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Characterizing far from equilibrium states of the one-dimensional nonlinear Schrödinger equation

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Abstract

We use the mathematical toolbox of the inverse scattering transform to study quantitatively the number of solitons in far from equilibrium one-dimensional systems described by the defocusing nonlinear Schrödinger equation. We present a simple method to identify the discrete eigenvalues in the Lax spectrum and provide an extensive benchmark of its efficiency. Our method can be applied in principle to all physical systems described by the defocusing nonlinear Schrödinger equation and allows to identify the solitons velocity distribution in numerical simulations and possibly experiments.

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17 **1 Introduction**

18 Transport phenomena in nonlinear dispersive media may involve shock waves or the propa-
 19 gation of gray solitons that can be observed in a wide variety of systems. They appear for
 20 example in the propagation of monochromatic light in nonlinear optical fibers [1–5] or atomic
 21 vapors [6], ultra-cold atoms confined in very elongated traps [7–9], polariton superfluids [10]
 22 and the propagation of surface waves in deep water [11]. This can be explained by the fact
 23 that all these experiments can be modeled by a common nonlinear wave propagation equation,
 24 namely the homogeneous one-dimensional (1D) nonlinear Schrödinger equation (1DNLSE):

$$i \frac{\partial \psi(\mathbf{z}, t)}{\partial t} = \left(-\frac{1}{2} \frac{\partial^2}{\partial \mathbf{z}^2} + g |\psi(\mathbf{z}, t)|^2 \right) \psi(\mathbf{z}, t), \quad (1)$$

25 written here in dimensionless form and where g quantifies the strength of the nonlinear term
 26 and $\psi(\mathbf{z}, t)$ is the wavefunction.

27 In particular equation (1) is well adapted to the description of weakly interacting 1D Bose
 28 gases held in tightly confining traps, in the mean field regime. Thanks to the fine control
 29 of ultracold quasi 1D atomic gases it is possible to study experimentally the stability of soli-
 30 tons [12], soliton scattering on impurities [13], soliton creation by phase imprinting [8, 14]
 31 or head-on soliton collisions [15]. This has motivated numerous theoretical works to predict
 32 the soliton dynamics, following a phase-imprinting [16, 17], in the presence of an external
 33 trap [18] or an obstacle [19, 20] and to study states with thermal-like correlations [21–24].

34 A wide variety of theoretical tools and analytical methods have been developed for the
 35 study of equation (1), among which the inverse scattering transform (IST) that evidenced
 36 the special role of solitons [25–29], Whitham’s modulation theory to capture the transport
 37 of dispersive shock waves [30, 31], as for example in the so-called "dam-break" problem [5],
 38 and Lagrangian models describing solitons as particles with an effective mass [32–34]. When
 39 equation (1) is used to model a system with periodic boundary conditions the stationary states
 40 are known [35] and a specific form of IST can be used [36–39], taking advantage of the spatial
 41 periodicity.

42 More recently, following the discovery of the generalized hydrodynamic equations applied
 43 to the Lieb-Liniger model for 1D interacting bosons [40–42], the hydrodynamic approach has
 44 been adapted to the study of soliton gases, offering a new insight in the study of these sys-
 45 tems [43–45]. The key ingredient in this approach is the knowledge of the distribution of
 46 soliton velocities [43], encoded in the discrete eigenvalues of the IST [28].

47 In this work we aim at contributing to the study of soliton gases by providing a accurate and
 48 robust method to identify propagating gray solitons in a far from equilibrium state evolving
 49 according to the defocusing nonlinear Schrödinger equation, Eq. (1) with $g > 0$, also known
 50 as the repulsive Gross-Pitaevskii equation. We consider a finite size system of length L with
 51 periodic boundary conditions, $\psi(\mathbf{z}, t) = \psi(\mathbf{z} + L, t)$ and we normalize the wavefunction to
 52 the total number of particles: $N = \int_0^L d\mathbf{z} |\psi(\mathbf{z}, t)|^2$. In the following we denote by $n_0 = N/L$
 53 the average density, $c = \sqrt{gn_0}$ the bare speed of sound and $\xi = 1/\sqrt{2gn_0}$ the bare healing
 54 length. These three parameters control the relevant physical scales associated to Eq. (1).
 55 We will focus on the regime of large non-linearity $gn_0 \gg 1$, such that $\xi \ll L$. To solve
 56 Eq. (1) we use a spectral method relying on a discrete grid with N_z points, see Appendix A for

57 details. In the following we will use $N_z = 512$ and a non-linearity $gn_0 = 2 \times 10^4$, such that
 58 $\delta z = L/N_z = 1.95 \times 10^{-3} < \xi = 5 \times 10^{-3}$.

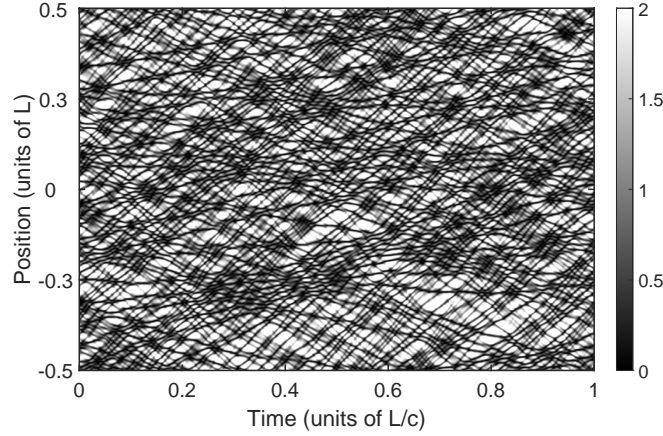


Figure 1: Density map of a far from equilibrium state evolving according to Eq. (1), exhibiting large density fluctuations, with many propagating solitons. The quantity plotted is $|\psi(\mathbf{z}, t)|^2/N$, normalized to the number of particles. The position \mathbf{z} is normalized to the length of the system L and the time to L/c where c is the bare speed of sound. We identify $N_s = 131.5 \pm 4.5$ solitons propagating in this particular realization. See text for details.

59 Figure 1 illustrates the problem we intend to solve. Given an arbitrary initial state, propa-
 60 gated in time according to Eq. (1), how can we identify the number of solitons and determine
 61 precisely their velocities? This is not an easy task: even if the propagation of many density
 62 dips is clearly seen in the space-time density map of Fig. 1, it is not possible to follow accurately
 63 the trajectory of an individual soliton, due to the multiple collisions with other gray solitons.

64 Instead of relying on data analysis tools to solve this issue [46], our idea is to use the well
 65 established tools of the inverse scattering transform method to measure the soliton velocity
 66 distribution. More precisely we are going to use the direct scattering transform to compute the
 67 Lax spectrum and build a soliton indicator to identify the solitons in the spectrum. By doing
 68 so we are adapting to the defocusing case a method that was successful for the study of the
 69 focusing NLSE [11]. Here however we are facing two main difficulties: (a) the creation of
 70 gray solitons is a threshold-less process [9, 47], and (b) the Lax spectrum is real, requiring a
 71 careful analysis to identify solitons.

72 This paper is organized as follows: section 2 introduces our soliton indicator, section 3
 73 presents a detailed benchmark of its efficiency, then section 4 discusses its applications and
 74 finally we conclude in section 5. We also provide three appendices that give additional details
 75 on the methods and a few extra examples.

76 2 Definition of a soliton indicator

77 The Lax operator corresponding to Eq. (1), for the defocusing case $g > 0$ reads [25, 28]:

$$\mathcal{L} = \frac{i}{2} \begin{pmatrix} \frac{\partial}{\partial z} & -\sqrt{g}\psi(\mathbf{z}, t) \\ \sqrt{g}\psi(\mathbf{z}, t)^* & -\frac{\partial}{\partial z} \end{pmatrix}. \quad (2)$$

78 Its spectrum can be computed analytically for a few simple cases, in particular for infinite size
 79 systems with well defined asymptotic densities $|\psi(\mathbf{z} \rightarrow \pm\infty, t)| = \sqrt{n_0}$, for which it is proven

80 that the spectrum is made of two continuous branches, separated by a gap, in which discrete
 81 eigenvalues can exist, each one corresponding to a gray soliton [25]. Since we are working
 82 with periodic boundary conditions and a finite size system, we first consider how this picture
 83 is modified. We can adapt the analytic one soliton solution of the infinite system to the case
 84 of periodic boundary conditions by considering:

$$\psi_1(z, t) = \sqrt{n_0} e^{i(k_0 z - \omega t)} \left(\cos \phi \tanh \left[\cos \phi \sqrt{g n_0} (z - \bar{z}(t)) \right] + i \sin \phi \right), \quad (3)$$

85 where $\omega = g n_0 + k_0^2/2$ and we introduced the phase gradient $e^{ik_0 z}$, that guarantees the
 86 compatibility with periodic boundary conditions if:

$$e^{ik_0 L} = \frac{i \sin \phi - \cos \phi \tanh \left[\cos \phi \sqrt{g n_0} L/2 \right]}{i \sin \phi + \cos \phi \tanh \left[\cos \phi \sqrt{g n_0} L/2 \right]}. \quad (4)$$

87 Equation (3) describes the propagation of a single gray soliton, parametrized by the angle
 88 $\phi \in [-\pi/2, \pi/2]$, moving at a constant velocity $\bar{z} = k_0 + c \sin \phi$. We note that Eq. (3) is only
 89 an approximate solution that is accurate only in the large non-linearity limit $L \gg \xi$. A more
 90 general solution can be found in terms of elliptic functions for arbitrary values of L/ξ [35].
 91 The angle ϕ sets both the speed of the soliton and the density depletion at its center, giving a
 92 relative contrast of $\cos[\phi]^2$.

