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ABSTRACT

In this paper, we consider the Korteweg-de Vries equation with time-dependent delay on the boundary or internal feedbacks. Under some assumptions on the time-dependent delay, on the weights of the feedbacks and on the length of the spatial domain, we prove the exponential stability results, using appropriate Lyapunov functionals. We finish by some numerical simulations that illustrate the stability results and the influence of the delay on the decay rate.

1. Introduction

In this work, we are interested in the effect of a time-varying delay in the boundary or internal stabilization of the Korteweg-de Vries equation (KdV). The KdV equation is given by $y_t + y_x + y_{xxx} + yy_x = 0$, this third-order nonlinear one-dimensional equation was introduced in Korteweg and de Vries (1895) to model the propagation of long water waves in a channel. In recent years, the controllability and stabilization properties of the KdV have been amply studied. We mention for instance the pioneer work Rosier (1997) where the boundary exact controllability was studied: the author showed that the KdV equation is not controllable by the right with one control if the length of the spatial domain L belongs to a countable set. This phenomena is related to the existence of solutions conserving the $L^2(0, L)$- energy. In Menzala, Vasconcellos and Zuazua (2002) the authors showed that if the length L is critical, an internal damping allows the exponential stability. This idea was then applied in several works for instance Pazoto (2005); Linares and Pazoto (2009); Parada, Crépeau and Prieur (2022b). We also refer to Cerpa (2014); Rosier and Zhang (2009) for a complete introduction about control of KdV equation.

Time delay phenomena appear in many applications, for example in biology, mechanics or engineering. Delays terms are unavoidable in practice due to measurement lag, analysis time, or computation time. Very active research has developed recently on stability problems of partial differential equations with delay. It is well known that even a small delay in the feedback mechanism can destabilize a system (see for example Datko (1988); Datko, Lagnese and Polis (1986)). But a delay term can also improve the performance of the system (Abdallah, Dorato, Benites-Read and Byrne (1993)). The problems of stability of systems with delay are of both theoretical and practical interest.

Recently, the problem of robustness with respect to constant time-delay of the KdV equation was studied in Baudouin, Crépeau and Valein (2019); Parada, Crépeau and Prieur (2022a); Valein (2022) using Lyapunov theory or deriving suitable observability inequalities. In the case where the KdV equation is in presence of memory terms, stability results were obtained in Chentouf (2021); Chentouf and Guesmia (2022). The stability of PDE's involving time-varying delays was analyzed in Nicaise, Valein and Fridman (2009) for one-dimensional heat and wave equations, in Nicaise and Pignotti (2011); Nicaise, Pignotti and Valein (2011) for wave equations in domains in $\mathbb{R}^n$ and in Fridman, Nicaise and Valein (2010) for general second-order evolution equations. We can also mention Park (2014) where a weak viscoelastic beam equation with time-varying delay was considered and the recent work Kong, Nonato, Liu, Dos Santos, Raposo and An (2022) studying exponential stability of piezoelectric beams. In our best knowledge, there is no work dealing with this problem for the KdV equation.

In this work, we are going to consider the two following systems

$$\begin{cases} (y_t + y_x + y_{xxx} + yy_x)(x,t) = 0, \ t > 0, \ x \in (0, L), \\ y(0,t) = y(L,t) = 0, \quad t > 0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0, t-\tau(t)), \quad t > 0, \\ y(x,0) = y_0(x), \quad x \in (0, L), \\ y_x(0, t-\tau(0)) = z_0(t-\tau(0)), \quad 0 < t < \tau(0), \end{cases} \tag{1}$$

and

$$\begin{cases} (y_t + y_x + y_{xxx} + yy_x)(x,t) + a(x)y(x,t) \\ \quad + b(x)y(x, t-\tau(t)) = 0, \quad t > 0, \ x \in (0, L), \\ y(0,t) = y(L,t) = y_x(L,t) = 0, \quad t > 0, \\ y(x,0) = y_0(x), \quad x \in (0, L), \\ y(x, t-\tau(0)) = z_0(x, t-\tau(0)), \\ \qquad 0 < t < \tau(0), \ x \in (0, L), \end{cases} \tag{2}$$

where $L > 0$ is the length of the spatial domain, $y(x,t)$ is the amplitude of the water wave at position x at time t. We assume that the delay τ is a function of time t, which satisfies

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the following conditions

$$0 < \tau_0 \leq \tau(t) \leq M, \qquad \forall t \geq 0, \tag{3}$$
$$\dot{\tau}(t) \leq d < 1, \qquad \forall t \geq 0, \tag{4}$$

where $0 \leq d < 1$, and

$$\tau \in W^{2,\infty}([0,T]), \qquad \forall T > 0. \tag{5}$$

Moreover, we assume that α, β, d in (1) are real constants satisfying

$$\text{The matrix } \Phi_{\alpha,\beta} = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(d-1) \end{pmatrix} \text{ is definite negative.} \tag{6}$$

In (2), $a = a(x)$ and $b = b(x)$ are nonnegative functions in $L^\infty(0, L)$. We will also assume that supp $b = \omega$ and

$$b(x) \geq b_0 > 0 \quad \text{in} \quad \omega, \tag{7}$$

where ω is an open nonempty subset of $(0, L)$. We assume that the coefficients a and b satisfy the following assumption:

$$\exists c_0 > 0, \quad \frac{2-d}{2-2d}\, b(x) + c_0 \leqslant a(x) \quad \text{in} \quad \omega. \tag{8}$$

Then $\omega = \text{supp } b \subset \text{supp } a$ and $a(x) \geq b_0 + c_0 > 0$ in ω. Some examples of functions satisfying these conditions are given in Section 4.

Remark 1.1. *We can note the following points on the coefficients of the boundary or internal feedback:*

- *A sufficient condition to obtain (6) is $|\alpha|+|\beta|+d < 1$. Indeed, on the one hand, we have $tr(\Phi_{\alpha,\beta}) = \alpha^2+\beta^2-1+|\beta|d < 0 \Leftrightarrow \alpha^2+\beta^2+|\beta|d < 1$, and $\alpha^2+\beta^2+|\beta|d < |\alpha|+|\beta|+d < 1$. On the other hand we have,*

$$\begin{aligned} \det(\Phi_{\alpha,\beta}) &= |\beta|(\beta^2 - 2|\beta| + 1 - \alpha^2 + d\alpha^2 - d + |\beta|d) \\ &= |\beta|((1-|\beta|)^2 + d(|\beta|-1) + \alpha^2(d-1)) \\ &= |\beta|((1-|\beta|)(1-|\beta|-d) + \alpha^2(d-1)) \\ &> |\beta|(\alpha^2 + \alpha^2 d - \alpha^2) = |\beta|\alpha^2 d > 0. \end{aligned}$$

  Then, $|\alpha| + |\beta| + d < 1$ implies that $\Phi_{\alpha,\beta}$ is definite negative.

- *If $d = 0$, (6) (resp. (8)) is equivalent to $|\alpha| + |\beta| < 1$ (resp. $b(x) + c_0 \leq a(x)$ in ω) which corresponds to the assumption for a constant time-delay given in Baudouin et al. (2019) (resp. Valein (2022)).*

- *If $d \to 1^-$, $\frac{2-d}{2-2d} \to +\infty$, and so we need that the weight a of the internal feedback without delay to be very large.*

In Baudouin et al. (2019), two different approaches for the exponential stability of the nonlinear KdV equation with boundary (constant) time-delay feedback were studied. The first was a Lyapunov functional approach with an estimation of the decay rate, but with a restrictive assumption on the length L of the spatial domain. The second one was an observability inequality approach without estimation on the decay rate and for any non-critical lengths (i.e. $L \notin \mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}^* \right\}$). The asymptotic stability of the nonlinear KdV equation with (constant) time-delay internal feedback was studied in Valein (2022). A semiglobal stability result for any lengths was proven in the case where the weight of the term with delay is smaller than the weight of the term without delay, using an observability inequality directly on the nonlinear system. A local exponential stability result was given in the case where the support of the term with delay is not included in the support of the term without delay.

The aim of our work is to extend these results in the case where the delay depends on the time. An important fact about systems (1) and (2) is that due to the effect of the time-varying delay, these systems are no longer invariant in time. Thus, the observability inequality approach does not work anymore, and we have to choose a new appropriate Lyapunov functional. An other difficulty, beyond the difficulty of dealing with a nonlinear equation, is that the first order linear operator depends on time (contrary to constant delay case) and the well-posedness is not trivial.

The outline of this paper is as follows. In Section 2, we prove the well-posedness results, firstly for the boundary case, secondly for the internal case. The stability results are proved in Section 3. Finally, we illustrate our results by some numerical simulations in Section 4.

2. Well-posedness results

The goal of this section is to prove appropriate well-posedness results of (1) and (2). We first prove the well-posedness result of the linearization around 0 of (1) (resp. (2)). The proof will be based on the semigroup theory and on introducing a new function for the delayed term. Then, we add a source term that plays the role of the nonlinearity. Finally, using a fixed-point approach, we show the well-posedness of the nonlinear systems (1) and (2).

2.1. Well-posedness result of (1)

The goal of this section is to prove appropriate local well-posedness result of (1).

