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# Formal stability in Hamiltonian fluid models for plasmas

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**Abstract.**

We review the progress made, during the last decade, on the analysis of formal stability for Hamiltonian fluid models for plasmas, carried out by means of the Energy-Casimir (EC) method. The review begins with a tutorial Section describing the essential concepts on the Hamiltonian formalism for fluid models and on the EC method, which will be frequently used in the article. Subsequently, a nonlinear stability analysis applied to reduced magnetohydrodynamics (MHD) is described, as paradigmatic example for the application of the EC method. The review of the recent results begins with the equilibrium and formal stability analysis of MHD with general helical symmetry, followed by the treatment of extended MHD. Applications of the EC method to a hybrid MHD-Vlasov model with pressure coupling and to a reduced fluid model accounting for electron temperature anisotropy are described next. The formal stability analysis of compressible reduced MHD is then presented and used to show the connection between the EC method and the classical  $\delta W$  method for MHD stability. The concept of negative energy mode (NEM) is also briefly reviewed and applied to a model for electron temperature gradient (ETG) instability. In the context of the search for equilibria by a variational procedure, which is part of the EC method, we discuss a recent interpretation of the classical tearing modes in terms of singular equilibria of MHD linearized about Beltrami equilibria. Finally, we mention some possible directions for future developments.

**1. Introduction**

Stability analysis is traditionally a subject of crucial importance in plasma physics. A main motivation for investigating the stability of plasma equilibria has, of course, its origins, in the research on magnetically confined plasmas such as in tokamaks, where predicting instabilities is of uttermost importance for the performance of a thermonuclear fusion device. Moreover, solar flares, coronal mass ejections and turbulent transport in accretion disks, provide classical examples, in the astrophysical context, of phenomena, the explanation of which is often given by invoking some plasma instabilities.

Especially in the early years of the research on both magnetically confined and astrophysical plasmas, a fluid description of plasma was frequently adopted. Such description often consisted in the single-fluid, MHD model. Although relatively rough, the MHD model made it possible to identify a number of important large scale linear instabilities such as, for instance, kink, interchange and tearing modes, occurring in magnetized plasmas.

In subsequent years, in parallel with the development of stability analyses based on kinetic and gyrokinetic models, a plethora of fluid models were derived, providing more refined versions of MHD, accounting for additional physical ingredients such as, for instance, two-fluid effects, Finite Larmor Radius effects, Landau damping and pressure anisotropy (an exhaustive referencing to all the fluid models developed in this context would be virtually impossible, so we prefer to limit ourselves to cite the fluid models of interest in the present review, which will be referenced to later in the article). Stability analyses based on such refined fluid models provided of course new insights on plasma instabilities. As an example of this process, one could consider how the analytical theory of tearing instability evolved from the pioneering work of Furth, Killeen and Rosenbluth [1], based on resistive MHD, to later results based on more refined fluid models, as for instance in Refs. [2, 3, 4, 5].

The increasing complexity of the refined fluid models, has often limited stability analyses to the investigation of spectral stability and has, in most cases, obliged to a numerical treatment, in particular for the case of equilibria depending non-trivially on more than one spatial coordinate. On the other hand, among the various approaches for investigating plasma stability by means of fluid models, the EC method emerged as a useful tool in the case of non-dissipative systems, in particular when analytical stability conditions are desired, also for equilibria depending on more than one dimension.

The EC method is, in general, a method for investigating stability in noncanonical Hamiltonian systems. Such systems possess, in addition to the Hamiltonian, as in the case of canonical Hamiltonian systems, also additional conserved quantities, referred to as Casimir invariants, originating from the degeneracy of the noncanonical Poisson bracket. Fluid models formulated in terms of Eulerian variables (as is the case of the fluid models for plasmas treated here), when taken in their non-dissipative limit, should possess a noncanonical Hamiltonian structure. Therefore, fluid models for plasmas provide a natural class of systems for which the EC method can be applied. We also point out that the EC method can of course be applied, and has actually successfully

been applied, also to kinetic models for plasmas. Although this subject lies outside the scope of the present paper, we mention, as examples in this context, the References [6, 7, 8, 9, 10, 11, 12], even though not in all these References the Hamiltonian formalism was used explicitly.

In its full formulation, as will be discussed later in Sec. 2.4, the EC method for fluid models can provide sufficient conditions for nonlinear stability of equilibria. However, the infinite-dimensional nature of the fluid models is often accompanied with issues related to the global existence in time of solutions of such systems, which can restrict the applicability of nonlinear stability conditions. However, one can limit the application of the EC method to the identification of conditions for *formal* stability, which implies linear stability. A precise definition of formal stability will be given in Sec. 2.3. Here we limit ourselves to recalling that an equilibrium of a dynamical system is formally stable if there exists a constant of motion for which the equilibrium point is a critical point, and the second variation of which, when evaluated at the equilibrium point, has definite sign. Because, for physical applications, linear stability is often an important enough information, in the plasma physics literature, most of the applications of the EC method are limited to the investigation of formal stability which, as above stated, guarantees linear stability. This is the kind of investigations that we will consider in this review, although, in order to provide an example, we treat in Sec. 3 the description of the complete nonlinear stability analysis for the case of reduced MHD.

The essence of the EC method for formal stability consists first of identifying critical points of a conserved *free energy functional*, given by a linear combination of the Hamiltonian with the Casimir invariants. Such critical points are equilibria of the system. If the second variation of the free energy functional, evaluated at one of such equilibria, has a definite sign, then such second variation provides a conserved norm for the perturbations of the corresponding linearized system, whose growth is thus bounded. This corresponds to formal stability and implies the linear stability of the equilibrium under consideration. In Ref. [10], in particular, one can find a proof that a positive

definite second variation evaluated at an equilibrium point is a constant of motion for the linearized system, thus providing the required conserved norm. Further details can also be found in Ref. [13].

Historically, as discussed in Refs. [13, 14], the EC method has a first antecedent in Lagrange's theorem [15], applicable to finite-dimensional canonical Hamiltonian systems, with a Hamiltonian function of separable form. In the case of one-degree-of-freedom systems, with coordinates of a phase space  $P$  given by the position  $q$  and the momentum  $p$ , a Hamiltonian  $H$  of separable form is of the type  $H(q, p) = p^2/(2m) + V(q)$ , where  $p^2/(2m)$  represents the kinetic energy of a body of mass  $m$ , and  $V(q)$  is the potential energy. Denoting with  $(q(t), p(t))$  the trajectories described by the dynamics as time  $t$  evolves, the equations of motion descending from the Hamiltonian  $H$  read

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad (1)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq}, \quad (2)$$

where the dot indicates time derivative. Critical points of  $H$  have coordinates  $(q, p) = (q_e, 0)$ , where  $q_e$  is a stationary point of  $V$ . Such critical points are equilibria of the system. Lagrange's theorem states that critical points for which  $q_e$  is a local minimum of  $V$  are stable. A generalization of this theorem, applicable to Hamiltonians not necessarily of separable form, is Dirichlet's theorem [16]. Denoting with  $z = (q, p)$  the set of phase space coordinates, critical points  $z = z_e$  of a Hamiltonian  $H(z)$  are equilibria of the system. According to Dirichlet's theorem, such equilibria are stable if the eigenvalues of the matrix with element  $\partial^2 H / \partial z^i \partial z^j |_{z=z_e}$ , with  $i, j = 1, 2$ , are non-zero and they all have the same sign. Comparing with the above short qualitative description of the EC method, one can see how Lagrange's and Dirichlet's theorems already contain the main ideas of the method: in the first place the identification of equilibria as critical points of a conserved quantity, and in the second place the condition of stability based on the definiteness of a quadratic form related to the second variation of the conserved

quantity, evaluated at the equilibrium of interest. These conditions ensure that, in phase space, the curves  $H(z) = \text{constant}$ , which correspond to the trajectories of the dynamics, are such that for any neighborhood  $N$  of the equilibrium point  $z_e$ , one can find a subneighborhood  $S \subset N$  such that, for initial conditions  $z(0) \in S$ , one has  $z(t) \in N$  for every  $t > 0$ , so that the trajectory never leaves  $N$  (see, for instance Ref. [17]). This corresponds to *nonlinear stability* of the equilibrium  $z_e$  [13]. For the case of fluid models of interest in this review, for which the phase space  $P$  is a normed space, the definition of nonlinear stability corresponds to the one provided later in Sec. 2.3. Given that the theorems by Lagrange and Dirichlet refer to canonical Hamiltonian systems, they rule out the presence of Casimir invariants, so that the free energy functional reduces, in this case, to the Hamiltonian. We also point out that, in the above formulations, Lagrange's and Dirichlet's theorems provide sufficient but not necessary conditions for stability. Nevertheless, with some additional assumptions on the differentiability of the potential energy  $V$ , Lagrange's theorem can be extended to yield both a necessary and sufficient condition for stability [13].

It is also important to mention the connection, for finite-dimensional systems, between the notion of formal stability expressed in Lagrange's, Dirichlet's and the EC method, and the second Lyapunov's method [18]. The latter states that, given a fixed point  $z_e$  of a finite-dimensional dynamical system  $\dot{z} = X(z)$  (Hamiltonian or not), if there exists a differentiable function  $V(z)$ , defined in a neighborhood  $W$  of  $z_e$ , such that

$$\begin{aligned} i) & V(z_e) = 0 \quad \text{and} \quad V(z) > 0, \quad \text{for } z \neq z_e, \quad \text{and} \\ ii) & \dot{V}(z) \leq 0 \quad \text{in} \quad W \setminus \{z_e\}, \end{aligned}$$

then  $z_e$  is stable (we omitted the part of the method concerning asymptotic stability, which is not relevant in the present Hamiltonian context). One can then see how, in finite dimension, the second Lyapunov's method generalizes Lagrange's, Dirichlet's theorems and the EC method, which correspond to the special case  $\dot{V}(z) = 0$ , that is when  $V$  is a constant of motion with a strict local minimum at the equilibrium point

(this corresponds, in particular, to positive definiteness of the free energy functional in the EC method).

An important breakthrough in the development of the EC method, in particular with regard to the application to fluids, came with the work of Arnol'd [19, 20, 21]. Indeed, the EC method is also referred to as "Arnol'd's method". We postpone the precise explanation of the method to Sec. 2 and here we would simply like to point out the connections with Lagrange's and Dirichlet's theorems. Indeed, in treating the two-dimensional (2D) Euler equation for an incompressible fluid, Arnol'd derived equilibria from extremizing a conserved functional given by the sum of the Hamiltonian of the system with other conserved quantities (Casimir invariants). Also, in order to obtain stability conditions, he required sign-definiteness of the second variation of the conserved functional, evaluated at the equilibrium of interest. A convexity argument was also introduced in order to overcome difficulties, due to the infinite-dimensional nature of the problem, in establishing nonlinear stability conditions. In infinite-dimensional systems, the above mentioned sign-definiteness, in general, is not a sufficient condition for an equilibrium to be a minimum of the free energy functional [14]. However, apart from the complications due to the infinite-dimensional nature of the fluid problem and to the existence of particular conserved quantities such the Casimir invariants, the spirit of the method is very similar to that of Dirichlet's (and, in turn, Lagrange's) theorem. Equilibria are indeed identified as critical points of a conserved quantity and the stability of such equilibria comes investigating the conserved quantity in a neighborhood of equilibrium points, in order to see whether these correspond to minima or maxima. This information requires the sign-definiteness of the second variation of the conserved functional at the equilibrium point in the fluid case and, analogously, the condition on the sign of the eigenvalues of the matrix  $\partial^2 H / \partial z^i \partial z^j |_{z=z_e}$  in Dirichlet's theorem.

A method related to the one conceived by Arnol'd for investigating fluid equilibria was used earlier in Ref. [22]. As will be discussed in Sec. 8, the so-called  $\delta W$  principle, formulated in Ref. [23] to provide stability conditions for static MHD equilibria, is also



intimately related to the EC method.

In the years following the work of Arnol'd, several applications of the method to plasma fluid models were developed. A considerable fraction of these were reviewed in Refs. [10, 13]. Other examples include application of the EC method to MHD equilibria with toroidal flows [24], Hall MHD [25] and nonlinear Alfvén waves in connection with spontaneous symmetry breaking [26]. Further References will be provided later in this paper.

The present review covers results obtained during essentially the last decade, where new applications emerged, in concomitance with the progress in the theoretical development of fluid modelling for plasmas, as well as with new motivations coming from experimental and observational results.

We proceed with describing the structure of the present review.

In order to make, at least to some extent, the exposition of the results self-contained, the review begins with Sec. 2, where the basics of the noncanonical Hamiltonian formalism for fluid models and of the EC method are recalled. In Sec. 3, in order to permit, the possibly unfamiliar Reader, a rather soft transition to the description of the most recent and complex results, we illustrate the application of the theory described in Sec. 2, to the relatively simple case of reduced MHD. In Sec. 4 we begin to review the most recent results. In such Section, in particular, we review the recent investigations providing the theory for equilibria and formal stability for MHD with a generic helical symmetry. Still in the MHD context, we consider the application of the EC method to extended MHD, a model which has recently regained interest also due to its possible applications to turbulence and magnetic reconnection. Extended MHD generalizes MHD by consistently including two-fluid effects such as Hall and electron inertia terms, while preserving a Hamiltonian structure. The equilibria and formal stability analysis of extended MHD is presented in Sec. 5. Coupling of MHD with kinetic equations has become, especially in recent years, an important tool for the description of energetic particles, for instance in tokamak plasmas. The Hamiltonian structure of hybrid MHD-

Vlasov models has been derived and the EC method applied to such models. In Sec. 6 we review the results of this application to a model characterized by a pressure coupling between the MHD and the kinetic particle species. Observation of coherent structures in the form of chain of magnetic islands in the solar wind motivated the stability analysis of such structures in the presence of electron temperature anisotropy. Formal stability conditions for such structures, in the cases of hot and cold ions, are discussed in Sec. 7, where plasma dynamics is described by means of a reduced fluid model. Section 8 on one hand, serves the purpose of deriving stability conditions for compressible reduced MHD (CRMHD) and, on the other hand, illustrates the connection between the EC method and the classical  $\delta W$  energy principle adopted in plasma physics. Furthermore, we recall how such results can be transferred to an analogous local model for thin accretion disks. A subject strictly related to the EC method is that of negative energy modes (NEMs), which are of importance for their potential of leading to instabilities in the presence of dissipation or nonlinearities. Their study in plasma physics has a long tradition and we review, in Sec. 9, their formulation in the Hamiltonian context and a more recent application to ETG driven instability. In Sec. 10 we review the bifurcation problem of Beltrami MHD equilibria and show how the classical tearing modes of plasma physics were recently given a new interpretation as singular equilibria coming from the extremization of the free energy functional for the linearized MHD system. We conclude in Sec. 11, where we also discuss some possible future directions.

## **2. Noncanonical Hamiltonian systems and the EC method**

As anticipated in Sec. 1, the EC method applies to systems possessing a noncanonical Hamiltonian structure, which is the case for fluid models for plasmas formulated from the Eulerian point of view. Before introducing the method, we find it then useful to recall the notion of noncanonical Hamiltonian field theory and some of its properties, which are relevant for the application of the EC method. The material presented in this Section is based, to a great extent, on Refs. [14, 13, 10], where further details can

also be found. Here, the exposition is reduced to the minimum necessary to make the present review relatively self-contained.

The Reader already familiar with basic elements of functional calculus and noncanonical Hamiltonian systems may skip Sec. 2 and go directly to Sec. 3.

### 2.1. Hamiltonian basics for fluid models

We consider a domain  $\mathcal{D} \subseteq \mathbb{R}^n$ , with  $n$  a positive integer, and a phase space  $P$ , consisting of the linear space of  $m$ -ples of fields  $\chi = (\chi^1(\mathbf{x}), \dots, \chi^m(\mathbf{x}))$ , where  $\chi^i : \mathcal{D} \rightarrow \mathbb{R}$ , for  $1 \leq i \leq m$ , and  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{D}$ . In the case of interest, i.e. fluid models,  $\mathcal{D}$  is the domain occupied by the fluid and we will typically consider a simply connected domain with  $n = 2$  or  $n = 3$  (although extensions to multiply connected domains are possible [10, 19] and such a case will be considered in Sec. 10). Once an additional dependence on the time parameter is introduced by the dynamics, the fields  $\chi^1, \dots, \chi^m$  will represent the dynamical variables evolving in the fluid models, such as, for instance, the mass density of the fluid, the components of velocity fields or of the magnetic field. We will assume that the fields  $\chi^i$  satisfy boundary conditions on  $\mathcal{D}$  such that boundary terms vanish when integrating by parts. This is accomplished, for instance, assuming periodic boundary conditions or vanishing boundary conditions for the fields and some of their derivatives, if required. We anticipate that, although the above described setting is sufficient for fluid models, for the case of hybrid kinetic-fluid models, treated in Sec. 6, the phase space will have to be extended to include distribution functions as dynamical field variables. In particular, the phase space  $P$  will be the linear space of  $m$ -ples of fields  $\chi = (\chi^1(\mathbf{x}), \dots, \chi^{m_f}(\mathbf{x}), \zeta^1(\mathbf{x}, \mathbf{p}), \dots, \zeta^{m_k}(\mathbf{x}, \mathbf{p}))$ , where the fields  $\zeta^1, \dots, \zeta^{m_k}$  are distribution functions depending on the space coordinates  $\mathbf{x}$  but also on the momenta  $\mathbf{p} = (p_1, \dots, p_{n'})$ , with  $n' \in \{1, 2, 3\}$ . The integer numbers  $m_f$  and  $m_k$  are such that  $m_f + m_k = m$  and they indicate the number of fluid and kinetic field variables, respectively. The kinetic fields  $\zeta^1, \dots, \zeta^{m_k}$  are defined on the domain  $\mathcal{D}_k = \{(\mathbf{x}, \mathbf{p}) : \mathbf{x} \in \mathcal{D}, \mathbf{p} \in \mathbb{R}^{n'}\}$  and are assumed to tend to zero faster than a polynomial

as  $p_i \rightarrow \infty$ , for  $i = 1, \dots, n'$ .

Because the extension of the subsequent notions of Hamiltonian theory for fluid models, to the hybrid case, is straightforward, the presentation will be carried out in the purely fluid setting. This will simplify the exposition at the expense of a little loss in generality.

In the treatment of Hamiltonian field theories for fluids, we will have to consider the set  $\Phi(P)$  of *observables* on the phase space  $P$ . More specifically, in our context, an observable  $G \in \Phi(P)$  is a functional  $G : P \rightarrow \mathbb{R}$  of the form

$$\begin{aligned} G(\chi^1, \dots, \chi^m) \\ = \int_D d^n x \Upsilon(\mathbf{x}, \chi^1, \dots, \chi^m, \partial^1 \chi^1, \dots, \partial^1 \chi^m, \partial^2 \chi^1, \dots, \partial^2 \chi^m, \dots, \partial^\alpha \chi^1, \dots, \partial^\alpha \chi^m, \dots) \end{aligned} \quad (3)$$

where  $\Upsilon$  is, in general, an operator, acting on the spatial coordinates  $\mathbf{x}$ , on the fields  $\chi^1, \dots, \chi^m$  and on all the derivatives of any order of the fields, with respect to  $x_1, \dots, x_n$ . In Eq. (3), in order to indicate all the derivatives of order  $\alpha$  of a function  $f$ , we adopted the multi-index notation  $\partial^\alpha f = \{(\partial^{\alpha_1}/\partial x_1^{\alpha_1})(\partial^{\alpha_2}/\partial x_2^{\alpha_2}) \dots (\partial^{\alpha_n}/\partial x_n^{\alpha_n})f \mid \alpha_1 + \alpha_2 + \dots + \alpha_n = \alpha\}$ , with  $\alpha, \alpha_1, \alpha_2, \dots, \alpha_n$  non-negative integers. The operator  $\Upsilon$  could also be of differential-integral type, which will be the case, for instance, of the Hamiltonian functional (54) of reduced MHD, where  $\phi[\omega]$  expresses the solution of a partial differential equation, depending on the field  $\omega$ , for the unknown function  $\phi$ .

The *first variation*  $\delta G$  of the functional  $G$ , at the point  $\chi = (\chi^1, \dots, \chi^m)$ , with respect to the variation  $\delta\chi = (\delta\chi^1, \dots, \delta\chi^m)$  is defined by

$$\begin{aligned} \delta G(\chi^1, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m) \\ = \lim_{\varepsilon \rightarrow 0} \frac{G(\chi^1 + \varepsilon\delta\chi^1, \chi^2, \dots, \chi^m) - G(\chi^1, \chi^2, \dots, \chi^m)}{\varepsilon} \\ + \lim_{\varepsilon \rightarrow 0} \frac{G(\chi^1, \chi^2 + \varepsilon\delta\chi^2, \dots, \chi^m) - G(\chi^1, \chi^2, \dots, \chi^m)}{\varepsilon} \\ + \dots + \lim_{\varepsilon \rightarrow 0} \frac{G(\chi^1, \chi^2, \dots, \chi^m + \varepsilon\delta\chi^m) - G(\chi^1, \chi^2, \dots, \chi^m)}{\varepsilon}. \end{aligned} \quad (4)$$

The variations  $\delta\chi^1, \dots, \delta\chi^m$  are functions  $\delta\chi^1, \dots, \delta\chi^m : \mathcal{D} \rightarrow \mathbb{R}$  determining the "direction" along which the functional  $G$  is varied.

For a fixed  $i$  such that  $1 \leq i \leq m$ , we will refer to the *functional derivative* of the functional  $G$  with respect to the field  $\chi^i(\mathbf{x})$ , as to the function (or, in general, the distribution)  $\delta G/\delta\chi^i(\mathbf{x})$  defined by the relation

$$\begin{aligned} & \int_D d^n x \frac{\delta G}{\delta\chi^i(\mathbf{x})} \delta\chi^i(\mathbf{x}) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{G(\chi^1(\mathbf{x}), \dots, \chi^i(\mathbf{x}) + \varepsilon\delta\chi^i(\mathbf{x}), \dots, \chi^m(\mathbf{x})) - G(\chi^1(\mathbf{x}), \dots, \chi^i(\mathbf{x}), \dots, \chi^m(\mathbf{x}))}{\varepsilon}, \end{aligned} \quad (5)$$

for all variations  $\delta\chi^i(\mathbf{x})$ .

We will assume that, in the cases under consideration, functional derivatives exist. In particular, this can enforce appropriate boundary conditions on the variations  $\delta\chi^1, \dots, \delta\chi^m$ . Also, in the definition (5) we indicated explicitly the dependence on the point  $\mathbf{x} \in \mathcal{D}$  but, in general, this will be omitted. In this respect, it is convenient to point out the case of functionals mapping fields to the value of the fields at one specific point. For instance, for a fixed  $i$ , this is the case of the functional

$$G(\chi^i)(\mathbf{x}) := \chi^i(\mathbf{x}) = \int_D d^n x' \delta(\mathbf{x}' - \mathbf{x}) \chi^i(\mathbf{x}'), \quad (6)$$

which associates to the field  $\chi^i$  its value at a fixed point  $\mathbf{x}$ . In the second step of Eq. (6), we cast the functional  $G$  in the form (3), with the help of the Dirac delta distribution  $\delta(\mathbf{x}' - \mathbf{x})$ . From the definition (5) it follows that

$$\frac{\delta G(\chi^i)(\mathbf{x})}{\delta\chi^j(\mathbf{x}')} = \delta_j^i \delta(\mathbf{x}' - \mathbf{x}), \quad j = 1, \dots, m, \quad (7)$$

where  $\delta_j^i$  is the Kronecker delta. In the left-hand side of Eq. (7), we explicitly indicated the dependence on  $\mathbf{x}$  and  $\mathbf{x}'$ , in order to emphasize the difference between the coordinate  $\mathbf{x}'$ , used in the integral representation of the functional  $G$ , and the fixed point  $\mathbf{x}$ , appearing in the definition of the functional.

We also recall that the expression (4) for the first variation, can be recast in a more geometric form as

$$\delta G(\chi; \delta\chi) = \left\langle \frac{\delta G}{\delta\chi}, \delta\chi \right\rangle. \quad (8)$$

In the expression (8),  $\delta\chi = (\delta\chi^1, \dots, \delta\chi^m) \in P$ , whereas we can identify  $\delta G/\delta\chi = (\delta G/\delta\chi^1, \dots, \delta G/\delta\chi^m)$  with an element of  $P'$ , which is the dual of the linear space  $P$  with respect to the pairing  $\langle, \rangle: P' \times P \rightarrow \mathbb{R}$  defined by

$$\langle f, g \rangle = \int_D d^n x \sum_{i=1}^m f^i g^i, \quad (9)$$

where  $f = (f^1, \dots, f^m) \in P'$  and  $g = (g^1, \dots, g^m) \in P$ .

A further operation that will play an important role in the EC method is the second variation. The first variation (4) is indeed again a functional of  $\chi$  and thus one can in turn take its first variation. The *second variation* of  $G$ , with respect to variations  $\delta\chi = (\delta\chi^1, \dots, \delta\chi^m)$  and  $\delta\chi' = (\delta\chi^{1'}, \dots, \delta\chi^{m'})$  is thus defined by

$$\begin{aligned} \delta^2 G(\chi^1, \dots, \chi^m; \delta\chi, \dots, \delta\chi^m, \delta\chi^{1'}, \dots, \delta\chi^{m'}) &= \\ &= \lim_{\eta \rightarrow 0} \frac{\delta G(\chi^1 + \eta\delta\chi^{1'}, \chi^2, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m) - \delta G(\chi^1, \chi^2, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m)}{\eta} \\ &+ \lim_{\eta \rightarrow 0} \frac{\delta G(\chi^1, \chi^2 + \eta\delta\chi^{2'}, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m) - \delta G(\chi^1, \chi^2, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m)}{\eta} \\ &+ \dots + \lim_{\eta \rightarrow 0} \frac{\delta G(\chi^1, \chi^2, \dots, \chi^m + \eta\delta\chi^{m'}; \delta\chi^1, \dots, \delta\chi^m) - \delta G(\chi^1, \chi^2, \dots, \chi^m; \delta\chi^1, \dots, \delta\chi^m)}{\eta} \end{aligned} \quad (10)$$

In particular, when  $\delta\chi' = \delta\chi$ , the second variation (that in this case, for brevity, we will denote as  $\delta^2 G(\chi; \delta\chi)$ ) can be seen as a quadratic form on vectors  $\delta\chi \in P$ .

In order to illustrate the above notions with an example, we consider a case in which  $n = 3$ ,  $\mathcal{D} = \mathbb{R}^3$ ,  $m = 2$  and the observable is  $G(\chi^1, \chi^2)$ , defined as

$$G(\chi^1, \chi^2) = \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left( (\chi^1)^2 \chi^2 + \left( \frac{\partial \chi^2}{\partial y} \right)^2 \right). \quad (11)$$

The functional  $G$  in Eq. (11) is of the form (3), with  $\Upsilon(\chi^1, \chi^2) = (\chi^1)^2 \chi^2 + (\partial \chi^2 / \partial y)^2$ , and  $\chi(x, y, z) = (\chi^1(x, y, z), \chi^2(x, y, z))$  is a 2-ple of fields (of no particular physical significance), depending on three Cartesian coordinates  $x, y$  and  $z$ .

Applying the definition (4) we have that the first variation of  $G$  is given by

$$\delta G(\chi^1, \chi^2; \delta\chi^1, \delta\chi^2) = \int_{\mathbb{R}^3} d^3x \left( \chi^1 \chi^2 \delta\chi^1 + \frac{(\chi^1)^2}{2} \delta\chi^2 - \frac{\partial^2 \chi^2}{\partial y^2} \delta\chi^2 \right) + \int_{\mathbb{R}^2} dx dz \left[ \frac{\partial \chi^2}{\partial y} \delta\chi^2 \right]_{y=-\infty}^{y=+\infty}, \quad (12)$$

where integrations by parts with respect to  $y$  were carried out. Making use of the boundary conditions  $\delta\chi^2 \rightarrow 0$  as  $y \rightarrow \pm\infty$ , the expression for the first variation reduces to

$$\delta G(\chi^1, \chi^2; \delta\chi^1, \delta\chi^2) = \int_{\mathbb{R}^3} d^3x \left( \chi^1 \chi^2 \delta\chi^1 + \left( \frac{(\chi^1)^2}{2} - \frac{\partial^2 \chi^2}{\partial y^2} \right) \delta\chi^2 \right). \quad (13)$$

From the definition (5), it also follows that the functional derivatives of  $G$  are given by

$$\frac{\delta G}{\delta \chi^1} = \chi^1 \chi^2, \quad \frac{\delta G}{\delta \chi^2} = \frac{(\chi^1)^2}{2} - \frac{\partial^2 \chi^2}{\partial y^2}. \quad (14)$$

On the other hand, from the definition (10), it follows that the second variation of  $G$  is given by

$$\begin{aligned} & \delta^2 G(\chi^1, \chi^2; \delta\chi^1, \delta\chi^2, \delta\chi^{1'}, \delta\chi^{2'}) \\ &= \int_{\mathbb{R}^3} d^3x \left( \chi^2 \delta\chi^1 \delta\chi^{1'} + \chi^1 \delta\chi^2 \delta\chi^{1'} + \chi^1 \delta\chi^1 \delta\chi^{2'} + \frac{\partial \delta\chi^2}{\partial y} \frac{\partial \delta\chi^{2'}}{\partial y} \right), \end{aligned} \quad (15)$$

where, again, integration by parts, assuming vanishing boundary terms, were carried out.

## 2.2. Hamiltonian field theories for fluids

We introduce the dependence on the time parameter  $t$  and consider curves in phase space,  $\chi(t; \mathbf{x}) : \mathbb{R} \rightarrow P$ , for  $\mathbf{x} \in \mathcal{D}$ . The curve  $\chi(t; \mathbf{x})$  is associated with an initial condition at  $t = 0$ , by  $\chi(0; \mathbf{x}) = \chi_0(\mathbf{x})$  for some  $\chi_0 \in P$ . The semicolon in the argument of  $\chi(t; \mathbf{x})$  is meant to emphasize that, in the dynamical system approach that we are following, the spatial variable  $\mathbf{x}$  is treated as a parameter labelling elements of the phase space.

In this context, a field theory is an infinite-dimensional dynamical system of the form

$$\dot{\chi}(t; \mathbf{x}) = X(\chi(t; \mathbf{x})), \quad (16)$$

or, in components,

$$\dot{\chi}^i(t; \mathbf{x}) = X^i(\chi^1(t; \mathbf{x}), \dots, \chi^m(t; \mathbf{x})), \quad i = 1, \dots, m, \quad (17)$$

for all  $\mathbf{x} \in \mathcal{D}$ . In Eq. (17) the dot indicates the derivative with respect to  $t$  and  $X : P \rightarrow P$  is a vector field mapping the phase space  $P$  (or a subset of it) to  $P$ . In the following, we will often omit the explicit dependence on  $t$  and  $\mathbf{x}$  of the curves in phase space which are solutions of the system. We remark that, although fluid models for plasmas are typically presented as systems of partial differential equations, with  $t$  and  $\mathbf{x}$  as independent variables, for the approach followed in our context, it is natural to reformulate such models as infinite-dimensional dynamical systems of the form (17).

We will say that a field theory of the form (17) possesses a *Hamiltonian structure* if it exists an observable  $H \in \Phi(P)$ , referred to as *Hamiltonian* (functional), and a *Poisson bracket*  $\{, \} : \Phi(P) \times \Phi(P) \rightarrow \Phi(P)$  such that the system can be written in the form

$$\dot{\chi}^i(t; \mathbf{x}) = \{ \chi^i(t; \mathbf{x}), H(\chi^1(t; \mathbf{x}), \dots, \chi^m(t; \mathbf{x})) \}, \quad i = 1, \dots, m. \quad (18)$$

From the physical point of view, as is well known, the Hamiltonian functional typically corresponds to the total energy of the system.

On the other hand, we recall that a Poisson bracket is an operator satisfying the following four properties:

- (i) bilinearity:  $\{F, G + K\} = \{F, G\} + \{F, K\}$ ,  $\{F + G, K\} = \{F, K\} + \{G, K\}$ ,  
 $\{\lambda F, G\} = \{F, \lambda G\} = \lambda\{F, G\}$ ,
- (ii) antisymmetry:  $\{F, G\} = -\{G, F\}$ ,
- (iii) Leibniz identity:  $\{FG, K\} = F\{G, K\} + \{F, K\}G$ ,
- (iv) Jacobi identity:  $\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$ ,

where  $\lambda$  is a constant and  $F, G, K \in \Phi(P)$  are generic observables.

Poisson brackets that will be relevant in our context will be those of the form

$$\{F, G\}(\chi^1, \dots, \chi^m) = \int_D d^m x \frac{\delta F}{\delta \chi^i} \mathcal{J}^{ij}(\chi^1, \dots, \chi^m) \frac{\delta G}{\delta \chi^j}, \quad (19)$$

where the sum over repeated indices  $i$  and  $j$  is understood. In Eq. (19),  $\mathcal{J}$  is an operator denoted as *cosymplectic form*. The antisymmetry property and the Jacobi identity, in



particular, constrain the admissible expressions for  $\mathcal{J}$ . We will not provide proofs that the Poisson brackets considered in this review satisfy the properties (i)–(iv), and for this matter, we refer the Reader to the corresponding original References. Nevertheless, as far as the Jacobi identity (which is in general the most difficult property to prove, among the four above) is concerned, we find it useful to mention two additional References. In Ref. [27], a useful Lemma is proved, showing that, if antisymmetry holds, in the quantity  $\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\}$  the contribution of the second order functional derivatives vanish. Thus, in evaluating the functional derivative of terms of the form  $\{G, K\}$ , one only has to consider the functional derivatives of the cosymplectic form. Reference [28], on the other hand, provides a practical procedure for proving the Jacobi identity for Poisson brackets of reduced fluid models, which are linear in the field variables, as for instance the bracket (55). In this case, the proof of the Jacobi identity can be reduced to the verification of commutation relations among matrices.

Using the expression (6) to rewrite  $\chi^i(t; \mathbf{x})$  on the right-hand side of Eq. (18), and making use also of the expression (19) for the Poisson bracket, we obtain that the Hamiltonian system (18) can be written as

$$\dot{\chi}^i(t; \mathbf{x}) = \mathcal{J}^{ij}(\chi^1(t; \mathbf{x}), \dots, \chi^m(t; \mathbf{x})) \frac{\delta H}{\delta \chi^j(t; \mathbf{x})}, \quad i = 1, \dots, m. \quad (20)$$

It can be shown that, for generic Hamiltonian systems, and in particular for Hamiltonian field theories systems of the form (18) (or, equivalently, of the form (20)), one has that every smooth observable  $F \in \Phi(P)$  evolves in time according to

$$\dot{F} = \{F, H\}. \quad (21)$$

Choosing, in particular,  $F = H$  in Eq. (21), implies  $\dot{H} = 0$ , by virtue of the antisymmetry of the Poisson bracket. This obviously amounts to the conservation of total energy in Hamiltonian systems.

On the other hand, further invariants can be inferred from the Poisson bracket. Indeed, if one considers observables  $C \in \Phi(P)$  such that

$$\{C, F\} = 0, \quad \forall F \in \Phi(P), \quad (22)$$

then, in particular  $\{C, H\} = 0$  so that, from Eq. (21), one has

$$\dot{C} = 0. \quad (23)$$

Observables  $C$  satisfying the property (22), are also conserved along the flow of Hamiltonian systems, and are referred to as *Casimir invariants*. In particular, the dynamics takes place on surfaces in the phase space, where the values of the Casimir invariants are constant and fixed by the initial conditions.

Casimir invariants play a crucial role in the stability method that we consider in this review. It is important to point out that non-trivial Casimir invariants only appear in Hamiltonian systems with a *noncanonical* Hamiltonian structure. In noncanonical Hamiltonian systems indeed it is not possible to identify globally on the whole phase space  $P$ , pairs of canonically conjugate dynamical variables, unlike what happens in the canonical Hamiltonian systems typically encountered in classical mechanics. In the specific context of fluid field theories considered here, canonical Hamiltonian systems occur when the number of fields  $m$  is even and the Hamiltonian system can be written in the form

$$\dot{\chi}^i = \frac{\delta H}{\delta \chi^{\frac{m}{2}+i}}, \quad \dot{\chi}^{\frac{m}{2}+i} = -\frac{\delta H}{\delta \chi^i}, \quad i = 1, \dots, \frac{m}{2}. \quad (24)$$

In this case, for a generic point  $\chi = (\chi^1, \dots, \chi^m)$  of the phase space, the field  $\chi^i$ , is canonically conjugated to the field  $\chi^{\frac{m}{2}+i}$ , with  $i = 1, \dots, m/2$ .

