^d} \tau_{\varepsilon \mathbf{v}}^{-1} (\mathbf{E}^\varepsilon f^\varepsilon)(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v} \sim \mathbf{I}^\varepsilon \pi^\varepsilon.$$

Therefore, we deduce

$$\pi^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \rho,$$

where ρ solves the following drift-diffusion-Poisson equation

$$(1.5) \quad \begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot [\mathbf{E} \rho] - \Delta_{\mathbf{x}} \rho = 0, \\ -\Delta_{\mathbf{x}} \phi = \rho - \rho_i, \quad \mathbf{E} = -\nabla_{\mathbf{x}} \phi, \end{cases}$$

Since we have $\pi^\varepsilon \sim \rho^\varepsilon$, this leads to the following formal result

$$f^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \rho \otimes \mathcal{M}.$$

As it turns out, the set of coordinates $(\mathbf{x} + \varepsilon \mathbf{v}, \mathbf{v})$ is very convenient to study the diffusive regime: it removes from the equation on ρ^ε the stiff dependence with respect to f^ε due to the transport operator, instead we end up with equation (1.3) which is consistent with the limiting model (1.5) as ε vanishes. Therefore, the distribution π^ε will play a key role in our analysis.

The present work consists in rigorously justifying this formal derivation. It is the continuation of a long process in order to justify the diffusive limit of the VFPF model. The story starts in [19], which provides local in time and weak convergence in dimension $d = 2$ and strong convergence on short time intervals without restriction on the dimension. The present article is somehow close to [19] since it uses the same functional framework. The case $d = 2$ in the latter result was then adapted in [11] to Newtonian interactions. The story goes on with [8] which achieves strong and local in time convergence without restriction on the dimension d . We also mention [14] in which a similar method is applied to the situation of a magnetized plasma with time dependent ion density. More recently, authors adapted hypocoercivity methods to the present situation in order to achieve global in time convergence. This is the case of [15] which prove global in time and strong convergence in weakly nonlinear regime (that is, under joint restrictions on the Debye length and the size of the initial data) when $d = 2$. We also mention [1], which proves that the linearized model associated (1.1) relaxes towards equilibrium at an exponential rate which is uniform with respect to ε . Authors deduce the same result on the nonlinear model in the case $d = 1$. Using similar methods, exponential relaxation as $t \rightarrow +\infty$ is proved in [13], in a weakly nonlinear setting. In perturbative settings, precise results are available [21], based on a spectral analysis of (1.1), treating simultaneously the diffusion limit with explicit convergence rates in ε and the long time behavior with optimal exponential rate in t .

In this article, we propose a strong convergence result in some L^2 space and prove that it occurs at rate $O(\varepsilon^\beta)$, with $\beta = (p-d)/(p-1)$ if f_0^ε lies in some L^p space: the (formal) optimal convergence rate is obtained as $p \rightarrow +\infty$. Our analysis is non-perturbative and it holds in any dimension $d \geq 1$. Our convergence estimate holds on bounded time intervals $[0, T^\varepsilon]$, for some explicitly given T^ε (see Theorem 2.1) which satisfies $T^\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. We point out that it should be possible to adapt our analysis to Newtonian interactions as long as the macroscopic model does not develop singularities but we do not follow this path to avoid these issues.

The article is organized as follows: in Section 2, we give our functional setting and state our main result, in Section 3 we derive uniform estimates for the solution to (1.1), then we conclude with Section 4 in which we prove our main result.

2. FUNCTIONAL SETTING AND MAIN RESULT

In the forthcoming analysis, we work with the following norm defined for all exponent $p \geq 1$

$$\|f\|_p = \left(\int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \frac{f}{\mathcal{M}} \right|^p \mathcal{M} \, d\mathbf{x} \, d\mathbf{v} \right)^{1/p},$$

and denote by $L^p(\mathcal{M})$ the set of all function whose latter norm is finite. Furthermore, we denote $\|\cdot\|_{L_{\mathbf{x}}^p}$ the usual norm over $L^p(\mathbb{K}^d)$.

Existence and uniqueness theory for (1.1) has been widely investigated and therefore it will not be our concern here. We mention [7] in which global classical solutions are constructed in dimension $d = 1, 2$, [17, 18] which extend this result to dimension $d = 3$ in both the friction and frictionless cases and notably prove regularizing effects and thus obtain infinite regularity, [3] in which existence and uniqueness of a global strong solution is obtained when $d = 3$ and with uniform bounds on the initial data and then [4] in which regularizing effects are proved for this weak solution, finally, we mention [6] which treats the case of an initial data in L^p and constructs solutions in dimensions $d = 3, 4$. Since this article does not require any constraint on the dimension, we consider a strong solution to (1.1) in dimension $d \geq 1$. Our main result reads as follows

Theorem 2.1. *Consider some exponent $p > d$ and set*

$$(\gamma, \beta) = \left(1 - \frac{d}{p}, \frac{p-d}{p-1} \right).$$

Suppose that the sequence $(f_0^\varepsilon)_{\varepsilon > 0}$ meets the following assumption

$$(2.1) \quad \|f_0^\varepsilon\|_p + \sup_{|\mathbf{x}_0| \leq 1} \left(|\mathbf{x}_0|^{-\beta} \|\tau_{\mathbf{x}_0} f_0^\varepsilon - f_0^\varepsilon\|_p \right) + \varepsilon^{-\beta} \|\rho_0^\varepsilon - \rho_0\|_{L_{\mathbf{x}}^p} \leq m_p,$$

for some positive constant m_p independent of ε . On top of that, suppose

$$(2.2) \quad \|\rho_0\|_{L_{\mathbf{x}}^p} + \|\rho_i\|_{L_{\mathbf{x}}^{p+1}} + \|\rho_0\|_{L_{\mathbf{x}}^\infty} + \|\rho_i\|_{L_{\mathbf{x}}^\infty} < +\infty.$$

Consider strong solutions $(f^\varepsilon)_{\varepsilon > 0}$ to (1.1) and ρ to (1.5) with initial data $(f_0^\varepsilon)_{\varepsilon > 0}$ and ρ_0 respectively. For all time $T > 0$, there exist two positive constants C_T and ε_T such that for all $\varepsilon \leq \varepsilon_T$, it holds

$$\|f^\varepsilon - \rho \mathcal{M}\|_{L^2([0,T], L^2(\mathcal{M}))} \leq C_T \varepsilon^\beta.$$

