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Control problems for the Navier-Stokes system with nonlocal spatial terms

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Abstract

We consider the local null controllability of a modified Navier-Stokes system where we include nonlocal spatial terms. We generalize a previous work where the nonlocal spatial term is given by the linearization of a Ladyzhenskaya model for a viscous incompressible fluid. Here the nonlocal spatial term is more complicated and we consider a control with one vanishing component. The proof of the result is based on a Carleman estimate where the main difficulty consists in handling the nonlocal spatial terms. One key point corresponds to a particular decomposition of the solution of the adjoint system that allows us to overcome regularity issues. With a similar approach, we also show the existence of insensitizing controls for the same system.

Keywords: Navier-Stokes system, controllability, Carleman estimates, nonlocal spatial terms

2010 Mathematics Subject Classification. 76D05, 93C20, 93B05, 93B07

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1 Introduction

Assume Ω is a domain of \mathbb{R}^d , $d = 2, 3$ with a smooth boundary $\partial\Omega$. We are interested in the control of the Navier-Stokes system with nonlocal spatial terms:

$$\left\{ \begin{array}{l} \partial_t z - \Delta z + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p + (z \cdot \nabla)z = f + \mathbf{1}_{\omega} u \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} z = 0 \quad \text{in } (0, T) \times \Omega, \\ z = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ z(0, \cdot) = z^0 \quad \text{in } \Omega. \end{array} \right. \quad (1.1)$$

The control u is localized in a nonempty domain $\omega \subset \Omega$. A similar system was considered in [16] for controllability issues, and in that article, the nonlocal spatial term appears from the linearization of a model considered by Ladyzhenskaya [19]. Some authors have also considered the controllability of partial differential equations with nonlocal spatial terms in [2, 13, 14, 17, 18, 20, 21, 22, 24]. Note in particular that in [14], the authors consider a general nonlocal term in the heat and in the wave equation of the form

$$\int_{\Omega} k(x, y)z(t, y) \, dy.$$

The nonlocal term in (1.1) can be written as above by setting $k(x, y) = \sum_{i=1}^n a^{(i)}(x)b^{(i)}(y)$. In [14], the authors do not need this particular form of the kernel and show the null-controllability of the heat equation but their method is based on a compactness-uniqueness argument that does not permit to deduce directly a controllability result on the nonlinear systems. Moreover, their kernel has to satisfy analytical conditions. Let us also emphasize that in [13], the authors consider a nonlinear heat equation with a nonlocal term similar to the one here. They first show the approximate controllability of the linearized system by using a compactness-uniqueness argument and then deduce the approximate controllability of the nonlinear system by using a Kakutani fixed point argument. They obtain the local exact controllability to trajectories with a passage to the limit. In [16], we consider a kernel of the above form but with $n = 1$. Here we deal with the case $n \geq 1$ and we use a control with one vanishing component. We also consider the insensitizing control problem.

Controllability of fluid systems with vanishing components for the control has been studied in several articles, see, for instance, [4], [3], [7] and [12]. Note that in [8], the authors obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components. Their method is different than the previous works and is based on a particular linearization and on results of Gromov. In the case of insensitizing controls, the problem associated with vanishing components for the control is tackled in [6] and in [5]. Let us remark that with the method developed here we slightly improve the result obtained in [6]. Indeed their observability results are obtained with more regularity on the source terms than what is needed here.

In order to study the controllability of (1.1), we linearized it as follows

$$\left\{ \begin{array}{l} \partial_t z - \Delta z + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p = f + \mathbf{1}_{\omega} u \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} z = 0 \quad \text{in } (0, T) \times \Omega, \\ z = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ z(0, \cdot) = z^0 \quad \text{in } \Omega, \end{array} \right. \quad (1.2)$$

and we show the observability of the corresponding adjoint system that writes as follows:

$$\left\{ \begin{array}{l} -\partial_t \varphi - \Delta \varphi + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla \pi = g \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi = 0 \quad \text{in } (0, T) \times \Omega, \\ \varphi = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \varphi(T, \cdot) = \varphi_T \quad \text{in } \Omega. \end{array} \right. \quad (1.3)$$

A standard method to obtain such an observability inequality on a the above system was introduced in [15] and is based on Carleman estimates (see, [10] for a general presentation and [11] for the use of Carleman estimate to deal with the Navier-Stokes system).

In what follows, we assume that $d = 2$ and we suppose that our control u in (1.1) satisfies the following condition

$$u \cdot e_2 = 0, \quad (1.4)$$

where (e_1, e_2) is the canonical basis of \mathbb{R}^2 . One can obtain the same results as the one state in this article for $d = 3$ or for $u \cdot e = 0$ for any vector $e \in \mathbb{R}^3$.

Let us present here our hypotheses on the functions $a^{(i)}$ and $b^{(i)}$, $i = 1, \dots, n$. First we the following regularity conditions on $a^{(i)}$ and $b^{(i)}$:

$$a^{(i)} \in H^2(0, T; L^2(\Omega)), \quad b^{(i)} \in H^2(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega)). \quad (1.5)$$

Then, we introduce

$$\check{b}^{(i)} := \left(\Delta b_1^{(i)} - \partial_{x_1} \operatorname{div} b^{(i)} \right). \quad (1.6)$$

Note that from Sobolev embedding (since $d = 2$, but it would also holds for $d = 3$), we have

$$\check{b}^{(i)} \in C^0([0, T] \times \bar{\Omega}).$$

We assume the existence of nonempty open sets ω_i ($i = 1, \dots, n$) and of a constant $c > 0$ such that

$$\bigcup_{i=1}^n \omega_i \Subset \omega \quad (1.7)$$

and such that for any $i = 1, \dots, n$,

$$\left| \check{b}^{(i)} \right| \geq c, \quad \check{b}^{(j)} \equiv 0 \quad (j > i) \quad \text{in } (0, T) \times \omega_i. \quad (1.8)$$

Notation. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leq CY$.

Let us state our first main result:

Theorem 1.1. *Let us consider $T > 0$ and assume (1.5), (1.8) and (1.7). There exist continuous functions $\sigma_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2, 3$, with $\sigma_i > 0$ in $[0, T)$, $\sigma_i(T) = 0$ such that any solution φ of (1.3) satisfies*

$$\|\sigma_3 \varphi\|_{L^2(0, T; L^2(\Omega))} + \|\varphi(0, \cdot)\|_{L^2(\Omega)} \lesssim \|\sigma_2 \varphi_1\|_{L^2(0, T; L^2(\omega))} + \|\sigma_1 g\|_{L^2(0, T; L^2(\Omega))}. \quad (1.9)$$

The functions $\sigma_1, \sigma_2, \sigma_3$ are defined precisely by (3.1), (3.2) in Section 3 by using the Carleman weights that we describe in Section 2.2. By standard methods (see, for instance, Section 4 in [16]), we deduce from Theorem 1.1 the following result:

Corollary 1.2. *Assume (1.5), (1.7) and (1.8). There exists a continuous function $\sigma_0 : [0, T] \rightarrow \mathbb{R}_+$, with $\sigma_0 > 0$ in $[0, T)$, $\sigma_0(T) = 0$ such that the following property holds: for any*

$$z^0 \in H_0^1(\Omega), \quad \operatorname{div} z^0 = 0, \quad \frac{f}{\sigma_3} \in L^2((0, T) \times \Omega),$$

there exists $u \in L^2(0, T; L^2(\omega))$ satisfying (1.4) and such that the solution z of (1.2) satisfies

$$\left\| \frac{z}{\sigma_0} \right\|_{L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))} \leq C \left(\left\| \frac{f}{\sigma_3} \right\|_{L^2((0, T) \times \Omega)} + \|z^0\|_{H^1(\Omega)} \right). \quad (1.10)$$

In particular, $z(T, \cdot) = 0$.

Assume moreover (3.5). Then there exists a constant c_0 such that if

$$\|z^0\|_{H^1(\Omega)} + \left\| \frac{f}{\sigma_3} \right\|_{L^2((0,T) \times \Omega)} \leq c_0,$$

there exists $u \in L^2(0, T; L^2(\omega))$ such that the solution z of (1.1) satisfies (1.10) and in particular, $z(T, \cdot) = 0$.

The precise definition of σ_0 is given by (3.3) by using the Carleman weights in Section 2.2.

Let us consider the existence of insensitizing controls for (1.1). More precisely, we assume that the initial condition contains uncertainties, and we consider the corresponding system:

$$\left\{ \begin{array}{l} \partial_t z_\tau - \Delta z_\tau + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot z_\tau \, dx \right) a^{(i)} + \nabla p_\tau + (z_\tau \cdot \nabla) z_\tau = f + 1_{\omega_{\natural}} u \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} z_\tau = 0 \quad \text{in } (0, T) \times \Omega, \\ z_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ z_\tau(0, \cdot) = z^0 + \tau \widehat{z}^0 \quad \text{in } \Omega, \end{array} \right. \quad (1.11)$$

where $\|\widehat{z}^0\|_{L^2(\Omega)} = 1$ and $\tau \in \mathbb{R}_+$ is assumed to be small. For $\tau = 0$, we write z instead of z_τ (so that z is the solution of (1.1) with ω replaced by ω_{\natural}). We have replaced ω by ω_{\natural} since we need here a control region larger than for the controllability result of the previous section. The system is observed by a sentinel \mathcal{J} chosen here as

$$\mathcal{J}(z_\tau) := \frac{1}{2} \iint_{(0,T) \times \mathcal{O}} |z_\tau|^2 \, dx \, dt,$$

where \mathcal{O} is a nonempty open subset of Ω . The insensitization problem consists in finding a control u such that

$$\left. \frac{d\mathcal{J}(z_\tau)}{d\tau} \right|_{\tau=0} = 0 \quad \forall \widehat{z}^0 \in L^2(\Omega), \quad \|\widehat{z}^0\|_{L^2(\Omega)} = 1. \quad (1.12)$$

However, by considering the system

$$\left\{ \begin{array}{l} -\partial_t w - \Delta w + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot w \, dx \right) b^{(i)} + \nabla q + (\nabla z)^* w - (z \cdot \nabla) w = z 1_{\mathcal{O}} \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} w = 0 \quad \text{in } (0, T) \times \Omega, \\ w = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ w(T, \cdot) = 0 \quad \text{in } \Omega, \end{array} \right. \quad (1.13)$$

one can write

$$\left. \frac{d\mathcal{J}(z_\tau)}{d\tau} \right|_{\tau=0} = \int_{\Omega} w(0, x) \cdot \widehat{z}^0(x) \, dx.$$

Consequently, it is equivalent to search for a control u such that the solution w of (1.13) satisfies $w(0, \cdot) = 0$. Since the right-hand side of w is $z 1_{\mathcal{O}}$ where z is the solution of (1.1), we deal with a controllability problem for a cascade system with a forward equation and a backward equation. It is usual to avoid some possible issues at $t = 0$ by assuming $z^0 = 0$. In order to show this null-controllability result, we consider the linearized system:

$$\left\{ \begin{array}{l} \partial_t z - \Delta z + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p = f^{(0)} + 1_{\omega_{\natural}} u \quad \text{in } (0, T) \times \Omega, \\ -\partial_t w - \Delta w + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot w \, dx \right) b^{(i)} + \nabla q = f^{(1)} + z 1_{\mathcal{O}} \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} z = \operatorname{div} w = 0 \quad \text{in } (0, T) \times \Omega, \\ z = w = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ z(0, \cdot) = 0 \quad \text{in } \Omega, \quad w(T, \cdot) = 0 \quad \text{in } \Omega, \end{array} \right. \quad (1.14)$$

and its adjoint system:

$$\left\{ \begin{array}{l} -\partial_t \varphi - \Delta \varphi + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla \pi_{\varphi} = \psi 1_{\mathcal{O}} + g^{(0)} \quad \text{in } (0, T) \times \Omega, \\ \partial_t \psi - \Delta \psi + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi \, dx \right) a^{(i)} + \nabla \pi_{\psi} = g^{(1)} \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi = \operatorname{div} \psi = 0 \quad \text{in } (0, T) \times \Omega, \\ \varphi = \psi = 0 \quad \text{on } (0, T) \times \partial \Omega, \\ \varphi(T, \cdot) = 0 \quad \text{in } \Omega, \quad \psi(0, \cdot) = \psi^0 \quad \text{in } \Omega. \end{array} \right. \quad (1.15)$$

We introduce

$$\check{a}^{(i)} := \left(\Delta a_1^{(i)} - \partial_{x_1} \operatorname{div} a^{(i)} \right). \quad (1.16)$$

We assume that $a^{(i)}$ and $b^{(i)}$ satisfy (1.5), (1.8) and we add the following hypotheses: first we suppose the additional regularity on $a^{(i)}$

$$a^{(i)} \in L^2(0, T; H^4(\Omega)). \quad (1.17)$$

Note that from Sobolev embeddings, (since $d = 2$, but it also holds for $d = 3$), we have

$$\check{a}^{(i)} \in C^0([0, T] \times \overline{\Omega}).$$

Then, we assume the existence of nonempty open sets $\widehat{\omega}_i$ ($i = 1, \dots, n$), $\widehat{\omega}$ and a constant $c > 0$ such that

$$\bigcup_{i=1}^n \widehat{\omega}_i \Subset \widehat{\omega} \Subset \mathcal{O}, \quad \omega_{\natural} := \omega \cup \widehat{\omega} \quad (1.18)$$

with the following properties

$$\check{b}_i \equiv 0 \quad \text{in } (0, T) \times \widehat{\omega} \quad (i = 1, \dots, n), \quad (1.19)$$

and for any $i = 1, \dots, n$,

$$\left| \check{a}^{(i)} \right| \geq c, \quad \check{a}^{(j)} \equiv 0 \quad (j > i) \quad \text{in } (0, T) \times \widehat{\omega}_i. \quad (1.20)$$

Remark 1.3. We don't need to have the same order as (1.8) and we can permute the index i in (1.20).

