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# Flow invariance for non densely defined Cauchy problems

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## Abstract

In this work, we find necessary and sufficient conditions in order that a family of sets  $\{C(t), t \in J\}$  be invariant for a Cauchy problem. We prove the existence and uniqueness of the solution using viability for time-dependent closed convex sets to the case of non densely defined Cauchy problem. Moreover, we propose several characterizations of conditions that lead to the viability theorem. Finally, a comparison principle for semilinear problems, when the nonlinear part is only defined in a closed convex set, is established.

**Keywords:** Viability, Hille-Yosida operator, Comparison theorem

## 1 Introduction

Given a nontrivial time interval  $J$  starting at zero and a Banach space  $X$ , we consider the following Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + F(t, u(t)), & t \in J \\ u(0) = x \in X \end{cases} \quad (1)$$

where  $A : D(A) \subset X \rightarrow X$  is an unbounded linear operator and  $F$  is a nonlinear operator on a subset of  $J \times X$ . In finite dimensions, differential systems of the type above are of great importance and the motivation for developing their theory comes from real-world problems. For instance, we refer for example to [1, 11, 20, 22].

In many situations of interest, the nonlinear term is defined only on a subset of the Banach space  $X$ . Typical examples of such problems are called flow invariance problems [15] or viability [5]. We recall that a subset  $C_0 \subset X$  is said viable with respect to (1) if, for each  $x \in C_0$ , at least one solution with initial condition  $x$  has values in  $C_0$ . It is said flow-invariant if for each  $x \in C_0$  all solutions with initial condition  $x$  have values in  $C_0$  ([2, 5]). Of course, both notions are equivalent when we have the uniqueness of the solution to (1). Such problems are related to the existence of solutions of differential equations or differential inclusions whose dynamics are restricted to closed subsets of the state space. We refer to [2] for a complete overview of the applications in biology, economy, and finance.

The flow invariance problem we are interested in has been studied by many authors using various frameworks and techniques. The first flow invariance result is Nagumo's viability theorem

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for ordinary differential equations [16]. This pioneering work has been rediscovered several times in the seventies among others by Brezis [4], Crandall [6], Hartman [9] and Martin [14]. Since then, many extensions were considered in the literature. Among comparison theorem, differential inequalities, and many other applications, invariant sets for abstract functional differential equations and reaction-diffusion systems have been studied in [10, 18]. In the monograph of Motreanu and Pavel [15], the authors studied the flow invariance of densely defined semilinear Cauchy problems. They detailed the positive invariance for general closed subset subjected to the tangency condition and also considered the positive invariance of the time-dependent closed subset. Such results are proved for semilinear differential inclusion problems and densely defined Cauchy problems.

For the time-independent closed convex subset, Thieme [21] extended the flow invariance for non-densely defined Cauchy problems with a Hille-Yosida linear operator. We mention that the later operator is perturbed by a Lipschitz continuous non-linear map. Magal *et al.* [13] generalize the work in [21] to prove a viability theorem for a Cauchy problem with respect to time-independent closed set and for the non-Hille-Yosida case. Motivated by these above works, here we investigate the non-densely defined Cauchy problems of type (1), with the Hille-Yosida operator defined on a time-dependent domain. In addition, we characterized the crucial tangency conditions and showed the applicability of our result, based on viability for the time-dependent closed convex sets to the case of non densely defined Cauchy problem. Such results include characterizations of conditions that lead to the viability theorem, the existence and uniqueness of the solution, and the comparison principle for semilinear problems when the nonlinear part is only defined in the positive cone of the Banach space  $X$ .

The rest of the work is organized as follows. Section 2 is a presentation of the main result, some basic notations, and materials on the flow invariance of the Cauchy problem. In Section 3, we find some flexible conditions to characterize the subtangential condition. We apply, in Section 4, the result on viability of the Cauchy problem to obtain a comparison principle in addition to the positivity of the solution.

## 2 Semilinear Cauchy problem on time-dependent closed sets

The main objective of this section is to recall the notion of a mild solution to the non-densely defined Cauchy problems.

Let  $A : D(A) \subset X \rightarrow X$  be a Hille-Yosida linear operator of type  $(M_A, \omega_A)$ , *i.e.*, the resolvent set  $\rho(A)$  of  $A$  contains a ray  $(\omega_A, +\infty)$  and  $R_\lambda(A) := (\lambda I - A)^{-1}$  satisfies

$$\|R_\lambda(A)^n\|_{\mathcal{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^n}, \quad \forall \lambda > \omega_A, \quad n = 1, 2, \dots$$

Let  $J \subset \mathbb{R}$  be a non-trivial interval with  $\min(J) = 0$ . Let  $C \subset X$  be a nonempty closed convex subset that meets  $\overline{D(A)}$ , and define

$$C_0 := C \cap \overline{D(A)}$$

which is also a closed convex subset of  $\overline{D(A)}$ . For all  $t \in J$ , we denote by  $C(t)$  a non empty closed such that  $C(t) \subseteq C_0$ , and define

$$\mathcal{C} = \{(t, x) \in J \times C_0 : x \in C(t)\}.$$

Let  $F : \mathcal{C} \subseteq J \times C_0 \rightarrow X$  and consider the Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + F(t, u(t)), & t \geq t_0 \text{ in } J \\ u(t_0) = x \in C(t_0) \end{cases} \quad (2)$$

where  $0 \leq t_0 < \sup(J)$ .

It is worth mentioning that subsets  $C(t)$ ,  $t \in J$ , are not necessarily convex but the construction of solutions to (2) required the convexity of  $C_0$ . Indeed, we describe a way of approximating solutions that blow up to a finite time. With a convex interpolation, the existence of such mild solutions holds. We can, however, easily release the convexity condition on  $C_0$  by assuming that each  $C(t)$  is arcwise connected in a certain sense given below (see [10, 18] for more details).

To proceed further, we need the following notion of integral solution for the Cauchy problem on time-dependent closed sets.

**Definition 2.1** *Let  $\tau > 0$  such that  $[t_0, t_0 + \tau] \subset J$ . A function  $u : [t_0, t_0 + \tau] \rightarrow C_0$  is called an integral solution to (2) with initial condition  $x \in C(t_0)$  at time  $t = t_0$  if  $u \in C([t_0, t_0 + \tau], C_0)$ ,  $u(t) \in C(t)$  for all  $t \in [t_0, t_0 + \tau]$  and*

$$\begin{cases} \int_{t_0}^t u(s) ds \in D(A), \quad \forall t \in [t_0, t_0 + \tau] \\ u(t) = x + A \int_{t_0}^t u(s) ds + \int_{t_0}^t F(s, u(s)) ds, \quad \forall t \in [t_0, t_0 + \tau]. \end{cases}$$

We also define the notion of mild solution to (2) which is equivalent to the notion of integral solution [11, 21]. Before proceeding, let us recall that the part  $A_0$  of  $A$  in  $\overline{D(A)}$  is the linear operator  $A_0 : D(A_0) \subset X \rightarrow X$  with

$$A_0 x = Ax, \quad \forall x \in D(A_0) := \{x \in D(A) : Ax \in \overline{D(A)}\}.$$

It is well known that [7, 11, 21] if  $A$  is a Hille-Yosida linear operator of type  $(M_A, \omega_A)$  the  $A_0$  generates a  $C_0$ -semigroup  $\{T_{A_0}(t)\}_{t \geq 0}$  on  $\overline{D(A)}$  with

$$\|T_{A_0}(t)x\| \leq M_A e^{\omega_A t} \|x\|, \quad \forall t \geq 0, \quad \forall x \in \overline{D(A)}.$$

**Definition 2.2** Let  $\tau > 0$  such that  $[t_0, t_0 + \tau] \subset J$ . A function  $u : [t_0, t_0 + \tau] \rightarrow C_0$  is called a mild solution to (2) with initial condition  $x \in C(t_0)$  at time  $t = t_0$  if  $u \in C([t_0, t_0 + \tau], C_0)$ ,  $u(t) \in C(t)$  for all  $t \in [t_0, t_0 + \tau]$  and

$$u(t) = T_{A_0}(t - t_0)x + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t - s) \lambda R_\lambda(A) F(s, u(s)) ds, \quad \forall t \in [t_0, t_0 + \tau].$$

To ensure the existence and uniqueness of the solution, we will require  $F$  to be regular enough. In fact, it is well known that with  $A = 0$  the existence of a solution for every continuous  $F$  means that the Banach space  $X$  is of finite dimension (see Godunov[8]). For these reasons, the nonlinear operator  $F$  is assumed to satisfy the following regularity condition :