93 Plugging the formula of Eq. (3) into the definition of Eq. (2), a lengthy but straightforward
 94 calculation shows that the Lax spectrum $\mathcal{L}v = \zeta v$ is made of two branches:

$$\zeta_q^\pm = -\frac{k_0}{4} \pm \frac{\sqrt{g n_0 + q^2}}{2}, \quad (5)$$

95 where $q \in 2\pi/L \times \mathbb{Z}$, corresponding to quasi plane-wave eigenvectors:

$$v_q^\pm(z, t) \propto e^{\pm i q z} \begin{pmatrix} e^{i \frac{k_0 z - \omega t}{2}} \left[i \frac{k_0 + 4\zeta_q^\pm \mp 2q}{2\sqrt{g n_0}} + \frac{\psi_1(z, t)}{\sqrt{n_0} e^{i(k_0 z - \omega t)}} \right] \\ e^{-i \frac{k_0 z - \omega t}{2}} \left[i \frac{k_0 + 4\zeta_q^\pm \pm 2q}{2\sqrt{g n_0}} - \frac{\psi_1(z, t)^*}{\sqrt{n_0} e^{-i(k_0 z - \omega t)}} \right] \end{pmatrix},$$

96 and a single eigenvalue:

$$\zeta_0 = -\frac{k_0}{4} - \frac{c}{2} \sin \phi, \quad (6)$$

97 lying in the gap between the two branches $\zeta_q^- < \zeta_0 < \zeta_q^+$ and associated to a localized eigen-
 98 vector:

$$v_0(z, t) \propto \operatorname{sech} \left[\cos \phi \sqrt{g n_0} (z - \bar{z}(t)) \right] \begin{pmatrix} e^{i \frac{k_0 z - \omega t}{2}} \\ -e^{-i \frac{k_0 z - \omega t}{2}} \end{pmatrix}.$$

99 Figure 2 shows the Lax spectrum numerically computed for the state of Eq. (3), evidencing
 100 the two branches and the isolated eigenvalue. We always sort the set of $2N_z$ eigenvalues
 101 by ascending order and focus on the central part of the spectrum, corresponding to the gap
 102 between the continuous branches, usually close to eigenvalue number N_z . The results are in
 103 very good agreement with the theoretical formulas presented above. The two main differences
 104 with the infinite system [25], come from the periodic boundary conditions: the Lax spectrum
 105 is globally shifted by a factor $-k_0/4$ and the two branches are made of many, closely spaced,
 106 discrete eigenvalues, because the wave-vector q is discrete. The gap between the two branches
 107 is $\zeta_{q=0}^+ - \zeta_{q=0}^- = c$, which corresponds to the speed of sound set by the uniform background
 108 density, while the sum $\zeta_{q=0}^+ + \zeta_{q=0}^- = -k_0/2$ reflects the velocity of the background flow.

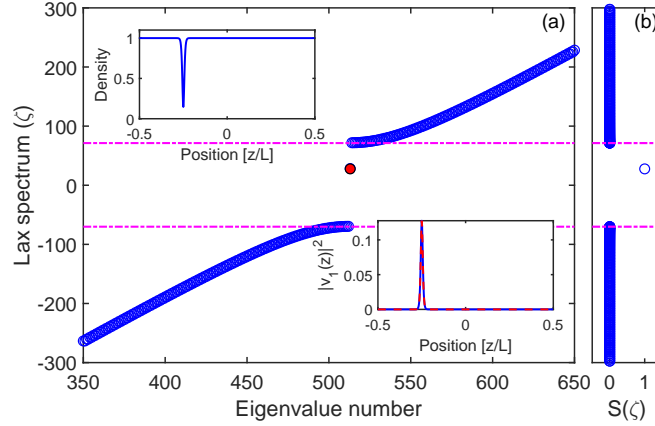


Figure 2: (color online) a) Lax spectrum for a single gray soliton state with angle $\phi = -\pi/8$: open blue circles indicate continuous branches of the spectrum, the single filled red circle is the discrete eigenvalue associated to a localized eigenstate and the two horizontal dashed magenta lines correspond to the gap boundaries at $-k_0/4 \pm c/2$. Upper inset: density of the single gray soliton state evidencing the density dip. Lower inset: square modulus of the eigenvector corresponding to the localized eigenvalue (blue solid line) compared to the analytical prediction (red dashed line). b) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for a threshold of $\epsilon = \pi^2/(4L^2 \sqrt{gn_0})$, see text for details. The only eigenvalue with an indicator equal to one corresponds to the gray soliton.

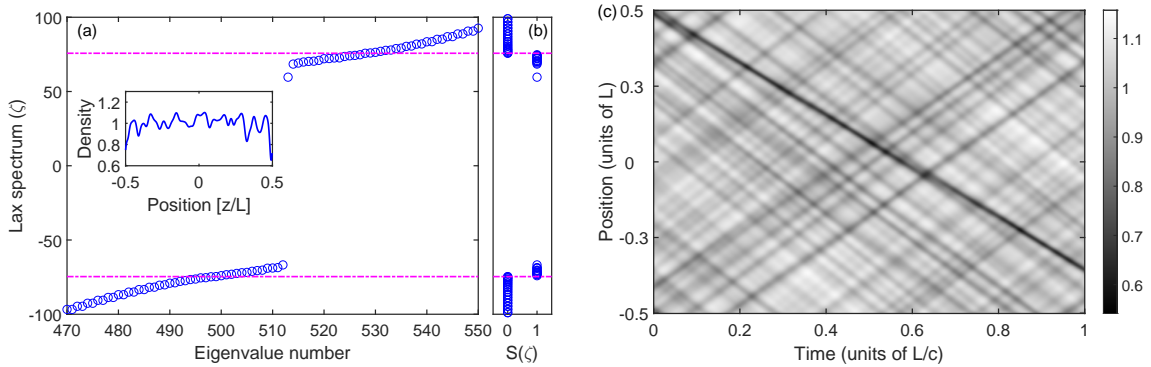


Figure 3: (color online) a) Lax spectrum for a weakly excited state (open blue circles), created using the protocol described in Appendix A. The two horizontal dashed-dotted magenta lines indicate the gap in the continuous spectrum, detected using the soliton indicator $S(\zeta)$. Inset: example of the density profile of this state (solid blue curve). b) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for a threshold of $\epsilon = \pi^2/(4L^2 \sqrt{gn_0})$, see text for details. c) Density colormap $|\psi(\mathbf{z}, t)|^2/N$ exhibiting a small number of fast propagating solitons for this particular realization.

109 While it is quite obvious to identify the isolated eigenvalue in the simple Lax spectrum
 110 shown in Fig. 2, it is in general a difficult problem, as for a discrete matrix operator the spec-
 111 trum is by essence discrete, and the notion of continuous branches is ill-defined. To illustrate
 112 this we consider an out-of-equilibrium state, corresponding to a small deviation with respect
 113 to a uniform background and yet resulting in a very rich dynamics, as shown on Fig. 3. We
 114 explain in Appendix A how we generate such an out-of-equilibrium state. On the one hand, the
 115 density map of Fig. 3c) shows many propagating density dips, moving at constant velocities
 116 across the sample, that can be identified as solitons by visual inspection, although it may be
 117 difficult already to count precisely the number of dips, see for example the inset of Fig. 3a). On
 118 the other hand, the associated Lax spectrum apparently displays two branches with a single
 119 isolated eigenvalue, leading to the obviously incorrect conclusion that there is only a single
 120 soliton propagating. In fact this isolated eigenvalue corresponds to the soliton with the highest
 121 contrast (or equivalently smallest velocity). This shows the need for an unambiguous method
 122 to identify and characterize the number of solitons in an excited state of Eq. (1), for a finite-size
 123 geometry.