Theorem 1.4. *Assume (1.5), (1.7), (1.8), (1.17), (1.18), (1.19) and (1.20). There exist continuous functions $\widehat{\sigma}_i : [0, T] \rightarrow \mathbb{R}_+$, $i = 1, 2, 3$, with $\widehat{\sigma}_i > 0$ in $(0, T]$, $\widehat{\sigma}_i(0) = 0$ such that any solution (φ, ψ) of (1.15) satisfies*

$$\begin{aligned} & \|\widehat{\sigma}_3 \varphi\|_{L^2(0, T; L^2(\Omega))} + \|\widehat{\sigma}_3 \psi\|_{L^2(0, T; L^2(\Omega))} \\ & \lesssim \|\widehat{\sigma}_2 \varphi_1\|_{L^2(0, T; L^2(\omega_{\natural}))} + \left\| \widehat{\sigma}_1 g^{(0)} \right\|_{L^2(0, T; L^2(\Omega))} + \left\| \widehat{\sigma}_1 g^{(1)} \right\|_{L^2(0, T; L^2(\Omega))}. \end{aligned} \quad (1.21)$$

We give the precise definitions of $\widehat{\sigma}_i$, $i = 1, 2, 3$ in Section 4.

Remark 1.5. As explained above, if we remove the nonlocal spatial terms of our system, the above result improves the corresponding result in [6] since in this previous work, one needs $g^{(1)} \in L^2(0, T; H^1(\Omega))$ with our approach we manage to remove this condition. A tool that we introduce here to deal with these regularity problems is an adequate decomposition of the solution of the adjoint system. In many articles devoted to the controllability of the Navier-Stokes system, the solution of the adjoint system is decomposed into two functions and here we show how to generalize such a decomposition by splitting the solution in several functions which allows to work with a regular solution for the Carleman estimate. A similar decomposition is already proposed in [1] but they need to modify the Carleman weights so that they are regular at $t = 0$. Such a strategy can not be applied in the context of insensitizing controls since we have to work with a forward system and a backward system. We thus need to consider another decomposition as in [1].

We deduce from the above result the existence of insensitizing controls for the linear and for the nonlinear problem:

Corollary 1.6. *Assume (1.5), (1.8), (1.7), (1.17), (1.20), (1.18), and (1.19). There exists a continuous function $\widehat{\sigma}_0 : [0, T] \rightarrow \mathbb{R}_+$, with $\widehat{\sigma}_0 > 0$ in $(0, T]$, $\widehat{\sigma}_0(0) = 0$ such that for any*

$$\frac{f^{(0)}}{\widehat{\sigma}_3}, \frac{f^{(1)}}{\widehat{\sigma}_3} \in L^2((0, T) \times \Omega),$$

there exists $u \in L^2(0, T; L^2(\omega))$ satisfying (1.4) and such that the solution (z, w) of (1.14) satisfies

$$\left\| \frac{(z, w)}{\widehat{\sigma}_0} \right\|_{L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))} \lesssim \left\| \frac{(f^{(0)}, f^{(1)})}{\widehat{\sigma}_3} \right\|_{L^2((0, T) \times \Omega)}. \quad (1.22)$$

In particular, $w(0, \cdot) = 0$.

Assume moreover (4.8). Then there exists a constant c_0 such that for any

$$\left\| \frac{f}{\widehat{\sigma}_3} \right\|_{L^2((0, T) \times \Omega)} \leq c_0,$$

there exists $u \in L^2(0, T; L^2(\omega))$ such that the solution (z, w) of (1.1)-(1.13) with $z^0 = 0$ satisfies

$$\left\| \frac{(z, w)}{\widehat{\sigma}_0} \right\|_{L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))} \lesssim \left\| \frac{f}{\widehat{\sigma}_3} \right\|_{L^2((0, T) \times \Omega)}. \quad (1.23)$$

In particular, $w(0, \cdot) = 0$.

The precise choice for $\widehat{\sigma}_0$ is given by (4.7) in Section 4.

The outline of the article is as follows: in the next section, we start by some well-posedness and regularity results for linear systems of the form (1.3) and (1.15). Then we introduce the Carleman weights we will use to show Theorem 1.1 and we recall some classical Carleman estimates. In Section 3, we show Theorem 1.1 by introducing our new decomposition for the solution of the adjoint system and then performing a Carleman estimate on one of the terms of this decomposition. Finally, Section 4 is devoted to the proof of Theorem 1.4 and Corollary 1.6 by using a similar decomposition as the one used in the proof of Theorem 1.1.

2 Preliminaries

2.1 Well-posedness results

Notation. To simplify the notation in the article, we write

$$\mathcal{L}(\psi, \pi) = (g, \psi^0) \quad (2.1)$$

if

$$\left\{ \begin{array}{l} \partial_t \psi - \Delta \psi + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi \, dx \right) a^{(i)} + \nabla \pi = g \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} \psi = 0 \quad \text{in } (0, T) \times \Omega, \\ \psi = 0 \quad \text{on } (0, T) \times \partial \Omega, \\ \psi(0, \cdot) = \psi^0 \quad \text{in } \Omega, \end{array} \right.$$

and

$$\mathcal{L}^*(\varphi, \pi) = (g, \varphi_T) \quad (2.2)$$

if

$$\left\{ \begin{array}{l} -\partial_t \varphi - \Delta \varphi + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla \pi = g \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi = 0 \quad \text{in } (0, T) \times \Omega, \\ \varphi = 0 \quad \text{on } (0, T) \times \partial \Omega, \\ \varphi(T, \cdot) = \varphi_T \quad \text{in } \Omega. \end{array} \right.$$

We also set

$$\begin{aligned} X_1 &:= L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ X_2 &:= L^2(0, T; H^4(\Omega)) \cap H^2(0, T; L^2(\Omega)), \\ X_3 &:= L^2(0, T; H^6(\Omega)) \cap H^3(0, T; L^2(\Omega)). \end{aligned}$$

Note that by interpolation, we have

$$X_k = \left(\bigcap_{j=0}^k H^j(0, T; H^{2(k-j)}(\Omega)) \right) \cap \left(\bigcap_{j=0}^{k-1} C^j([0, T]; H^{2(k-j)-1}(\Omega)) \right).$$

We have the following results that can be obtained in a standard way:

Proposition 2.1. *Assume (1.5). Then for any $\varphi_T \in H_0^1(\Omega)$, $\operatorname{div} \varphi_T = 0$ in Ω and $g \in L^2(0, T; L^2(\Omega))$, there exists a unique solution of (2.2) $(\varphi, \pi) \in X_1 \times L^2(0, T; H^1(\Omega)/\mathbb{R})$ and we have the estimate*

$$\|\varphi\|_{X_1} + \|\nabla \pi\|_{L^2(0, T; L^2(\Omega))} \lesssim \|\varphi_T\|_{H^1(\Omega)} + \|g\|_{L^2(0, T; L^2(\Omega))}.$$

Assume $g \in X_1$, with $g(T, \cdot) = 0$. Then the solution of $\mathcal{L}^*(\varphi, \pi) = (g, 0)$ satisfies $\varphi \in X_2$ with the estimate

$$\|\varphi\|_{X_2} \lesssim \|g\|_{X_1}.$$

Assume $g \in X_2$, with $g(T, \cdot) = \partial_t g(T, \cdot) = 0$. Then the solution of $\mathcal{L}^*(\varphi, \pi) = (g, 0)$ satisfies $\varphi \in X_3$ with the estimate

$$\|\varphi\|_{X_3} \lesssim \|g\|_{X_2}.$$

Proposition 2.2. *Assume (1.5) and (1.17). Then, for any $\psi^0 \in H_0^1(\Omega)$ such that $\operatorname{div} \psi^0 = 0$ in Ω and for any $g \in L^2(0, T; L^2(\Omega))$, there exists a unique solution of (2.1) $(\psi, \pi) \in X_1 \times L^2(0, T; H^1(\Omega)/\mathbb{R})$ and we have the estimate*

$$\|\psi\|_{X_1} + \|\nabla \pi\|_{L^2(0, T; L^2(\Omega))} \lesssim \|\psi^0\|_{H^1(\Omega)} + \|g\|_{L^2(0, T; L^2(\Omega))}.$$

Assume $g \in X_1$, with $g(0, \cdot) = 0$. Then the solution of $\mathcal{L}(\psi, \pi) = (g, 0)$ satisfies $\psi \in X_2$ with the estimate

$$\|\psi\|_{X_2} \lesssim \|g\|_{X_1}.$$

Assume $g \in X_2$, with $g(0, \cdot) = \partial_t g(0, \cdot) = 0$. Then the solution of $\mathcal{L}(\psi, \pi) = (g, 0)$ satisfies $\psi \in X_3$ with the estimate

$$\|\psi\|_{X_3} \lesssim \|g\|_{X_2}.$$

2.2 Some standard Carleman estimates

We recall here some Carleman estimates that were obtained in previous articles. Let us first introduce the weights functions. Using [15] (see also [23, Theorem 9.4.3, p.299]), there exist for any $i = 1, \dots, n$, $\eta_i \in C^2(\overline{\Omega})$ satisfying

$$\eta_i > 0 \text{ in } \Omega, \quad \eta_i = 0 \text{ on } \partial \Omega, \quad \max_{\Omega} \eta_i = 1, \quad \nabla \eta_i \neq 0 \text{ in } \overline{\Omega \setminus \omega_i}. \quad (2.3)$$

We use η_i to define the following standard functions:

$$\alpha_i(t, x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m + \eta_i(x))\}}{t^m(T-t)^m}, \quad \xi_i(t, x) = \frac{\exp\{\lambda(2m + \eta_i(x))\}}{t^m(T-t)^m}, \quad (2.4)$$

where

$$m \geq 11, \quad \lambda > 1.$$

The minimum and the maximum of these functions with respect to x do not depend on i :

$$\alpha_{\#}(t) = \max_{x \in \bar{\Omega}} \alpha_i(t, x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{2\lambda m\}}{t^m(T-t)^m}, \quad \xi_{\#}(t) = \min_{x \in \bar{\Omega}} \xi_i(t, x) = \frac{\exp\{2\lambda m\}}{t^m(T-t)^m}, \quad (2.5)$$

$$\alpha_{\flat}(t) = \min_{x \in \bar{\Omega}} \alpha_i(t, x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m+1)\}}{t^m(T-t)^m}, \quad \xi_{\flat}(t) = \max_{x \in \bar{\Omega}} \xi_i(t, x) = \frac{\exp\{\lambda(2m+1)\}}{t^m(T-t)^m}, \quad (2.6)$$

so that

$$\xi_{\#} \leq \xi_i \leq \xi_{\flat}, \quad e^{-s\alpha_{\#}} \leq e^{-s\alpha_i} \leq e^{-s\alpha_{\flat}} \quad (i = 1, \dots, n). \quad (2.7)$$

We also have the following useful formula: for any $\varepsilon \in (0, 1)$, if $\lambda \geq -\ln \varepsilon$, then

$$e^{-s\alpha_i} \leq e^{-s(1-\varepsilon)\alpha_{\#}} \leq e^{-s(1-\varepsilon)\alpha_j} \quad (i, j \in \{1, \dots, n\}). \quad (2.8)$$

Note that we have the following useful relations: for any $j \geq 1$, there exists a constant $C > 0$

$$\left| \frac{d^j \alpha_{\#}}{dt^j} \right| + \left| \frac{d^j \xi_{\#}}{dt^j} \right| \leq CT^j (\xi_{\#})^{1+j/m}. \quad (2.9)$$

We also have

$$\xi_{\#} \geq \frac{4^m}{T^{2m}}. \quad (2.10)$$

In particular, for $s_0 > 0$, if $s \geq s_0 T^{2m}$,

$$s\xi_{\#} \geq s_0 4^m. \quad (2.11)$$

We can now state a Carleman estimate for the gradient operator (see, for instance, [7, Lemma 3]):

Lemma 2.3. *Assume $i \in \{1, \dots, n\}$ and $r \in \mathbb{R}$. There exists $C > 0$ depending on the geometry and on η_i such that for any $\lambda \geq C$, $s \geq CT^{2m}$ and $u \in L^2(0, T; H^1(\Omega))$,*

$$\begin{aligned} & \iint_{(0, T) \times \Omega} e^{-2s\alpha_i} (s\lambda\xi_i)^{r+2} |u|^2 dx dt \\ & \lesssim \iint_{(0, T) \times \Omega} (s\lambda\xi_i)^r e^{-2s\alpha_i} |\nabla u|^2 dx dt + \iint_{(0, T) \times \omega_i} (s\lambda\xi_i)^{r+2} e^{-2s\alpha_i} |u|^2 dx dt. \end{aligned}$$

In particular, for any $u \in L^2(0, T)$,

$$\iint_{(0, T) \times \Omega} e^{-2s\alpha_i} |u|^2 dx dt \lesssim \iint_{(0, T) \times \omega_i} e^{-2s\alpha_i} |u|^2 dx dt. \quad (2.12)$$

The second Carleman estimate we recall here corresponds to the Laplace operator (see, for instance, [7, Lemma 4]):