(F) The non-linearity  $F$  is continuous from  $J \times C_0$  to  $X$ . We assume in addition that for each  $\tau_0 \in J$  and  $r_0 > 0$ , there exists  $\kappa := \kappa(r_0, \tau_0) > 0$  such that for all  $t \in [0, \tau_0]$

$$\|F(t, x) - F(t, y)\| \leq \kappa \|x - y\|, \quad \forall x, y \in C_0, \quad \|x\| \leq r_0, \quad \text{and} \quad \|y\| \leq r_0.$$

A central role in the present work is played by a subtangential condition and some of its characterizations. In fact, when  $C_0$  is closed and  $\text{dom}(F(t, \cdot)) = C_0$ , the Lipschitz condition on the nonlinearity  $F$  is not enough to guarantee the existence of solution with value in  $C_0$  in both finite and infinite dimensions. Indeed, when  $C_0$  is closed, the "vector field"  $F$  must be tangent to  $C_0$  in a certain sense. More precisely, it is proved in [20] that when  $C(t) = C_0$ , for all  $t \in J$ , and (F) is satisfied then the following subtangential condition is a necessary and sufficient condition for the existence of a solution to (2) with value in  $C_0$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)F(t, x); C_0) = 0 \quad (3)$$

where  $S_A(t) : X \rightarrow \overline{D(A)}$  is the bounded linear operator given by

$$S_A(t)y := \lim_{\lambda \rightarrow +\infty} \int_0^t T_{A_0}(t - s) \lambda R_\lambda(A) y ds, \quad \forall y \in X, \quad \forall t \geq 0. \quad (4)$$

Note that when  $A$  is densely defined, then (4) takes the following form

$$S_A(t)y := \int_0^t T_{A_0}(t - s) y ds, \quad \forall y \in X, \quad \forall t \geq 0.$$

Therefore, in the case of the densely defined operator  $A$ , equality (3) leads to the classical subtangential condition given in [15]. Henceforth, for more convenience, we will use the notation

$$X_0 = \overline{D(A)}.$$

The subtangential condition (3) is quite tricky. However, in Section 3, we find a way of overcoming obstacles in some configurations. For instance, we will prove that if  $C$  is a distance set or convex and the two below items are satisfied

- $X = H$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,
- $\lambda(\lambda I - A)^{-1}C \subset C$  for all large  $\lambda > \omega_A$ ,

then the subtangential condition (3) implies the following assertion : if  $w \in X$  is a normal vector to  $C$  at  $x$  (in a sense to be precised later) then  $\langle w, F(\ell, x) \rangle \leq 0$ .

The above assertion is somehow helpful to better motivate the term **subtangential**. Thanks to the following result, see [20, Lemma 2.9], which provides a sufficient condition for obtaining the subtangential condition (3).

**Lemma 2.3** *Let  $C$  be a nonempty convex subset. Under condition (F), if the following are verified*

i)  $\lambda(\lambda I - A)^{-1}(C) \subset C$  for all large  $\lambda > \omega_A$ ;

ii) for each  $(\ell, x) \in J \times C_0$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tF(\ell, x); C) = 0, \quad (5)$$

then the subtangential condition (3) is satisfied.

**Remark 2.4** *When  $C$  is open in  $X$  then (5) is always satisfied. This can be obtained by simple arguments. Therefore, only the tangency on the boundary of  $C$  is important. Equivalent characterizations to (5) will be given in Section 3.*

To obtain the existence of integral solution to (2), we will need a generalization of the subtangential condition (3) for time-dependent sets. This reads as follows

(S1) For each  $\ell \in J$  and  $x \in C(\ell)$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)F(\ell, x); C(\ell + t)) = 0$$

where  $S_A(t)$  is defined in (4).

(S2) If the sequence  $(t_k, x_k)_{k \geq 0} \subset J \times C_0$  with  $x_k \in C(t_k)$  converges to  $(t^*, x^*) \in J \times C_0$  then  $x^* \in C(t^*)$ .

Note that condition (S2) is a somehow closedness of the family of non empty sets  $C(t)$ ,  $t \in J$ . Of course when the family of sets is closed and does not depends on  $t \in J$  then (S2) is automatically satisfied. The following theorem is essentially inspired by [15] where the proof has been done in the case of a densely defined Cauchy problem. Let us also mention [10, 18] where such a problem has been applied respectively to the system of reaction-diffusion equations with infinite delay. Here is the main result of this paper.

**Theorem 2.5** *Let  $\{C(t) \subseteq C_0 : t \in J\}$  be a family of closed sets such that  $C(t)$  is non empty for each  $t \in J$  and  $C_0$  is closed and convex. Assume that conditions (F), (S1) and (S2) are satisfied. Then for each  $x \in C(t_0)$  and  $0 \leq t_0 < \sup(J)$ , there exists a unique maximally defined integral (mild) solution to (2) with initial condition  $x$  at time  $t = t_0$ .*

To throw some light on the definition of the family of subsets  $C(t)$ ,  $t \in J$  and some existing literature, we mention that when  $C(t) = C_0$ , for all  $t \in J$ , Theorem 2.5 has been proved in [20] while [13] obtained a more general version of this theorem by assuming that  $A$  is not Hille-Yosida

and  $C_0$  is only closed. However, note that the result in [13] is based on the fact that the non-linear map  $F$  is defined in the whole  $J \times X$ . But here, sets  $C(t)$ ,  $t \in J$ , are only assumed to be closed subsets of a convex set. Let us show the necessity of the condition **(S1)** first. It is a consequence of the following result.

**Lemma 2.6** *Let  $\{C(t) \subseteq C_0 : t \in J\}$  be a family of closed sets such that  $C(t)$  is non empty for each  $t \in J$ . Assume that condition **(F)** is satisfied. If for each  $x \in C(t_0)$  and  $0 \leq t_0 < \sup(J)$ , there exists a unique maximally defined integral (mild) solution to (2) with initial condition  $x$  at time  $t = t_0$  then **(S1)** is verified.*

*Proof.* Let  $x \in C(\ell)$  with  $\ell \in J$ . Then there exists  $\tau > 0$  such that  $u \in C([\ell, \ell + \tau], C_0)$  is a solution to (2). This means that  $u(\ell + t) \in C(\ell + t)$  for all  $t \in [0, \tau]$  and

$$u(\ell + t) = T_{A_0}(t)x + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell + t - s) \lambda (\lambda I - A)^{-1} F(s, u(s)) ds, \quad \forall t \in [0, \tau].$$

Hence, for all  $t \in (0, \tau]$

$$\text{dist}(T_{A_0}(t)x + S_A(t)F(\ell, x); C(\ell + t)) \leq \|u(\ell + t) - T_{A_0}(t)x + S_A(t)F(\ell, x)\|,$$

that is

$$\frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)F(\ell, x); C(\ell + t)) \leq \frac{1}{t} \|v(t)\|$$

with

$$v(t) := \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell + t - s) \lambda (\lambda I - A)^{-1} [F(s, u(s)) - F(\ell, x)] ds.$$

Since

$$\|v(t)\| \leq M_A^2 \int_{\ell}^{\ell+t} e^{\omega_A(\ell+t-s)} \|F(s, u(s)) - F(\ell, x)\|, \quad \forall t \in (0, \tau]$$

it follows that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)F(\ell, x); C(\ell + t)) = 0.$$

■

The rest of the proof of our main Theorem will be given in Section 5.

### 3 Characterization of the subtangential condition

In this section, we give several characterizations of (5) that will be useful in the sequel. These equivalent characterizations will be used in Section 4 to show how to derive general conditions for the monotony of the semiflow generated by (2) and comparison principles. To begin with, we first introduce some definitions.

**Definition 3.1** *A subset  $C \subseteq X$  is called a distance set if for each  $x \in X \setminus C$  there exists  $y \in C$  such that*

$$\text{dist}(x; C) = \|x - y\|.$$

**Remark 3.2** *It is worth noting that a distance set is closed.*

**Definition 3.3** Let  $C \subseteq X$  be a closed subset. A vector  $v \in X$  is said to be tangent to  $C$  at  $x \in C$  if

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tv; C) = 0.$$

The above notion of tangency is usually called tangency in the sense of Federer. The tangency in the sense of Bouligand is defined similarly by replacing the "lim" by "lim inf". We can notice that if  $v \in X$  is tangent to  $C$  at  $x \in C$  in the sense of Federer, then it is tangent to  $C$  at  $x \in C$  in the sense of Bouligand. We now prove an equivalent definition to the tangent vector that turns out to be useful in the development of this section. This reads as follows.