More precisely, there exists a constant C only depending on exponent p and dimension d such that

$$\|f^\varepsilon - \rho \mathcal{M}\|_{L^2([0,t], L^2(\mathcal{M}))} \leq \varepsilon^\beta \left(C m_p^{2p'} e^{C_{\rho_0, \rho_i} C t} + e^{C_{\rho_0, \rho_i}^{-1} C} C m_p^2 e^{C_{\rho_0, \rho_i} C t} \right),$$

for all time t less than T^ε , with T^ε is given by

$$T^\varepsilon = \frac{1}{C_{\rho_0, \rho_i} C} \ln \left(1 + \frac{C_{\rho_0, \rho_i}}{4 \left(\|f_0^\varepsilon\|_p^2 + C m_p^2 \varepsilon^{\gamma(1+\frac{2}{p-1})} \right)} \varepsilon^{-\gamma} \right),$$

where $p' = p/(p-1)$ and where the constant C_{ρ_0, ρ_i} is given by

$$C_{\rho_0, \rho_i} = \max \left(\|\rho_0\|_{L_{\mathbf{x}}^p}^2, \|\rho_i\|_{L_{\mathbf{x}}^{p+1}}^{2-2/p^2}, \|\rho_0\|_{L_{\mathbf{x}}^\infty}, \|\rho_i\|_{L_{\mathbf{x}}^\infty} \right).$$

The main difficulty consists in estimating the nonlinear term $\mathbf{E}^\varepsilon f^\varepsilon$ in (1.1). Indeed, if we follow the same method as in [19] and take directly its $\|\cdot\|_p$ -norm, we end up with the following differential inequality for $\|f^\varepsilon\|_p$, which blows up in finite time

$$\frac{d}{dt} \|f^\varepsilon\|_p^p \lesssim \|f^\varepsilon\|_p^{p+2}.$$

The key point is therefore to include the re-scaled marginal π^ε in our computations. More precisely, we perform the following decomposition

$$\mathbf{E}^\varepsilon = (\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon) + (\mathbf{I}^\varepsilon - \mathbf{E}) + \mathbf{E},$$

where \mathbf{E} and \mathbf{I}^ε are given in (1.5) and (1.4) respectively. We rely on a functional inequality to prove that $(\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon)$ is of order $\varepsilon^\gamma \|\|f^\varepsilon\|_p$ and estimate $(\mathbf{I}^\varepsilon - \mathbf{E})$ and \mathbf{E} thanks to the properties of drift-diffusion equation (1.5). It enables to derive the following differential inequality

$$\frac{d}{dt} \|\|f^\varepsilon\|_p^p \lesssim \varepsilon^\gamma \|\|f^\varepsilon\|_p^{p+2}.$$

From the latter inequality, we bound $\|\|f^\varepsilon\|_p$ on time intervals with size of order $|\ln \varepsilon|$; this provides a global in time estimate in the limit $\varepsilon \rightarrow 0$.

3. *A priori* ESTIMATES

The main object of this section consists in deriving *a priori* estimates for (1.1) in $L^p(\mathcal{M})$ (see (1) in Proposition 3.3). Building on this key estimate, we deduce that π^ε given by (1.3) converges towards the solution ρ to the macroscopic model (1.5) (see (2) in Proposition 3.3) and prove equicontinuity for solutions to (1.1) (see Proposition 3.5). Let us first introduce some notations and recall a functional inequality that will be used in the proofs of this section. For all function $\rho \in L^p \cap L^1(\mathbb{K}^d)$ with $p > 1$, which in the case $\mathbb{K} = \mathbb{T}$ meets the compatibility assumption

$$\int_{\mathbb{K}^d} \rho \, d\mathbf{x} = 0,$$

there exists a unique solution $\Delta_{\mathbf{x}}^{-1} \rho$ in $W^{p,2}(\mathbb{K}^d)$ to the Poisson equation (see [10] Section 9.6)

$$\begin{cases} \Delta_{\mathbf{x}} \phi = \rho, \\ \int_{\mathbb{K}^d} \phi \, d\mathbf{x} = 0, \quad \text{if } \mathbb{K} = \mathbb{T}. \end{cases}$$

Furthermore, thanks to Morrey's inequality, we have the following estimate

$$(3.1) \quad \|\|\nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1} \rho\|_{\mathcal{C}^{0,\gamma}} \leq m_{d,p} \|\rho\|_{L_{\mathbf{x}}^p},$$

for all exponent $p > d$, where constant $m_{d,p}$ only depends on (d,p) and where $\mathcal{C}^{0,\gamma}$ stands for the set of bounded, γ -Hölder functions, with $\gamma = 1 - d/p$. In this section, we denote by Δ_p the dissipation of L^p -norms due to the Laplace operator

$$\Delta_p[\rho] = (p-1) \int_{\mathbb{K}^d} |\nabla_{\mathbf{x}} \rho|^2 |\rho|^{p-2} \, d\mathbf{x},$$

and by \mathcal{D}_p the dissipation of $L^p(\mathcal{M})$ -norms due to the Fokker Planck operator

$$\mathcal{D}_p[f] = (p-1) \int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \nabla_{\mathbf{v}} \left(\frac{f}{\mathcal{M}} \right) \right|^2 \left| \frac{f}{\mathcal{M}} \right|^{p-2} \mathcal{M} \, d\mathbf{x} \, d\mathbf{v}.$$

We start with the following intermediate result which will be essential in order to propagate $L^p(\mathcal{M})$ -norms.

Lemma 3.1. *Consider a smooth solution f^ε to equation (1.1). For all exponent $p > d$ and all positive ε , it holds*

$$\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_{\mathbf{x}}^\infty} \leq C_{d,p} \varepsilon^\gamma \|\|f^\varepsilon\|_p, \quad \forall t \in \mathbb{R}^+,$$

where exponent γ is given by $\gamma = 1 - d/p$, and where $C_{d,p}$ is a constant only depending the dimension d and exponent p . In the latter estimates, the electric fields \mathbf{E}^ε and \mathbf{I}^ε are given by (1.1) and (1.4).

Proof. We consider some positive ε and some $(t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{K}^d$. We have

$$(\rho^\varepsilon - \pi^\varepsilon)(t, \mathbf{x}) = \int_{\mathbb{R}^d} (f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon)(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}.$$

Applying the operator $\nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1}$ to the latter relation and taking the supremum over all \mathbf{x} in \mathbb{K}^d , we obtain

$$\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty} \leq \int_{\mathbb{R}^d} \|\nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1} [f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon](t, \cdot, \mathbf{v})\|_{L_x^\infty} \, d\mathbf{v}.$$

To estimate $(f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon)(t, \cdot, \mathbf{v})$, we notice that for each $\mathbf{v} \in \mathbb{R}^d$, it holds

$$\|\nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1} [f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon](t, \cdot, \mathbf{v})\|_{L_x^\infty} \leq |\varepsilon \mathbf{v}|^\gamma \|\nabla_{\mathbf{x}} \Delta_{\mathbf{x}}^{-1} f^\varepsilon(t, \cdot, \mathbf{v})\|_{\mathcal{C}^{0,\gamma}},$$

and therefore apply Morrey's inequality (3.1) which yields

$$\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty} \leq m_{d,p} \varepsilon^\gamma \int_{\mathbb{R}^d} \|f^\varepsilon(t, \cdot, \mathbf{v})\|_{L_x^p} |\mathbf{v}|^\gamma \, d\mathbf{v}.$$