Lemma 2.4. *Assume $i \in \{1, \dots, n\}$ and $r \in \mathbb{R}$. There exists $C > 0$ depending on the geometry and on η_i such that $\lambda \geq C$, $s \geq CT^{2m}$ and $u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$,*

$$\begin{aligned} & \iint_{(0, T) \times \Omega} s^{r+4} \lambda^{r+6} \xi_i^{r+4} e^{-2s\alpha_i} |u|^2 dx dt + \iint_{(0, T) \times \Omega} s^{r+2} \lambda^{r+4} \xi_i^{r+2} e^{-2s\alpha_i} |\nabla u|^2 dx dt \\ & \lesssim \iint_{(0, T) \times \Omega} s^{r+1} \lambda^{r+2} \xi_i^{r+1} e^{-2s\alpha_i} |\Delta u|^2 dx dt + \iint_{(0, T) \times \omega_i} s^{r+4} \lambda^{r+6} \xi_i^{r+4} e^{-2s\alpha_i} |u|^2 dx dt. \end{aligned} \quad (2.13)$$

Finally, we recall a Carleman estimate for the heat equation with Neumann boundary conditions:

$$\begin{cases} \partial_t u + \Delta u = f^{(1)} & \text{in } (0, T) \times \Omega, \\ -\frac{\partial u}{\partial n} = f^{(2)} & \text{on } (0, T) \times \partial\Omega. \end{cases} \quad (2.14)$$

The following lemma is obtained in [9] (see also [7, Lemma 5]):

Lemma 2.5. *Assume $i \in \{1, \dots, n\}$. There exists $C > 0$ depending on the geometry and on η_i such that for any*

$$\begin{aligned} \lambda &\geq C, \quad s \geq C(T^{2m} + T^m), \\ f^{(1)} &\in L^2(0, T; L^2(\Omega)), \quad f^{(2)} \in L^2(0, T; L^2(\partial\Omega)), \end{aligned}$$

and $u \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ weak solution of (2.14),

$$\begin{aligned} &\iint_{(0, T) \times \Omega} s^3 \lambda^4 \xi_i^3 e^{-2s\alpha_i} |u|^2 \, dx \, dt + \iint_{(0, T) \times \Omega} s \lambda^2 \xi_i e^{-2s\alpha_i} |\nabla u|^2 \, dx \, dt \\ &\quad \lesssim \iint_{(0, T) \times \Omega} e^{-2s\alpha_i} |f^{(1)}|^2 \, dx \, dt + \iint_{(0, T) \times \partial\Omega} s \lambda \xi_i e^{-2s\alpha_i} |f^{(2)}|^2 \, d\gamma \, dt \\ &\quad \quad \quad + \iint_{(0, T) \times \omega_i} s^3 \lambda^4 \xi_i^3 e^{-2s\alpha_i} |u|^2 \, dx \, dt. \end{aligned} \quad (2.15)$$

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The functions $\sigma_1, \sigma_2, \sigma_3$ are defined below by using the weights defined by Section 2.2. More precisely, let us consider $N_1, N_2 > 0$. We write $N := N_1 + N_2$. We define

$$\sigma_1 := \begin{cases} e^{-N_2 s \alpha_\#} & \text{in } \left[\frac{T}{2}, T \right] \\ [e^{-N_2 s \alpha_\#}] \left(\frac{T}{2} \right) & \text{in } \left[0, \frac{T}{2} \right] \end{cases}, \quad \sigma_2 := \begin{cases} e^{-(N+1/2)s\alpha_\#} & \text{in } \left[\frac{T}{2}, T \right] \\ [e^{-(N+1/2)s\alpha_\#}] \left(\frac{T}{2} \right) & \text{in } \left[0, \frac{T}{2} \right] \end{cases}, \quad (3.1)$$

$$\sigma_3 := \begin{cases} s^3 \lambda^4 \xi_\#^3 e^{-(N+1)s\alpha_\#} & \text{in } \left[\frac{T}{2}, T \right] \\ [s^3 \lambda^4 \xi_\#^3 e^{-(N+1)s\alpha_\#}] \left(\frac{T}{2} \right) & \text{in } \left[0, \frac{T}{2} \right] \end{cases}. \quad (3.2)$$

Note that $\sigma_i(T) = 0$ $i = 1, 2, 3$.

We are going to show the existence of $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$, Theorem 1.1 with σ_i defined as above.

Then for Corollary 1.2, we assume $N_2 > 1$ and we define

$$\sigma_0 := \begin{cases} e^{-(N_2-1)s\alpha_\#} & \text{in } \left[\frac{T}{2}, T \right] \\ [e^{-(N_2-1)s\alpha_\#}] \left(\frac{T}{2} \right) & \text{in } \left[0, \frac{T}{2} \right] \end{cases}. \quad (3.3)$$

Note that

$$\frac{\sigma_2}{\sigma_0}, \frac{\sigma_3}{\sigma_0}, \frac{\sigma_0' \sigma_1}{\sigma_0^2} \in L^\infty(0, T) \quad (3.4)$$

and if

$$N_2 \geq N_1 + 3, \quad (3.5)$$

then, we have

$$\frac{\sigma_0^2}{\sigma_3} \in L^\infty(0, T).$$

These properties allow us to deduce Corollary 1.2 from Theorem 1.1 (see, for instance, Section 4 in [16]) by again assuming $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$ for some $s_0 > 0$ and $\lambda_0 > 0$.

3.1 Decomposition of the adjoint system

In order to prove Theorem 1.1, we decompose the solution of (1.3) into three parts. We recall that $N_1, N_2 > 0$ and that $N = N_1 + N_2$. Then we set

$$\rho_i := e^{-N_i s \alpha_\#} \quad (i = 1, 2) \quad (3.6)$$

and we consider the following decomposition of the solution (φ, π) of (1.3):

$$\rho_1 \rho_2 \varphi = \rho_1 \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)}, \quad (3.7)$$

where

$$\mathcal{L}^*(\varphi^{(1)}, \pi^{(1)}) = (\rho_2 g, 0), \quad (3.8)$$

$$\mathcal{L}^*(\varphi^{(2)}, \pi^{(2)}) = (N_2 s \alpha'_\# \rho_1 \varphi^{(1)}, 0), \quad (3.9)$$

$$\mathcal{L}^*(\varphi^{(3)}, \pi^{(3)}) = (N s \alpha'_\# (\varphi^{(2)} + \varphi^{(3)}), 0). \quad (3.10)$$

We are going to use Proposition 2.1 to estimate $\varphi^{(1)}$ and $\varphi^{(2)}$ and we show a Carleman estimate on $\varphi^{(3)}$. In order to state this estimate, we introduce to quantities associated with $\varphi^{(3)}$:

$$\begin{aligned} I(\varphi^{(3)}) := & \iint_{(0, T) \times \Omega} e^{-2s\alpha_1} \left(s\lambda^2 \xi_1 \left| \nabla^2 \Delta \varphi_1^{(3)} \right|^2 + s^3 \lambda^4 \xi_1^3 \left| \nabla \Delta \varphi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0, T) \times \Omega} e^{-2s\alpha_1} \left(s^5 \lambda^6 \xi_1^5 \left| \Delta \varphi_1^{(3)} \right|^2 + s^6 \lambda^8 \xi_1^6 \left| \nabla \varphi_1^{(3)} \right|^2 + s^8 \lambda^{10} \xi_1^8 \left| \varphi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0, T) \times \Omega} e^{-2s\alpha_\#} s^6 \lambda^8 \xi_\#^6 \left| \varphi^{(3)} \right|^2 dx dt \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} J(\varphi^{(3)}) := & \left\| \theta_1 \varphi^{(3)} \right\|_{L^2(0, T; H^2(\Omega))}^2 + \left\| \theta_1 \partial_t \varphi^{(3)} \right\|_{L^2(0, T; L^2(\Omega))}^2 \\ & + \left\| \theta_2 \varphi^{(3)} \right\|_{L^2(0, T; H^4(\Omega))}^2 + \left\| \theta_2 \partial_t \varphi^{(3)} \right\|_{L^2(0, T; H^2(\Omega))}^2 + \left\| \theta_2 \partial_t^2 \varphi^{(3)} \right\|_{L^2(0, T; L^2(\Omega))}^2 \\ & + \left\| \theta_3 \varphi^{(3)} \right\|_{L^2(0, T; H^6(\Omega))}^2 + \left\| \theta_3 \partial_t \varphi^{(3)} \right\|_{L^2(0, T; H^4(\Omega))}^2 + \left\| \theta_3 \partial_t^2 \varphi^{(3)} \right\|_{L^2(0, T; H^2(\Omega))}^2 + \left\| \theta_3 \partial_t^3 \varphi^{(3)} \right\|_{L^2(0, T; L^2(\Omega))}^2, \end{aligned} \quad (3.12)$$

where

$$\theta_1 := s^{2-1/m} \lambda^4 (\xi_\#)^{2-1/m} e^{-s\alpha_\#}, \quad \theta_2 := s^{1-2/m} \lambda^4 (\xi_\#)^{1-2/m} e^{-s\alpha_\#}, \quad \theta_3 := s^{-3/m} \lambda^4 (\xi_\#)^{-3/m} e^{-s\alpha_\#}. \quad (3.13)$$

With the above notation, we can state the Carleman estimates we obtain for $\varphi^{(3)}$:

Proposition 3.1. *Let us consider $T > 0$ and assume (1.5), (1.8) and (1.7). There exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$ and for any solution φ of (1.3) can be decomposed as in (3.7), (3.8), (3.9) and (3.10) and the functions $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ satisfy*

$$\left\| \varphi^{(1)} \right\|_{X_1} + \left\| \varphi^{(2)} \right\|_{X_2} \lesssim \|\rho_2 g\|_{L^2((0, T) \times \Omega)}, \quad (3.14)$$

and

$$I(\varphi^{(3)}) + J(\varphi^{(3)}) \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \iint_{(0,T)\times\omega} e^{-s\alpha_\#} \left(\varphi_1^{(3)}\right)^2 dx dt. \quad (3.15)$$

In particular, φ verifies the following estimate:

$$\begin{aligned} & \iint_{(0,T)\times\Omega} s^6 \lambda^8 \xi_\#^6 e^{-2(N+1)s\alpha_\#} |\varphi|^2 dx dt \\ & \lesssim \iint_{(0,T)\times\Omega} e^{-2N_2 s\alpha_\#} |g|^2 dx dt + \iint_{(0,T)\times\omega} e^{-s(2N+1)\alpha_\#} |\varphi_1|^2 dx dt. \end{aligned} \quad (3.16)$$

We deduce Theorem 1.1 from Proposition 3.1 in a standard way by using the well-posedness of (1.3) (see Proposition 2.1). In the remainder of this section, we prove the above proposition.

Applying Proposition 2.1, we have

$$\left\| \varphi^{(1)} \right\|_{X_1} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}. \quad (3.17)$$

Using (2.9), we deduce that for $s \geq T^m$,

$$\left\| s\alpha'_\# \rho_1 \right\|_{C^1([0,T]\times\bar{\Omega})} \lesssim 1.$$

The above estimate and (3.17) imply

$$\left\| N_2 s\alpha'_\# \rho_1 \varphi^{(1)} \right\|_{X_1} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)} \quad (3.18)$$

and combining this with Proposition 2.1, we deduce

$$\left\| \varphi^{(2)} \right\|_{X_2} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)} \quad (3.19)$$

and we have proved (3.14). Then, from (3.10), we deduce

$$\mathcal{L}^*(\theta_1 \varphi^{(3)}, \theta_1 \pi^{(3)}) = \left(N s \theta_1 \alpha'_\# \left(\varphi^{(2)} + \varphi^{(3)} \right) - \theta'_1 \varphi^{(3)}, 0 \right), \quad (3.20)$$

$$\mathcal{L}^*(\theta_2 \varphi^{(3)}, \theta_2 \pi^{(3)}) = \left(N s \theta_2 \alpha'_\# \left(\varphi^{(2)} + \varphi^{(3)} \right) - \theta'_2 \varphi^{(3)}, 0 \right), \quad (3.21)$$

$$\mathcal{L}^*(\theta_3 \varphi^{(3)}, \theta_3 \pi^{(3)}) = \left(N s \theta_3 \alpha'_\# \left(\varphi^{(2)} + \varphi^{(3)} \right) - \theta'_3 \varphi^{(3)}, 0 \right). \quad (3.22)$$

From (3.13)

$$\theta'_1 = \left(2 - \frac{1}{m} \right) s^{2-1/m} \lambda^4 (\xi_\#)^{1-1/m} \xi'_\# e^{-s\alpha_\#} - s\alpha'_\# \theta_1.$$

From (2.9) and (2.11), for $s \geq T^m + T^{2m}$,

$$|N s \theta_1 \alpha'_\#| + |\theta'_1| \lesssim s^3 \lambda^4 \xi_\#^3 e^{-s\alpha_\#}. \quad (3.23)$$

Using the above estimate and (3.19), and applying Proposition 2.1 to (3.20), we deduce

$$\left\| \theta_1 \varphi^{(3)} \right\|_{X_1}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2.$$

Combining this estimate with (3.23), we obtain

$$\left\| \theta_1 \varphi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \theta_1 \partial_t \varphi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2. \quad (3.24)$$

From (3.13), we have the following formulas:

$$\theta'_2 = \left(1 - \frac{2}{m}\right) s^{1-2/m} \lambda^4 (\xi_{\#})^{-2/m} \xi'_{\#} e^{-s\alpha_{\#}} - s\alpha'_{\#} \theta_2,$$

$$\begin{aligned} \theta''_2 &= \left(1 - \frac{2}{m}\right) \left(-\frac{2}{m}\right) s^{1-2/m} \lambda^4 (\xi_{\#})^{-1-2/m} (\xi'_{\#})^2 e^{-s\alpha_{\#}} + \left(1 - \frac{2}{m}\right) s^{1-2/m} \lambda^4 (\xi_{\#})^{-2/m} \xi''_{\#} e^{-s\alpha_{\#}} \\ &\quad - 2s^{2-2/m} \lambda^4 \left(1 - \frac{2}{m}\right) (\xi_{\#})^{-2/m} \alpha'_{\#} \xi'_{\#} e^{-s\alpha_{\#}} - s\alpha''_{\#} \theta_2 + s^2 (\alpha'_{\#})^2 \theta_2, \\ &\quad (s\alpha'_{\#} \theta_2)' = s\alpha''_{\#} \theta_2 + s\alpha'_{\#} \theta'_2. \end{aligned}$$