**Lemma 3.4** Let  $C \subseteq X$  be a closed subset. Then the following properties are equivalent:

i) The vector  $v \in X$  satisfies

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tv; C) = 0.$$

ii) There exists a map  $\theta : [0, +\infty) \rightarrow X$  such that

$$x + tv + t\theta(t) \in C, \forall t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \theta(t) = 0.$$

iii) For each sequence  $(t_n)_{n \geq 0} \subset [0, +\infty)$  such that  $\lim_{n \rightarrow +\infty} t_n = 0$ , there exists a sequence  $(\theta_n)_{n \geq 0} \subset X$  such that

$$x + t_n v + t_n \theta_n \in C, \forall n \geq 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \theta_n = 0.$$

*Proof.* It is obvious that ii) implies iii). We prove in the following that i) is equivalent to ii) and iii) implies i). Assume i). Then for each  $t > 0$  there exists  $u(t) \in C$  such that

$$\text{dist}(x + tv; C) \leq \|u(t) - x - tv\| < \text{dist}(x + tv; C) + t^2.$$

Therefore,

$$\lim_{t \rightarrow 0^+} \frac{\|u(t) - x - tv\|}{t} = 0.$$

Thus, ii) follows by setting

$$\theta(t) := \frac{u(t) - x - tv}{t}, \forall t > 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \theta(t) = 0.$$

Assume ii). Then we have for each  $t \geq 0$

$$\begin{aligned} \text{dist}(x + tv; C) &\leq \|[x + tv] - [x + tv + t\theta(t)]\| \\ &\leq t\|\theta(t)\| \end{aligned}$$

and i) follows. The implication iii)  $\implies$  i) can be done similarly by using the sequential characterization of the limit. ■

We now make a link between the tangent vector to  $C$  at  $x \in C$  and the normal vector to  $C$  at  $x \in C$ . Before proceeding, we introduce the following definition due to Bony [3].

**Definition 3.5 (Bony)** A vector  $w \in X$  is an outer normal vector to  $C$  at  $x \in C$  if the following properties are satisfied:



- i) There exists  $x_0 \in X \setminus C$  and  $r > 0$  such that  $B(x_0, r) \subset X \setminus C$ ;
- ii)  $\overline{B(x_0, r)} \cap C = \{x\}$  ;
- iii)  $w = \frac{1}{h}(x_0 - x)$  for some  $h > 0$ .

The following lemma gives another characterization of an outer normal vector to a set.

**Lemma 3.6** *Let  $w \in X$  and  $x \in C \subset X$  be given. The following properties are equivalent:*

- i) There exists  $h > 0$  such that  $\overline{B(x + hw, h\|w\|)} \cap C = \{x\}$ ;
- ii)  $w \in X$  is an outer normal vector to  $C$  at  $x \in C$ .

*Proof.* Assume that i) is satisfied with  $h > 0$ . Since  $x \notin B(x + hw, h\|w\|)$  it follows that  $B(x + hw, h\|w\|) \subset X \setminus C$ . Setting  $x_0 = x + hw$  (i.e.  $w = \frac{1}{h}(x_0 - x)$ ) and  $r = h\|w\|$ , we conclude that ii) holds true.

Assume that ii) is satisfied. Note that  $x_0 = x + hw$  and  $B(x_0, r) \subset X \setminus C$ . Since  $x \in C$ , we have  $\|x - x_0\| = h\|w\| \geq r$ . Moreover, using the fact that  $x \in \overline{B(x_0, r)}$  there exists  $(z_n) \subset B(x_0, r)$  such that  $z_n \rightarrow x$  when  $n \rightarrow +\infty$ . As a consequence  $\|z_n - x_0\| = \|z_n - hw - x\| < r$  and by letting  $n \rightarrow +\infty$  we obtain  $h\|w\| \leq r$ . This shows that  $h\|w\| = r$  and the proofs are completed.

■

In preparation for the lemma that gives the relationship between a tangent vector to a set and an outer normal vector, we recall the following result.

**Lemma 3.7** ([19]) *Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous such that  $f(a) = 0$ . Assume that there exists  $c \geq 0$  such that*

$$\liminf_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h} \leq cf(t), \quad \forall t \in [a, b)$$

*with the possible exception of some countable set  $D \subset (a, b)$ . Then  $f(t) \leq 0$  for all  $t \in [a, b]$ .*

Using ideas from Crandall [6] together with Lemma 3.7 we can prove the following lemma.

**Lemma 3.8 (Crandall)** *Let  $X$  be a real Hilbert space with a norm induced by the inner product  $\langle \cdot, \cdot \rangle$ . Let  $C$  be a distance set or convex and  $x \in C$  be fixed. Then the following properties are equivalent:*

- i)  $v \in X$  is a tangent vector to  $C$  at  $x$  i.e.

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tv; C) = 0;$$

- ii) For each outer normal vector  $w \in X$  to  $C$  at  $x \in C$  we have  $\langle w, v \rangle \leq 0$ .

*Proof.* Since any closed convex set in a Hilbert space is a distance set, we only need to prove that the lemma holds true when  $C$  is a distance set. Assume that condition i) is satisfied. Then by Lemma 3.4, there exists a map  $\theta : (0, +\infty) \rightarrow X$  such that

$$x + tv + t\theta(t) \in C, \quad \forall t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \theta(t) = 0.$$

Let  $w \in X$  be an outer normal vector to  $C$  at  $x \in C$ . Thus there exists  $h > 0$  such that  $B(x + hw, h\|w\|) \cap C = \emptyset$ . From where we obtain

$$\|x + tv + t\theta(t) - [x + hw]\| \geq h\|w\|, \quad \forall t \geq 0$$

hence

$$t\|v + \theta(t)\|^2 - 2\langle v + \theta(t), hw \rangle \geq 0, \quad \forall t \geq 0.$$

The property ii) follows by letting  $t \rightarrow 0^+$ . Next, we assume that ii) is satisfied. Recall that  $C$  is a distance set. Then there exists a map  $y : [0, +\infty) \rightarrow C$  such that

$$\text{dist}(x + tv; C) = \|x + tv - y(t)\| \tag{6}$$

that is

$$\|x + tv - z\| \geq \|x + tv - y(t)\|, \quad \forall z \in C, \quad \forall t \geq 0. \tag{7}$$

Thus setting

$$w(t) = x + tv - y(t), \quad \forall t \geq 0, \tag{8}$$

it follows from (7) that

$$\|w(t) + y(t) - z\| \geq \|w(t)\|, \quad \forall z \in C, \quad \forall t \geq 0,$$

which means that  $w(t)$  is an outer normal vector to  $C$  at  $y(t)$ . Therefore, we have by assumption

$$\langle w(t), v \rangle \leq 0, \quad \forall t \geq 0. \tag{9}$$

From (6) and (8), we see that to prove item i), it is enough to show that  $\frac{1}{t}\|w(t)\| \rightarrow 0$  when  $t \rightarrow 0^+$ . To do this, we note that for  $t, h \geq 0$ , using the fact that  $y(t) \in C$  we have from (7) that

$$\begin{aligned} \|w(t+h)\|^2 &\leq \|x + (t+h)v - y(t)\|^2 = \|hv + w(t)\|^2 \\ &= h^2\|v\|^2 + \|w(t)\|^2 + 2h\langle w(t), v \rangle \end{aligned}$$

hence

$$\|w(t+h)\|^2 - \|w(t)\|^2 \leq h^2\|v\|^2 + 2h\langle w(t), v \rangle, \quad \forall t, h \geq 0..$$

Using (9) we have  $2h\langle w(t), v \rangle \leq 0$  for  $t, h \geq 0$  so that

$$\|w(t+h)\|^2 - \|w(t)\|^2 \leq h^2\|v\|^2, \quad \forall t, h \geq 0.$$

To complete the proof, we note that the map  $t \mapsto \|w(t)\| = \text{dist}(x+tv, C)$  is continuous providing that  $t \mapsto \|w(t)\|^2$  is continuous. Moreover  $\|w(0)\|^2 = 0$  and

$$\liminf_{h \rightarrow 0^+} \frac{\|w(t+h)\|^2 - \|w(t)\|^2}{h} \leq 0, \quad \forall t \geq 0. \tag{10}$$

Lemma 3.7 implies that  $\|w(t)\|^2 \leq 0$  for all  $t \geq 0$  that is  $\|w(t)\|^2 = 0$  for all  $t \geq 0$ . The proof is completed. ■

**Remark 3.9** *If there exists  $h > 0$  such that  $\overline{B(x + hw, h\|w\|)} \cap C = \{x\}$  then we have*

$$\overline{B(x + \epsilon w, \epsilon\|w\|)} \cap C = \{x\}, \quad \forall \epsilon \in [0, h].$$

*This is a consequence of the fact that  $\|x + \epsilon w - y\| \geq \|x + hw - y\| - \|(\epsilon - h)w\| \geq h\|w\| - (h - \epsilon)\|w\| = \epsilon\|w\|$  for all  $\epsilon \in [0, h]$  and  $y \in C$ .*

The next lemma gives another characterization of the outer normal vector which will motivate the subsequent lemmas.

