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# On Finsler manifolds with hyperbolic geodesic flows

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## Abstract

Let  $(M, F)$  be a closed  $C^\infty$  Finsler manifold and  $\varphi$  its geodesic flow. In the case that  $\varphi$  is Anosov, we extend to the Finsler setting a Riemannian vanishing result of M. Gromov about the  $L^\infty$ -cohomology.

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*Keywords.* Finsler metric, Kähler hyperbolicity, Anosov flow.

## 1 Introduction

Let  $M$  be a closed  $C^\infty$  manifold of dimension  $n$  and  $TM_0 = TM - \{0\}$ . A *Finsler metric* on  $M$  is a function  $F : TM \rightarrow \mathbb{R}^+$  satisfying:

- (a)  $F(tv) = tF(v)$  for any  $v \in TM$  and  $t > 0$ ;
- (b)  $F$  is strictly positive and  $C^\infty$  on  $TM_0$ ; and
- (c) in standard local coordinates  $(x_i, y_i)$  on  $TM$ , the matrix  $(\frac{\partial^2 F^2}{\partial y_i \partial y_j})_{1 \leq i, j \leq n}$  is positively definite. For simplicity, we are denoting  $F(v)$  instead of  $F(x, v)$ , where  $(x, v) \in TM$ .

A Finsler metric  $F$  on  $M$  is said to be *Riemannian* if for any  $x \in M$ , there exists an inner product  $g_x$  on  $T_x M$  such that  $F_x^2(v) = g_x(v, v) \quad \forall v \in T_x M$ . There exist lots of non-Riemannian Finsler metrics, for example the so-called Randers metrics (see [5]). Given a Finsler manifold  $(M, F)$ , let  $c : [a, b] \rightarrow M$  be a piecewise  $C^\infty$  curve in  $M$ . Since  $F$  is positively homogeneous, the length of  $c$  is well-defined as  $L(c) = \int_a^b F(c'(t))dt$ . For any  $p, q \in M$ , we define  $d(p, q) = \inf_c L(c)$ , where the infimum is taken over all piecewise  $C^\infty$  curves  $c$  issuing from  $p$  to  $q$ . The *geodesics* of  $(M, F)$  are characterized as the constant speed curves minimizing locally the length, which are then necessarily  $C^\infty$  curves. The Finsler metric  $F$  is said to be *reversible* if  $F(-v) = F(v)$  for any  $v \in TM$ . In this article, we shall consider general Finsler metrics without the reversibility assumption. So generally speaking,

for any  $p, q \in M$ , we do not necessarily have  $d(p, q) = d(q, p)$ , i.e. the distance function  $d$  is asymmetric. This kind of asymmetry occurs naturally in many applications and offers one of the most interesting aspects in Finsler geometry (see [1]).

M. Gromov showed in [10] that if  $M$  is a closed Riemannian manifold of negative sectional curvature, then every closed bounded form of degree  $\geq 2$  on  $\widetilde{M}$  has a bounded primitive. This means that the  $L^\infty$ -cohomology of  $\widetilde{M}$  vanishes in degree  $\geq 2$ .

Given a closed Riemannian manifold  $(M, g)$  of negative curvature, it is well-known that its geometric hyperbolicity resulting from the negative curvature assumption implies that its geodesic flow defined on the sphere bundle  $SM$  is Anosov, i.e. dynamically hyperbolic. Therefore, the geometric hyperbolicity can be considered as stronger than the dynamical hyperbolicity. In this perspective, X. Cheng extended in [4] the vanishing result above to closed Riemannian manifolds with Anosov geodesic flows. In [2], K. Burns and G. Paternain obtained the same vanishing result for Riemannian manifolds with Anosov magnetic flows. In this paper, we generalize this Riemannian result to the Finsler setting:

**Theorem 1.1.** *If  $M$  is a closed Finsler manifold with Anosov geodesic flow, then every closed bounded form of degree  $\geq 2$  on  $\widetilde{M}$  has a bounded primitive.*

The vanishing of the  $L^\infty$ -cohomology in degree  $\geq 2$  is closely related to the following Riemannian question of S. S. Chern: let  $M$  be a  $2m$ -dimensional closed Riemannian manifold of negative sectional curvature. Is it true that the signals of the Euler number  $\chi(M)$  and  $(-1)^m$  are the same, i.e.  $(-1)^m \cdot \chi(M) > 0$ ? M. Gromov proved in [10] this conjecture in the Kähler hyperbolic case.