From the above relations, (2.9) and (2.11), we have for $s \geq T^m + T^{2m}$,

$$|Ns\alpha'_{\#} \theta_2| + |\theta'_2| \lesssim \theta_1, \quad |Ns(\alpha'_{\#} \theta_2)'| + |\theta''_2| \lesssim s^3 \lambda^4 \xi_{\#}^3 e^{-s\alpha_{\#}}. \quad (3.25)$$

Thus, from (3.24)

$$\left\| Ns\theta_2 \alpha'_{\#} (\varphi^{(2)} + \varphi^{(3)}) - \theta'_2 \varphi^{(3)} \right\|_{X_1}^2 \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \left\| s^3 \lambda^4 (\xi_{\#})^3 e^{-s\alpha_{\#}} \varphi^{(3)} \right\|_{L^2((0,T) \times \Omega)}^2.$$

Applying Proposition 2.1 to (3.21) and using the above estimate yield

$$\left\| \theta_2 \varphi^{(3)} \right\|_{X_2}^2 \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \left\| s^3 \lambda^4 (\xi_{\#})^3 e^{-s\alpha_{\#}} \varphi^{(3)} \right\|_{L^2((0,T) \times \Omega)}^2.$$

Using (3.23) and (3.24), we deduce

$$\begin{aligned} \left\| \theta_2 \varphi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^2 + \left\| \theta_2 \partial_t \varphi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \theta_2 \partial_t^2 \varphi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \left\| s^3 \lambda^4 (\xi_{\#})^3 e^{-s\alpha_{\#}} \varphi^{(3)} \right\|_{L^2((0,T) \times \Omega)}^2. \end{aligned} \quad (3.26)$$

From (3.13), we have the following formulas:

$$\theta'_3 = -s^{-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi'_{\#} e^{-s\alpha_{\#}} - s\alpha'_{\#} \theta_3,$$

$$\begin{aligned} \theta''_3 &= -s^{-3/m} \lambda^4 \frac{3}{m} \left(-1 - \frac{3}{m}\right) (\xi_{\#})^{-2-3/m} (\xi'_{\#})^2 e^{-s\alpha_{\#}} - s^{-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi''_{\#} e^{-s\alpha_{\#}} \\ &\quad + s^{1-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi'_{\#} \alpha'_{\#} e^{-s\alpha_{\#}} - s\alpha''_{\#} \theta_3 - s\alpha'_{\#} \theta'_3, \end{aligned}$$

$$\begin{aligned} \theta'''_3 &= -s^{-3/m} \lambda^4 \frac{3}{m} \left(1 + \frac{3}{m}\right) \left(2 + \frac{3}{m}\right) (\xi_{\#})^{-3-3/m} (\xi'_{\#})^3 e^{-s\alpha_{\#}} + s^{-3/m} \lambda^4 \frac{9}{m} \left(1 + \frac{3}{m}\right) (\xi_{\#})^{-2-3/m} \xi'_{\#} \xi''_{\#} e^{-s\alpha_{\#}} \\ &\quad - 2s^{1-3/m} \lambda^4 \frac{3}{m} \left(1 + \frac{3}{m}\right) (\xi_{\#})^{-2-3/m} (\xi'_{\#})^2 \alpha'_{\#} e^{-s\alpha_{\#}} - s^{-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi'''_{\#} e^{-s\alpha_{\#}} \\ &\quad + 2s^{1-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi''_{\#} \alpha'_{\#} e^{-s\alpha_{\#}} + s^{1-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi'_{\#} \alpha''_{\#} e^{-s\alpha_{\#}} \\ &\quad - s^{2-3/m} \lambda^4 \frac{3}{m} (\xi_{\#})^{-1-3/m} \xi'_{\#} (\alpha'_{\#})^2 e^{-s\alpha_{\#}} - s\alpha'''_{\#} \theta_3 - s\alpha''_{\#} \theta'_3 - s\alpha'_{\#} \theta''_3 - s\alpha'_{\#} \theta'_3, \end{aligned}$$

$$\begin{aligned}\frac{d}{dt} (s\alpha'_\# \theta_3) &= s\alpha''_\# \theta_3 + s\alpha'_\# \theta'_3, \\ \frac{d^2}{dt^2} (s\alpha'_\# \theta_3) &= s\alpha'''_\# \theta_3 + 2s\alpha''_\# \theta'_3 + s\alpha'_\# \theta''_3.\end{aligned}$$

From the above relations, (2.9) and (2.11), we have for $s \geq T^{2m}$,

$$|Ns\alpha'_\# \theta_3| + |\theta'_3| \lesssim \theta_2, \quad |Ns(\alpha'_\# \theta_3)'| + |\theta''_3| \lesssim \theta_1, \quad |Ns(\alpha'_\# \theta_3)''| + |\theta'''_3| \lesssim s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#}. \quad (3.27)$$

Thus, using (3.24) and (3.26), we deduce

$$\left\| Ns\theta_3\alpha'_\# \left(\varphi^{(2)} + \varphi^{(3)} \right) - \theta'_3\varphi^{(3)} \right\|_{X_2}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2.$$

Applying Proposition 2.1 to (3.22), we obtain

$$\left\| \theta_3 \varphi^{(3)} \right\|_{X_3}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2.$$

Using (3.27), (3.26), (3.24), we deduce

$$\begin{aligned}\left\| \theta_3 \varphi^{(3)} \right\|_{L^2(0,T;H^6(\Omega))}^2 + \left\| \theta_3 \partial_t \varphi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^2 + \left\| \theta_3 \partial_t^2 \varphi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \theta_3 \partial_t^3 \varphi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2.\end{aligned} \quad (3.28)$$

Gathering (3.28), (3.26) and (3.24) and recalling (3.12) yield

$$J(\varphi^{(3)}) \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\| s^3 \lambda^4 (\xi_\#)^3 e^{-s\alpha_\#} \varphi^{(3)} \right\|_{L^2((0,T)\times\Omega)}^2. \quad (3.29)$$

3.2 Carleman estimates

Taking the divergence of the first equation of (3.10), we obtain the following relation for $\pi^{(3)}$:

$$\Delta \pi^{(3)} = - \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right) \operatorname{div} b^{(i)}. \quad (3.30)$$

In particular, recalling that $\check{b}^{(i)}$ is given by (1.6), we can get rid of the pressure in the first equation of (3.10) by applying the operator $\nabla \Delta$ on its first component:

$$-\partial_t \nabla \Delta \varphi_1^{(3)} - \Delta \nabla \Delta \varphi_1^{(3)} + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right) \nabla \check{b}^{(i)} = Ns\alpha'_\# \left(\nabla \Delta \varphi_1^{(2)} + \nabla \Delta \varphi_1^{(3)} \right) \quad \text{in } (0,T) \times \Omega. \quad (3.31)$$

Then we apply Lemma 2.5 to the above system with $i = 1$,

$$\begin{aligned}\iint_{(0,T)\times\Omega} s^3 \lambda^4 \xi_1^3 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt + \iint_{(0,T)\times\Omega} s \lambda^2 \xi_1 e^{-2s\alpha_1} \left| \nabla^2 \Delta \varphi_1^{(3)} \right|^2 dx dt \\ \lesssim \iint_{(0,T)\times\Omega} s^2 (\alpha'_\#)^2 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt + \iint_{(0,T)\times\Omega} s^2 (\alpha'_\#)^2 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(2)} \right|^2 dx dt \\ + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt + \iint_{(0,T)\times\partial\Omega} s \lambda \xi_\# e^{-2s\alpha_\#} \left| \frac{\partial}{\partial n} \nabla \Delta \varphi_1^{(3)} \right|^2 d\gamma dt \\ + \iint_{(0,T)\times\omega_1} s^3 \lambda^4 \xi_1^3 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt.\end{aligned} \quad (3.32)$$

Note that, from (2.9) and (2.7), for $s \geq T^m$,

$$|s\alpha'_\# e^{-s\alpha_1}| \lesssim (s\xi_1)^{1+1/m} e^{-s\alpha_1}.$$

In particular, using (2.11) and (3.19), there exists $s_0 > 0$ such that for $s \geq s_0 (T^{2m} + T^m)$,

$$\begin{aligned} & \iint_{(0,T) \times \Omega} s^3 \lambda^4 \xi_1^3 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} s \lambda^2 \xi_1 e^{-2s\alpha_1} \left| \nabla^2 \Delta \varphi_1^{(3)} \right|^2 dx dt \\ & \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt \\ & + \iint_{(0,T) \times \partial\Omega} s \lambda \xi_\# e^{-2s\alpha_\#} \left| \frac{\partial}{\partial n} \nabla \Delta \varphi_1^{(3)} \right|^2 d\gamma dt + \iint_{(0,T) \times \omega_1} s^3 \lambda^4 \xi_1^3 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.33)$$

Then, we apply Lemma 2.3 and Lemma 2.4 (with $i = 1$) to deduce

$$\begin{aligned} & \iint_{(0,T) \times \Omega} s^8 \lambda^{10} \xi_1^8 e^{-2s\alpha_1} \left| \varphi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} s^6 \lambda^8 \xi_1^6 e^{-2s\alpha_1} \left| \nabla \varphi_1^{(3)} \right|^2 dx dt \\ & + \iint_{(0,T) \times \Omega} s^5 \lambda^6 \xi_1^5 e^{-2s\alpha_1} \left| \Delta \varphi_1^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} s^3 \lambda^4 \xi_1^3 e^{-2s\alpha_1} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt \\ & + \iint_{(0,T) \times \omega_1} s^5 \lambda^6 \xi_1^5 e^{-2s\alpha_1} \left| \Delta \varphi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \omega_1} s^8 \lambda^{10} \xi_1^8 e^{-2s\alpha_1} \left| \varphi_1^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.34)$$

Moreover, using the Poincaré inequality and the divergence condition, we also obtain

$$\iint_{(0,T) \times \Omega} s^6 \lambda^8 \xi_\#^6 e^{-2s\alpha_\#} \left| \varphi_2^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} s^6 \lambda^8 \xi_1^6 e^{-2s\alpha_1} \left| \nabla \varphi_1^{(3)} \right|^2 dx dt.$$

Combining the above estimate with (3.33) and (3.34) and recalling (3.11), we deduce that

$$\begin{aligned} I(\varphi^{(3)}) & \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt \\ & + \iint_{(0,T) \times \omega_1} e^{-2s\alpha_1} \left(s^3 \lambda^4 \xi_1^3 \left| \nabla \Delta \varphi_1^{(3)} \right|^2 + s^5 \lambda^6 \xi_1^5 \left| \Delta \varphi_1^{(3)} \right|^2 + s^8 \lambda^{10} \xi_1^8 \left| \varphi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0,T) \times \partial\Omega} s \lambda \xi_\# e^{-2s\alpha_\#} \left| \frac{\partial}{\partial n} \nabla \Delta \varphi_1^{(3)} \right|^2 d\gamma dt. \end{aligned}$$

The above estimate and (3.29) yield

$$\begin{aligned} I(\varphi^{(3)}) + J(\varphi^{(3)}) & \lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt \\ & + \iint_{(0,T) \times \omega_1} e^{-2s\alpha_1} \left(s^3 \lambda^4 \xi_1^3 \left| \nabla \Delta \varphi_1^{(3)} \right|^2 + s^5 \lambda^6 \xi_1^5 \left| \Delta \varphi_1^{(3)} \right|^2 + s^8 \lambda^{10} \xi_1^8 \left| \varphi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0,T) \times \partial\Omega} s \lambda \xi_\# e^{-2s\alpha_\#} \left| \frac{\partial}{\partial n} \nabla \Delta \varphi_1^{(3)} \right|^2 d\gamma dt. \end{aligned} \quad (3.35)$$

From the trace theorem, interpolation results, (2.10) and the fact that $m \geq 9$, we also have

$$\iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\#} s \lambda \xi_\# \left| \frac{\partial}{\partial n} \nabla \Delta \varphi_1^{(3)} \right|^2 d\gamma dt \lesssim \lambda^{-7} \left\| \theta_2 \varphi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^{3/2} \left\| \theta_3 \varphi^{(3)} \right\|_{L^2(0,T;H^6(\Omega))}^{1/2}$$

Therefore, the above estimate and (3.35) imply the existence of constants $\lambda_0 > 0$ and $C > 0$ such that for $\lambda \geq \lambda_0$,

$$\begin{aligned} I(\varphi^{(3)}) + J(\varphi^{(3)}) &\lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt \\ &\quad + \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1} \left(s^3 \lambda^4 \xi_1^3 \left| \nabla \Delta \varphi_1^{(3)} \right|^2 + s^5 \lambda^6 \xi_1^5 \left| \Delta \varphi_1^{(3)} \right|^2 + s^8 \lambda^{10} \xi_1^8 \left| \varphi_1^{(3)} \right|^2 \right) dx dt. \end{aligned} \quad (3.36)$$

3.3 Removing the nonlocal spatial terms

We set

$$\delta := \frac{1}{4^5(n+1)}.$$

Using (2.8) and Lemma 2.3, there exists $C > 0$ such that if $\lambda \geq C$ and $s \geq CT^{2m}$,

$$\begin{aligned} \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(n)} \cdot \varphi^{(3)} dx \right)^2 dx dt &\leq \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\alpha_n} \left(\int_{\Omega} a^{(n)} \cdot \varphi^{(3)} dx \right)^2 dx dt \\ &\lesssim \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\int_{\Omega} a^{(n)} \cdot \varphi^{(3)} dx \right)^2 dx dt. \end{aligned} \quad (3.37)$$