2.1.1. Well-posedness result of the linear system

In this part, we focus on the study of linearization around 0 of (1), that is

$$\begin{cases} (y_t + y_x + y_{xxx})(x,t) = 0, \quad t > 0,\ x \in (0,L), \\ y(0,t) = y(L,t) = 0, \quad t > 0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0, t-\tau(t)), \quad t > 0, \\ y(x,0) = y_0(x), \quad x \in (0,L), \\ y_x(0, t-\tau(0)) = z_0(t-\tau(0)), \quad 0 < t < \tau(0). \end{cases} \tag{9}$$

Now, classically, we introduce a new variable that takes into account the delay term (see, for instance, Nicaise et al.

(2009)). Let $z(\rho,t)=y_x(0,t-\tau(t)\rho)$ for $\rho\in(0,1)$ and $t>0$. Then, z verifies the following transport equation

$$\begin{cases} \tau(t)z_t(\rho,t)+(1-\dot\tau(t)\rho)z_\rho(\rho,t)=0,\ t>0,\ \rho\in(0,1),\\ z(0,t)=y_x(0,t), \quad t>0,\\ z(\rho,0)=z_0(-\tau(0)\rho), \quad \rho\in(0,1). \end{cases}$$

Define $U=(y,z)^T$, then U satisfies

$$U_t=\begin{pmatrix} y_t\\ z_t\end{pmatrix}=\begin{pmatrix} -y_x-y_{xxx}\\ \dfrac{\dot\tau(t)\rho-1}{\tau(t)}z_\rho\end{pmatrix}.$$

This problem can be rewritten as the following first-order evolution equation

$$\begin{cases} U_t(t)=\mathcal{A}(t)U(t), \qquad t>0,\\ U(0)=(y_0,z_0(-\tau(0)\cdot))^T=:U_0, \end{cases} \tag{10}$$

where the time-dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} y\\ z\end{pmatrix}=\begin{pmatrix} -y_x-y_{xxx}\\ \dfrac{\dot\tau(t)\rho-1}{\tau(t)}z_\rho\end{pmatrix}, \quad \text{with domain}$$

$$D(\mathcal{A}(t))=\Big\{(y,z)\in\big(H^3(0,L)\cap H_0^1(0,L)\big)\times H^1(0,1),\\ z(0)=y_x(0),\ y_x(L)=\alpha y_x(0)+\beta z(1)\Big\}.$$

Note that the domain of the operator $\mathcal{A}(t)$ is independent of time t, i.e., $D(\mathcal{A}(t))=D(\mathcal{A}(0))$, $t>0$. Now, we introduce the Hilbert space $H=L^2(0,L)\times L^2(0,1)$, equipped with the usual inner product

$$\left\langle\begin{pmatrix} y\\ z\end{pmatrix},\begin{pmatrix} \tilde y\\ \tilde z\end{pmatrix}\right\rangle=\int_0^L y\tilde y dx+\int_0^1 z\tilde z d\rho,$$

endowed with the norm $\|\cdot\|_H$. To prove the well-posedness of (10) we follow Nicaise et al. (2009). The proof is based on showing that the triplet $\{\mathcal{A},H,\mathcal{Y}\}$, with $\mathcal{A}=\{\mathcal{A}(t):t\in[0,T]\}$, for some $T>0$ fixed and $\mathcal{Y}=D(\mathcal{A}(0))$, forms a constant domain system (CD-system), see Kato (1970, 1985). The following theorem gives the existence and uniqueness results and is proved in Kato (1970):

Theorem 2.1. *Assume that*

1. $\mathcal{Y}=D(\mathcal{A}(0))$ *is a dense subset of* H,
2. $D(\mathcal{A}(t))=D(\mathcal{A}(0))$, *for all* $t>0$,
3. *for all* $t\in[0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* H *and the family* $\mathcal{A}=\{\mathcal{A}(t):t\in[0,T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e. the semigroup* $(S_t(s))_{s\geq0}$ *generated by* $\mathcal{A}(t)$ *satisfies* $\|S_t(s)U\|_H\leq Ce^{ms}\|U\|_H$, *for all* $U\in H$ *and* $s\geq0$*),*
4. $\partial_t\mathcal{A}(t)$ *belongs to* $L^\infty_*([0,T],B(\mathcal{Y},H))$, *the space of equivalent classes of essentially bounded, strongly measure functions from* $[0,T]$ *into the set* $B(\mathcal{Y},H)$ *of bounded operators from* $\mathcal{Y}$ *into* H.

Then, problem (10) *has a unique solution* $U\in C([0,T],\mathcal{Y})\cap C^1([0,T],H)$ *for any initial datum in* $\mathcal{Y}$.

In particular, we are going to prove the following result.

Theorem 2.2. *Assume* (3)-(6). *Let* $U_0\in H$, *then there exists a unique solution* $U\in C([0,+\infty),H)$ *to* (10). *Moreover, if* $U_0\in D(\mathcal{A}(0))$ *then* $U\in C([0,+\infty),D(\mathcal{A}(0)))\cap C^1([0,+\infty),H)$.

Proof: Clearly, the space $\mathcal{Y}=D(\mathcal{A}(0))$ is a dense subset of H and, by definition, $D(\mathcal{A}(t))=D(\mathcal{A}(0))$, for all $t>0$. Now, to prove the third point of Theorem 2.1, we introduce the following time-dependent inner product on H to use the variable norm technique of Kato

$$\left\langle\begin{pmatrix} y\\ z\end{pmatrix},\begin{pmatrix} \tilde y\\ \tilde z\end{pmatrix}\right\rangle_t=\int_0^L y\tilde y dx+|\beta|\tau(t)\int_0^1 z\tilde z d\rho,$$

with associated norm denoted by $\|\cdot\|_t$. By (3), the norms $\|\cdot\|_H$ and $\|\cdot\|_t$ are equivalent in H: $\forall t\geq0$, $\forall(y,z)\in H$,

$$(1+|\beta|\tau_0)\|(y,z)\|_H^2\leq\|(y,z)\|_t^2\\ \leq(1+|\beta|M)\|(y,z)\|_H^2. \tag{11}$$

We first observe that, following (Fridman et al., 2010, Theorem 2.4),

$$\frac{\|U\|_t}{\|U\|_s}\leq e^{\frac{c}{2\tau_0}|t-s|}, \quad \forall t,s\in[0,T], \tag{12}$$

where $U=(y,z)\in H$ and c is a positive constant. Now, we calculate $\langle\mathcal{A}(t)U,U\rangle_t$ for a fixed $t\in[0,T]$. Take $U=(y,z)^T\in D(\mathcal{A}(0))$, then

$$\langle\mathcal{A}(t)U,U\rangle_t=\left\langle\begin{pmatrix} -y_x-y_{xxx}\\ \dfrac{\dot\tau(t)\rho-1}{\tau(t)}z_\rho\end{pmatrix},\begin{pmatrix} y\\ z\end{pmatrix}\right\rangle_t\\ =\int_0^L(-y_x-y_{xxx})ydx+|\beta|\int_0^1(\dot\tau(t)\rho-1)z_\rho z d\rho.$$

By integrating by parts in space and in ρ, we have

$$\langle\mathcal{A}(t)U,U\rangle_t=\frac12[y_x^2]_0^L-\frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2d\rho\\ +\frac{|\beta|}{2}[(\dot\tau(t)\rho-1)z^2]_0^1.$$

Moreover using the boundary conditions, we obtain

$$\langle\mathcal{A}(t)U,U\rangle_t=\frac12\big(\alpha y_x(0)+\beta z(1)\big)^2-\frac12y_x^2(0)\\ -\frac{|\beta|}{2}\dot\tau(t)\int_0^1 z^2d\rho+\frac{|\beta|}{2}(\dot\tau(t)-1)z^2(1)+\frac{|\beta|}{2}y_x^2(0).$$

Now, by (4) we derive

$$\langle\mathcal{A}(t)U,U\rangle_t-\kappa(t)\langle U,U\rangle_t\leq\frac12\begin{pmatrix} y_x(0)\\ z(1)\end{pmatrix}^T\Phi_{\alpha,\beta}\begin{pmatrix} y_x(0)\\ z(1)\end{pmatrix},$$

where $\kappa(t) = \dfrac{(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)}$ and where $\Phi_{\alpha,\beta}$ is defined by (6). Finally, using (6), we get

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \le 0.$$

The above inequality proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle\cdot,\cdot\rangle_t$.

Let us prove that for all $t \in [0,T]$, $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$.