124 We intend to solve this problem by defining a soliton indicator $S(\zeta)$ equal to $\mathbf{1}$ if the eigen-
 125 value ζ corresponds to a soliton or $\mathbf{0}$ on the contrary. To achieve this we make use of a funda-
 126 mental property of integrable systems: the interaction between solitons results only in delays
 127 during their propagation, without altering their properties [28, 43]. Now taking advantage of
 128 periodic boundary conditions, consider an extended system containing two copies of the same
 129 state, as sketched on Fig. 4a). We then expect to find twice as many solitons in the extended
 130 system, with respect to the original one, each soliton being present twice, with exactly the same
 131 velocity. For the continuous branches, we expect a different behavior: the quasi plane-waves
 132 will extend over the interval of length $2L$, resulting in a denser spectrum, see for example
 133 Eq. (5).

134 Figure 4b) shows how the Lax eigenvalues for the initial and extended systems are dis-
 135 tributed: the horizontal axis is shifted and rescaled to emphasize the fact that each eigenval-
 136 ues corresponding to a soliton is doubly degenerate in the extended set. In the continuous
 137 branches the analysis is less obvious as the spectrum is denser.

138 Based on this physical intuition we build a soliton indicator by comparing the degeneracy
 139 of each eigenvalue of the Lax spectrum in the initial and extended systems: a eigenvalue with
 140 a degeneracy increased by a factor of two will correspond to a soliton. As we will show below,
 141 this allows to identify properly the propagating solitons in a arbitrary out-of equilibrium state
 142 of Eq. (1). To implement this, we compute the set of eigenvalues $\{\zeta_i\}$ of the initial system and
 143 the set of eigenvalues $\{\zeta'_i\}$ of the extended system. We then define for each eigenvalue ζ_i the
 144 soliton indicator as:

$$S(\zeta_i) = \frac{\text{card}\{\zeta'_j \text{ such that } |\zeta_i - \zeta'_j| < \epsilon\}}{\text{card}\{\zeta_j \text{ such that } |\zeta_i - \zeta_j| < \epsilon\}} - \mathbf{1}, \quad (7)$$

145 which is the ratio of the number of eigenvalues close to ζ_i in the extended set and original
 146 set, respectively, minus one. We use here the **card** notation to denote the cardinality of a set
 147 of eigenvalues. If the degeneracy of eigenvalue ζ_i is doubled in the extended set, we find
 148 $S(\zeta_i) = \mathbf{1}$, while if the degeneracy does not change $S(\zeta_i) = \mathbf{0}$. In the following we define the
 149 number of solitons in a particular state N_{sol} as the number of eigenvalues with indicator equal
 150 to one.

151 Equation (7) introduces a parameter ϵ that defines the threshold between degenerate
 152 and non-degenerate eigenvalues. Because we have in mind to apply this method to nu-
 153 merical simulations with finite grid size and discrete approximations of the operators, there-
 154 fore introducing some level of numerical error, we have to choose a finite threshold value.
 155 Looking at the analytical formula corresponding to the single soliton solution, we may use

Eq. (5) to estimate the minimal distance between two eigenvalues at the gap edge ($q \rightarrow 0$): $\delta\zeta \simeq \delta q^2 / (4\sqrt{gn_0}) = \pi^2 / (L^2 \sqrt{gn_0})$, for the initial system, where $\delta q = 2\pi/L$. In the extended system this value is divided by a factor of 4 and it seems then reasonable to choose a value $\epsilon \leq \pi^2 / (4L^2 \sqrt{gn_0})$ to separate two distinct eigenvalues in the extended set.

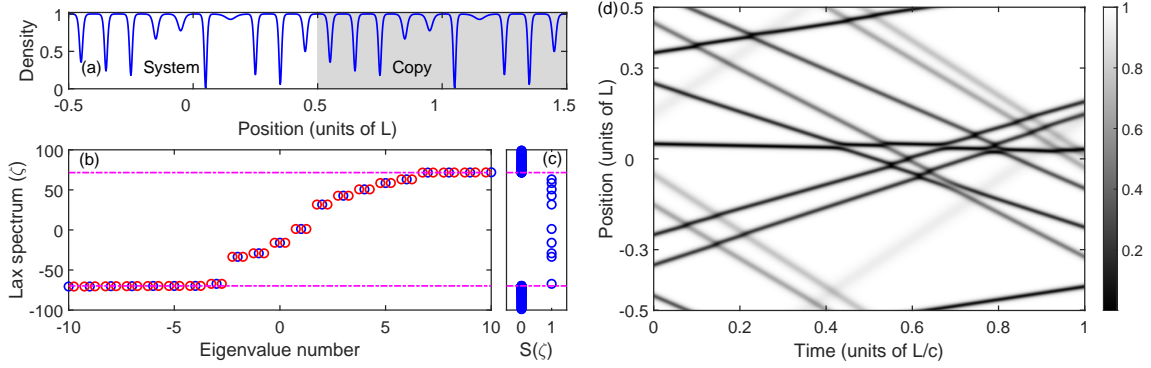


Figure 4: (color online) a) Density as a function of position for a state with $N_{\text{imp}} = 10$ solitons using the analytical formula described in Eq. (8). The solitons can be clearly identified as density dips by visual inspection. The gray shaded area represents the system copy we use to define the extended system. b) Lax spectrum for the initial (open blue circles) and extended (open red circles) systems, respectively. The two dashed-dotted magenta lines indicate the gap in the continuous part of the spectrum detected using the soliton indicator $S(\zeta)$. c) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for a threshold of $\epsilon = \pi^2 / (4L^2 \sqrt{gn_0})$. d) Density colormap $|\psi(z, t)|^2 / N$ of the propagating solitons for this particular realization.

Figure 4 illustrates this concept of soliton indicator, using an artificially created (see section 3) state for which solitons can be clearly identified as density dips by visual inspection, see the initial density profile shown in Fig. 4a) and the density map of Fig. 4d). For this particular realization the soliton indicator, computed for a threshold value of $\epsilon = \pi^2 / (4L^2 \sqrt{gn_0})$, identifies 10 eigenvalues with $S(\zeta) = 1$, corresponding to the 10 imprinted solitons. Similarly, Fig. 2b) and 3b) show the value of $S(\zeta)$, computed with the same value of $\epsilon = \pi^2 / (4L^2 \sqrt{gn_0})$, for a simple state with a single soliton and an out-of-equilibrium state, respectively. It seems to behave as anticipated, identifying a few eigenvalues as solitons inside the gap between the two continuous branches. In section 3 we benchmark how the soliton indicator behaves with the choice of threshold and give a robust definition of the number of solitons.

Interestingly we also find with the indicator the two gap edges $\zeta_{q=0}^{\pm}$, corresponding to the last (first) eigenvalue of the quasi continuous lower (upper) branch. In analogy with the analytical formula of the single soliton solution, see above, we may interpret the amplitude of the gap $\zeta_{q=0}^+ - \zeta_{q=0}^- = c_{\text{eff}}$ as the effective speed of sound in the sample and deduce the effective background flow: $k_{\text{eff}} = -2(\zeta_{q=0}^+ + \zeta_{q=0}^-)$.

3 Benchmark of the soliton indicator

In this section we study systematically how the choice of the threshold ϵ in the soliton indicator of Eq. (7) affects the efficiency of the soliton detection. To do so, we study the Lax spectrum of states defined by the following analytical formula:

$$\psi_M(z) = \sqrt{n_0} \prod_{j=1}^M e^{ik_j z} (\cos \phi_j \tanh[\cos \phi_j \sqrt{gn_0}(z - z_j)] + i \sin \phi_j), \quad (8)$$

179 where $M > 0$ is an integer. For $M = 1$, Eq. (8) is simply Eq. (3) evaluated at $t = 0$, thus
 180 describing a single gray soliton with angle ϕ_1 located at position z_1 , provided that k_1 is chosen
 181 according to Eq. (4) to fulfill periodic boundary conditions. For $M > 1$ it is a reasonable
 182 assumption that Eq. (8) describes a state with M solitons, provided that they are initially
 183 located far from each other. In other words, we use Eq. (8) as a guess state containing a dilute
 184 gas of M gray solitons that we will use to benchmark our soliton detection method. We will
 185 proceed as follows. First we choose a value of $M \geq 1$, then we draw randomly M phases
 186 $\phi_j \in [-\pi/2, \pi/2]$ and build the state of Eq. (8) by distributing the M solitons at regular
 187 spacings $z_{j+1} - z_j = L/M$ over the interval $[0, L]$. We then compute the Lax spectrum for
 188 this state and study how the soliton indicator changes with the choice of threshold ϵ . We
 189 repeat this procedure many times to sample the different possible soliton gas configurations
 190 and analyze the results to define a probability of success for our soliton detection method.