Using (3.30) and applying the operator Δ on the first component of the first equation of (3.10) we deduce

$$-\partial_t \Delta \varphi_1^{(3)} - \Delta^2 \varphi_1^{(3)} + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right) \check{b}^{(i)} = N s \alpha'_\# \left(\Delta \varphi_1^{(2)} + \Delta \varphi_1^{(3)} \right) \quad \text{in } (0, T) \times \Omega, \quad (3.38)$$

where we recall that $\check{b}^{(i)}$ is given by (1.6). Using (1.8) for $i = n$, we deduce from (3.37) and the above equation that

$$\begin{aligned} \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_{\Omega} a^{(n)} \cdot \varphi^{(3)} dx \right)^2 dx dt &\lesssim \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\partial_t \Delta \varphi_1^{(3)} \right)^2 dx dt \\ &\quad + \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\Delta^2 \varphi_1^{(3)} \right)^2 dx dt + \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(s \alpha'_\# \Delta \varphi_1^{(3)} \right)^2 dx dt \\ &\quad + \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \sum_{i=1}^{n-1} \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\alpha_n} \left(\int_{\Omega} a^{(i)} \cdot \varphi^{(3)} dx \right)^2 dx dt \end{aligned} \quad (3.39)$$

Then we can proceed by induction, using (1.8), and we deduce from (3.36) that

$$\begin{aligned} I(\varphi^{(3)}) + J(\varphi^{(3)}) &\lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 \\ &\quad + \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1} \left(s^3 \lambda^4 \xi_1^3 \left| \nabla \Delta \varphi_1^{(3)} \right|^2 + s^5 \lambda^6 \xi_1^5 \left| \Delta \varphi_1^{(3)} \right|^2 + s^8 \lambda^{10} \xi_1^8 \left| \varphi_1^{(3)} \right|^2 \right) dx dt \\ &\quad + \sum_{i=1}^n \iint_{(0,T)\times\omega_i} e^{-2(1-(n-i+1)\delta)s\alpha_i} \left(\left(\partial_t \Delta \varphi_1^{(3)} \right)^2 + \left(\Delta^2 \varphi_1^{(3)} \right)^2 + \left(s \alpha'_\# \Delta \varphi_1^{(3)} \right)^2 \right) dx dt. \end{aligned} \quad (3.40)$$

Using (1.7), let us consider open subsets $\tilde{\omega}^{(i)}$ $i = 1, \dots, 5$, such that

$$\bigcup_{i=1}^n \omega_i \Subset \tilde{\omega}^{(1)} \Subset \dots \Subset \tilde{\omega}^{(5)} \Subset \omega. \quad (3.41)$$

We also consider smooth functions

$$\chi^{(i)} \in C^\infty(\mathbb{R}^2; [0, 1]), \quad \chi^{(i)} \equiv 1 \text{ in } \tilde{\omega}^{(i)}, \quad \text{supp } \chi^{(i)} \subset \tilde{\omega}^{(i+1)} \quad (i = 1, \dots, 4).$$

We deduce from (3.40) and (2.8) that

$$\begin{aligned} I(\varphi^{(3)}) + J(\varphi^{(3)}) &\lesssim \|\rho_2 g\|_{L^2((0,T) \times \Omega)}^2 \\ &+ \iint_{(0,T) \times \tilde{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\left| \nabla \Delta \varphi_1^{(3)} \right|^2 + \left| \Delta \varphi_1^{(3)} \right|^2 + \left| \varphi_1^{(3)} \right|^2 + \left(\partial_t \Delta \varphi_1^{(3)} \right)^2 + \left(\Delta^2 \varphi_1^{(3)} \right)^2 \right) dx dt. \end{aligned} \quad (3.42)$$

We have

$$\begin{aligned} &\iint_{(0,T) \times \tilde{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\Delta^2 \varphi_1^{(3)} \right)^2 dx dt \leq \iint_{(0,T) \times \tilde{\omega}^{(2)}} \chi^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\Delta^2 \varphi_1^{(3)} \right)^2 dx dt \\ &= - \iint_{(0,T) \times \tilde{\omega}^{(2)}} \chi^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \nabla \Delta^2 \varphi_1^{(3)} \cdot \nabla \Delta \varphi_1^{(3)} dx dt - \iint_{(0,T) \times \tilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \Delta^2 \varphi_1^{(3)} \nabla \chi^{(1)} \cdot \nabla \Delta \varphi_1^{(3)} dx dt \\ &\lesssim \left\| \theta_3 \varphi^{(3)} \right\|_{L^2(0,T; H^5(\Omega))} \left(\iint_{(0,T) \times \tilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\#} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt \right)^{1/2}. \end{aligned}$$

We can proceed similarly to estimate successively

$$\begin{aligned} &\iint_{(0,T) \times \tilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\#} \left| \nabla \Delta \varphi_1^{(3)} \right|^2 dx dt, \quad \iint_{(0,T) \times \tilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \varphi_1^{(3)} \right|^2 dx dt, \\ &\iint_{(0,T) \times \tilde{\omega}^{(4)}} e^{-2s(1-\frac{1}{4^2})\alpha_\#} \left| \nabla \varphi_1^{(3)} \right|^2 dx dt. \end{aligned}$$

For the last term, we integrate by parts in time:

$$\begin{aligned} &\iint_{(0,T) \times \tilde{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\partial_t \Delta \varphi_1^{(3)} \right)^2 dx dt = - \iint_{(0,T) \times \tilde{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\partial_t^2 \Delta \varphi_1^{(3)} \right) \left(\Delta \varphi_1^{(3)} \right) dx dt \\ &\quad + \frac{1}{2} \iint_{(0,T) \times \tilde{\omega}^{(1)}} \left(e^{-2s(1-\frac{1}{4^5})\alpha_\#} \right)'' \left(\Delta \varphi_1^{(3)} \right)^2 dx dt \\ &\lesssim \left\| \theta_3 \partial_t^2 \varphi^{(3)} \right\|_{L^2(0,T; H^2(\Omega))} \left(\iint_{(0,T) \times \tilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \varphi_1^{(3)} \right|^2 dx dt \right)^{1/2} \\ &\quad + \iint_{(0,T) \times \tilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \varphi_1^{(3)} \right|^2 dx dt. \end{aligned}$$

Using the above computations, we deduce (3.15) from (3.42). We obtain (3.16) by combining (3.15) with (3.7) and (3.14). This ends the proof of Proposition 3.1 (and thus of Theorem 1.1).

4 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. First we need to define the functions $\hat{\sigma}_i$, $i = 1, 2, 3$. As in Section 2.2, there exist (see [15] or [23, Theorem 9.4.3, p.299]) for any $i = 1, \dots, n$, $\hat{\eta}_i \in C^2(\bar{\Omega})$ satisfying

$$\hat{\eta}_i > 0 \text{ in } \Omega, \quad \hat{\eta}_i = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \hat{\eta}_i = 1, \quad \nabla \hat{\eta}_i \neq 0 \text{ in } \overline{\Omega \setminus \tilde{\omega}_i}. \quad (4.1)$$

We use $\hat{\eta}_i$ to define the following standard functions:

$$\hat{\alpha}_i(t, x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m + \hat{\eta}_i(x))\}}{t^m(T-t)^m}, \quad \hat{\xi}_i(t, x) = \frac{\exp\{\lambda(2m + \hat{\eta}_i(x))\}}{t^m(T-t)^m}, \quad (4.2)$$

where $m \geq 9$, $\lambda > 1$. Recalling the definitions (2.5) and (2.6), we have

$$\xi_{\sharp} \leq \widehat{\xi}_i \leq \xi_b, \quad e^{-s\alpha_{\sharp}} \leq e^{-s\widehat{\alpha}_i} \leq e^{-s\alpha_b} \quad (i = 1, \dots, n). \quad (4.3)$$

Let us consider $M_1, M_2 > 0$ and let us write $M := M_1 + M_2$. We recall that N_1, N_2 and N are defined at the beginning of Section 3. We assume

$$N_2 \geq M + 1 \quad (4.4)$$

and we define

$$\widehat{\sigma}_1 := \begin{cases} e^{-M_2 s \alpha_{\sharp}} & \text{in } \left[0, \frac{T}{2}\right] \\ [e^{-M_2 s \alpha_{\sharp}}] \left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}, \quad \widehat{\sigma}_2 := \begin{cases} e^{-M s \alpha_{\sharp}} & \text{in } \left[0, \frac{T}{2}\right] \\ [e^{-M s \alpha_{\sharp}}] \left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}, \quad (4.5)$$

$$\widehat{\sigma}_3 := \begin{cases} e^{-(N+1)s\alpha_{\sharp}} & \text{in } \left[0, \frac{T}{2}\right] \\ [e^{-(N+1)s\alpha_{\sharp}}] \left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}. \quad (4.6)$$

Note that $\widehat{\sigma}_i(0) = 0$ $i = 1, 2, 3$.

We are going to show the existence of $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$, Theorem 1.4 holds with the weights defined as above.

For Corollary 1.6, we assume $M_2 > 1$ and we define

$$\widehat{\sigma}_0 := \begin{cases} e^{-(M_2-1)s\alpha_{\sharp}} & \text{in } \left[0, \frac{T}{2}\right] \\ [e^{-(M_2-1)s\alpha_{\sharp}}] \left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}. \quad (4.7)$$

Note that

$$\frac{\widehat{\sigma}_2}{\widehat{\sigma}_0}, \frac{\widehat{\sigma}_3}{\widehat{\sigma}_0}, \frac{\widehat{\sigma}'_0 \widehat{\sigma}_1}{\widehat{\sigma}_0^2} \in L^\infty(0, T)$$

and if

$$2M_2 \geq N + 3. \quad (4.8)$$

then, we have

$$\frac{\widehat{\sigma}_0^2}{\widehat{\sigma}_3} \in L^\infty(0, T). \quad (4.9)$$

We are going to show Corollary 1.6 with the above choices and for $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$, where $s_0 > 0$ and $\lambda_0 > 0$ are large enough.

Remark 4.1. There exist N_1, N_2, M_1 and M_2 satisfying (3.5), (4.4) and (4.8). For instance one can take

$$N_1 = 2, \quad N_2 = 13, \quad M_1 = 2, \quad M_2 = 9.$$

4.1 Decomposition of the adjoint system

As for the proof of Theorem 1.4, we first start by decomposing the solution of (1.15) into three parts. We only need to focus on ψ since we have already obtained a Carleman estimate for φ in Proposition 3.1. We recall that $M_1, M_2 > 0$ and that $M = M_1 + M_2$. We set

$$\widehat{\rho}_i := e^{-M_i s \alpha^*} \quad (i = 1, 2). \quad (4.10)$$

From (1.15) and using the notation (2.1), we see that ψ is solution of

$$\mathcal{L}(\psi, \pi_\psi) = (g^{(1)}, \psi^0), \quad (4.11)$$

Let us consider the following decomposition of ψ :

$$\widehat{\rho}_1 \widehat{\rho}_2 \psi = \widehat{\rho}_1 \psi^{(1)} + \psi^{(2)} + \psi^{(3)}, \quad (4.12)$$

where

$$\mathcal{L}(\psi^{(1)}, \pi_{\psi^{(1)}}) = (\widehat{\rho}_2 g^{(1)}, 0), \quad (4.13)$$

$$\mathcal{L}(\psi^{(2)}, \pi_{\psi^{(2)}}) = \left(-M_2 s \alpha'_\# \widehat{\rho}_1 \psi^{(1)}, 0 \right), \quad (4.14)$$

$$\mathcal{L}(\psi^{(3)}, \pi_{\psi^{(3)}}) = \left(-M s \alpha'_\# \left(\psi^{(2)} + \psi^{(3)} \right), 0 \right). \quad (4.15)$$

We estimate $\psi^{(1)}$ and $\psi^{(2)}$ by using Proposition 2.2 and their regularity properties allow us to perform a Carleman estimate on $\psi^{(3)}$. Let us introduce the following quantities associated with $\psi^{(3)}$:

$$\begin{aligned} \widehat{I}(\psi^{(3)}) := & \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(s\lambda^2 \widehat{\xi}_1 \left| \nabla^3 \Delta \psi_1^{(3)} \right|^2 + s^3 \lambda^4 \widehat{\xi}_1^3 \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(s^5 \lambda^6 \widehat{\xi}_1^5 \left| \nabla \Delta \psi_1^{(3)} \right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0,T) \times \Omega} e^{-2s\alpha_\#} s^7 \lambda^8 \xi_\#^7 \left| \psi^{(3)} \right|^2 dx dt \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \widehat{J}(\psi^{(3)}) := & \left\| \widehat{\theta}_1 \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \widehat{\theta}_1 \partial_t \psi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ & + \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^2 + \left\| \widehat{\theta}_2 \partial_t \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \widehat{\theta}_2 \partial_t^2 \psi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \\ & + \left\| \widehat{\theta}_3 \psi^{(3)} \right\|_{L^2(0,T;H^6(\Omega))}^2 + \left\| \widehat{\theta}_3 \partial_t \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^2 + \left\| \widehat{\theta}_3 \partial_t^2 \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\| \widehat{\theta}_3 \partial_t^3 \psi^{(3)} \right\|_{L^2(0,T;L^2(\Omega))}^2, \end{aligned} \quad (4.17)$$

where we have set

$$\begin{aligned} \widehat{\theta}_1 := & s^{5/2-1/m} \lambda^4 (\xi_\#)^{5/2-1/m} e^{-s\alpha_\#}, \quad \widehat{\theta}_2 := s^{3/2-2/m} \lambda^4 (\xi_\#)^{3/2-2/m} e^{-s\alpha_\#}, \\ & \widehat{\theta}_3 := s^{1/2-3/m} \lambda^4 (\xi_\#)^{1/2-3/m} e^{-s\alpha_\#}. \end{aligned} \quad (4.18)$$