Let $t \in [0,T]$ be fixed, and $(f,h)^T \in H$. We look for $U = (y,z)^T \in D(\mathcal{A}(t))$ solution of $(\lambda I - \mathcal{A}(t))U = (f,h)^T$, that is

$$\begin{cases} \lambda y + y_x + y_{xxx} = f, \\ \lambda z + \left(\dfrac{1-\dot\tau(t)\rho}{\tau(t)}\right) z_\rho = h, \\ y(0) = y(L) = 0,\ y_x(0) = z(0), \\ y_x(L) = \alpha y_x(0) + \beta z(1). \end{cases} \tag{13}$$

Following Nicaise et al. (2009), if we find y with the appropriate regularity, then we can obtain z, given by

$$z(\rho) = \begin{cases} y_x(0)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)} + e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)} \\ \times\displaystyle\int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma} e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)} d\sigma, & \text{if } \dot\tau(t)\neq 0, \\ y_x(0)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho} \\ \times\displaystyle\int_0^\rho e^{\lambda\tau(t)\sigma} h(\sigma)d\sigma, & \text{if } \dot\tau(t) = 0. \end{cases}$$

In particular $z(1) = y_x(0)g_0(t) + g_h(t)$, where

$$g_0(t) = \begin{cases} e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}, & \text{if } \dot\tau(t)\neq 0, \\ e^{-\lambda\tau(t)}, & \text{if } \dot\tau(t) = 0, \end{cases}$$

$$g_h(t) = \begin{cases} e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\displaystyle\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma} \\ \times e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)} d\sigma, & \text{if } \dot\tau(t)\neq 0, \\ \tau(t)e^{-\lambda\tau(t)}\displaystyle\int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma)d\sigma, & \text{if } \dot\tau(t) = 0. \end{cases}$$

This implies that y must satisfy

$$\begin{cases} \lambda y + y_x + y_{xxx} = f, \\ y(0) = y(L) = 0, \\ y_x(L) = (\alpha + \beta g_0(t))y_x(0) + \beta g_h(t). \end{cases}$$

Consider now $\psi(x) = \dfrac{\beta x(x-L)g_h(t)}{L(1+\alpha+\beta g_0(t))}$ (t is fixed here) and $\tilde\alpha = \tilde\alpha(t) = \alpha + \beta g_0(t)$. After some computations, we can observe that $\varphi = y - \psi$ solves

$$\begin{cases} \lambda\varphi + \varphi_x + \varphi_{xxx} = \tilde f := f - \lambda\psi - \psi_x - \psi_{xxx}, \\ \varphi(0) = \varphi(L) = 0, \\ \varphi_x(L) = \tilde\alpha\varphi_x(0). \end{cases} \tag{14}$$

As t is fixed, the problem can be seen as $(\lambda I - A_{\tilde\alpha})\varphi = \tilde f$ where the operator $A_{\tilde\alpha}$ is defined by $A_{\tilde\alpha}\varphi = -\varphi' - \varphi'''$, with $D(A_{\tilde\alpha}) = \{\varphi \in H^3(0,L)\cap H_0^1(0,L), \varphi'(L) = \tilde\alpha\varphi'(0)\}$ and where $\tilde f \in L^2(0,L)$ (since $\varphi \in C^\infty([0,L])$ and $f \in L^2(0,L)$). To conclude this part, we show that under the condition, (6) the operator $A_{\tilde\alpha}$ is maximal.

Lemma 2.3. *If $|\tilde\alpha| < 1$, then the operator $A_{\tilde\alpha}$ is maximal.*

Proof: Consider $|\tilde\alpha| < 1$, clearly $A_{\tilde\alpha}$ is closed. Let us prove that $A_{\tilde\alpha}$ and $A_{\tilde\alpha}^*$ are dissipative. Let $\varphi \in D(A_{\tilde\alpha})$, then we get

$$\begin{aligned}(A_{\tilde\alpha}\varphi,\varphi)_{L^2(0,L)} &= \int_0^L (-\varphi' - \varphi''')\varphi dx \\ &= \frac{1}{2}(\tilde\alpha^2 - 1)(\varphi'(0))^2 \le 0.\end{aligned}$$

The dual of the operator $A_{\tilde\alpha}$ is defined by $A_{\tilde\alpha}^*\zeta = \zeta' + \zeta'''$ with domain $D(A_\alpha^*) = \{\zeta \in H^3(0,L)\cap H_0^1(0,L), \zeta'(0) = \tilde\alpha\zeta'(L)\}$. Similarly, for $\zeta \in D(A_{\tilde\alpha}^*)$ we have $(A_{\tilde\alpha}^*\zeta,\zeta)_{L^2(0,L)} = \frac{1}{2}(\tilde\alpha^2 - 1)(\zeta'(L))^2 \le 0$. Thus, by Pazy (1983), $A_{\tilde\alpha}$ is the generator of a C_0 semigroup of contraction on $L^2(0,L)$. By the Lumer-Phillips theorem (see (Pazy, 1983, Thm. 4.3)), $A_{\tilde\alpha}$ is a maximal operator. □

Thus, it is enough to check that $|\tilde\alpha| < 1$: as $|g_0(t)| < 1$, then $|\tilde\alpha| \le |\alpha| + |\beta| < 1$ by (6) and Remark 1.1. Therefore, by Lemma 2.3, we have the existence of $\varphi \in D(A_{\tilde\alpha})$ solution of (14) and hence $(y,z)^T \in D(\mathcal{A}(t))$ solution of (13).

We have then shown that $\lambda I - \mathcal{A}(t)$ is surjective. Then, as $\kappa(t) > 0$, we have that $\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t \in [0,T]$. We conclude that $\tilde{\mathcal{A}}(t)$ generates a strongly semigroup on H and $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t), t \in [0,T]\}$ is a stable family of generators in H with a stability constant independent of t, using (12), by (Kato, 1970, Prop 3.4) (see also Kato (1985)).

Finally, $\dot\kappa(t) = \frac{\ddot\tau(t)\dot\tau(t)}{2\tau(t)(\dot\tau(t)^2+1)^{1/2}} - \frac{\dot\tau(t)(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T > 0$ (by (5)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ \dfrac{\ddot\tau(t)\tau(t)\rho - \dot\tau(t)(\dot\tau(t)\rho - 1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with $\dfrac{\ddot\tau(t)\tau(t)\rho - \dot\tau(t)(\dot\tau(t)\rho - 1)}{\tau(t)^2}$ bounded on $[0,T]$ by (5). Thus,

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0,T], B(D(\mathcal{A}(0)), H)),$$

which proves the fourth point of Theorem 2.1. Therefore, all assumptions of Theorem 2.1 are verified, thus the problem

$$\begin{cases} \tilde U_t(t) = \tilde{\mathcal{A}}(t)\tilde U(t), & t > 0, \\ \tilde U(0) = U_0 \end{cases}$$

has a unique solution $\tilde U \in C([0,\infty), H)$ and $\tilde U \in C([0,\infty), D(\mathcal{A}(0))) \cap C^1([0,\infty), H)$ if $U_0 \in D(\mathcal{A}(0))$. Lastly, we can check that our solution of (9) is $U(t) =$

$e^{\int_0^t \kappa(s)ds}\tilde{U}(t)$. Indeed,

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)ds}\tilde{U}_t(t) \\ &= \kappa(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)ds}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\int_0^t \kappa(s)ds}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\int_0^t \kappa(s)ds}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\int_0^t \kappa(s)ds}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. □

2.1.2. Well-posedness of the linear system with extra source term

Consider now (9) with a source term f in the y-equation

$$\begin{cases} (y_t + y_x + y_{xxx})(x,t) = f(x,t),\ t>0,\ x\in(0,L), \\ \tau(t)z_t(\rho,t) + (1-\dot{\tau}(t)\rho)z_\rho(\rho,t) = 0, \\ \qquad t>0, \quad \rho\in(0,1), \\ y(0,t) = y(L,t) = 0, \quad t>0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta z(1,t), \quad t>0, \\ z(0,t) = y_x(0,t), \quad t>0, \\ y(x,0) = y_0(x), \quad x\in(0,L), \\ z(\rho,0) = z_0(-\tau(0)\rho), \quad \rho\in(0,1). \end{cases} \tag{15}$$

Proposition 2.4. *Assume that (3)-(6) hold. Let $U_0 = (y_0, z_0) \in H$ and $f \in L^1((0,\infty), L^2(0,L))$. Then there exists a unique solution $U = (y,z) \in C([0,+\infty), H)$ to (15). Moreover, for $T>0$, the following estimates hold*

$$\|(y,z)\|_{C([0,T],H)} \le C\left(\|U_0\|_H + \|f\|_{L^1((0,T),L^2(0,L))}\right), \tag{16}$$

$$\|y\|_{L^2((0,T),H^1(0,L))} \le C\left(\|U_0\|_H + \|f\|_{L^1((0,T),L^2(0,L))}\right). \tag{17}$$

Proof: The above system can be written as $U_t(t) = \mathcal{A}(t)U(t) + (f,0)$. Using (Kato, 1975, Th 2) we can show that if $U_0 \in H$ and $f \in L^1((0,\infty), L^2(0,L))$, then there exists a unique solution $U \in C([0,\infty), H)$. Furthermore, $U \in C([0,\infty), D(\mathcal{A}(0))) \cap C^1([0,\infty), H)$ if $U_0 \in D(\mathcal{A}(0))$ and $f \in C([0,\infty), L^2(0,L)) \cap L^1((0,\infty), D(\mathcal{A}(0)))$. Now take $U = (y,z)$ a classical solution of (15). Let us choose the following energy

$$E(t) = \frac{1}{2}\int_0^L y^2(x,t)dx + \frac{|\beta|\tau(t)}{2}\int_0^1 z^2(\rho,t)d\rho \tag{18}$$

corresponding to the time-dependent norm $\|\cdot\|_t$ on H. Differentiating (18), we obtain

$$\dot{E}(t) = \int_0^L yy_t dx + \frac{|\beta|\dot{\tau}(t)}{2}\int_0^1 z^2 d\rho + |\beta|\tau(t)\int_0^1 zz_t d\rho.$$

Now, using (15) and integrations by parts, we derive

$$\begin{aligned} \dot{E}(t) = \frac{1}{2}\big[&(\alpha^2 - 1 + |\beta|)y_x^2(0,t) + 2\alpha\beta y_x(0,t)z(1,t) \\ &+(\beta^2 - |\beta|(1-\dot{\tau}(t)))z^2(1,t)\big] + \int_0^L yf dx. \end{aligned}$$

Using (4)-(6) we get

$$\dot{E}(t) + \begin{pmatrix} y_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}\Phi_{\alpha,\beta}\right)\begin{pmatrix} y_x(0,t) \\ z(1,t) \end{pmatrix} \le \int_0^L yf dx.$$