191 As mentioned in the previous section, an annoying consequence of working with periodic
 192 boundary conditions is the extra phase gradient proportional to k_j that must be included in
 193 the definition of the state and that results in a shift of the gap edges. One way to avoid this
 194 issue is to consider a special case with a gas of pairs of solitons having opposite phases: all
 195 phase gradients then compensate and the Lax spectrum becomes symmetric, all eigenvalues
 196 coming by pairs of opposite sign. We will first discuss the case of a dilute gas of soliton pairs
 197 before generalizing to a dilute gas of solitons. In the following, $N_{\text{imp}} = M$ will be the number
 198 of solitons we intend to imprint, using Eq. (8) and N_{sol} the number of solitons detected for a
 199 particular threshold and realization.

200 3.1 Dilute gas of soliton pairs

201 A first generic feature that we observe is that $S(\zeta)$ is always 0 , except for a small number of
 202 eigenvalues, in the central part of the spectrum (assuming that the eigenvalues are sorted by
 203 increasing value). The number of eigenvalues corresponding to $S(\zeta) = 1$, that we denote N_{sol}
 204 is always smaller than N_{imp} for small thresholds and tends to increase with the value of the
 205 threshold.

206 Figure 5a) illustrates this by showing the value of $S(\zeta)$ for $N_{\text{imp}} = 20$, corresponding
 207 to 10 imprinted pairs, averaged over 500 realizations with phases ϕ_j drawn randomly, as a
 208 function of the threshold ϵ and the index of the eigenvalue in the spectrum N_ζ . Figure 5b)
 209 shows the relative difference between the average number of detected and imprinted solitons
 210 $\Delta = (N_{\text{sol}} - N_{\text{imp}})/N_{\text{imp}}$, as a function of the threshold and for initial states with different
 211 numbers of pairs. For each curves the results are averaged over an ensemble sampling at least
 212 10000 random phases. From both figures we see that a low threshold value leads to an under-
 213 estimation of the number of solitons, while a value above 5×10^{-2} induces an overestimation.

214 In order to provide a reasonable estimate of the number of solitons in a single realization
 215 we define empirically two thresholds: $\epsilon_- = 1 \times 10^{-3}$ and $\epsilon_+ = 5 \times 10^{-2}$, that give a lower
 216 (N_{sol}^-) and upper (N_{sol}^+) bound respectively on the number of solitons. We have chosen those
 217 particular values such that the average magnitude of Δ remains below 0.1, corresponding to
 218 an error of less than 10%, on average. This allows us to measure the number of soliton in a
 219 single realization $N_{\text{sol}} = \bar{N}_{\text{sol}} \pm \delta N_{\text{sol}}$, where $\bar{N}_{\text{sol}} = (N_{\text{sol}}^+ + N_{\text{sol}}^-)/2$ and $\delta N_{\text{sol}} = (N_{\text{sol}}^+ - N_{\text{sol}}^-)/2$.

220 Figure 5c) reports the average value of the relative number of detected solitons with the
 221 average uncertainty, as a function of the number of imprinted solitons. It shows that with our
 222 choice of thresholds ϵ_\pm , \bar{N}_{sol} tends to underestimate the number of imprinted solitons by 2
 223 to 5%. We also report the relative number of imprinted solitons with phases below a certain
 224 value, in magnitude: $|\phi| < \kappa\pi/2$, with $\kappa \in [0, 1]$. We find that our method gives a average
 225 number of solitons that agrees very well with the number of imprinted solitons with phases
 226 below $0.97 \times \pi/2$, within self-consistently estimated uncertainties. The agreement is less good
 227 for higher numbers of pairs, which we attribute to the fact that the analytical formula is not

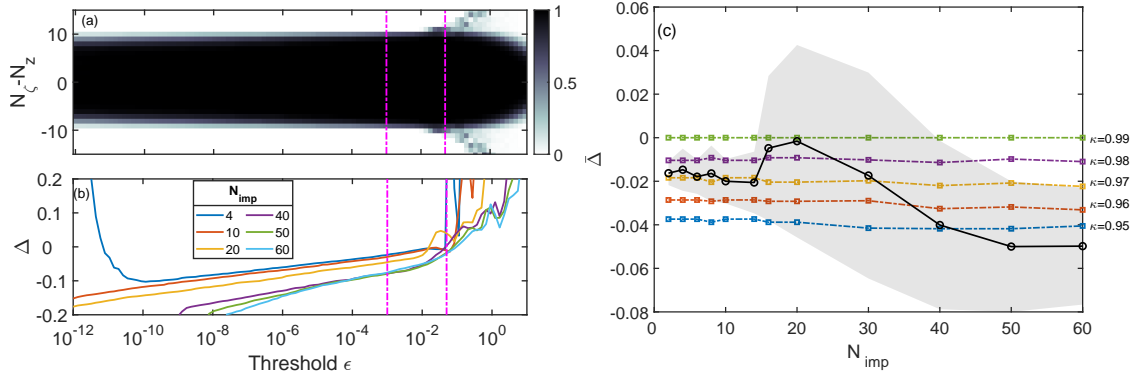


Figure 5: (color online) a) Average value of the soliton indicator over **500** realizations for $N_{\text{imp}} = 20$, corresponding to **10** imprinted pairs of solitons as a function of threshold ϵ and the index of the eigenvalue in the spectrum N_{ζ} . We perform the numerical simulation with grid size $N_z = 1024$ and nonlinearity $gn_0 = 2 \times 10^4$. The two vertical dashed-dotted magenta lines corresponds to the two thresholds $\epsilon_- = 1 \times 10^{-3}$ and $\epsilon_+ = 5 \times 10^{-2}$. b) The relative difference between the average number of detected and imprinted solitons $\Delta = (N_{\text{sol}} - N_{\text{imp}})/N_{\text{imp}}$ as a function of thresholds for small to large number of imprinting solitons. For each curves, the results are averaged over an ensemble sampling at least **10000** random phases. c) The relative difference between the average number of detected and imprinted solitons $\bar{\Delta} = (\bar{N}_{\text{sol}} - N_{\text{imp}})/N_{\text{imp}}$ (black open markers with solid black line) with average uncertainty represented by the gray shaded area as a function of the number of imprinted solitons N_{imp} . The different dashed-dotted lines with square markers represent the value of Δ as a function of N_{imp} with phases below a certain value, $|\phi| < \kappa\pi/2$. Here we only plot five different κ values ranging from $\kappa = 0.95$ to $\kappa = 0.99$.

228 accurate anymore because the solitons tend to overlap at higher density. However we think
 229 that our method can still faithfully estimate the number of solitons in the sample with a rea-
 230 sonable uncertainty. We note that solitons with phases greater than $0.97 \times \pi/2$ correspond
 231 to dips in the density profile with contrast below 3×10^{-3} , using a naive estimation based on
 232 Eq. (3), that are hardly visible anyway. We stress also that the results presented in Fig. 5 are
 233 ensemble averaged, in order to calibrate the relevant thresholds and the efficiency of the pro-
 234 tocol. However when applied to particular states our method can identify exactly the number of
 235 solitons, when the contrast of solitons is sufficiently high, even if they partially overlap.

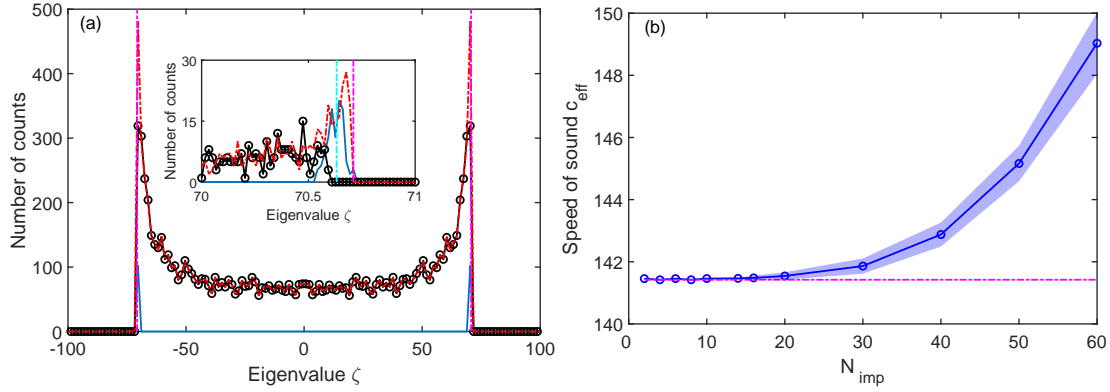


Figure 6: (color online) a) Histogram of soliton eigenvalues, computed for a state with 5 imprinted pairs ($N_{\text{imp}} = 10$), averaged over 1000 realizations. The black solid line with open circles shows the histogram computed for eigenvalues identified by the threshold ϵ_- , the blue solid line the histogram for eigenvalues identified only by the threshold ϵ_+ and the dashed-dotted magenta line the expected distribution of imprinted solitons. The inset evidences the behavior near the gap edge where the solitons are more difficult to detect. The dash-dotted vertical magenta lines correspond to $\pm c/2$, while the dash-dotted cyan lines correspond to $\pm \sin(0.97\pi/2)c/2$. b) Effective speed of sound c_{eff} (open blue circles) found from the gap edges as a function of the number of imprinted solitons. The solid blue line is a guide to the eye. The light blue shaded area indicates the uncertainty on the speed of sound, extracted from the gap edges computed with the thresholds ϵ_{\pm} , see text for details. The horizontal dash-dotted magenta line indicates the bare speed of sound c .