Using (1.18), we consider an open set $\widehat{\omega}_0$ such that

$$\bigcup_{i=1}^n \widehat{\omega}_i \Subset \widehat{\omega}_0 \Subset \widehat{\omega}. \quad (4.19)$$

With the above notation, we can state the Carleman estimates we obtain for $\psi^{(3)}$:

Proposition 4.2. *Assume (1.5), (1.7), (1.8), (1.17), (1.18), (1.19) and (1.20). There exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ and $s \geq s_0(T^m + T^{2m})$, the solution ψ of (4.11), can be decomposed as in (4.12), (4.13), (4.14) and (4.15) and the functions $\psi^{(1)}$, $\psi^{(2)}$ and $\psi^{(3)}$ satisfy*

$$\left\| \psi^{(1)} \right\|_{X_1} + \left\| \psi^{(2)} \right\|_{X_2} \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}, \quad (4.20)$$

and

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \iint_{(0,T) \times \widehat{\omega}_0} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt. \quad (4.21)$$

In particular, any solution (φ, ψ) of (1.15) verifies the following estimate:

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2(N+1)s\alpha_\#} |\varphi|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2(M+1)s\alpha_\#} |\psi|^2 dx dt \\ & \lesssim \iint_{(0,T) \times \Omega} e^{-2M_2s\alpha_\#} \left| g^{(0)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2M_2s\alpha_\#} \left| g^{(1)} \right|^2 dx dt \\ & \quad + \iint_{(0,T) \times (\omega \cup \widehat{\omega})} e^{-2sM\alpha_\#} |\varphi_1|^2 dx dt. \end{aligned} \quad (4.22)$$

We now prove the above proposition. Applying Proposition 2.2, we obtain

$$\left\| \psi^{(1)} \right\|_{X_1} \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}, \quad (4.23)$$

$$\left\| M_2 s \alpha'_\# \widehat{\rho}_1 \psi^{(1)} \right\|_{X_1} \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}, \quad (4.24)$$

$$\left\| \psi^{(2)} \right\|_{X_2} \lesssim \left\| M_2 s \alpha'_\# \widehat{\rho}_1 \psi^{(1)} \right\|_{X_1} \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}. \quad (4.25)$$

We thus deduce (4.20). Then, from (4.15), we deduce

$$\mathcal{L}(\widehat{\theta}_1 \psi^{(3)}, \widehat{\theta}_1 \pi_\psi^{(3)}) = \left(-Ms \widehat{\theta}_1 \alpha'_\# \left(\psi^{(2)} + \psi^{(3)} \right) + \widehat{\theta}'_1 \psi^{(3)}, 0 \right), \quad (4.26)$$

$$\mathcal{L}(\widehat{\theta}_2 \psi^{(3)}, \widehat{\theta}_2 \pi_\psi^{(3)}) = \left(-Ms \widehat{\theta}_2 \alpha'_\# \left(\psi^{(2)} + \psi^{(3)} \right) + \widehat{\theta}'_2 \psi^{(3)}, 0 \right), \quad (4.27)$$

$$\mathcal{L}(\widehat{\theta}_3 \psi^{(3)}, \widehat{\theta}_3 \pi_\psi^{(3)}) = \left(-Ms \widehat{\theta}_3 \alpha'_\# \left(\psi^{(2)} + \psi^{(3)} \right) + \widehat{\theta}'_3 \psi^{(3)}, 0 \right). \quad (4.28)$$

By similar computations as Section 3.1 (see (3.29)), and using Proposition 2.2, one can show that $\widehat{J}(\psi^{(3)})$ defined by (4.17) satisfies

$$\widehat{J}(\psi^{(3)}) \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \left\| s^{7/2} \lambda^4 (\xi_\#)^{7/2} e^{-s\alpha_\#} \psi^{(3)} \right\|_{L^2((0,T) \times \Omega)}^2. \quad (4.29)$$

4.2 Carleman estimates

Taking the divergence of the first equation of (4.15), we obtain the following relation for $\pi_\psi^{(3)}$:

$$\Delta \pi_\psi^{(3)} = - \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right) \operatorname{div} a^{(i)}. \quad (4.30)$$

Then, applying the operators Δ and $\nabla^2 \Delta$ on the first component of the first equation of (4.15), we deduce

$$\partial_t \Delta \psi_1^{(3)} - \Delta^2 \psi_1^{(3)} + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right) \check{a}^{(i)} = -Ms \alpha'_\# \left(\Delta \psi_1^{(2)} + \Delta \psi_1^{(3)} \right) \quad \text{in } (0, T) \times \Omega \quad (4.31)$$

and

$$\begin{aligned} \partial_t \nabla^2 \Delta \psi_1^{(3)} - \Delta \nabla^2 \Delta \psi_1^{(3)} + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right) \nabla^2 \check{a}^{(i)} \\ = -Ms \alpha'_\# \left(\nabla^2 \Delta \psi_1^{(2)} + \nabla^2 \Delta \psi_1^{(3)} \right) \quad \text{in } (0, T) \times \Omega \end{aligned} \quad (4.32)$$

where $\check{\alpha}^{(i)}$ is defined by (1.16). We continue by following the steps in Section 3.2: first, we apply Lemma 2.5 to (4.32):

$$\begin{aligned}
& \iint_{(0,T) \times \Omega} s^3 \lambda^4 \widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1} \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} s \lambda^2 \widehat{\xi}_1 e^{-2s\widehat{\alpha}_1} \left| \nabla^3 \Delta \psi_1^{(3)} \right|^2 dx dt \\
& \lesssim \iint_{(0,T) \times \Omega} s^2 (\alpha'_{\#})^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3 \Delta \psi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} s^2 (\alpha'_{\#})^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3 \Delta \psi_1^{(2)} \right|^2 dx dt \\
& + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right)^2 dx dt + \iint_{(0,T) \times \partial\Omega} s \lambda \xi_{\#} e^{-2s\alpha_{\#}} \left| \frac{\partial}{\partial n} \nabla^2 \Delta \psi_1^{(3)} \right|^2 d\gamma dt \\
& + \iint_{(0,T) \times \omega_1} s^3 \lambda^4 \widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1} \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 dx dt. \quad (4.33)
\end{aligned}$$

The term

$$\iint_{(0,T) \times \Omega} s^2 (\alpha'_{\#})^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3 \Delta \psi_1^{(3)} \right|^2 dx dt,$$

can be absorbed by the right-hand side of (4.33) and by using (4.25), we have

$$\iint_{(0,T) \times \Omega} s^2 (\alpha'_{\#})^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3 \Delta \psi_1^{(2)} \right|^2 dx dt \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2.$$

Then, we apply Lemma 2.3 to deduce

$$\begin{aligned}
& \iint_{(0,T) \times \Omega} s^7 \lambda^8 \widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1} \left| \Delta \psi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} s^5 \lambda^6 \widehat{\xi}_1^5 e^{-2s\widehat{\alpha}_1} \left| \nabla \Delta \psi_1^{(3)} \right|^2 dx dt \\
& \lesssim \iint_{(0,T) \times \Omega} s^3 \lambda^4 \widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1} \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \widehat{\omega}_1} s^5 \lambda^6 \widehat{\xi}_1^5 e^{-2s\widehat{\alpha}_1} \left| \nabla \Delta \psi_1^{(3)} \right|^2 dx dt \\
& + \iint_{(0,T) \times \widehat{\omega}_1} s^7 \lambda^8 \widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1} \left| \Delta \psi_1^{(3)} \right|^2 dx dt. \quad (4.34)
\end{aligned}$$

Using the ellipticity of the Laplace operator with Dirichlet boundary conditions, we deduce that

$$\iint_{(0,T) \times \Omega} s^7 \lambda^8 \xi_{\#}^7 e^{-2s\alpha_{\#}} \left(\left| \psi_1^{(3)} \right|^2 + \left| \nabla \psi_1^{(3)} \right|^2 \right) dx dt \lesssim \iint_{(0,T) \times \Omega} s^7 \lambda^8 \widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1} \left| \Delta \psi_1^{(3)} \right|^2 dx dt. \quad (4.35)$$

Moreover, using the Poincaré inequality and the divergence condition, we also obtain

$$\iint_{(0,T) \times \Omega} s^7 \lambda^8 \xi_{\#}^7 e^{-2s\alpha_{\#}} \left| \psi_2^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} s^7 \lambda^8 \xi_{\#}^7 e^{-2s\alpha_{\#}} \left| \nabla \psi_1^{(3)} \right|^2 dx dt.$$

Combining the above estimate with (4.35), (4.34) and (4.33), we deduce that $\widehat{I}(\psi^{(3)})$ defined by (4.16) satisfies

$$\begin{aligned}
\widehat{I}(\psi^{(3)}) & \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right)^2 dx dt \\
& + \iint_{(0,T) \times \widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3 \lambda^4 \widehat{\xi}_1^3 \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 + s^5 \lambda^6 \widehat{\xi}_1^5 \left| \nabla \Delta \psi_1^{(3)} \right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt \\
& + \iint_{(0,T) \times \partial\Omega} s \lambda \xi_{\#} e^{-2s\alpha_{\#}} \left| \frac{\partial}{\partial n} \nabla^2 \Delta \psi_1^{(3)} \right|^2 d\gamma dt.
\end{aligned}$$

Combining the above inequality with (4.29) we deduce

$$\begin{aligned}
\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) &\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right)^2 dx dt \\
&+ \iint_{(0,T) \times \widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3 \lambda^4 \widehat{\xi}_1^3 \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 + s^5 \lambda^6 \widehat{\xi}_1^5 \left| \nabla \Delta \psi_1^{(3)} \right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt \\
&+ \iint_{(0,T) \times \partial\Omega} s \lambda \xi_{\#} e^{-2s\alpha_{\#}} \left| \frac{\partial}{\partial n} \nabla^2 \Delta \psi_1^{(3)} \right|^2 d\gamma dt. \quad (4.36)
\end{aligned}$$

Then, from trace theorems and interpolation inequalities

$$\int_{\partial\Omega} \left| \frac{\partial}{\partial n} \nabla^2 \Delta \psi_1^{(3)} \right|^2 d\gamma \lesssim \left\| \psi^{(3)} \right\|_{H^5(\Omega)} \left\| \psi^{(3)} \right\|_{H^6(\Omega)} \lesssim \left\| \psi^{(3)} \right\|_{H^4(\Omega)}^{1/2} \left\| \psi^{(3)} \right\|_{H^6(\Omega)}^{3/2}.$$

Thus, using (4.18) and (2.10), we have for $m \geq 11$, and $s \geq T^{2m} + T^m$,

$$\iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_{\#}} s \lambda \xi_{\#} \left| \frac{\partial}{\partial n} \nabla^2 \Delta \psi_1^{(3)} \right|^2 d\gamma dt \lesssim \lambda^{-7} \left\| \widehat{\theta}^2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))}^{1/2} \left\| \widehat{\theta}^3 \psi^{(3)} \right\|_{L^2(0,T;H^6(\Omega))}^{3/2}.$$

Combining the above estimate and (4.36), we deduce the existence of $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$,

$$\begin{aligned}
\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) &\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^n \iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right)^2 dx dt \\
&+ \iint_{(0,T) \times \widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3 \lambda^4 \widehat{\xi}_1^3 \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 + s^5 \lambda^6 \widehat{\xi}_1^5 \left| \nabla \Delta \psi_1^{(3)} \right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt. \quad (4.37)
\end{aligned}$$

4.3 Removing the nonlocal spatial terms

We set

$$\delta := \frac{1}{4^5(n+1)}.$$

Using (2.8) and Lemma 2.3, there exists $C > 0$ such that if $\lambda \geq C$ and $s \geq CT^{2m}$,

$$\begin{aligned}
\iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(n)} \cdot \psi^{(3)} dx \right)^2 dx dt &\lesssim \iint_{(0,T) \times \Omega} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_{\Omega} b^{(n)} \cdot \psi^{(3)} dx \right)^2 dx dt \\
&\lesssim \iint_{(0,T) \times \widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_{\Omega} b^{(n)} \cdot \psi^{(3)} dx \right)^2 dx dt. \quad (4.38)
\end{aligned}$$

Using (1.20) for $i = n$, (4.31) and the above equation, we deduce that

$$\begin{aligned}
\iint_{(0,T) \times \Omega} e^{-2s\widehat{\alpha}_1} \left(\int_{\Omega} b^{(n)} \cdot \psi^{(3)} dx \right)^2 dx dt &\lesssim \iint_{(0,T) \times \widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\partial_t \Delta \psi_1^{(3)} \right)^2 dx dt \\
&+ \iint_{(0,T) \times \widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\Delta^2 \psi_1^{(3)} \right)^2 dx dt + \iint_{(0,T) \times \widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(s \alpha'_{\#} \Delta \psi_1^{(3)} \right)^2 dx dt \\
&+ \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \sum_{i=1}^{n-1} \iint_{(0,T) \times \Omega} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_{\Omega} b^{(i)} \cdot \psi^{(3)} dx \right)^2 dx dt \quad (4.39)
\end{aligned}$$

Then we can proceed by induction by using (4.20), and we deduce from (4.37) that

$$\begin{aligned} \widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) &\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 \\ &+ \iint_{(0,T) \times \widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3 \lambda^4 \widehat{\xi}_1^3 \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 + s^5 \lambda^6 \widehat{\xi}_1^5 \left| \nabla \Delta \psi_1^{(3)} \right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt \\ &+ \sum_{i=1}^n \iint_{(0,T) \times \widehat{\omega}_i} e^{-2(1-(n-i+1)\delta)s\widehat{\alpha}_i} \left(\left(\partial_t \Delta \psi_1^{(3)} \right)^2 + \left(\Delta^2 \psi_1^{(3)} \right)^2 + \left(s\alpha'_\# \Delta \psi_1^{(3)} \right)^2 \right) dx dt. \end{aligned} \quad (4.40)$$