**Definition 1.2.** A closed Kähler manifold  $(M, g, J)$  is said to be *Kähler hyperbolic* if the lifted Kähler form  $\tilde{\omega}$  to the universal covering space  $\widetilde{M}$  admits a bounded primitive.

More precisely, M. Gromov proved the following result:

**Theorem 1.3.** *(M. Gromov [10]) Let  $M$  be a  $2m$ -dimensional closed Kähler manifold. If  $M$  is Kähler hyperbolic, then the Euler number  $\chi(M)$  does not vanish and satisfies  $(-1)^m \cdot \chi(M) > 0$ .*

Thus he obtained the following result: if  $M$  is a  $2m$ -dimensional closed  $C^\infty$  Riemannian manifold of negative sectional curvature, and  $M$  is homotopy equivalent to a closed Kähler manifold, then the Euler number  $\chi(M)$

does not vanish and satisfies  $(-1)^m \cdot \chi(M) > 0$ . We deduce from Theorems 1.1 and 1.3 the following:

**Corollary 1.** *Let  $M$  be a  $2m$ -dimensional closed  $C^\infty$  Finsler manifold with Anosov geodesic flow. If  $M$  is homotopy equivalent to a closed Kähler manifold, then the Euler number  $\chi(M)$  does not vanish and satisfies  $(-1)^m \cdot \chi(M) > 0$ .*

## 2 Symplectic formulation of Finsler geodesic flows

In this section, we recall some definitions concerning Finsler geodesic flows. See, for instance [11] or [9] for more details. Let  $(M, F)$  be a closed  $C^\infty$  Finsler manifold and  $\pi : TM_0 \rightarrow M$  be the canonical projection. The *potential* of  $(M, F)$  is defined as

$$A_v(\eta) = \frac{1}{2} \frac{d}{dt} (F^2(v + d\pi_v(\eta))) \big|_{t=0}, \quad \forall v \in TM_0, \eta \in T_v(TM_0),$$

which is a  $C^\infty$  1-form on  $TM_0$ . In the case of a Riemannian Finsler metric  $\sqrt{g}$ , we have

$$A_v(\eta) = g(d\pi_v(\eta), v),$$

which is the so-called Liouville 1-form of the Riemannian metric  $g$ .

Let  $E : TM_0 \rightarrow \mathbb{R}$  be defined by  $E(v) = \frac{1}{2}(F(v))^2$ . Let  $\omega$  be the symplectic form on  $TM_0$  obtained by pulling back the canonical symplectic form of  $T^*M_0$  via the Legendre transform, then we have  $\omega = dA$  (see [11, Section 2]). Let  $X$  be the Hamiltonian vector field of  $E$  with respect to  $\omega$ , i.e.

$$i_X \omega = -dE.$$

The Hamiltonian flow  $\Phi$  generated by  $X$  preserves the map  $E$  and the symplectic form  $\omega$ . The projections of the orbits of  $\Phi$  to  $M$  are just the geodesics of  $(M, F)$ . The unit sphere bundle with respect to  $F$  is defined as

$$S_F M = E^{-1}\left(\frac{1}{2}\right) = \{v \in TM \mid F(v) = 1\}.$$

The restriction of the Hamiltonian flow on  $S_F M$  is said to be the geodesic flow of  $(M, F)$ , denoted by  $\varphi$ . Since  $\omega = dA$ ,  $\varphi$  integrates the Reeb field of the contact 1-form  $A|_{S_F M}$ .

### 3 Anosov flows and transversality

Let us first recall some definitions: let  $N$  be a closed  $C^\infty$  manifold and  $\psi : N \rightarrow N$  a  $C^\infty$  flow generated by the vector field  $X$ . We say that  $\psi$  is an *Anosov flow* if there exists a  $D\psi$ -invariant splitting of the tangent bundle

$$TN = E^{ss} \oplus \mathbb{R}X \oplus E^{su},$$

a Riemannian metric on  $N$ , and two positive numbers  $a$  and  $b$  such that

$$\| D\psi_t(v) \| \leq a \cdot e^{-bt} \| v \| \quad \forall v \in E^{ss}, t \geq 0,$$

$$\| D\psi_{-t}(v) \| \leq a \cdot e^{-bt} \| v \| \quad \forall v \in E^{su}, t \geq 0.$$