Notice that $-\Phi_{\alpha,\beta}$ is a symmetric positive definite matrix. Then there exists $C>0$ such that

$$\dot{E}(t) + y_x^2(0,t) + z^2(1,t) \le C\int_0^L yf dx.$$

Now take $0 \le s \le T$ and integrate the above expression on $[0,s]$ to obtain

$$\begin{aligned} &E(s) + \int_0^s y_x^2(0,t)dt + \int_0^s z^2(1,t)dt \\ &\le C\left(\int_0^s\int_0^L yf dxdt + E(0)\right). \end{aligned} \tag{19}$$

Thus, by (11) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\|(y(\cdot,s), z(\cdot,s))\|_H^2 \\ &\le C\left(\|U_0\|_H^2 + \|f\|_{L^1((0,T),L^2(0,L))}\|(y,z)\|_{C([0,T],H)}\right). \end{aligned}$$

Taking the maximum for $s\in[0,T]$ and using the Young inequality, we conclude (16). In addition, taking $s=T$ in (19) and using (16) we obtain the following hidden regularity:

$$\begin{aligned} &\int_0^T y_x^2(0,t)dt + \int_0^T z^2(1,t)dt \\ &\le C\left(\|U_0\|_H^2 + \|f\|^2_{L^1((0,T),L^2(0,L))}\right). \end{aligned} \tag{20}$$

Multiplying $y-$equation of (15) by xy, integrating on $(0,T)\times(0,L)$ and performing integration by parts, we get

$$\begin{aligned} \frac{1}{2}\int_0^L xy^2(x,T)dx + \frac{3}{2}\int_0^T\int_0^L y_x^2 dxdt &= \frac{1}{2}\int_0^L xy_0^2 dx \\ &+\frac{1}{2}\int_0^T\int_0^L y^2 dxdt + \frac{1}{2}\int_0^T Ly_x^2(L,t)dt \\ &+\int_0^T\int_0^L xfydxdt, \end{aligned}$$

and then

$$\begin{aligned} \|y_x\|^2_{L^2((0,L)\times(0,T))} &\le C\left(\int_0^L y_0^2 dx + \int_0^T\int_0^L y^2 dxdt\right. \\ &\left.+\int_0^T y_x^2(0,t)dt + \int_0^T z^2(1,t)dt + \int_0^T\int_0^L fydxdt\right). \end{aligned}$$

Finally, using (16) and (20) we derive (17). □

2.1.3. Well-posedness of the nonlinear system

Now we are ready to prove the local well-posedness result for the nonlinear system (1). Let $T>0$ and introduce the space $B = C([0,T], L^2(0,L)) \cap L^2((0,T), H^1(0,L))$ endowed with the norm

$$\|y\|_B = \|y\|_{C([0,T],L^2(0,L))} + \|y\|_{L^2((0,T),H^1(0,L))}.$$

Now, to consider the nonlinearity yy_x, the next proposition will be crucial; its proof can be founded in Rosier (1997).

Proposition 2.5. *Let $y \in L^2((0,T),H^1(0,L))$. Then $yy_x \in L^1((0,T),L^2(0,L))$ and the map*

$$y \in L^2((0,T),H^1(0,L)) \mapsto yy_x \in L^1((0,T),L^2(0,L))$$

is continuous. Moreover, there exists $K>0$ such that, for any y, $\tilde{y} \in L^2((0,T),H^1(0,L))$, we have

$$\int_0^T \|yy_x - \tilde{y}\tilde{y}_x\|_{L^2(0,L)} \le K\|y-\tilde{y}\|_{L^2((0,T),H^1(0,L))} \times \left(\|y\|_{L^2((0,T),H^1(0,L))} + \|\tilde{y}\|_{L^2((0,T),H^1(0,L))}\right).$$

Theorem 2.6. *Let $T>0$, $L>0$ and assume that (3)-(6) hold. Then there exist $r, C>0$ such that for every $(y_0,z_0) \in H$ satisfying $\|(y_0,z_0)\|_H \le r$, there exists a unique solution $y \in B$ of the system (1) verifying $\|y\|_B \le C\|(y_0,z_0)\|_H$.*

Proof: Let $(y_0,z_0) \in H$ such that $\|(y_0,z_0)\|_H \le r$ for $r>0$ chosen small enough later. Take $\tilde{y} \in B$ and consider the map $P : B \to B$, defined by $P(\tilde{y}) = y$, where y is the solution of

$$\begin{cases} (y_t + y_x + y_{xxx})(x,t) = (-\tilde{y}\tilde{y}_x)(x,t),\ t>0,\ x\in(0,L),\\ y(0,t) = y(L,t) = 0, \quad t>0,\\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0,t-\tau(t)), \quad t>0,\\ y(x,0) = y_0(x), \quad x\in(0,L),\\ y_x(0,t-\tau(0)) = z_0(t-\tau(0)), \quad 0<t<\tau(0). \end{cases}$$

Clearly, $y \in B$ is a solution of (1) if and only if y is a fixed point of P. Now from Proposition 2.4 we find that the map P is well-defined and from Proposition 2.5, (16)-(17), we get

$$\|P(\tilde{y})\|_B = \|y\|_B \le C\left(\|(y_0,z_0)\|_H + \|\tilde{y}\tilde{y}_x\|_{L^1((0,T),L^2(0,L))}\right) \le C\left(\|(y_0,z_0)\|_H + \|\tilde{y}\|_B^2\right).$$

Following the same arguments, we can show that

$$\|P(\tilde{y}_1) - P(\tilde{y}_2)\|_B \le C\left(\|\tilde{y}_1\|_B + \|\tilde{y}_2\|_B\right)\|\tilde{y}_1 - \tilde{y}_2\|_B.$$

Now we restrict P to the closed ball $\{\tilde{y} \in B,\ \|\tilde{y}\|_B \le R\}$, where $R>0$ to be chosen later. Then,

$$\|P(\tilde{y})\|_B \le C(r+R^2),$$
$$\|P(\tilde{y}_1) - P(\tilde{y}_2)\|_B \le 2CR\|\tilde{y}_1 - \tilde{y}_2\|_B.$$

Finally, it is enough to consider $R < \frac{1}{2C}$ and $r < \frac{R}{2C}$. With this choice, P maps the closed ball $\{\tilde{y} \in B,\ \|\tilde{y}\| \le R\}$ into itself and $\|P(\tilde{y}_1) - P(\tilde{y}_2)\|_B \le 2CR\|\tilde{y}_1 - \tilde{y}_2\|_B$ with $2CR<1$. Lastly, we deduce the well-posedness result by invoking the Banach fixed point theorem on the map P. □

2.2. Well-posedness result of (2)

The goal of this section is to prove appropriate global well-posedness result of (2). We adopt the same methodology as in subsection 2.1, so we skip here some details.

2.2.1. Well-posedness of the linear system

In this subsection, we will study the well-posedness result of the KdV equation (2) linearized around 0, that is

$$\begin{cases} (y_t + y_x + y_{xxx})(x,t) + a(x)y(x,t)\\ \quad +b(x)y(x,t-\tau(t)) = 0, \quad t>0,\ x\in(0,L),\\ y(0,t) = y(L,t) = y_x(L,t) = 0, \quad t>0,\\ y(x,0) = y_0(x), \quad x\in(0,L),\\ y(x,t-\tau(0)) = z_0(x,t-\tau(0)),\ 0<t<\tau(0),\ x\in(0,L). \end{cases}$$

As previously, we introduce $z(x,\rho,t) = y_{|\omega}(x,t-\tau(t)\rho)$ for any $x\in\omega$, $\rho\in(0,1)$ and $t>0$, and define $U=(y,z)^T$. This problem can be rewritten as the following first-order evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}_2(t)U(t), \qquad t>0,\\ U(0) = (y_0, z_0(\cdot,-\tau(0)\cdot))^T =: U_0, \end{cases} \tag{21}$$

where the time-dependent operator $\mathcal{A}_2(t)$ is defined by

$$\mathcal{A}_2(t)\begin{pmatrix} y\\ z\end{pmatrix} = \begin{pmatrix} -y_x - y_{xxx} - ay - b\tilde{z}(.,1)\\ \dfrac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

where $\tilde{z}(.,1) \in L^2(0,L)$ is the extension of $z(.,1)$ by zero outside ω, with domain

$$D(\mathcal{A}_2(t)) = \left\{(y,z) \in H^3(0,L)\times L^2(\omega,H^1(0,1)),\ y(0) = y(L) = y_x(L) = 0,\ z(x,0) = y_{|\omega}(x)\right\}.$$

The domain of the operator $\mathcal{A}_2(t)$ is independent of the time t, i.e $D(\mathcal{A}_2(t)) = D(\mathcal{A}_2(0))$, $t>0$. The Hilbert space $H = L^2(0,L)\times L^2(\omega\times(0,1))$, is provided with the time-dependent inner product

$$\left\langle \begin{pmatrix} y\\ z\end{pmatrix}, \begin{pmatrix} \tilde{y}\\ \tilde{z}\end{pmatrix} \right\rangle_t = \int_0^L y\tilde{y}dx + \tau(t)\int_\omega\int_0^1 \xi(x)z\tilde{z}d\rho dx,$$

where ξ is a nonnegative function in $L^\infty(0,L)$ such that $\operatorname{supp}\xi = \operatorname{supp} b = \omega$ and

$$\frac{1}{1-d}\,b(x) + c_0 \leqslant \xi(x) \leqslant 2a(x) - b(x) - c_0 \quad \text{in} \quad \omega. \tag{22}$$

This choice of ξ is possible due to (8).
It is clear that the norm $\|\cdot\|_t$ is equivalent to the usual norm $\|\cdot\|_H$ on H: $\forall t\ge 0$, $\forall (y,z)\in H$,

$$(1+\tau_0 b_0)\|(y,z)\|_H^2 \le \|(y,z)\|_t^2 \le (1+2M\|a\|_\infty)\|(y,z)\|_H^2, \tag{23}$$

using (3) and (7). The following theorem gives the existence and uniqueness results of (21).