The corresponding Poisson bracket reads

$$\{F, G\} = \int_D d^n x \sum_{i=1}^{m/2} \left( \frac{\delta F}{\delta \chi^i} \frac{\delta G}{\delta \chi^{\frac{m}{2}+i}} - \frac{\delta F}{\delta \chi^{\frac{m}{2}+i}} \frac{\delta G}{\delta \chi^i} \right), \quad (25)$$

and the  $m \times m$  matrix, associated with the corresponding cosymplectic form, is given by

$$\mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (26)$$

where we indicated with  $I$  the  $(m/2) \times (m/2)$  identity matrix. Note, in particular, that, in this case, the cosymplectic form does not depend on  $\chi$ . Applying the definition of Casimir invariant (22), to the canonical Poisson bracket (25), one finds that, in this

case, Casimir invariants have to satisfy  $\delta C/\delta\chi^i = 0$ , for  $i = 1, \dots, m$ . Considering observables of the form (3), it follows that the only Casimir invariants in the canonical case are the trivial ones, corresponding to constants.

We finally recall that the relation (22), defining Casimir invariants of a Poisson bracket, is clearly associated with the properties of the cosymplectic form. In particular, Casimir invariants satisfy

$$\mathcal{J}^{ij}(\chi) \frac{\delta C}{\delta\chi^j} = 0, \quad i = 1, \dots, m, \quad (27)$$

and are thus clearly related to the kernel of the cosymplectic form  $\mathcal{J}$ . Because of the dependence of  $\mathcal{J}$  on the phase space coordinates  $\chi$ , the dimension of the kernel of the cosymplectic form may change as  $\chi$  varies. This gives rise to different kinds of solutions depending on what portion of the phase space one is considering. This issue becomes particularly relevant and intricate namely in infinite-dimensional systems. For nonlinear systems, singular Casimir elements can appear, as discussed in Ref. [29], with focus on the Euler equation for an incompressible fluid, and in Ref. [30], where an application to 2D MHD is treated. In the present context, one also remarks the presence of singular elements (hyperfunctions) in the kernel of the cosymplectic form for linearized systems, which is connected with the appearance of tearing modes, as will be discussed in Sec. 10.

### 2.3. Different types of stability

Here we summarize, following essentially Refs. [10, 13], different notions of stability relevant for our purposes, and the relations between them. We will specialize to the case of field theories (although not necessarily Hamiltonian, at this stage) of the form (17), although the concepts can of course be applied to a much wider class of dynamical systems.

We begin by recalling that an *equilibrium* point  $\chi_e(\mathbf{x}) = (\chi_e^1(\mathbf{x}), \dots, \chi_e^m(\mathbf{x})) \in P$  for a system of the form (17) is an element of the phase space such that

$$X^i(\chi_e^1, \dots, \chi_e^m) = 0, \quad i = 1, \dots, m. \quad (28)$$

An equilibrium point can be stable according to different notions of stability. Here we take into account the following ones:

*Spectral stability.* We consider the linearization of the system (16) about an equilibrium point  $\chi_e$ , obtained by setting  $\chi(t; \mathbf{x}) = \chi_e(\mathbf{x}) + \epsilon \delta\chi(t; \mathbf{x})$  and retaining only terms linear in the parameter  $\epsilon$ . The resulting linear system can be written, in compact form, as

$$\delta\dot{\chi} = DX(\chi_e) \cdot \delta\chi. \quad (29)$$

The system (29) is formulated on the phase space  $P$  consisting of  $m$ -ples of perturbations  $\delta\chi(\mathbf{x}) = (\delta\chi^1(\mathbf{x}), \dots, \delta\chi^m(\mathbf{x}))$  with the appropriate boundary conditions on  $\mathcal{D}$ . Solutions of the system (29) correspond to curves  $\delta\chi(t; \mathbf{x}) = (\delta\chi^1(t; \mathbf{x}), \dots, \delta\chi^m(t; \mathbf{x}))$  associating an element of  $P$  to every time  $t$  at any given point  $\mathbf{x} \in \mathcal{D}$ .

The equilibrium point  $\chi_e$  is said to be spectrally stable if the spectrum of the linear operator  $DX(\chi_e)$  has no strictly positive real part. In particular, if the system (29) is Hamiltonian, spectral stability corresponds to *neutral* stability, i.e. to the case where the spectrum of  $DX(\chi_e)$  has purely imaginary eigenvalues. In this case, the amplitude of the perturbation  $\delta\chi$  oscillates in time.

*Linear stability.* The equilibrium  $\chi_e$  is said to be linearly stable if it is possible to find a norm  $\|\cdot\| : P \rightarrow \mathbb{R}$ , such that, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, if  $\|\delta\chi(0; \mathbf{x})\| < \delta$ , then  $\|\delta\chi(t; \mathbf{x})\| < \epsilon$  for all  $t > 0$ . Here,  $\delta\chi(t; \mathbf{x})$  indicates solutions of the *linearized* system (29).

*Formal (or energy) stability.* An equilibrium point  $\chi_e$  of the system (17) is said to be formally (or, equivalently, energy) stable if there exists a constant of motion  $F(\chi)$  such that

$$\delta F(\chi_e; \delta\chi) = 0, \quad \forall \delta\chi \in P, \quad (30)$$

and

$$\delta^2 F(\chi_e; \delta\chi, \delta\chi) > 0 \quad \text{or} \quad < 0, \quad \forall \delta\chi \in P \setminus \{0\}. \quad (31)$$

In other words, formal stability means that there exists a constant of motion  $F : P \rightarrow \mathbb{R}$  satisfying the following two properties: its first variation  $\delta F$  vanishes at the equilibrium

point, for all admissible perturbations  $\delta\chi$  (strictly speaking, the admissible perturbations  $\delta\chi$  might belong to a subset  $V \subset P$ , consisting of perturbations satisfying boundary conditions in such a way that functional derivatives exist, as discussed in Sec. 2.1). Also, the quadratic form,  $\delta^2 F(\chi_e; \delta\chi, \delta\chi)$ , associated with the second variation evaluated at equilibrium, must be positive (or negative) definite, for all admissible perturbations excluded the zero perturbation  $\delta\chi = 0$ .

*Nonlinear stability.* An equilibrium  $\chi_e$  of the system (17) is nonlinearly stable (or, simply, stable) if there exists a norm  $\| \cdot \| : P \rightarrow \mathbb{R}$  such that, for every  $\epsilon > 0$ , there exists a  $\delta > 0$ , so that, if  $\|\chi(0; \mathbf{x}) - \chi_e(\mathbf{x})\| < \delta$ , then  $\|\chi(t; \mathbf{x}) - \chi_e(\mathbf{x})\| < \epsilon$  for all  $t > 0$ . In this case,  $\chi(t; \mathbf{x})$  is a solution of the system (17), which is, in general, a nonlinear system. This definition corresponds, in a normed space, to the definition of Lyapunov stability usually provided for a space with a given topology (see, e.g. Ref. [31] for the finite-dimensional case). Comparing the definition of nonlinear stability with the above definition of linear stability, one can see that linear stability amounts to nonlinear stability of the linearized system.

The above four types of stability are connected by implication relations that we summarize here:

- linear stability implies spectral stability but the converse is not generally true,
- formal stability implies linear stability but the converse is not generally true,
- nonlinear stability implies spectral stability but the converse is not generally true,
- formal stability does not generally imply nonlinear stability (it does in finite dimension).

These implications are discussed, with various counterexamples, as well as historical remarks, in Refs. [13, 10, 14]. In particular, we find it useful to emphasize that, contrary

to what one might expect, we have that

- nonlinear stability *does not* imply linear stability.

In order to see this, one can consider the canonical Hamiltonian system with Hamiltonian  $H(q, p) = p^2/2 + q^4/4$ . This Hamiltonian is of separable form and, by Lagrange's theorem, one can conclude that the equilibrium point  $(q_e, p_e) = (0, 0)$  (which is a critical point of the Hamiltonian) is nonlinearly stable, because the point  $q = 0$  is a local minimum of  $V(q) = q^4/4$ . On the other hand, the equations of motion of the system, linearized about  $(q, p) = (0, 0)$ , read

$$\delta\dot{q} = \delta p,$$

$$\delta\dot{p} = 0.$$

The solution of this system is given by  $\delta q(t) = \delta p(0)t + \delta q(0)$ ,  $\delta p(t) = \delta p(0)$ . It is thus clear that, given its linear dependence on  $t$ , the trajectory of  $\delta q(t)$  (if  $\delta p(0) \neq 0$ ) will leave, after a sufficiently long time, any neighborhood of the equilibrium point at the origin, thus preventing linear stability.

#### 2.4. The EC method

We describe here the EC method applied to Hamiltonian field theories, which will play a central role for the stability analyses considered in this review. We consider a noncanonical Hamiltonian field theory in the form (18) and suppose that the corresponding Poisson bracket  $\{, \}$  possesses  $M$  independent Casimir invariants denoted as  $C^1, \dots, C^M$ . Because of the property (27), combined with Eq. (20), the equations of motion of the system can be written as

$$\dot{\chi}^i = \mathcal{J}^{ij} \frac{\delta \mathcal{F}}{\delta \chi^j}, \quad i = 1, \dots, m. \quad (32)$$

where  $\mathcal{F} : P \rightarrow \mathbb{R}$  is an observable, sometimes denoted as *free energy functional*, defined by

$$\mathcal{F} = H + \sum_{\alpha=1}^M \lambda_{\alpha} C^{\alpha}, \quad (33)$$

where  $\lambda_1, \dots, \lambda_M$  are arbitrary constants. Because both the Hamiltonian and the Casimir invariants are preserved by the dynamics, also  $\mathcal{F}$  is. In particular, the additional contributions of the Casimir invariants do not modify the dynamics generated by the Hamiltonian  $H$  and the Poisson bracket  $\{, \}$ . We also point out that, in principle, constants of motion other than the Casimir invariants can be added to build the free energy functional. In the case of noncanonical Hamiltonian systems, the choice of the Casimir invariants follows naturally, as these can be derived from the noncanonical Poisson bracket, according to the definition (22).

From the definition (28) and from Eq. (32), it follows that points in phase space  $\chi_e = (\chi_e^1, \dots, \chi_e^m)$ , satisfying

$$\left. \frac{\delta \mathcal{F}}{\delta \chi^i} \right|_{\chi=\chi_e} = 0, \quad i = 1, \dots, m, \quad (34)$$

are equilibria for the Hamiltonian system (18). Because, from Eq. (8), one has that

$$\delta \mathcal{F}(\chi^1, \dots, \chi^m; \delta \chi^1, \dots, \delta \chi^m) = \int_D d^n x \sum_{i=1}^m \frac{\delta \mathcal{F}}{\delta \chi^i} \delta \chi^i, \quad (35)$$

it follows that equilibria  $\chi_e$  satisfying Eq. (34) are points at which the *first variation of  $\mathcal{F}$  vanishes*, for all perturbations  $\delta \chi = (\delta \chi^1, \dots, \delta \chi^m)$ , i.e.

$$\delta \mathcal{F}(\chi_e; \delta \chi) = 0, \quad (36)$$

for all  $\delta \chi \in P$ . This generalizes, to the noncanonical case, the property of equilibria of canonical Hamiltonian systems of being extremal points of the Hamiltonian. In the noncanonical case, the relations (34) determining the equilibria, are enriched by the presence of the Casimir invariants. On the other hand, we point out that *not all* the equilibria of a noncanonical Hamiltonian field theory can be found by solving the system (34). Plasma physics, in particular, can provide examples where the solutions of

Eqs. (34) do not exhaust all the possible equilibria of the system. We also anticipate, that unlike the case (33), where we considered a finite number of Casimir invariants, fluid models can possess infinite families of Casimir invariants, depending on arbitrary functions (this is typically the case when the system possesses a translational symmetry along one coordinate). In such cases, the arbitrary constants  $\lambda_\alpha$ , will be omitted.

If an equilibrium  $\chi_e$  satisfies the relation (36) and the second variation of  $\mathcal{F}$  is such that

$$\delta^2 \mathcal{F}(\chi_e; \delta\chi, \delta\chi) > 0 \quad \text{or} \quad < 0, \quad (37)$$

for all  $\delta\chi \in P \setminus \{0\}$ , then the equilibrium  $\chi_e$  is formally (or energy) stable. To verify this, it is sufficient to compare the relations (36) and (37) with the definition of formal stability given in Sec.2.3. The constant of motion  $F$ , in the case of the EC method, is provided by the free energy functional  $\mathcal{F}$ .

As anticipated in Sec. 1, in the present review we will consider results related to the analysis of formal stability carried out by means of the procedure that we just finished to describe. However, in its full formulation, the EC method provides also a procedure for determining conditions for nonlinear stability. The steps required for determining nonlinear stability originated namely in the context of hydrodynamic stability, in particular in Refs. [19, 20]. For completeness, we find it appropriate to summarize here also the additional steps used for nonlinear stability and briefly comment about possible related limitations. These steps consist, first, in the identification of two quadratic functionals  $Q_1$  and  $Q_2$  such that

$$Q_1(\Delta\chi) \leq H(\chi_e + \Delta\chi) - H(\chi_e) - \delta H(\chi_e; \Delta\chi) \quad (38)$$

and

$$Q_2(\Delta\chi) \leq C(\chi_e + \Delta\chi) - C(\chi_e) - \delta C(\chi_e; \Delta\chi), \quad (39)$$

for all  $\Delta\chi$ , where  $\Delta\chi = \chi - \chi_e \in P$  is a finite perturbation of the equilibrium  $\chi_e$ . Note that we are considering here the case of a free energy functional  $\mathcal{F} = H + C$ , but the extension to the general form (33) is straightforward.



We recall that sufficient condition for (38) and (39) to hold, are given by

$$Q_1(v) \leq \delta^2 H(u; v), \quad (40)$$

and

$$Q_2(v) \leq \delta^2 C(u; v), \quad (41)$$

respectively, for all  $u$  and  $v$  in  $P$ .

Then one defines the norm

$$\|\Delta\chi\|^2 = Q_1(\Delta\chi) + Q_2(\Delta\chi), \quad (42)$$

by imposing

$$Q_1(\Delta\chi) + Q_2(\Delta\chi) > 0, \quad (43)$$

for all  $\Delta\chi \neq 0$ . Note that, due to Eq. (36), one has  $\delta H(\chi_e; \Delta\chi) + \delta C(\chi_e; \Delta\chi) = 0$ , and thus

$$Q_1(\Delta\chi) + Q_2(\Delta\chi) \leq H(\chi_e + \Delta\chi) + C(\chi_e + \Delta\chi) - H(\chi_e) - C(\chi_e). \quad (44)$$

Moreover, for a solution  $\chi(t; \mathbf{x})$ , upon defining  $\Delta\chi(t; \mathbf{x}) = \chi(t; \mathbf{x}) - \chi_e(\mathbf{x})$ , one has

$$\begin{aligned} \|\Delta\chi(t; \mathbf{x})\|^2 &= Q_1(\Delta\chi(t; \mathbf{x})) + Q_2(\Delta\chi(t; \mathbf{x})) \\ &\leq H(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) + C(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) \\ &\quad - H(\chi_e(\mathbf{x})) - C(\chi_e(\mathbf{x})) \\ &= \mathcal{F}(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x})), \end{aligned} \quad (45)$$

at any time  $t$ . In Eq. (45) we made use of the fact that  $H$  and  $C$  are constants of motion, and thus their values at a time  $t$  are equal to their value at the time  $t = 0$ . We also voluntarily resorted to a heavier notation in these steps, by indicating the explicit dependence on the spatial and temporal coordinate, the latter in particular, being crucial at this stage. If the solution  $\chi(t; \mathbf{x})$  exists for all time  $t > 0$  and the free energy functional  $\mathcal{F}$  is continuous at  $\chi_e$ , with respect to the distance in  $P$  induced by the norm  $\|\cdot\|$ , then the equilibrium  $\chi_e$  is nonlinearly stable. To see this, we first recall that, according to the definition of nonlinear stability provided in Sec. 2.3, the equilibrium

$\chi_e$  is nonlinearly stable if, for every  $\epsilon > 0$ , there exists  $\delta$  such that, if  $\|\Delta\chi(0; \mathbf{x})\| < \delta$ , then  $\|\Delta\chi(t; \mathbf{x})\| < \epsilon$  for all  $t > 0$ . By the hypothesis of continuity of  $\mathcal{F}$  at  $\chi_e$ , one has that, for every  $\bar{\epsilon} = \epsilon^2 > 0$ , there exists  $\bar{\delta} > 0$  such that, for  $\chi \in P$ , if  $\|\chi(\mathbf{x}) - \chi_e(\mathbf{x})\| < \bar{\delta}$ , then  $|\mathcal{F}(\chi(\mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x}))| < \bar{\epsilon}$ . Thus, if one chooses  $\delta = \bar{\delta}$ , so that the initial condition  $\chi(0)$  is sufficiently close to  $\chi_e$  (i.e.  $\|\chi(0; \mathbf{x}) - \chi_e(\mathbf{x})\| < \bar{\delta}$ ) then, using the relation (45), it follows that

$$\begin{aligned} \|\Delta\chi(t; \mathbf{x})\|^2 &\leq \mathcal{F}(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x})) = \mathcal{F}(\chi(0; \mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x})) \\ &\leq |\mathcal{F}(\chi(0; \mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x}))| < \epsilon^2, \end{aligned} \quad (46)$$

for all  $t > 0$ , which implies nonlinear stability of  $\chi_e$ .

We recall that a sufficient condition for the continuity of  $\mathcal{F}$  at  $\chi_e$  is the existence of two constants  $C_1$  and  $C_2$  such that

$$H(\chi_e + \Delta\chi) - H(\chi_e) - \delta H(\chi_e; \Delta\chi) \leq C_1 \|\Delta\chi\|^2, \quad (47)$$

$$C(\chi_e + \Delta\chi) - C(\chi_e) - \delta C(\chi_e; \Delta\chi) \leq C_2 \|\Delta\chi\|^2. \quad (48)$$

Combining the relations (42), (44), (47) and (48) one indeed obtains

$$\begin{aligned} \|\Delta\chi(t; \mathbf{x})\|^2 &= Q_1(\Delta\chi(t; \mathbf{x})) + Q_2(\Delta\chi(t; \mathbf{x})) \\ &\leq H(\chi_e(\mathbf{x}) + \Delta\chi(t; \mathbf{x})) + C(\chi_e(\mathbf{x}) + \Delta\chi(t; \mathbf{x})) - H(\chi_e(\mathbf{x})) - C(\chi_e(\mathbf{x})) \\ &= H(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) + C(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) - H(\chi_e(\mathbf{x})) - C(\chi_e(\mathbf{x})) \\ &= \mathcal{F}(\chi_e(\mathbf{x}) + \Delta\chi(0; \mathbf{x})) - \mathcal{F}(\chi_e(\mathbf{x})) \leq (C_1 + C_2) \|\Delta\chi(0; \mathbf{x})\|^2, \end{aligned} \quad (49)$$

expressing how the distance of the solution at any time  $t$ , as measured by  $\|\Delta\chi(t; \mathbf{x})\|^2$ , is controlled by a constant depending on the initial condition.

We remark that, requiring the sufficient conditions (40)-(41), together with the inequality (43), implies

$$\delta^2 \mathcal{F}(u; v) \geq Q_1(v) + Q_2(v) > 0, \quad (50)$$

for all  $u, v \in P$ . This condition is stronger than the condition of formal stability. Therefore, although in general nonlinear stability does not imply formal stability, when applying the EC method to infinite-dimensional systems, it can happen that the

resulting conditions for nonlinear stability are stronger than those on formal stability. In such cases, also linear stability is thus implied.

Due to the general scarcity of results on the existence and uniqueness of solutions of fluid systems for plasmas, the results on nonlinear stability obtained from the EC method are, most of the times, of conditional type. The solution is assumed to exist up to a certain time  $T_*$  and the stability condition holds for  $t \leq T_*$ . Also, the nonlinear stability criterion, as above seen, depends on the norm, which implies that nonlinear stability might be ascertained with respect to one norm but not with respect to another. Some limitations due to the application of the EC method for nonlinear stability were pointed out, in the case of models for ordinary fluids, in Refs. [32] and [33]. In particular, in the former Reference, it is pointed out, in the framework of a quasi-geostrophic model, that, in the presence of boundary conditions with a continuous symmetry, nonlinearly stable flows must also possess such symmetry. In the latter Reference, it was shown that no non-trivial nonlinearly stable equilibria of the 2D incompressible Euler equation can be found by means of the EC method in the two following cases: when the domain is topologically equivalent to a sphere or when the domain is bounded, simply connected and the flow possesses zero net vorticity. A relevant example which is not ruled out by this limitation is that of the so-called Kelvin-Stuart "cat's eyes" flow, whose nonlinear stability is investigated in Ref. [34].

In the context of plasmas, we mention that the EC method has been applied to derive sufficient conditions, for nonlinear stability for MHD flows with symmetry, in Refs. [35, 10] and in a number of other References, although not making explicit use of the noncanonical Hamiltonian structure, as in, e.g., Refs. [36, 37, 38, 39].

### 3. An introductory example : 2D reduced MHD

Although not belonging to the period of time covered in general by the present review, we think that the stability analysis of reduced MHD, described in Ref. [35], might serve as an illustrative example for the application of the EC method.

The system of reduced MHD was derived in Refs. [40, 41] for describing low-frequency plasma dynamics in the presence of a strong guide field, the latter mimicking the intense toroidal component of the magnetic field in tokamaks. We are considering here, in particular, the low- $\beta$  version of reduced MHD, where  $\beta$  is the ratio between the kinetic plasma pressure and the magnetic pressure associated with the guide field. Also, we consider the translationally symmetric case in the cylindrical approximation, valid for large aspect-ratio tokamaks. Therefore the dynamical variables are functions of only two cylindrical coordinates  $r$  and  $\theta$ , describing the cross-section of the tokamak, and no dependence on the  $z$  coordinate, associated with the direction of the guide field, is assumed. Although the original Ref. [35] treated the helically symmetric case, this slight modification in the geometry of the system, could help in building more easily the connections with the cases that will be treated in the next Sections. As will become clear also with the cases treated later, the EC method is indeed effective in particular for systems possessing some spatial symmetry. In this case, a greater (typically infinite) number of Casimir invariants is expected. Considering that, as explained in Sec. 2.4, Casimir invariants are of help for identifying equilibria, if a great number of Casimir invariants is available, a large class of equilibria can, in general, be analyzed, thus making the method particularly fruitful.

Given the above considerations, the corresponding system of reduced MHD equations, in normalized Alfvénic variables, reads

$$\frac{\partial \psi}{\partial t} = -[\phi, \psi]_r, \quad (51)$$

$$\frac{\partial \omega}{\partial t} = -[\phi, \omega]_r + [\psi, \Delta_{\perp} \psi]_r. \quad (52)$$

We proceed with situating Eqs. (51)-(52) in the general framework depicted in Sec. 2.2. We first note that we are here in the case  $m = 2$  and  $n = 2$ . Indeed, we identify two dynamical field variables corresponding to the magnetic flux function  $\chi^1 = \psi(r, \theta, t)$  and to the vorticity  $\chi^2 = \omega(r, \theta, t) = \Delta_{\perp} \phi(r, \theta, t)$ , where  $r$  and  $\theta$  are two of the coordinates of a cylindrical coordinate system  $(r, \theta, z)$ . We also indicated with the symbol  $\Delta_{\perp}$  the "perpendicular" Laplacian operator, which does not contain derivatives

with respect to the guide field coordinate, and which in cylindrical coordinates is defined as  $\Delta_{\perp}f = (\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_{\theta}^2)f$ , for a function  $f$ . The dynamical field variables do not depend on  $z$ , reflecting the fact that the system, as above anticipated, does not vary along the direction of the toroidal guide field, identified by the unit vector  $\hat{z}$ . We consider the dynamical field variables  $\psi$  and  $\Delta_{\perp}\phi$  on a domain  $\mathcal{D} = \{(r, \theta) : 0 \leq r \leq R, 0 \leq \theta < 2\pi\}$ , with  $R > 0$  and assuming periodicity in  $\theta$ , so that  $\psi(r, \theta + 2\pi, t) = \psi(r, \theta, t)$  and  $\phi(r, \theta + 2\pi, t) = \phi(r, \theta, t)$ . We also impose  $\phi(0, \theta) = \psi(0, \theta) = 0$ , with  $\phi, \psi = O(r)$  as  $r \rightarrow 0$ , and  $\phi(R, \theta) = \psi(R, \theta) = 0$ . The elements of the phase space  $P$  consist then of 2-ples  $(\psi, \omega)$ , where the functions  $\psi, \omega : \mathcal{D} \rightarrow \mathbb{R}$  satisfy the boundary conditions we just described. The fields  $\psi$  and  $\phi$  are associated with  $\mathbf{B}_{\perp}$ , which is the planar magnetic field perpendicular to the guide field direction, and with the plasma single-fluid velocity  $\mathbf{v}$ , by the relations  $\mathbf{B}_{\perp} = \nabla\psi \times \hat{z}$  and  $\mathbf{v} = \hat{z} \times \nabla\phi$ , respectively (note that, in order to facilitate the comparison with other models that will be presented in this review, we changed a sign convention, so that  $\psi$  in the present review corresponds to  $-\psi$  of Ref. [35]). We also point out that these relations are actually exact only up to higher order terms, such as the perturbations along the toroidal direction, which are neglected in the reduced MHD ordering.

In Eqs. (51), (52), we introduced the canonical bracket

$$[f, g]_r = \frac{1}{r} \left( \frac{\partial f}{\partial r} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial r} \right), \quad (53)$$

where we adopted the subscript  $r$  to symbolize the use of cylindrical coordinates and to distinguish it from the bracket in Cartesian coordinates that will be introduced later in the paper.

The reduced MHD equations (51)-(52) provide a simplified version of the original ideal MHD equations. In particular, the number of dynamical field variables is reduced to two (against, for instance, the eight field variables present in the full MHD system (75)-(78)). We specify that Eq. (51) expresses the frozen-in condition of the magnetic field in the velocity field  $\mathbf{v}$ , whereas Eq. (52) is the evolution equation for the vorticity, subject to the Lorentz force, represented by the second term on the right-hand side.

The Hamiltonian formulation of reduced MHD was presented in Ref. [42]. It consists of the Hamiltonian functional

$$H(\psi, \omega) = \frac{1}{2} \int_D d\tau (-\psi \Delta_{\perp} \psi - \omega \phi[\omega]), \quad (54)$$

and of the Poisson bracket

$$\{F, G\} = \int_D d\tau (\omega [F_{\omega}, G_{\omega}]_r + \psi ([F_{\psi}, G_{\omega}]_r + [F_{\omega}, G_{\psi}]_r)), \quad (55)$$

where  $d\tau = r dr d\theta$  and where the subscripts on functionals denote functional derivatives, so that, for instance,

$$F_{\omega} = \frac{\delta F}{\delta \omega}. \quad (56)$$

The Hamiltonian  $H \in \Phi(P)$  in Eq. (54) is an observable of the form (3). Note that, on the right-hand side, we wrote  $\phi[\omega]$  to indicate the solution on  $\mathcal{D}$ , with the prescribed boundary conditions, of the equation  $\Delta_{\perp} \phi = \omega$ , for a given  $\omega$ . To simplify the notation, we will later omit this kind of specification, unless required to avoid ambiguities. Integrating by parts, making use of the boundary conditions, the Hamiltonian (54) can be written as

$$H(\psi, \omega) = \frac{1}{2} \int_D d\tau (|\nabla_{\perp} \psi|^2 + |\nabla_{\perp} \phi|^2) = \frac{1}{2} \int_D d\tau (|\mathbf{B}_{\perp}|^2 + |\mathbf{v}|^2), \quad (57)$$

where  $\nabla_{\perp} f$  is the "perpendicular" gradient, which, in cylindrical coordinates, is defined by  $\nabla_{\perp} f = \partial_r f \mathbf{e}_r + (1/r) \partial_{\theta} f \mathbf{e}_{\theta}$ , with  $\mathbf{e}_r$  and  $\mathbf{e}_{\theta}$  orthonormal unit vectors. The form (57) is more perspicuous from the physical point of view, as it shows how the Hamiltonian is given by the sum of the magnetic and of the kinetic energy, corresponding to the first and second term on the farthest right-hand side, respectively. It can be verified by direct computation (taking advantage from the fact that boundary terms vanish when integrating by parts) that the Hamiltonian (54) is a conserved quantity for the reduced MHD equations (51)-(52).

The Poisson bracket (55), on the other hand, can be shown to satisfy the above properties (i)-(iv) defining Poisson brackets. In particular, this bracket has a semi-direct product structure [13, 14, 43].

Applying the general expression (18) with  $\chi^1 = \psi$ ,  $\chi^2 = \omega$ , the Hamiltonian (54) and the Poisson bracket (55), one obtains indeed the reduced MHD equations (51)-(52). In order to see this, one makes use of integration by parts, of the formula (7) and of the functional derivatives of the Hamiltonian, which read

$$H_\psi = -\Delta_\perp \psi, \quad H_\omega = -\phi. \quad (58)$$

The expressions (58) can be obtained from the definition (5), again, making use of integration by parts. Notice also that, a useful formula for Poisson brackets involving canonical "inner" brackets such as  $[\cdot, \cdot]_r$  in Eq. (55) is given by

$$\int_D d\tau f[g, h]_r = \int_D d\tau g[h, f]_r = \int_D d\tau h[f, g]_r, \quad (59)$$

for functions  $f, g$  and  $h$ .

From the definition (22), and making use of the identities (59), one finds that the Casimir invariants of the Poisson bracket (55) consist of the two infinite families of observables

$$C_1(\psi) = \int_D d\tau \mathcal{J}(\psi), \quad C_2(\omega, \psi) = \int_D d\tau \omega \mathcal{I}(\psi), \quad (60)$$

where  $\mathcal{J}$  and  $\mathcal{I}$  are arbitrary functions. Making use of the expressions (54) and (60), from the definition (33) we can build the free energy functional, which reads

$$\mathcal{F} = H + C_1 + C_2 = \int_D d\tau \left( -\frac{\psi \Delta_\perp \psi}{2} - \frac{\omega \phi}{2} + \mathcal{J}(\psi) + \omega \mathcal{I}(\psi) \right). \quad (61)$$

Equilibria for reduced MHD can be found by means of the variational procedure described in Sec. 2.4, which consists of extremizing the first variation of  $\mathcal{F}$  by solving

$$\delta \mathcal{F}(\psi, \omega; \delta\psi, \delta\omega) = 0, \quad (62)$$

for arbitrary perturbations  $\delta\psi$  and  $\delta\omega$ . From Eq. (61) it follows that

$$\begin{aligned} & \delta \mathcal{F}(\psi, \omega; \delta\psi, \delta\omega) \\ &= \int_D d\tau \left( (-\Delta_\perp \psi + \mathcal{J}'(\psi) + \omega \mathcal{I}'(\psi)) \delta\psi + (\mathcal{I}(\psi) - \phi) \delta\omega \right), \end{aligned} \quad (63)$$

where the prime denotes derivative with respect to the argument (in the original Ref. [35] variations  $\delta\phi$  are used instead of  $\delta\omega$ , but this difference has no consequences if the solution of  $\Delta_{\perp}\delta\phi = \delta\omega$ , for any given  $\delta\omega$ , is unique). The extremization condition (62) thus implies that solutions of the system

$$-\Delta_{\perp}\psi + \mathcal{J}'(\psi) + \omega\mathcal{I}'(\psi) = 0, \quad (64)$$

$$\mathcal{I}(\psi) - \phi = 0, \quad (65)$$

for any given choice of the arbitrary functions  $\mathcal{J}$  and  $\mathcal{I}$ , are equilibria of reduced MHD. Equation (64) is indeed a generalization, in cylindrical geometry and accounting for equilibrium plasma flow, of the Grad-Shafranov equation describing axisymmetric magnetostatic equilibria (see, e.g. Refs. [44, 45]). On the other hand, recalling that  $\mathbf{B}_{\perp} = \nabla\psi \times \hat{z}$  and  $\mathbf{v} = \hat{z} \times \nabla\phi$ , Eq. (65) implies  $\mathbf{v}_e = -\mathcal{I}'(\psi_e)\mathbf{B}_e$ , for equilibrium solutions  $(\psi, \omega) = (\psi_e, \omega_e)$  with  $\Delta_{\perp}\phi_e = \omega_e$ ,  $\mathbf{B}_{e\perp} = \nabla\psi_e \times \hat{z}$  and  $\mathbf{v}_e = \hat{z} \times \nabla\phi_e$ . This expresses the fact that, at equilibrium, the velocity and the magnetic field are collinear. In Ref. [35], two special cases are considered. Case I corresponds to the choice  $\mathcal{I}(\psi) = 0$ , which leads to

$$-\Delta_{\perp}\psi + \mathcal{J}'(\psi) = 0, \quad (66)$$

$$\phi = 0. \quad (67)$$

Here one can recognize a Grad-Shafranov equation in Eq. (66) and the condition of no equilibrium flow in Eq. (67). Case II corresponds to  $\mathcal{I}(\psi) = -\psi$  and leads to the equilibrium equations

$$-\Delta_{\perp}\psi + \mathcal{J}'(\psi) - \omega = 0, \quad (68)$$

$$\psi = -\phi. \quad (69)$$

This situation corresponds to a family of equilibria (depending on the choice of  $\mathcal{J}$ ) with a flow propagating at the local Alfvén speed, as Eq. (69) implies  $\mathbf{v}_e = \mathbf{B}_{e\perp}$ . The relation (69), inserted into Eq. (68), also implies  $\mathcal{J} = \text{constant}$ .

The second variation of  $\mathcal{F}$ , evaluated at equilibrium solutions of (64)-(65) can be



written as

$$\begin{aligned}
& \delta^2 \mathcal{F}(\psi_e, \omega_e; \delta\psi, \delta\omega) \\
&= \int_D d\tau \left( -\delta\psi \Delta_{\perp} \delta\psi - \delta\phi \delta\omega + 2\mathcal{I}'(\psi_e) \delta\omega \delta\psi \right. \\
&\quad \left. + (\omega_e \mathcal{I}''(\psi_e) + \mathcal{J}''(\psi_e)) (\delta\psi)^2 \right) \\
&= \int_D d\tau \left( |\nabla_{\perp} \delta\phi - \nabla_{\perp} (\mathcal{I}'(\psi_e) \delta\psi)|^2 + (1 - \mathcal{I}'^2(\psi_e)) |\nabla_{\perp} \delta\psi|^2 \right. \\
&\quad \left. + (\mathcal{I}'(\psi_e) \Delta_{\perp} \mathcal{I}'(\psi_e) + \omega_e \mathcal{I}''(\psi_e) + \mathcal{J}''(\psi_e)) (\delta\psi)^2 \right).
\end{aligned} \tag{70}$$

As discussed in Sec. 2.4, formal stability of the equilibrium  $(\psi_e, \omega_e)$  is attained if  $\delta^2 \mathcal{F}(\psi_e, \omega_e; \delta\psi, \delta\omega)$  has a definite sign for every non-vanishing perturbation  $(\delta\psi, \delta\omega)$ . If we consider the above mentioned particular equilibria we see that, in case I, the equilibrium is energy stable if

$$\mathcal{J}''(\psi_e) > 0. \tag{71}$$

Considering that  $-\Delta_{\perp} \psi_e = J_e(\psi_e)$  is the equilibrium current density, from Eqs. (66) and (71), it follows that the stability condition can be reformulated as

$$\frac{dJ_e}{d\psi_e} < 0, \tag{72}$$

thus implying a monotonicity condition on the equilibrium current density.

In the case II of Alfvénic equilibria, one has  $\mathcal{I}'(\psi_e) = -1$  and  $\mathcal{J}$  is a constant function. Therefore, the second variation reduces to

$$\delta^2 \mathcal{F}(\psi_e, \omega_e; \delta\psi, \delta\omega) = \int_D d\tau |\nabla_{\perp} \delta\phi - \nabla_{\perp} \delta\psi|^2. \tag{73}$$

The equilibrium is thus formally stable and the second variation in Eq. (73) provides a conserved quantity for the linearized dynamics. One sees that, in this case, the difference between the gradients of the perturbations  $\delta\psi$  and  $\delta\phi$  remains bounded in time, although the amplitude of the perturbations can grow. We remark that formal stability for a Hamiltonian refined version of reduced MHD, accounting for density fluctuations, electron inertia and electron diamagnetic drift effects, was investigated in Ref. [46].