Then we rewrite the latter inequality as follows

$$\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty} \leq m_{d,p} \varepsilon^\gamma \int_{\mathbb{R}^d} \left(\int_{\mathbb{K}^d} \left| \frac{f^\varepsilon(t, \mathbf{x}, \mathbf{v})}{\mathcal{M}(\mathbf{v})} \right|^p \, d\mathbf{x} \right)^{\frac{1}{p}} |\mathbf{v}|^\gamma \mathcal{M}(\mathbf{v}) \, d\mathbf{v}.$$

Applying Hölder's inequality to the latter relation, we deduce the result

$$\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty} \leq m_{d,p} \varepsilon^\gamma \left(\int_{\mathbb{R}^d} |\mathbf{v}|^{\frac{p\gamma}{p-1}} \mathcal{M}(\mathbf{v}) \, d\mathbf{v} \right)^{\frac{p-1}{p}} \|f^\varepsilon\|_p.$$

□

We now provide estimates for the macroscopic model (1.5)

Proposition 3.2. *Consider a smooth solution ρ to equation (1.5). For all exponent p , lying in $(1, +\infty)$, it holds*

$$\|\rho(t)\|_{L_x^p} \leq \max\left(\|\rho_0\|_{L_x^p}, \|\rho_i\|_{L_x^{p+1}}^{1-1/p^2}\right), \quad \forall t \in \mathbb{R}^+.$$

Furthermore, it holds

$$\|\rho(t)\|_{L_x^\infty} \leq \max(\|\rho_0\|_{L_x^\infty}, \|\rho_i\|_{L_x^\infty}), \quad \forall t \in \mathbb{R}^+.$$

where ρ_i is given in (1.5).

We postpone the proof of this result to Appendix A since it is not the main point in our analysis.

We turn to the main result of this section, in which we provide estimates in $L^p(\mathcal{M})$ for the solution f^ε to (1.1). As a direct consequence, we derive convergence estimates for π^ε and f^ε respectively towards ρ and the local equilibrium $\rho^\varepsilon \mathcal{M}$.

Proposition 3.3. *Consider some exponent $p > d$ and set $\gamma = 1 - d/p$. Let $(f^\varepsilon)_{\varepsilon > 0}$ be a sequence of smooth solutions to (1.1) whose initial conditions meet assumption (2.1) and ρ be a smooth solution to (1.5) whose initial condition meets (2.2). There exists a constant C only depending on exponent p and dimension d such that for all $\varepsilon \leq 1$, and all t less than T^ε given in Theorem 2.1*

(1) it holds

$$\|f^\varepsilon\|_p \leq 2 \left(\|f_0^\varepsilon\|_p + \varepsilon^{\gamma \left(\frac{1}{2} + \frac{1}{p-1}\right)} C_{d,p} m_p \right) e^{C_{\rho_0, \rho_i} C_{d,p} t},$$

where C_{ρ_0, ρ_i} and m_p are given Theorem 2.1;

(2) it holds

$$\|\pi^\varepsilon - \rho\|_{L_x^2} (t) \leq \varepsilon^\beta C m_p^{2p'} e^{C_{\rho_0, \rho_i} C t},$$

where $\beta = (p - d) / (p - 1)$ and $1/p' + 1/p = 1$;

(3) it holds

$$\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2([0,t], L^2(\mathcal{M}))} \leq \varepsilon C_{d,p} m_p^2 e^{C_{\rho_0, \rho_i} C_{d,p} t}.$$

Proof. The core of this proof consists in deriving item (1) in Proposition 3.3. To do so, we consider some positive ε and for $t \in \mathbb{R}^+$ define u as follows

$$u(t) = \|f^\varepsilon(t)\|_p^2 + \varepsilon^{-\alpha} \|(\pi^\varepsilon - \rho)(t)\|_{L_x^p}^2,$$

for some positive α which needs to be determined. Our strategy consists in estimating each one of the term composing $u(t)$ separately and then to propose a combination of these two estimates which allow us to close the estimate on u . In order to simplify notations, we omit the dependence with respect to $(t, \mathbf{x}, \mathbf{v})$ when the context is clear. Furthermore, we denote by $C_{d,p}$ a positive constant depending only on exponent p and dimension d in this proof.

We start by estimating $\|f^\varepsilon(t)\|_p$. We multiply equation (1.1) by $(f^\varepsilon/\mathcal{M})^{p-1}$ and integrate over $\mathbb{K}^d \times \mathbb{R}^d$. After an integration by part with respect to \mathbf{v} , it yields

$$\frac{1}{p} \frac{d}{dt} \|f^\varepsilon\|_p^p + \frac{1}{\varepsilon^2} \mathcal{D}_p[f^\varepsilon] = \frac{p-1}{\varepsilon} \int_{\mathbb{K}^d \times \mathbb{R}^d} \frac{f^\varepsilon}{\mathcal{M}} \nabla_{\mathbf{v}} \left(\frac{f^\varepsilon}{\mathcal{M}} \right) \mathbf{E}^\varepsilon \left| \frac{f^\varepsilon}{\mathcal{M}} \right|^{p-2} \mathcal{M} \, d\mathbf{x} \, d\mathbf{v}.$$

Therefore, applying Young's inequality, we deduce

$$(3.2) \quad \frac{1}{p} \frac{d}{dt} \|f^\varepsilon\|_p^p + \frac{1-\eta}{\varepsilon^2} \mathcal{D}_p[f^\varepsilon] \leq \frac{p-1}{4\eta} \|\mathbf{E}^\varepsilon\|_{L_x^\infty}^2 \|f^\varepsilon\|_p^p,$$

for all positive η . Then we replace $\|\mathbf{E}^\varepsilon\|_{L_x^\infty}^2$ as follows in the latter estimate

$$\|\mathbf{E}^\varepsilon\|_{L_x^\infty}^2 \leq 3 \left(\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty}^2 + \|\mathbf{I}^\varepsilon - \mathbf{E}\|_{L_x^\infty}^2 + \|\mathbf{E}\|_{L_x^\infty}^2 \right),$$

and estimate the norm $\|\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon\|_{L_x^\infty}$ applying Lemma 3.1, the quantities $\|\mathbf{I}^\varepsilon - \mathbf{E}\|_{L_x^\infty}$, $\|\mathbf{E}\|_{L_x^\infty}$ applying Morrey's inequality (3.1). It yields for all $\eta > 0$

$$(3.3) \quad \frac{1}{p} \frac{d}{dt} \|f^\varepsilon\|_p^p + \frac{1-\eta}{\varepsilon^2} \mathcal{D}_p[f^\varepsilon] \leq \frac{C_{d,p}}{\eta} \left(\varepsilon^{2\gamma} \|f^\varepsilon\|_p^2 + \|\pi^\varepsilon - \rho\|_{L_x^p}^2 + \|\rho\|_{L_x^p}^2 \right) \|f^\varepsilon\|_p^p.$$