**Lemma 3.10** *Let  $X$  be a real Hilbert space with a norm induced by the inner product  $\langle \cdot, \cdot \rangle$ . Assume that  $C$  is convex and let  $x \in C$  be given. Then the following properties are equivalent:*

- i)  $w \in X$  and  $\langle w, x - y \rangle \geq 0$  for all  $y \in C$ ;*
- ii)  $w \in X$  is an outer normal vector to  $C$  at  $x$ .*

*Proof.* Assume that i) is satisfied. Then we have for all  $y \in C$

$$\|w + x - y\|^2 = \|w\|^2 + \|x - y\|^2 + 2\langle w, x - y \rangle \geq \|w\|^2,$$

that is  $y \notin B(w + x, \|w\|)$ . Since  $x \in \overline{B(w + x, \|w\|)}$  it follows that  $\overline{B(w + x, \|w\|)} \cap C = \{x\}$ . This proves ii). Let us now assume that ii) holds true. Thus, there exists  $h > 0$  such that  $\overline{B(x + hw, h\|w\|)} \cap C = \{x\}$  and  $B(x + hw, h\|w\|) \cap C = \emptyset$ . From where we deduce that

$$\|y - x - hw\| \geq h\|w\|, \quad \forall y \in C. \quad (11)$$

In light of Remark 3.9, we can assume that  $h \in (0, 1]$  so that for each  $\epsilon \in (0, 1)$ , the convexity of  $C$  ensures that

$$x + \epsilon h(y - x) \in C, \quad \forall y \in C \quad (12)$$

Therefore, using (11) and (12) we obtain

$$\|x + \epsilon h(y - x) - x - hw\| \geq h\|w\|, \quad \forall \epsilon \in (0, 1), \quad \forall y \in C$$

that is

$$\|\epsilon(y - x) - w\| \geq \|w\|, \quad \forall \epsilon \in (0, 1), \quad \forall y \in C$$

hence

$$\epsilon\|y - x\|^2 + 2\langle w, x - y \rangle \geq 0, \quad \forall \epsilon \in (0, 1), \quad \forall y \in C.$$

The result follows by letting  $\epsilon \rightarrow 0^+$ . ■

From the above Lemma 3.10 we can see that if  $X$  is a real Hilbert space and  $C$  is convex then  $w \in X$  is an outer normal vector to  $C$  at  $x \in C$  if and only if

$$\langle w, x \rangle = \sup_{y \in C} \langle w, y \rangle.$$

This motivates the following lemma where a specific case has been proved in [17] when  $C$  is a positive cone. We show that it still holds true when  $C$  is a closed convex set.

**Lemma 3.11** *Assume that  $C$  is closed and convex. Let  $v \in X$  and  $x \in C$  be given and fixed. Then the following properties are equivalent:*

- i) *There exists  $\theta : [0, +\infty) \rightarrow X$  such that  $x + tv + t\theta(t) \in C$  for all  $t \geq 0$  and  $\theta(t) \rightarrow 0$  when  $t \rightarrow 0^+$ ;*
- ii)  $\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tv; C) = 0$ ;
- iii) *For each  $x^* \in X^*$  with  $x^*(x) = \sup_{y \in C} x^*(y)$  we have  $x^*(v) \leq 0$ ;*
- iv) *For each  $x^* \in X^*$  with  $x^*(x) = \inf_{y \in C} x^*(y)$  we have  $x^*(v) \geq 0$ .*

*Proof.* The equivalence between i) and ii) is already obtained in Lemma 3.4. The equivalence between iii) and iv) follows from the equivalence

$$x^*(x) = \sup_{y \in C} x^*(y) \iff -x^*(x) = \inf_{y \in C} -x^*(y).$$

Next, we prove that ii) is equivalent to iv). Assume that ii) is satisfied. Then there exists  $\theta : [0, +\infty) \rightarrow X$  such that  $x + tv + t\theta(t) \in C$  for all  $t \geq 0$  and  $\theta(t) \rightarrow 0$  when  $t \rightarrow 0^+$ . Let  $x^* \in X^*$  with  $x^*(x) = \inf_{z \in C} x^*(z)$ . Then we have

$$x^*(x + tv + t\theta(t)) \geq x^*(x), \quad \forall t \geq 0 \implies x^*(v + \theta(t)) \geq 0, \quad \forall t \geq 0$$

so that

$$x^*(v) \geq 0.$$

Assume that iv) is satisfied. Let us note that the convexity of  $C$  implies that for each  $h > 0$  and  $y \in C$  we have

$$x + \frac{t}{h}(y - x) \in C, \quad \forall t \in [0, h] \tag{13}$$

so that

$$\text{dist}(x + tv; C) \leq \|x + tv - (x + \frac{t}{h}(y - x))\| = \frac{t}{h} \|x + hv - y\|, \quad t \in [0, h], \quad \forall y \in C. \tag{14}$$

In the following, we show that there is a contradiction if ii) does not hold true. This will be done by using the Hann-Banach separation theorem for convex sets. Assume that ii) does not hold true. This means that there exists  $\epsilon > 0$  and a sequence  $(t_n)_{n \geq 0} \subset (0, +\infty)$  such that

$$\lim_{n \rightarrow +\infty} t_n = 0 \quad \text{and} \quad \text{dist}(x + t_n v; C) \geq t_n \epsilon, \quad \forall n \geq 0. \tag{15}$$

Consider the set

$$D_0 = \{y \in X \mid \text{there exists } h > 0 \text{ such that } \|x + hv - y\| < h\epsilon\}.$$

The set  $D_0$  is non empty because  $x + hv \in D_0$  for each  $h > 0$ . Moreover,  $D_0$  is convex and open in  $X$ . We now show that  $D_0 \cap C = \emptyset$ . In fact if  $y \in D_0 \cap C$  then there exists  $h > 0$  such that  $\|x + hv - y\| < \epsilon h$ . Moreover for  $n$  large enough we have  $t_n \in [0, h]$  and (13) implies that  $\text{dist}(x + t_n v; C) < t_n \epsilon$  which contradict (15). Therefore, the Hann-Banach separation theorem for convex sets ensures that there exists  $x^* \in X^*$  and  $\alpha \in \mathbb{R}$  such that

$$x^*(z_1) < \alpha \leq x^*(z_2), \quad \text{for all } z_1 \in D_0 \text{ and } z_2 \in C.$$

Therefore,  $x^*(x) \geq \alpha$  and

$$x + hv \in D_0, \forall h > 0 \implies x^*(x + hv) < \alpha, \forall h > 0$$

hence letting  $h \rightarrow 0^+$  we obtain  $x^*(x) \leq \alpha$  providing that  $x^*(x) = \alpha$ . As a consequence, we obtain  $x^*(x) = \inf_{z \in C} x^*(z)$ . Moreover, we have

$$x + hv \in D_0, x^*(x + hv) < \alpha \text{ and } x^*(x + hv) = x^*(x) + hx^*(v) = \alpha + hx^*(v)$$

providing that

$$\alpha + hx^*(v) < \alpha, h > 0 \implies hx^*(v) < 0, h > 0 \implies x^*(v) < 0$$

which is a contradiction to iii). ■

## 4 Some consequence of Theorem 2.5 in comparison theory

In this section, we will give conditions to obtain the nonnegativity of the solutions as well as the monotony with respect to the initial condition of the semilinear Cauchy problem (2) when  $C(t) = X_{0+}$  for all  $t \in J$ , where  $X_0 = \overline{D(A)}$ . We also derive the comparison of solutions of semilinear problems, upper and lower solutions. Henceforth, we assume that the Banach space  $X$  has a positive cone  $X_+$ . Let us note that similar results have been obtained in [12] for the non-densely defined Cauchy problem by using a different approach. We recall that  $X_+$  is a positive cone of  $X$  if it is a closed convex subset of  $X$  such that  $X_+ \cap (-X_+) = 0$  and; for all  $\lambda \in \mathbb{R}_+$ ,  $x, y \in X_+$  we have  $x + y \in X_+$ ,  $\lambda x \in X_+$ . The positive cone  $X_+$  induced a partial order in  $X$  that is for all  $x, y \in X$  we have

$$y \geq x \iff y - x \in X_+ \iff x \leq y.$$

Note that  $X_{0+} := X_0 \cap X_+ = \overline{D(A)} \cap X_+$  is also a positive cone of the Banach space  $X_0$ .