236 So far we have only considered the number of detected solitons, we now study how the
 237 eigenvalues are distributed. To do so we again use the state of Eq. (8) and identify for each
 238 realization of the random phases the eigenvalues corresponding to solitons. More precisely we
 239 build a first histogram of the eigenvalues identified by the lower threshold ϵ_- and a second one
 240 for the extra ones identified only by the second threshold ϵ_+ . Figure 6a) shows the result for
 241 5 pairs, averaged over 1000 realizations, and compare it to the expected histogram, obtained
 242 from the knowledge of all the phases and using Eq. (6) with the bare speed of sound. The
 243 measured histogram shows a very good agreement with the expected one, except near the
 244 edges (see the inset), as could be anticipated.

245 We have varied the number of imprinted pairs between 1 and 30, always sampling roughly
 246 10000 phases (in total) and observed qualitatively a similar behavior. However for number
 247 of pairs larger than 10 we start to observe significant deviations. To confirm this we extract
 248 from each realization an estimation of the speed of sound $\bar{c}_{\text{eff}} \pm \delta c_{\text{eff}}$ (from the gap edges
 249 corresponding to the two thresholds, see Sec. 2) and plot the averaged effective speed of
 250 sound and uncertainty as a function of the number of pairs on Fig. 6b). We attribute the fact
 251 that the speed of sound deviates from the background value $c = \sqrt{gn_0}$ to the failure of Eq. (8)

252 to describe a dense soliton gas. However we think that the method we explained here still
 253 gives a reasonable estimate of the soliton gas properties.

254 3.2 Dilute gas of solitons

255 We now briefly confirm our choice of thresholds by studying a dilute gas of solitons, each phase
 256 being drawn independently. Figure 7a) reports the average relative error on the number of
 257 solitons that remains consistent with the results of Fig. 6b) (note that 30 solitons is equivalent
 258 to 15 pairs): on average solitons with phases of magnitude up to $0.97 \times \pi/2$ are correctly
 259 detected. Figure 7b) evidences that the distribution of detected eigenvalues indeed follows
 260 the expected law. However, one notable difference is the distribution of extra eigenvalues
 261 (those detected only in between ϵ_- and ϵ_+), near the edges of the histogram that is much
 262 broader. This is not due to a lesser precision of the detection method, but reflects the random
 263 shifts of the gap edges in the continuous spectrum due to the background phase, as mentioned
 264 above (see Eqs. (5), (6) and (8)).

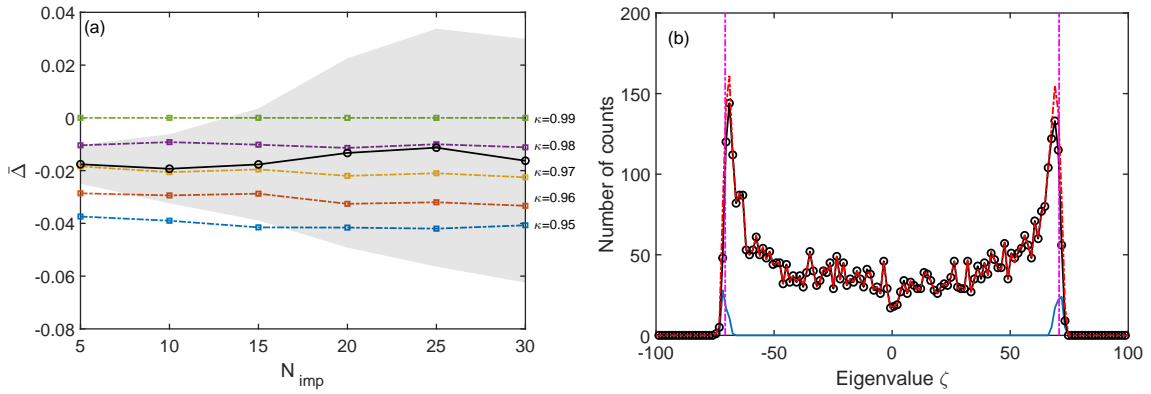


Figure 7: (color online) a) The relative difference between the average number of detected and imprinted solitons $\bar{\Delta}$ (black open markers with solid black line) with average uncertainty represented by the gray shaded area as a function of the number of imprinted solitons N_{imp} . The different dashed-dotted lines with square markers represent the value of Δ as a function of N_{imp} with phases below a certain value, $|\phi| < \kappa\pi/2$. Here we only plot five different κ values ranging from $\kappa = 0.95$ to $\kappa = 0.99$. b) Histogram of soliton eigenvalues, computed for a state with 10 imprinted solitons $N_{\text{imp}} = 10$, averaged over 500 realizations. The black solid line with open circles shows the histogram computed for eigenvalues identified by the threshold ϵ_- , the blue solid line the histogram for eigenvalues identified only by the threshold ϵ_+ and the dashed-dotted magenta line the expected distribution of imprinted solitons.

265 For example, when applied to very simple states, as shown on Fig. 2 for a single soliton or
 266 on Fig. 4 for a state with 10 clearly visible solitons, our soliton detection method gives an exact
 267 result: $N_{\text{sol}} = 1 \pm 0$ or 10 ± 0 , respectively, which can be simply confirmed in that case by a visual
 268 inspection of the space-time density map. Finally our method enables a simple and efficient
 269 identification of solitons for arbitrary excited states, as shown in Fig. 1 and 3, resulting in
 270 $N_{\text{sol}} = 131.5 \pm 4.5$ and 25.5 ± 2.5 respectively, with a guarantee that the uncertainty concerns
 271 only the shallowest (fastest) solitons. Moreover we emphasize that our method give also access
 272 to the full distribution of Lax eigenvalues corresponding to the soliton gas, a key ingredient in
 273 the generalized hydrodynamics approach [43].

274 3.3 Comparison to an analytical result

275 Finally to validate our findings we compare the outcome of our method with a simple analytical
 276 result. A initial state with a hyperbolic tangent wavefunction is known to generate a odd
 277 number of solitons with a specific distribution of velocities in a infinite size system [9]. We
 278 can adapt this result to the case of periodic boundary conditions, by using the initial state:

$$\psi_{\text{ana}}(z) = \sqrt{n_0} \tanh(\sqrt{gn_0}\alpha z) e^{ik_0 z} \quad (9)$$

279 where the phase gradient $e^{ik_0 z}$ is there to compensate the phase jump at $z = 0$ and ensure com-
 280 patibility with periodic boundary conditions. The parameter α controls the width of the initial
 281 density dip and the distribution of generated solitons. Using this initial wavefunction, the Lax
 282 operator of Eq. (2) can be diagonalized exactly and the resulting eigenvalues of the discrete
 283 spectrum are given by $\zeta_0 = -\frac{k_0}{4}$, $\zeta_{2j} = -\frac{k_0}{4} + \frac{c}{2} \sqrt{1 - (1 - j\alpha)^2}$, $\zeta_{2j+1} = -\frac{k_0}{4} - \frac{c}{2} \sqrt{1 - (1 - j\alpha)^2}$
 284 where $j = 1, 2, \dots, N_0$ and N_0 is the largest integer such that $N_0 < 1/\alpha$. These formula show
 285 that for arbitrary α , the initial wavefunction profile of the form of Eq. (9) always produces a
 286 dark soliton at $z = 0$ and additional N_0 pairs of symmetric gray solitons corresponding to the
 287 nonzero eigenvalues. As a result, the total number of eigenvalues and thus the total number
 288 of solitons is $2N_0 + 1$ and depends on the value of α . We now benchmark our soliton detection
 289 method using Eq. 9 for different values of α , using the two thresholds $\epsilon_- = 1 \times 10^{-3}$ and
 290 $\epsilon_+ = 5 \times 10^{-2}$.