In Ref. [35], nonlinear stability for reduced MHD is also investigated, by means of the procedure described in Sec. 2.4. It might be interesting to note that, on the basis of this analysis, the conclusion for nonlinear stability for Alfvénic equilibria corresponds to that for formal stability. On the other hand, for the equilibria of case I, the condition (71) is replaced by

$$0 < s \leq 2\mathcal{J}''(\psi_e) \leq S < \infty, \quad (74)$$

where  $s$  and  $S$  are two constants. Thus, for Grad-Shafranov equilibria, nonlinear stability requires stronger conditions than formal stability.

#### 4. MHD with helical symmetry

The full (as opposed to the reduced version considered in Sec. 3) MHD description, according to which a plasma behaves as a single conducting fluid, is one of the most frequently adopted descriptions for large-scale plasma dynamics. As anticipated in Sec. 3, most of the applications of the EC method (including those to MHD) refer to the case where the dynamical field variables do not depend on one of the coordinates of the adopted coordinate system. In the MHD context, depending on the application, different symmetries can be relevant. As already mentioned in Sec. 3, in systems with a strong and straight guide field, such as tokamaks in the large aspect-ratio approximation, the coordinate  $z$  associated with the guide field direction in a cylindrical coordinate system  $(r, \theta, z)$ , can be taken as the ignorable one. For modelling astrophysical jets, on the other hand, MHD field variables are often taken to depend on the radial coordinate  $r$  and on a helical coordinate  $u = m\theta + kz$  (with  $m$  a positive integer and  $k \neq 0$  a constant), again ignoring the explicit dependence on  $z$ .

In a number of recent works [47, 48, 49, 50, 51] the Authors carried out a thorough analysis of the Hamiltonian structure of ideal MHD in the presence of a generalized helical symmetry. Also, they studied the corresponding variational equilibria, as well as their stability, using different methods, such as an energy principle based on the Lagrangian description, the method of dynamically accessible perturbations and the

EC method, which is the one of interest here. This analysis provides a very general framework for MHD stability analysis in the presence of symmetry, and unifies the analyses carried out for cases of special symmetries. We summarize here part of the main results of their analysis, focusing on those most relevant for the present review.

We begin by introducing the following system of ideal MHD equations (this time in dimensional units) not possessing any particular spatial symmetry:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \quad (75)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \frac{v^2}{2} + U + \frac{p}{\rho} \right) - (\nabla \times \mathbf{v}) \times \mathbf{v} + T \nabla s + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (76)$$

$$\frac{\partial s}{\partial t} = -\mathbf{v} \cdot \nabla s, \quad (77)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (78)$$

The system is completed with the divergence-free condition  $\nabla \cdot \mathbf{B} = 0$ , which is preserved by the dynamics, once it is prescribed as initial condition. In the system (75)-(78) we indicated with  $\rho$  the mass density, with  $s$  the entropy per unit mass, with  $T$  the temperature, with  $p$  the pressure and with  $U = U(\rho, s)$  the internal energy of the plasma. The magnetic and velocity fields are indicated with  $\mathbf{B}$  and  $\mathbf{v}$ , respectively. All such fields depend on three coordinates  $(r, \theta, z)$  of a cylindrical coordinate system. However, note that these fields are not all independent. Indeed, once a form for the internal energy  $U$  is prescribed, the pressure and the temperature follow from the relations  $p = \rho^2 \partial U / \partial \rho$  and  $T = \partial U / \partial s$ , respectively. Equation (75) corresponds to the mass continuity equation, Eq. (76) is the velocity equation, Eq. (77) expresses entropy advection in a dissipationless fluid, whereas Eq. (78) corresponds to the frozen-in condition for the magnetic field. If we refer to the setting introduced in Sec. 2.1, we are in the case  $n = 3$  and  $m = 8$ , with  $\chi = (\rho, v^r, v^\theta, v^z, B^r, B^\theta, B^z, s)$ . The MHD system (75)-(78) was shown to possess a noncanonical Hamiltonian structure in Ref. [52]. In Ref. [47], the Hamiltonian structure for the axisymmetric (i.e. assuming independence of the field variables on  $\theta$ ) version of the MHD system was derived, together with an analysis of the equilibria obtained from the variational principle described in Sec.

2.4. An analogous analysis was carried out, in the same Reference, also for MHD in Cartesian coordinates  $(x, y, z)$  with translational symmetry with respect to  $z$ . In Ref. [48], on the other hand, the case of MHD in the presence of a helical symmetry was considered. More precisely, it was assumed that the field variables in the system (75)-(78) depend only on the radial coordinate  $r$  and on a generalized helical coordinate  $u$ , where  $u = \theta[l] \sin \alpha + z \cos \alpha$ , with  $l$  indicating a scale length and  $\alpha$  a fixed helical angle. One sees that the choice  $\alpha = 0$ , leading to  $u = z$ , corresponds to axisymmetry, whereas  $\alpha = \pi/2$ , which implies  $u = \theta[l]$ , yields to field variables independent on  $z$ . For  $0 < \alpha < \pi/2$ , one finally obtains a non-degenerate helical symmetry. This allows to formulate MHD in the presence of a generic spatial symmetry.

In the presence of such helical symmetry, a generic field  $f$  satisfies the relation  $\mathbf{h} \cdot \nabla f = 0$ , where

$$\mathbf{h} = kr \nabla r \times \nabla u = -kr \cos \alpha \hat{\theta} + k[l] \sin \alpha \hat{z}, \quad (79)$$

is a divergence-free unit vector pointing in the direction along which the field is invariant. In Eq. (79),  $\hat{\theta}$  is a unit vector along  $\theta$  and

$$k = \frac{1}{\sqrt{[l]^2 \sin^2 \alpha + r^2 \cos^2 \alpha}} \quad (80)$$

is a metric factor. The helically symmetric velocity and magnetic fields can be represented as

$$\mathbf{v}(r, u) = \mathbf{v}_\perp(r, u) + v_h(r, u)\mathbf{h}, \quad (81)$$

$$\mathbf{B}(r, u) = \nabla \psi(r, u) \times k\mathbf{h} + B_h(r, u)\mathbf{h}, \quad (82)$$

where the vector field  $\mathbf{v}_\perp$  possess two components, directed along  $\nabla r$  and  $\nabla u$ , respectively. Note that, with the representation (82), using  $\nabla \cdot \mathbf{h} = 0$ , the condition  $\nabla \cdot \mathbf{B} = 0$  is automatically satisfied.

The introduction of the magnetic flux function  $\psi$  allows to decrease the number of dynamical variables by one. Thus, in terms of the seven variables  $\chi =$

$(\rho, \mathbf{v}_\perp, v_h, \psi, B_h, s)$ , the helical symmetric MHD system can be written as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}_\perp), \quad (83)$$

$$\begin{aligned} \frac{\partial \mathbf{v}_\perp}{\partial t} = & -\nabla \left( \frac{v_\perp^2}{2} + \frac{v_h^2}{2} + U + \frac{p}{\rho} \right) - (\nabla \times \mathbf{v}_\perp) \times \mathbf{v}_\perp + kv_h \nabla \left( \frac{v_h}{k} \right) \\ & + k^2[l] \sin(2\alpha) v_h \mathbf{h} \times \mathbf{v}_\perp + T \nabla s \end{aligned} \quad (84)$$

$$\begin{aligned} & - \frac{1}{4\pi\rho} \nabla \cdot (k^2 \nabla \psi) \nabla \psi - \frac{kB_h}{4\pi\rho} \nabla \left( \frac{B_h}{k} \right) - k^3[l] \sin(2\alpha) \frac{B_h}{4\pi\rho} \nabla \psi, \\ \frac{\partial v_h}{\partial t} = & -k \mathbf{v}_\perp \cdot \nabla \left( \frac{v_h}{k} \right) - \frac{k}{4\pi\rho} \left[ \psi, \frac{B_h}{k} \right]_h, \end{aligned} \quad (85)$$

$$\frac{\partial s}{\partial t} = -\mathbf{v}_\perp \cdot \nabla s, \quad (86)$$

$$\frac{\partial \psi}{\partial t} = -\mathbf{v}_\perp \cdot \nabla \psi, \quad (87)$$

$$\frac{\partial B_h}{\partial t} = -\frac{1}{k} \nabla \cdot (kB_h \mathbf{v}_\perp) + k^3[l] \sin(2\alpha) \nabla \psi \cdot \mathbf{v}_\perp - \frac{1}{k} [\psi, kv_h]_h. \quad (88)$$

We note that in the original Ref. [48], the system and its Hamiltonian structure were formulated using two alternative variables  $\mathbf{M}_\perp = \rho \mathbf{v}_\perp$  and  $\sigma = \rho s$ , instead of  $\mathbf{v}_\perp$  and  $\rho$ . However, in our context we found it convenient to reformulate the system using  $\mathbf{v}_\perp$  and  $\rho$  as variables, given that these variables will be those adopted for the stability analysis.

In Eqs. (85) and (88) we introduced the bracket  $[\cdot, \cdot]_h$  for helically symmetric system, defined by

$$[f, g]_h = (\nabla f \times \nabla g) \cdot k \mathbf{h}, \quad (89)$$

for two functions  $f$  and  $g$ .

The Hamiltonian and the Poisson bracket generating the symmetric MHD system (83)-(88) are given by

$$H(\rho, \mathbf{v}_\perp, v_h, \psi, B_h, s) = \int_D d^3r \left( \rho \left( \frac{|\mathbf{v}_\perp|^2}{2} + \frac{v_h^2}{2} \right) + \rho U + k^2 \frac{|\nabla \psi|^2}{8\pi} + \frac{B_h^2}{8\pi} \right) \quad (90)$$

and

$$\begin{aligned}
\{F, G\} = & - \int_D d^3r \left( F_{\mathbf{v}_\perp} \cdot \nabla \left( G_\rho - \frac{\mathbf{v}_\perp}{\rho} \cdot G_{\mathbf{v}_\perp} - \frac{v_h}{\rho} G_{v_h} \right) - G_{\mathbf{v}_\perp} \cdot \nabla \left( F_\rho - \frac{\mathbf{v}_\perp}{\rho} \cdot F_{\mathbf{v}_\perp} - \frac{v_h}{\rho} F_{v_h} \right) \right. \\
& + \frac{v_h}{k} \left[ F_{\mathbf{v}_\perp} \cdot \nabla \left( k \frac{G_{v_h}}{\rho} \right) - G_{\mathbf{v}_\perp} \cdot \nabla \left( k \frac{F_{v_h}}{\rho} \right) \right] + k^2 [l] \sin(2\alpha) \frac{v_h}{\rho} \mathbf{h} \cdot (F_{\mathbf{v}_\perp} \times G_{\mathbf{v}_\perp}) \\
& + \mathbf{v}_\perp \cdot \left[ (F_{\mathbf{v}_\perp} \cdot \nabla) \left( \frac{G_{\mathbf{v}_\perp}}{\rho} \right) - (G_{\mathbf{v}_\perp} \cdot \nabla) \left( \frac{F_{\mathbf{v}_\perp}}{\rho} \right) \right] + \frac{\nabla s}{\rho} \cdot (F_s G_{\mathbf{v}_\perp} - G_s F_{\mathbf{v}_\perp}) \quad (91) \\
& + k \frac{B_h}{\rho} \left[ F_{\mathbf{v}_\perp} \cdot \nabla \left( \frac{G_{B_h}}{k} \right) - G_{\mathbf{v}_\perp} \cdot \nabla \left( \frac{F_{B_h}}{k} \right) \right] + \frac{\psi}{\rho} (F_{\mathbf{v}_\perp} \cdot \nabla G_\psi - G_{\mathbf{v}_\perp} \cdot \nabla F_\psi) \\
& - \psi \left[ F_\psi \nabla \cdot \left( \frac{G_{\mathbf{v}_\perp}}{\rho} \right) - G_\psi \nabla \cdot \left( \frac{F_{\mathbf{v}_\perp}}{\rho} \right) \right] \\
& - k^3 [l] \sin(2\alpha) \nabla \psi \cdot \left( F_{B_h} \frac{G_{\mathbf{v}_\perp}}{\rho} - G_{B_h} \frac{F_{\mathbf{v}_\perp}}{\rho} \right) + \psi \left( \left[ \frac{G_{B_h}}{k}, k \frac{F_{v_h}}{\rho} \right]_h - \left[ \frac{F_{B_h}}{k}, k \frac{G_{v_h}}{\rho} \right]_h \right),
\end{aligned}$$

respectively.

Although the expression (91) is rather complicated, one can get some insight on it by making connection, at least at an intuitive level, with simpler Poisson brackets obtained in particular cases. For instance, let us consider the case  $\alpha = \pi/2$ , corresponding, as above said, to translational invariance along  $z$ . In this case, one has  $\mathbf{h} = \hat{z}$  and  $k = 1/l$ . If one then evaluates the bracket (91) choosing  $F$  and  $G$  functionals only of the variables  $(\mathbf{v}_\perp, \psi)$ , in the resulting expression all terms involving functional derivatives with respect to  $\rho, B_h, v_h$  and  $s$  vanish. Introducing the vorticity  $\boldsymbol{\omega} = \omega \hat{z} = \nabla \times \mathbf{v}_\perp$ , one can obtain the relation

$$F_{\mathbf{v}_\perp} = \nabla \times (\bar{F}_\omega \hat{z}), \quad (92)$$

where  $F(\mathbf{v}_\perp, \psi) = \bar{F}(\omega, \psi)$ . The relation (92) is an example of application of the *functional chain rule*, which allows to determine the expression of functional derivatives in terms of a new set of dynamical variables. In the case under consideration, the relation (92) is obtained from the relation

$$\int_D d\tau (F_{\mathbf{v}_\perp} \cdot \delta \mathbf{v}_\perp + F_\psi \delta \psi) = \int_D d\tau (\bar{F}_\omega \delta \omega + \bar{F}_{\psi'} \delta \psi'), \quad (93)$$

following from the change of variables  $(\hat{z} \cdot \nabla \times \mathbf{v}_\perp, \psi) = (\omega, \psi')$ . Inserting the relations  $\hat{z} \cdot \nabla \times \delta \mathbf{v}_\perp = \delta \omega$  and  $\delta \psi = \delta \psi'$  into Eq. (93) and carrying out an integration by parts,

one obtains the relation (92). For a detailed introduction to the use of the functional chain rule, the Reader can consult Ref. [13].

Once the functional derivatives with respect to  $\mathbf{v}_\perp$  are replaced using Eq. (92), one can assume uniform mass density imposing  $\rho = \text{constant}$  (which implies incompressibility, i.e.  $\nabla \cdot \mathbf{v} = 0$ ). Applying the appropriate normalization, the resulting expression for  $\{F, G\}$ , corresponds to the Poisson bracket (55) of 2D reduced MHD. Note also that the Hamiltonian (54) of reduced MHD can be obtained from the Hamiltonian (90) of helically symmetric MHD by setting  $v_h = B_h = 0$ ,  $\mathbf{v}_\perp = \hat{z} \times \nabla\phi$ , imposing  $\rho$  to be constant and assuming that  $U$  does not depend on  $s$  (the internal energy becomes then a constant function which plays no role in the dynamical equations, and can therefore be set equal to zero). Thus, one can see, how 2D reduced MHD connects with helically symmetric MHD and how its Hamiltonian structure can be retrieved. On the other hand, we stress that, the procedure we outlined to obtain the Hamiltonian structure of 2D reduced MHD is only partially rigorous. In particular, imposing  $\rho$  to be a constant, directly in the Poisson bracket, is an operation that, in principle, does not automatically preserve the Jacobi identity. We can verify that the Jacobi identity is not violated, in this case, as the resulting bilinear form is known to be a Poisson bracket. For a more rigorous approach to the imposition of the incompressibility condition while preserving a Hamiltonian structure, one can make use of Dirac's theory of constraints. In Ref. [53] an application of such method is carried out both in the canonical Hamiltonian framework of the Lagrangian variables, and in the noncanonical Eulerian description of fluids and magnetofluids. The connection between the two approaches is also unveiled and analyzed in detail. In the Eulerian approach, an earlier application of Dirac's theory of constraints for imposing incompressibility is described in Ref. [54].

The application of the EC method to helically symmetric MHD continues with the identification of the Casimir invariants, which are required in order to build the free energy functional. The Casimir invariants of the bracket (91) are found in Ref. [48] in the general case. The stability analysis, on the other hand, focuses on the case in

which  $s = \mathcal{S}(\psi)$ , for some function  $\mathcal{S}$ , i.e. the entropy is supposed to be constant on magnetic flux surfaces. By considering Eq. (87), one immediately sees that Eq. (86) is automatically satisfied if  $s = \mathcal{S}(\psi)$ . In this limit, the following four infinite families of Casimir invariants are found:

$$C_1(\rho, \psi) = \int_D d^3r \rho \mathcal{J}(\psi), \quad (94)$$

$$C_2(B_h, \psi) = \int_D d^3r (k B_h \mathcal{H}(\psi) + k^4 [l] \sin(2\alpha) \mathcal{H}^-(\psi)), \quad (95)$$

$$C_3(\rho, v_h, \psi) = \int_D d^3r \frac{\rho}{k} v_h \mathcal{G}(\psi), \quad (96)$$

$$C_4(\mathbf{v}_\perp, v_h, B_h, \psi) = \int_D d^3r (\mathbf{v}_\perp \cdot (\nabla\psi \times k\mathbf{h}) + v_h B_h) \mathcal{K}(\psi), \quad (97)$$

where  $\mathcal{J}$ ,  $\mathcal{H}$ ,  $\mathcal{G}$ ,  $\mathcal{K}$  are arbitrary functions and  $\mathcal{H}^-(\psi) = \int_{\psi_0}^{\psi} \mathcal{H}(\psi') d\psi'$ , with  $\psi_0$  labelling a magnetic surface of reference. Some of these Casimir invariants can be connected with known conserved quantities. For instance, for  $\mathcal{H}(\psi) = 2\psi$ , the invariant  $C_2$  corresponds to magnetic helicity, whereas, for  $\mathcal{K}(\psi) = 1$ ,  $C_4$  reduces to cross-helicity.

Combining Eqs. (90) and (94)-(97), one obtains the free energy functional

$$\begin{aligned} \mathcal{F}(\rho, \mathbf{v}_\perp, v_h, \psi, B_h, s) = \int_D d^3r \left( \rho \left( \frac{|\mathbf{v}_\perp|^2}{2} + \frac{v_h^2}{2} \right) + \rho U + k^2 \frac{|\nabla\psi|^2}{8\pi} + \frac{B_h^2}{8\pi} - \rho \mathcal{J}(\psi) \right. \\ \left. - k B_h \mathcal{H}(\psi) - k^4 [l] \sin(2\alpha) \mathcal{H}^-(\psi) - \frac{\rho}{k} v_h \mathcal{G}(\psi) - \mathbf{v}_\perp \cdot (\nabla\psi \times k\mathbf{h}) \mathcal{K}(\psi) - v_h B_h \mathcal{K}(\psi) \right). \end{aligned} \quad (98)$$

Note that, in Eq. (98), following Ref. [48], the Casimir invariants are subtracted, instead of being added, to the Hamiltonian, unlike in our definition (33). However, due to the arbitrariness of the free functions in the Casimir invariants, this has no consequences on the physical meaning of the resulting equilibrium states.



Setting  $\delta\mathcal{F} = 0$  yields the following equilibrium equations:

$$\rho\mathbf{v}_\perp - \mathcal{K}(\psi)\mathbf{B}_\perp = 0, \quad (99)$$

$$\rho v_h - \mathcal{K}(\psi)B_h - \frac{\rho}{k}\mathcal{G}(\psi) = 0, \quad (100)$$

$$\frac{|\mathbf{v}_\perp|^2}{2} + \frac{v_h^2}{2} + U + \frac{p}{\rho} - \mathcal{J}(\psi) - \frac{v_h}{k}\mathcal{G}(\psi) = 0, \quad (101)$$

$$\frac{B_h}{4\pi} - k\mathcal{H}(\psi) - v_h\mathcal{K}(\psi) = 0, \quad (102)$$

$$\begin{aligned} & - \nabla \cdot \left( \frac{k^2}{4\pi} \nabla \psi \right) + \rho T \mathcal{S}'(\psi) - \rho \mathcal{J}'(\psi) - kB_h \mathcal{H}'(\psi) \\ & - k^4[l] \sin(2\alpha) \mathcal{H}(\psi) - \frac{\rho}{k} v_h \mathcal{G}'(\psi) - \mathbf{v} \cdot \mathbf{B} \mathcal{K}'(\psi) \\ & + \nabla \cdot (\mathcal{K}(\psi) k \mathbf{h} \times \mathbf{v}_\perp) = 0. \end{aligned} \quad (103)$$

In particular, Eqs. (99)-(103) follow from setting to zero in  $\delta\mathcal{F}$ , the coefficients of  $\delta\mathbf{v}_\perp$ ,  $\delta v_h$ ,  $\delta\rho$ ,  $\delta B_h$  and  $\delta\psi$ , respectively.

A simple but relevant case can be obtained by choosing  $\mathcal{G} = 0$  and  $\mathcal{K} = \text{constant}$ , which, from Eqs. (99)-(100), leads to field-aligned flows. From Eqs. (100) and (102) one also derives the expressions

$$v_h = \left( 4\pi k \mathcal{H}(\psi) \frac{\mathcal{K}(\psi)}{\rho} + \frac{\mathcal{G}(\psi)}{k} \right) (1 - M^2)^{-1}, \quad (104)$$

$$B_h = \left( 4\pi k \mathcal{H}(\psi) + 4\pi \mathcal{K}(\psi) \frac{\mathcal{G}(\psi)}{k} \right) (1 - M^2)^{-1}, \quad (105)$$

where the Alfvénic Mach number  $M^2 = 4\pi\mathcal{K}^2/\rho$  was introduced. Equations (104) and (105) show the presence of Alfvén singularities at magnetic surfaces where  $M^2 = 1$ . In order for equilibrium solutions to be regular, the choice of the free functions  $\mathcal{K}$ ,  $\mathcal{G}$  and  $\mathcal{H}$  must then be such that, at magnetic surfaces where  $M^2 = 1$ , one has  $4\pi k \mathcal{H} \mathcal{K} / \rho + \mathcal{G} / k = 0$  and  $k\mathcal{H} + \mathcal{K}\mathcal{G}/k = 0$ .

Equation (101), on the other hand, is a generalization of the Bernoulli equation for a fluid in equilibrium, whereas Eq. (103) is a generalized Grad-Shafranov equation, accounting for equilibrium flows.

As was pointed out in Ref. [48], making use *a posteriori*, of the equilibrium relations (99), (102) and (104)-(105) directly into the expression (98), one can obtain a variational

principle for the equilibria, based on a free energy functional only depending on  $\psi$  and  $\rho$  (and in principle depending even only on  $\psi$ , although  $\rho$  can be related to  $\psi$  in an implicit way).

The second variation of the free energy functional, evaluated at equilibrium quantities, reads

$$\begin{aligned}
\delta^2 \mathcal{F}(\chi_e; \delta\chi) &= \int_D d^3r \left( \rho_e |\delta\mathbf{v}|^2 + 2 \left( \mathbf{v}_e - \frac{\mathcal{G}(\psi_e)}{k} \mathbf{h} \right) \delta\mathbf{v} \delta\rho + \left( \rho_e \frac{\partial^2 U}{\partial \rho^2} + 2 \frac{\partial U}{\partial \rho} \right) (\delta\rho)^2 \right. \\
&\quad - 2\mathcal{K}(\psi_e) \delta\mathbf{v} \cdot \delta\mathbf{B} + \frac{|\delta\mathbf{B}|^2}{4\pi} - 2 \left( \mathcal{K}'(\psi_e) \mathbf{B} + \frac{\rho}{k} \mathcal{G}'(\psi_e) \mathbf{h} \right) \cdot \delta\mathbf{v} \delta\psi \\
&\quad \left. + 2 \left( \rho_e \frac{\partial^2 U}{\partial \rho \partial s} \mathcal{S}'(\psi_e) + \frac{\partial U}{\partial s} \mathcal{S}'(\psi_e) - \mathcal{J}'(\psi_e) - \frac{v_{he}}{k} \mathcal{G}'(\psi_e) \right) \delta\rho \delta\psi - 2(\mathcal{K}'(\psi_e) \mathbf{v}_e + k \mathcal{H}'(\psi_e) \mathbf{h}) \cdot \delta\mathbf{B} \delta\psi \right. \\
&\quad \left. + \left( \rho_e T \mathcal{S}''(\psi) + \rho_e \frac{\partial^2 U}{\partial s^2} \mathcal{S}'^2(\psi_e) - \rho_e \mathcal{J}''(\psi_e) - k B_{he} \mathcal{H}''(\psi_e) - k^4 [l] \sin(2\alpha) \mathcal{H}'(\psi_e) \right. \right. \\
&\quad \left. \left. - \frac{\rho_e}{k} v_{he} \mathcal{G}''(\psi_e) - \mathbf{v}_e \cdot \mathbf{B}_e \mathcal{K}''(\psi_e) \right) (\delta\psi)^2 \right), \tag{106}
\end{aligned}$$

where the short-hand notations  $\delta\mathbf{B} = \nabla\delta\psi \times k\mathbf{h} + \delta B_h \mathbf{h}$  and  $\delta\mathbf{v} = \delta\mathbf{v}_\perp + \delta v_h \mathbf{h}$  were used in order to write the second variation in a more compact form.

The expression (106) is rather involved and no obvious conditions for its positive (or negative) definiteness emerge. However, we can again make contact with simpler models, such as 2D reduced MHD. In particular, the terms  $\int_D d^3r (\rho_e/2) |\delta\mathbf{v}_\perp|^2$  and  $\int_D d^3r |\delta\mathbf{B}_\perp|^2 / (4\pi)$  (contained in the terms  $\int_D d^3r (\rho_e/2) |\delta\mathbf{v}|^2$  and  $\int_D d^3r |\delta\mathbf{B}|^2 / (4\pi)$ , respectively) in Eq. (106) are reminiscent, for constant  $\rho_e$ , of the positive definite terms  $\int_D d\tau |\nabla\delta\phi|^2 / 2$  and  $\int_D d\tau |\nabla\delta\psi|^2 / 2$ , that one obtains from the first two terms in the second line of Eq. (70), after integration by parts. Also, the term  $-\int_D d^3r \rho_e \mathcal{J}''(\psi_e) (\delta\psi)^2$  in Eq. (106) is analogous to the term  $\int_D d\tau \mathcal{J}''(\psi_e) (\delta\psi)^2$  of Eq. (70). Finally, the term  $\int_D d\tau 2\omega_e \mathcal{I}''(\psi_e) (\delta\psi)^2$  of Eq. (70) can be obtained from the term  $-\int_D d^3r 2(\mathcal{K}'(\psi_e) \mathbf{v}_e) \cdot \delta\mathbf{B} \delta\psi$  of Eq. (106), after an integration by parts and noticing that, upon a comparison between Eq. (65) and Eq. (99), the function  $-\mathcal{I}'$  for 2D reduced MHD plays a role analogous to  $\mathcal{K}$  for symmetric MHD. Elements for the stability analysis of 2D reduced MHD can thus be identified.

In spite of the complexity of the formula (106), it was shown in Refs. [49, 50] that

such expression can be simplified to

$$\delta^2 \mathcal{F}(Z_e; \delta Z) = \int_D d^3 r \left( a_1 |\delta \mathbf{S}|^2 + a_2 (\delta Q)^2 + a_3 (\delta R_h)^2 + a_4 |\delta \mathbf{R}_\perp|^2 + a_5 (\delta \psi)^2 \right), \quad (107)$$

where  $\delta \mathbf{S}$ ,  $\delta Q$ ,  $\delta R_h$ ,  $\delta \mathbf{R}_\perp$  are perturbations of the quantities

$$\mathbf{S} = \rho \mathbf{v} - \mathcal{K}(\psi) \mathbf{B} - \frac{\rho}{k} \mathcal{G}(\psi) \mathbf{h}, \quad (108)$$

$$Q = \frac{B^2}{2\rho^2} \mathcal{K}^2(\psi) + U + \frac{p}{\rho} - \mathcal{J}(\psi) - \frac{\mathcal{G}^2(\psi)}{2k^2}, \quad (109)$$

$$R_h = \frac{1 - M^2}{4\pi} B_h - k \mathcal{H}(\psi) - \frac{\mathcal{K}(\psi) \mathcal{G}(\psi)}{k}, \quad (110)$$

$$\mathbf{R}_\perp = \frac{1 - M^2}{4\pi} \nabla \psi \times k \mathbf{h}, \quad (111)$$

and

$$a_1 = \frac{1}{\rho}, \quad a_2 = \frac{\rho}{c_s^2 - M^2 c_a^2}, \quad (112)$$

$$a_3 = \frac{4\pi(c_s^2 - M^2 c_a^2)}{c_s^2 - M^2(c_s^2 + c_a^2) + (M^4/(4\pi\rho))B_\perp^2}, \quad (113)$$

$$a_4 = \frac{4\pi}{(1 - M^2)} \frac{c_s^2 - M^2(c_s^2 + c_a^2) + (M^4/(4\pi\rho))B_\perp^2}{c_s^2 - M^2(c_s^2 + c_a^2)}, \quad (114)$$

$$a_5 = -\Upsilon - a_1 \left| \frac{\delta \mathbf{S}}{\delta \psi} \right|_{\tilde{Z}_S}^2 - a_2 \left| \frac{\delta Q}{\delta \psi} \right|_{\tilde{Z}_S}^2 - a_3 \left| \frac{\delta R_h}{\delta \psi} \right|_{\tilde{Z}_S}^2 - a_4 \left| \frac{\delta \mathbf{R}_\perp}{\delta \psi} \right|_{\tilde{Z}_S}^2, \quad (115)$$

$$c_a^2 = \frac{B^2}{4\pi\rho}, \quad c_s^2 = \frac{\partial p}{\partial \rho}, \quad (116)$$

where

$$\begin{aligned} \Upsilon = & - \left( \rho T \mathcal{S}''(\psi) + \rho \frac{\partial^2 U}{\partial S^2} \mathcal{S}'(\psi)^2 - \rho \mathcal{J}''(\psi) - k B_h \mathcal{H}''(\psi) \right. \\ & \left. - k^4 [l] \sin(2\alpha) \mathcal{H}'(\psi) - \frac{\rho}{k} v_h \mathcal{G}''(\psi) - \mathbf{v} \cdot \mathbf{B} \mathcal{K}''(\psi) \right). \end{aligned} \quad (117)$$

In Eq. (107) the new set of variables  $Z = (\mathbf{S}, Q, R_h, \mathbf{R}_\perp, \psi)$  was introduced. The symbol  $|_{\tilde{Z}_S}$  indicates that functional derivatives with respect to  $\psi$  are taken as if  $\psi$  and its spatial derivatives were independent variables. The advantage of the variables  $Z$  is that, in terms of them, equilibrium states are particularly easy to express. Indeed, at

equilibrium, one has

$$\mathbf{S} = 0, \quad (118)$$

$$Q = 0, \quad (119)$$

$$R_h = 0. \quad (120)$$

On the other hand, the variables forming the set  $Z$  are not all independent. In fact, in defining them in terms of the original variables  $\chi$ , the fields  $\mathbf{B}$  and  $\psi$  were considered as independent variables. The variations  $\delta\mathbf{S}$ ,  $\delta Q$  and  $\delta R_h$  and  $\delta\mathbf{R}_\perp$  are determined accordingly, so that, for instance, the perturbation

$$\delta\mathbf{S} = \rho\delta\mathbf{v} + \mathbf{v}\delta\rho - \mathcal{K}'(\psi)\mathbf{B}\delta\psi - \mathcal{K}(\psi)\delta\mathbf{B} - (\mathcal{G}(\psi)\delta\rho + \mathcal{G}'(\psi)\rho\delta\psi)\frac{\mathbf{h}}{k}, \quad (121)$$

is expressed without making use of the relation  $\nabla\delta\psi \times k\mathbf{h} = \delta\mathbf{B}_\perp$ . With this approach, one can establish, from Eq. (107), a sufficient condition for formal stability by imposing

$$a_i > 0, \quad i = 1, \dots, 5. \quad (122)$$

Note, in particular, that  $a_1$  is always positive, whereas the conditions on  $a_2$ ,  $a_3$  and  $a_4$  explicitly depend on the local Alfvén velocity  $c_a$ , sound speed  $c_s$  and Mach number  $M$ .

Although the conditions (122) already provide a stability criterion, they are not optimal, as they were obtained treating  $\delta\mathbf{B}_\perp$  and  $\delta\psi$  as independent perturbations. However, as above recalled, they are constrained by the relation  $\nabla\delta\psi \times k\mathbf{h} = \delta\mathbf{B}_\perp$ . The perturbations considered for obtaining the condition (122) are therefore more general, and consequently, such stability condition is only sufficient. Taking into account the relation between  $\delta\mathbf{B}$  and  $\delta\psi$ , in Refs. [48, 50] a softer stability condition is found. In particular, if  $a_2$  and  $a_3$  are positive, it is shown that  $\delta^2\mathcal{F}$  is minimized when the functional

$$\begin{aligned} \delta^2\mathcal{F}(Z_e; \delta\mathbf{S} = 0, \delta Q = 0, \delta R_h = 0; \delta\psi) &= \int_D d^3r \left( b_1(\mathbf{e}_\psi \cdot \nabla\delta\psi)^2 \right. \\ &\left. + b_2(\delta\psi)^2 + (b_1 + b_3)|\nabla\delta\psi \times \mathbf{e}_\psi|^2 \right) \end{aligned} \quad (123)$$

is minimized. In Eq. (123) we introduced the symbol  $\mathbf{e}_\psi = \nabla\psi_e/|\nabla\psi_e|$ . Note that the functional  $\delta^2\mathcal{F}$  in Eq. (123) only depends on the variation  $\delta\psi$ . The functional (123) is minimized when the following Euler-Lagrange equation holds

$$\nabla \cdot [b_1 I + b_3(I - \mathbf{e}_\psi \mathbf{e}_\psi)] \cdot \nabla \delta\psi - b_2 \delta\psi = 0, \quad (124)$$

where  $I - \mathbf{e}_\psi \mathbf{e}_\psi$  is the projector on planes tangent to the surfaces  $\psi = \text{constant}$ .

In Ref. [51] the stability analysis is specialized to two examples. The first case refers to thermal convection in static equilibria of a fluid against gravity, in the presence and in absence of a magnetic field. The second example concerns a rotating magnetized pinch. We remark that, as above anticipated, in Refs. [49, 51], in addition to the EC method, other two different methods are applied for investigating stability properties of symmetric MHD. In one case, stability is investigated for static equilibria of symmetric MHD expressed in terms of Lagrangian variables, instead of Eulerian variables, which was the case with the EC method. In the second case, the method of dynamically accessible perturbations, in the Eulerian framework, is adopted [13]. With the latter method, the class of perturbations is restricted to those that preserve the constraints imposed by the Casimir invariants. In Ref. [49] it is concluded that, when considering spatially symmetric perturbations, the stability by the EC method implies the stability by the two other methods, as it considers a more general class of perturbations. However, in the absence of spatial symmetries, as pointed out in Ref. [55], the method of dynamically accessible perturbations is more general, as it does not suffer from the limitations of the EC method due to restricting equilibria to critical points of the free energy functional.

## 5. Symmetric extended MHD

The MHD system (75)-(78) fails to be accurate at scales of the order of the ion skin depth and, even more, at the scale of the electron skin depth. For phenomena occurring at such scales a more general model, referred to as extended MHD, is more appropriate.

The normalized equations of extended MHD read

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \quad (125)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \frac{v^2}{2} + U + \frac{p}{\rho} + d_e^2 \frac{J^2}{2\rho^2} \right) - \boldsymbol{\omega} \times \mathbf{v} + \mathbf{J} \times \mathbf{B}^*, \quad (126)$$

$$\frac{\partial \mathbf{B}^*}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B}^* - d_i \frac{\mathbf{J} \times \mathbf{B}^*}{\rho} + d_e^2 \frac{\mathbf{J} \times \boldsymbol{\omega}}{\rho} \right), \quad (127)$$

where  $d_i$  and  $d_e$  are the normalized ion and electron skin depth, respectively, and

$$\mathbf{B}^* = \mathbf{B} - d_e^2 \nabla \times \left( \frac{\nabla \times \mathbf{B}}{\rho} \right) \quad (128)$$

is the "generalized" magnetic field. Note in fact that, in spite of the presence of the additional contribution relevant on scales of the order of the electron skin depth, and corresponding to the second term on the right-hand side of Eq. (128), the vector field  $\mathbf{B}^*$  satisfies  $\nabla \cdot \mathbf{B}^* = 0$ . We also indicated with  $\mathbf{J} = \nabla \times \mathbf{B}$  the current density and with  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$  the vorticity. When compared to the usual MHD system (75)-(78), one sees that the continuity equation (125) remains identical, whereas the velocity equation (126) acquires new contributions due to electron inertia and associated with terms proportional to  $d_e^2$  (a coefficient which is indeed proportional to the electron mass). Similarly, the induction equation (127) also acquires a contribution due to electron inertia, and also accounts for the Hall term, represented by the second term on the right-hand side, and associated with the ion skin depth  $d_i$ . In particular, one notices that, due to electron inertia, the magnetic field  $\mathbf{B}$  is no longer a frozen-in quantity. We also remark that the version of extended MHD considered here assumes a barotropic closure. Therefore the internal energy is assumed not to depend on the specific entropy  $s$ . This explains the absence of the dynamical variable  $s$  in this system.