### 4.1 Existence of nonnegative solutions

In this section, we will give criteria for the existence of a nonnegative solution to the semilinear Cauchy problem (2). Since  $X_{0+}$  is closed convex, the existence of a nonnegative solution to (2) is a direct consequence of the positive invariance of  $X_{0+}$ . The main contribution of this section is to give general and flexible sufficient conditions. We start first by stating the theorem which is a consequence of the Theorem 2.5, follows by a discussion of these sufficient conditions.

**Theorem 4.1** *Under the condition (F), the following properties are equivalent:*

- i) For each  $x \in X_{0+}$  there exists a unique maximally defined nonnegative solution to (2)
- ii) For each  $\ell \in J$  and  $x \in X_{0+}$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)F(\ell, x); X_{0+}) = 0. \quad (16)$$

Using Lemma 2.3 one knows that a sufficient condition to satisfy (16) is that there exists  $\omega > 0$  such that  $(\lambda I - A)^{-1}X_+ \subseteq X_+$  for all  $\lambda > \omega$  and

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tF(\ell, x); X_+) = 0, \quad \forall (\ell, x) \in J \times X_{0+}. \quad (17)$$

In the following, we give an equivalent condition to (17). In fact, this will be obtained by using the results in the Section 2. To do so, we recall that the dual cone  $X_+^*$  is defined by

$$X_+^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in X_+\}.$$

The following lemma holds true.

**Lemma 4.2** *For each  $x \in X_{0+}$  the following properties are equivalent:*

- i)  $\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tF(\ell, x); X_+) = 0$  for all  $\ell \in J$ ;
- ii) For each  $x^* \in X_+^*$  with  $x^*(x) = 0$  we have  $x^*(F(\ell, x)) \geq 0$ , for all  $\ell \in J$ .

*Proof.* Taking advantage of Lemma 3.11 to prove the equivalence between i) and ii), we can prove that ii) is equivalent to

$$\text{If } x^* \in X^* \text{ with } x^*(x) = \inf_{y \in X_+} x^*(y) \text{ then } x^*(F(\ell, x)) \geq 0, \quad \forall \ell \in J. \quad (18)$$

Assume that (18) is satisfied. Let  $x^* \in X_+^*$  with  $x^*(x) = 0$  be given. Then we have

$$\inf_{y \in X_+} x^*(y) = 0 = x^*(x)$$

and (18) implies that  $x^*(F(\ell, x)) \geq 0$ . Assume that ii) is satisfied. Let  $x^* \in X^*$  be given such that  $x^*(x) = \inf_{y \in X_+} x^*(y)$ . Then we have

$$0 = x^*(x) \quad \text{and} \quad 0 \leq x^*(y), \quad \forall y \in X_+$$

that is  $x^* \in X_+^*$  and  $x^*(x) = 0$  so that  $x^*(F(\ell, x)) \geq 0$ . ■

**Remark 4.3** *In addition to the condition  $\lambda(\lambda I - A)^{-1}X_+ \subseteq X_+$  for large  $\lambda$ , we may have the following condition*

**(CM)** *For each  $r > 0$  and  $\tau_0 > 0$  there exists  $\lambda > 0$  such that*

$$F(t, x) + \lambda x \in X_+, \quad \forall x \in X_{0+}, \|x\| \leq r, t \in [0, \tau_0].$$

*It is easy to see that if the condition (CM) is satisfied then condition ii) of Lemma 4.2 is satisfied but the converse is not true even in finite dimension. The following example from Walter [23] make it clear. For  $X = \mathbb{R}^3$  and  $X_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq x_3^2 \text{ and } x_3 \geq 0\}$  let*

$$F(x_1, x_2, x_3) = (-x_2, x_1, 0).$$

*We clearly have*

$$F(1, 0, 1) + \lambda(1, 0, 1) = (\lambda, 1, \lambda) \notin X_+, \quad \forall \lambda > 0.$$

*Let  $y = (y_1, y_2, y_3) \in X^* = \mathbb{R}^3 \setminus \{0\}$  and  $(x_1, x_2, x_3) \in X_+$  such that  $\langle y, x \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$ . Then we have  $y = h(-x_1, -x_2, x_3)$  with  $h > 0$  so that*

$$\langle y, F(x) \rangle = -h x_2 x_1 + h x_1 x_2 + h x_3 0 = 0.$$

We conclude this section by giving the following theorem.

**Theorem 4.4** *Suppose that condition **(F)** holds and there exists  $\omega > \omega_A$  such that  $(\lambda I - A)^{-1}X_+ \subset X_+$  for all  $\lambda > \omega$ . Assume in addition that one of the following condition hold:*

- i) For each  $(\ell, x) \in J \times X_{0+}$  we have  $\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tF(\ell, x); X_+) = 0$ ;*
- ii) For each  $x^* \in X_+^*$  with  $x^*(x) = 0$  we have  $x^*(F(\ell, x)) \geq 0$ , for all  $\ell \in J$ ;*
- iii) For each  $r > 0$  and  $\tau_0 > 0$  there exists  $\lambda > 0$  such that  $F(t, x) + \lambda x \in X_+$  for every  $x \in X_{0+}$  and  $\ell \in [0, \tau]$  with  $\|x\| \leq r$ ;*

*Then for each  $x \in X_{0+}$  there exists a unique nonnegative maximally defined integral (mild) solution to (2).*

## 4.2 Comparison of solutions and monotony

The comparison theorem and the monotony are very helpful in studying the boundedness of the solutions as well as their asymptotic behavior. We will see that this is again a consequence of the theory developed in Section 4. Let  $A : D(A) \subset X \rightarrow X$  be a Hille-Yosida linear operator of type  $(M_A, \omega_A)$  possibly with non dense domain and  $G : [0, +\infty) \times X_0 \rightarrow X$  be given. Consider the semilinear Cauchy problems

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + G(t, u(t)), & t \geq t_0 \\ u(t_0) = x \in X_{0+}. \end{cases} \quad (19)$$

We make the following assumption.

**Assumption 4.5** *We assume that:*

- i)  $G$  is Lipschitz continuous on bounded sets of  $[0, +\infty) \times X_{0+}$ ;*
- ii) There exists  $\omega > \omega_A$  such that  $\lambda(\lambda I - A)^{-1}X_+ \subset X_+$  for all  $\lambda > \omega$  and*

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tG(\ell, x); X_+) = 0, \quad \forall (\ell, x) \in [0, +\infty) \times X_{0+}.$$

Then, the following result holds

**Theorem 4.6** *Let  $X$  be a Banach lattice with  $X_+$  a normal cone. Let Assumption 4.5 be satisfied. Assume in addition that there exists  $v \in C([0, +\infty), X_{0+})$  satisfying*

$$v(t) \geq T_{A_0}(t - t_0)v(t_0) + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t - s)\lambda R_\lambda(A)G(s, v(s))ds, \quad t \geq t_0$$

*and for all  $\ell \geq 0$ ,  $x \in [0, v(\ell)]$  we have*

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(v(\ell) - x + t[G(\ell, v(\ell)) - G(\ell, x)]; X_+) = 0. \quad (20)$$

*Then for each  $t_0 \geq 0$  and  $x \in [0, v(t_0)]$  there exists a unique integral (mild) solution  $u \in C([t_0, +\infty), X_{0+})$  to (19) with*

$$0 \leq u(t) \leq v(t), \quad \forall t \geq t_0.$$

*Proof.* Consider the following time-dependent closed sets

$$C(t) := \{x \in X_{0+} : 0 \leq x \leq v(t)\}, \quad \forall t \geq 0.$$

The subsets  $C(t)$ ,  $t \geq 0$  clearly satisfy closedness condition that is for a convergent sequence  $(t_n, x_n)$  to  $(t^*, x^*)$  with  $x_n \in C(t_n)$  we have  $x^* \in C(t^*)$ . Let  $x \in C(\ell)$  be given for some  $\ell \geq 0$  i.e  $0 \leq x \leq v(\ell)$ . Since  $X$  is a Banach lattice it follows that  $X_0$  is also a Banach lattice with positive cone  $X_{0+}$ . Thus, we have from [10, Lemma 2.1] that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(t)x + S_A(t)G(\ell, x), C(t+l)) &\leq \lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(w_1(t, \ell, x), X_{0+}) \\ &\quad + \lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(w_2(t, \ell, x), X_{0+}) \end{aligned}$$

with

$$w_1(t, \ell, x) := T_{A_0}(t)x + S_A(t)G(\ell, x)$$

and

$$w_2(t, \ell, x) := v(\ell+t) - T_{A_0}(t)x - S_A(t)G(\ell, x).$$

Thanks to Assumption 4.5 and Lemma 2.3 we have

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(w_1(t, \ell, x), X_{0+}) = 0.$$