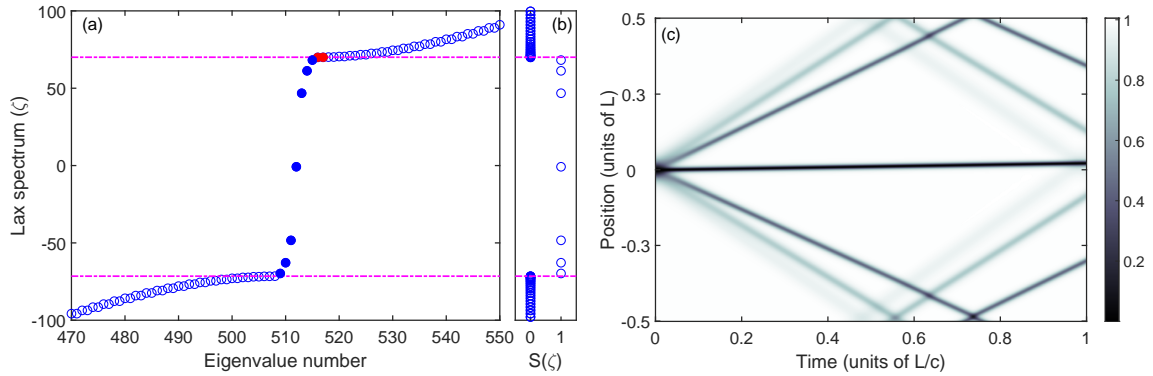


Figure 8: (color online) a) Lax spectrum as a function of the eigenvalue number for $\alpha = 0.26$ (open blue circles) using the initial state of Eq. (9). The two horizontal dashed-dotted magenta lines indicate the gap in the continuous spectrum, detected using the soliton indicator $S(\zeta)$. The filled blue circles denote the eigenvalues corresponding to the ϵ_- threshold, while the two filled red circles denote the eigenvalue detected only by the ϵ_+ threshold. b) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for the threshold ϵ_- . c) Density colormap showing the propagation of the initial state, with seven gray solitons propagating.

291 Figure 8a) shows a typical Lax spectrum we obtain for the state of Eq. (9), here for the
 292 particular value $\alpha = 0.26$. Our soliton indicator method identifies correctly the gap between
 293 the two continuous branches, and $N_{\text{sol}} = 8 \pm 1$ solitons. The analytical formula predicts exactly
 294 7 solitons. By comparing the predicted eigenvalues to the detected ones, we find an excellent
 295 agreement: all expected solitons are correctly detected and the two extra ones correspond to
 296 very shallow features close to the gap. This result is consistent with our analysis of the dilute
 297 soliton gas. In Fig. 8c), we plot the density colormap corresponding to the propagation of the
 298 initial state. Here it can be checked that indeed seven density dips (gray lines) are propagating,
 299 although the fastest (shallowest) ones are barely visible on this color scale. It is interesting to
 300 note also that the solitons moving upwards (relative to the orientation of the plot) are faster

301 than those moving downwards: the symmetry is broken by the background phase gradient
 302 e^{ik_0z} .

303 In order to evaluate the accuracy of our soliton indicator we repeat the same study for 200
 304 different values of $\alpha \in [0.05, 0.90]$. Fig. 9 reports the result of this analysis, in the form of the
 305 histogram of eigenvalues corresponding to solitons. We compare this histogram to the one we
 306 expect based on the analytical formulas. We find an excellent agreement, except near the edges
 307 where our choice of upper threshold tends to lead to "false positive" soliton detection, as was
 308 observed for the previous study. Here we limit ourselves to $\alpha > 0.05$ to avoid distortion of the
 309 hyperbolic tangent shape due to periodic boundary conditions: when $\alpha < 0.05$ the effective
 310 speed of sound we measure deviates significantly from the bare value c , which provides an
 311 indication that the background is not flat anymore.

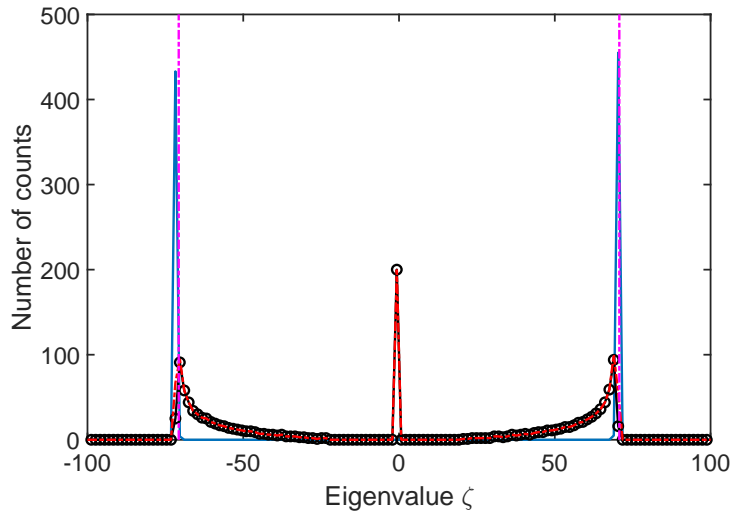


Figure 9: (color online) Histogram of the soliton eigenvalues computed for a range of α values. The black solid line with open circles shows the histogram computed for eigenvalues identified by the threshold ϵ_- , and the blue solid line is the histogram for eigenvalues identified only by the threshold ϵ_+ . The red dashed line shows the histogram computed for eigenvalues for the soliton as described in Eq. (9). The two dashed-dotted vertical magenta lines indicate the gap in the Lax spectrum at the positions $\pm c/2$.

312 To summarize, we have shown in section 3 that it is possible to use the soliton indicator of
 313 Eq. (7) with two thresholds to estimate an upper and a lower bound of the soliton number, the
 314 effective speed of sound with an uncertainty and finally the distribution of soliton velocities. By
 315 using the two benchmarking methods we demonstrate that we are able to identify all solitons
 316 and that when extra solitons are detected it is always near the gap edges, which leads only to
 317 a small overestimation of the speed of sound. Importantly our method can then be applied to
 318 identify solitons in an arbitrary initial state.

319 4 Discussion

320 In this last section we discuss our results and highlight a few open questions.

321 We have checked how the choice of the thresholds ϵ_{\pm} was related to the value of the non-
 322 linearity. To do so, we have repeated the benchmarking procedure for different values of the
 323 nonlinearity gn_0 and we have found that choosing the thresholds as $\epsilon_{\pm} \rightarrow \epsilon_{\pm} \times \sqrt{2e4/(gn_0)}$
 324 provide similar performances, keeping in mind that our benchmarking protocol using the for-

325 mula of Eq. (8) requires that $L \gg \xi$ (or $gn_0 \gg 1$). Similarly, we have also checked that our
 326 method is robust with respect to the grid size, provided that $\delta z = L/N_z < \xi$, by testing dif-
 327 ferent values of $N_z \in \{256, 512, 1024\}$. Even for $N_z = 256$ (meaning that $\delta z \simeq \xi$) we find
 328 that the soliton indicator correctly identifies the discrete eigenvalues in a dilute soliton gas
 329 initial state, see Eq. (8). However, we note that with this lower grid size the time propagation
 330 induces numerical errors and that the Lax spectrum evolve slightly with time, indicating that
 331 the discretization is too sparse. This is in fact a very convenient way to check whether the time
 332 evolution of Eq. (1) is computed with a sufficiently high accuracy. Indeed, if the continuous
 333 model is correctly mapped onto a discrete grid, we find that it remains approximately inte-
 334 grable, in the sense that the Lax spectrum is indeed time-independent. However, it may not be
 335 the case, for example if the number of modes is too small ($\delta z \geq \xi$) or if the time propagation
 336 introduces errors.

337 While varying the grid size N_z or the nonlinearity gn_0 we noticed that for some trial
 338 functions the soliton indicator could take values other than 0 or 1 , depending on the threshold
 339 value ϵ . This always occurs near the edges of the continuous branches, where, as discussed
 340 above it is difficult to distinguish very shallow solitons. For the purpose of our analysis we
 341 have considered that these few eigenvalues with "anomalous" indicator (meaning $S(\zeta) \neq 0$ or
 342 1) belonged to the same class as their neighboring eigenvalues. We interpret this phenomenon
 343 as a consequence of the fact that gray solitons are created without threshold in the defocusing
 344 1DNLSE.

345 We emphasize that our method gives access to an estimation of the effective speed of sound
 346 in the system (see the end of section 2), which can be much larger than the bare estimate
 347 $c = \sqrt{gn_0}$. To the best of our knowledge there is no other method to access directly this
 348 quantity for a arbitrary initial state. For example, in Fig. 1 it can be seen that some propagating
 349 features travel much faster than the bare speed of sound, and their speed is compatible with
 350 the measured speed of sound $c_{\text{eff}} = 268 \pm 4$ (in dimensionless units). Moreover we obtain the
 351 whole distribution of soliton velocities and it would be very interesting to apply our method
 352 to a set of thermal states, generated by a stochastic classical field model [24]. In principle it
 353 should be possible to obtain the equation of state $c_{\text{eff}}(\mu, T)$, relating the speed of sound to the
 354 chemical potential μ and the temperature T , and compare it to analytical predictions [48].