The vector bundles  $E^{ss}$  and  $E^{su}$  are called the *strong stable* and *strong unstable* distributions of  $\psi$ . They are both integrable to Hölder continuous foliations with  $C^\infty$  leaves, denoted respectively by  $\mathcal{F}^{ss}$  and  $\mathcal{F}^{su}$ . The vector bundles  $E^{ss} \oplus \mathbb{R}X$  and  $E^{su} \oplus \mathbb{R}X$  are called the *stable* and *unstable* distributions of  $\psi$ . They are also integrable to Hölder continuous foliations with  $C^\infty$  leaves, denoted respectively by  $\mathcal{F}^s$  and  $\mathcal{F}^u$ . For any  $x \in N$ , the leaves of  $\mathcal{F}^{ss}$ ,  $\mathcal{F}^{su}$ ,  $\mathcal{F}^s$ , and  $\mathcal{F}^u$  containing  $x$  are denoted respectively by  $W^{ss}(x)$ ,  $W^{su}(x)$ ,  $W^s(x)$ , and  $W^u(x)$ .

Let  $(M, F)$  be a closed  $C^\infty$  Finsler manifold of negative flag curvature (see [5]), it is well-known that its geodesic flow defined on  $S_F M$  is Anosov ([8]). See, for instance, [7] for a general study of Finsler geodesic flows in negative flag curvature. According to Theorem 2 and the comments concerning the Anosov hypothesis in [12], we have the following proposition:

**Proposition 1.** ([12]) *Let  $(M, F)$  be a closed  $C^\infty$  Finsler manifold of dimension  $n$  and  $\varphi$  its geodesic flow defined on  $S_F M$ . If  $\varphi$  is Anosov, then its stable foliation  $\mathcal{F}^s$  is transverse to the fibers of the sphere bundle  $\pi : S_F M \rightarrow M$ .*

The proof of the following proposition is given only for the convenience of the reader, since its arguments are known.

**Proposition 2.** *Consider the lifted stable foliation  $\tilde{\mathcal{F}}^s$  on the covering space  $S_{\tilde{F}} \tilde{M}$ . If the geodesic flow  $\varphi$  is Anosov, then each leaf of  $\tilde{\mathcal{F}}^s$  intersects each fiber of the sphere bundle  $\tilde{\pi} : S_{\tilde{F}} \tilde{M} \rightarrow \tilde{M}$  exactly once. In addition,  $\tilde{M}$  is diffeomorphic to  $\mathbb{R}^n$ , where  $n = \dim(M)$ .*

*Proof.* Let  $v \in S_{\tilde{F}} \tilde{M}$  and  $\tilde{W}^s(v)$  the leaf of  $\tilde{\mathcal{F}}^s$  containing  $v$ . We deduce from the above proposition that the foliation  $\tilde{\mathcal{F}}^s$  is transverse to the fibers

of the sphere bundle  $\tilde{\pi} : S_{\tilde{F}}\tilde{M} \rightarrow \tilde{M}$ . Since the fibers are compact, a result of C. Ehresmann (see [3]) implies that the map

$$\tilde{\pi} |_{\tilde{W}^s(v)} : \tilde{W}^s(v) \rightarrow \tilde{M}$$

is a covering map. Since  $\tilde{M}$  is simply connected,  $\tilde{\pi} |_{\tilde{W}^s(v)}$  is a diffeomorphism. Therefore,  $\tilde{W}^s(v)$  intersects each fiber of the bundle  $S_{\tilde{F}}\tilde{M} \rightarrow \tilde{M}$  at exactly one point.

Take  $v \in S_{\tilde{F}}\tilde{M}$  such that  $\tilde{W}^s(v)$  contains no lifts of  $\varphi$ -periodic orbits. It is well-known that  $\tilde{W}^s(v)$  is diffeomorphic to  $\mathbb{R}^n$ . We deduce that  $\tilde{M}$  is diffeomorphic to  $\mathbb{R}^n$ .  $\square$

## 4 Proof of Theorem 1.1

Let  $(M, F)$  be a closed  $C^\infty$  Finsler manifold with Anosov geodesic flow  $\varphi$ . Let  $v \in S_F M$  and  $\tilde{W}^s(v)$  be the leaf containing  $v$  of the foliation  $\tilde{\mathcal{F}}^s$ . We denote by  $\tilde{X}$  the tangent vector field of the lifted flow  $\tilde{\varphi}$  over  $S_{\tilde{F}}\tilde{M}$ . By Proposition 2,  $\tilde{\pi}$  sends  $\tilde{W}^s(v)$  diffeomorphically onto  $\tilde{M}$ . We define

$$\hat{E}^{ss} = D\tilde{\pi}(\tilde{E}^{ss} |_{\tilde{W}^s(v)}) \quad \text{and} \quad \hat{X} = D\tilde{\pi}(\tilde{X} |_{\tilde{W}^s(v)}).$$