We now estimate $\|(\pi^\varepsilon - \rho)(t)\|_{L_x^p}$. To do so, we consider the difference between equations (1.3) and (1.5) multiplied by $(\pi^\varepsilon - \rho) |\pi^\varepsilon - \rho|^{p-2}$ and integrated over \mathbb{K}^d . After an integration by part, it yields

$$\frac{1}{p} \frac{d}{dt} \|\pi^\varepsilon - \rho\|_{L_x^p}^p + \Delta_p[\pi^\varepsilon - \rho] = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,$$

where \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{A}_3 are given by

$$\begin{cases} \mathcal{A}_1 = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}}(\pi^\varepsilon - \rho) \int_{\mathbb{R}^d} (\tau_{\varepsilon \mathbf{v}}^{-1} \mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon) g^\varepsilon \, d\mathbf{v} |\pi^\varepsilon - \rho|^{p-2} \, d\mathbf{x}, \\ \mathcal{A}_2 = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}}(\pi^\varepsilon - \rho) (\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon) \pi^\varepsilon |\pi^\varepsilon - \rho|^{p-2} \, d\mathbf{x}, \\ \mathcal{A}_3 = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}}(\pi^\varepsilon - \rho) (\mathbf{I}^\varepsilon \pi^\varepsilon - \mathbf{E} \rho) |\pi^\varepsilon - \rho|^{p-2} \, d\mathbf{x}. \end{cases}$$

\mathcal{A}_1 and \mathcal{A}_2 are error terms which we estimate using Sobolev inequalities whereas we estimate \mathcal{A}_3 using the properties of the limiting macroscopic equation (1.5).

Let us start with \mathcal{A}_1 , which we estimate using Young inequality

$$\mathcal{A}_1 \leq \eta \Delta_p[\pi^\varepsilon - \rho] + \frac{p-1}{4\eta} \mathcal{A}_{11},$$

for all positive η and where \mathcal{A}_{11} is given by

$$\mathcal{A}_{11} = \int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} (\tau_{\varepsilon \mathbf{v}}^{-1} \mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon) g^\varepsilon d\mathbf{v} \right|^2 |\pi^\varepsilon - \rho|^{p-2} d\mathbf{x}.$$

Thanks to Morrey inequality (3.1), we have for all $\mathbf{v} \in \mathbb{R}^d$

$$\|\tau_{\varepsilon \mathbf{v}}^{-1} \mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon\|_{L^\infty} \leq C_{d,p} |\varepsilon \mathbf{v}|^\gamma \|\rho^\varepsilon\|_{L_x^p}.$$

After applying Jensen inequality to estimate $\|\rho^\varepsilon\|_{L_x^p}$, this yields

$$\|\tau_{\varepsilon \mathbf{v}}^{-1} \mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon\|_{L^\infty} \leq C_{d,p} |\varepsilon \mathbf{v}|^\gamma \|f^\varepsilon\|_p.$$

We substitute $\tau_{\varepsilon \mathbf{v}}^{-1} \mathbf{E}^\varepsilon - \mathbf{E}^\varepsilon$ with the latter estimate in the definition of \mathcal{A}_{11} and deduce

$$\mathcal{A}_{11} = C_{d,p} \varepsilon^{2\gamma} \|f^\varepsilon\|_p^2 \int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} |\mathbf{v}|^\gamma g^\varepsilon d\mathbf{v} \right|^2 |\pi^\varepsilon - \rho|^{p-2} d\mathbf{x}.$$

Applying Hölder's inequality, we obtain

$$\mathcal{A}_{11} \leq C_{d,p} \varepsilon^{2\gamma} \|f^\varepsilon\|_p^2 \|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2} \left(\int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} |\mathbf{v}|^\gamma g^\varepsilon d\mathbf{v} \right|^p d\mathbf{x} \right)^{2/p}.$$

To estimate the integral in the latter inequality, we apply Hölder inequality

$$\left(\int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} |\mathbf{v}|^\gamma g^\varepsilon d\mathbf{v} \right|^p d\mathbf{x} \right)^{2/p} \leq \left(\int_{\mathbb{R}^d} |\mathbf{v}|^{p'\gamma} \mathcal{M} d\mathbf{v} \right)^{2/p'} \|g^\varepsilon\|_p^2.$$

Then we notice that $\|g^\varepsilon\|_p = \|f^\varepsilon\|_p$ and deduce

$$\mathcal{A}_{11} \leq C_{d,p} \varepsilon^{2\gamma} \|f^\varepsilon\|_p^4 \|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2}.$$

In the end, we deduce the following bound for \mathcal{A}_1

$$\mathcal{A}_1 \leq \eta \Delta_p [\pi^\varepsilon - \rho] + \frac{C_{d,p}}{\eta} \varepsilon^{2\gamma} \|f^\varepsilon\|_p^4 \|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2}.$$

To estimate \mathcal{A}_2 , we follow the same method as before excepted that we apply Lemma 3.1 to bound $\mathbf{E}^\varepsilon - \mathbf{I}^\varepsilon$, it yields

$$\mathcal{A}_2 \leq \eta \Delta_p [\pi^\varepsilon - \rho] + \frac{C_{d,p}}{\eta} \varepsilon^{2\gamma} \|f^\varepsilon\|_p^4 \|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2}.$$

We turn to the last term \mathcal{A}_3 , which decomposes as follows

$$\mathcal{A}_3 = \mathcal{A}_{31} + \mathcal{A}_{32} + \mathcal{A}_{33},$$

where the latter terms are given by

$$\begin{cases} \mathcal{A}_{31} = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}} (\pi^\varepsilon - \rho) (\mathbf{I}^\varepsilon - \mathbf{E}) (\pi^\varepsilon - \rho) |\pi^\varepsilon - \rho|^{p-2} d\mathbf{x}, \\ \mathcal{A}_{32} = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}} (\pi^\varepsilon - \rho) (\mathbf{I}^\varepsilon - \mathbf{E}) \rho |\pi^\varepsilon - \rho|^{p-2} d\mathbf{x}, \\ \mathcal{A}_{33} = (p-1) \int_{\mathbb{K}^d} \nabla_{\mathbf{x}} (\pi^\varepsilon - \rho) \mathbf{E} (\pi^\varepsilon - \rho) |\pi^\varepsilon - \rho|^{p-2} d\mathbf{x}. \end{cases}$$

We start with \mathcal{A}_{31} , which rewrites as follows after an integration by part

$$\mathcal{A}_{31} = -\frac{p-1}{p} \int_{\mathbb{K}^d} |\pi^\varepsilon - \rho|^p \nabla_{\mathbf{x}} \cdot (\mathbf{I}^\varepsilon - \mathbf{E}) d\mathbf{x}.$$