The extended MHD model was originally formulated in Ref. [56] and has recently re-gained interest for its capability of incorporating two-fluid effects into a single-fluid model. The model was shown to be energy-conserving in Ref. [57]. An action principle formulation that allows to derive the extended MHD model in Eulerian form, from an action principle formulated in the Lagrangian picture, was given in Ref. [58]. Two

Eulerian action principle formulations of relativistic extended MHD, on the other hand, were provided in Ref. [59]. Topological properties of the related invariants, in connection also with MHD and Hall MHD were analyzed in Ref. [60], whereas exact nonlinear solutions obtained from generalized Beltrami states were derived in Ref. [61]. In terms of applications, the extended MHD model was adopted for the investigation of turbulent cascades in Refs. [62, 63] and of 2D magnetic reconnection in Refs. [64] and [65]. In the latter Reference, the Hamiltonian structure of the 2D incompressible version of extended MHD was also discussed.

The Hamiltonian structure of the extended MHD model (125)-(127) was provided in Ref. [66]. In terms of the dynamical variables  $\chi = (\rho, \mathbf{v}, \mathbf{B}^*)$ , such structure is given by the Hamiltonian

$$H(\rho, \mathbf{v}, \mathbf{B}^*) = \int_D d^3x \left( \rho \frac{v^2}{2} + \rho U(\rho) + \mathbf{B}[\rho, \mathbf{B}^*] \cdot \mathbf{B}^* \right) \quad (129)$$

and by the Poisson bracket

$$\begin{aligned} \{F, G\} = & - \int_D d^3x \left( F_\rho \nabla \cdot G_{\mathbf{v}} - G_\rho \nabla \cdot F_{\mathbf{v}} - \frac{\nabla \times \mathbf{v}}{\rho} \cdot (F_{\mathbf{v}} \times G_{\mathbf{v}}) \right. \\ & - \frac{\mathbf{B}^*}{\rho} \cdot [F_{\mathbf{v}} \times (\nabla \times G_{\mathbf{B}^*}) - G_{\mathbf{v}} \times (\nabla \times F_{\mathbf{B}^*})] + d_i \frac{\mathbf{B}^*}{\rho} \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \\ & \left. - d_e^2 \frac{\nabla \times \mathbf{v}}{\rho} \cdot [(\nabla \times F_{\mathbf{B}^*}) \times (\nabla \times G_{\mathbf{B}^*})] \right). \end{aligned} \quad (130)$$

A derivation of the noncanonical Poisson bracket (130) from an action principle formulated in terms of Lagrangian coordinates was provided in Ref. [67]. Note that, in the Hamiltonian (129), the magnetic field  $\mathbf{B}$  has to be seen as an operator acting on the dynamical variables  $\rho$  and  $\mathbf{B}^*$  by solving Eq. (128). For the sake of determining the expression for  $H_{\mathbf{B}^*}$ , required to determine the equations of motion, we point out that, one practical way to proceed is to make use of the variables  $(\rho, \mathbf{v}, \mathbf{B})$  and write the Hamiltonian  $H$  in terms of such variables, which, using the relation (128), allows for an explicit expression  $\bar{H}(\rho', \mathbf{v}', \mathbf{B})$ , where  $\rho' = \rho$  and  $\mathbf{v}' = \mathbf{v}$ . From such expression, one obtains

$$\bar{H}_{\mathbf{B}} = \mathbf{B} + d_e^2 \nabla \times \left( \nabla \times \left( \frac{\mathbf{B}}{\rho'} \right) \right). \quad (131)$$

On the other hand, making use of the functional chain rule, one has

$$\bar{F}_{\rho'} = F_{\rho}, \quad \bar{F}_{\mathbf{v}'} = F_{\mathbf{v}}, \quad \bar{F}_{\mathbf{B}} = F_{\mathbf{B}^*} + d_e^2 \nabla \times \left( \nabla \times \left( \frac{F_{\mathbf{B}^*}}{\rho} \right) \right), \quad (132)$$

for two functionals  $F$  and  $\bar{F}$  such that  $\bar{F}(\rho', \mathbf{v}', \mathbf{B}) = F(\rho, \mathbf{v}, \mathbf{B}^*)$ . Comparing Eq. (131) with the third relation on the right-hand side of Eq. (132), where one replaces  $\bar{F}$  with  $\bar{H}$  and  $F$  with  $H$ , it follows that the latter relation is solved by taking

$$H_{\mathbf{B}^*} = \mathbf{B}, \quad (133)$$

which yields the required expression for  $H_{\mathbf{B}^*}$ .

We also remark that Casimir invariants of extended MHD correspond to the observables

$$C_1(\rho) = \int_D d^3x \rho, \quad (134)$$

$$C_2(\mathbf{v}, \mathbf{B}^*) = \int_D d^3x \mathbf{P}_+^* \cdot (\nabla \times \mathbf{P}_+^*), \quad (135)$$

$$C_3(\mathbf{v}, \mathbf{B}^*) = \int_D d^3x \mathbf{P}_-^* \cdot (\nabla \times \mathbf{P}_-^*), \quad (136)$$

with

$$\mathbf{P}_{\pm}^* = \mathbf{A}^* + \gamma_{\pm} \mathbf{v}, \quad (137)$$

where

$$\gamma_{\pm} = \frac{d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2}, \quad (138)$$

and  $\mathbf{A}^*$  is a generalized vector potential satisfying  $\nabla \times \mathbf{A}^* = \mathbf{B}^*$ . Clearly, the Casimir invariant  $C_1$  reflects conservation of the total mass. On the other hand,  $C_{2,3}$  correspond to generalized helicities, accounting for the presence of two species in the plasma. In Ref. [65] it was pointed out how, in the 2D case and in the small mass ratio limit  $d_e^2/d_i^2 \rightarrow 0$ , the fields  $\mathbf{P}_{\pm}$  correspond to the canonical momenta of the ion and electron fluids in the plasma, respectively. A connection between the Casimir invariants  $C_{1,2,3}$  and those of ideal MHD can also be established. In the case of  $C_1$  the connection is trivial because  $C_1$  is already a Casimir invariant for ideal MHD. However, in general,



retrieving Casimir invariants of ideal MHD from those of extended MHD by taking the limits  $d_e, d_i \rightarrow 0$  is not so obvious [66]. In Ref. [68] it is shown how this can be achieved in the translationally symmetric case, taking first the  $d_e \rightarrow 0$  limit (which leads to Hall MHD) and then the  $d_i \rightarrow 0$  limit, with an appropriate rescaling of the free functions present in the Casimir invariants. In the case of  $C_{2,3}$  we notice first that, for  $d_e \rightarrow 0$ , one has

$$C_3 \rightarrow \int_D d^3x \mathbf{A} \cdot \mathbf{B}, \quad (139)$$

which corresponds to magnetic helicity, a Casimir invariant of ideal MHD. On the other hand, taking  $d_e \rightarrow 0$  and then  $d_i \rightarrow 0$  in  $C_2$ , one has

$$C_2 = \int_D d^3x (\mathbf{A} \cdot \mathbf{B} + 2d_i \mathbf{v} \cdot \mathbf{B} + \mathcal{O}(d_i^2)). \quad (140)$$

Thus, one retrieves, at the zero order in  $d_i$ , again magnetic helicity, which is conserved in its own, and, at the first order, cross-helicity, another Casimir invariant of ideal MHD.

Also for extended MHD, we consider the case where the system possesses a symmetry, which makes the application of the EC method fruitful. In Ref. [69], the analysis of generic helically symmetric equilibria of extended MHD was carried out. Here we consider the axisymmetric case, for which Ref. [55] provides a complete stability analysis, including also the use of the above mentioned methods of dynamical accessible and of Lagrangian perturbations. Considering again a cylindrical coordinate system  $(r, \theta, z)$ , we assume that the dynamical field variables do not depend on the  $\theta$  coordinate. It is thus convenient to introduce the following representation for the velocity and magnetic fields:

$$\mathbf{v} = rv_\theta \nabla \theta + \nabla \chi \times \nabla \theta + \nabla \Upsilon, \quad (141)$$

$$\mathbf{B} = rB_\theta \nabla \theta + \nabla \psi \times \nabla \theta. \quad (142)$$

In this way, the velocity field  $\mathbf{v}$  and the magnetic field  $\mathbf{B}$  are expressed in terms of the scalar fields  $v_\theta, \chi, \Upsilon, B_\theta$  and  $\psi$ , all depending only on the  $r$  and  $z$  spatial coordinates.

The Hamiltonian (129) and the Poisson bracket (130) are then expressed in terms of the reduced set of variables  $\hat{\chi} = (\rho, v_\theta, \chi, \Upsilon, B_\theta^*, \psi^*)$ . The resulting Casimir invariants

for the axisymmetric system correspond to the families

$$C_1(B_\theta^*, \chi, \psi^*, v_\theta) = \int_D d^2r \left( \frac{B_\theta^*}{r} + \gamma\Omega[\chi] \right) \mathcal{K}(\psi^* + \gamma r v_\theta), \quad (143)$$

$$C_2(B_\theta^*, \chi, \psi^*, v_\theta) = \int_D d^2r \left( \frac{B_\theta^*}{r} + \mu\Omega[\chi] \right) \mathcal{G}(\psi^* + \mu r v_\theta), \quad (144)$$

$$C_3(\rho, \psi^*, v_\theta) = \int_D d^2r \rho \mathcal{M}(\psi^* + \gamma r v_\theta), \quad (145)$$

$$C_4(\rho, \psi^*, v_\theta) = \int_D d^2r \rho \mathcal{N}(\psi^* + \mu r v_\theta), \quad (146)$$

where  $\psi^* = \psi - d_e^2 \rho^{-1} \Delta^* \psi$ ,  $B_\theta^* = B_\theta - d_e^2 r \nabla \cdot [r^{-2} \rho^{-1} \nabla(r B_\theta)]$ ,  $\mathbf{v}_\perp = \nabla \chi \times \nabla \theta + \nabla \Upsilon$ ,  $\Omega = (\nabla \times \mathbf{v}_\perp) \cdot \nabla \theta$ , with  $\Delta^* = r^2 \nabla \cdot (r^{-2} \nabla)$  indicating the so-called Shafranov operator. Also,  $\gamma$  and  $\mu$  are parameters defined by  $(\gamma, \mu) = (\gamma_+, \gamma_-) = (d_i \pm \sqrt{d_i^2 + 4d_e^2})/2$ , whereas  $\mathcal{K}$ ,  $\mathcal{G}$ ,  $\mathcal{M}$  and  $\mathcal{N}$  are arbitrary functions.

The first variation of the free energy functional thus reads

$$\begin{aligned} & \delta \mathcal{F}(\rho, v_\theta, \chi, \Upsilon, B_\theta^*, \psi^*; \delta \rho, \delta v_\theta, \delta \chi, \delta \Upsilon, \delta B_\theta^*, \delta \psi^*) \\ &= \int_D d^2r \left[ (h(\rho) - \mathcal{M}(\psi^* + \gamma r v_\theta) - \mathcal{N}(\psi^* + \mu r v_\theta) \right. \\ & \quad \left. + \frac{v_\theta^2}{2} + \frac{|\mathbf{v}_\perp[\chi, \Upsilon]|^2}{2} + \frac{d_e^2}{2r^2 \rho^2} ((\Delta^* \psi)^2 + |\nabla(r B_\theta)|^2)) \right] \delta \rho \\ & \quad + \left( B_\theta [B_\theta^*, \rho] - \frac{\mathcal{K}(\psi^* + \gamma r v_\theta) + \mathcal{G}(\psi^* + \mu r v_\theta)}{r} \right) \delta B_\theta^* \\ & \quad + (\rho \mathbf{v}_\perp - \gamma \nabla \mathcal{K}(\psi^* + \gamma r v_\theta) \times \nabla \theta - \mu \nabla \mathcal{G}(\psi^* + \mu r v_\theta) \times \nabla \theta) \cdot \delta \mathbf{v}_\perp [\delta \chi, \delta \Upsilon] \\ & \quad + \left( \rho v_\theta - \gamma r \left( \frac{B_\theta^*}{r} + \gamma \Omega[\chi] \right) \mathcal{K}'(\psi^* + \gamma r v_\theta) - \mu r \left( \frac{B_\theta^*}{r} + \mu \Omega[\chi] \right) \mathcal{G}'(\psi^* + \mu r v_\theta) \right. \\ & \quad \left. - \gamma r \rho \mathcal{M}'(\psi^* + \gamma r v_\theta) - \mu r \rho \mathcal{N}'(\psi^* + \mu r v_\theta) \right) \delta v_\theta \\ & \quad - \left( \frac{\Delta^* \psi [\psi^*, \rho]}{r^2} + \left( \frac{B_\theta^*}{r} + \gamma \Omega[\chi] \right) \mathcal{K}'(\psi^* + \gamma r v_\theta) + \left( \frac{B_\theta^*}{r} + \mu \Omega[\chi] \right) \mathcal{G}'(\psi^* + \mu r v_\theta) \right. \\ & \quad \left. + \rho \mathcal{M}'(\psi^* + \gamma r v_\theta) + \rho \mathcal{N}'(\psi^* + \mu r v_\theta) \right) \delta \psi^* \Big], \end{aligned} \quad (147)$$

where we indicated with  $h(\rho) = (\rho U(\rho))'$  the plasma enthalpy. Note that, also in this case, although the first variation  $\delta \mathcal{F}$  is expressed in terms of the set of variables specified in the first line of Eq. (147), some short-hand notations were used. For instance, it turned out to be practical to use, in some cases, the variable  $\psi$ , although the latter is not independent on  $\psi^*$  and  $\rho$ . The convenience of the use of this short-hand notation

comes again, in this case, from the expression of the functional derivative (133) but in the axisymmetric case. As the Reader might have noticed with the cases treated so far, such practice of adopting short-hand notation is rather common in the problems considered here. In order to simplify the expressions, from now on, we will omit the explicit dependence on the variables declared as independent ones in a given expression. So, for instance,  $\mathbf{v}_\perp[\chi, \Upsilon]$  would be indicated simply as  $\mathbf{v}_\perp$ .

Extremizing the free energy functional leads to the following equilibrium equations [69]:

$$h(\rho) - \mathcal{M}(\varphi) - \mathcal{N}(\xi) + \frac{v_\theta^2}{2} + \frac{|\mathbf{v}_\perp|^2}{2} + \frac{d_e^2}{2r^2\rho^2}((\Delta^*\psi)^2 + |\nabla(rB_\theta)|^2) = 0, \quad (148)$$

$$B_\theta - \frac{\mathcal{K}(\varphi) + \mathcal{G}(\xi)}{r} = 0, \quad (149)$$

$$-\nabla \cdot (\rho \nabla \Upsilon) + (\nabla \chi \times \nabla \rho) \cdot \nabla \theta = 0, \quad (150)$$

$$-\nabla \cdot \left( \frac{\rho}{r^2} \nabla \chi \right) + (\nabla \rho \times \nabla \Upsilon) \cdot \nabla \theta + \gamma \frac{\Delta^* \mathcal{K}(\varphi)}{r^2} + \mu \frac{\Delta^* \mathcal{G}(\xi)}{r^2} = 0, \quad (151)$$

$$\rho v_\theta - \gamma r \left( \frac{B_\theta^*}{r} + \gamma \Omega \right) \mathcal{K}'(\varphi) - \mu r \left( \frac{B_\theta^*}{r} + \mu \Omega \right) \mathcal{G}'(\xi) - \gamma r \rho \mathcal{M}'(\varphi) - \mu r \rho \mathcal{N}'(\xi) = 0, \quad (152)$$

$$\frac{\Delta^* \psi}{r^2} + \left( \frac{B_\theta^*}{r} + \gamma \Omega \right) \mathcal{K}'(\varphi) + \left( \frac{B_\theta^*}{r} + \mu \Omega \right) \mathcal{G}'(\xi) + \rho \mathcal{M}'(\varphi) + \rho \mathcal{N}'(\xi) = 0, \quad (153)$$

where we denoted  $\varphi = \psi^* + \gamma v_\theta / r$  and  $\xi = \psi^* + \mu v_\theta / r$ . Note that the relations (150) and (151) follow from setting equal to zero, in  $\delta \mathcal{F}$ , the coefficients of  $\delta \Upsilon$  and  $\delta \chi$ , respectively, which were not appearing explicitly in the formulation (147).

Equation (148) is a generalized Bernoulli equation, accounting in particular for a term, proportional to  $d_e^2$ , due to the contribution of electron inertia to the energy balance. Equation (149) shows how the free functions  $\mathcal{K}$  and  $\mathcal{G}$ , associated with the Casimir invariants, determine the equilibrium toroidal magnetic field. Equations (150), (151) and (152) can be seen as equations determining the three functions  $\Upsilon$ ,  $\chi$  and  $v_\theta$  identifying the equilibrium velocity field, whereas Eq. (153) is the generalized Grad-Shafranov equation for the magnetic flux function  $\psi$ .

In Ref. [55], the general condition for stability, obtained from the second variation of  $\mathcal{F}$ , was derived. We report this result here, first indicating that the second variation of

the free energy functional, evaluated at equilibrium solutions (denoted by the subscript  $e$ ), can be expressed as:

$$\begin{aligned}
& \delta^2 \mathcal{F}(\rho_e, v_{\theta e}, \chi_e, \Upsilon_e, B_{\theta e}^*, \psi_e^*; \delta\rho, \delta v_\theta, \delta\chi, \delta\Upsilon, \delta B_\theta^*, \delta\psi^*) \\
&= \int_D d^2r \left[ \frac{d_e^2}{\rho r^2} |\nabla(r\delta B_\theta)|^2 + \frac{|\nabla\delta\psi|^2}{r^2} + \frac{d_e^2 r^2}{\rho} \left( \nabla \cdot \left( \frac{\nabla\psi}{r^2} \right) \right)^2 \right. \\
&+ \rho \left( \delta v_\theta + \frac{v_\theta}{\rho} \delta\rho \right)^2 + \rho \left| \delta\mathbf{v}_\perp + \frac{\mathbf{v}_\perp}{\rho} \delta\rho \right|^2 - 2 \frac{d_e^2}{r^2 \rho} \nabla(\delta\mathcal{K} + \delta\mathcal{G}) \cdot \nabla(r\delta B_\theta) \\
&\left. + 2 \frac{d_e^2}{r^2 \rho^2} \nabla(\delta\mathcal{K} + \delta\mathcal{G}) \cdot \nabla(rB_\theta) \delta\rho - 2((\gamma\nabla\delta\mathcal{K} + \mu\nabla\delta\mathcal{G}) \times \nabla\theta) \cdot \delta\mathbf{v}_\perp \right] + \mathcal{Q}, \tag{154}
\end{aligned}$$

where  $\mathcal{Q}$  is the following symmetric quadratic form

$$\mathcal{Q} = \int_D d^2r (\delta B_\theta, \delta\varphi, \delta\xi, \delta\rho) \mathcal{A} (\delta B_\theta, \delta\varphi, \delta\xi, \delta\rho)^T, \tag{155}$$

which turns out to be conveniently expressed in terms of the variations  $(\delta B_\theta, \delta\varphi, \delta\xi, \delta\rho)$ .

In the expression (155), the matrix  $\mathcal{A}$  is given by

$$\mathcal{A} = \begin{pmatrix} 1 & A_{\varphi B_\theta} & A_{\xi B_\theta} & 0 \\ A_{\varphi B_\theta} & A_{\varphi\varphi} & 0 & A_{\varphi\rho} \\ A_{\xi B_\theta} & 0 & A_{\xi\xi} & A_{\xi\rho} \\ 0 & A_{\varphi\rho} & A_{\xi\rho} & A_{\rho\rho} \end{pmatrix}, \tag{156}$$

where

$$A_{\varphi\varphi} = - \left( \frac{B_\theta^*}{r} + \gamma\Omega \right) \mathcal{K}''(\varphi) - \rho \mathcal{M}''(\varphi), \tag{157}$$

$$A_{\xi\xi} = - \left( \frac{B_\theta^*}{r} + \mu\Omega \right) \mathcal{G}''(\xi) - \rho \mathcal{N}''(\xi), \tag{158}$$

$$A_{\varphi B_\theta} = - \frac{\mathcal{K}'(\varphi)}{r}, \quad A_{\xi B_\theta} = - \frac{\mathcal{G}'(\xi)}{r}, \tag{159}$$

$$A_{\varphi\rho} = - \mathcal{M}'(\varphi), \quad A_{\xi\rho} = - \mathcal{N}'(\xi), \tag{160}$$

$$A_{\rho\rho} = \frac{1}{\rho} \left( c_s^2 - v_\theta^2 - |\mathbf{v}_\perp|^2 - \frac{d_e^2}{\rho^2} \left( r^2 \left[ \nabla \cdot \left( \frac{\nabla\psi}{r^2} \right) \right]^2 \right) + \frac{|\nabla(rB_\theta)|^2}{r^2} \right), \tag{161}$$

with  $c_s^2 = \rho h'(\rho)$ .

A sufficient condition for positive definiteness of  $\delta^2 \mathcal{F}$  is that both  $\mathcal{Q} > 0$  and all the remaining terms in  $\delta^2 \mathcal{F}$  be positive.

A necessary and sufficient condition for  $\mathcal{Q}$  to be positive is that the principal minors of the matrix  $\mathcal{A}$  all be positive. This occurs if

$$A_{\varphi\varphi} - A_{\varphi B_\theta}^2 > 0, \quad (162)$$

$$A_{\xi\xi}(A_{\varphi\varphi} - A_{\varphi B_\theta}^2) - A_{\varphi\varphi}A_{\xi B_\theta}^2 > 0, \quad (163)$$

$$\begin{aligned} & A_{\rho\rho}[A_{\xi\xi}(A_{\varphi\varphi} - A_{\varphi B_\theta}^2) - A_{\varphi\varphi}A_{\xi B_\theta}^2] \\ & + (A_{\varphi B_\theta}A_{\xi\rho} - A_{\xi B_\theta}A_{\varphi\rho})^2 - A_{\xi\xi}A_{\varphi\rho}^2 - A_{\varphi\varphi}A_{\xi\rho}^2 > 0. \end{aligned} \quad (164)$$

Also, the first five terms on the right-hand side of the expression (154) are non-negative. On the other hand, the remaining terms in  $\delta^2\mathcal{F}$  have no definite sign.

Despite the presence of a symmetry, the inclusion of two-fluid effects and equilibrium flows, makes the stability analysis of extended MHD significantly more complicated, with respect to conventional MHD, and general stability conditions are very hard to derive. Nevertheless, as discussed in Ref. [55], explicit stability conditions can be found in some special cases. One of these is the case of equilibria with purely toroidal flows and no toroidal magnetic field (and thus purely toroidal current). Such equilibria correspond to  $\chi_e = \Upsilon_e = B_{\theta e} = 0$  and can be attained with the choice  $\mathcal{K} = \mathcal{G} = 0$ . With this constraint, from Eq. (154), it follows that  $\mathcal{Q} > 0$  implies  $\delta^2\mathcal{F} > 0$ . Thus, stability is attained if the conditions (162)-(164) are fulfilled. More explicitly, for purely toroidal equilibrium current and flows, stability is attained if

$$\mathcal{M}''(\varphi_e) < 0, \quad (165)$$

$$\mathcal{N}''(\xi_e) < 0, \quad (166)$$

$$\begin{aligned} & \mathcal{M}''(\varphi_e)\mathcal{N}''(\xi_e) \left( c_{s_e}^2 - v_{\theta e}^2 - \frac{d_e^2}{\rho_e^2} \left( r^2 \left[ \nabla \cdot \left( \frac{\nabla\psi_e}{r^2} \right) \right]^2 \right) \right) \\ & + \mathcal{N}''(\xi_e)\mathcal{M}'^2(\varphi_e) + \mathcal{M}''(\varphi_e)\mathcal{N}'^2(\xi_e) > 0. \end{aligned} \quad (167)$$

Once the conditions (165)-(166), which concern the convexity of the functions  $\mathcal{M}$  and  $\mathcal{N}$ , are applied to the condition (167), it follows that, for the inequality (167) to be satisfied, it is necessary that the term multiplying  $\mathcal{M}''(\varphi_e)\mathcal{N}''(\xi_e)$  be positive. This implies an

upper bound on the equilibrium toroidal velocity, corresponding to the equilibrium sound speed  $c_s$ , diminished by a contribution due to electron inertia.

A further special case which leads to explicit solutions in terms of equilibrium quantities, for arbitrary perturbations, is that of Hall MHD with purely toroidal equilibrium flows. This corresponds to setting  $d_e = 0$  (which implies  $\mu = 0$ ) and  $\mathcal{K}' = 0$ . In this way, the terms in  $\delta^2 \mathcal{F}$ , whose definiteness could not be ascertained, are removed, and the stability conditions, following from Eqs. (162)-(164), are given by

$$\mathcal{M}''(\varphi_e) < 0, \quad (168)$$

$$\frac{\mathcal{G}(\psi_e)\mathcal{G}''(\xi_e)}{r^2} + \rho_e \mathcal{N}'''(\psi_e) + \frac{\mathcal{G}'^2(\psi_e)}{r^2} < 0, \quad (169)$$

$$\begin{aligned} & [\mathcal{M}''(\varphi_e)(c_{s_e}^2 - v_{\theta_e}^2) + \mathcal{M}'^2(\varphi_e)] \left( \frac{\mathcal{G}(\psi_e)\mathcal{G}''(\psi_e)}{r^2} + \rho_e \mathcal{N}'''(\psi_e) + \frac{\mathcal{G}'^2(\psi_e)}{r^2} \right) \\ & + \rho_e \mathcal{M}''(\varphi_e) \mathcal{N}'^2(\psi_e) > 0. \end{aligned} \quad (170)$$

Taking into account the conditions (168) and (169) into the inequality (170), it emerges that a necessary condition for the latter relation to be satisfied, is that  $v_{\theta_e}^2 < c_{s_e}^2$ , which sets an upper bound for the toroidal velocity. Note that, by comparing to the corresponding stability condition found for the previous case of purely toroidal flow with no toroidal magnetic field, one sees that the presence of electron inertia introduces a stronger requirement for stability.

In Ref. [55], the conditions (168)-(170) were also applied to examine stability in the case of a domain corresponding to a poloidal section of an ITER-like tokamak and assuming an adiabatic equation of state to fix the function  $h(\rho)$ . Prescribing a polynomial form for the free functions  $\mathcal{G}$ ,  $\mathcal{M}$  and  $\mathcal{N}$ , and solving numerically the equilibrium equations, it emerged that, whereas the conditions (168) and (169) can easily be satisfied over the whole domain, by properly choosing the parameters in the polynomial ansätze for the free functions, the condition (170) is harder to be fulfilled. In particular, only for  $\beta < 1\%$  (with  $\beta$  indicating here the ratio between plasma pressure and magnetic pressure) an equilibrium satisfying all the three stability conditions was found. Stability attained for low  $\beta$  suggested that the condition (170) might be related

to the suppression of pressure-driven modes.

In the same framework, an investigation of the role of the  $d_i$  parameter, showed that increasing  $d_i$  favoured the achievement of stability over the whole domain.

A further way, adopted in Ref. [55], to infer stability conditions in a special case is that of considering a sub-class of constrained variations. Similarly to the procedure adopted in Sec. 4, where not-independent variations were considered, one can introduce relations between the variations. In particular, one introduces a relation between  $\delta B_\theta$ ,  $\delta \mathbf{v}_\perp$ ,  $\delta \mathcal{K}$  and  $\delta \mathcal{G}$ , obtained by extremizing the functional  $\delta^2 \mathcal{F}$ , seen as a functional of the variations  $\delta B_\theta$  and  $\delta \mathbf{v}_\perp$ . This yields the relations

$$\delta B_\theta = \frac{\delta \mathcal{K} + \delta \mathcal{G}}{r}, \quad (171)$$

$$\delta \mathbf{v}_\perp = -\mathbf{v}_\perp \frac{\delta \rho}{\rho} + \frac{(\gamma \nabla \delta \mathcal{K} + \mu \nabla \delta \mathcal{G}) \times \nabla \theta}{\rho} \quad (172)$$

(note that, due to a misprint, Eq. (172) differs from the corresponding Eq. (37) of Ref. [55]). Then, we restrict to incompressible perturbations, by setting  $\delta \rho = 0$ . Inserting this constraint, together with the relations (171)-(172), into the expression (154), leads to the expression

$$\begin{aligned} \delta^2 \tilde{\mathcal{F}} = & \int_D d^2 r \left( \frac{|\nabla \delta \psi|^2}{r^2} + \frac{d_e^2 r^2}{\rho} \left( \nabla \cdot \left( \frac{\nabla \psi}{r^2} \right) \right)^2 \right. \\ & \left. + \rho (\delta v_\theta)^2 + \frac{(\delta \mathcal{K} - \delta \mathcal{G})^2}{r^2} \right) + \tilde{\mathcal{Q}}, \end{aligned} \quad (173)$$

where

$$\begin{aligned} \tilde{\mathcal{Q}} = & \int_D d^2 r \left( A_{\varphi\varphi} (\delta\varphi)^2 + A_{\xi\xi} (\delta\xi)^2 \right) \\ & - \int_D d^2 r \left[ \frac{\gamma^2 + d_e^2}{r^2 \rho} \left( (\mathcal{K}')^2 |\nabla \delta\varphi|^2 + 2\mathcal{K}' \delta\varphi \nabla \mathcal{K}' \cdot \nabla \delta\varphi \right) \right. \\ & \left. + \frac{\mu^2 + d_e^2}{r^2 \rho} \left( (\mathcal{G}')^2 |\nabla \delta\xi|^2 + 2\mathcal{G}' \delta\xi \nabla \mathcal{G}' \cdot \nabla \delta\xi \right) \right. \\ & \left. + \left( 2\frac{\mathcal{K}'^2}{r^2} + \frac{\gamma^2 + d_e^2}{r^2 \rho} |\nabla \mathcal{K}'|^2 \right) (\delta\varphi)^2 \right. \\ & \left. + \left( 2\frac{\mathcal{G}'^2}{r^2} + \frac{\mu^2 + d_e^2}{r^2 \rho} |\nabla \mathcal{G}'|^2 \right) (\delta\xi)^2 \right]. \end{aligned} \quad (174)$$

Note that the functional  $\tilde{\mathcal{Q}}$  does not include the same terms of  $\mathcal{Q}$  in Eq. (154).

From the expression (173), it follows that stability is attained if  $\tilde{\mathcal{Q}} > 0$ . Analogously to the functional (107) of helical MHD, the functional  $\tilde{\mathcal{Q}}$  can also be conveniently cast in the diagonal form

$$\tilde{\mathcal{Q}} = \int_D d^2r \left( \tilde{A}_{\varphi\varphi}(\delta\varphi)^2 + \tilde{A}_{\xi\xi}(\delta\xi)^2 \right), \quad (175)$$

where

$$\begin{aligned} \tilde{A}_{\varphi\varphi} = & - \left( \frac{B_\theta^*}{r} + \gamma\Omega \right) \mathcal{K}'' - \rho\mathcal{M}'' - 2\frac{\mathcal{K}'^2}{r^2} \\ & - \frac{\gamma^2 + d_e^2}{\rho r^2} (|\nabla\mathcal{K}'|^2 + (\mathcal{K}')^2|\mathbf{k}_\varphi|^2 + \mathbf{k}_\varphi \cdot \nabla(\mathcal{K}')^2), \end{aligned} \quad (176)$$

$$\begin{aligned} \tilde{A}_{\xi\xi} = & - \left( \frac{B_\theta^*}{r} + \mu\Omega \right) \mathcal{G}'' - \rho\mathcal{N}'' - 2\frac{\mathcal{G}'^2}{r^2} \\ & - \frac{\mu^2 + d_e^2}{\rho r^2} (|\nabla\mathcal{G}'|^2 + (\mathcal{G}')^2|\mathbf{k}_\xi|^2 + \mathbf{k}_\xi \cdot \nabla(\mathcal{G}')^2), \end{aligned} \quad (177)$$

with

$$\mathbf{k}_\varphi = \frac{\nabla\delta\varphi}{\delta\varphi}, \quad \mathbf{k}_\xi = \frac{\nabla\delta\xi}{\delta\xi}. \quad (178)$$

Stability is thus attained if  $\tilde{A}_{\varphi\varphi} > 0$  and  $\tilde{A}_{\xi\xi} > 0$ . Applying the Cauchy-Schwarz inequality to the scalar products in the last terms of Eqs. (176) and (177), one can see that a sufficient condition for this to happen is that both the inequality

$$a_\varphi|\mathbf{k}_\varphi|^2 + b_\varphi|\mathbf{k}_\varphi| + c_\varphi > 0 \quad (179)$$

and

$$a_\xi|\mathbf{k}_\xi|^2 + b_\xi|\mathbf{k}_\xi| + c_\xi > 0, \quad (180)$$

hold. In Eqs. (179) and (180) we introduced

$$\begin{aligned} & a_\varphi|\mathbf{k}_\varphi|^2 + b_\varphi|\mathbf{k}_\varphi| + c_\varphi \\ & = - \left( \frac{B_\theta^*}{r} + \gamma\Omega \right) \mathcal{K}'' - \rho\mathcal{M}'' - 2\frac{\mathcal{K}'^2}{r^2} \\ & - \frac{\gamma^2 + d_e^2}{\rho r^2} (|\nabla\mathcal{K}'|^2 + (\mathcal{K}')^2|\mathbf{k}_\varphi|^2 + |\mathbf{k}_\varphi||\nabla(\mathcal{K}')^2|) \end{aligned} \quad (181)$$



and

$$\begin{aligned}
& a_\xi |\mathbf{k}_\xi|^2 + b_\xi |\mathbf{k}_\xi| + c_\xi \\
&= - \left( \frac{B_\theta^*}{r} + \mu\Omega \right) \mathcal{G}'' - \rho \mathcal{N}''' - 2 \frac{\mathcal{G}'^2}{r^2} \\
& - \frac{\mu^2 + d_e^2}{\rho r^2} (|\nabla \mathcal{G}'|^2 + (\mathcal{G}')^2 |\mathbf{k}_\xi|^2 + |\mathbf{k}_\xi| |\nabla (\mathcal{G}')^2|).
\end{aligned} \tag{182}$$

In order for the conditions (179)-(180) to be satisfied, the polynomial in  $|\mathbf{k}_\varphi|$  and  $|\mathbf{k}_\xi|$ , in the expressions (181) and (182), must each have at least one positive real root. Because  $a_\varphi, b_\varphi < 0$  and  $a_\xi, b_\xi < 0$ , one root of each of the polynomials will always be negative.

Therefore, in order for the remaining roots to be positive, it is required that

$$c_\varphi = - \left( \frac{B_\theta^*}{r} + \gamma\Omega \right) \mathcal{K}'' - \rho \mathcal{M}''' - 2 \frac{\mathcal{K}'^2}{r^2} - \frac{\gamma^2 + d_e^2}{\rho r^2} |\nabla \mathcal{K}'|^2 > 0, \tag{183}$$

$$c_\xi = - \left( \frac{B_\theta^*}{r} + \mu\Omega \right) \mathcal{G}'' - \rho \mathcal{N}''' - 2 \frac{\mathcal{G}'^2}{r^2} - \frac{\mu^2 + d_e^2}{\rho r^2} |\nabla \mathcal{G}'|^2 > 0. \tag{184}$$

When the conditions (183)-(184) are fulfilled, we denote with  $\mathbf{k}_\varphi^+$  and  $\mathbf{k}_\xi^+$ , respectively, the positive roots of the two polynomials in Eqs. (179)-(180). The two conditions (183)-(184) imply also that the two polynomials are positive for  $|\mathbf{k}_\varphi| = 0$  and  $|\mathbf{k}_\xi| = 0$ , respectively. Therefore, the conditions (179)-(180) are satisfied, when  $c_\varphi, c_\xi > 0$ , for  $0 \leq |\mathbf{k}_\varphi| < \mathbf{k}_\varphi^+$  and  $0 \leq |\mathbf{k}_\xi| < \mathbf{k}_\xi^+$ . On the other hand, Poincaré's inequality

$$\int_D d^2r |\mathbf{k}_\varphi|^2 (\delta\varphi)^2 = \int_D d^2r |\nabla \delta\varphi|^2 \geq C^{-1} \int_D d^2r (\delta\varphi)^2, \tag{185}$$

implies a lower positive bound on  $|\mathbf{k}_\varphi|$  (and analogously for  $|\mathbf{k}_\xi|$ ). In the expression (185), we indicated with  $C$  the Poincaré constant, which depends on the domain  $\mathcal{D}$ .