Let  $\tau$  be the  $C^\infty$  flow on  $\tilde{M}$  generated by  $\hat{X}$ . Therefore, for any  $t \in \mathbb{R}$ ,

$$\tau_t = \tilde{\pi} \circ \tilde{\varphi}_t \circ \tilde{\pi}^{-1}$$

and  $\hat{E}^{ss}$  is  $\tau$ -invariant. Since by Proposition 1 the distribution  $E^s$  is transverse to the vertical bundle  $V(S_F M)$ , there exists  $C_1 > 0$  such that for any  $\xi \in \tilde{E}^s(v)$ ,

$$\frac{1}{C_1} \tilde{F}(D\tilde{\pi}(\xi)) \leq |\xi| \leq C_1 \tilde{F}(D\tilde{\pi}(\xi)),$$

where  $|\xi|$  is calculated with respect to the lift of an arbitrarily chosen Riemannian metric on  $S_F M$ .

**Proposition 3.** *There exist positive constants  $C$  and  $b$  such that for any  $x \in \tilde{M}$  and any  $u \in \hat{E}^{ss}(x)$ , we have that for any  $t \geq 0$ ,*

$$\tilde{F}(D\tau_t(u)) \leq Ce^{-bt} \tilde{F}(u).$$

*Proof.* Since  $\varphi$  is Anosov, there exist  $a, b > 0$  such that for any  $\xi \in \widetilde{E}^{ss}$ ,

$$|D\widetilde{\varphi}_t(\xi)| \leq ae^{-bt} |\xi|$$

for any  $t \geq 0$ . Let  $\xi \in \widetilde{E}^{ss}$  such that  $d\widetilde{\pi}(\xi) = u$ . We have

$$\begin{aligned} \widetilde{F}(D\tau_t(u)) &= \widetilde{F}(D\widetilde{\pi}(D\widetilde{\varphi}_t(\xi))) \leq C_1 |D\widetilde{\varphi}_t(\xi)| \\ &\leq C_1 ae^{-bt} |\xi| \leq C_1^2 ae^{-bt} \widetilde{F}(u) \end{aligned}$$

for any  $t \geq 0$ . We set  $C = C_1^2 a$ .  $\square$

**Definition 4.1.** A  $C^\infty$   $k$ -form  $\alpha$  on  $\widetilde{M}$  is said to be *bounded* if

$$\|\alpha\|_\infty = \sup_{x \in \widetilde{M}} \|\alpha\|_x < +\infty,$$

where  $\|\alpha\|_x = \sup_{e_1, \dots, e_k \in T_x \widetilde{M} \setminus \{0\}} \frac{|\alpha(e_1, \dots, e_k)|}{\widetilde{F}(e_1) \dots \widetilde{F}(e_k)}$ .

Now let us prove Theorem 1.1: we suppose that  $k \geq 2$ , let  $\alpha$  be a closed bounded  $C^\infty$   $k$ -form on  $\widetilde{M}$ . Since  $\widetilde{M}$  is diffeomorphic to  $\mathbb{R}^n$ ,  $\alpha$  is exact. Let us prove that there exists a bounded 1-form  $\beta$  such that  $\alpha = d\beta$ . Let  $T$  be an arbitrary, fixed real number, and  $\tau$  be the flow on  $\widetilde{M}$  generated by the vector field  $\hat{X}$ . Then we define

$$\beta_T = - \int_0^T \tau_s^*(i_{\hat{X}} \alpha) ds.$$

Since  $\alpha$  is closed, we have

$$\alpha - \tau_T^* \alpha = - \int_0^T \frac{d}{ds} \tau_s^* \alpha ds = - \int_0^T \tau_s^* \mathcal{L}_{\hat{X}} \alpha ds = - \int_0^T \tau_s^* di_{\hat{X}} \alpha ds = d\beta_T.$$