Theorem 2.7. *Assume (3)-(5), that a and b are nonnegative functions belonging to $L^\infty(0,L)$ satisfying (7)-(8) and that $U_0 \in H$. Then there exists a unique mild solution $U \in C([0,+\infty),H)$ to (21). Moreover, if $U_0 \in D(\mathcal{A}_2(0))$ then $U \in C([0,+\infty),D(\mathcal{A}_2(0))) \cap C^1([0,+\infty),H)$.*

Proof: As for Theorem 2.2, we prove the four assumptions of Theorem 2.1. We have, for all $t > 0$, $D(\mathcal{A}_2(t)) = D(\mathcal{A}_2(0))$, which is a dense subset of H. Let $t \in [0,T]$ be fixed. To prove 3. of Theorem 2.1, we start by computing $\langle \mathcal{A}_2(t)U, U\rangle_t$.
Let $U = (y,z)^T \in D(\mathcal{A}_2(0))$. Similarly to the proof of Theorem 2.2, integrating by parts in space and in ρ, we obtain

$$\langle \mathcal{A}_2(t)U,U\rangle_t = \frac{1}{2}[y_x^2]_0^L - \int_0^L a(x)y^2dx - \int_\omega b(x)z(x,1)y(x)dx + \frac{1}{2}\int_\omega \xi(x)(\dot\tau(t)-1)z^2(x,1)dx + \frac{1}{2}\int_\omega \xi(x)z^2(x,0)dx - \frac{1}{2}\dot\tau(t)\int_\omega\int_0^1 \xi(x)z^2d\rho dx.$$

Since we have

$$-\int_\omega b(x)z(x,1)y(x)dx \leqslant \frac{1}{2}\int_\omega b(x)z^2(x,1)dx + \frac{1}{2}\int_\omega b(x)y^2dx,$$

then

$$\langle \mathcal{A}_2(t)U,U\rangle_t \leq -\frac{1}{2}y_x^2(0) + \int_\omega\left(-a(x)+\frac{b(x)}{2}+\frac{\xi(x)}{2}\right)y^2(x)dx - \int_{(0,L)\setminus\omega} a(x)y^2dx + \int_\omega\left(\frac{b(x)}{2}+\frac{\xi(x)(\dot\tau(t)-1)}{2}\right)z^2(x,1)dx - \frac{1}{2}\dot\tau(t)\int_\omega\int_0^1\xi(x)z^2d\rho dx.$$

Taking ξ such that (22) is satisfied and from (4), we get $-a(x)+\frac{b(x)}{2}+\frac{\xi(x)}{2} < 0$ and $\frac{b(x)}{2}+\frac{\xi(x)(\dot\tau(t)-1)}{2} \leqslant \frac{b(x)}{2}+\frac{\xi(x)(d-1)}{2} < 0$.
Hence,

$$\langle \mathcal{A}_2(t)U,U\rangle_t - \kappa(t)\langle U,U\rangle_t \leqslant 0, \text{ where } \kappa(t) = \frac{(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)},$$

which means that the operator $\widetilde{\mathcal{A}_2}(t) := \mathcal{A}_2(t) - \kappa(t)I$ is dissipative.
Now we will compute $\langle \mathcal{A}_2(t)^*U,U\rangle_t$, where $\mathcal{A}_2(t)^*$ is the adjoint of $\mathcal{A}_2(t)$. The adjoint $\mathcal{A}_2(t)^*$ is defined by

$$\mathcal{A}_2(t)^*\begin{pmatrix} y\\ z\end{pmatrix} = \begin{pmatrix} y_x + y_{xxx} - ay + \xi(x)\tilde z(.,0)\\ \dfrac{1-\dot\tau(t)\rho}{\tau(t)}z_\rho - \dfrac{\dot\tau(t)}{\tau(t)}z\end{pmatrix},$$

with domain

$$D(\mathcal{A}_2(t)^*) = \Big\{(y,z)\in H^3(0,L)\times L^2(\omega,H^1(0,1)),$$
$$y(0)=y(L)=y_x(0)=0,\ z(x,1) = \frac{-b(x)}{\xi(x)(1-\dot\tau(t))}y_{|\omega}(x)\Big\}.$$

Then, for all $U=(y,z)^T \in D(\mathcal{A}_2(t)^*)$, we get integrating by parts in space and in ρ,

$$\langle \mathcal{A}_2(t)^*U,U\rangle_t = -\frac{1}{2}[y_x^2]_0^L - \int_0^L a(x)y^2dx + \int_\omega \xi(x)y(x)z(x,0)dx + \frac{1}{2}\int_\omega\int_0^1 \xi(x)\dot\tau(t)z^2d\rho dx + \frac{1}{2}\int_\omega \xi(x)[(1-\dot\tau(t)\rho)z^2]_0^1dx - \int_\omega\int_0^1\xi(x)\dot\tau(t)z^2d\rho dx.$$

Then, using the boundary conditions, we have

$$\langle \mathcal{A}_2(t)^*U,U\rangle_t = -\frac{1}{2}y_x^2(L,t) - \int_0^L a(x)y^2dx + \int_\omega \xi(x)y(x)z(x,0)dx - \frac{\dot\tau(t)}{2}\int_\omega\int_0^1\xi(x)z^2d\rho dx + \frac{1}{2}\int_\omega \frac{b^2(x)}{\xi(x)(1-\dot\tau(t))}y^2(x)dx - \frac{1}{2}\int_\omega\xi(x)z^2(x,0)dx.$$

Using Young's inequality, we obtain

$$\langle \mathcal{A}_2(t)^*U,U\rangle_t \leq -\int_{(0,L)\setminus\omega} a(x)y^2dx - \frac{\dot\tau(t)}{2}\int_\omega\int_0^1\xi(x)z^2d\rho dx - \frac{1}{2}\int_\omega\left(2a(x)-\xi(x)-\frac{b^2(x)}{\xi(x)(1-\dot\tau(t))}\right)y^2dx.$$

By (22) and (4), we have

$$\xi(x) \leq 2a(x) - b(x) - c_0 \leq 2a(x) - \frac{b^2(x)}{\xi(x)(1-d)} - c_0 \leq 2a(x) - \frac{b^2(x)}{\xi(x)(1-\dot\tau(t))},$$

since $b(x) \leq \xi(x)(1-d)$ (see (22)). Consequently $2a(x) - \xi(x) - \frac{b^2(x)}{\xi(x)(1-\dot\tau(t))} \geq 0$. Hence,

$$\langle \mathcal{A}_2(t)^*U,U\rangle_t - \kappa(t)\langle U,U\rangle_t \leqslant 0,$$

which means that the operator $\widetilde{\mathcal{A}_2}(t)^* = \mathcal{A}_2(t)^* - \kappa(t)I$ is dissipative.
Since $\widetilde{\mathcal{A}_2}(t)$ and $\widetilde{\mathcal{A}_2}(t)^*$ are dissipative and $\widetilde{\mathcal{A}_2}(t)$ is a densely defined closed linear operator, then $\widetilde{\mathcal{A}_2}(t)$ is the infinitesimal generator of a C_0 semigroup of contraction on H (see Pazy (1983)) for any $t\in[0,T]$ be fixed. As the proof of Theorem 2.2, we can easily prove (12). Consequently, for all $t \in [0,T]$, $\widetilde{\mathcal{A}_2}(t)$ generates a strongly continuous semigroup on H and the family $\widetilde{\mathcal{A}_2} = \{\widetilde{\mathcal{A}_2}(t) : t\in[0,T]\}$ is stable with stability constants C and m independent of t (see Proposition 3.4 of Kato (1970)). These mean that 3. of Theorem 2.1 is satisfied. Finally, we can also prove, similarly to the proof of Theorem 2.2, that

$$\frac{d}{dt}\widetilde{\mathcal{A}_2}(t) \in L^\infty_*([0,T], B(D(\mathcal{A}_2(0)), H)).$$

Since all assumptions of Theorem 2.1 are verified, then the problem

$$\begin{cases} \tilde U_t(t) = \tilde{\mathcal{A}}_2(t)\tilde U,\\ \tilde U(0) = U_0,\end{cases}$$

has a unique solution $\tilde U \in C([0,+\infty), D(\mathcal{A}_2(0))) \cap C^1([0,+\infty), H)$ for $U_0 \in D(\mathcal{A}_2(0))$. The requested solution of (21) is then given by $U(t) = e^{\int_0^t \kappa(s)ds}\tilde U(t)$, similarly to the proof of Theorem 2.2. □

2.2.2. *Well-posedness of the linear system with a source term*

In this subsection, we will study the well-posedness of the following linear KdV equation with a source term

$$\begin{cases} y_t(x,t) + y_x(x,t) + y_{xxx}(x,t) + a(x)y(x,t) \\ \quad +b(x)y(x,t-\tau(t)) = f(x,t),\ t>0,\ x\in(0,L), \\ y(0,t) = y(L,t) = y_x(L,t) = 0, \quad t>0, \\ y(x,0) = y_0(x), \quad x\in(0,L), \\ y(x,t-\tau(0)) = z_0(x,t-\tau(0)), \\ \quad 0<t<\tau(0),\ x\in(0,L). \end{cases} \tag{24}$$

Proposition 2.8. *Assume* (3)-(5) *and that a and b are nonnegative functions belonging to* $L^\infty(0,L)$ *satisfying* (7)-(8). *For any* $(y_0, z_0(.,-\tau(0).)) \in H$ *and* $f \in L^1(0,T,L^2(0,L))$ *there exists a unique mild solution* $(y, y(.,t-\tau(t).)) \in B \times C([0,T],H)$ *to* (24). *Moreover, there exists* $C>0$ *independent of* T *such that* (16) *and* (17) *hold.*

Proof: The proof is similar to the proof of Proposition 2.4 and is left to the readers (see also Valein (2022)). □

2.2.3. *Well-posedness of the nonlinear system* (2)

Finally, we will show the global well-posedness result of the nonlinear system (2).