Thus, the admissible values of  $|\mathbf{k}_\varphi|$  and  $|\mathbf{k}_\xi|$  for which  $\tilde{Q} > 0$ , and thus stability under constrained variations is attained, are those for which

$$c_\varphi > 0, \quad c_\xi > 0 \tag{186}$$

and

$$\int_D d^2r (|\mathbf{k}_\varphi|^2 - C^{-1}) (\delta\varphi)^2 \geq 0, \tag{187}$$

$$\int_D d^2r (|\mathbf{k}_\xi|^2 - C^{-1}) (\delta\xi)^2 \geq 0. \tag{188}$$

The conditions (187)-(188) are fulfilled if  $\min(|\mathbf{k}_\varphi|) \geq C^{-1}$  and  $\min(|\mathbf{k}_\xi|) \geq C^{-1}$ , which makes the conditions (186) sufficient, if  $|\mathbf{k}_\varphi| < \mathbf{k}_\varphi^+$  and  $|\mathbf{k}_\xi| < \mathbf{k}_\xi^+$ .

Also in the case of extended MHD, stability analyses adopting a Lagrangian description of the fluid and the method of dynamically accessible variations, have been carried out [55].

## 6. Hybrid Vlasov-MHD model

In some circumstances one particle species in a plasma requires a kinetic description, whereas the remaining species can adequately be described following a fluid approach. This can be the case, for instance, of fast particles in tokamaks, for which an accurate modelling requires the knowledge of the distribution function in order to capture wave-particle interactions, whereas the rest of the plasma can be described using a fluid, e.g. MHD, approach. Weakly collisional space plasmas also often require a kinetic description of the ion dynamics, whereas the electron species can approximately treated as a fluid. In these circumstances, an effective tool comes from hybrid kinetic-fluid models, in which the dynamical field variables are a set of distribution functions, related to the kinetic particle species and a set of fluid variables, describing the remaining species. Hybrid models represent nowadays a very common tool. However, in spite of their frequent use, the investigation of the Hamiltonian structure of such models is relatively recent and was initiated in Ref. [70], where the Hamiltonian structure of the fundamental hybrid Vlasov-MHD models was provided. In particular, the two approaches adopted for coupling the kinetic and the fluid dynamics, i.e. through the pressure tensor or through the current density, were analyzed. The importance of the Hamiltonian structure, and in particular of energy conservation, for the spectral stability properties of equilibria of such models was put in evidence in Ref. [71]. Indeed, in this Reference it was shown that the lack of energy conservation present in some hybrid models discussed in the literature, could lead to instabilities whose physical origin is uncertain. Such instabilities are removed when the terms restoring the Hamiltonian structure in the model are taken into

account. The possibility of building hybrid Hamiltonian models, coupling a dynamics based on a gyrokinetic description with that based on a reduced fluid description, is discussed in Ref. [72]. Recently, Hamiltonian hybrid models describing kinetic ions coupled with thermal ions and electrons modelled by Hall MHD with anisotropic electron pressure, have been presented in Ref. [73].

From the point of view of the EC stability, an investigation of hybrid Vlasov-MHD models described in Ref. [70] was carried out, in the 2D limit, in Refs. [74, 75]. This permitted, in particular, to see how the presence of a kinetic species modifies the conditions found for the EC stability of 2D incompressible MHD. Also, we mention the application of the EC method for the formal stability analysis of a Hamiltonian hybrid model for the description of non-neutral plasmas [76]. In this case, the stability analysis permitted to determine the impact of the presence of massive charged particles, described kinetically by virtue of their large Larmor radius, on the stability properties of a pure electron plasma in the presence of a uniform and constant magnetic field.

Here we proceed with reviewing the stability analysis of the hybrid Vlasov-MHD model, based on the pressure coupling scheme, carried out in Refs. [74, 75]. This will provide a paradigmatic example of the application of the EC method to hybrid systems.

We first introduce the equations of the hybrid model with pressure coupling scheme. Also in this case, in order to facilitate the comparison with other models discussed in this review, we slightly modify the notation, with respect to Ref. [75].

In dimensionless variables, the pressure coupling hybrid model reads

$$\frac{\partial \psi}{\partial t} = -[\phi, \psi]_x, \quad (189)$$

$$\frac{\partial \omega}{\partial t} = -[\phi, \omega]_x + [\psi, \Delta_{\perp} \psi]_x + \hat{z} \cdot \nabla \times (\nabla_{\perp} \cdot \mathbb{P}_{\perp}), \quad (190)$$

$$\begin{aligned} \frac{\partial f}{\partial t} &= -[\phi, f]_x - \mathbf{p}_{\perp} \cdot \nabla_{\perp} f \\ &- ((\mathbf{p}_{\perp} \cdot \hat{z} \times \nabla) \nabla \phi + p_z \nabla_{\perp} \psi) \cdot \frac{\partial f}{\partial \mathbf{p}_{\perp}} + (\mathbf{p}_{\perp} \cdot \nabla_{\perp} \psi) \frac{\partial f}{\partial p_z}. \end{aligned} \quad (191)$$

This model describes the dynamics of a species of "hot" particles with distribution function  $f$ , coupled with the dynamics of a bulk plasma described by 2D incompressible

MHD. The distribution function  $f = f(\mathbf{x}, \mathbf{p})$  depends on the spatial Cartesian coordinates  $\mathbf{x} = (x, y)$  and on the momentum coordinates  $\mathbf{p} = (p_x, p_y, p_z)$ . The system is thus independent on the coordinate  $z$  and its spatial domain  $\mathcal{D}$  can be taken to be, for instance, a periodic box in the  $xy$  plane. The MHD variables correspond to the vorticity  $\omega = \Delta_{\perp} \phi$  and to the magnetic flux function  $\psi$ , which is related to the planar magnetic field by  $\mathbf{B} = \nabla \psi \times \hat{z}$ . The canonical bracket  $[\cdot, \cdot]_x$  in Cartesian coordinates is defined by

$$[f, g]_x = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}, \quad (192)$$

for two functions  $f$  and  $g$ . The coupling between the kinetic dynamics, described by the Vlasov equation (191) and the MHD dynamics, governed by the ideal Ohm's law (189) and the vorticity equation (190), is mediated by the perpendicular second order moment  $\mathbb{P}_{\perp}$ , defined by

$$\mathbb{P}_{\perp} = \int_{\mathbb{R}^3} d^3 p \mathbf{p}_{\perp} \mathbf{p}_{\perp} f, \quad (193)$$

where  $\mathbf{p}_{\perp} = (p_x, p_y)$ . The tensor  $\mathbb{P}_{\perp}$  is related to the pressure exerted by the kinetic species on the bulk plasma. In the absence of such pressure coupling term on the right-hand side of Eq. (190), Eqs. (189) and (190) reduce to ordinary 2D incompressible MHD which corresponds to Eqs. (51)-(52) of 2D reduced MHD, apart from the choice of the coordinate system and the domain. In order to make contact with the general setting introduced in Sec. 2.1, we note that we are in the case with  $n = 2$ ,  $n' = 3$ ,  $m = 3$ ,  $m_f = 2$ ,  $m_k = 1$ , where  $(\chi^1, \chi^2) = (\psi(x, y), \omega(x, y))$  and  $\zeta^1 = f(x, y, p_x, p_y, p_z)$ .

The Hamiltonian structure consists of the Hamiltonian observable

$$H(\psi, \omega, f) = \frac{1}{2} \int_D d^2 x (-\psi \Delta_{\perp} \psi - \omega \phi) + \frac{1}{2} \int_D d^2 x \int_{\mathbb{R}^3} d^3 p f |\mathbf{p}|^2 \quad (194)$$

and of the Poisson bracket

$$\begin{aligned} \{F, G\} &= \int_D d^2 x (\omega [F_{\omega}, G_{\omega}]_x + \psi ([F_{\psi}, G_{\omega}]_x + [F_{\omega}, G_{\psi}]_x)) \\ &\quad - \int_D d^2 x \int_{\mathbb{R}^3} d^3 p f ([F_f, \mathbf{p} \cdot \hat{z} \times \nabla G_{\omega}]_p + [\mathbf{p} \cdot \hat{z} \times \nabla F_{\omega}, G_f]_p) \\ &\quad + \int_D d^2 x \int_{\mathbb{R}^3} d^3 p f \left( [F_f, G_f]_p + \nabla \psi \cdot \left( \frac{\partial G_f}{\partial p_z} \frac{\partial F_f}{\partial \mathbf{p}_{\perp}} - \frac{\partial F_f}{\partial p_z} \frac{\partial G_f}{\partial \mathbf{p}_{\perp}} \right) \right), \end{aligned} \quad (195)$$

where we introduced a second canonical bracket

$$[g, h]_p = \nabla_{\perp} g \cdot \frac{\partial h}{\partial \mathbf{p}_{\perp}} - \nabla_{\perp} h \cdot \frac{\partial g}{\partial \mathbf{p}_{\perp}}, \quad (196)$$

acting on two functions  $g$  and  $h$  depending on both spatial and momentum coordinates.

The Hamiltonian (194) consists of the sum of magnetic and fluid kinetic energy, corresponding to the first two terms on the right-hand side (which also correspond to the Hamiltonian (54)) with the kinetic energy of the hot species, given by the last term on the right-hand side. With regard to the Poisson bracket, one notices that the first three terms on the right-hand side of Eq. (195) yield the Poisson bracket (55). The fourth and the fifth term introduce the coupling between the kinetic species and the fluid, whereas the last three terms account for the remaining contributions in the Vlasov equation.

Casimir invariants of the Poisson bracket (195) are given by the infinite families

$$C_1(\psi) = \int_D d^2x \mathcal{J}(\psi), \quad (197)$$

$$C_2(\omega, f, \psi) = \int_D d^2x \bar{\omega} \mathcal{I}(\psi), \quad (198)$$

$$C_3(f) = \int_D d^2x \int_{\mathbb{R}^3} d^3p \mathcal{K}(f), \quad (199)$$

where  $\mathcal{J}$ ,  $\mathcal{I}$  and  $\mathcal{K}$  are arbitrary functions and

$$\bar{\omega} = \omega - \hat{z} \cdot \nabla \times \mathbf{K}, \quad (200)$$

with

$$\mathbf{K} = \int_{\mathbb{R}^3} d^3p \mathbf{p}_{\perp} f, \quad (201)$$

is a generalized vorticity, which accounts also for the vorticity generated by the perpendicular velocity of the "hot" particle population. The Casimir invariant  $C_2$  thus generalizes the reduced MHD invariant  $C_2$  of Eq. (60), providing a conservation law for a hybrid fluid-kinetic cross-helicity. The invariants  $C_1$  and  $C_3$  reflect, on the other hand, the frozen-in condition of ideal reduced MHD, and the Casimir invariants of the Vlasov equation, respectively, the latter expressing also entropy conservation.

Combining the expressions (194) and (197)-(199), one can build the free energy functional

$$\mathcal{F} = H + C_1 + C_2 + C_3, \quad (202)$$

whose first variation reads

$$\begin{aligned} & \delta\mathcal{F}(\psi, \omega, f; \delta\psi, \delta\omega, \delta f) \\ &= \int_D d^2x [(-\Delta_\perp\psi + \mathcal{J}'(\psi) + \bar{\omega}\mathcal{I}'(\psi))\delta\psi + (\mathcal{I}(\psi) - \phi)\delta\omega] \\ &+ \int_D d^2x \int_{\mathbb{R}^3} d^3p \left( \frac{|\mathbf{p}|^2}{2} + \mathcal{K}'(f) - \mathcal{I}'(\psi)\mathbf{p}_\perp \cdot \nabla\psi \times \hat{z} \right) \delta f. \end{aligned} \quad (203)$$

Setting  $\delta\mathcal{F} = 0$  yields the equilibrium equations

$$-\Delta_\perp\psi + \mathcal{J}'(\psi) + \bar{\omega}\mathcal{I}'(\psi) = 0, \quad (204)$$

$$\mathcal{I}(\psi) - \phi = 0, \quad (205)$$

$$\mathcal{K}'(f) + \frac{|\mathbf{p}|^2}{2} - \mathcal{I}'(\psi)\mathbf{p}_\perp \cdot \nabla\psi \times \hat{z} = 0. \quad (206)$$

Again, it is natural to compare such equilibrium relations with the corresponding relations (64)-(65) of 2D reduced MHD. The hybrid Grad-Shafranov equation (204) has the same form of the MHD Grad-Shafranov equation (64), but with the vorticity  $\omega$  replaced by the hybrid vorticity  $\bar{\omega}$ . Alternatively [74], the hybrid Grad-Shafranov equation (204) can be recast as

$$-\Delta_\perp\psi + \mathcal{J}'(\psi) + \mathcal{I}'(\psi)\nabla_\perp \cdot ((1 + n_{kin})\nabla_\perp\mathcal{I}(\psi)) = 0. \quad (207)$$

The formulation (207) puts in evidence explicitly the role of the particle density of the kinetic population, defined as

$$n_{kin} = \int_{\mathbb{R}^3} d^3p f. \quad (208)$$

Equation (205) is identical to Eq. (65), implying that the presence of a kinetic population does not alter the property of the equilibrium MHD flow of being collinear to the magnetic field on constant- $\psi$  contour lines. Finally, using Eq. (204), from Eq. (206) we obtain the equilibrium relation

$$\frac{1}{2}|\mathbf{p} + \mathbf{v}_e|^2 - \frac{1}{2}|\mathbf{v}_e|^2 + \mathcal{K}'(f_e) = 0, \quad (209)$$

where we recall that  $\mathbf{v}_e = \hat{z} \times \nabla \phi_e$  is the equilibrium MHD velocity. Assuming that the function  $\mathcal{K}'$  be invertible, Eq. (209) yields

$$f_e = (\mathcal{K}')^{-1} \left( \frac{1}{2} |\mathbf{p} + \mathbf{v}_e|^2 - \frac{1}{2} |\mathbf{v}_e|^2 \right), \quad (210)$$

which generalizes the expression for a Vlasov equilibrium distribution function, by taking into account the presence of the MHD flow.

With regard to the stability conditions, one can express [74] the second variation of  $\mathcal{F}$  as

$$\begin{aligned} & \delta^2 \mathcal{F}(\psi_e, \omega_e, f_e; \delta\psi, \delta\omega, \delta f) \\ &= \delta^2 \mathcal{F}_{MHD} + \int_D d^2x \int_{\mathbb{R}^3} d^3p (\hat{z} \cdot \mathbf{p}_\perp \times \nabla_\perp f_e) \mathcal{I}''(\psi_e) (\delta\psi)^2 \\ &+ \frac{1}{2} \int d^2x \int_{\mathbb{R}^3} d^3p \mathcal{K}''(f_e) \left( \delta f - \frac{2}{\mathcal{K}''(f_e)} (\nabla_\perp \mathcal{I}'(\psi_e) \cdot \hat{z} \times \mathbf{p}_\perp) \delta\psi \right)^2 \\ &+ \frac{1}{2} \int d^2x \int_{\mathbb{R}^3} d^3p \mathcal{K}''(f_e) \left( \delta f - \frac{2}{\mathcal{K}''(f_e)} (\mathcal{I}'(\psi_e) \nabla_\perp \delta\psi \cdot \hat{z} \times \mathbf{p}_\perp) \right)^2 \quad (211) \\ &- 2 \int_D d^2x (\text{Tr } \Pi_{\perp e}) |\nabla_\perp \mathcal{I}'(\psi_e)|^2 (\delta\psi)^2 + 2 \int_D d^2x \int_{\mathbb{R}^3} d^3p \frac{1}{\mathcal{K}''(f_e)} (\mathbf{p}_\perp \cdot \nabla_\perp \mathcal{I}'(\psi_e))^2 (\delta\psi)^2 \\ &- 2 \int_D d^2x (\text{Tr } \Pi_{\perp e}) \mathcal{I}'^2(\psi_e) |\nabla_\perp \delta\psi|^2 + 2 \int_D d^2x \int_{\mathbb{R}^3} d^3p \frac{1}{\mathcal{K}''(f_e)} (\mathbf{p}_\perp \cdot \nabla_\perp \delta\psi)^2 \mathcal{I}'^2(\psi_e) \end{aligned}$$

where

$$\Pi_{\perp e} = \int_{\mathbb{R}^3} d^3p \frac{1}{\mathcal{K}''(f_e)} \mathbf{p}_\perp \mathbf{p}_\perp \quad (212)$$

is assumed to be well defined and where

$$\begin{aligned} & \delta^2 \mathcal{F}_{MHD}(\psi_e, \omega_e; \delta\psi, \delta\omega) \\ &= \int_D d^2x \left( |\nabla_\perp \delta\phi - \nabla_\perp (\mathcal{I}'(\psi_e) \delta\psi)|^2 + (1 - \mathcal{I}'^2(\psi_e)) |\nabla_\perp \delta\psi|^2 \right) \quad (213) \\ &+ (\mathcal{I}'(\psi_e) \Delta_\perp \mathcal{I}'(\psi_e) + \omega_e \mathcal{I}''(\psi_e) + \mathcal{J}''(\psi_e)) (\delta\psi)^2 \end{aligned}$$

is the second variation of the MHD contribution to  $\mathcal{F}$ , corresponding to Eq. (70).

Inspecting the sign of  $\delta^2 \mathcal{F}$ , it follows that stability is attained if the conditions

$$|\mathcal{I}'(\psi_e)|^2 < \frac{1}{1 + 2 \text{Tr } \Pi_{\perp e}}, \quad (214)$$

$$\bar{\omega}_e \mathcal{I}''(\psi_e) + \mathcal{J}''(\psi_e) + \mathcal{I}'(\psi_e) \Delta_\perp \mathcal{I}'(\psi_e) - 2 |\nabla_\perp \mathcal{I}'(\psi_e)|^2 \text{Tr } \Pi_{\perp e} > 0, \quad (215)$$

$$\mathcal{K}''(f_e) > 0, \quad (216)$$

are fulfilled.

Because of the relation  $\mathbf{v}_e = -\mathcal{I}'(\psi_e)\mathbf{B}_e$ , and accounting for the normalization of the variables, one sees that the condition (214) implies an upper bound on the amplitude of the equilibrium velocity. In the absence of a kinetic particle population, the condition would correspond to the equilibrium flow being sub-Alfvénic, with respect to the local Alfvén speed. The presence of the "hot" particles implies a stronger condition, thus reflecting the potential destabilizing role of the kinetic population. A further possible destabilizing role of the kinetic particles is associated with the last term on the left-hand side of Eq. (215). This condition can also be reformulated as

$$\begin{aligned} & \frac{1}{|\mathbf{B}_e|^2} \left[ \left( 1 - \frac{|\mathbf{v}_e|^2}{|\mathbf{B}_e|^2} \right) \mathbf{B}_e \times \hat{z} \cdot \nabla_{\perp} J_e - \left( \mathbf{B}_e \times \hat{z} \cdot \nabla_{\perp} \frac{|\mathbf{v}_e|^2}{|\mathbf{B}_e|^2} \right) \left( \mathbf{B}_e \times \hat{z} \cdot \frac{\nabla_{\perp} |\mathbf{B}_e|^2}{2|\mathbf{B}_e|^2} \right) \right] \\ & + \frac{|\mathbf{v}_e|}{|\mathbf{B}_e|} \frac{\mathbf{B}_e \times \hat{z}}{|\mathbf{B}_e|^2} \cdot \nabla_{\perp} (\hat{z} \cdot \nabla \times \mathbf{K}_e) - 2 \left| \nabla_{\perp} \left( \frac{|\mathbf{v}_e|}{|\mathbf{B}_e|} \right) \right|^2 \text{Tr} \Pi_{\perp_e} > 0, \end{aligned} \quad (217)$$

where we recall that  $J_e = -\Delta_{\perp} \psi_e$  corresponds to the equilibrium current density. The two terms on the first line of Eq. (217) descend from  $\delta^2 \mathcal{F}_{MHD}$  and are associated with the kink and interchange instabilities, respectively (see also Sec. 8). The remaining terms, on the other hand, are peculiar to the hybrid model. In particular, the first term on the second line is analogous to the kink instability term, but referred to the flow associated with the kinetic species. The last term provides a destabilizing contribution when the stability condition (216) is fulfilled.

Finally, from the inequality (216), we infer that Gaussian (with respect to the argument indicated in Eq. (210)) equilibrium distribution functions satisfy the stability condition.

## 7. Stability of magnetic island chains in the presence of electron temperature anisotropy

A useful feature of the EC method, with respect to spectral stability methods usually applied in plasma physics, is its capability of providing sufficient stability conditions for 2D equilibria. This feature turns out to be useful for investigating equilibria with non-trivial spatial structures, as is the case of chains of magnetic islands. Such structures



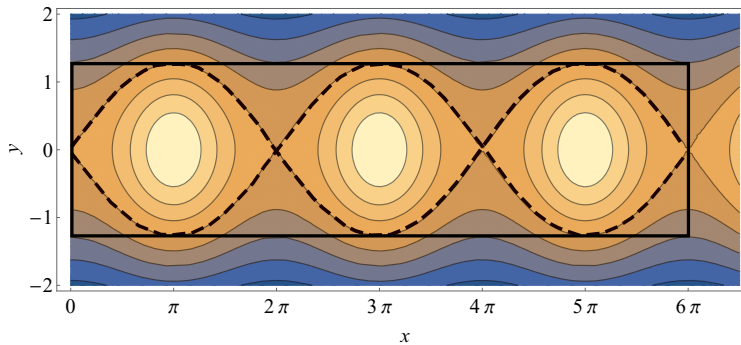


Figure 1: The figure shows a surface plot and some contour lines of the "cat's eyes" function  $\psi_e$  of Eq. (218). The domain  $\mathcal{D}$ , enclosed by the separatrixes, indicated with black dotted curves, and the domain  $R_n$ , corresponding to the rectangle enclosed by black solid lines, are also shown. The figure refers to the case  $n = 3$  and  $a = 1.12$ . After Ref. [79].

have long been studied in plasma physics and are often modelled by means of the classical Kelvin-Stuart "cat's eyes" [77, 78] solution. In terms of a magnetic flux function  $\psi$ , such equilibrium solution reads

$$\psi_e(x, y) = -\log(a \cosh y + \sqrt{a^2 - 1} \cos x), \quad (218)$$

where  $a > 1$ . The field lines of the associated magnetic field  $\mathbf{B}_{e\perp} = \nabla\psi_e \times \hat{z}$  correspond to the contour lines of  $\psi_e$ , an example of which is depicted in Fig. 1. The study of the stability of such magnetic island chains began already some decades ago [80, 81, 82, 83, 84]. These studies were based on the MHD description and were motivated mainly by the need to understand the stability of arrays of magnetic islands forming in both laboratory and space plasmas, as a consequence of magnetic reconnection. Always in the MHD context, the application of the EC method to study the stability of magnetic island chains in ideal MHD, was considered in Ref. [10]. A further application of the EC method to the related problem of the interchange instability of vortex chains is presented in Ref. [85].

More recently, solar wind observations [86, 87] as well as analytical modelling [88] motivated the investigation of the stability of magnetic island chains on the basis of models accounting for properties more specific to solar wind plasmas. In this context, reduced gyrofluid models, valid under the assumption of a magnetic field with

a strong guide field component, can provide a valuable tool. Gyrofluid models [89] are derived from gyrokinetic systems and describe the evolution of moments of the distribution functions of the particle gyrocenters, unlike traditional fluid models which evolve moments of particle distribution functions. In Ref. [79] a reduced gyrofluid model was derived and applied, in two specific limits, to derive stability conditions for magnetic island chains in the presence of electron temperature anisotropy. Indeed, collisionless plasmas as the solar wind, often exhibit this kind of anisotropy.

In the following, we summarize the results described in Ref. [79] about the formal stability of magnetic island chains, in the two limits of hot and cold ions, respectively. These two limits correspond to  $\tau_{\perp i} \gg 1$  and  $\tau_{\perp i} \ll 1$ , respectively, where  $\tau_{\perp i} = T_{0\perp i}/T_{0\perp e}$  is the ratio between ion and electron perpendicular temperatures, with respect to the magnetic field, in a bi-Maxwellian distribution function which is taken as equilibrium distribution function. The analysis is carried out in the 2D limit, assuming invariance along the  $z$  axis of a Cartesian coordinate system, which is also the axis along which the guide field is directed. Indeed, the magnetic field, in dimensionless units, is taken of the form

$$\mathbf{B}(x, y, t) = \nabla\psi(x, y, t) \times \hat{z} + (1 + B_{\parallel}(x, y, t))\hat{z} \quad (219)$$

where we indicated explicitly the dependence on the independent variables in order to identify the perturbations  $\psi$  and  $B_{\parallel}$  and separate them from the guide field which, in normalized variables, is the unit vector along  $\hat{z}$ .

The analysis is carried out in the domain

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi n, \quad |y| \leq \cosh^{-1} \left( 1 + \frac{\sqrt{a^2 - 1}}{a} (1 - \cos x) \right) \right\}, \quad (220)$$

with  $a > 1$  and  $n$  a positive integer. The domain is depicted in Fig. 1, where one can see that it corresponds to the space enclosed by the separatrices of  $n$  islands. The choice of the domain is inspired by Ref. [34], where nonlinear stability for "cat's eyes" solutions in the case of an incompressible 2D fluid is treated. Such domain indeed allows for a natural choice of boundary conditions.

In the following, the two cases of hot and cold ions are discussed, beginning directly with the presentation of the corresponding model equations. We refer the Reader to Ref. [79] for the derivation of the more general gyrofluid model as well as for the derivation of the models in the two limits.

### 7.1. Hot-ion limit

In the limit  $\tau_{\perp i} \gg 1$ , assuming sufficiently small ion gyrocenter density and parallel velocity fluctuations along the guide field, the 2D plasma dynamics for electrons with constant but anisotropic temperature can be described by the set of equations

$$\frac{\partial N_e}{\partial t} - b_{\star}[\psi, \Delta_{\perp} \psi]_x = 0, \quad (221)$$

$$\frac{\partial \psi}{\partial t} - \kappa[N_e, \psi]_x = 0. \quad (222)$$

In Eqs. (221)-(222) we indicated with  $N_e = N_e(x, y, t)$  the gyrocenter electron density fluctuations, whereas  $b_{\star}$  and  $\kappa$  are two parameters defined by

$$b_{\star} = \frac{2}{\beta_{\perp e}} + 1 - \frac{1}{\Theta_e}, \quad (223)$$

$$\kappa = \frac{2}{\beta_{\perp e}} + \frac{1}{\Theta_e} - 1. \quad (224)$$

In the definitions of  $b_{\star}$  and  $\kappa$  we introduced  $\beta_{\perp e} = 8\pi n_0 T_{0\perp e} / B_0^2$  and  $\Theta_e = T_{0\perp e} / T_{0\parallel e}$ , where  $n_0$  is the equilibrium constant density for both species,  $B_0$  is the dimensional amplitude of the guide field and  $T_{0\parallel e}$  is the electron parallel equilibrium temperature. The parameter  $\beta_{\perp e}$  represents the ratio between the equilibrium electron perpendicular temperature and the magnetic pressure associated with the guide field. The parameter  $\Theta_e$ , on the other hand, is a measure of the electron temperature anisotropy.

Equation (221) descends from the electron continuity equation, whereas Eq. (222) can be seen as an ideal Ohm's law, descending from the parallel component of the electron momentum equation. The model is valid on scales much larger than the electron thermal gyroradius, so that electron finite Larmor radius effects are neglected. Note that, in the hot-ion limit, the following relations

$$B_{\parallel} = -N_e, \quad \phi = -\frac{2}{\beta_{\perp e}} N_e. \quad (225)$$

hold, where we indicated with  $\phi$  the normalized electrostatic potential. Once the system (221)-(222) is solved for  $N_e$  and  $\psi$ , one can thus retrieve the parallel magnetic and the electrostatic potential fluctuations from Eqs. (225).

We impose that the fields  $N_e$  and  $\psi$  satisfy the boundary conditions

$$\psi|_{\partial\mathcal{D}} = a_\psi, \quad (226)$$

$$N_e|_{\partial\mathcal{D}} = a_N, \quad (227)$$

where  $\partial\mathcal{D}$  is the boundary of  $\mathcal{D}$  and  $a_\psi, a_N$  are constants. Condition (226) implies that  $\mathbf{B}_\perp \cdot \mathbf{n} = 0$ , with  $\mathbf{n}$  the outward unit vector normal to  $\partial\mathcal{D}$ , so that the magnetic field in the plane perpendicular to the guide field is tangent to the boundary. On the other hand, from Eq. (225) it follows that the condition (227) implies  $\mathbf{U}_\perp \cdot \mathbf{n} = 0$ , where  $\mathbf{U}_\perp = \hat{z} \times \nabla(\phi - B_\parallel) = (1 - 2/\beta_\perp) \hat{z} \times \nabla N_e$  is the perpendicular velocity field advecting  $\psi$  and  $N_e$  [79], and given by the sum of  $\mathbf{E} \times \mathbf{B}$  and grad  $B$  drifts. Such velocity field is therefore tangent to the boundary.

The model (221)-(222) can be given a noncanonical Hamiltonian formulation choosing  $(\chi^1, \chi^2) = (N_e, \psi)$  as dynamical variables, with

$$H(N_e, \psi) = \int_D d^2x \left( b_\star \frac{|\nabla\psi|^2}{2} + \kappa \frac{N_e^2}{2} \right) \quad (228)$$

as Hamiltonian and

$$\{F, G\} = \int_D d^2x (N_e [F_{N_e}, G_{N_e}]_x + \psi ([F_\psi, G_{N_e}]_x + [F_{N_e}, G_\psi]_x)). \quad (229)$$

as Poisson bracket.

Because the bracket (229) has the same structure as the bracket (60) of reduced MHD, we infer that the Casimir invariants are also analogous and read

$$C_1(\psi) = \int_D d^2x \mathcal{J}(\psi), \quad C_2(N_e, \psi) = \int_D d^2x N_e \mathcal{I}(\psi), \quad (230)$$

with  $\mathcal{J}$  and  $\mathcal{I}$  arbitrary functions.

Note that, in order to show that the observables  $H, C_1$  and  $C_2$  are conserved by the dynamics of Eqs. (221)-(222) on the domain  $\mathcal{D}$  with boundary conditions (226)-(227),

one has to resort to the relations

$$\int_D d^2x f \Delta_{\perp} g = - \int_D d^2x \nabla f \cdot \nabla g + \int_{\partial \mathcal{D}} f \frac{\partial g}{\partial n} ds, \quad (231)$$

$$\int_D d^2x f [g, h]_x = \int_D d^2x h [f, g]_x - \int_{\partial \mathcal{D}} h f \nabla g \cdot d\mathbf{l}, \quad (232)$$

for functions  $f, g$  and  $h$  and where  $d\mathbf{l}$  is an arc element along  $\partial \mathcal{D}$ . Actually, the Poisson bracket (229) is a valid Poisson bracket for functionals  $F$  and  $G$  such that  $\nabla F_{\psi, N_e}$  and  $\nabla G_{\psi, N_e}$  are normal to the boundary  $\partial \mathcal{D}$ . In any case, the functionals  $H, C_1$  and  $C_2$  are conserved independently on the validity of the Poisson bracket (229). In fact, we recall that, as pointed out in Sec. 2.4, in general the free energy functional can be built by adding known constants of motion, not necessarily corresponding to Casimir invariants.

We define

$$\mathcal{F}(N_e, \psi) = \int_D d^2x \left( b_{\star} \frac{|\nabla \psi|^2}{2} + \kappa \frac{N_e^2}{2} + N_e \mathcal{I}(\psi) + \mathcal{J}(\psi) \right). \quad (233)$$

From setting to zero the first variation

$$\begin{aligned} \delta \mathcal{F}(N_e, \psi; \delta N_e, \delta \psi) = & \quad (234) \\ \int_D d^2x \left( (-b_{\star} \Delta_{\perp} \psi + \mathcal{I}'(\psi) N_e + \mathcal{J}'(\psi)) \delta \psi + (\kappa N_e + \mathcal{I}(\psi)) \delta N_e \right), \end{aligned}$$

one readily obtains the equilibrium equations

$$- \Delta_{\perp} \psi + \frac{\mathcal{J}'(\psi)}{b_{\star}} + \frac{\mathcal{I}'(\psi) N_e}{b_{\star}} = 0, \quad (235)$$

$$\mathcal{I}(\psi) + \kappa N_e = 0. \quad (236)$$

The system (235)-(236) closely resembles the system (64)-(65) of reduced MHD. Equation (235) is a modified Grad-Shafranov equation, with the electron gyrocenter density playing the role of the vorticity. Recalling the proportionality relations (225) and the expression for  $\mathbf{U}_{\perp e}$ , Eq. (236) implies that, at equilibrium,  $\mathbf{U}_{\perp e}$  is proportional to the magnetic field, on contour lines of  $\psi$ . Static solutions correspond to the choice  $\mathcal{I} = 0$ . Choosing  $\mathcal{I}(\psi) = \pm \sqrt{2/\beta_{\perp e}} \psi$  yields the analogous of Alfvénic solutions, with an Alfvén speed modified by the presence of electron temperature anisotropy.

Because we are interested in the stability of magnetic island chains, we need to consider the choice of the arbitrary functions  $\mathcal{I}$  and  $\mathcal{J}$  yielding (218) as equilibrium solution for  $\psi$ . It is well known that the "cat's eyes" function (218) is solution of the Liouville equation

$$\Delta_{\perp}\psi = -e^{2\psi}. \quad (237)$$

We can thus obtain a magnetic island chain as equilibrium solution by choosing  $\mathcal{J}$  and  $\mathcal{I}$  such that Eq. (235) coincides with Eq. (237). This is accomplished by imposing

$$\mathcal{J}(\psi) = -\frac{b_{\star}}{2}e^{2\psi} + \frac{\mathcal{I}^2(\psi)}{2\kappa} + c_1, \quad (238)$$

where  $c_1$  is an arbitrary constant. One is thus left with a family of equilibria, depending on the free function  $\mathcal{I}(\psi_e)$ , all of which possess the function (218) as solution for the magnetic flux function.

The second variation of the free energy functional, evaluated at a generic equilibrium solution  $(N_e, \psi) = (N_{ee}, \psi_e)$ , is given by

$$\begin{aligned} \delta^2\mathcal{F}(\psi_e, N_{ee}; \delta\psi, \delta N_e) &= \int_D d^2x \left( b_{\star}|\nabla\delta\psi|^2 + (\mathcal{I}''(\psi_e)N_{ee} + \mathcal{J}''(\psi_e) - \mathcal{I}'^2(\psi_e))(\delta\psi)^2 \right. \\ &\quad \left. + (\kappa - 1)(\delta N_e)^2 + (\mathcal{I}'(\psi_e)\delta\psi + \delta N_e)^2 \right). \end{aligned} \quad (239)$$

Clearly, if  $b_{\star} > 0$  and  $\kappa > 1$ , only the second term on the right-hand side of Eq. (239) can provide a negative contribution and thus indefiniteness. In particular, when specifying for  $\mathcal{J}$  the constraint (238), such contribution turns out to be indeed negative (a similar feature is present also in the "cat's eyes" solutions for the 2D Euler equation of an incompressible fluid [90]).