Recall that for each  $t > 0$  we have

$$S_A(t)y = \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)y ds, \quad \forall y \in X.$$

Thus, for each  $t > 0$  we have

$$\begin{aligned} v(\ell+t) &\geq T_{A_0}(t)v(\ell) + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)G(s, v(s))ds \\ &\geq T_{A_0}(t)v(\ell) + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(s, v(s)) - G(\ell, v(\ell))]ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)G(\ell, v(\ell))ds \end{aligned}$$

so that

$$\begin{aligned} w_2(t, \ell, x) &\geq T_{A_0}(t)[v(\ell) - x] \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(s, v(s)) - G(\ell, v(\ell))]ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(\ell, v(\ell)) - G(\ell, x)]ds \\ &\geq T_{A_0}(t)[v(\ell) - x] + S_A(t)[G(\ell, v(\ell)) - G(\ell, x)] \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(s, v(s)) - G(\ell, v(\ell))]ds. \end{aligned}$$

Hence, it follows that for each  $t > 0$

$$\begin{aligned} \text{dist}(w_2(t, \ell, x), X_{0+}) &\leq \text{dist}(T_{A_0}(t)[v(\ell) - x] + S_A(t)[G(\ell, v(\ell)) - G(\ell, x)], X_{0+}) \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} \|T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(s, v(s)) - G(\ell, v(\ell))]\| ds. \end{aligned}$$



Next, we note that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \lim_{\lambda \rightarrow +\infty} \int_{\ell}^{\ell+t} \|T_{A_0}(\ell+t-s)\lambda R_{\lambda}(A)[G(s, v(s)) - G(\ell, v(\ell))]\| ds = 0$$

and infer from (25) and Lemma 2.3 that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(T_{A_0}(\ell)[v(\ell) - x] + S_A(t)[G(\ell, v(\ell)) - G(\ell, x)], X_{0+}) = 0.$$

Thus, using Theorem 2.5 one knows that for each  $t_0 \geq 0$  and each  $x \in C(t_0)$  there exists a maximally defined integral solution  $u \in C([t_0, t_0 + \tau), X_{0+})$  to (19) such that  $u(t) \in C(t)$  for all  $t \in [t_0, t_0 + \tau)$ . Finally using the fact the the cone  $X_+$  is normal and the inequality  $0 \leq u(t) \leq v(t)$  for all  $t \in [t_0, t_0 + \tau)$  we deduce that the solution is globally defined in  $[0, +\infty)$ , and this ends the proof of the theorem.  $\blacksquare$

Using similar arguments as in the proof of Theorem 4.6 we obtain the following result.

**Theorem 4.7** *Let  $X$  be a Banach lattice. Let Assumption 4.5 be satisfied. Assume in addition that there exists  $v \in C([0, +\infty), X_{0+})$  satisfying*

$$v(t) \leq T_{A_0}(t - t_0)v(t_0) + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t - s)\lambda R_{\lambda}(A)G(s, v(s))ds, \quad t \geq t_0$$

and for all  $\ell \geq 0, x \in [v(\ell), +\infty)$  we have

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x - v(\ell) + t[G(\ell, x) - G(\ell, v(\ell))]; X_+) = 0. \quad (21)$$

Then for each  $t_0 \geq 0$  and  $x \in [v(t_0), +\infty)$  there exists a unique maximally defined integral (mild) solution  $u \in C([t_0, t_0 + \tau), X_{0+})$ ,  $\tau > 0$ , to (19) with

$$v(t) \leq u(t), \quad \forall t \in [t_0, t_0 + \tau).$$

Moreover, as consequences of Theorems 4.6 and 4.7, we have the following estimates

**Theorem 4.8** *Let  $X$  be a Banach lattice with  $X_+$  a normal cone. Let Assumption 4.5 be satisfied.*

i) *Let  $b \in D(A) \cap X_{0+}$  such that*

$$Ab + G(t, b) \leq 0, \quad \forall t \geq 0 \quad (22)$$

and for all  $\ell \geq 0, x \in [0, b]$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(b - x + t[G(\ell, b) - G(\ell, x)]; X_+) = 0. \quad (23)$$

Then for each  $t_0 \geq 0$  and  $x \in [0, b]$  there exists a unique integral (mild) solution  $u \in C([t_0, +\infty), X_{0+})$  to (19) with,  $0 \leq u(t) \leq b, \forall t \geq t_0$ .

ii) *Let  $a \in D(A) \cap X_{0+}$  such that*

$$Aa + G(t, a) \geq 0, \quad \forall t \geq 0 \quad (24)$$

and for all  $\ell \geq 0$ ,  $x \in [a, +\infty)$

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x - a + t[G(\ell, x) - G(\ell, a)], X_+) = 0. \quad (25)$$

Then for each  $t_0 \geq 0$  and  $x \in [a, +\infty)$  there exists a unique maximally defined integral (mild) solution  $u \in C([t_0, t_0 + \tau], X_{0+})$ ,  $\tau > 0$ , to (19) with,  $a \leq u(t)$ ,  $\forall t \in [t_0, t_0 + \tau)$ .

*Proof.* The proof of item i) of the theorem relies on a suitable application of Theorem 4.6. To do so, let us set  $v(t) = b$  for all  $t \geq 0$  and note that

$$v'(t) = Av(t) + G(t, v(t)) - Ab - G(t, b), \quad \forall t \geq 0$$

hence for all  $t \geq t_0$  we have

$$\begin{aligned} v(t) &= T_{A_0}(t - t_0)v(t_0) + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t - s)\lambda R_\lambda(A)[G(s, v(s)) - Ab - G(t, b)]ds \\ &\geq T_{A_0}(t - t_0)v(t_0) + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t - s)\lambda R_\lambda(A)G(s, v(s))ds. \end{aligned}$$

The result follows from Theorem 4.6.

Finally, a similar argument applies for the proof of item ii) as a direct application of Theorem 4.7. ■

Now, let us give a comparison of two mild solutions to Cauchy problems. Let  $H : [0, +\infty) \times X_{0+} \rightarrow X$  be given and consider the semilinear Cauchy problem

$$\begin{cases} \frac{dv(t)}{dt} = Au(t) + H(t, v(t)), & t \geq t_0 \\ v(t_0) = y \in X_{0+}. \end{cases} \quad (26)$$

Concerning the map  $H$ , we will require the following assumption.

**Assumption 4.9** *We assume that:*

- i)  $H$  is Lipschitz continuous on bounded sets of  $[0, +\infty) \times X_{0+}$ ;
- ii) For all  $(\ell, x) \in [0, +\infty) \times X_{0+}$  we have

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x + tH(\ell, x), X_+) = 0.$$

**Theorem 4.10** *Let Assumptions 4.5 and 4.9 be satisfied. Assume in addition that one of the below conditions holds:*

- i) For each  $\ell \in J$ ,  $x, y \in X_{0+}$  with  $x - y \in X_{0+}$  we have,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \text{dist}(x - y + t(G(\ell, x) - H(t, y)); X_+) = 0;$$

- ii) For each  $x^* \in X_+^*$  with  $x^*(x) = x^*(y)$  and  $x \geq y$  we have,  $x^*(G(\ell, x)) \geq x^*(H(\ell, y))$ , for all  $\ell \in J$ ;

- iii) For each  $r > 0$  and  $\tau_0 > 0$ , there exists  $\lambda > 0$  such that, if  $x, y \in X_{0+}$  and  $\ell \in [0, \tau_0]$  with  $\|x\|, \|y\| \leq r$  and  $x \geq y$  then,

$$H(\ell, x) - G(\ell, y) + \lambda(x - y) \in X_+;$$

therefore, for all  $t_0 \geq 0$  and  $x, y \in X_{0+}$  with  $x - y \in X_{0+}$  the mild solutions  $u$  and  $v$ , respectively of (19) and (26), satisfy  $u(t) - v(t) \in X_{0+}$  in the common interval of existence.