355 Similarly, we may use our soliton indicator tool to test the accuracy of the recently proposed
 356 generalized hydrodynamics equation for gray soliton gases [43]. Indeed, once the distribution
 357 of Lax eigenvalues corresponding to solitons is known, we may select a particular one as a test
 358 "tracer" soliton and follow its trajectory over time. To do so we may look at the position of
 359 the maximum of the associated eigenvector, see the lower inset of Fig. 2a). By comparing the
 360 initial and final position associated to the same eigenvalue (recall that the Lax spectrum is
 361 time invariant) we obtain a direct measure of the effective velocity of the soliton, that can be
 362 compared to the hydrodynamic effective velocity [43].

363 A natural extension of our work would be to find a way to extend our soliton detection
 364 method to non-integrable cases, that are highly relevant in the context of atomtronics: for
 365 example introduce a localized barrier that will change the soliton velocities [49, 50] or enable
 366 the nucleation of new solitons [19, 20]. In the limit of large barriers, that is considering hard
 367 wall boundary conditions the Lax spectrum can be readily computed using a mirror image
 368 technique [51]. It would be also very relevant to consider the case of a harmonic poten-
 369 tial: although it breaks formally the integrability of Eq. (1) it sustains long-lived soliton like
 370 solutions [9]. For all these examples we believe that our method can be adapted to study
 371 quantitatively how the external potential term breaks integrability, in the spirit of the study
 372 reported in [11] for the focusing (attractive) case. More generally it would be interesting to
 373 extend our results to other integrable equations similar to Eq. (1) [52, 53].

374 Finally, we emphasize that the method we introduce in this work is rather simple as it

375 requires only Fourier transforms and the diagonalization of a simple matrix, see Appendix B
 376 for details. Therefore it can be readily applied to any simulation relying on Eq. (1). We provide
 377 in Appendix C two examples of specific states with peculiar behavior that can be identified
 378 only with the use of the soliton indicator. We also think that it can even be used to analyze
 379 experimental data, provided that one can measure the amplitude and the phase of the field
 380 $\psi(\mathbf{z}, t)$ at a given time, with a spatial resolution of at least ξ , and calibrate the non-linearity
 381 parameter g . This is within reach in experiments dealing with the propagation of light pulses
 382 in nonlinear fibers [54] or atomic vapors [6].

383 5 Conclusion

384 We have reported a detailed study of the use of the inverse scattering transform tools to identify
 385 the number of gray solitons in the defocusing one-dimensional nonlinear Schrödinger equa-
 386 tion. We define a self consistent soliton indicator that allows a study of the soliton distribution
 387 and demonstrate through a extensive benchmark its reliability. More precisely we are able to
 388 count the solitons and find their velocities within a given margin of error, where the errors
 389 always concern fast, shallow, gray solitons. Moreover it provides a accurate measurement of
 390 the effective speed of sound in a arbitrary excited state. We think that our method is very
 391 relevant to the analysis of gray soliton gases, in the context of a generalized hydrodynamics
 392 approach.

393 Acknowledgements

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398 A Methods

399 We provide here details on the numerical methods we use to study the 1DNLSE. To numerically
 400 solve Eq. (1), we use a spectral method relying on fast Fourier transforms to evaluate exactly
 401 the kinetic energy term [55], with a regular grid of $N_z = 512$ points and a dimensionless non-
 402 linear parameter $gN = 2 \times 10^4$, well within the mean field regime [50]. The grid introduces a
 403 natural cut-off for the wavevectors at $k_{\max} = \pi N_z/L$ and to avoid aliasing in the computation
 404 of nonlinear terms we use a projector onto the low k region: $|k| < k_{\text{cut}} = 2k_{\max}/3$ [56].

405 The groundstate of Eq. (1) corresponds to a flat density profile: $\psi_0 = \sqrt{n_0}$, where $n_0 = N/L$,
 406 fixing the value of the bare speed of sound $c = \sqrt{gn_0}$ and healing length $\xi = 1/\sqrt{2gn_0}$.

407 To drive the system to out-of-equilibrium states we use a simple excitation protocol: we
 408 introduce a Gaussian potential that we stir back and forth along the z axis [57–61]. To do so,
 409 we add to Eq. (1) the excitation potential in the form of a moving Gaussian obstacle:

$$V_{\text{stirr}}(t, z) = V_b(t) \exp[-(z - z_c(t))^2/\sigma^2],$$

410 where $V_b(t)$ is the time-dependent barrier height, $\sigma = 4\xi$ is the width of the barrier and
 411 $z_c(t) = \delta z \cos(\omega_{\text{exc}} t)$ is the position of the barrier. The amplitude of the motion is set to
 412 $\delta z = L/4$, the barrier is turned on in a time $t_{\text{on}} = L/c$, kept at its maximum value $V_0 = gn_0$

413 for $T_{\text{exc}} = 8 \times L/c$ and turned off in a time $t_{\text{off}} = L/c$. V_0 is chosen such that the density
 414 is nearly completely depleted at the peak of the barrier which facilitates the creation of soli-
 415 tons, while t_{on} and t_{off} are slow enough to prevent the creation of excitations if the barrier
 416 is not moving (i.e. for $\omega_{\text{exc}} = 0$ or $\delta z = 0$). We then vary ω_{exc} to control the amount of
 417 excitation created in the final state. For example the state of Fig. 1 was generated with a
 418 fast oscillation $\omega_{\text{exc}} = 4.5 \times c/L$, while the state of Fig. 3 corresponds to a slower frequency
 419 $\omega_{\text{exc}} = 0.384 \times c/L$. At the end of the excitation phase, when the barrier amplitude is turned
 420 off, we record the wavefunction and compute its Lax spectrum. Our analysis protocol allows
 421 then to extract for each simulation the number of solitons.

422 B Computation of the Lax spectrum

423 The Lax eigenvalue equation we want to solve is $\mathcal{L}\mathbf{v} = \zeta\mathbf{v}$, with $\mathbf{v} = (v_1, v_2)^T$ a two-component
 424 vector. In order to compute this equation in momentum space, we take the Fourier transform,
 425 $\hat{\mathcal{L}} * \hat{\mathbf{v}} = \zeta \hat{\mathbf{v}}$, where $*$ is the convolution and $\hat{\mathbf{v}}$ is the Fourier transform of \mathbf{v} . The equation then
 426 reads:

$$\begin{pmatrix} -\frac{k}{2}\hat{v}_1 - i\frac{\sqrt{g}}{2}\hat{\psi} * \hat{v}_2 \\ i\frac{\sqrt{g}}{2}\hat{\psi} * \hat{v}_1 + \frac{k}{2}\hat{v}_2 \end{pmatrix} = \zeta \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix}.$$

427 Now to compute the convolution, corresponding to off-diagonal terms in \mathcal{L} , we have to write it
 428 using the discrete Fourier transform: $\hat{\psi}(\mathbf{k}) \rightarrow \hat{\psi}_q = \sum_{n=0}^{N_z-1} \psi_n e^{-ik_q x_n}$ where x_n or k_q belong
 429 to the discrete grid in position or momentum space. Then the discrete convolution reads:

$$[\hat{\psi} * \hat{\mathbf{v}}]_q = \sum_{p=0}^{N_z-1} \hat{\psi}_{q-p} \hat{v}_p,$$

430 where the index $q - p$ in the sum is taken modulo N_z , as $\hat{\psi}_q = \hat{\psi}_{q+N_z}$. This operation can be
 431 written in a matrix form:

$$C \hat{\mathbf{v}} = \begin{pmatrix} \hat{\psi}_0 & \hat{\psi}_1 & \cdots & \hat{\psi}_{N_z-2} & \hat{\psi}_{N_z-1} \\ \hat{\psi}_{N_z-1} & \hat{\psi}_0 & \cdots & \hat{\psi}_{N_z-3} & \hat{\psi}_{N_z-2} \\ & & \cdots & & \\ \hat{\psi}_2 & \hat{\psi}_3 & \cdots & \hat{\psi}_0 & \hat{\psi}_1 \\ \hat{\psi}_1 & \hat{\psi}_2 & \cdots & \hat{\psi}_{N_z-1} & \hat{\psi}_0 \end{pmatrix} \hat{\mathbf{v}}$$

432 from which it is clear that the convolution matrix has a Toeplitz structure.