Theorem 2.9. *Let* $L>0$ *and assume* (3)-(5) *and that a and b are nonnegative functions belonging to* $L^\infty(0,L)$ *satisfying* (7) *and* (8). *Then for any* $(y_0, z_0(.,-\tau(0).)) \in H$, *there exists a unique* $y \in B$ *solution of system* (2).

Proof: Following Menzala et al. (2002), we can get the global existence of the solution by showing the local (in time) existence and using the decay of the energy. Let $\tilde{y} \in B$, we consider the map $\Psi : B \longrightarrow B$ defined by $\Psi(\tilde{y}) = y$ where y is the solution of the following system

$$\begin{cases} y_t(x,t) + y_x(x,t) + y_{xxx}(x,t) + a(x)y(x,t) \\ \quad +b(x)y(x,t-\tau(t)) = -\tilde{y}(x,t)\tilde{y}_x(x,t),\ t>0,\ x\in(0,L), \\ y(0,t) = y(L,t) = y_x(L,t) = 0, \quad t>0, \\ y(x,0) = y_0(x), \quad x\in(0,L), \\ y(x,t-\tau(0)) = z_0(x,t-\tau(0)),\ 0<t<\tau(0),\ x\in(0,L). \end{cases}$$

We can prove similarly to the proof of (Valein, 2022, Proposition 4) (see also Theorem 2.6) that Ψ is a contraction on the closed ball $\{y \in B / \|y\|_B \le R\}$ for some chosen R. Hence, from the Banach fixed point theorem, the map Ψ has a unique fixed point $y \in B$ which is the solution of the nonlinear system (2). □

3. Exponential stability results

In this section, we prove the exponential stability results, firstly with the boundary damping, secondly with the internal damping.

3.1. Boundary stability result

We start this section showing that for a solution of (1) the energy is a not-increasing function of time. We recall that the energy of (1) is defined by

$$E(t) = \frac{1}{2}\int_0^L y^2(x,t)dx + \frac{|\beta|\tau(t)}{2}\int_0^1 y_x^2(0,t-\tau(t)\rho)d\rho. \tag{25}$$

Proposition 3.1. *Suppose that* (3)-(6) *be satisfied. Then for all regular solution of* (1), *the energy defined by* (25) *is not increasing and satisfies*

$$\dot{E}(t) \le \frac{1}{2}Y^T\Phi_{\alpha,\beta}Y \le 0, \tag{26}$$

where $Y = (y_x(0,t), y_x(0,t-\tau(t)))^T$.

Proof: It is enough to follow the proof of Proposition 2.4 and notice that for $y \in H_0^1(0,L)$, $\int_0^L y^2 y_x dx = 0$. □

Consider the following new Lyapunov candidate

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \tag{27}$$

where E is defined by (25), μ_1, $\mu_2 > 0$ and

$$V_1(t) = \int_0^L x y^2(x,t)dx, \tag{28}$$

$$V_2(t) = \tau(t)\int_0^1 (1-\rho)y_x^2(0,t-\tau(t)\rho)d\rho. \tag{29}$$

Note that V_1 is classical for the KdV equation and V_2 comes from the delay term depending on time.

Theorem 3.2. *Suppose that* (3)-(6) *are satisfied and assume that the length* L *fulfills* $L < \pi\sqrt{3}$. *Then, there exists* $r>0$ *such that, for every* $(y_0,z_0) \in H$ *satisfying* $\|(y_0,z_0)\|_0 \le r$, *the energy of the system* (1) *decays exponentially. More precisely, there exist two positive constants* γ *and* C *such that*

$$E(t) \le Ce^{-2\gamma t}E(0),\ \forall t>0, \tag{30}$$

with

$$\gamma \le \min\left\{\frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{3L^2(1+2L\mu_1)}, \frac{(1-d)\mu_2}{M(2\mu_2+|\beta|)}\right\}, \tag{31}$$

$$C \le 1 + \max\left\{L\mu_1, \frac{2\mu_2}{|\beta|}\right\},$$

where, μ_1 *and* μ_2 *are taken such that*

$$\mu_2 \le \min\left\{\frac{1-\alpha^2-\beta^2-|\beta|d}{2}, \frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)}{2(1-|\beta|-d)}, \frac{|\beta|-\alpha^2(d-1)-\beta^2}{2|\beta|}\right\}, \tag{32}$$

$$\mu_1 \le \min\left\{\frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)+2\mu_2(|\beta|+d-1)}{2L(|\beta|-\alpha^2(d-1)-\beta^2-2\mu_2|\beta|)}, \frac{1-\alpha^2-\beta^2-|\beta|d+2\mu_2}{2L(\alpha^2+\beta^2)}\right\}. \tag{33}$$

Remark 3.3. *We note that the decay rate γ decreases when the upper bound M of the delay $\tau(t)$ increases, as shown in the estimation of the decay rate* (31)*. We can also observe the same phenomenom when d tends to* 1.

Proof: Note that the function V is equivalent to the energy E. More precisely, for every $t > 0$,

$$E(t) \leq V(t) \leq \left(1 + \max\left\{L\mu_1, \frac{2\mu_2}{|\beta|}\right\}\right) E(t). \quad (34)$$

Thus, it suffices to show that V decays exponentially. Let $\gamma > 0$ to fix later, we are going to prove that $\dot{V}(t)+2\gamma V(t) \leq 0$. Let y solution of (1) with $(y_0, z_0)^T \in D(\mathcal{A}(0))$ such that $\|(y_0, z_0)\|_0 \leq r$ with $r > 0$ chosen later. First, we differentiate V_1 and use integration by parts to have

$$\begin{aligned}\dot{V}_1(t) &= L\alpha^2 y_x^2(0,t) + 2L\alpha\beta y_x(0,t)y_x(0,t-\tau(t))\\ &+L\beta^2 y_x^2(0,t-\tau(t)) - 3\int_0^L y_x^2 dx + \int_0^L y^2 dx \\ &+\frac{2}{3}\int_0^L y^3 dx.\end{aligned} \quad (35)$$

Similarly, we differentiate V_2:

$$\begin{aligned}\dot{V}_2(t) &= \dot{\tau}(t)\int_0^1 (1-\rho)y_x^2(0,t-\tau(t)\rho)d\rho\\ &+2\tau(t)\int_0^1 (1-\rho)y_x(0,t-\tau(t)\rho)\partial_t y_x(0,t-\tau(t)\rho)d\rho.\end{aligned}$$

Noting that $-\tau(t)\partial_t y_x(0,t-\tau(t)\rho) = (1-\dot{\tau}(t)\rho)\partial_\rho y_x(0,t-\tau(t)\rho)$ and performing integration by parts, we get

$$\dot{V}_2(t) = -\int_0^1 (1-\dot{\tau}(t)\rho)y_x^2(0,t-\tau(t)\rho)d\rho + y_x^2(0,t). \quad (36)$$

Joining (26), (35) and (36) we have

$$\begin{aligned}\dot{V}(t) + 2\gamma V(t) &\leq Y^T\left[\frac{1}{2}\Phi_{\alpha,\beta} + \Psi_{\alpha,\beta}\right]Y - 3\mu_1\int_0^L y_x^2 dx\\ &+\frac{2}{3}\mu_1\int_0^L y^3 dx + (\mu_1+\gamma+2L\mu_1\gamma)\int_0^L y^2 dx\\ &+(\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d))\int_0^1 y_x^2(0,t-\tau(t)\rho)d\rho,\end{aligned}$$

where the matrix $\Psi_{\alpha,\beta}$ is defined by

$$\Psi_{\alpha,\beta} = \begin{pmatrix} L\mu_1\alpha^2 + \mu_2 & \mu_1\alpha\beta L \\ \mu_1\alpha\beta L & \mu_1\beta^2 L\end{pmatrix}.$$

Then, as $\Phi_{\alpha,\beta}$ is definite negative and by the continuity of the trace and the determinant, we find that for μ_1 and μ_2 small enough, the matrix $\frac{1}{2}\Phi_{\alpha,\beta} + \Psi_{\alpha,\beta}$ is negative definite. More precisely, following (Baudouin et al., 2019, Remark 5), we can take μ_1 and μ_2 as in (32)-(33) . For the term involving $\int_0^L y^3 dx$, note that

$$\begin{aligned}\int_0^L y^3 dx &\leq \|y\|_{L^\infty(0,L)}^2 \int_0^L |y| dx\\ &\leq \|y\|_{L^\infty(0,L)}^2 \|y\|_{L^2(0,L)}\sqrt{L}.\end{aligned}$$