*Proof.* The proof will be done by applying Theorem 2.5. To do this, let us consider the semilinear Cauchy problem

$$\begin{cases} \frac{dw(t)}{dt} = Aw(t) + F(t, w(t)), & t \geq t_0 \\ w(t_0) = w_0 \in X_{0+} \end{cases} \quad (27)$$

where  $F : J \times X_{0+} \rightarrow X$  is given by

$$F(t, z) = G(t, v(t) + z) - H(t, v(t)), \quad \forall (t, z) \in [0, +\infty) \times X_{0+}.$$

It is clear that  $F$  satisfies the condition of Theorem 4.4. Therefore for each initial condition  $w_0 \in X_{0+}$  at time  $t = t_0$ , there exists a unique maximally defined nonnegative integral solution to (27). The proof is completed by observing that  $w$  is a solution to (27) with initial condition  $x - y$  if and only if  $u = w + v$ .  $\blacksquare$

## 5 Proof of Theorem 2.5

The proof of Theorem 2.5 is lengthy and will be done throughout several steps and lemmas. To begin with, let us fix some conditions that will be assumed satisfied in this section. In the sequel, we will only prove local existence since the proof of the maximality is similar to [18, 10, 21]. Let  $x_0 \in C_0$  and  $t_0 \in J$  be given and fixed. Without loss of generality, we set  $t_0 = 0$ . Let  $r_0 > 0$  be fixed such that  $\|x_0\| < r_0$ . Let  $0 < \hat{\tau} < \sup(J)$ . From condition (F) there exists  $\kappa := \kappa(r_0, \hat{\tau}) > 0$  such that

$$\|F(t, x) - F(t, y)\| \leq \kappa \|x - y\|, \quad (28)$$

for all  $t \in [0, \hat{\tau}]$ ,  $x, y \in C_0$  and  $\|x\| \leq r_0$ ,  $\|y\| \leq r_0$ . Next, set

$$M_0 := 2\kappa r_0 + \sup_{t \in [0, \hat{\tau}]} \|F(t, x_0)\|,$$

so that

$$\|F(t, x)\| \leq M_0, \quad \forall t \in [0, \hat{\tau}], \quad x \in C_0, \quad \text{with } \|x\| \leq r_0. \quad (29)$$

We next prove that there exists an integral solution  $u \in C([0, \tau], C_0)$  to (2) with  $\tau \in [0, \hat{\tau}]$  and  $u(t) \in C(t)$  for all  $t \in [0, \tau]$ . More precisely let us fix  $\tau > 0$  small enough such that

$$\sup_{0 \leq t \leq \tau} \|T_{A_0}(t)x_0 - x_0\| + \int_0^\tau e^{\omega_A^+(\tau-s)} [M_A^2 M + M_A] ds + \|x_0\| \leq r_0 \quad (30)$$

with  $\omega_A^+ = \max\{\omega_A, 0\}$ .

## 5.1 Construction of the knots

**Proposition 5.1** *Let  $\epsilon \in (0, 1)$  be given. There exists a sequence  $(t_k, y_k)_{k \geq 0} \in [0, \tau] \times C_0$  such that  $y_k \in C(t_k)$  and the set*

$$\mathcal{I}_k := \left\{ \eta \in (0, \epsilon) : \begin{cases} \|F(t, y) - F(t_k, y_k)\| \leq \epsilon, \\ y \in C_0, t \in [t_k, t_k + \eta] \text{ with } \|y - y_k\| \leq \eta \\ \frac{1}{\eta} \text{dist}(T_{A_0}(\eta)y_k + S_A(\eta)F(t_k, y_k), C(t_k + \eta)) < \frac{\epsilon}{2} \\ \sup_{0 \leq t \leq \eta} \|T_{A_0}(t)y_k - y_k\| \leq \epsilon \end{cases} \right\} \quad (31)$$

is nonempty for every  $k \geq 0$  and the following properties hold true:

- i) For every  $k \geq 0$  we have  $t_{k+1} = \min(t_k + \hat{\eta}_k, \tau)$  with  $\hat{\eta}_k \in (\eta_k/2, \eta_k)$  and  $\eta_k = \sup(\mathcal{I}_k) \in (0, \epsilon]$ ;
- ii)  $y_0 = x_0, t_0 = 0$  and for every  $k \geq 0$  we have  $y_k \in C(t_k)$  with

$$\begin{cases} y_{k+1} = T_{A_0}(t_{k+1} - t_k)y_k + S_A(t_{k+1} - t_k)F(t_k, y_k) + (t_{k+1} - t_k)H_k, \forall k \geq 0 \\ H_k \in X_0 \text{ and } \|H_k\| \leq \frac{\epsilon}{2}, \forall k \geq 0. \end{cases} \quad (32)$$

*Proof.* Fix  $t_0 = 0$ . Note that for  $k = 0$  and  $y_0 = x_0 \in C_0$  the following set

$$\mathcal{I}_k := \left\{ \eta \in (0, \epsilon) : \begin{cases} \|F(t, y) - F(t_k, y_k)\| \leq \epsilon, \\ y \in C_0, t \in [t_k, t_k + \eta] \text{ with } \|y - y_k\| \leq \eta \\ \frac{1}{\eta} \text{dist}(T_{A_0}(\eta)y_k + S_A(\eta)F(t_k, y_k), C(t_k + \eta)) < \frac{\epsilon}{2} \\ \sup_{0 \leq t \leq \eta} \|T_{A_0}(t)y_k - y_k\| \leq \epsilon \end{cases} \right\} \quad (33)$$

is nonempty and bounded so that

$$\eta_0 := \sup(\mathcal{I}_0) > 0,$$

is well defined. Let  $\hat{\eta}_0 \in (\eta_0/2, \eta_0)$  and set

$$t_1 = \min(t_0 + \hat{\eta}_0, \tau).$$

Since  $t_1 - t_0 \in [0, \hat{\eta}_0]$  it follows that

$$\frac{1}{t_1 - t_0} \text{dist}(T_{A_0}(t_1 - t_0)y_0 + S_A(t_1 - t_0)F(t_0, y_0), C(t_1)) < \frac{\epsilon}{2}$$

providing that there exists  $y_1 \in C(t_1)$  such that

$$\frac{1}{t_1 - t_0} \|T_{A_0}(t_1 - t_0)y_0 + S_A(t_1 - t_0)F(t_0, y_0) - y_1\| \leq \frac{\epsilon}{2}.$$

Thus, setting

$$H_0 := \frac{1}{t_1 - t_0} [y_1 - T_{A_0}(t_1 - t_0)y_0 - S_A(t_1 - t_0)F(t_0, y_0)] \in X_0 \quad (34)$$

it follows that

$$\|H_0\| \leq \frac{\epsilon}{2}$$

and

$$y_1 = T_{A_0}(t_1 - t_0)y_0 + S_A(t_1 - t_0)F(t_0, y_0) + (t_1 - t_0)H_0 \in C_0.$$

The result follows by induction on  $k \geq 1$ . ■

**Lemma 5.2** *Let  $(t_k, y_k)_{k \geq 0} \in [0, \tau] \times C_0$  with  $y_k \in C(t_k)$  be the sequence provided in Proposition 5.1. Then we have the following properties:*

i) For all  $k > m \geq 0$  we have

$$\begin{aligned} y_k &= T_{A_0}(t_k - t_m)y_m + \sum_{i=m}^{k-1} \lim_{\lambda \rightarrow +\infty} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - s)\lambda R_\lambda(A)F(t_i, y_i)ds \\ &\quad \sum_{i=m}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - t_{i+1})H_i ds \end{aligned} \quad (35)$$

ii) For all  $k \geq 0$  we have  $\|y_k\| \leq r_0$

iii) For all  $k > m \geq 0$  we have

$$\|y_k - T_{A_0}(t_k - t_m)y_m\| \leq M_A^2 \int_{t_m}^{t_k} e^{\omega_A(t_k - s)}(M + M_A e^{\omega_{A^S}})ds \quad (36)$$

iv) For all  $k \geq 0$  we have

$$\|y_{k+1} - y_k\| \leq M_0 \epsilon \quad (37)$$

with

$$M_0 := 1 + e^{\omega_A^+} [M_A^2 M + M_A]. \quad (38)$$

*Proof.* We first prove item i). Setting

$$L_k y := T_{A_0}(t_{k+1} - t_k)y, \quad \forall k \geq 0, \quad \forall y \in X_0$$

and

$$f_k := S_A(t_{k+1} - t_k)F(t_k, y_k) + (t_{k+1} - t_k)H_k, \quad \forall k = 0, 1, \dots,$$

it follows that

$$y_{k+1} = L_k y_k + f_k, \quad k = 0, 1, \dots$$

Using the discrete-time variation of constants formula together with the semigroup properties we obtain for all  $k > m$

$$y_k = T_{A_0}(t_k - t_m)y_m + \sum_{i=m}^{k-1} T_{A_0}(t_k - t_{i+1})[S_A(t_{i+1} - t_i)F(t_i, y_i) + (t_{i+1} - t_i)H_i].$$