Therefore, replacing \mathbf{I}^ε and \mathbf{E} according to equations (1.4) and (1.5), we deduce the following relation

$$\mathcal{A}_{31} = -\frac{p-1}{p} \int_{\mathbb{K}^d} |\pi^\varepsilon - \rho|^p (\pi^\varepsilon - \rho) d\mathbf{x}.$$

Since π^ε has positive values and taking the absolute value of ρ , we obtain

$$\mathcal{A}_{31} \leq \frac{p-1}{p} \|\rho\|_{L_x^\infty} \|\pi^\varepsilon - \rho\|_{L_x^p}^p.$$

Remark 3.4. *It is the only time we use the L^∞ -norm of ρ in our analysis.*

To estimate \mathcal{A}_{32} and \mathcal{A}_{33} , we use the same techniques as the ones already used to estimate \mathcal{A}_1 and \mathcal{A}_2 . Therefore, we do not detail the computations. In the end it yields

$$\mathcal{A}_{32} + \mathcal{A}_{33} \leq \eta \Delta_p [\pi^\varepsilon - \rho] + \frac{C_{d,p}}{\eta} \|\rho\|_{L_x^p}^2 \|\pi^\varepsilon - \rho\|_{L_x^p}^p,$$

for all positive η . Gathering latter computations and taking η small enough, we obtain the following estimate

$$\frac{1}{p} \frac{d}{dt} \|\pi^\varepsilon - \rho\|_{L_x^p}^p \leq C_{d,p} \left(\varepsilon^{2\gamma} \|f^\varepsilon\|_p^4 \|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2} + \left(\|\rho\|_{L_x^p}^2 + \|\rho\|_{L_x^\infty} \right) \|\pi^\varepsilon - \rho\|_{L_x^p}^p \right),$$

for some constant $C_{d,p}$ only depending on d and p . Dividing by $\|\pi^\varepsilon - \rho\|_{L_x^p}^{p-2}$ the latter estimate, this yields

$$(3.4) \quad \frac{1}{2} \frac{d}{dt} \|\pi^\varepsilon - \rho\|_{L_x^p}^2 \leq C_{d,p} \left(\varepsilon^{2\gamma} \|f^\varepsilon\|_p^4 + \left(\|\rho\|_{L_x^p}^2 + \|\rho\|_{L_x^\infty} \right) \|\pi^\varepsilon - \rho\|_{L_x^p}^2 \right).$$

It is now possible to obtain item (1) in Proposition 3.3: we set η to 1 in (3.3) and take the sum between estimate (3.3) divided by $\|f^\varepsilon\|_p^{p-2}$ and estimate (3.4), we deduce that u verifies the following differential inequality

$$\frac{1}{2} \frac{d}{dt} u(t) \leq C_{d,p} \left((\varepsilon^{2\gamma-\alpha} + \varepsilon^{2\gamma}) \|f^\varepsilon\|_p^4 + \|\pi^\varepsilon - \rho\|_{L_x^p}^2 \|f^\varepsilon\|_p^2 + C_\rho \left(\varepsilon^{-\alpha} \|\pi^\varepsilon - \rho\|_{L_x^p}^2 + \|f^\varepsilon\|_p^2 \right) \right),$$

where C_ρ is given by

$$C_\rho = \|\rho\|_{L_x^p}^2 + \|\rho\|_{L_x^\infty}.$$

Hence, taking $\alpha = \gamma$ and applying Lemma 3.2 to estimate C_ρ , we deduce

$$\frac{d}{dt} u(t) \leq C_{d,p} \left(\varepsilon^\gamma u(t)^2 + C_{\rho_0, \rho_i} u(t) \right),$$

where C_{ρ_0, ρ_i} is given by

$$C_{\rho_0, \rho_i} = \max \left(\|\rho_0\|_{L_x^p}^2, \|\rho_i\|_{L_x^{p+1}}^{2-2/p^2}, \|\rho_0\|_{L_x^\infty}, \|\rho_i\|_{L_x^\infty} \right).$$

We divide the latter estimate by $(\varepsilon^\gamma u^2 + C_{\rho_0, \rho_i} u)$ and notice that

$$\frac{1}{\varepsilon^\gamma u^2 + C_{\rho_0, \rho_i} u} = \frac{1}{C_{\rho_0, \rho_i}} \left(\frac{1}{u} - \frac{\varepsilon^\gamma}{\varepsilon^\gamma u + C_{\rho_0, \rho_i}} \right),$$

therefore, we obtain

$$\frac{d}{dt} \ln \left(\frac{u(t)}{\varepsilon^\gamma u(t) + C_{\rho_0, \rho_i}} \right) \leq C_{\rho_0, \rho_i} C_{d,p}.$$

Integrating between 0 and t and taking the exponential in the latter estimate, it yields

$$(3.5) \quad u(t) \leq u_0 \left(1 - \varepsilon^\gamma \frac{u_0}{C_{\rho_0, \rho_i}} (e^{C_{\rho_0, \rho_i} C_{d,p} t} - 1) \right)^{-1} e^{C_{\rho_0, \rho_i} C_{d,p} t},$$

for all time t verifying

$$t < \frac{1}{C_{\rho_0, \rho_i} C_{d,p}} \ln \left(1 + \frac{C_{\rho_0, \rho_i}}{u_0} \varepsilon^{-\gamma} \right).$$

To conclude this step, we estimate u_0 by applying the triangular inequality

$$(3.6) \quad u_0 \leq \|f_0^\varepsilon\|_p^2 + 2\varepsilon^{-\gamma} \left(\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_x^p}^2 + \|\rho_0^\varepsilon - \rho_0\|_{L_x^p}^2 \right).$$

To estimate $\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_{\mathbf{x}}^p}$, we apply Hölder's inequality to the right hand side of the following relation

$$\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_{\mathbf{x}}^p} = \left(\int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} \frac{(\tau_{\varepsilon \mathbf{v}}^{-1} f_0^\varepsilon - f_0^\varepsilon)(\mathbf{x}, \mathbf{v})}{\mathcal{M}(\mathbf{v}) |\varepsilon \mathbf{v}|^\beta} |\varepsilon \mathbf{v}|^\beta \mathcal{M}(\mathbf{v}) d\mathbf{v} \right|^p d\mathbf{x} \right)^{1/p},$$

which yields

$$\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_{\mathbf{x}}^p} \leq C_{d,p} \varepsilon^\beta \left(\int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \frac{(\tau_{\varepsilon \mathbf{v}}^{-1} f_0^\varepsilon - f_0^\varepsilon)(\mathbf{x}, \mathbf{v})}{\mathcal{M}(\mathbf{v}) |\varepsilon \mathbf{v}|^\beta} \right|^p \mathcal{M}(\mathbf{v}) d\mathbf{v} d\mathbf{x} \right)^{1/p}.$$