This problem can be overcome, as done in Ref. [34], by resorting to a Poincaré inequality, which turns out to be effective for finding lower bounds to functionals such as  $\delta^2\mathcal{F}$ . In particular, we consider the Poincaré inequality

$$\int_D d^2x |\nabla\delta\psi|^2 \geq k_{min}^2 \int_D d^2x (\delta\psi)^2, \quad (240)$$

with  $\delta\psi|_{\partial\mathcal{D}} = 0$  and  $k_{min}^2$  the minimal eigenvalue of the operator  $-\Delta_{\perp}$  acting on the functions defined over  $\mathcal{D}$  and vanishing on  $\partial\mathcal{D}$ . Still following Ref. [34], one can define

the domain

$$R = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2n\pi, \quad |y| \leq l = \cosh^{-1} \left( 1 + 2 \frac{\sqrt{a^2 - 1}}{a} \right) \right\} \quad (241)$$

and take advantage of the inequality  $k_{min}^2 > k_R^2$ , where  $k_R^2$  is the minimum eigenvalue of the operator  $-\Delta_\perp$  acting on functions defined on  $R$  and vanishing on the boundary of  $R$ . The domain  $R \supset \mathcal{D}$ , an example of which is provided in Fig. 1, is a rectangle of height  $2l$ , equal to the island width, and enclosing the chain of magnetic islands. The constant  $k_R^2$  can be determined exactly. In particular, one has

$$k_R^2 = \frac{1}{4n^2} + \frac{\pi^2}{4l^2}. \quad (242)$$

Thus, assuming  $b_\star > 0$  and making use of the Poincaré inequality (240), one obtains the relation

$$\begin{aligned} \delta^2 \mathcal{F}(\psi_e, -\mathcal{I}(\psi_e)/\kappa; \delta\psi, \delta N_e) &\geq \int_D d^2x \left( (b_\star k_R^2 - 2b_\star e^{2\psi_e} + (1/\kappa - 1)\mathcal{I}'^2(\psi_e)) (\delta\psi)^2 \right. \\ &\quad \left. + (\kappa - 1)(\delta N_e)^2 + (\mathcal{I}'(\psi_e)\delta\psi + \delta N_e)^2 \right), \end{aligned} \quad (243)$$

where we also used Eq. (236). Specializing to the "cat's eyes" solution (218), and recalling that

$$\min_{(x,y) \in \mathcal{D}} (-2b_\star e^{2\psi_e(x,y)}) = -2b_\star e^{2\psi_e(\pi,0)} = -\frac{2b_\star}{(a - \sqrt{a^2 - 1})^2}, \quad (244)$$

one obtains the following inequality, valid for  $(x, y) \in \mathcal{D}$ :

$$\begin{aligned} &b_\star k_R^2 - 2b_\star e^{2\psi_e(x,y)} + \left( \frac{1}{\kappa} - 1 \right) \mathcal{I}'^2(\psi_e(x, y)) \\ &\geq b_\star \left( k_R^2 - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) + \left( \frac{1}{\kappa} - 1 \right) \mathcal{I}'^2(\psi_e(x, y)). \end{aligned} \quad (245)$$

Making use of the relations (243), (245) and recalling the above mentioned conditions  $b_\star > 0$  and  $\kappa > 1$ , one obtains that the second variation of the free energy functional for the "cat's eyes" solution is positive if the conditions

$$b_\star > 0, \quad (246)$$

$$\kappa > 1, \quad (247)$$

$$b_\star \left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} - \frac{2}{(a - \sqrt{a^2 - 1})^2} \right) \geq \max_{(x,y) \in \mathcal{D}} \left( 1 - \frac{1}{\kappa} \right) \mathcal{I}'^2(\psi_e(x, y)) \quad (248)$$

are fulfilled.

Conditions (246) and (247) are valid for any choice of  $\mathcal{I}$  (and thus for any equilibrium for the velocity  $\mathbf{U}_{\perp e}$ ). Such two conditions imply

$$\Theta_e > \frac{\beta_{\perp e}}{2 + \beta_{\perp e}}, \quad \text{if } 0 < \beta_{\perp e} \leq 1, \quad (249)$$

$$\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} < \Theta_e < \frac{\beta_{\perp e}}{2(\beta_{\perp e} - 1)} \quad \text{if } 1 < \beta_{\perp e} < 4. \quad (250)$$

One thus sees that, the stability conditions (249)-(250) imply an upper bound for  $\beta_{\perp e}$ , corresponding to  $\beta_{\perp e} = 4$ . Also, in both conditions one finds the inequality  $\Theta_e > \beta_{\perp e}/(2 + \beta_{\perp e})$ , thus retrieving the well known condition for avoiding the firehose instability due to temperature anisotropy (see, e.g. Ref. [91]).

The condition (248), on the other hand, depends on the choice of the free function  $\mathcal{I}$ , as well as on specific parameters of the magnetic islands, such as  $n$  and  $a$ . Making use of the expression of  $l$  in terms of  $a$ , following from the definition of the domain  $R$ , such condition can indeed be reformulated as

$$\begin{aligned} b_{\star} & \left( \frac{1}{4n^2} + \frac{\pi^2}{4 \left( \cosh^{-1} \left( 1 + 2 \frac{\sqrt{a^2-1}}{a} \right) \right)^2} - \frac{2}{(a - \sqrt{a^2-1})^2} \right) \\ & \geq \max_{(x,y) \in \mathcal{D}} \left( 1 - \frac{1}{\kappa} \right) \mathcal{I}^2(\psi_e(x, y)). \end{aligned} \quad (251)$$

As discussed in Ref. [79], in the absence of perpendicular velocity equilibrium (i.e. for the choice  $\mathcal{I} = 0$ ), in the case  $n = 1$  of a single island, the condition (248), for  $b_{\star} > 0$ , is fulfilled if

$$1 < a < 1.026..., \quad (252)$$

which implies a limitation on the size of the island.

With the choice  $\mathcal{I}(\psi) = V_1 \psi$ , for constant  $V_1$ , one obtains a one-parameter family of equilibria including the Alfvénic flows. In this case, the relation (248) becomes

$$\begin{aligned} b_{\star} & \left( \frac{1}{4n^2} + \frac{\pi^2}{4 \left( \cosh^{-1} \left( 1 + 2 \frac{\sqrt{a^2-1}}{a} \right) \right)^2} - \frac{2}{(a - \sqrt{a^2-1})^2} \right) \\ & \geq \left( 1 - \frac{1}{\kappa} \right) V_1^2. \end{aligned} \quad (253)$$



This condition can be interpreted as an upper bound on the square of the equilibrium flow amplitude, which is proportional to  $V_1^2$ . One thus retrieves another modified version of the sub-Alfvénic condition for stability such as the inequality (214) encountered for the hybrid kinetic-MHD model.

### 7.2. Cold-ion limit

We consider here the opposite limit  $\tau_{\perp i} \ll 1$ . This regime is typically less relevant for solar wind plasmas but can be of interest for laboratory plasmas and is useful for comparison with other similar reduced models.

In the cold-ion limit we consider the plasma dynamics as governed by the system [79]

$$\frac{\partial N_e}{\partial t} + [\phi, N_e]_x - b_*[\psi, \Delta_{\perp} \psi]_x = 0, \quad (254)$$

$$\frac{\partial \psi}{\partial t} + [\phi, \psi]_x + \lambda[N_e, \psi]_x = 0, \quad (255)$$

where the same notation of the hot-ion case is used and where the parameter  $\lambda$  is defined by

$$\lambda = \frac{\beta_{\perp e}}{2 + \beta_{\perp e}} - \frac{1}{\Theta_e}. \quad (256)$$

We point out that, in the derivation of the model, the contributions of the ion gyrocenter density and parallel velocity have been neglected. Although this follows from an asymptotic limit for  $\tau_{\perp i} \gg 1$ , this is not the case for  $\tau_{\perp i} \ll 1$ . Therefore, in the cold-ion limit, neglecting such contributions has to be taken just as a simplifying assumption. The coupling of the system with the evolution equations for the ion gyrocenter density and parallel velocity should in principle be taken into account in a consistent asymptotic limit  $\tau_{\perp i} \rightarrow 0$  for finite  $\beta_{\perp e}$ .

In the cold-ion limit, the parallel magnetic perturbations and the electrostatic potential are related to the dynamical variables by

$$B_{\parallel} = -\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} N_e, \quad \Delta_{\perp} \phi = N_e. \quad (257)$$

Note that, when inserting the relation  $N_e = \Delta_\perp \phi$  into the system (254)-(255), the continuity equation (254) takes the form of a vorticity equation with the coefficient of the Lorentz force (last term on the left-hand side) modified by the presence of electron temperature anisotropy [92]. On the other hand, Ohm's law (255) includes, in addition to the  $\mathbf{E} \times \mathbf{B}$  advection term already present in reduced MHD and corresponding to the second term on the left-hand side, the additional term, identified by the  $\lambda$  coefficient, which originates from the divergence of the anisotropic electron pressure tensor. A similar term, in the isotropic, low- $\beta$  limit, is present in the reduced fluid model of Ref. [93]. Note that, despite the presence of this additional term, the magnetic flux function  $\psi$  is still purely advected, given that Eq. (255) can be written as

$$\frac{\partial \psi}{\partial t} + \mathbf{v}_\lambda \cdot \nabla \psi = 0, \quad (258)$$

where

$$\mathbf{v}_\lambda = \hat{z} \times \nabla(\phi + \lambda \Delta_\perp \phi). \quad (259)$$

The perpendicular magnetic field  $\mathbf{B}_\perp$  is thus frozen into a fluid moving with velocity  $\mathbf{v}_\lambda$ .

The following boundary conditions are adopted:

$$\psi|_{\partial \mathcal{D}} = a_\psi, \quad (260)$$

$$\phi|_{\partial \mathcal{D}} = a_\phi, \quad (261)$$

with constants  $a_\psi$  and  $a_\phi$ . In addition to the condition (260), imposing  $\mathbf{B}_\perp$  tangent to the boundary, we have the condition (261), implying that the  $\mathbf{E} \times \mathbf{B}$  velocity field  $\mathbf{U}_{\mathbf{E} \times \mathbf{B}} = \hat{z} \times \nabla \phi$  is also tangent to the boundary of  $\mathcal{D}$ .

With such boundary conditions the cold-ion model admits the following conserved functional

$$H(N_e, \psi) = \int_D d^2x \left( b_\star \frac{|\nabla \psi|^2}{2} + \frac{|\nabla \phi|^2}{2} - \lambda \frac{N_e^2}{2} \right), \quad (262)$$

which, in concomitance with the same Poisson bracket (229) of the hot-ion model, provides the Hamiltonian structure.

The same functionals (230) of the hot-ion case are also conserved. Thus, one can define the free energy functional

$$\mathcal{F}(N_e, \psi) = \int_D d^2x \left( b_\star \frac{|\nabla\psi|^2}{2} + \frac{|\nabla\phi|^2}{2} - \lambda \frac{N_e^2}{2} + N_e \mathcal{I}(\psi) + \mathcal{J}(\psi) \right). \quad (263)$$

Extremizing the first variation

$$\begin{aligned} \delta\mathcal{F}(N_e, \psi; \delta N_e, \delta\psi) = \\ \int_D d^2x \left( (-b_\star \Delta_\perp \psi + \mathcal{I}'(\psi) N_e + \mathcal{J}'(\psi)) \delta\psi + (\mathcal{I}(\psi) - \lambda N_e - \phi) \delta N_e \right) \end{aligned} \quad (264)$$

leads to the equilibrium relations

$$\Delta_\perp \psi = \frac{\mathcal{I}'(\psi) N_e}{b_\star} + \frac{\mathcal{J}'(\psi)}{b_\star}, \quad (265)$$

$$\mathcal{I}(\psi) = \phi + \lambda N_e. \quad (266)$$

The constraint that  $\psi$  obeys Liouville's equation (237) then leads to

$$\mathcal{J}'(\psi) = -N_e \mathcal{I}'(\psi) - b_\star e^{2\psi}. \quad (267)$$

Also in this case one has to investigate a family of equilibria with a magnetic island chain and depending on a free function  $\mathcal{I}$ . We summarize the analysis of Ref. [79] which separates two cases. The first case corresponds to  $\mathcal{I}'(\psi) \neq 0$ , so that, from Eqs. (237) and (265), one obtains

$$N_e = -\frac{b_\star e^{2\psi} + \mathcal{J}'(\psi)}{\mathcal{I}'(\psi)}. \quad (268)$$

The relation (268) implies that  $N_e = N_e(\psi)$ . From Eq. (266) it also follows then, that  $\phi = \phi(\psi)$ . In particular, from combining the relation  $\phi = \phi(\psi)$  with Eq. (267) and with the relation  $N_e = \Delta_\perp \phi$ , it follows (details are provided in Ref. [79]), that  $\phi$  has to be of the form

$$\phi = K_1 \psi + K_2, \quad (269)$$

with constants  $K_1$  and  $K_2$ . Using the relation  $N_e = \Delta_\perp \phi$ , it follows that the equilibrium solutions supporting magnetic island chains, for the case  $\mathcal{I}'(\psi) \neq 0$ , are given by

$$\psi = \psi_e = -\log(a \cosh y + \sqrt{a^2 - 1} \cos x), \quad (270)$$

$$N_e = K_1 \Delta_\perp \psi_e = -\frac{K_1}{(a \cosh y + \sqrt{a^2 - 1} \cos x)^2}, \quad (271)$$

with  $K_1 \neq 0$ . The free function  $\mathcal{I}$  and  $\mathcal{J}$ , in this case, are given by

$$\mathcal{I}(\psi) = -\lambda K_1 e^{2\psi} + K_1 \psi + K_2, \quad (272)$$

$$\mathcal{J}(\psi) = -\frac{\lambda K_1^2}{2} e^{4\psi} + \frac{K_1^2 - b_\star}{2} e^{2\psi} + G_1, \quad (273)$$

with arbitrary constant  $G_1$ .

From the proportionality relation (269), it follows that, at equilibrium, the  $\mathbf{E} \times \mathbf{B}$  velocity is proportional to the local Alfvén velocity, whereas  $N_{ee}$  and  $B_{\parallel e}$  are both proportional to the equilibrium current density, which is equal to  $-\Delta_\perp \psi_e$ .

In the case  $\mathcal{I}'(\psi) = 0$ , one has

$$\mathcal{I}(\psi) = F_1, \quad (274)$$

with constant  $F_1$ , and Eqs. (265) and (266) get decoupled. In particular, the electrostatic potential gets determined solely by Eq. (266). We note that, for  $\lambda \neq 0$ , one can have a non-trivial solution for  $\phi$ , and thus a non-trivial  $\mathbf{E} \times \mathbf{B}$  equilibrium velocity field. This establishes a difference between this model and models, such as reduced MHD, where the magnetic flux function is purely advected by the  $\mathbf{E} \times \mathbf{B}$  flow. In reduced MHD, for instance, the choice  $\mathcal{I} = \text{constant}$ , leads to  $\nabla \phi_e = 0$ , as it follows from Eq. (65), and consequently to a vanishing  $\mathbf{E} \times \mathbf{B}$  equilibrium flow. This difference reflects the presence of finite electron temperature effects in our model.

For the case  $\mathcal{I}'(\psi) \neq 0$  the second variation of the free energy functional is given by

$$\begin{aligned} \delta^2 \mathcal{F}(\psi, N_e; \delta\psi, \delta N_e) &= \int_D d^2x \left( b_\star |\nabla \delta\psi|^2 + |\nabla \delta\phi|^2 \right. \\ &\quad \left. - \lambda (\delta N_e)^2 + 2\mathcal{I}'(\psi) \delta N_e \delta\psi + (\mathcal{J}''(\psi) + N_e \mathcal{I}''(\psi)) (\delta\psi)^2 \right). \end{aligned} \quad (275)$$

When evaluated at the equilibria with magnetic island chains this second variation can be written as [79]

$$\begin{aligned} \delta^2 \mathcal{F}(\psi_e, K_1 \Delta_\perp \psi_e; \delta\psi, \delta N_e) &= \\ &\int_D d^2x \left( (b_\star - K_1^2 (1 - 2\lambda e^{2\psi_e})^2) |\nabla \delta\psi|^2 + |\nabla \delta\phi - K_1 (1 - 2\lambda e^{2\psi_e}) \nabla \delta\psi|^2 \right. \\ &\quad \left. + (K_1^2 - b_\star + 4\lambda K_1^2 |\nabla \psi_e|^2 (2\lambda e^{2\psi_e} - 1) - 4\lambda^2 K_1^2 e^{4\psi_e}) 2e^{2\psi_e} (\delta\psi)^2 - \lambda (\delta N_e)^2 \right), \end{aligned} \quad (276)$$

Definite positiveness of  $\delta^2\mathcal{F}$  can be attained by imposing that the coefficients of  $|\nabla\delta\psi|^2$ ,  $(\delta\psi)^2$  and  $(\delta N_e)^2$  be positive. In order for the coefficient of  $(\delta N_e)^2$  to be positive, one requires  $\lambda < 0$ . The coefficient of  $|\nabla\delta\psi|^2$  is positive if  $b_\star > \max_{(x,y)\in\mathcal{D}} K_1^2(1 - 2\lambda e^{2\psi_e(x,y)})^2$ , which, for  $\lambda < 0$ , occurs for

$$b_\star > K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})^2} \right)^2. \quad (277)$$

The coefficient of  $(\delta\psi)^2$ , on the other hand, can be made positive, for  $\lambda < 0$ , by setting  $b_\star < K_1^2(1 - 4\lambda^2 \exp(4\psi_e))$ . However, the latter condition is not compatible with the condition (277). Also in this case the problem can be circumvented by resorting to Poincaré inequality, thanks to which we can obtain the bound

$$\begin{aligned} \delta^2\mathcal{F}(\psi_e, K_1\Delta_\perp\psi_e; \delta\psi, \delta N_e) &\geq \int_D d^2x \left( |\nabla\delta\psi - K_1(1 - 2\lambda e^{2\psi_e})\nabla\delta\psi|^2 \right. \\ &+ (k_R^2(b_\star - K_1^2(1 - 2\lambda e^{2\psi_e}))^2) + (K_1^2 - b_\star + 4\lambda K_1^2|\nabla\psi_e|^2(2\lambda e^{2\psi_e} - 1) \\ &\left. - 4\lambda^2 K_1^2 e^{4\psi_e})2e^{2\psi_e}(\delta\psi)^2 - \lambda(\delta N_e)^2 \right). \end{aligned} \quad (278)$$

By virtue of the relation

$$\begin{aligned} &k_R^2(b_\star - K_1^2(1 - 2\lambda e^{2\psi_e}))^2 + (K_1^2 - b_\star - 4\lambda^2 K_1^2 e^{4\psi_e})2e^{2\psi_e} \\ &\geq k_R^2(b_\star - \max_{(x,y)\in\mathcal{D}} \{K_1^2(1 - 2\lambda e^{2\psi_e(x,y)})^2\}) + \min_{(x,y)\in\mathcal{D}} \{(K_1^2 - b_\star - 4\lambda^2 K_1^2 e^{4\psi_e})2e^{2\psi_e}\} \quad (279) \\ &= k_R^2 \left( b_\star - K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})^2} \right)^2 \right) - 2 \frac{b_\star - K_1^2}{(a - \sqrt{a^2 - 1})^2} - 8 \frac{\lambda^2 K_1^2}{(a - \sqrt{a^2 - 1})^6}. \end{aligned} \quad (280)$$

one can then conclude that the second variation is positive definite if the following three

conditions

$$b_\star > K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})^2} \right)^2, \quad (281)$$

$$\lambda < 0, \quad (282)$$

$$\begin{aligned} & \left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} \right) \left( b_\star - K_1^2 \left( 1 - \frac{2\lambda}{(a - \sqrt{a^2 - 1})^2} \right)^2 \right) \\ & > 2 \frac{b_\star - K_1^2}{(a - \sqrt{a^2 - 1})^2} + 8 \frac{\lambda^2 K_1^2}{(a - \sqrt{a^2 - 1})^6}. \end{aligned} \quad (283)$$

hold.

The first condition (281) can be interpreted as an upper limit to the amplitude  $K_1$  of the  $\mathbf{E} \times \mathbf{B}$  equilibrium flow and also as a condition on the suppression of the firehose instability (which we recall to occur, at least for equilibria without flow, for  $b_\star < 0$ ). The second condition (282) can be recast as  $\Theta_e < 1 + 2/\beta_{\perp e}$  and provides an upper bound to the temperature anisotropy, similarly to the condition (247) of the hot-ion case. Finally, the relation (283) can be seen as a condition on  $a$ , which controls the size of the magnetic islands.

For the case  $\mathcal{I}'(\psi) = 0$  the second variation of the free energy functional, evaluated at the equilibria of interest, is given by

$$\begin{aligned} \delta^2 \mathcal{F}(\psi_e, \Delta_\perp \phi_{eq}; \delta\psi, \delta N_e) = \\ \int_D d^2x \left( b_\star |\nabla \delta\psi|^2 + |\nabla \delta\phi|^2 - 2b_\star e^{2\psi_e} (\delta\psi)^2 - \lambda (\delta N_e)^2 \right). \end{aligned} \quad (284)$$

The expression (284) corresponds to the second variation (276) in the limit  $K_1 = 0$ . Therefore, stability conditions can be obtained directly from Eqs. (281)-(283) specializing to the case  $K_1 = 0$ , which yields

$$\frac{\beta_{\perp e}}{2 + \beta_{\perp e}} < \Theta_e < 1 + \frac{2}{\beta_{\perp e}}, \quad (285)$$

$$\left( \frac{1}{4n^2} + \frac{\pi^2}{4l^2} \right) > \frac{2}{(a - \sqrt{a^2 - 1})^2}. \quad (286)$$

The condition (285) sets a lower and an upper bound to temperature anisotropy, whereas condition (286) can be seen as a condition on the parameter  $a$ , with consequences analogous to those of the condition (251) for hot ions, but with no equilibrium perpendicular flow.

## 8. Connection with the $\delta W$ energy principle for compressible reduced MHD

One of the classical methods used to ascertain stability of static plasma equilibria, in the fluid and dissipationless description, is the so called  $\delta W$  energy principle [23], (some extensions of which to the case of equilibria with flows were formulated later in Ref. [94]). We refer the Reader to References such as [95, 96, 44] for detailed expositions. Here we are content to recall the basic elements of this method, which are relevant in our context. Loosely speaking, the energy principle states that, if the functional  $\delta W$ , which represents the variation of potential energy as a consequence of a displacement from equilibrium, is non-negative for every allowed displacement, then the equilibrium is linearly stable (a more quantitative expression of the energy principle will be provided below). Given the variational nature of the  $\delta W$  energy principle, a natural question concerns its possible relation with the EC method. The compatibility between the results of the energy principle and the EC method was pointed out in Ref. [10]. The precise correspondence between the two methods for the case of reduced MHD was, on the other hand, shown in Ref. [97]. Here we review the results of Ref. [98], where the relation between the energy principle and the EC method is investigated for the case of 2D CRMHD [99, 100], a model which generalizes 2D reduced MHD by accounting for perturbations of the electron pressure and of the velocity parallel to the guide field, as well as for magnetic field curvature. This model, although formulated in a slab geometry with Cartesian coordinates, can indeed be considered as a model describing locally the dynamics of a tokamak plasma.

We proceed by first introducing the CRMHD system in normalized form, which

reads

$$\frac{\partial \psi}{\partial t} = -[\phi, \psi]_x, \quad (287)$$

$$\frac{\partial \omega}{\partial t} = -[\phi, \omega]_x + [\psi, \Delta_{\perp} \psi]_x + 2[p, h]_x, \quad (288)$$

$$\frac{\partial v}{\partial t} = -[\phi, v]_x - [p, \psi]_x, \quad (289)$$

$$\frac{\partial p}{\partial t} = -[\phi, p]_x - \beta[v, \psi]_x + 2\beta[h, \phi]_x. \quad (290)$$

In this model the magnetic field is assumed to be given by  $\mathbf{B} = \epsilon \nabla \psi \times \hat{z} + [(1 + \epsilon x)^{-1} + \epsilon \bar{b}] \hat{z} + \mathcal{O}(\epsilon^2)$ , where  $\epsilon = a/R_0 \ll 1$  is the small ratio between the characteristic scale  $a$  of the poloidal plane and the major radius  $R_0$  of a tokamak. The field  $\bar{b} = \bar{b}(x, y)$  accounts for parallel magnetic perturbations.

In Eqs. (287)-(290), we indicated with  $\phi$  the electrostatic potential, and with  $\omega = \Delta_{\perp} \phi$  the corresponding vorticity. The pressure perturbations (assumed to be equal for the electron and ion species) are denoted with  $p$ , whereas  $v$  indicates the ion velocity perturbations along the direction of the magnetic field which, at the leading order, corresponds to the direction  $\hat{z}$  of the guide (toroidal) field. The quantity  $h = x$  accounts for the curvature of the background static magnetic field (with  $\nabla h$  pointing in the direction opposite to that of the magnetic curvature), whereas the parameter  $\beta$  corresponds to the ratio between kinetic plasma pressure and the magnetic pressure based on the guide field. The four dynamical variables  $\chi = (\psi, \omega, v, p)$  are defined on a 2D domain  $\mathcal{D}$  described by the  $x$  and  $y$  coordinates.

Equation (287) descends from ideal Ohm's law, combined with Faraday's law. Equation (288) is the vorticity equation. Equations (287) and (288) can be directly compared with Eqs. (51) and (52) of reduced MHD. The difference with the latter ones emerges due to the presence of the last term on the right-hand side of Eq. (288) which, through magnetic curvature, introduces a coupling with the pressure perturbations. Eqs. (289) and (290), on the other hand, can be derived from the component along the magnetic field of the ion fluid equation, and from the electron pressure equation under the assumption of isothermal electrons.



We now apply the EC method to CRMHD. The Hamiltonian structure of the system (287)-(290) [101, 102] is given by the Hamiltonian functional

$$H(\psi, \omega, v, p) = \frac{1}{2} \int_D d^2x \left( -\psi \Delta_{\perp} \psi - \omega \phi + v^2 + p^2/\beta \right) \quad (291)$$

and by the Poisson bracket

$$\begin{aligned} \{F, G\} = & \int_D d^2x \left( \omega [F_{\omega}, G_{\omega}]_x + \psi ([F_{\psi}, G_{\omega}]_x + [F_{\omega}, G_{\psi}]_x - \beta [F_p, G_v]_x - \beta [F_v, G_p]_x) \right. \\ & \left. + v ([F_{\omega}, G_v]_x + [F_v, G_{\omega}]_x) + (p + 2\beta h) ([F_{\omega}, G_p]_x + [F_p, G_{\omega}]_x) \right). \end{aligned} \quad (292)$$

Also in this case, one can consider the reduced MHD Hamiltonian (54) as reference, and notice that the Hamiltonian of CRMHD includes two additional sources of energy, corresponding to the last two terms on the right-hand side of Eq. (291) and associated with parallel ion velocity and electron pressure perturbations, respectively.

The Casimir invariants of the Poisson bracket (292) are

$$\begin{aligned} C_1(\psi) &= \int_D d^2x \mathcal{J}(\psi), \\ C_2(v, \psi) &= \int_D d^2x v \mathcal{N}(\psi), \\ C_3(\psi, p) &= \int_D d^2x \mathcal{L}(\psi) (p/\beta + 2h), \\ C_4(\omega, \psi, v, p) &= \int_D d^2x (\omega \mathcal{I}(\psi) - v \mathcal{I}'(\psi) (p/\beta + 2h)), \end{aligned} \quad (293)$$

with  $\mathcal{J}$ ,  $\mathcal{N}$ ,  $\mathcal{L}$  and  $\mathcal{I}$  arbitrary functions. Comparing with Eq. (60) one sees that the invariant  $C_1$  is shared with reduced MHD. The Casimir invariants  $C_2$  and  $C_4$  descend from the conservation of the cross-helicity  $\int_D d^2x \mathbf{v} \cdot \mathbf{B}$  of ideal MHD. This follows from considering that, in CRMHD [100], the parallel magnetic perturbations  $\bar{b}$  are taken to be proportional to  $-p$ , which assures perpendicular pressure balance. By the same argument, one can see that the Casimir  $C_3$  is related to the conservation of magnetic helicity in ideal MHD.

The free energy functional can be written as  $\mathcal{F} = H + C_1 + C_2 + C_3 + C_4$  and its

first variation reads

$$\begin{aligned}
& \delta\mathcal{F}(\psi, \omega, v, p; \delta\psi, \delta\omega, \delta v, \delta p) \\
&= \int_D d^2x \left( (\mathcal{I}(\psi) - \phi) \delta\omega + \left( -\Delta_{\perp}\psi + \mathcal{J}'(\psi) + v \mathcal{N}'(\psi) + \left(\frac{p}{\beta} + 2h\right) \mathcal{L}'(\psi) \right. \right. \\
&+ \left. \left. \omega \mathcal{I}'(\psi) - v \mathcal{I}''(\psi) \left(\frac{p}{\beta} + 2h\right) \right) \delta\psi + \left( v + \mathcal{N}(\psi) - \mathcal{I}'(\psi) \left(\frac{p}{\beta} + 2h\right) \right) \delta v \right. \\
&+ \left. \frac{1}{\beta} (p + \mathcal{L}(\psi) - v \mathcal{I}'(\psi)) \delta p \right). \tag{294}
\end{aligned}$$

Extremals of the functional  $\mathcal{F}$  correspond to

$$\begin{aligned}
& -\Delta_{\perp}\psi + \mathcal{J}'(\psi) + v \mathcal{N}'(\psi) + \left(\frac{p}{\beta} + 2h\right) \mathcal{L}'(\psi) \\
&+ \omega \mathcal{I}'(\psi) - v \mathcal{I}''(\psi) \left(\frac{p}{\beta} + 2h\right) = 0, \tag{295}
\end{aligned}$$

$$\mathcal{I}(\psi) - \phi = 0, \tag{296}$$

$$v + \mathcal{N}(\psi) - \mathcal{I}'(\psi) \left(\frac{p}{\beta} + 2h\right) = 0, \tag{297}$$

$$p + \mathcal{L}(\psi) - v \mathcal{I}'(\psi) = 0. \tag{298}$$

As in the case of reduced MHD, Eq. (296) expresses the fact that the equilibrium poloidal flow is a function of the magnetic potential  $\psi$ . However, in the subsequent analysis, we focus on the case of equilibria with no poloidal flow, corresponding to the choice  $\mathcal{I} = 0$ . In this limit, from Eqs. (297) and (298) it follows immediately that also the parallel ion flow and the electron pressure becomes flux functions, i.e.  $v = v(\psi) = -\mathcal{N}(\psi)$  and  $p = p(\psi) = -\mathcal{L}(\psi)$ . In this limit Eq. (295) becomes

$$-\Delta_{\perp}\psi + \mathcal{J}'(\psi) - v'(\psi)v(\psi) - p'(\psi) \left(\frac{p(\psi)}{\beta} + 2h\right) = 0, \tag{299}$$

which provides a further generalization of the Grad-Shafranov equation in slab geometry, different from those that we previously encountered, in particular as it accounts for magnetic field curvature.

Although only equilibria without poloidal flow are considered in this case, it is worth recalling that the following transformation exists [97, 103]

$$\chi = \int^{\psi} d\bar{\psi} \sqrt{1 - \mathcal{I}'^2(\bar{\psi})}, \tag{300}$$

which allows to construct equilibrium solutions with poloidal flow, starting from equilibrium solutions with  $\phi_e = 0$ .

For simplicity, we will also impose  $\mathcal{N} = 0$ , which, from Eq. (297), amounts to imposing  $v_e = 0$ , i.e. no toroidal flow. We are thus restricting to the case of static equilibria in the presence of magnetic curvature. Taking into account this choice for the free functions of the Casimir invariants, the second variation of  $\mathcal{F}$  is given by

$$\begin{aligned} \delta^2 \mathcal{F}(\psi_e, p_e; \delta\psi, \delta\omega, \delta v, \delta p) &= \int_D d^2x \left[ |\nabla\delta\psi|^2 + |\nabla\delta\phi|^2 + (\delta v)^2 + \frac{(\delta p)^2}{\beta} \right. \\ &\left. + \left( \mathcal{J}''(\psi_e) + \left( \frac{p_e}{\beta} + 2h \right) \mathcal{L}''(\psi_e) \right) (\delta\psi)^2 + 2\mathcal{L}'(\psi_e) \frac{\delta p}{\beta} \delta\psi \right], \end{aligned} \quad (301)$$

where, in order to write as the sum of two squares the first two terms on the right-hand side, we carried out an integration by parts and used the relation  $\delta\omega = \Delta_\perp \delta\phi$ .

Making use of Eq. (298), the second variation  $\delta^2 \mathcal{F}$  can be rewritten as

$$\begin{aligned} \delta^2 F(\psi_e, p_e; \delta\psi, \delta\omega, \delta v, \delta p) &= \int_D d^2x \left[ |\nabla\delta\phi|^2 + (\delta v)^2 + |\nabla\delta\psi|^2 \right. \\ &+ \frac{1}{\beta} \left( \delta p - \frac{\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla p}{|\mathbf{B}_{e\perp}|^2} \delta\psi \right)^2 - \frac{\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla J_e}{|\mathbf{B}_{e\perp}|^2} (\delta\psi)^2 \\ &\left. + 2 \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla p_e)}{|\mathbf{B}_{e\perp}|^2} \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla h)}{|\mathbf{B}_{e\perp}|^2} (\delta\psi)^2 \right], \end{aligned} \quad (302)$$

where we recall that  $J_e = -\Delta_\perp \psi_e$ . Two possible sources of instability exist, corresponding to the last two terms on the right-hand side of Eq. (302), which have indefinite sign. The formulation (302) puts in evidence that the sign of these terms depends on the directions of the pressure gradient, of the poloidal magnetic field  $\mathbf{B}_{e\perp}$ , of the gradient of the current density  $J_e$  and of the magnetic curvature.

We consider now the corresponding result that the energy principle provides for the case of static MHD equilibria. According to the formulation of Ref. [104], the  $\delta W$  energy principle states that an equilibrium  $(\mathbf{B}, \mathbf{v}, p) = (\mathbf{B}_e, 0, p_e)$  of compressible ideal MHD (an equation of state  $\partial_t(p/\rho) + \mathbf{v} \cdot \nabla(p/\rho) = 0$  is assumed) is linearly stable if and

only if the functional

$$\begin{aligned} \delta W(\mathbf{B}_e, p_e; \boldsymbol{\xi}) = & \frac{1}{2} \int_D d^2x \left( \left( \mathbf{Q} - \frac{\boldsymbol{\xi} \cdot \nabla p_e}{|\mathbf{B}_e|^2} \mathbf{B}_e \right)^2 + p_e (\nabla \cdot \boldsymbol{\xi})^2 \right. \\ & \left. - \frac{j_{\parallel e}}{|\mathbf{B}_e|} (\boldsymbol{\xi} \times \mathbf{B}_e) \cdot \mathbf{Q} + 2(\boldsymbol{\xi} \cdot \nabla p_e)(\boldsymbol{\xi} \cdot \nabla h) \right). \end{aligned} \quad (303)$$

is such that  $\delta W \geq 0$  for all allowable displacements  $\boldsymbol{\xi}(x, y, t)$ . The value of a displacement  $\boldsymbol{\xi}(x, y, t)$  corresponds to the position of the fluid element that was at the position  $(x, y)$  at equilibrium, i.e. at  $t = 0$ . The displacement vector field  $\boldsymbol{\xi}$  is related to the perturbation of the MHD velocity field  $\delta \mathbf{v}(x, y, t)$  by  $\partial_t \boldsymbol{\xi}(x, y, t) = \delta \mathbf{v}(x, y, t)$ . In the  $\delta W$  functional (303), the perturbations are therefore encoded in the displacement  $\boldsymbol{\xi}$ . The parallel equilibrium current density  $j_{\parallel e}$  is defined by  $j_{\parallel e} = (\mathbf{B}_e/|\mathbf{B}_e|) \cdot \nabla \times \mathbf{B}_e$ . We also indicated with  $\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_e)$  the perturbation of the magnetic field, as it can be inferred from the induction equation  $\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B})$  linearized about a static equilibrium.

Inspecting the expression (303) it follows that the stability condition  $\delta W \geq 0$  is satisfied if the last two terms on the right-hand side are positive. Such two terms are indeed responsible for two known instabilities occurring in ideal MHD [104]. The first of these terms is associated with the kink instability, due to the interaction between the equilibrium magnetic field and the current density gradient. The second term accounts for the interchange instability, which is triggered when the equilibrium pressure gradient and the magnetic curvature are parallel. In general, there will always be some displacement  $\boldsymbol{\xi}$ , for which such term is destabilizing, except when  $\boldsymbol{\xi}$  and  $\nabla h$  are exactly parallel.

Following Ref. [98], we will show that, applying the appropriate approximations, the two destabilizing terms in  $\delta W$ , correspond to the two destabilizing terms previously detected in the expression (302) for  $\delta^2 \mathcal{F}$ .

First of all, we will identify the MHD velocity field  $\mathbf{v}$  with the velocity field  $\hat{z} \times \nabla \phi + v \hat{z}$  of CRMHD, although in the latter, strictly speaking, the parallel velocity  $v$  is the ion velocity. However, by virtue of the small electron/ion mass ratio, this

identification is not unreasonable. In the second place, we identify the CRMHD electron (or, equivalently, ion) pressure, with the total pressure of MHD.