Using (4.19), we can consider open subsets $\check{\omega}^{(i)}$ $i = 1, \dots, 3$, such that

$$\bigcup_{i=1}^n \widehat{\omega}_i \Subset \check{\omega}^{(1)} \Subset \check{\omega}^{(2)} \Subset \check{\omega}^{(3)} = \widehat{\omega}_0 \quad (4.41)$$

and smooth functions

$$\check{\chi}^{(i)} \in C^\infty(\mathbb{R}^2; [0, 1]), \quad \check{\chi}^{(i)} \equiv 1 \text{ in } \check{\omega}^{(i)}, \quad \text{supp } \check{\chi}^{(i)} \subset \check{\omega}^{(i+1)} \quad (i = 1, 2).$$

We deduce from (4.40) that

$$\begin{aligned} \widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) &\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 \\ &+ \iint_{(0,T) \times \check{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 + \left(\partial_t \Delta \psi_1^{(3)} \right)^2 + \left| \nabla \Delta \psi_1^{(3)} \right|^2 + \left| \Delta \psi_1^{(3)} \right|^2 \right) dx dt. \end{aligned} \quad (4.42)$$

By integrating by parts and by using (4.18), we have

$$\begin{aligned} &\iint_{(0,T) \times \check{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\Delta^2 \psi_1^{(3)} \right)^2 dx dt \leq \iint_{(0,T) \times \check{\omega}^{(2)}} \check{\chi}^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\Delta^2 \psi_1^{(3)} \right)^2 dx dt \\ &= - \iint_{(0,T) \times \check{\omega}^{(2)}} \check{\chi}^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \nabla \Delta^2 \psi_1^{(3)} \cdot \nabla \Delta \psi_1^{(3)} dx dt - \iint_{(0,T) \times \check{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \Delta^2 \psi_1^{(3)} \nabla \check{\chi}^{(1)} \cdot \nabla \Delta \psi_1^{(3)} dx dt \\ &\lesssim \left\| \widehat{\theta}_3 \psi^{(3)} \right\|_{L^2(0,T; H^5(\Omega))} \left(\iint_{(0,T) \times \check{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\#} \left| \nabla \Delta \psi_1^{(3)} \right|^2 dx dt \right)^{1/2}. \end{aligned}$$

We can proceed similarly to estimate

$$\iint_{(0,T) \times \check{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\#} \left| \nabla \Delta \psi_1^{(3)} \right|^2 dx dt, \quad \iint_{(0,T) \times \check{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left| \nabla^2 \Delta \psi_1^{(3)} \right|^2 dx dt.$$

For the last term, we integrate by parts in time:

$$\begin{aligned} &\iint_{(0,T) \times \check{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\partial_t \Delta \psi_1^{(3)} \right)^2 dx dt = - \iint_{(0,T) \times \check{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\#} \left(\partial_t^2 \Delta \psi_1^{(3)} \right) \left(\Delta \psi_1^{(3)} \right) dx dt \\ &\quad + \frac{1}{2} \iint_{(0,T) \times \check{\omega}^{(1)}} \left(e^{-2s(1-\frac{1}{4^5})\alpha_\#} \right)'' \left(\Delta \psi_1^{(3)} \right)^2 dx dt \\ &\lesssim \left\| \widehat{\theta}_3 \partial_t^2 \psi^{(3)} \right\|_{L^2(0,T; H^2(\Omega))} \left(\iint_{(0,T) \times \check{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt \right)^{1/2} \\ &\quad + \iint_{(0,T) \times \check{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt. \end{aligned}$$

Using the above computations, (4.42) yields (4.21). Recalling (4.16), (4.12), and (4.20), we have

$$\iint_{(0,T) \times \Omega} e^{-2s\alpha_\#} s^7 \lambda^8 \xi_\#^7 |\widehat{\rho}_1 \widehat{\rho}_2 \psi|^2 dx dt \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \widehat{I}(\psi^{(3)}).$$

On the other hand, since φ satisfies (1.15), it is solution of (1.3) with

$$g := \psi 1_{\mathcal{O}} + g^{(0)}.$$

Applying (3.16) yields

$$\begin{aligned} \left\| s^3 \lambda^4 \xi_\#^3 e^{-(N+1)s\alpha_\#} \varphi \right\|_{L^2((0,T) \times \Omega)}^2 &\lesssim \left\| e^{-N_2 s \alpha_\#} \psi \right\|_{L^2((0,T) \times \Omega)}^2 + \left\| e^{-N_2 s \alpha_\#} g^{(0)} \right\|_{L^2((0,T) \times \Omega)}^2 \\ &\quad + \iint_{(0,T) \times \omega} e^{-s(2N+1)\alpha_\#} |\varphi_1|^2 dx dt \end{aligned}$$

and since $N_2 \geq M + 1$, we can combine this relation with (4.21) and deduce

$$\begin{aligned} \widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) + \left\| s^3 \lambda^4 \xi_\#^3 e^{-(N+1)s\alpha_\#} \varphi \right\|_{L^2((0,T) \times \Omega)}^2 &+ \left\| s^{7/2} \lambda^4 \xi_\#^{7/2} e^{-(M+1)s\alpha_\#} \psi \right\|_{L^2((0,T) \times \Omega)}^2 \\ &\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \left\| \rho_2 g^{(0)} \right\|_{L^2((0,T) \times \Omega)}^2 \\ &+ \iint_{(0,T) \times \omega} e^{-s(2N+1)\alpha_\#} |\varphi_1|^2 dx dt + \iint_{(0,T) \times \widehat{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt. \end{aligned} \quad (4.43)$$

4.4 Removing the local term in ψ

Using (4.19), there exists $\check{\chi} \in C^\infty(\mathbb{R}^2; [0, 1])$, such that

$$\check{\chi} \equiv 1 \text{ in } \check{\omega}^{(3)} = \widehat{\omega}_0, \quad \text{supp } \check{\chi} \subset \widehat{\omega}$$

and we can write

$$\begin{aligned} \iint_{(0,T) \times \widehat{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt &\leq \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt \\ &= \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \Delta \psi_1 - \widehat{\rho}_1 \Delta \psi_1^{(1)} - \Delta \psi_1^{(2)} \right) dx dt. \end{aligned} \quad (4.44)$$

Applying the operator Δ on the first component of the first equation of (1.15) and using (1.19), we deduce

$$-\partial_t \Delta \varphi_1 - \Delta^2 \varphi_1 = \Delta \psi_1 + \Delta g_1^{(0)} \quad \text{in } (0, T) \times \widehat{\omega}.$$

Combining the above relation with (4.44), we find

$$\begin{aligned} \iint_{(0,T) \times \widehat{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left| \Delta \psi_1^{(3)} \right|^2 dx dt &\leq - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_\#} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \partial_t \Delta \varphi_1 \right. \\ &\quad \left. + \widehat{\rho}_1 \widehat{\rho}_2 \Delta^2 \varphi_1 + \widehat{\rho}_1 \widehat{\rho}_2 \Delta g_1^{(0)} + \widehat{\rho}_1 \Delta \psi_1^{(1)} + \Delta \psi_1^{(2)} \right) dx dt. \end{aligned} \quad (4.45)$$

Now, we estimate each term of right-hand side of the above relation. First,

$$\begin{aligned}
& - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \partial_t \Delta \varphi_1 \right) dx dt \\
& = \iint_{(0,T) \times \widehat{\omega}} \left(e^{-s(2+M-\frac{2}{4^3})\alpha_{\sharp}} \right)' \left(\Delta \check{\chi} \Delta \psi_1^{(3)} + 2 \nabla \check{\chi} \cdot \nabla \Delta \psi_1^{(3)} + \check{\chi} \Delta^2 \psi_1^{(3)} \right) \varphi_1 dx dt \\
& \quad + \iint_{(0,T) \times \widehat{\omega}} e^{-s(2+M-\frac{2}{4^3})\alpha_{\sharp}} \left(\Delta \check{\chi} \partial_t \Delta \psi_1^{(3)} + 2 \nabla \check{\chi} \cdot \partial_t \nabla \Delta \psi_1^{(3)} + \check{\chi} \partial_t \Delta^2 \psi_1^{(3)} \right) \varphi_1 dx dt
\end{aligned}$$

so that, using (4.18),

$$\begin{aligned}
& - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \partial_t \Delta \varphi_1 \right) dx dt \\
& \lesssim \left(\left\| \widehat{\theta}_3 \partial_t \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} + \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} \right) \left(\iint_{(0,T) \times \widehat{\omega}} e^{-2s(1+M-\frac{1}{4^2})\alpha_{\sharp}} |\varphi_1|^2 dx dt \right)^{1/2}. \quad (4.46)
\end{aligned}$$

Similarly,

$$\begin{aligned}
& - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \Delta^2 \varphi_1 \right) dx dt \\
& \lesssim \left\| \widehat{\theta}_3 \psi^{(3)} \right\|_{L^2(0,T;H^6(\Omega))} \left(\iint_{(0,T) \times \widehat{\omega}} e^{-2s(1+M-\frac{1}{4^2})\alpha_{\sharp}} |\varphi_1|^2 dx dt \right)^{1/2}, \quad (4.47)
\end{aligned}$$

$$\begin{aligned}
& - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \widehat{\rho}_2 \Delta g_1^{(0)} \right) dx dt \\
& \lesssim \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} \left(\iint_{(0,T) \times \widehat{\omega}} e^{-2s(1+M-\frac{1}{4^2})\alpha_{\sharp}} \left| g_1^{(0)} \right|^2 dx dt \right)^{1/2} \\
& \lesssim \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} \left\| \widehat{\rho}_1 \widehat{\rho}_2 g^{(0)} \right\|_{L^2(0,T;L^2(\Omega))}, \quad (4.48)
\end{aligned}$$

and

$$\begin{aligned}
& - \iint_{(0,T) \times \widehat{\omega}} \check{\chi} e^{-2s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right) \left(\widehat{\rho}_1 \Delta \psi_1^{(1)} + \Delta \psi_1^{(2)} \right) dx dt \\
& \lesssim \left(\iint_{(0,T) \times \widehat{\omega}} e^{-4s(1-\frac{1}{4^3})\alpha_{\sharp}} \left(\Delta \psi_1^{(3)} \right)^2 dx dt \right)^{1/2} \left(\left\| \psi^{(1)} \right\|_{L^2(0,T;H^2(\Omega))} + \left\| \psi^{(2)} \right\|_{L^2(0,T;H^2(\Omega))} \right) \\
& \lesssim \left\| \widehat{\theta}_1 \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))} \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}. \quad (4.49)
\end{aligned}$$

Gathering (4.43), (4.45), (4.46)–(4.49) and using $N_2 \geq M + 1$ imply

$$\begin{aligned}
& \widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) + \left\| s^3 \lambda^4 \xi_{\sharp}^3 e^{-(N+1)s\alpha_{\sharp}} \varphi \right\|_{L^2((0,T) \times \Omega)}^2 + \left\| s^{7/2} \lambda^4 \xi_{\sharp}^{7/2} e^{-(M+1)s\alpha_{\sharp}} \psi \right\|_{L^2((0,T) \times \Omega)}^2 \\
& \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T) \times \Omega)}^2 + \left\| \widehat{\rho}_1 \widehat{\rho}_2 g^{(0)} \right\|_{L^2((0,T) \times \Omega)}^2 \\
& \quad + \iint_{(0,T) \times \omega} e^{-s(2N+1)\alpha_{\sharp}} |\varphi_1|^2 dx dt + \iint_{(0,T) \times \widehat{\omega}} e^{-2s(1+M-\frac{1}{4^2})\alpha_{\sharp}} |\varphi_1|^2 dx dt. \quad (4.50)
\end{aligned}$$

The above relation implies (4.22) and this concludes the proof of Proposition 4.2.