Then we split the integral in the right hand side of the latter inequality by distinguishing the case where $|\varepsilon \mathbf{v}| \leq 1$ from the case where $|\varepsilon \mathbf{v}| > 1$, this yields

$$\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_{\mathbf{x}}^p} \leq C_{d,p} \varepsilon^\beta \left(\|f_0^\varepsilon\|_p + \sup_{|\mathbf{x}_0| \leq 1} |\mathbf{x}_0|^{-\beta} \|\tau_{\mathbf{x}_0} f_0^\varepsilon - f_0^\varepsilon\|_p \right).$$

We substitute $\|\pi_0^\varepsilon - \rho_0^\varepsilon\|_{L_{\mathbf{x}}^p}$ according to the latter estimate in (3.6) and apply assumption (2.1), it yields

$$u_0 \leq \|f_0^\varepsilon\|_p^2 + C_{d,p} m_p^2 \varepsilon^{\gamma(1 + \frac{2}{p-1})},$$

where m_p is the constant in (2.1). Replacing u_0 in (3.5) thanks to the latter inequality, we deduce item (1) of Proposition 3.3, that is

$$\|f^\varepsilon\|_p \leq 2 \left(\|f_0^\varepsilon\|_p + \varepsilon^{\gamma(\frac{1}{2} + \frac{1}{p-1})} C_{d,p} m_p \right) e^{C_{\rho_0, \rho_i} C_{d,p} t},$$

for all time t less than T^ε , where T^ε is given by

$$T^\varepsilon = \frac{1}{C_{\rho_0, \rho_i} C_{d,p}} \ln \left(1 + \frac{C_{\rho_0, \rho_i}}{4 \left(\|f_0^\varepsilon\|_p^2 + \varepsilon^{\gamma(1 + \frac{2}{p-1})} C_{d,p} m_p^2 \right)} \varepsilon^{-\gamma} \right).$$

In order to prove item (2), we consider relation (3.4), replace $\|f^\varepsilon(t)\|_p$ with the estimate given by item (1) in Proposition 3.3 and apply Lemma 3.2 to bound ρ , this yields

$$\frac{d}{dt} \|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^p}^2 \leq C_{d,p} \left(\varepsilon^{2\gamma} m_p^4 e^{C_{\rho_0, \rho_i} C_{d,p} t} + C_{\rho_0, \rho_i} \|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^p}^2 \right),$$

for all time t less than T^ε , where m_p and C_{ρ_0, ρ_i} are given in Theorem 2.1. Multiplying the latter estimate by $e^{-C_{\rho_0, \rho_i} C_{d,p} t}$, integrating between 0 and t and taking the square root on both sides of the inequality, we obtain

$$\|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^p} \leq e^{C_{\rho_0, \rho_i} C_{d,p} t} \left(\|\pi_0^\varepsilon - \rho_0\|_{L_{\mathbf{x}}^p} + \varepsilon^\gamma C_{d,p} m_p^2 t \right).$$

Since we already proved in the first step that $\|\pi_0^\varepsilon - \rho_0\|_{L_{\mathbf{x}}^p} \leq C_{d,p} m_p \varepsilon^{\gamma(1 + \frac{2}{p-1})}$, and since $1 \leq m_p$, we deduce

$$\|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^p} \leq \varepsilon^\gamma C_{d,p} m_p^2 e^{C_{\rho_0, \rho_i} C_{d,p} t}.$$

Item (2) in Proposition 3.3 is obtained applying the following interpolation inequality in the latter estimate

$$\|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^2} \leq \|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^1}^{\frac{p-2}{p-1}} \|\pi^\varepsilon - \rho\|_{L_{\mathbf{x}}^{p'}}^{1/p'},$$

where p' is given by $1/p' + 1/p = 1$ and noticing that $p' \gamma = \beta$.

To prove item (3) in Proposition 3.3, we set $p = 2$ in (3.2), apply Morrey's inequality (3.1) to estimate \mathbf{E}^ε and Jensen's inequality, which ensures $\|f^\varepsilon\|_2 \leq \|f^\varepsilon\|_p$. This yields

$$\frac{1}{2} \frac{d}{dt} \|f^\varepsilon\|_2^2 + \frac{1-\eta}{\varepsilon^2} \mathcal{D}_2[f^\varepsilon] \leq \frac{C_{d,p}}{\eta} \|f^\varepsilon\|_p^4.$$

Therefore, taking $\eta = 1/2$, integrating the latter estimate between 0 and t and replacing $\|f^\varepsilon\|_p$ by the estimate given in item (1) in Proposition 3.3 we obtain

$$\int_0^t \mathcal{D}_2[f^\varepsilon] \, ds \leq \varepsilon^2 \left(\|f_0^\varepsilon\|_p^2 + C_{d,p} \int_0^t m_p^4 e^{C_{\rho_0, \rho_i} C_{d,p} s} \, ds \right),$$

for all time t between 0 and T^ε . Hence, applying the Gaussian-Poincaré inequality which ensures $\|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_2^2 \leq \mathcal{D}_2[f^\varepsilon]$, we deduce

$$\int_0^t \|f^\varepsilon - \rho^\varepsilon \mathcal{M}\|_2^2 \, ds \leq C_{d,p} \varepsilon^2 m_p^4 e^{C_{\rho_0, \rho_i} C_{d,p} t},$$

which yields the result by taking the square root of the latter estimate. \square

Building on Proposition 3.3, we are able to prove equicontinuity estimates for f^ε in $L^p(\mathcal{M})$.

Proposition 3.5. *Consider some exponent p greater than the dimension d and set $\beta = (p-d)/(p-1)$. Under assumptions (2.2) on (ρ_0, ρ_i) and (2.1) on the sequence of initial conditions $(f_0^\varepsilon)_{\varepsilon>0}$, consider a sequence of solutions $(f^\varepsilon)_{\varepsilon>0}$ to (1.1). There exists a constant C only depending on exponent p and dimension d such that for all positive ε less than 1, it holds*

$$\sup_{|\mathbf{x}_0| \leq 1} |\mathbf{x}_0|^{-\beta} \|f^\varepsilon - \tau_{\mathbf{x}_0} f^\varepsilon\|_2(t) \leq e^{C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t}} \sup_{|\mathbf{x}_0| \leq 1} |\mathbf{x}_0|^{-\beta} \|f_0^\varepsilon - \tau_{\mathbf{x}_0} f_0^\varepsilon\|_p.$$

for all time t less than T^ε , where T^ε , m_p and C_{ρ_0, ρ_i} are given in Theorem 2.1.