We neglect parallel flow perturbations, i.e. we set  $\boldsymbol{\xi} \cdot \hat{z} = 0$ , which, in the CRMHD context, amounts to  $\delta v = 0$ . Next, we consider the incompressibility assumption in the energy principle, setting  $\nabla \cdot \boldsymbol{\xi} = 0$  and define  $Q_{\parallel} = \mathbf{Q} \cdot \mathbf{B}_e / |\mathbf{B}_e|$ , which leads to the decomposition  $\mathbf{Q} = \mathbf{Q}_{\perp} + Q_{\parallel} \mathbf{B}_e / |\mathbf{B}_e|$ . Making use of the relation [104]  $Q_{\parallel} = |\mathbf{B}_e| (2\boldsymbol{\xi} \cdot \nabla h) + (1/|\mathbf{B}_e|) \boldsymbol{\xi} \cdot \nabla p_e$ , one can write

$$\begin{aligned} \left( \mathbf{Q} - \frac{\boldsymbol{\xi} \cdot \nabla p_e}{|\mathbf{B}_e|^2} \mathbf{B}_e \right)^2 &= (\mathbf{Q}_{\perp} + 2(\boldsymbol{\xi} \cdot \nabla h) \mathbf{B}_e)^2 \\ &\simeq |\mathbf{Q}_{\perp}|^2 + 4(\boldsymbol{\xi} \cdot \nabla h)^2. \end{aligned} \quad (304)$$

The approximation in the second line of Eq. (304) descends from assuming that  $\mathbf{B}_e$  is, at the leading order, given by  $\mathbf{B}_e = \hat{z}$ , which corresponds to the strong guide field assumption of CRMHD.

The perpendicular magnetic perturbation  $\mathbf{Q}_{\perp}$ , on the other hand, can be expressed in terms of the perturbation of the magnetic flux  $\delta\psi$  function so that

$$\mathbf{Q}_{\perp} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_e) = \nabla \times (\delta\psi \hat{z}). \quad (305)$$

Using Eq. (304) it follows then that

$$\left( \mathbf{Q} - \frac{\boldsymbol{\xi} \cdot \nabla p_e}{|\mathbf{B}_e|^2} \mathbf{B}_e \right)^2 \simeq |\nabla \delta\psi|^2 + 4(\boldsymbol{\xi} \cdot \nabla h)^2. \quad (306)$$

This allows for a rewriting of the first term in the expression (303) for  $\delta W$ , which puts in evidence the positive contribution  $|\nabla \delta\psi|^2$ , related to magnetic field line bending, and present also in the expression (302) for  $\delta^2 \mathcal{F}$ .

The second term in Eq. (303) vanishes for purely incompressible flows and, in any event, is a positive definite term, that cannot contribute to instabilities. Note that the analogy between  $\delta W$  and EC methods for CRMHD can actually go beyond the incompressible assumption, as the second term in the right-hand side of Eq. (303) has a counterpart in the fourth term in the right-hand side of Eq. (301) [98]. The third

term in Eq. (303) can be rewritten, using the above approximations (see Ref. [98] for details) as

$$- \int_D d^2x \frac{j_{\parallel e}}{|\mathbf{B}_e|} (\boldsymbol{\xi} \times \mathbf{B}_e) \cdot \mathbf{Q} \simeq \int_D d^2x \delta\psi \boldsymbol{\xi} \cdot \nabla j_{\parallel e}. \quad (307)$$

From Eq. (299) one can express the parallel equilibrium current density as

$$J_e = -\mathcal{J}'(\psi_e) + p'(\psi_e) \left( \frac{p(\psi_e)}{\beta} + 2h \right), \quad (308)$$

from which one can derive the relation

$$\begin{aligned} & \mathcal{J}''(\psi_e) - p''(\psi_e) \frac{p(\psi_e)}{\beta} - \frac{p'^2(\psi_e)}{\beta} - 2p''(\psi_e)h \\ &= -\frac{\hat{z} \times \mathbf{B}_{e\perp}}{|\mathbf{B}_{e\perp}|^2} \cdot \nabla J_e + 2 \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla p_e)}{|\mathbf{B}_{e\perp}|^2} \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla h)}{|\mathbf{B}_{e\perp}|^2}. \end{aligned} \quad (309)$$

Upon identifying (again, by considering that  $\mathbf{B}_e \simeq \hat{z}$  at the leading order) the parallel current density  $j_{\parallel e}$  present in the expression for  $\delta W$ , with  $J_e$ , occurring in the expression for  $\delta^2 \mathcal{F}$  one finds, from Eq. (307)

$$\begin{aligned} & - \int_D d^2x \frac{j_{\parallel e}}{|\mathbf{B}_e|} (\boldsymbol{\xi} \times \mathbf{B}_e) \cdot \mathbf{Q} \\ & \simeq \int_D d^2x \left( (\delta\psi)^2 \left( \mathcal{J}''(\psi_e) - p''(\psi_e) \frac{p(\psi_e)}{\beta} - \frac{p'^2(\psi_e)}{\beta} - 2p''(\psi_e)h \right) - 2\boldsymbol{\xi} \cdot \nabla p_e \boldsymbol{\xi} \cdot \nabla h \right), \end{aligned} \quad (310)$$

where we also made use of the equation

$$\delta\psi = -\boldsymbol{\xi} \cdot \nabla \psi_e, \quad (311)$$

descending from Eq. (305) and relating the poloidal magnetic perturbation with the fluid MHD displacement. Inserting the relation (309) into Eq. (310), one finally obtains

$$\begin{aligned} & - \int_D d^2x \frac{j_{\parallel e}}{|\mathbf{B}_e|} (\boldsymbol{\xi} \times \mathbf{B}_e) \cdot \mathbf{Q} \\ & \simeq \int_D d^2x \left( (\delta\psi)^2 \left( -\frac{\hat{z} \times \mathbf{B}_{e\perp}}{|\mathbf{B}_{e\perp}|^2} \cdot \nabla J_e + 2 \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla p_e)}{|\mathbf{B}_{e\perp}|^2} \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla h)}{|\mathbf{B}_{e\perp}|^2} \right) - 2\boldsymbol{\xi} \cdot \nabla p_e \boldsymbol{\xi} \cdot \nabla h \right). \end{aligned} \quad (312)$$

With the help of Eqs. (306) and (312) the expression (303) can be written as

$$\begin{aligned} \delta W & \simeq \int_D d^2x (|\nabla \delta\psi|^2 + 4(\boldsymbol{\xi} \cdot \nabla h)^2 \\ & - \frac{\hat{z} \times \mathbf{B}_{e\perp}}{|\mathbf{B}_{e\perp}|^2} \cdot \nabla J_e (\delta\psi)^2 + 2 \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla p_e)}{|\mathbf{B}_{e\perp}|^2} \frac{(\hat{z} \times \mathbf{B}_{e\perp} \cdot \nabla h)}{|\mathbf{B}_{e\perp}|^2} (\delta\psi)^2). \end{aligned} \quad (313)$$

Comparing the expression (313) with Eq. (302), it emerges that the two sources of instability in  $\delta W$ , associated with the kink and interchange instabilities, are *namely the two destabilizing terms in  $\delta^2 \mathcal{F}$* . The stability predictions based on the EC method and on the energy principle are thus strictly related. Note also that the terms containing  $|\nabla \delta \phi|^2$  and  $(\delta v)^2$  in  $\delta^2 \mathcal{F}$  contribute to the second variation of the kinetic energy and thus have no counterpart in  $\delta W$ , which involves the variation in potential energy only (in fact, as pointed out in Ref. [13],  $\delta W$  refers rather to a second variation in the potential energy). The first two terms in Eq. (313), on the other hand, have a stabilizing effect and are related, as above anticipated, to field line bending and compressibility.

In Ref. [98], the above analysis is extended to the case with equilibrium toroidal flow, showing that this more general class of equilibria does not introduce further instabilities other than those of kink and interchange type. In the same Reference, the EC method is applied to a model [105] describing locally the MHD dynamics in a thin accretion disk. The analysis takes advantage from an analogy between such model and CRMHD. The stability results obtained from CRMHD can thus be transferred to the model for accretion disks. In particular, it turns out that the counterpart, in accretion disks, of the CRMHD interchange instability, is the magnetorotational instability, where the roles of pressure gradient and magnetic curvature in tokamaks are played by shear velocity gradient and the Coriolis term, respectively, in accretion disks.

## 9. Negative energy modes

An important concept, in connection with the EC method, and with stability problems in general, is that of *negative energy mode*. Roughly speaking, a NEM corresponds to a stable mode of oscillations of linearized conservative system, providing a negative contribution to the total energy of the system. The importance of NEMs is due to the fact that, when dissipation is added, a NEM can become unstable. Intuitively, as dissipation tends to decrease the total energy, it can contribute to increase the amplitude of a mode providing a negative contribution to the energy, i.e. a NEM.

Thus, an instability is triggered. Also, a linear system with NEMs can develop finite-time singularities when nonlinearities are added to the system. Detecting NEMs of an equilibrium state can thus offer important information about the dynamics of the system.

The investigation of NEMs has a long tradition, dating back to early work of Refs. [106, 107, 108]. A classical paradigmatic example of a nonlinear Hamiltonian system that can develop a finite-time singularity due to the presence of NEMs was provided in Ref. [109]. This example is also discussed in detail in Ref. [13]. Examples of applications to fluid models in plasma physics include streaming instabilities [110], MHD instabilities [111, 112, 113, 114], instabilities due to wave-wave interaction [115, 116], magnetorotational instability [117, 118], magnetosonic waves in the solar atmosphere [119], collisionless magnetic reconnection [120], ion and electron temperature gradient driven instabilities [121, 122]. The presence of NEMs in relation with anomalous transport, in the framework of Maxwell-drift kinetic theory, has been investigated in Refs. [123, 124]. Further examples involving kinetic and drift-kinetic models, are indicated in Ref. [122]. Not all such investigations of NEMs, on the other hand, followed the same approach. The Hamiltonian formalism provides an effective and natural approach to the analysis of NEMs, in particular in connection with the EC method. Indeed, on one hand, the knowledge of the Hamiltonian structure of a model provides an unambiguous expression for the energy of the system, corresponding namely to the Hamiltonian functional. Moreover, as we saw, formal stability is attained if  $\delta^2\mathcal{F}$  has a definite sign. It turns out that the indefiniteness in the sign of  $\delta^2\mathcal{F}$  implies that either the equilibrium is unstable, or it is spectrally stable but possesses NEMs. Also, the Hamiltonian approach provides a clear identification of the NEMs of a system, when the latter is expressed in its *normal form*. Detailed expositions of the Hamiltonian approach to the analysis of NEMs, oriented towards applications to fluids and plasmas, can be found in Refs. [13, 11]. Here we content to briefly review the main elements of such an approach in the case of finite-dimensional systems and illustrate an application



to ETG driven modes, based on Ref. [122].

### 9.1. Basic elements of mode signature for Hamiltonian systems with finite degrees of freedom

We consider a canonical linear Hamiltonian system with  $N$  degrees of freedom. We denote the dynamical variables with  $z = (q^1, \dots, q^N, p_1, \dots, p_N)$ . Such system can be written in the form

$$\dot{z}^i = \{z^i, H_L\}, \quad (314)$$

where

$$H_L(z) = \frac{1}{2} \sum_{i,j=1}^{2N} A_{ij} z^i z^j, \quad (315)$$

is the Hamiltonian and

$$\{f, g\} = \sum_{i=1}^N \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \quad (316)$$

is a *canonical* Poisson bracket. In Eq. (315), we indicated with  $A_{ij}$  the elements of a real  $2N \times 2N$  matrix  $A$ . On the other hand, the canonical Poisson bracket (316) can be written as

$$\{f, g\} = \sum_{i,j=1}^{2N} \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}, \quad (317)$$

where

$$J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (318)$$

is a  $2N \times 2N$  matrix of the kind that we already introduced in Eq. (26). Note indeed, that the canonical bracket (316) corresponds to a finite-dimensional version of the canonical bracket for fields written in Eq. (25). Although the above described system deals with finite dimension and a canonical Poisson bracket, it will turn out that it contains most of the properties required to deal with NEMs in noncanonical Hamiltonian fluid models with discrete spectrum.

Making use of Eqs. (315), (316) and (318), the system (314) can be rewritten as

$$\dot{z} = J_c A z. \quad (319)$$

If one assumes a temporal dependence of the form

$$z = \tilde{z}e^{i\omega t} + \tilde{z}^*e^{-i\omega t}, \quad (320)$$

where  $*$  denotes complex conjugate, the system (319) leads to the eigenvalue problem

$$i\omega_\alpha z_\alpha = J_c A z_\alpha, \quad \alpha = 1, \dots, N, \quad (321)$$

where the index  $\alpha$  labels the eigenvalues  $\omega_\alpha$  and the eigenvectors  $z_\alpha$ . We assume that the eigenvalues are real and distinct. We are thus considering *spectrally stable* modes of oscillations. Remark that, when compared to the definition of spectral stability given in Sec. 2.3, due to the temporal dependence assumed in Eq. (320), spectral (neutral) stability occurs when the eigenvalues have no imaginary part. Note also that only  $N$  equations are indicated in Eq. (321). Indeed, because the dynamical variables  $z$  are real, the remaining eigenvalues and eigenvectors are determined by the relations  $\omega_{-\alpha} = -\omega_\alpha$  and  $z_{-\alpha} = z_\alpha^*$ , respectively.

We introduce the quantity

$$h(\alpha, \beta) := i\omega_\alpha z_\beta^T J_c^{-1} z_\alpha = z_\beta^T A z_\alpha, \quad (322)$$

for which the properties

$$h(\alpha, \beta) - h(\beta, \alpha) = 0, \quad (323)$$

$$h(\alpha, \beta) = 0, \quad \text{if } \beta \neq -\alpha, \quad (324)$$

hold. Considering the expression (315) for the Hamiltonian (i.e. the total energy) of the system, it follows that

$$h(-\alpha, \alpha) = z_{-\alpha}^T A z_\alpha = z_\alpha^{*T} A z_\alpha = i\omega_\alpha z_\alpha^{*T} J_c^{-1} z_\alpha, \quad (325)$$

can be defined as the energy associated with the mode  $(z_\alpha, \omega_\alpha; z_\alpha^*, -\omega_\alpha)$ .

Choosing the appropriate normalization of the eigenvectors, the purely imaginary quantity  $z_\alpha^{*T} J_c^{-1} z_\alpha$  can be made equal to  $2i$  or to  $-2i$ , depending on whether, for the

corresponding  $\alpha$ , its imaginary part is positive or negative. If the eigenvalue  $\omega_\alpha$  of an eigenvector  $z_\alpha$  is positive and

$$z_\alpha^{*T} J_c^{-1} z_\alpha = -2i, \quad (326)$$

then the energy associated with that mode, according to Eq. (325), is given by

$$h(-\alpha, \alpha) = i\omega_\alpha z_\alpha^{*T} J_c^{-1} z_\alpha = 2\omega_\alpha > 0. \quad (327)$$

The mode  $\alpha$  is thus referred to as a *positive energy mode* (PEM). The case in which  $h(-\alpha, \alpha) > 0$ , due to  $\omega_\alpha < 0$  and  $z_\alpha^{*T} J_c^{-1} z_\alpha = 2i$ , corresponds to the same mode, because of the above mentioned properties  $\omega_{-\alpha} = \omega_\alpha$  and  $z_{-\alpha} = z_\alpha^*$ . When  $h(-\alpha, \alpha) < 0$  the  $\alpha$  mode is a NEM. We also note that, due to Sylvester's theorem (see, for instance, Ref. [125], p. 577), the number of PEMs and NEMs of a system, which is referred to as the mode signature of the system, is independent on the choice of coordinates. This property becomes particularly useful because, as will be seen in the case treated in Sec. 9.2, it permits to identify NEMs even if the system is not formulated in canonical coordinates.

In the Hamiltonian setting, the identification of the mode signature can become even more transparent. Indeed, for the stable modes, there exists a transformation  $T(Q_1, \dots, Q_N, P_1, \dots, P_N) = (q^1, \dots, q^N, p_1, \dots, p_N)$  which casts the Hamiltonian of the system in the form

$$H_L = \frac{1}{2} \sum_{\alpha=1}^N \sigma_\alpha \omega_\alpha (P_\alpha^2 + Q_\alpha^2), \quad (328)$$

with  $\omega_\alpha$  indicating the positive eigenvalues, whereas  $\sigma_i \in \{-1, 1\}$  indicates the signature of the mode. The form (328) is referred to as the *normal form* of the Hamiltonian. When written in its normal form, the Hamiltonian is expressed as the sum of Hamiltonians of harmonic oscillators with characteristic frequencies given by the eigenvalues  $\omega_\alpha$  and with a characteristic sign given by the signature of the mode. PEMs, corresponding to  $\sigma_\alpha = +1$ , provide a positive contribution to the total energy, whereas NEMs, for which  $\sigma_\alpha = -1$ , subtract, as expected, energy to the system. As above anticipated, Sylvester's theorem guarantees that the number of positive and negative eigenvalues of

the symmetric matrix associated with the quadratic form (328) is independent on the choice of the basis. Therefore, in order to identify NEMs and PEMs of a quadratic Hamiltonian, it is not necessary to cast the latter into its normal form.

The algorithm for constructing the canonical transformation  $T$  consists of identifying the  $N$  eigenvectors  $z_1, z_2, \dots, z_N$  that satisfy the relation (326). Then, the  $2N \times 2N$  matrix associated with the transformation  $T$  is given by

$$T = \text{col}(\text{Re } z_1, \text{Re } z_2 \dots \text{Re } z_N, \text{Im } z_1, \text{Im } z_2, \dots, \text{Im } z_N), \quad (329)$$

which is the matrix whose  $i$ th column is given by the elements of  $\text{Re } z_i$ .

### 9.2. Mode signature in a model for ETG turbulence

We consider the following reduced fluid model [126]

$$\frac{\partial}{\partial t}(1 - \nabla^2)\phi = [\phi, \nabla^2\phi + x]_x + \left[ \frac{p}{\sqrt{r}}, \sqrt{r}x \right]_x, \quad (330)$$

$$\frac{\partial}{\partial t} \frac{p}{\sqrt{r}} = \left[ \frac{p}{\sqrt{r}}, \phi \right]_x + [\sqrt{r}x, \phi]_x. \quad (331)$$

The model is formulated in Cartesian coordinates  $(x, y)$  on a 2D domain  $\mathcal{D}$ . Equation (330) is an equation of the Charney-Hasegawa-Mima type for the normalized electrostatic potential  $\phi$ , with the additional last term on the right-hand side, which introduces the coupling with the evolution equation (331) for the electron pressure  $p$ .

The parameter

$$r = \frac{L_n^2}{L_B L_P}, \quad (332)$$

depends on the characteristic scale lengths of background profiles of density, pressure and magnetic field, indicated by  $L_n$ ,  $L_P$  and  $L_B$ , respectively. The parameter  $r$  also introduces the ETG instability, which is suppressed if  $r \rightarrow 0$ , when one linearizes Eqs. (330)-(331) about the equilibrium  $\phi_e = p_e = 0$ . The model is described in more detail in Ref. [126].

The Hamiltonian formulation of the system (330)-(331) was provided in Ref. [127]. Such structure is conveniently expressed in terms of the alternative set of dynamical

variables  $\chi^1 = \Lambda$  and  $\chi^2 = \Pi$  defined by

$$\Lambda = \phi - \nabla^2 \phi, \quad \Pi = \frac{p}{\sqrt{r}} + \sqrt{r}x. \quad (333)$$

In terms of these variables Eqs. (330)-(331) read

$$\frac{\partial}{\partial t} \Lambda = -[\mathcal{L}^{-1} \Lambda, \Lambda - x] + [\Pi, \sqrt{r}x], \quad (334)$$

$$\frac{\partial}{\partial t} \Pi = [\Pi, \mathcal{L}^{-1} \Lambda], \quad (335)$$

where we introduced the operator  $\mathcal{L}$ , and its inverse  $\mathcal{L}^{-1}$ , formally defined by  $\mathcal{L}f = f - \nabla^2 f$ , and  $\mathcal{L}^{-1} \mathcal{L}f = \mathcal{L} \mathcal{L}^{-1} f = f$ , for a function  $f$ .

The corresponding Hamiltonian structure is given by the Hamiltonian

$$H(\Lambda, \Pi) = \frac{1}{2} \int_D d^2x (\Lambda \mathcal{L}^{-1}(\Lambda) - \Pi^2 + 2\sqrt{r}\Pi x), \quad (336)$$

and by the Poisson bracket

$$\{F, G\} = \int_D d^2x ((x - \Lambda)[F_\Lambda, G_\Lambda] - \Pi([F_\Lambda, G_\Pi] + [F_\Pi, G_\Lambda])). \quad (337)$$

Note that the Poisson bracket (337) is analogous to the Poisson bracket (55) of reduced MHD. Consequently, we can easily deduce the Casimir invariants, which are given by the families

$$C_1(\Pi) = \int_D d^2x \mathcal{J}(\Pi), \quad C_2(\Lambda, \Pi) = \int_D d^2x (\Lambda - x) \mathcal{I}(\Pi), \quad (338)$$

with  $\mathcal{J}$  and  $\mathcal{I}$  arbitrary functions.

We now proceed to the application of the EC method for the ETG model. From setting equal to zero the first variation of the free energy functional  $\mathcal{F}(\Lambda, \Pi) = H(\Lambda, \Pi) + C_1(\Pi) + C_2(\Lambda, \Pi)$ , we obtain the equilibrium equations

$$\mathcal{L}^{-1} \Lambda + \mathcal{I}(\Pi) = 0, \quad (339)$$

$$-\Pi + \sqrt{r}x + \mathcal{J}'(\Pi) + (\Lambda - x) \mathcal{I}'(\Pi) = 0. \quad (340)$$

In particular, if one reinterprets Eq. (339) in terms of the original variables  $\phi$  and  $p$ , it emerges that the choice  $\mathcal{I} = 0$  corresponds to  $\phi_e = 0$  and thus to no equilibrium flow.

The second variation of  $\mathcal{F}$ , on the other hand, can be rearranged as

$$\begin{aligned} & \delta^2 \mathcal{F}(\Lambda_e, \Pi_e; \delta\Lambda, \delta\Pi) \\ &= \int_D d^2x \left[ (\mathcal{L}^{-1}\delta\Lambda)^2 + |\mathcal{L}^{-1}\nabla\delta\Lambda|^2 + \mathcal{I}'(\Pi_e)(\delta\Lambda + \delta\Pi)^2 \right. \\ & \quad \left. - \mathcal{I}'(\Pi_e)(\delta\Lambda)^2 + \left( \mathcal{I}''(\Pi_e) - 1 + (\Lambda_e - x)\mathcal{I}''(\Pi_e) - \mathcal{I}'(\Pi_e) \right) (\delta\Pi)^2 \right]. \end{aligned} \quad (341)$$

From Eq. (341) it follows that a sufficient condition for stability is given by

$$\mathcal{I}(\Pi_e) = 0, \quad \mathcal{I}''(\Pi_e) > 1. \quad (342)$$

As above remarked, this corresponds to a stability condition for equilibrium with no flow. In particular, we will consider here, a family of no-flow equilibria given by

$$\Lambda_e = 0, \quad \Pi_e = \alpha_\Pi x, \quad (343)$$

where  $\alpha_\Pi$  is a parameter. This will be the equilibrium states for which we intend to investigate the mode signature, and in particular the occurrence of NEMs.

From Eqs. (339)-(340) it follows that the one-parameter family of equilibria (343) corresponds to the following choice for the free functions of the Casimir invariants:

$$\mathcal{I}(\Lambda) = 0, \quad \mathcal{J}(\Pi) = \frac{1}{2} \left( 1 - \frac{\sqrt{r}}{\alpha_\Pi} \right) \Pi^2. \quad (344)$$

We consider the linearization of the ETG model (330)-(331) about the equilibrium (343). The dynamical variables are thus written as the sum of a corresponding equilibrium with a perturbation, in the following way:  $\Lambda = \tilde{\Lambda}$  and  $\Pi = \alpha_\Pi x + \tilde{\Pi}$ . The resulting linear system reads

$$\dot{\tilde{\Lambda}} = -\frac{\partial}{\partial y} \mathcal{L}^{-1} \tilde{\Lambda} - \sqrt{r} \frac{\partial}{\partial y} \tilde{\Pi}, \quad (345)$$

$$\dot{\tilde{\Pi}} = \alpha_\Pi \frac{\partial}{\partial y} \mathcal{L}^{-1} \tilde{\Lambda}. \quad (346)$$

Assuming periodicity along  $x$  and  $y$  of the perturbations amplitudes, we can expand these as Fourier series. Denoting  $\mathbf{k} = (k_x, k_y)$  we write:  $\tilde{\Lambda} = \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\Lambda}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}$ ,

$\tilde{\Pi} = \sum_{\mathbf{k}=-\infty}^{+\infty} \tilde{\Pi}_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}$ . From Eqs. (345)-(346) one then obtains the system

$$\dot{\tilde{\Lambda}}_{\mathbf{k}} = i\frac{k_y}{1+k_{\perp}^2}\tilde{\Lambda}_{\mathbf{k}} + i\sqrt{r}k_y\tilde{\Pi}_{\mathbf{k}}, \quad (347)$$

$$\dot{\tilde{\Pi}}_{\mathbf{k}} = -i\alpha_{\Pi}\frac{k_y}{1+k_{\perp}^2}\tilde{\Lambda}_{\mathbf{k}}. \quad (348)$$

Assuming, for the spectral amplitudes  $\tilde{\Lambda}_{\mathbf{k}}$  and  $\tilde{\Pi}_{\mathbf{k}}$  a time dependence of the form  $\tilde{\Lambda}_{\mathbf{k}}(t), \tilde{\Pi}_{\mathbf{k}}(t) \sim e^{i\omega t}$ , one obtains from Eqs. (347)-(348), the following dispersion relation

$$\omega^2 - \frac{k_y}{1+k_{\perp}^2}\omega + \alpha_{\Pi}\frac{\sqrt{r}k_y^2}{1+k_{\perp}^2} = 0, \quad (349)$$

where  $k_{\perp} = \sqrt{k_x^2 + k_y^2}$ . The dispersion relation (349) possesses the two solutions [127]

$$\omega_s^k = \frac{k}{2(1+k_{\perp}^2)}(1 - \sqrt{1 - 4(1+k_{\perp}^2)\alpha_{\Pi}\sqrt{r}}), \quad (350)$$

$$\omega_f^k = \frac{k}{2(1+k_{\perp}^2)}(1 + \sqrt{1 - 4(1+k_{\perp}^2)\alpha_{\Pi}\sqrt{r}}), \quad (351)$$

where we denoted  $k_y = k$ . The two solutions  $\omega_s^k$  and  $\omega_f^k$  correspond, given their phase velocities, to a slow and a fast mode, respectively. According to the definition of spectral stability (or, in this case, of neutral stability), provided in Sec. 9.1, it follows from the expressions (350)-(351) that a necessary and sufficient condition for spectral stability of the equilibria (343) is given by

$$\alpha_{\Pi} < \frac{1}{4(1+k_{\perp}^2)\sqrt{r}}. \quad (352)$$

This condition amounts to an upper bound on the steepness of the pressure gradient. Note also that, as  $r \rightarrow 0$ , the condition (352) is satisfied for any finite  $\alpha_{\Pi}$  and  $k_{\perp}$ . This reflects the above mentioned suppression of the ETG instability.

The eigenvectors corresponding to the eigenvalues  $\omega_{s,f}^k$ , for  $k > 0$ , are given by

$$\tilde{\Lambda}_{s,f}^k = -\omega_{s,f}^k \frac{1+k_{\perp}^2}{\alpha_{\Pi}k} \tilde{\Pi}_{s,f}^k = -\frac{1}{2\alpha_{\Pi}}(1 \pm \sqrt{1 - 4(1+k_{\perp}^2)\alpha_{\Pi}\sqrt{r}})\tilde{\Pi}_{s,f}^k. \quad (353)$$

As recalled in Sec. 9.1, the remaining eigenvalues, for each fixed  $k > 0$ , are given by  $\omega_{-s,-f}^k = -\omega_{s,f}^k$  and are associated with the eigenvectors  $\tilde{\Lambda}_{s,f}^{k*}$ , with  $\tilde{\Lambda}_{s,f}^k$  given by Eq. (353).

Equations (347)-(348) give a linear system which is amenable to the application of the procedure depicted in Sec. 9.1, for the identification of the mode signature. Indeed, although the system (347)-(348) is infinite-dimensional, it can still be cast in Hamiltonian form with a discrete (although infinite) set of dynamical variables. These correspond to the Fourier amplitudes  $\bar{z} = (\tilde{\Lambda}_k, \tilde{\Pi}_k)$ , labelled by the wave number  $k = k_y$ . The Hamiltonian of the system is

$$H_L(\tilde{\Lambda}_k, \tilde{\Pi}_k) = \sum_{k=1}^{+\infty} H_L^k = 2\pi \sum_{k=1}^{+\infty} \left( \frac{|\tilde{\Lambda}_k|^2}{1+k_\perp^2} - \frac{\sqrt{r}}{\alpha_\Pi} |\tilde{\Pi}_k|^2 \right). \quad (354)$$

Note that the  $k_x$  component of the wave vector appears in the system only through  $k_\perp$ , which can be treated as a parameter. The Hamiltonian (354) is indeed of the form (315), although with  $N = +\infty$ .

Equations (347)-(348), with fixed  $k_x$ , can be obtained from the Hamiltonian (354), combined with the Poisson bracket

$$\begin{aligned} \{F, G\} = & \sum_{k=1}^{+\infty} \frac{ik}{2\pi} \left[ \left( \frac{\partial F}{\partial \tilde{\Lambda}_k} \frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}} \frac{\partial G}{\partial \tilde{\Lambda}_k} \right) \right. \\ & \left. - \alpha_\Pi^2 \left( \frac{\partial F}{\partial \tilde{\Lambda}_k} \frac{\partial G}{\partial \tilde{\Pi}_{-k}} + \frac{\partial F}{\partial \tilde{\Pi}_k} \frac{\partial G}{\partial \tilde{\Lambda}_{-k}} - \frac{\partial F}{\partial \tilde{\Pi}_{-k}} \frac{\partial G}{\partial \tilde{\Lambda}_k} - \frac{\partial F}{\partial \tilde{\Lambda}_{-k}} \frac{\partial G}{\partial \tilde{\Pi}_k} \right) \right]. \end{aligned} \quad (355)$$

As anticipated in Sec. 9.1, due to Sylvester's theorem, the Hamiltonian is in principle sufficient to detect mode signature, even though the system is not expressed in terms of canonical coordinates yet. Indeed, one can evaluate the energy of the slow and fast mode, for a given  $k$ , by inserting the corresponding eigenvalues (350)-(351) and eigenvectors (353) into the expression (354), which yields

$$H_{L_{s,f}}^k = 2\pi \left( \frac{1 - 4\sqrt{r}\alpha_\Pi - 2k_\perp^2\alpha_\Pi\sqrt{r} \pm \sqrt{1 - 4(1+k_\perp^2)\alpha_\Pi\sqrt{r}}}{2\alpha_\Pi^2} \right) |\tilde{\Pi}_{s,f}^k|^2. \quad (356)$$

In the two expressions of Eq. (356) associated with the  $\pm$  symbol, the  $+$  and  $-$  sign refer to the fast and slow mode, respectively.

As the above definition of PEMS and NEMs holds for stable modes only, we restrict to the case where the condition (352) is fulfilled. Even restricting to this case, the sign of the expressions (356) depends on parameters. An effective way to detect mode signature



in this case, is to identify it in a special limit. The modes will then preserve their sign as parameters are varied continuously, as long as an instability does not occur. In this case we consider the limit  $k_{\perp} \rightarrow 0$ . One sees that, for the fast mode, in this limit  $H_{L_f}^k$  is positive for any value of  $\alpha_{\Pi}$  corresponding to stability (recall the condition (352)). Thus, the fast mode is a PEM and such will remain as  $k_{\perp}$  is varied until the stability condition holds. For the slow mode, again considering the  $k_{\perp} \rightarrow 0$  limit, one has positive  $H_{L_s}^k$  for  $\alpha_{\Pi} < 0$ . In this case the slow mode is thus a PEM. When  $0 < \alpha_{\Pi} < 1/4\sqrt{r}$ , on the other hand,  $H_{L_s}^k$  becomes negative and the slow mode is a NEM, one which is fragile to instabilities induced by dissipation. We remark that an explicit destabilization by dissipation for ion temperature gradient modes and kinetic ballooning modes was shown in the linear theory of a Hamiltonian gyrofluid model in Ref. [121]. The slow mode preserves this signature as parameters are varied in the stability region. When the instability threshold (352) is crossed, a bifurcation takes place. This bifurcation is of the Krein type [13], which is one of the types of bifurcations that can occur in Hamiltonian systems, and always involves a NEM.

The situation is exemplified in Fig. 2. For  $\alpha_{\Pi} < 0$  the equilibrium is spectrally stable for all values of  $k_{\perp}$  and both the slow and the fast modes are PEMs. For  $\alpha_{\Pi} > 0$  and  $k_{\perp}$  below the instability threshold, occurring at around  $k_{\perp} = 1.22$  in this case, two stable modes, a PEM and a NEM, coexist. At the bifurcation point the two modes merge to yield an instability for  $k_{\perp} > 1.22$ .

If we compare with the criterion (342) obtained from the EC method, we note that, for the equilibria (343), using Eq. (342), the EC method predicts formal stability for

$$\alpha_{\Pi} < 0. \tag{357}$$

This is namely the condition providing spectral stability and two PEMs. As above seen, when  $\alpha_{\Pi}$  becomes positive, one of the PEMs turns into a NEM, producing an indefiniteness in the sign of  $\delta^2\mathcal{F}$ . For  $k_{\perp}$  sufficiently small, the corresponding modes are still stable modes of oscillations, though, but could be destabilized by dissipation or nonlinearities. This exemplifies the connection between NEMs and the EC method for

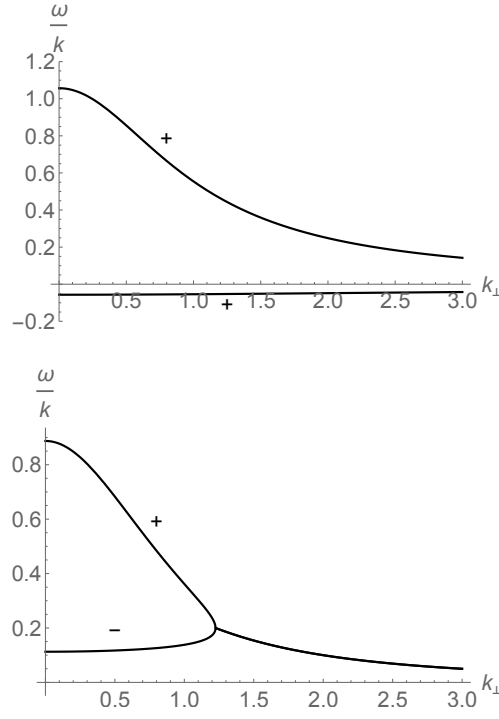


Figure 2: Plots illustrating the mode signature of the slow and fast modes (350) and (351). The upper plot corresponds to a case with  $\alpha_{\Pi} < 0$ . The equilibrium is formally and spectrally stable, with two PEMs. The plot at the bottom refers to a case with  $\alpha_{\Pi} > 0$ . In this case the pressure gradient, for  $0 < k_{\perp} < 1.22$ , is such to turn a PEM into a NEM, corresponding to the slow mode. These modes are spectrally stable but the condition (357), for formal stability, is not fulfilled. At  $k_{\perp} = 1.22$  a Kreĭn bifurcation takes place, where the two real eigenvalues merge and become complex conjugates for  $k_{\perp} > 1.22$ , leading to an instability. The values of the parameters are  $\alpha_{\Pi} = -0.3$  for the top plot,  $\alpha_{\Pi} = 0.5$  for the bottom plot and  $\sqrt{r} = 0.2$  for both plots (we remark a misprint in the caption of Fig. 1 of Ref. [122], which is analogous to the present Fig. 2, and in which the value of  $\alpha_{\mathcal{P}}$  is actually equal to  $-2$ , instead of  $-0.3$ ).

formal stability.

Following the procedure described in Sec. 9.1, one can also cast the Hamiltonian (354) into normal form. This is accomplished by first passing from the variables  $\bar{z} = (\tilde{\Lambda}_k, \tilde{\Pi}_k)$  to the *real* canonical variables

$$\begin{aligned}
 q_k^1 &= \sqrt{\frac{\pi}{k\alpha_{\Pi}^2}} (\tilde{\Pi}_k + \alpha_{\Pi} \tilde{\Lambda}_k + \tilde{\Pi}_{-k} + \alpha_{\Pi} \tilde{\Lambda}_{-k}), \\
 p_k^1 &= -i \sqrt{\frac{\pi}{k\alpha_{\Pi}^2}} (\tilde{\Pi}_k + \alpha_{\Pi} \tilde{\Lambda}_k - \tilde{\Pi}_{-k} - \alpha_{\Pi} \tilde{\Lambda}_{-k}), \\
 q_k^2 &= \sqrt{\frac{\pi}{k}} (\tilde{\Lambda}_k + \tilde{\Lambda}_{-k}), \quad p_k^2 = i \sqrt{\frac{\pi}{k}} (\tilde{\Lambda}_k - \tilde{\Lambda}_{-k}).
 \end{aligned} \tag{358}$$

In terms of these variables  $z = (q_k^1, q_k^2, p_k^1, p_k^2)$ , the Hamiltonian reads

$$H_L = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{i,j=1}^4 A_{ij}^k z^{ki} z^{kj}, \quad (359)$$

which, apart from the infinite number of degrees of freedom, is namely of the form (315).