4.5 Controllability results

First let us deduce Theorem 1.4 from Proposition 4.2:

Proof of Theorem 1.4. Since $N_2 \geq M + 1$,

$$N + 1 > M > M_2$$

and there exists $P > 2$ such that

$$\left(\frac{1}{4} \frac{P^2}{P-1} \right)^m < \frac{N+1}{M} < \frac{N+1}{M_2}.$$

In particular, from (2.5)

$$-(N+1)s\alpha_{\sharp}(T/2) + M_2s\alpha_{\sharp}(T/P) < 0, \quad -(N+1)s\alpha_{\sharp}(T/2) + Ms\alpha_{\sharp}(T/P) < 0. \quad (4.51)$$

We consider $\chi_0 \in C^\infty([0, T])$ such that

$$\chi_0 \equiv 0 \quad \text{in } [0, T/P], \quad \chi_0 \equiv 1 \quad \text{in } [T/2, T], \quad |\chi'_0| \lesssim 1/T.$$

Then we deduce from (1.15) that

$$\begin{cases} \mathcal{L}^*(\chi_0\varphi, \chi_0\pi_\varphi) = (\chi_0\psi 1_{\mathcal{O}} + \chi_0g^{(0)} - \chi'_0\varphi, 0), \\ \mathcal{L}(\chi_0\psi, \chi_0\pi_\psi) = (\chi_0g^{(1)} + \chi'_0\psi, 0). \end{cases} \quad (4.52)$$

Then, combining Proposition 2.1 and Proposition 2.2, we deduce

$$\begin{aligned} \|\chi_0\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\chi_0\psi\|_{L^2(0,T;L^2(\Omega))} &\lesssim \|\chi_0g^{(0)}\|_{L^2(0,T;L^2(\Omega))} + \|\chi_0g^{(1)}\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + \|\chi'_0\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\chi'_0\psi\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

In particular, from (4.5), (4.6) and (4.51),

$$\begin{aligned} &\|\widehat{\sigma}_3\varphi\|_{L^2(T/2,T;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(T/2,T;L^2(\Omega))} \\ &\lesssim e^{-(N+1)s\alpha_{\sharp}(T/2)+M_2s\alpha_{\sharp}(T/P)} \left(\|\widehat{\sigma}_1g^{(0)}\|_{L^2(T/P,T;L^2(\Omega))} + \|\widehat{\sigma}_1g^{(1)}\|_{L^2(T/P,T;L^2(\Omega))} \right) \\ &\quad + \frac{1}{T} e^{-(N+1)s\alpha_{\sharp}(T/2)+Ms\alpha_{\sharp}(T/P)} \left(\|\widehat{\sigma}_2\varphi\|_{L^2(T/P,T/2;L^2(\Omega))} + \|\widehat{\sigma}_2\psi\|_{L^2(T/P,T/2;L^2(\Omega))} \right) \\ &\lesssim \|\widehat{\sigma}_1g^{(0)}\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_1g^{(1)}\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_2\varphi\|_{L^2(T/P,T/2;L^2(\Omega))} + \|\widehat{\sigma}_2\psi\|_{L^2(T/P,T/2;L^2(\Omega))}. \end{aligned} \quad (4.53)$$

Then, we deduce from (4.22) that

$$\begin{aligned} &\|\widehat{\sigma}_3\varphi\|_{L^2(0,T/2;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(0,T/2;L^2(\Omega))} \\ &\lesssim \|\widehat{\sigma}_1g^{(0)}\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_1g^{(1)}\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_2\varphi_1\|_{L^2(0,T;L^2(\omega \cup \widehat{\omega}))}. \end{aligned}$$

Combining the above relation and (4.53) yields the result. \square

The proof of Corollary 1.6 is completely standard and we only present the main ideas to prove it from Theorem 1.4. First, we define the space

$$\mathcal{X}_0 := \left\{ (\varphi, \pi_\varphi, \psi, \pi_\psi) \in C^\infty([0, T] \times \overline{\Omega}) : \operatorname{div} \varphi = \operatorname{div} \psi = 0, \quad \varphi = \psi = 0 \text{ on } (0, T) \times \partial\Omega, \right. \\ \left. \int_{\Omega} \pi_\varphi \, dx = \int_{\Omega} \pi_\psi \, dx = 0 \right\},$$

the operators

$$\begin{aligned} L^* \varphi &:= -\partial_t \varphi - \Delta \varphi + \sum_{i=1}^n \left(\int_{\Omega} a^{(i)} \cdot \varphi \, dx \right) b^{(i)} \\ L \psi &:= \partial_t \psi - \Delta \psi + \sum_{i=1}^n \left(\int_{\Omega} b^{(i)} \cdot \psi \, dx \right) a^{(i)}, \end{aligned}$$

and the bilinear symmetric form

$$\begin{aligned} \langle (\varphi, \pi_\varphi, \psi, \pi_\psi), (\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi) \rangle_{\mathcal{X}} &:= \iint_{(0,T) \times \Omega} \widehat{\sigma}_1^2 (L^* \varphi + \nabla \pi_\varphi - \psi 1_{\mathcal{O}}) \cdot (L^* \check{\varphi} + \nabla \check{\pi}_\varphi - \check{\psi} 1_{\mathcal{O}}) \, dx \, dt \\ &+ \iint_{(0,T) \times \Omega} \widehat{\sigma}_1^2 (L \psi + \nabla \pi_\psi) \cdot (L \check{\psi} + \nabla \check{\pi}_\psi) \, dx \, dt + \iint_{(0,T) \times (\omega \cup \widehat{\omega})} \widehat{\sigma}_2^2 \varphi_1 \check{\varphi}_1 \, dx \, dt. \end{aligned}$$

From (1.21), we deduce that

$$\| \widehat{\sigma}_3 \varphi \|_{L^2(0,T;L^2(\Omega))} + \| \widehat{\sigma}_3 \psi \|_{L^2(0,T;L^2(\Omega))} \lesssim \| (\varphi, \pi_\varphi, \psi, \pi_\psi) \|_{\mathcal{X}} := \langle (\varphi, \pi_\varphi, \psi, \pi_\psi), (\varphi, \pi_\varphi, \psi, \pi_\psi) \rangle_{\mathcal{X}}^{1/2}. \quad (4.54)$$

and thus $\| \cdot \|_{\mathcal{X}}$ is a norm and we can define the completion \mathcal{X} of \mathcal{X}_0 for this norm.

We also define

$$\ell((\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi)) := \iint_{(0,T) \times \Omega} (f^{(0)} \cdot \check{\varphi} + f^{(1)} \cdot \check{\psi}) \, dx \, dt,$$

From (4.54), we deduce that ℓ is a linear continuous form of \mathcal{X} and

$$\| \ell \|_{\mathcal{X}'} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)}.$$

Thus from the Riesz theorem, there exists a unique $(\varphi, \pi_\varphi, \psi, \pi_\psi) \in \mathcal{X}$ such that

$$\forall (\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi) \in \mathcal{X}, \quad \langle (\varphi, \pi_\varphi, \psi, \pi_\psi), (\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi) \rangle_{\mathcal{X}} = \ell((\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi)). \quad (4.55)$$

We set

$$z := \widehat{\sigma}_1^2 (L^* \varphi + \nabla \pi_\varphi - \psi 1_{\mathcal{O}}), \quad w := \widehat{\sigma}_1^2 (L \psi + \nabla \pi_\psi), \quad u := -\widehat{\sigma}_2^2 \varphi_1 e_1, \quad (4.56)$$

and from (4.55), we deduce that

$$\left\| \frac{z}{\widehat{\sigma}_1} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{w}{\widehat{\sigma}_1} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{u}{\widehat{\sigma}_2} \right\|_{L^2((0,T) \times (\omega \cup \widehat{\omega}))} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} \quad (4.57)$$

and that

$$\begin{aligned} &\iint_{(0,T) \times \Omega} z \cdot (L^* \check{\varphi} + \nabla \check{\pi}_\varphi - \check{\psi} 1_{\mathcal{O}}) \, dx \, dt + \iint_{(0,T) \times \Omega} w \cdot (L \check{\psi} + \nabla \check{\pi}_\psi) \, dx \, dt \\ &= \iint_{(0,T) \times \omega_1} u \cdot \check{\varphi} \, dx \, dt + \iint_{(0,T) \times \Omega} f^{(0)} \cdot \check{\varphi} \, dx \, dt + \iint_{(0,T) \times \Omega} f^{(1)} \cdot \check{\psi} \, dx \, dt. \end{aligned} \quad (4.58)$$

The last relation yields that (z, w) is a weak solution of (1.14). We recall that $\widehat{\sigma}_0$ is defined by (4.7). We can check that

$$\begin{cases} \mathcal{L} \left(\frac{z}{\widehat{\sigma}_0}, \frac{p}{\widehat{\sigma}_0} \right) = \left(\frac{f^{(0)}}{\widehat{\sigma}_0} - \frac{\widehat{\sigma}'_0 z}{\widehat{\sigma}_0^2}, 0 \right), \\ \mathcal{L}^* \left(\frac{w}{\widehat{\sigma}_0}, \frac{q}{\widehat{\sigma}_0} \right) = \left(\frac{f^{(1)}}{\widehat{\sigma}_0} + \frac{z}{\widehat{\sigma}_0} 1_{\mathcal{O}} + \frac{\widehat{\sigma}'_0 w}{\widehat{\sigma}_0^2}, 0 \right), \end{cases} \quad (4.59)$$

and thus using (3.4), Proposition 2.1 and Proposition 2.2, we deduce that

$$\left\| \frac{z}{\widehat{\sigma}_0} \right\|_{X_1} + \left\| \frac{w}{\widehat{\sigma}_0} \right\|_{X_1} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)}. \quad (4.60)$$

This implies in particular that $w(0, \cdot) = 0$.

In order to prove the existence of insensitizing controls for (1.1) and (1.13), we define

$$\mathcal{F}_3 := \left\{ \left(f^{(0)}, f^{(1)} \right) ; \frac{f^{(0)}}{\widehat{\sigma}_3}, \frac{f^{(1)}}{\widehat{\sigma}_3} \in L^2((0, T) \times \Omega) \right\}$$

and the mapping

$$\mathcal{N} : \mathcal{F}_3 \rightarrow \mathcal{F}_3, \quad \left(f^{(0)}, f^{(1)} \right) \mapsto \left(f - (z \cdot \nabla)z, -(\nabla z)^* w + (z \cdot \nabla) w \right)$$

where (z, w) is the above solution, that is given by (4.56) and that satisfies (4.60) and (4.58).

Using (4.9), one can check that the map \mathcal{N} is well-defined and from (4.60), we can also show that if

$$\left\| \frac{f}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} \leq r$$

and if r is small enough, the closed ball

$$\mathcal{B}_3 := \left\{ f \in \mathcal{F}_3 ; \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T) \times \Omega)} \leq 2r \right\}$$

is invariant by \mathcal{N} and is a strict contraction on this set. This yields the existence of a fixed point for \mathcal{N} . The corresponding solution (z, w) satisfies (1.1) and (1.13), and (1.23). This concludes the proof of Corollary 1.6.

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References

- [1] Jon Asier Bárcena-Petisco, Sergio Guerrero, and Ademir F. Pazoto. Local null controllability of a model system for strong interaction between internal solitary waves. *Commun. Contemp. Math.*, 24(2):30, 2022. Id/No 2150003.
- [2] Umberto Biccari and Víctor Hernández-Santamaría. Null controllability of linear and semilinear nonlocal heat equations with an additive integral kernel. *SIAM J. Control Optim.*, 57(4):2924–2938, 2019.
- [3] Nicolás Carreño. Local controllability of the N -dimensional Boussinesq system with $N - 1$ scalar controls in an arbitrary control domain. *Math. Control Relat. Fields*, 2(4):361–382, 2012.
- [4] Nicolás Carreño and Sergio Guerrero. Local null controllability of the N -dimensional Navier-Stokes system with $N - 1$ scalar controls in an arbitrary control domain. *J. Math. Fluid Mech.*, 15(1):139–153, 2013.
- [5] Nicolás Carreño, Sergio Guerrero, and Mamadou Gueye. Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system. *ESAIM Control Optim. Calc. Var.*, 21(1):73–100, 2015.

- [6] Nicolás Carreño and Mamadou Gueye. Insensitizing controls with one vanishing component for the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 101(1):27–53, 2014.
- [7] Jean-Michel Coron and Sergio Guerrero. Null controllability of the N -dimensional Stokes system with $N - 1$ scalar controls. *J. Differential Equations*, 246(7):2908–2921, 2009.
- [8] Jean-Michel Coron and Pierre Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3):833–880, 2014.
- [9] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. *ESAIM Control Optim. Calc. Var.*, 12(3):442–465, 2006.
- [10] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.*, 45(4):1399–1446, 2006.
- [11] Enrique Fernández-Cara, Sergio Guerrero, Oleg Yu. Imanuvilov, and Jean-Pierre Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
- [12] Enrique Fernández-Cara, Sergio Guerrero, Oleg Yu. Imanuvilov, and Jean-Pierre Puel. Some controllability results for the N -dimensional Navier-Stokes and Boussinesq systems with $N - 1$ scalar controls. *SIAM J. Control Optim.*, 45(1):146–173, 2006.
- [13] Enrique Fernández-Cara, Juan Límaco, Dany Nina-Huaman, and Miguel R. Núñez Chávez. Exact controllability to the trajectories for parabolic PDEs with nonlocal nonlinearities. *Math. Control Signals Systems*, 31(3):415–431, 2019.
- [14] Enrique Fernández-Cara, Qi Lü, and Enrique Zuazua. Null controllability of linear heat and wave equations with nonlocal spatial terms. *SIAM J. Control Optim.*, 54(4):2009–2019, 2016.
- [15] Andreĭ V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [16] Sergio Guerrero and Takéo Takahashi. Controllability to trajectories of a Ladyzhenskaya model for a viscous incompressible fluid. *C. R. Math. Acad. Sci. Paris*, 359:719–732, 2021.
- [17] Víctor Hernández-Santamaría and Kévin Le Balc’h. Local controllability of the one-dimensional nonlocal Gray-Scott model with moving controls. *J. Evol. Equ.*, 21(4):4539–4574, 2021.
- [18] Víctor Hernández-Santamaría and Kévin Le Balc’h. Local null-controllability of a nonlocal semilinear heat equation. *Appl. Math. Optim.*, 84(2):1435–1483, 2021.
- [19] Olga A. Ladyženskaja. New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems. *Trudy Mat. Inst. Steklov.*, 102:85–104, 1967.
- [20] Juan Límaco, Miguel R. Núñez Chávez, and Dany Nina Huaman. Exact controllability for nonlocal and nonlinear hyperbolic PDEs. *Nonlinear Anal.*, 214:Paper No. 112569, 24, 2022.
- [21] Pierre Lissy and Enrique Zuazua. Internal controllability for parabolic systems involving analytic non-local terms. *Chinese Ann. Math. Ser. B*, 39(2):281–296, 2018.
- [22] Sorin Micu and Takéo Takahashi. Local controllability to stationary trajectories of a Burgers equation with nonlocal viscosity. *J. Differential Equations*, 264(5):3664–3703, 2018.
- [23] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.
- [24] Xiuxiang Zhou. Integral-type approximate controllability of linear parabolic integro-differential equations. *Systems Control Lett.*, 105:44–47, 2017.