Proof. We consider $\varepsilon > 0$ and $(t, \mathbf{x}_0) \in \mathbb{R}^+ \times \mathbb{K}^d$ such that $|\mathbf{x}_0| \leq 1$. We first compute the equation solved by $h^\varepsilon := f^\varepsilon - \tau_{\mathbf{x}_0} f^\varepsilon$. It is given by the difference between equation (1.1) and equation (1.1) translated by \mathbf{x}_0 with respect to the spatial variable, that is

$$\partial_t h^\varepsilon + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} h^\varepsilon + \frac{1}{\varepsilon} (\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon) \cdot \nabla_{\mathbf{v}} f^\varepsilon + \frac{1}{\varepsilon} \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon \cdot \nabla_{\mathbf{v}} h^\varepsilon = \frac{1}{\varepsilon^2} \nabla_{\mathbf{v}} \cdot [\mathbf{v} h^\varepsilon + \nabla_{\mathbf{v}} h^\varepsilon].$$

Then we compute the norm of h^ε by multiplying the latter equation by $(h^\varepsilon/\mathcal{M})|h^\varepsilon/\mathcal{M}|^{p-2}$ and integrating over $\mathbb{K}^d \times \mathbb{R}^d$. After integrating by part with respect to \mathbf{v} , this yields

$$\frac{1}{p} \frac{d}{dt} \|h^\varepsilon\|_p^p + \frac{1}{\varepsilon^2} \mathcal{D}_p[h^\varepsilon] = \frac{p-1}{\varepsilon} \int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \frac{h^\varepsilon}{\mathcal{M}} \right|^{p-2} \nabla_{\mathbf{v}} \left(\frac{h^\varepsilon}{\mathcal{M}} \right) \cdot (f^\varepsilon (\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon) + h^\varepsilon \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon) \, d\mathbf{x} \, d\mathbf{v}.$$

We apply Young's inequality to estimate the right hand side in the latter inequality and deduce

$$\frac{1}{p} \frac{d}{dt} \|h^\varepsilon\|_p^p \leq \frac{p-1}{2} \int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \frac{h^\varepsilon}{\mathcal{M}} \right|^{p-2} \left(\left| \frac{f^\varepsilon}{\mathcal{M}} \right|^2 |\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon|^2 + \left| \frac{h^\varepsilon}{\mathcal{M}} \right|^2 |\tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon|^2 \right) \mathcal{M} \, d\mathbf{x} \, d\mathbf{v}.$$

After taking the uniform norms of $\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon$ and \mathbf{E}^ε and applying Hödler's inequality to estimate the cross product between f^ε and h^ε , it yields

$$\frac{1}{p} \frac{d}{dt} \|h^\varepsilon\|_p^p \leq \frac{p-1}{2} \left(\|\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon\|_{L_x^\infty}^2 \|h^\varepsilon\|_p^{p-2} \|f^\varepsilon\|_p^2 + \|\mathbf{E}^\varepsilon\|_{L_x^\infty}^2 \|h^\varepsilon\|_p^p \right).$$

We apply Morrey's inequality (3.1) to estimate the uniform norms of $\mathbf{E}^\varepsilon - \tau_{\mathbf{x}_0} \mathbf{E}^\varepsilon$ and \mathbf{E}^ε , it yields

$$\frac{1}{p} \frac{d}{dt} \|h^\varepsilon\|_p^p \leq C_{d,p} \|h^\varepsilon\|_p^p \|f^\varepsilon\|_p^2.$$

To conclude, we divide the latter estimate by $\|h^\varepsilon\|_p^{p-1}$, multiply it by $e^{-C_{d,p} \int_0^t \|f^\varepsilon\|_p^2 \, ds}$, integrate between 0 and t and replace $\|f^\varepsilon\|_p$ with the estimate in item (1) of Proposition 3.3, it yields

$$\|h^\varepsilon\|_p \leq \exp \left(C_{\rho_0, \rho_i}^{-1} \left(\|f_0^\varepsilon\|_p + C_{d,p} m_p \varepsilon^{\frac{\gamma}{2}} \right)^2 e^{C_{d,p} C_{\rho_0, \rho_i} t} \right) \|h_0^\varepsilon\|_p,$$

for all time t less than T^ε , where T^ε is given in Theorem 2.1. We obtain the result dividing the latter estimate by $|\mathbf{x}_0|^\beta$, taking the supremum over all \mathbf{x}_0 with norm less than 1 and since according to Jensen's inequality it holds $\|h^\varepsilon\|_2 \leq \|h^\varepsilon\|_p$. \square

4. PROOF OF THEOREM 2.1

We consider the following decomposition of the quantity that we need to estimate

$$\|f^\varepsilon - \rho \mathcal{M}\|_{L^2([0,t], L^2(\mathcal{M}))} \leq (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)(t),$$

where \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{E}_3 are given by

$$\begin{cases} \mathcal{E}_1(t) = \|f_0^\varepsilon - \rho^\varepsilon \mathcal{M}\|_{L^2([0,t], L^2(\mathcal{M}))}, \\ \mathcal{E}_2(t) = \|\rho^\varepsilon - \pi^\varepsilon\|_{L^2([0,t], L_{\mathbf{x}}^2(\mathbb{R}^d))}, \\ \mathcal{E}_3(t) = \|\pi^\varepsilon - \rho\|_{L^2([0,t], L_{\mathbf{x}}^2(\mathbb{R}^d))}. \end{cases}$$

We estimate \mathcal{E}_1 and \mathcal{E}_3 thanks to Proposition 3.3. Indeed, according to item (3), it holds

$$\mathcal{E}_1(t) = C \varepsilon m_p^2 e^{C_{\rho_0, \rho_i} C t},$$

and according to item (2), it holds

$$\mathcal{E}_3(t) \leq C \varepsilon^\beta m_p^{2p'} e^{C_{\rho_0, \rho_i} C t},$$

for all time t less than T^ε , where T^ε is given in Theorem 2.1. We turn to the last term \mathcal{E}_2 . To estimate $\|\rho^\varepsilon - \pi^\varepsilon\|_{L_{\mathbf{x}}^2}$, we apply Cauchy-Schwarz inequality to the right hand side of the following relation

$$\|\rho^\varepsilon - \pi^\varepsilon\|_{L_{\mathbf{x}}^2} = \left(\int_{\mathbb{K}^d} \left| \int_{\mathbb{R}^d} \frac{(f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon)(t, \mathbf{x}, \mathbf{v})}{\mathcal{M}(\mathbf{v}) |\varepsilon \mathbf{v}|^\beta} |\varepsilon \mathbf{v}|^\beta \mathcal{M}(\mathbf{v}) d\mathbf{v} \right|^2 d\mathbf{x} \right)^{1/2}.$$

It yields

$$\|\pi^\varepsilon - \rho^\varepsilon\|_{L_{\mathbf{x}}^2} \leq \varepsilon^\beta \left(\int_{\mathbb{K}^d \times \mathbb{R}^d} \left| \frac{(f^\varepsilon - \tau_{\varepsilon \mathbf{v}}^{-1} f^\varepsilon)(t, \mathbf{x}, \mathbf{v})}{\mathcal{M}(\mathbf{v}) |\varepsilon \mathbf{v}|^\beta} \right|^2 \mathcal{M}(\mathbf{v}) d\mathbf{v} d\mathbf{x} \right)^{1/2},$$

where we used that $\int_{\mathbb{R}^d} |v|^{2\beta} \mathcal{M} d\mathbf{v} \leq 1$. Then we split the integral in the right hand side of the latter inequality by distinguishing the case where $|\varepsilon \mathbf{v}| \leq 1$ from the case where $|\varepsilon \mathbf{v}| > 1$, this yields

$$\|\pi^\varepsilon - \rho^\varepsilon\|_{L_{\mathbf{x}}^2} \leq \varepsilon^\beta \left(2 \|f^\varepsilon\|_2 + \sup_{|\mathbf{x}_0| \leq 1} |\mathbf{x}_0|^{-\beta} \|\tau_{\mathbf{x}_0} f^\varepsilon - f^\varepsilon\|_2 \right).$$