433 Finally we diagonalize the $2N_z \times 2N_z$ matrix:

$$\hat{L} = \frac{i}{2} \begin{pmatrix} ik & -\sqrt{g}C \\ \sqrt{g}C^\dagger & -ik \end{pmatrix},$$

434 where \mathbf{k} stands for the diagonal matrix of discrete wave-vectors. All operations are imple-
 435 mented in Octave/Matlab language using built-in functions.

436 As \hat{L} is a generic hermitian matrix, the typical algorithmic complexity cost of its diagonal-
 437 ization is $\mathcal{O}(N_z^3)$ [62]. For the grid sizes we used in this work we have not found any significant
 438 difference in computation time between computing only the eigenvalues or the eigenvalues
 439 and the eigenvectors. However the latter requires more available memory to store the matrix of
 440 eigenvectors. Our soliton detection algorithm requires to diagonalize first a $2N_z \times 2N_z$ matrix
 441 and then a $4N_z \times 4N_z$ matrix, and to identify the soliton positions we need the eigenvectors of
 442 the first matrix. In our Matlab implementation we indeed observe that the diagonalization of

443 the second matrix takes roughly 8 times longer than the first one. For a grid size of $N_z = 512$
 444 it amounts to about 1 second and 8 seconds, respectively, on a standard laptop computer.
 445 Our algorithm also require pairs of fast direct and inverse Fourier transforms with complexity
 446 scaling as $\mathcal{O}(N_z \log N_z)$ that have a negligible impact on the total computation time.

447 C Examples of peculiar non-stationary states

448 In this last appendix we report two examples of initial states leading to a non-trivial dynamics
 449 that can be understood correctly only with the analysis of the Lax spectrum thanks to the
 450 soliton indicator.

451 We first consider the state of Eq. (9), with $\alpha = 2$ and a non-linearity $gn_0 = 2 \times 10^4$. Since
 452 the initial density dip is very narrow, we use a grid size of $N_z = 1024$ to correctly compute the
 453 time-evolution. In that case, the analytical result predicts that there is a single dark soliton at
 454 $\zeta_0 = -\frac{k_0}{4}$ in the gap. Our soliton indicator method detects $N_{\text{sol}} = 10 \pm 7$ solitons and a effective
 455 speed of sound $c_{\text{eff}} = 142.9 \pm 1.3$ very close to the bare speed of sound c . Figure 10a) and
 456 b) show the Lax spectrum structure, evidencing clearly the isolated eigenvalue corresponding
 457 to the dark soliton, and a bunch of discrete eigenvalues very close to the gap edges, that are
 458 within the uncertainty of our detection method.

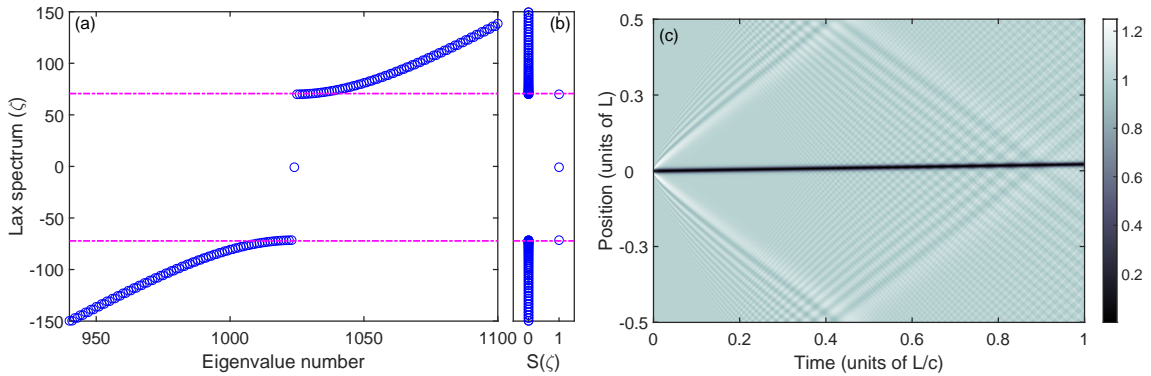


Figure 10: (color online) a) Lax spectrum as a function of the eigenvalue number for $\alpha = 2$ (open blue circles) using the initial state of Eq. (9). The two horizontal dashed-dotted magenta lines indicate the gap in the continuous spectrum, detected using the soliton indicator $S(\zeta)$. b) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for the threshold ϵ_- . c) Density colormap showing the propagation of the initial state.

459 Figure 10c) displays the density colormap of the time-evolution for this initial state. One
 460 can clearly see a dark soliton slowly moving due to the background phase gradient $\mathbf{k}_0 \simeq \pi/L$,
 461 and several bright features propagating at a speed significantly faster than the bare speed of
 462 sound. We interpret this as a Bogoliubov excitation propagating at a group velocity $v_g = \frac{d\omega}{dk}$
 463 larger than the speed of sound, which do not appear as a soliton in the Lax spectrum. We note
 464 that an analysis based only on the apparent speed of propagating features in Fig. 10c) would
 465 lead to an incorrect estimation of the speed of sound. We have checked that a initial state
 466 with a small Bogoliubov excitation wave-packet on top of a homogeneous background results
 467 indeed in a similar dynamics, except for the slow moving soliton that is absent.

468 Finally we present a second puzzling case, showing that it is possible to construct a non sta-
 469 tionary state with a non trivial Lax spectrum exhibiting several continuous branches, separated
 470 by multiple gaps. To do so we consider the initial state:

$$\psi_k(\mathbf{z}) = \sqrt{n_0} (1 + \eta \cos k\mathbf{z}),$$

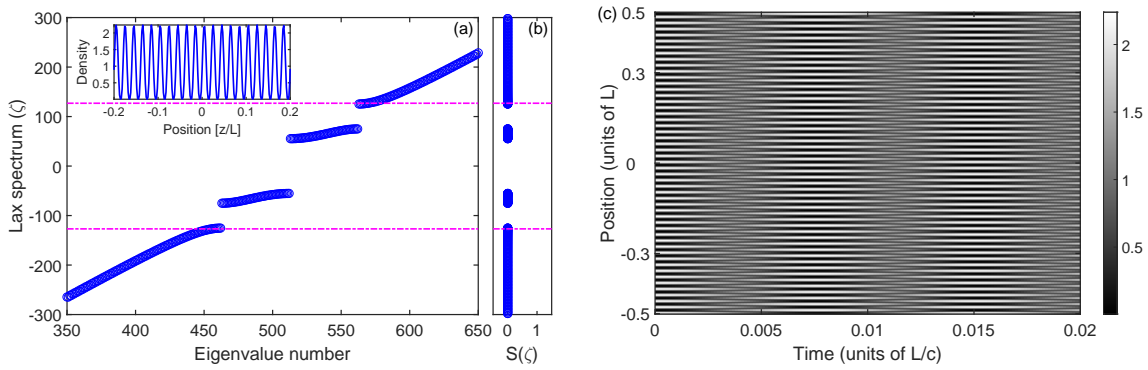


Figure 11: (color online) a) Lax spectrum as a function of eigenvalue number for a highly periodic initial state (open blue circles). Inset: initial density profile as a function of the position, zoomed in around $z = 0$. b) Soliton indicator $S(\zeta)$ for each eigenvalue, computed for the threshold ϵ_- . In a) and b) the two horizontal dashed-dotted magenta lines indicate the edges of the main gap in the continuous spectrum. c) Density colormap of the periodic state for this particular realization (note the small time scale).

471 with $n_0 = 1$, $\eta = 0.5$, $k = 50 \times 2\pi/L$ and nonlinearity $gn_0 = 2 \times 10^4$.

472 Figure 11 shows the Lax spectrum, the soliton indicator and the typical time evolution of
 473 the density profile, for this initial state. The Lax spectrum displays a non trivial structure,
 474 with many eigenvalues inside the gap. However, the analysis of our soliton indicator indicates
 475 that these eigenvalues do not correspond to solitons but rather behave as plane waves, in
 476 term of degeneracy. We interpret this as a spectrum with four continuous branches, separated
 477 by three gaps. Here since the soliton indicator is always 0 we define the gap based on the
 478 analysis of the distance between consecutive eigenvalues. This structure seems to arise from
 479 the highly periodic pattern of the initial state, that give rise also to a time periodic evolution
 480 at a high frequency. We also note that the density evolution do not display the propagation
 481 of many gray solitons. We have carefully checked that this is not the result of a numerical
 482 artifact. Although we do not think that this state is of particular significance for the study of
 483 the physical properties of the 1DNLSE, its existence supports the need for a careful analysis of
 484 Lax spectrum to detect eigenvalues corresponding to solitons. We leave for a future work the
 485 question of the possibility of engineering a state with both continuous branches and soliton
 486 eigenvalues inside the gaps.

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