As we have seen above,  $\widetilde{\pi} : \widetilde{W}^s(v) \rightarrow \widetilde{M}$  is a  $C^\infty$  diffeomorphism. Moreover,  $\hat{E}^{ss} = D\widetilde{\pi}(\hat{E}^{ss} |_{\widetilde{W}^s(v)})$  and  $\hat{X} = D\widetilde{\pi}(\hat{X} |_{\widetilde{W}^s(v)})$ . We deduce that

$$T\widetilde{M} = \hat{E}^{ss} \oplus \mathbb{R} \cdot \hat{X}.$$

Now let  $x \in \widetilde{M}$  and  $w \in T_x \widetilde{M}$ . Thus there exist  $w' \in \hat{E}^{ss}(x)$  and  $w''$  colinear with  $\hat{X}(x)$  such that  $w = w' + w''$ . Since the distribution  $E^{ss}$  on  $S_F M$  is transverse to the vector field  $X$ , there exists a positive constant  $C_2$  such that

$$\widetilde{F}(w') \leq C_2 \widetilde{F}(w).$$

Therefore, for any  $w_1, \dots, w_{k-1} \in T_x \widetilde{M}$ , we have, by Proposition 3, the following:

$$\begin{aligned}
|(\tau_s^*(i_{\hat{X}}\alpha))_x(w_1, \dots, w_{k-1})| &= |\alpha_{\tau_s(x)}(\hat{X}(\tau_s(x)), D_x \tau_s(w_1), \dots, D_x \tau_s(w_{k-1}))| \\
&= |\alpha_{\tau_s(x)}(\hat{X}(\tau_s(x)), D_x \tau_s(w'_1), \dots, D_x \tau_s(w'_{k-1}))| \\
&\leq \|\alpha\|_{\infty} \tilde{F}(\hat{X}(\tau_s(x))) \tilde{F}(D_x \tau_s(w'_1)) \cdots \tilde{F}(D_x \tau_s(w'_{k-1})) \\
&\leq \|\alpha\|_{\infty} C_1 \|X\|_{\infty} C^{k-1} e^{-(k-1)bs} \tilde{F}(w'_1) \cdots \tilde{F}(w'_{k-1}) \\
&\leq \|\alpha\|_{\infty} C_1 \|X\|_{\infty} C^{k-1} e^{-(k-1)bs} C_2^{k-1} \tilde{F}(w_1) \cdots \tilde{F}(w_{k-1}),
\end{aligned}$$

where  $\|X\|_{\infty}$  is calculated with respect to an arbitrarily chosen Riemannian metric on  $S_F M$ , and we are considering, without loss of generality, that all  $k-1$  terms of type  $C_2$  concerning  $w'_i$  are equal,  $i = 1, \dots, k-1$ .

Hence the form  $\beta_T$  converges as  $T \rightarrow +\infty$  to a form  $\beta$  such that

$$|\beta_x(w_1, \dots, w_{k-1})| \leq \frac{\|\alpha\|_{\infty} \|X\|_{\infty} C_1 C^{k-1} C_2^{k-1}}{(k-1)b} \tilde{F}(w_1) \cdots \tilde{F}(w_{k-1}),$$

thus  $\beta$  is bounded. Moreover, since  $\tau_T^* \alpha$  tends to zero,  $\alpha = d\beta$ . The proof of Theorem 1.1 is complete.

The proof of Corollary 1 is given for the convenience of the reader (see [4]): suppose that  $\Psi : M \rightarrow M_1$  is a homotopy equivalence of  $M$  to a closed Kähler manifold  $M_1$ . By an approximation if necessary, we can assume that  $\Psi$  is  $C^\infty$ . Let  $\pi : \widetilde{M} \rightarrow M$  and  $\pi_1 : \widetilde{M}_1 \rightarrow M_1$  be the universal covering maps. Let  $\omega$  be the Kähler form of  $M_1$ , which is certainly bounded. Therefore, with respect to the lifted Finsler metric  $\tilde{F}$ , the pull back  $(\Psi \circ \pi)^*(\omega) = \pi^*(\Psi^*(\omega))$  is also a bounded 2-form on  $\widetilde{M}$ . Thus by Theorem 1.1, there exists a bounded 1-form  $\beta$  on  $\widetilde{M}$  such that

$$(\Psi \circ \pi)^*(\omega) = d\beta.$$

Then we can deduce as in [6] that  $\pi_1^*(\omega)$  also admits a bounded primitive, i.e.  $M_1$  is Kähler hyperbolic. So by Theorem 1.3, we have  $(-1)^m \cdot \chi(M_1) > 0$ . Since the Euler number is a homotopy invariant, we get  $(-1)^m \cdot \chi(M) > 0$ .

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