Then we apply item (1) in Proposition 3.3 to estimate the norm of f^ε and Proposition 3.5 to estimate the norm of $\tau_{\mathbf{x}_0} f^\varepsilon - f^\varepsilon$. It yields

$$\|\pi^\varepsilon - \rho^\varepsilon\|_{L_{\mathbf{x}}^2} \leq \varepsilon^\beta (C m_p e^{C_{\rho_0, \rho_i} C t} + \exp(C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t})),$$

for all time t less than T^ε , where T^ε is given in Theorem 2.1. Taking the L^2 norm of the latter inequality over the time interval $[0, t]$, it yields

$$\|\pi^\varepsilon - \rho^\varepsilon\|_{L^2([0,t], L_{\mathbf{x}}^2(\mathbb{R}^d))} \leq \varepsilon^\beta \left(C_{\rho_0, \rho_i}^{-1/2} m_p e^{C_{\rho_0, \rho_i} C t} + \exp(C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t}) \right),$$

where we used the following inequality to estimate the time integral of the double exponential term

$$\exp(C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t}) \leq \frac{1}{C^2 m_p^2} \frac{d}{dt} \exp(C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t}).$$

Gathering the estimate on \mathcal{E}_1 , \mathcal{E}_2 and \mathcal{E}_3 , we obtain the estimate in Theorem 2.1, that is

$$\|f^\varepsilon - \rho \mathcal{M}\|_{L^2([0,t], L^2(\mathcal{M}))} \leq \varepsilon^\beta \left(C m_p^{2p'} e^{C_{\rho_0, \rho_i} C t} + \exp(C_{\rho_0, \rho_i}^{-1} C m_p^2 e^{C_{\rho_0, \rho_i} C t}) \right),$$

for all time t less than T^ε , where T^ε is given in Proposition 3.3.

5. CONCLUSION

We have proposed a method in order to treat the diffusive scaling for the VFPF model which provides non-perturbative strong convergence results with explicit rates. Our approach may be regarded as an alternative to compactness methods relying on averaging lemmas widely used in this context [8, 14, 16, 20] with the advantage that it provides explicit convergence rates. An interesting and challenging improvement would consist in conciliating this approach with hypocoercivity methods [15, 1, 13] which present the advantage of providing global in time convergence estimates but which fail, for now, to provide non-perturbative results in the physical case. To be noted that up to our knowledge, non-perturbative results [5, 2] treating the long time behavior of (1.1) rely on compactness arguments and are thus non quantitative. Therefore, associating hypocoercivity methods with the one presented in this article might be a way to treat simultaneously the diffusive regime $\varepsilon \rightarrow 0$ and the long time behavior $t \rightarrow +\infty$ in a non-perturbative setting and with explicit rates. Other natural questions arise from this work such as adapting this method in order to analyze other asymptotic regimes [12] or to check its robustness on other collision operators such as linearized Boltzmann operators.

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APPENDIX A. PROOF OF PROPOSITION 3.2

Let us start with the case where exponent p is strictly less than $+\infty$. We compute the time derivative of $\|\rho\|_{L_x^p}^p$ by multiplying equation (1.5) by ρ^{p-1} and integrating with respect to \mathbf{x} . After an integration by part, we obtain

$$\frac{1}{p} \frac{d}{dt} \|\rho\|_{L_x^p}^p + \Delta_p[\rho] = \frac{p-1}{p} \mathcal{D},$$

where \mathcal{D} is given by

$$\mathcal{D} = \int_{\mathbb{K}^d} \mathbf{E} \cdot \nabla_{\mathbf{x}} \rho^p \, d\mathbf{x}.$$

Integrating by part and according to equation (1.5), we rewrite \mathcal{D} as follows

$$\mathcal{D} = \int_{\mathbb{K}^d} (\rho_i - \rho) \rho^p \, d\mathbf{x} = \int_{\mathbb{K}^d} \rho_i \rho^p \, d\mathbf{x} - \|\rho\|_{L_x^{p+1}}^{p+1}.$$

To estimate the \mathcal{D} , we apply Hölder's inequality

$$\mathcal{D} \leq \|\rho_i\|_{L_x^{p+1}} \|\rho\|_{L_x^{p+1}}^p - \|\rho\|_{L_x^{p+1}}^{p+1},$$

and therefore deduce

$$\mathcal{D} \leq \|\rho_i\|_{L_x^{p+1}}^{p+1} \mathbf{1}_{\|\rho\|_{L_x^{p+1}} \leq \|\rho_i\|_{L_x^{p+1}}}.$$

Furthermore, we use that ρ is a probability measure to apply Jensen inequality and we deduce

$$\|\rho\|_{L_x^p} \leq \|\rho\|_{L_x^{p+1}}^{(p^2-1)/p^2}.$$

Injecting this inequality in the latter estimate on \mathcal{D} , we obtain

$$\mathcal{D} \leq \|\rho_i\|_{L_x^{p+1}}^{p+1} \mathbf{1}_{\|\rho\|_{L_x^p} \leq \|\rho_i\|_{L_x^{p+1}}^{(p^2-1)/p^2}}.$$

Therefore, we obtain

$$\frac{d}{dt} \|\rho\|_{L_x^p}^p \leq (p-1) \|\rho_i\|_{L_x^{p+1}}^{p+1} \mathbf{1}_{\|\rho\|_{L_x^p} \leq \|\rho_i\|_{L_x^{p+1}}^{(p^2-1)/p^2}}.$$

One can check that for any positive η , the constant

$$C_\eta = \max\left(\|\rho_0\|_{L_x^p}^p, \|\rho_i\|_{L_x^{p+1}}^{(p^2-1)/p}\right) + \eta$$

is a super solution to the latter differential inequality. Therefore, it holds for all $\eta > 0$

$$\|\rho\|_{L_x^p}^p \leq C_\eta.$$

Hence, taking $\eta \rightarrow 0$, we obtain the expected result

$$\|\rho\|_{L_x^p} \leq \max\left(\|\rho_0\|_{L_x^p}, \|\rho_i\|_{L_x^{p+1}}^{1-1/p^2}\right),$$

for all time $t \geq 0$. The case $p = +\infty$ is obtained taking the limit $p \rightarrow +\infty$ in the latter estimate.

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