By the injection of $H_0^1(0,L)$ into $L^\infty(0,L)$ we know that $\|y\|_{L^\infty(0,L)} \leq \sqrt{L}\|y_x\|_{L^2(0,L)}$, then

$$\int_0^L y^3 dx \leq L^{3/2}\|y_x\|_{L^2(0,L)}^2\|y\|_{L^2(0,L)}.$$

Finally, using Proposition 3.1 we can obtain $\|y\|_{L^2(0,L)} \leq r$ and hence invoking Poincaré's inequality

$$\begin{aligned}&\dot{V}(t) + 2\gamma V(t)\\ &\leq \left(\frac{L^2}{\pi^2}(\mu_1+\gamma+2L\mu_1\gamma) + \frac{2}{3}L^{3/2}r\mu_1 - 3\mu_1\right)\int_0^L y_x^2 dx\\ &+(\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d))\int_0^1 y_x^2(0,t-\tau(t)\rho)d\rho.\end{aligned}$$

Now, following Baudouin et al. (2019), as $L < \pi\sqrt{3}$, it is possible to choose r small enough to have $r < \frac{3(3\pi^2-L^2)}{2L^{3/2}\pi^2}$. Then, we can choose $\gamma > 0$ such that

$$\begin{aligned}&\tfrac{L^2}{\pi^2}(\mu_1+\gamma+2L\mu_1\gamma) + \frac{2}{3}L^{3/2}r\mu_1 - 3\mu_1 \leq 0,\\ &\gamma|\beta|M + 2\mu_2\gamma M - \mu_2(1-d) < 0.\end{aligned}$$

Thus, we can easily obtain (31). Therefore, we have $\dot{V}(t) + 2\gamma V(t) \leq 0$ and hence $V(t) \leq V(0)e^{-2\gamma t}$ for all $t > 0$. Using (34) we obtain (30). Since $D(\mathcal{A}(0))$ is dense in H, we can take $(y_0, z_0) \in H$. □

3.2. Internal stability result

In this section, we will study the local stability of (2) using some Lyapunov functional. We consider the following definition of the energy of the nonlinear system (2)

$$\begin{aligned}E(t) &= \frac{1}{2}\int_0^L y^2(x,t)dx\\ &+\frac{\tau(t)}{2}\int_\omega\int_0^1 \xi(x)y^2(x,t-\tau(t)\rho)d\rho dx,\end{aligned} \quad (37)$$

where ξ is defined by (22). In the following proposition, we will prove the decay of the energy of the nonlinear system (2).

Proposition 3.4. *Assume* (3)-(5) *and that a and b are non-negative functions belonging to $L^\infty(0,L)$ satisfying* (7) *and* (8)*. Then, for any regular solution of* (2)*, the energy E defined by* (37) *is non-increasing and satisfies*

$$\begin{aligned}\dot{E}(t) &\leq -\frac{1}{2}y_x^2(0,t) - \int_{(0,L)\setminus\omega} a(x)y^2(x,t)dx\\ &-\frac{1}{2}\int_\omega (2a(x) - b(x) - \xi(x))y^2(x,t)dx\\ &-\frac{1}{2}\int_\omega (\xi(x)(1-d) - b(x))y^2(x,t-\tau(t))dx \leq 0.\end{aligned}$$

Proof: The proof is similar to the proof of the dissipativity of $\mathcal{A}_2(t)$, noting that $\int_0^L y^2 y_x dx = 0$ (see also the proof of

Proposition 2.4). □

Now we take the following Lyapunov functional

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_3(t), \tag{38}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are fixed constants taken small enough, E is the energy defined by (37), V_1 by (28) and V_3 is defined by

$$V_3(t) = \tau(t) \int_\omega \int_0^1 (1-\rho) y^2(x, t-\tau(t)\rho) d\rho dx. \tag{39}$$

From the definition of $V(t)$ and $E(t)$, we have for any $t > 0$,

$$E(t) \le V(t) \le \left(1 + \max\left\{ L\mu_1, \frac{\mu_2}{b_0} \right\}\right) E(t). \tag{40}$$

In the following theorem, we will prove that the energy of the nonlinear system (2) decays exponentially.

Theorem 3.5. *Assume (3)-(5) and that a and b are non-negative functions belonging to $L^\infty(0, L)$ that satisfy (7) and (8), and assume that the length L satisfies $L < \pi\sqrt{3}$. Then, there exists $r > 0$ small enough, such that, for every $(y_0, z_0) \in H$ satisfying $\|(y_0, z_0)\|_0 \le r$, the energy of the nonlinear system (2) decays exponentially. More precisely, there exist two positive constants γ and C such that*

$$E(t) \le Ce^{-2\gamma t} E(0), \ \forall t > 0,$$

with

$$\gamma \le \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2} r\pi^2)\mu_1}{3L^2(1+2L\mu_1)}, \frac{(1-d)\mu_2}{M(2\mu_2 + \|\xi\|_{L^\infty(0,L)})} \right\}, \tag{41}$$

$$C \le 1 + \max\left\{ L\mu_1, \frac{2\mu_2}{b_0} \right\},$$

where, μ_1 and μ_2 are taken such that

$$\mu_1 \le \inf_{x\in\omega}\left\{ \frac{2a(x) - b(x) - \xi(x)}{2Lb(x)}, \frac{(1-d)\xi(x) - b(x)}{2Lb(x)} \right\}, \tag{42}$$

$$\mu_2 \le \inf_{x\in\omega}\left\{ \frac{2a(x) - b(x) - \xi(x) - 2\mu_1 Lb(x)}{2} \right\}. \tag{43}$$

Remark 3.6. *We note that the decay rate γ decreases when the upper bound M of the delay $\tau(t)$ increases, as shown in the estimation of the decay rate (41). We can also observe the same phenomenon when d tends to* 1.

Proof: Since E and V are equivalent from (40), we will prove that V decays exponentially, so we will prove that $\dot V(t) + 2\gamma V(t) \le 0$ for all $t > 0$. Assume that y is a solution of (2) with $(y_0, z_0(., -\tau(0).))^T \in D(\mathcal{A}_2(0))$ satisfying $\|(y_0, z_0(., -\tau(0).))\|_0 \le r$. We start by differentiating V_1 and integrating by parts, we get

$$\begin{aligned}\dot V_1(t) &= -3\int_0^L y_x^2(x,t)dx + \int_0^L y^2(x,t)dx \\ &+\frac{2}{3}\int_0^L y^3(x,t)dx - 2\int_0^L xa(x)y^2(x,t)dx \\ &-2\int_0^L xb(x)y(x,t)y(x,t-\tau(t))dx.\end{aligned}$$

Now, we differentiate V_3 and integrating by parts, we get, using $-\tau(t)\partial_t y(x, t-\tau(t)\rho) = (1-\dot\tau(t)\rho)\partial_\rho y(x, t-\tau(t)\rho)$,

$$\dot V_3(t) = \int_\omega y^2(x,t)dx - \int_\omega \int_0^1 (1-\dot\tau(t)\rho) y^2(x, t-\tau(t)\rho) d\rho dx.$$

Then

$$\begin{aligned}&\dot V(t) + 2\gamma V(t) \\ &\le \frac{1}{2}\int_\omega (-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2) y^2(x,t)dx \\ &+\frac{1}{2}\int_\omega (b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x)) y^2(x, t-\tau(t))dx \\ &-\int_{(0,L)\setminus\omega} a(x)y^2(x,t)dx + (\mu_1 + \gamma + 2\gamma\mu_1 L)\int_0^L y^2(x,t)dx \\ &-3\mu_1\int_0^L y_x^2(x,t)dx + \frac{2}{3}\mu_1\int_0^L y^3(x,t)dx \\ &+\int_\omega\int_0^1 (\gamma\xi(x)\tau(t) + 2\gamma\mu_2\tau(t) - \mu_2(1-d)) y^2(x, t-\tau(t)\rho)d\rho dx.\end{aligned}$$

Using Poincaré's inequality, we obtain

$$\begin{aligned}&\dot V(t) + 2\gamma V(t) \\ &\le \frac{1}{2}\int_\omega (-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2) y^2(x,t)dx \\ &+\frac{1}{2}\int_\omega (b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x)) y^2(x, t-\tau(t))dx \\ &+\left(\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1\right)\int_0^L y_x^2(x,t)dx + \frac{2}{3}\mu_1\int_0^L y^3(x,t)dx \\ &+\int_\omega\int_0^1 (\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2) y^2(x, t-\tau(t)\rho)d\rho dx.\end{aligned}$$

From (22), we can choose μ_1 and μ_2 small enough to get $-2a(x) + b(x) + \xi(x) + 2\mu_1 Lb(x) + 2\mu_2 \le 0$ and $b(x) - (1-d)\xi(x) + 2\mu_1 Lb(x) \le 0$ in ω. More precisely, by (22), we can take μ_1 and μ_2 as in (42)-(43). From the Cauchy-Schwarz inequality and Proposition 3.4, we get, as in the proof of Theorem 3.2,

$$\int_0^L y^3(x,t)dx \le L\sqrt{Lr}\|y_x(.,t)\|^2_{L^2(0,L)}.$$

Finally, we obtain

$$\begin{aligned}\dot V(t) + 2\gamma V(t) &\le \left(\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 + \frac{2rL^{3/2}\mu_1}{3}\right)\int_0^L y_x^2(x,t)dx \\ &+\int_\omega\int_0^1 (\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2) y^2(x, t-\tau(t)\rho)d\rho dx.\end{aligned}$$