In Eq. (359) one has

$$A^k = \begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -c & b \end{pmatrix}, \quad z^k = (q_1^k, q_2^k, p_1^k, p_2^k) \quad (360)$$

where  $a = -\sqrt{r}\alpha_{\Pi}k$ ,  $b = k(1/(1+k_{\perp}^2) - \sqrt{r}\alpha_{\Pi})$ , and  $c = \sqrt{r}|\alpha_{\Pi}|k$ . The Poisson bracket (355), on the other hand, takes the canonical form

$$\{F, G\} = \sum_{k=1}^{\infty} \frac{\partial F}{\partial q_1^k} \frac{\partial G}{\partial p_1^k} - \frac{\partial F}{\partial p_1^k} \frac{\partial G}{\partial q_1^k} + \frac{\partial F}{\partial q_2^k} \frac{\partial G}{\partial p_2^k} - \frac{\partial F}{\partial p_2^k} \frac{\partial G}{\partial q_2^k}. \quad (361)$$

The transformation  $T^k : (Q_1^k, Q_2^k, P_1^k, P_2^k) \rightarrow (q_1^k, q_2^k, p_1^k, p_2^k)$  which, for a fixed  $k$ , casts  $H_L$  into normal form, is associated with the matrix

$$T^k = \begin{pmatrix} \frac{1}{D_-} & \frac{1}{D_+} & 0 & 0 \\ -\frac{B_-}{D_-} & -\frac{B_+}{D_+} & 0 & 0 \\ 0 & 0 & \frac{1}{D_-} & -\frac{1}{D_+} \\ 0 & 0 & \frac{B_-}{D_-} & -\frac{B_+}{D_+} \end{pmatrix}, \quad (362)$$

where

$$B_{\pm} = \frac{b+a \pm \sqrt{(b+a)^2 - 4c^2}}{2c}, \\ D_+ = \sqrt{B_+^2 - 1}, \quad D_- = \sqrt{1 - B_-^2}.$$

Applying this transformation (restricted to the  $k$  corresponding to real eigenvalues), the Hamiltonian becomes

$$H'_L = \frac{1}{2} \sum'_k \omega_f^k (Q_2^{k^2} + P_2^{k^2}) - \omega_s^k (Q_1^{k^2} + P_1^{k^2}), \quad (363)$$

where the prime symbol indicates that the sum is restricted to the stable modes. The normal form (363) clearly expresses how the slow modes, when  $\omega_s^k > 0$ , provide a negative contribution to the total energy, and are thus NEMs. This confirms what was

previously deduced without resorting to the canonical variables. Further details on the calculations leading to the transformation into normal form are provided in Ref. [122].

The above reviewed problem showed how the issue of the mode signature, and in particular, of NEMs, is related to the EC method. We point out, however, that the identification of the mode signature, by means of the normal form, encompasses a significantly vaster area of research. In particular, the above example dealt with the case of discrete, although infinite, eigenvalues. However, in general, the spectrum of a linear operator, can contain a continuous component, in the presence of which the normal form (328) acquires an integral representation. This situation, which is significantly more complex to treat than the case of the discrete spectrum, can typically arise in plasma fluid models when the system is linearized about an equilibrium with shear of a velocity or of the magnetic field. The tools for handling the continuous spectrum case were developed in Refs. [128, 129]. In this context, the Reader could also consider Refs. [122, 113, 114, 130] for relevant examples.

A further remark concerns the connection with the issue of singular equilibria, which was already mentioned in Sec. 2.2. As mentioned in Sec. 9.1, in the case of real and distinct eigenvalues in finite dimension, if  $\omega_\alpha$  is a solution of the eigenvalue problem (321), then  $-\omega_\alpha$  is also a solution. This is indeed the case for the ETG problem treated in this Section. More in general, for noncanonical Hamiltonian systems linearized about an equilibrium point where the rank of the cosymplectic form (a matrix in the finite-dimensional case) does not change, the eigenvalues can be complex and satisfy the following symmetries: if  $\lambda$  is an eigenvalue, then  $-\lambda$  and  $\lambda^*$  are also eigenvalues (see, for instance, Ref. [13]). However, when a Hamiltonian system is linearized about a singular equilibrium, i.e. an equilibrium where the rank of the cosymplectic form changes, such symmetry properties of the eigenvalues are lost [131, 132]. In Ref. [133] it was pointed out that shear flows are singular equilibria of the 3D Euler equation for an incompressible fluid in Fourier space. For such equilibria the EC method cannot be applied because these equilibria are not extremals of the free energy functional.

## 10. Tearing modes for Beltrami equilibria

The search for equilibria by extremizing the free energy functional is, as we have seen, an essential part of the EC method. This process is intimately related to the analysis of the kernel of the cosymplectic form  $\mathcal{J}$ . Indeed, from Eq. (27), one has that the functional derivatives of Casimir invariants namely belong to the kernel of  $\mathcal{J}$ . In the cases reviewed in the previous Sections, the functional derivatives of the Casimir invariants were typically smooth functions. However, in infinite-dimensional systems, *singular* elements of the kernel of  $\mathcal{J}$  can also appear. Interestingly, as shown in Refs. [134, 30], a manifestation of this phenomenon has an interpretation in plasma physics and corresponds to the appearance of the so called tearing modes [1], as perturbations of Beltrami equilibria in MHD. In the following part of this Section, the analysis of linearized MHD about Beltrami states, in the Hamiltonian approach, and in particular the connection between tearing modes and singular elements in the kernel of a cosymplectic form, are reviewed. This topic is actually only partially relevant to the EC method, as it only concerns the search for equilibria. Nevertheless, we believe it could be useful to describe it in detail in this Review. Indeed, it provides a recent example of how, in addition to smooth solutions, also non-trivial and physically relevant singular equilibrium solutions can be found, in the process of applying the EC method to fluid plasma models. Still in the context of reconnecting modes described by means on noncanonical Hamiltonian systems, we mention the theory for fast nonlinear collisionless reconnection developed in Refs. [135, 136, 137].

We begin by recalling the normalized set of incompressible MHD equations as formulated in Ref. [30] (in Ref. [134] a barotropic closure is assumed but, without essential modifications, the analysis of the incompressible case can be adapted to the barotropic case):

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathcal{P}_\sigma[\mathbf{v} \cdot \nabla \mathbf{v}] + \mathcal{P}_\sigma[(\nabla \times \mathbf{B}) \times \mathbf{B}], \quad (364)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (365)$$

where  $\mathbf{v}$  and  $\mathbf{B}$  are the velocity and magnetic fields, respectively, as in Secs. 4 and 5. In Ref. [30], the MHD system (364)-(365) is formulated in a precise analytical setting. The domain  $\mathcal{D} \in \mathbb{R}^3$  is assumed to be smoothly bounded and, in general multiply connected. The fields  $\mathbf{v}$  and  $\mathbf{B}$  are assumed to have vanishing normal components on the boundaries, so that  $\mathbf{v} \cdot \mathbf{n} = 0$  and  $\mathbf{B} \cdot \mathbf{n} = 0$  on the boundary  $\partial\mathcal{D}$  of  $\mathcal{D}$ . The dynamical variables  $\chi = (\mathbf{v}, \mathbf{B})$  belong to the phase space  $P = L^2_\sigma(\mathcal{D}) \times L^2_\sigma(\mathcal{D})$ , in such a way that both  $\mathbf{v}$  and  $\mathbf{B}$  belong to the Hilbert space

$$L^2_\sigma(\mathcal{D}) := \{\mathbf{u} \in L^2(\mathcal{D}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0\}, \quad (366)$$

where  $L^2(\mathcal{D})$  is the Hilbert space of Lebesgue measurable, square-integrable vector fields on  $\mathcal{D}$ , equipped with the inner product  $\langle \mathbf{a}, \mathbf{b} \rangle = \int_D d^3x \mathbf{a} \cdot \mathbf{b}$ . In Eq. (364) we indicated with  $\mathcal{P}_\sigma$  the projector onto the space  $L^2_\sigma(\mathcal{D})$ . The use of this projector removes the pressure gradient term, as this belongs to the non-divergence-free component of the equation.

The Hamiltonian functional of Eqs. (364)-(365) is given by

$$H(\mathbf{v}, \mathbf{B}) = \frac{1}{2} \int_D d^3x (|\mathbf{v}|^2 + |\mathbf{B}|^2), \quad (367)$$

whereas the Poisson bracket can be written as

$$\{F, G\} = \int_D d^3x (F_{\mathbf{v}}, F_{\mathbf{B}}) \mathcal{J}(\mathbf{v}, \mathbf{B}) (G_{\mathbf{v}}, G_{\mathbf{B}})^T, \quad (368)$$

where

$$\mathcal{J}(\mathbf{v}, \mathbf{B}) = \begin{pmatrix} -\mathcal{P}_\sigma[(\nabla \times \mathbf{v}) \times \circ] & \mathcal{P}_\sigma(\nabla \times \circ) \times \mathbf{B} \\ \nabla \times (\circ \times \mathbf{B}) & 0 \end{pmatrix}. \quad (369)$$

To write the Poisson bracket, we adopted the formulation (368), which differs from the one adopted in the previous Sections, since we wanted to emphasize the expression for the cosymplectic operator  $\mathcal{J}$  (compare with Eq. (19)), which has a major role in the present discussion. Note also that  $\mathcal{P}_\sigma[(\nabla \times \mathbf{v}) \times \mathbf{v}] = \mathcal{P}_\sigma[(\mathbf{v} \cdot \nabla) \mathbf{v}]$ .

Casimir invariants of the Poisson bracket (368) are known to correspond to the magnetic and cross-helicity, given by

$$C_1(\mathbf{B}) = \frac{1}{2} \int_D d^3x \mathbf{A} \cdot \mathbf{B}, \quad C_2(\mathbf{v}, \mathbf{B}) = \frac{1}{2} \int_D d^3x \mathbf{v} \cdot \mathbf{B}, \quad (370)$$

respectively, where  $\mathbf{A}$  is the magnetic vector potential, satisfying  $\nabla \times \mathbf{A} = \mathbf{B}$ .

The following free energy functional can thus be constructed:

$$\mathcal{F}(\mathbf{v}, \mathbf{B}) = \frac{1}{2} \int_D d^3x (|\mathbf{v}|^2 + |\mathbf{B}|^2) - \frac{\mu}{2} \int_D d^3x \mathbf{A} \cdot \mathbf{B} - \frac{\nu}{2} \int_D d^3x \mathbf{v} \cdot \mathbf{B}, \quad (371)$$

where  $\mu$  and  $\nu$  are two constants. Setting  $\delta\mathcal{F} = 0$  yields the equilibrium equations

$$\mathbf{v} - \nu\mathbf{B} = 0, \quad (372)$$

$$\nabla \times \mathbf{B} - \mu\mathbf{B} - \nu\nabla \times \mathbf{v} = 0. \quad (373)$$

The present analysis focuses on equilibria with  $\mathbf{v}_e = 0$ , which correspond to choosing  $\nu = 0$ . The purely magnetic component of the equilibrium is thus determined by

$$\nabla \times \mathbf{B} - \mu\mathbf{B} = 0. \quad (374)$$

Solutions of Eq. (374) are referred to as *Beltrami equilibria* and play a fundamental role in the modelling of astrophysical and laboratory plasmas, as in the case of Taylor's relaxation theory [138, 139].

Tearing modes can be seen as singular equilibrium states of the MHD equations (364)-(365) linearized about Beltrami equilibria. In order to discuss this matter, it is appropriate to first discuss, as in Refs. [134, 30], some properties of Beltrami equilibria. As above anticipated, we are considering, in this Section a domain  $\mathcal{D}$  which is multiply connected. Assuming the domain  $\mathcal{D}$  possesses  $s$  "handles" (or "holes" in the 2D case), we define with  $\mathcal{D}_S = \mathcal{D} / \cup_{l=1}^s \Sigma_l$  the simply connected domain corresponding to  $\mathcal{D}$ , devoid of the union of  $s$  cuts  $\Sigma_1, \Sigma_2, \dots, \Sigma_s$  across its handles. The magnetic fluxes  $\Phi_l(\mathbf{B})$  defined by

$$\Phi_l(\mathbf{B}) = \int_{\Sigma_l} \mathbf{B} \cdot d\boldsymbol{\sigma}, \quad (375)$$

where  $\boldsymbol{\sigma}$  is the surface element on  $\Sigma_l$ , are constants of motion due to the fact that  $\mathbf{v} \cdot \mathbf{n} = 0$  on each surface  $\Sigma_l$ . A magnetic field  $\mathbf{B} \in L^2_\sigma(\mathcal{D})$ , and in particular a Beltrami field  $\mathbf{B}_\mu$ , can be written, making use of the Hodge-Kodaira decomposition, as

$$\mathbf{B}_\mu = \mathbf{B}_\Sigma + \mathbf{B}_H, \quad (376)$$

where  $\mathbf{B}_\Sigma \in L^2_\Sigma(\mathcal{D})$  and  $\mathbf{B}_H \in L^2_H(\mathcal{D})$ , with

$$L^2_\Sigma(\mathcal{D}) = \{\mathbf{u} \in L^2(\mathcal{D}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0, \Phi_l(\mathbf{u}) = 0 \text{ for } l = 1, \dots, s\}, \quad (377)$$

$$L^2_H(\mathcal{D}) = \{\mathbf{u} \in L^2(\mathcal{D}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0, \nabla \times \mathbf{u} = 0\}. \quad (378)$$

The space  $L^2_\Sigma(\mathcal{D})$  therefore includes the component of the magnetic field associated with a current density, whereas the space  $L^2_H(\mathcal{D})$  includes the "vacuum" component.

By virtue of the decomposition (376), the equilibrium equation (374) can be written as

$$(\text{curl} - \mu)\mathbf{B}_\Sigma = \mu\mathbf{B}_H, \quad (379)$$

where we used the notation 'curl' for the curl operator  $\nabla \times$ . Thus, the seemingly homogeneous equation (374), once it is cast in the form (379), can be seen as an inhomogeneous equation for the unknown  $\mathbf{B}_\Sigma$ , for given  $\mathbf{B}_H$  and  $\mu$ . The constant values of the fluxes  $\Phi_1, \dots, \Phi_s$ , determine uniquely the vacuum component  $B_H$ .

The solutions of the inhomogeneous equation (379) exhibit a bifurcation, depending on the value of the parameter  $\mu$ . In order to describe the bifurcation of the solutions it is appropriate to introduce some definitions. We indicate with  $\mathcal{S} : H^1_{\Sigma\Sigma}(\mathcal{D}) \rightarrow L^2_\Sigma(\mathcal{D})$  the *self-adjoint* curl operator, defined by  $\mathcal{S}\mathbf{u} = \nabla \times \mathbf{u}$ , for every  $\mathbf{u} \in H^1_{\Sigma\Sigma}(\mathcal{D})$ , where

$$H^1_{\Sigma\Sigma}(\mathcal{D}) = \{\mathbf{u} \in L^2_\Sigma(\mathcal{D}) \cap H^1(\mathcal{D}) : \nabla \times \mathbf{u} \in L^2_\Sigma(\mathcal{D})\}. \quad (380)$$

The property of self-adjointness of the curl operator was first formulated and proved in Ref. [140]. In the definition of the domain  $H^1_{\Sigma\Sigma}(\mathcal{D})$  we indicated with  $H^1$  the Sobolev space of order 1, consisting of the Hilbert space of square integrable vector fields, whose derivatives are also square integrables. The point spectrum of  $\mathcal{S}$ , denoted with  $\sigma_P(\mathcal{S})$ , is real and does not contain the zero eigenvalue (note that the domain  $H^1_{\Sigma\Sigma}(\mathcal{D})$  of  $\mathcal{S}$  does not contain elements of  $L^2_H(\mathcal{D})$ , which are curl-free). In particular, the operator  $\mathcal{S}$  is invertible. The self-adjoint curl  $\mathcal{S}$  can be extended to the *non self-adjoint* curl operator  $\mathcal{T}$ , the range of which includes also the "vacuum" fields in  $L^2_H(\mathcal{D})$ . More precisely, denoting with  $\mathbf{A}_H \in L^2_\Sigma(\mathcal{D})$  the vector potential of an element  $\mathbf{B}_H \in L^2_H(\mathcal{D})$ ,



so that  $\nabla \times \mathbf{A}_H = \mathbf{B}_H$ , we have that  $\mathcal{T} : H_{\Sigma\sigma}^1(\mathcal{D}) \rightarrow L_\sigma^2(\mathcal{D})$  and the operator  $\mathcal{T}$  acts as  $\mathcal{T}\mathbf{u} = \nabla \times \mathbf{u}$  for every  $\mathbf{u} \in H_{\Sigma\sigma}^1(\mathcal{D})$ , where

$$H_{\Sigma\sigma}^1(\mathcal{D}) = \{\mathbf{u} \in L_\Sigma^2(\mathcal{D}) \cap H^1(\mathcal{D}) : \nabla \times \mathbf{u} \in L_\sigma^2(\mathcal{D})\}. \quad (381)$$

Equation (379) can then be reformulated as

$$(\mathcal{T} - \mu)\mathbf{B}_\Sigma = \mu\mathbf{B}_H. \quad (382)$$

The following results can then be proved [140, 134, 30]:

- If  $\mu \notin \sigma_P(\mathcal{S})$  the inhomogeneous equation (379) has a unique solution

$$\mathbf{B}_\Sigma = (\mathcal{T} - \mu I)^{-1} \mu \mathbf{B}_H. \quad (383)$$

These solutions are referred to as branch-(A) solutions and they occur when  $\mu$  is not an eigenvalue of  $\mathcal{S}$ . A branch-(A) solution, when inserted into the expression for  $C_1$  in Eq. (370), yields a corresponding magnetic helicity, depending on  $\mu$ , which we denote as  $C_1 = \mathcal{C}_A(\mu)$ . Because the magnetic helicity is a Casimir invariant, its value remains constant and is prescribed by the initial condition. Denoting with  $c_1$  the prescribed value of  $C_1$ , from the relation  $c_1 = \mathcal{C}_A(\mu)$  one can find the value of  $\mu$  corresponding to a particular surface of constant magnetic helicity in phase space.

- If  $\mu = \lambda_j \in \sigma_P(\mathcal{S})$ , with eigenfunction  $\boldsymbol{\omega}_j$ , so that  $\mathcal{S}\boldsymbol{\omega}_j = \lambda_j\boldsymbol{\omega}_j$ , the homogeneous part of Eq. (379) has a non-trivial solution  $\mathbf{B}_\Sigma = \boldsymbol{\omega}_j$ . Moreover, if and only if

$$\langle \mathbf{A}_H, \boldsymbol{\omega}_j \rangle = 0, \quad (384)$$

Eq. (379) has a particular solution  $\mathbf{B}_\Sigma = \mathbf{G} \in L_\Sigma^2(\mathcal{D})$  such that  $\langle \mathbf{G}, \boldsymbol{\omega}_j \rangle = 0$ . In this case, the inhomogeneous Beltrami equation (379) has the general solution

$$\mathbf{B}_\Sigma = \alpha \boldsymbol{\omega}_j + \mathbf{G}, \quad (385)$$

with  $\alpha$  arbitrary constant. Solutions of this type are referred to as branch-(B) solutions. It is these solutions that bifurcate from the branch-(A) solutions, when the parameter

$\mu$  "crosses" the spectrum of  $\mathcal{S}$ . For solutions of branch-(B) the prescribed value of the magnetic helicity determines the constant  $\alpha$ , instead of the value of  $\mu$  (which is fixed equal to  $\lambda_j$ ) as was the case for the branch-(A) solutions. Indeed, a branch-(B) solution yields a magnetic helicity  $C_1 = \mathcal{C}_B(\alpha)$  and one can retrieve the value of  $\alpha$  corresponding to a given helicity from the relation  $c_1 = \mathcal{C}_B(\alpha)$ .

In order to see how the tearing mode emerges in this context, we need to consider the MHD system (364)-(365) linearized about an equilibrium  $(\mathbf{v}_e, \mathbf{B}_e) = (0, \mathbf{B}_\mu)$ , where  $\mathbf{B}_\mu$  is a Beltrami field. The linearized system reads

$$\frac{\partial \delta \mathbf{v}}{\partial t} = \mathcal{P}_\sigma [(\nabla \times \mathbf{B}_\mu) \times \delta \mathbf{B} + (\nabla \times \delta \mathbf{B}) \times \mathbf{B}_\mu], \quad (386)$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\delta \mathbf{v} \times \mathbf{B}_\mu), \quad (387)$$

where  $\delta \mathbf{v}$  and  $\delta \mathbf{B}$  are the perturbations of the velocity and magnetic fields, respectively, with  $\delta \mathbf{B} \in L^2_\Sigma(\mathcal{D})$ . The system (386)-(387) has the form given in Eq. (29) and is also Hamiltonian, with noncanonical structure consisting of the Hamiltonian

$$H(\delta \mathbf{v}, \delta \mathbf{B}) = \frac{1}{2} \int_D d^3x (|\delta \mathbf{v}|^2 + |\delta \mathbf{B}|^2 - \delta \mathbf{B} \mu \mathcal{S}^{-1} \delta \mathbf{B}) \quad (388)$$

and of the Poisson bracket

$$\{F, G\} = \int_D d^3x (F_{\delta \mathbf{v}}, F_{\delta \mathbf{B}}) \mathcal{J}_L (G_{\delta \mathbf{v}}, G_{\delta \mathbf{B}})^T, \quad (389)$$

where

$$\mathcal{J}_L = \begin{pmatrix} 0 & \mathcal{P}_\sigma [(\nabla \times \circ) \times \mathbf{B}_\mu] \\ \nabla \times (\circ \times \mathbf{B}_\mu) & 0 \end{pmatrix} \quad (390)$$

is the cosymplectic form. Note that, unlike the cosymplectic form (369) of the original MHD system, the operator  $\mathcal{J}_L$  is constant, as it does not depend on the dynamical variables, which in this case are  $\delta \mathbf{v}$  and  $\delta \mathbf{B}$ . The derivation of the Hamiltonian structure of a system obtained from the linearization of a Hamiltonian system with Lie-Poisson structure is discussed in Ref. [10].

Being the linearized system (386)-(387) a noncanonical Hamiltonian system, one can in turn build a free energy functional and extremize it in order to find equilibrium

states of the linearized system. Casimir invariants of the Poisson bracket (389) are of the form

$$C_v(\delta\mathbf{v}) = \int_D d^3x \delta\mathbf{v} \cdot \bar{\mathbf{v}}, \quad C_b(\delta\mathbf{B}) = \int_D d^3x \delta\mathbf{B} \cdot \bar{\mathbf{b}}, \quad (391)$$

where  $\bar{\mathbf{v}}$  and  $\bar{\mathbf{b}}$  are such that  $(\bar{\mathbf{v}}, 0)^T$  and  $(0, \bar{\mathbf{b}})^T$  belong to the kernel of  $\mathcal{J}_L$ . From the expression (390), it follows that  $\bar{\mathbf{v}}$  and  $\bar{\mathbf{b}}$  must satisfy

$$\nabla \times (\mathbf{B}_\mu \times \bar{\mathbf{v}}) = 0, \quad \nabla \cdot \bar{\mathbf{v}} = 0, \quad (392)$$

$$\mathbf{B}_\mu \times (\nabla \times \bar{\mathbf{b}}) = 0. \quad (393)$$

Here we restrict to free energy functionals of the form

$$\mathcal{F}(\delta\mathbf{v}, \delta\mathbf{B}) = H(\delta\mathbf{v}, \delta\mathbf{B}) - \beta C_b(\delta\mathbf{B}), \quad (394)$$

with  $\beta$  an arbitrary constant. Extremization of the free energy functional (394) yields equilibria with  $\delta\mathbf{v}_e = 0$ . We focus then on the possible solutions for  $\bar{\mathbf{b}}$ , in order to determine the Casimir invariant  $C_b$ . In particular, one easily sees that Eq. (393) is satisfied by  $\bar{\mathbf{b}}$  such that

$$\nabla \times \bar{\mathbf{b}} = \eta(\mathbf{x})\mathbf{B}_\mu, \quad (395)$$

for some scalar function  $\eta(\mathbf{x})$ . A necessary condition for Eq. (395) to hold, is that

$$\nabla\eta \cdot \mathbf{B}_\mu = 0. \quad (396)$$

The choice  $\bar{\mathbf{b}} = \mathbf{B}_\mu$ , with  $\eta = \mu$ , provides a regular, although trivial solution. More in general, the condition (396) implies that  $\eta$  be constant along field lines of  $\mathbf{B}_\mu$ . For the existence of magnetic field lines we require that  $\mathbf{B}_\mu$  possess a spatial symmetry. One particular case (which can be extended to more general spatial symmetries [134]) is the one in which  $\mathbf{B}_\mu$  only depends on the  $x$  coordinate, so that  $\mathbf{B}_\mu(x) = (0, B_y(x), B_z(x))^T$ . One can find other regular solutions choosing  $\bar{\mathbf{b}}$  to be of the form  $\bar{\mathbf{b}}(x) = (0, \bar{b}_y(x), \bar{b}_z(x))^T$ . However, one can also find *singular* solutions. This occurs, if, for instance, one looks for solutions of the form

$$\bar{\mathbf{b}} = \begin{pmatrix} 0 \\ ik_y\theta(x) \\ ik_z\theta(x) \end{pmatrix} e^{i(k_y y + k_z z)}, \quad (397)$$

with constants  $k_y$  and  $k_z$ . The condition (393) then implies

$$(k_y B_y(x) + k_z B_z(x)) \frac{\partial \theta(x)}{\partial x} = 0. \quad (398)$$

Equation (398) is solved in terms of a *singular* solution

$$\theta(x) = a_1 + a_2 Y(x - x^\dagger), \quad (399)$$

where  $a_1, a_2$  are arbitrary complex constants,  $Y$  is the Heaviside step function and  $x^\dagger$  is a real number satisfying the resonance condition

$$k_y B_y(x^\dagger) + k_z B_z(x^\dagger) = 0. \quad (400)$$

The locus of points satisfying  $x = x^\dagger$  identifies the *resonant surface*, which, in linear tearing mode theory, identifies the position of the current sheet in ideal MHD, where magnetic reconnection takes place, in the presence of some non-ideal effect.

Making use of the solution (399), one obtains

$$\eta(\mathbf{x}) = i \frac{k_y}{B_z} e^{i(k_y y + k_z z)} \delta(x - x^\dagger), \quad (401)$$

where  $\delta$  is the Dirac delta distribution, and also

$$\bar{\mathbf{b}} = \begin{pmatrix} 0 \\ ik_y \theta(x) \\ ik_z \theta(x) \end{pmatrix} (a_1 + a_2 Y(x - x^\dagger)) e^{i(k_y y + k_z z)}. \quad (402)$$

Solutions for  $\bar{\mathbf{b}}$  such as (402) correspond to singular elements  $(0, \bar{\mathbf{b}})^T$  in the kernel of  $\mathcal{J}_L$ .

The equilibria following from extremizing the free energy functional (394), with  $\bar{\mathbf{b}}$  given by Eq. (402) are determined by the equilibrium equation

$$(\mathcal{S} - \mu) \mathcal{S}^{-1} \delta \mathbf{B} - \beta \mathcal{P}_\Sigma \bar{\mathbf{b}} = 0, \quad (403)$$

where  $\mathcal{P}_\Sigma$  is the projector from  $L^2(\mathcal{D})$  onto  $L^2_\Sigma(\mathcal{D})$ . If  $\mu = \lambda_j \in \sigma_P(\mathcal{S})$ , then Eq. (403) admits, as solution,  $\delta \mathbf{B} = a \boldsymbol{\omega}_j$ , with  $a$  an arbitrary constant. This case also implies  $\beta = 0$ . The value of the arbitrary constant  $a$ , on the other hand, can be fixed from

the condition  $a \langle \boldsymbol{\omega}_j, \bar{\mathbf{b}} \rangle = C_b(\delta\mathbf{B}) = c_1$ , analogously to the helicity matching above discussed for the Beltrami equilibria.

We restrict to the case  $\mu > 0$  (the case  $\mu < 0$  will be analogous but involving the negative side of the spectrum  $\sigma_P(\mathcal{S})$ ). If  $\mu \notin \sigma_P(\mathcal{S})$ , the equilibrium solution is given by  $\delta\mathbf{B} = \beta\mathcal{S}(\mathcal{S} - \mu)^{-1}\mathcal{P}_\Sigma\bar{\mathbf{b}}$ . Such equilibrium solutions correspond namely to the *tearing modes*. In order to better identify these solutions with the tearing modes as usually presented in the plasma physics literature, it is helpful to note that, upon making use of Gauss formula, the equilibrium equation coming from  $\delta\mathcal{F} = 0$  can be reformulated as

$$\nabla \times \delta\mathbf{B} - \mu\delta\mathbf{B} = 0 \quad \text{in } \mathcal{D} \setminus \Gamma^\dagger, \quad (404)$$

$$[\hat{x} \times \delta\mathbf{B}]_{x^\dagger} - \beta a_2 \hat{x} \times \nabla e^{ik_y y + k_z z} = 0, \quad (405)$$

where  $\Gamma^\dagger$  indicates the resonant surface  $x = x^\dagger$  and where we introduced the symbol  $[f]_{x^\dagger} = \lim_{\epsilon \rightarrow 0} f(x^\dagger + \epsilon) - f(x^\dagger - \epsilon)$ , to indicate the 'jump' of a function  $f$  across  $x = x^\dagger$ . If  $\mu = \lambda_j \in \sigma_P(\mathcal{S})$ , one sees that  $\beta = 0$ , and the solution  $\delta\mathbf{B} = \boldsymbol{\omega}_j$  has no jump across the resonant surface. On the other hand, if  $\mu \notin \sigma_P(\mathcal{S})$ , the solutions for  $\delta\mathbf{B}$  have to be constructed separately on the two regions of  $\mathcal{D}$  divided by  $\Gamma^\dagger$ , and the global solution has a 'jump' at  $x = x^\dagger$ . This is namely what occurs when determining the "outer" solution, in the ideal MHD region, of the classical tearing mode problem [1, 141, 142].

Note that, due to the conservation of the Casimir invariant  $C_b(\delta\mathbf{B}) = \langle \delta\mathbf{B}, \bar{\mathbf{b}} \rangle$  (denoted as helical-flux Casimir), the component of any perturbation  $\delta\mathbf{B}$  such that  $\langle \delta\mathbf{B}, \bar{\mathbf{b}} \rangle \neq 0$  remains constant. In particular, this stabilizes the tearing mode, which, on the other hand, can get destabilized by the presence of some non-ideal effect, which breaks the constraint imposed by the helical-flux Casimir.

We remark also that, similarly to the branch-(B) solution discussed above, which consists of the superposition of a Beltrami field  $\mathbf{G}$  with an eigenfunction  $a\boldsymbol{\omega}_j$ , one could consider the superposition of a Beltrami field  $\mathbf{B}_\mu$  with a tearing mode  $\mathbf{B}$ , as long as one restricts to a neighborhood of the equilibrium Beltrami field (given that the tearing mode refers to the linearized dynamics). However, while the branch-(B) solution bifurcates

only at discrete values of  $\mu$ , corresponding to the eigenvalues of  $\mathcal{S}$ , the tearing mode exists for continuous values of  $\mu$ . Also, whereas a branch-(A) and a branch-(B) solution belong to the same surface of constant helicity (by properly choosing the constants  $\mu$  and  $\alpha$ ), the superposition of  $\mathbf{B}_\mu$  with a tearing mode belongs to a surface with a different helicity. An analysis based on the Hamiltonian (388) associated with the tearing mode, also permits to retrieve another well known result of classical tearing mode theory. Namely, the property that  $\Delta' > 0$  yields a tearing instability, where  $\Delta'$  is the parameter measuring the 'jump', across the resonant surface [1], of the logarithmic derivative of the perturbed magnetic flux. Such instability is made possible by the presence of a negative potential energy that can be estimated from Eq. (388) in the case of a tearing mode, and which turns out to be proportional to  $\Delta'$  [134]. However, the instability requires a non-ideal mechanism that breaks the conservation of the helical-flux Casimir. Although the presence of a negative potential energy breaks the coerciveness of the quadratic form that can ensure stability of a Beltrami equilibrium [143], this is only a sufficient but not necessary condition for stability.

Although not corresponding to a full application of the EC method for stability (no investigation of the second variation of  $\mathcal{F}$  was made), the above discussion also shows how the free energy functional, which the EC method is based upon, is related to concepts, such as the tearing mode, which emerged in plasma physics in a different context, namely that of non-ideal instabilities.

## 11. Conclusions and perspectives

We reviewed recent applications of the EC method for finding formal stability conditions for fluid plasma equilibria. The review hopefully showed how diverse can be the applications of the method, ranging from extended MHD with general helical symmetries, to hybrid MHD-kinetic models and reduced fluid models. In fact, the method is very versatile and flexible, as it applies to any Hamiltonian system with Casimir invariants. The potential of its applications is therefore very vast, in particular

in plasma physics, where a great number of noncanonical Hamiltonian models are frequently used. The review also recalled how, in fact, different concepts such as the  $\delta W$  energy principle, NEMs and tearing modes, which emerged independently from different contexts, are all somehow related to the EC method.

On the other hand, the growing need, for instance in fusion plasmas research, of models favouring adherence to the experimental reality over simplicity, has made the EC method less popular than other, often numerical, methods for ascertaining stability conditions for complex plasma experiments. The limitations on the boundary conditions represent one of the drawbacks of the method, in this respect. Also, the EC method does not provide information about the growth rate of instabilities, which is often an important information for predicting which instabilities are likely to be the most relevant ones, for instance for tokamak confinement. Sufficient conditions found by means of the EC method might also be too strong to be used for suppressing a certain instability. Finally, the Hamiltonian nature of the method, of course rules out applications to dissipative phenomena.

In spite of these limitations, we believe that the method is a conceptually powerful tool which provides a unifying view for the equilibria and stability conditions of many models of use in plasma physics. In the review we indeed tried to emphasize how equilibria of different models can be somehow classified in terms of the Casimir invariants. Moreover, the method remains one of the few available that can provide relevant analytical conditions for two-dimensional equilibria.

In terms of future developments we mention here a few possible directions that we identified. In terms of stronger connections with experiments, we remark the analysis of Ref. [55] (whose results were briefly summarized in Sec. 5), where the conditions for positive definiteness of the second variation of the free energy functional were used to identify regions of stability in the cross section of a tokamak with ITER-like parameters. A similar analysis was carried out in Ref. [144], where MHD equilibria with pressure anisotropy were considered. Such an approach, which assumes a non-crucial role played

by boundary values of the inner regions where conditions for sign definiteness of  $\delta^2\mathcal{F}$  are fulfilled, is promising also in terms of extensions to other experimental facilities and numerical simulations.

Further connections between the EC method and other analytical stability methods found in plasma physics might be found. For instance, it could be interesting to investigate the relation between the stability criterium of Refs. [145, 146, 144] and the EC method. In particular, it could be interesting to see how the sufficient stability conditions found in Refs. [146] for "cat's eyes" magnetic island chains and in Ref. [144] for equilibria with pressure anisotropy, compare with the results of Ref. [79] described in Sec. 7.

As shown also in this review, the EC method can effectively be applied to Hamiltonian reduced fluid models. A number of such models, can be derived (see, for instance, Refs. [72, 147] ) from parent Hamiltonian drift-kinetic or gyrokinetic models, in the so-called  $\delta f$  approximation, which assumes the particles distribution function be close to equilibrium. Studying how the inclusion of the evolution of higher order moments can affect formal stability conditions is a natural question that can be addressed by means of the EC method. A concrete example, in this context, would be the extension of the stability analysis of magnetic island chains described in Sec. 7.2, to include ion gyrocenter density and parallel velocity fluctuations, which are relevant when  $\beta_{\perp e}$  is of order unity.

It could also be interesting to investigate the influence of stochastic noise on plasma equilibria extending to plasma models the stochastic EC method presented in Ref. [148].

Finally, from a more mathematical point of view, it is worth recalling that rigorous linear and nonlinear stability analyses of plasma equilibria based on fluid models more refined than MHD, are still largely missing. A mathematically rigorous application of the EC method in this context, as done, for instance, in the case of the Vlasov-Poisson system [12, 149, 150], provides a vast domain to be explored.



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