It is sufficient to have $\frac{L^2(\mu_1 + \gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 + \frac{2rL^{3/2}\mu_1}{3} \le 0$ and $\gamma\xi(x)M + 2\gamma\mu_2 M - (1-d)\mu_2 \le 0$. Hence, we take γ as in (41) where r can be chosen such that $9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2 > 0$ which means that $0 < r < \frac{9\pi^2 - 3L^2}{2L^{3/2}\pi^2}$, and which is possible since $0 < L < \sqrt{3}\pi$.
Finally, we integrate $\dot V(t) + 2\gamma V(t) \le 0$ over $(0, t)$ to obtain $V(t) \le V(0)e^{-2\gamma t}$, for all $t > 0$. From (40), we obtain

$$E(t) \le E(0)\left(1 + \max\{L\mu_1, \frac{2\mu_2}{b_0}\}\right)e^{-2\gamma t}, \ \forall t > 0.$$

Since $D(\mathcal{A}_2(0))$ is dense in H, we can take $(y_0, z_0(., -\tau(0).)) \in H$. □

4. Numerical simulations

The aim of this section is to illustrate the stability results obtained in this work with some numerical simulations that adapt the schemes used in Baudouin et al. (2019); Colin and Gisclon (2001); Parada et al. (2022a). We choose a final time T and build a uniform spatial and time discretization of $N_x + 1$ and $N_t + 1$ points, respectively, separated by the steps $\Delta x = L/N_x$ and $\Delta t = T/N_t$. We present now the numerical scheme in the case of boundary delay. The internal case follows similar ideas, (see Parada et al. (2022a) for a similar scheme in the case of constant delay in a network). We choose the delay step $\Delta\rho = 1/N_\rho$. Now we introduce the notation $y(i\Delta x, n\Delta t) = y_i^n$ and $z(k\Delta\rho, n\Delta t) = z_k^n$ for $i = 0, \cdots, N_x$, $k = 0, \cdots, N_\rho$ and $n = 0, \cdots, N_t$. We use the following approximation for the derivatives:

$$\mathrm{D}_x^+ y_i = \frac{y_{i+1} - y_i}{\Delta x}, \quad \mathrm{D}_x^- y_i = \frac{y_i - y_{i-1}}{\Delta x},$$
$$\mathrm{D}_x y_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x}, \quad \mathrm{D}_\rho^+ z_k = \frac{z_{k+1} - z_k}{\Delta\rho}.$$

To approximate the term of third order ∂_x^3, we use $\mathrm{D}_x^+\mathrm{D}_x^+\mathrm{D}_x^-$. To approximate the nonlinear term, we use explicit approximation $y_i^n \mathrm{D}_x^+ y_i^n$. Note now that by the boundary conditions we have that $y_{N_x}^n = y_0^n = 0$, $z_0^n = y_1^n/\Delta x$ and $y_{N_x-1}^n = -\alpha y_1^n - \beta\Delta x z_{N_\rho}^n$ for all $n = 0, \cdots, N_t$. Then, taking $C = \mathrm{D}_x^+\mathrm{D}_x^+\mathrm{D}_x^- + \mathrm{D}_x$, $\tau^n = \tau(n\Delta t)$ and $\dot\tau^n = \dot\tau(n\Delta t)$, our scheme can be seen as

$$\begin{cases} \dfrac{y_i^{n+1} - y_i^n}{\Delta t} + (Cy_i^{n+1})_i + y_i^n \mathrm{D}_x^+ y_i^n = 0, & i = 1, \cdots, N_x - 1, \\ & n = 1, \cdots, N_t - 1, \\ \tau^{n+1}\left(\dfrac{z_k^{n+1} - z_k^n}{\Delta t}\right) + (1 - \dot\tau^{n+1} k\Delta\rho)(\mathrm{D}_\rho^+ z_k^{n+1}) = 0, & k = 1, \cdots, N_\rho - 1, \\ & n = 1, \cdots, N_t - 1, \\ y_{N_x}^n = y_0^n = 0, \quad n = 1, \cdots, N_t, \\ z_0^n = y_1^n/\Delta x, \quad n = 1, \cdots, N_t, \\ y_{N_x-1}^n = -\alpha y_1^n - \beta\Delta x z_{N_\rho}^n, \quad n = 1, \cdots, N_t, \\ y_i^0 = y_0(i\Delta x), \quad i = 1, \cdots, N_x, \\ z_k^0 = z_0(-\tau(0)k\Delta\rho), \quad k = 1, \cdots, N_\rho. \end{cases}$$

Now, we use this scheme with the following parameters $L = 1$ and $T = 10$. For the discretization, we use $N_x = 100$, $N_\rho = 100$ and $N_t = 100$. The initial conditions are $y_0(x) = 0.5(1 - \cos(2\pi x))$, $z_0(\rho) = -0.5\sin(2\pi\rho)$ and the delay is $\tau(t) = d(1.5 + \sin(t))$.

For Figure 1 we use $\alpha = 0.1$ and $\beta = 0.1$. We can observe how the decay rate depends on the size of d, as mentioned in Remark 3.3. In particular, in the case $d = 1.3$ which does not satisfy (4), the energy is not decreasing. For Figure 2 we consider the internal delay where the feedback terms are constant in their support supp a = supp $b = (0, L/2)$, $a(x) = 2$, $b(x) = 1$ and $\xi(x) = 2.1$. The initial conditions are $y_0(x) = 1 - \cos(2\pi x)$, $z_0(x, \rho) = (1 - \cos(2\pi x))\cos(2\pi\rho)$ and the delay is $\tau(t) = M + \frac{\sin(t)}{2}$. We can observe how the decay rate depends on how large M is, as explained in Remark 3.6. Finally, in Figure 3 we present a comparison between the action of time-varying delay and constant delay for boundary and internal feedbacks. We take $\tau(t) = d(1.5 + \sin(t))$, $\tau_{max} = 2.5d$ and $\tau_{min} = 0.5d$. In both figures, we see how the energy associates to time-varying delay is oscillating between the associated to $\tau_{max} = 2.5d$ and $\tau_{min} = 0.5d$.

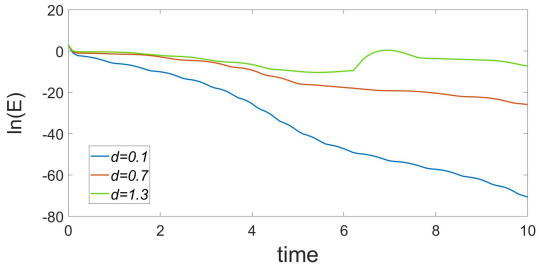

Figure 1: Time-evolution of $t \mapsto \ln(E(t))$ for different values of d (boundary delay).

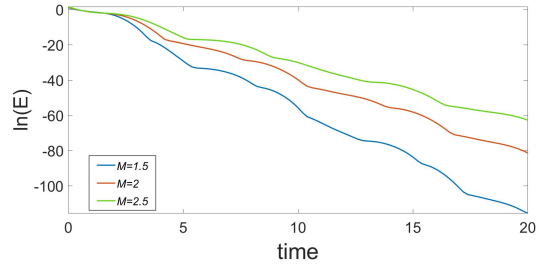

Figure 2: Time-evolution of $t \mapsto \ln(E(t))$ for different values of M (internal delay).

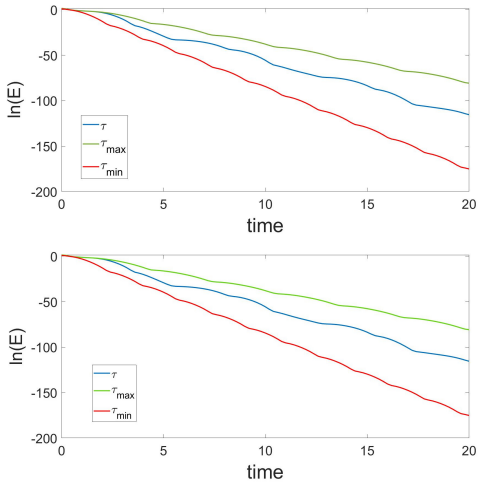

Figure 3: Time-evolution of $t \mapsto \ln(E(t))$ in the case of constant and varying (up) boundary delay (down) internal delay.

5. Conclusion

In this paper, we presented some boundary and internal stability results for the nonlinear KdV equation with time-varying delay. We proved appropriate well-posedness results, and we studied the local stability using some Lyapunov functionals. Finally, numerical simulations were presented to illustrate the results obtained. We could improve this paper in the following directions: remove the assumption $\operatorname{supp} b \subset \operatorname{supp} a$, consider the case L non critical (and not only $L < \sqrt{3}\pi$) and the case $\tau_0 = 0$ in (3). These questions remain open, since, for the two first, the system is not invariant by translation in time contrary to constant delay (see Capistrano-Filho and Gonzalez Martinez (2023); Valein (2022)), and for the last one, we need more regularity of the solutions (see Nicaise et al. (2011)). Moreover the condition $L < \sqrt{3}\pi$ is a technical one and comes from the choice of the multiplier x in the expression of V_1. To find a better multiplier is an open problem as far as we know.

We mention here some possible future research: the cases of mixed boundary and internal damping with time-varying delay, time- and spatially-varying delay as in Lhachemi, Prieur and Shorten (2021) or study the stabilization problem when the delay (constant or variable) is in the nonlinear term as in Liu (2002); Zhu (2014) for Burger's and Kuramoto-Sivashinsky equations, respectively.

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