Next, observe that for  $i = m, \dots, k-1$

$$\begin{aligned} T_{A_0}(t_k - t_{i+1})S_A(t_{i+1} - t_i)F(t_i, y_i) &= \\ &= \lim_{\lambda \rightarrow +\infty} T_{A_0}(t_k - t_{i+1}) \int_0^{t_{i+1} - t_i} T_{A_0}(t_{i+1} - t_i - s)\lambda R_\lambda(A)F(t_i, y_i)ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_0^{t_{i+1} - t_i} T_{A_0}(t_k - t_i - s)\lambda R_\lambda(A)F(t_i, y_i)ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - s)\lambda R_\lambda(A)F(t_i, y_i)ds \end{aligned}$$

so that

$$\begin{aligned}
y_k &= T_{A_0}(t_k - t_m)y_m + \sum_{i=m}^{k-1} \lim_{\lambda \rightarrow +\infty} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - s) \lambda R_\lambda(A) F(t_i, y_i) ds \\
&\quad \sum_{i=m}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - t_{i+1}) H_i ds.
\end{aligned} \tag{39}$$

To prove ii), we argue by induction. By construction  $y_0 = x_0$  and  $\|x_0\| \leq r_0$ . For  $k \geq 1$ , assume that  $\|y_i\| \leq r_0$  for  $i = 0, \dots, k-1$ . Since  $\epsilon \in (0, 1)$ ,  $\|H_i\| \leq \epsilon \leq 1$  and  $\|F(t_i, y_i)\| \leq M$  for all  $i = m, \dots, k-1$  it follows that for each  $s \in [t_i, t_{i+1}]$

$$\|T_{A_0}(t_k - s) \lambda R_\lambda(A) F(t_i, y_i)\| \leq M_A e^{\omega_A^+(t_k - s)} \frac{\lambda M_A}{\lambda - \omega_A} M, \quad i = m, \dots, k-1.$$

and

$$\|T_{A_0}(t_k - t_{i+1}) H_i\| \leq M_A e^{\omega_A(t_k - t_{i+1})} \leq M_A e^{\omega_A^+(t_k - s)}.$$

Therefore for  $k > m$  it follows from (39) that

$$\begin{aligned}
\|y_k - T_{A_0}(t_k - t_m)y_m\| &\leq \sum_{i=m}^{k-1} \int_{t_i}^{t_{i+1}} e^{\omega_A^+(t_k - s)} [M_A^2 M + M_A] ds \\
&\leq \int_{t_m}^{t_k} e^{\omega_A^+(t_k - s)} [M_A^2 M + M_A] ds
\end{aligned} \tag{40}$$

and for  $m = 0$  in (40) we obtain

$$\|y_k - T_{A_0}(t_k)x_0\| \leq \int_0^{t_k} e^{\omega_A^+(t_k - s)} [M_A^2 M + M_A] ds \leq \int_0^\tau e^{\omega_A^+(\tau - s)} [M_A^2 M + M_A] ds.$$

Hence

$$\begin{aligned}
\|y_k\| &\leq \|y_k - T_{A_0}(t_k)x_0\| + \|T_{A_0}(t_k)x_0 - x_0\| + \|x_0\| \\
&\leq \sup_{0 \leq t \leq \tau} \|T_{A_0}(t)x_0 - x_0\| + \int_0^\tau e^{\omega_A^+(\tau - s)} [M_A^2 M + M_A] ds + \|x_0\|.
\end{aligned}$$

and we infer from (30) that  $\|y_k\| \leq r_0$ . The proof of iii) is already included the proof of ii) by (40).

Now we proceed to the proof of iv). Note that

$$y_{k+1} - y_k = y_{k+1} - T_{A_0}(t_{k+1} - t_k)y_k + T_{A_0}(t_{k+1} - t_k)y_k - y_k, \quad \forall k \geq 0$$

and by construction

$$\|T_{A_0}(t_{k+1} - t_k)y_k - y_k\| \leq \epsilon.$$

Using (40) we have

$$\begin{aligned}
\|y_{k+1} - T_{A_0}(t_{k+1} - t_k)y_k\| &\leq \int_{t_k}^{t_{k+1}} e^{\omega_A^+(t_{k+1} - s)} [M_A^2 M + M_A] ds \\
&\leq (t_{k+1} - t_k) e^{\omega_A^+(t_{k+1} - t_k)} [M_A^2 M + M_A]
\end{aligned}$$

and since  $0 \leq t_{k+1} - t_k \leq \eta_k \leq \epsilon$  we obtain

$$\|y_{k+1} - y_k\| \leq \epsilon + \epsilon e^{\omega_A^+ \epsilon} [M_A^2 M + M_A].$$

■

**Lemma 5.3** *There exists  $(t^*, y^*) \in [0, \tau] \times C(t^*)$  such that  $t_k \rightarrow t^*$  and  $y_k \rightarrow y^*$  as  $k \rightarrow +\infty$ .*

*Proof.* To prove the lemma, we notice that by construction the sequence  $(t_k)$  is non decreasing and bounded above so that there exists  $t^* \in [0, \tau]$  such that  $t_k \rightarrow t^*$  as  $k \rightarrow +\infty$ . To prove that  $(y_k)$  is a convergent sequence, we will show that it is a Cauchy sequence. Indeed, using (36) we have for all  $k \geq j > m$

$$\begin{aligned} \|y_k - y_j\| &\leq \|y_k - T_{A_0}(t_k - t_m)y_m\| + \|T_{A_0}(t_j - t_m)y_m - y_j\| \\ &\quad + \|T_{A_0}(t_k - t_m)y_m - T_{A_0}(t_j - t_m)y_m\| \\ &\leq \int_{t_m}^{t_k} e^{\omega_A^+(t_k-s)} [M_A^2 M + M_A] ds + \int_{t_m}^{t_j} e^{\omega_A^+(t_j-s)} [M_A^2 M + M_A] ds \\ &\quad + \|T_{A_0}(t_k - t_m)y_m - T_{A_0}(t_j - t_m)y_m\| \end{aligned}$$

so that

$$\limsup_{k, j \rightarrow +\infty} \|y_k - y_j\| \leq 2 \int_{t_m}^{t^*} e^{\omega_A^+(t^*-s)} [M_A^2 M + M_A] ds$$

Since  $m$  is arbitrary and  $t_m \rightarrow t^*$  when  $m \rightarrow +\infty$  it follows that  $(y_k)$  is a Cauchy sequence. Recalling that  $C_0$  is closed, we deduce that there exists  $y^* \in C_0$  such that  $y_k \rightarrow y^*$  when  $k \rightarrow +\infty$ . Therefore, we conclude that  $(t^*, y^*) \in [0, \tau] \times C(t^*)$ .  $\blacksquare$

**Lemma 5.4** *The sequence  $(t_k, y_k)_{k \geq 0}$  is stationary. That is there exists  $n_\epsilon \geq 1$  such that  $(t_k, y_k) = (t_{n_\epsilon}, y_{n_\epsilon}) = (\tau, y^*)$  for all  $k \geq n_\epsilon$ .*

*Proof.* To prove the lemma, we will argue by contradiction. Let us first note that if there exists  $n_\epsilon \geq 1$  such that  $t_{n_\epsilon} = \tau$  then for each  $k \geq n_\epsilon$  we have  $t_{k+1} = \min(t_k + \hat{\eta}_k/2, \tau) = \tau$ . Thus, to argue by contradiction let us assume that  $t_k < \tau$  for all  $k \geq 1$ . Therefore, using the definition of the sequence  $(t_k)_{k \geq 0}$  it follows that

$$t_{k+1} = \min(t_k + \hat{\eta}_k/2, \tau) = t_k + \hat{\eta}_k/2 < \tau, \quad \forall k \geq 0. \quad (41)$$

Let  $(t^*, y^*) \in [0, 0 + \tau] \times C_0$  be given by Lemma 5.3 such that  $t_k \rightarrow t^*$  and  $y_k \rightarrow y^*$  when  $k \rightarrow +\infty$ . Thus, we have

$$\lim_{k \rightarrow +\infty} \hat{\eta}_k = \lim_{k \rightarrow +\infty} 2(t_{k+1} - t_k) = 0$$

and since  $\hat{\eta}_k \in (\eta_k/2, \eta_k)$  we obtain

$$\lim_{k \rightarrow +\infty} \eta_k = 0. \quad (42)$$

Moreover, using the fact that  $y^* \in C(t^*)$  we also have

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \text{dist}(T_{A_0}(h)y^* + S_A(h)F(t^*, y^*), C(t^*)) = 0. \quad (43)$$

Let  $K$  be the compact set defined by

$$K := \{y_k, F(t_k, y_k), y^*, F(t^*, y^*) : k \geq 0\}. \quad (44)$$

Next, using (43) and (44) it follows that

$$\mathcal{I}^* := \left\{ \eta \in (0, \epsilon) : \begin{cases} \|F(t, y) - F(t^*, y^*)\| \leq \frac{\epsilon}{2}, \\ y \in C_0, t \in [0, \tau] \text{ with } |t - t^*| \leq 3\eta, \|y - y^*\| \leq 3\eta \\ \frac{1}{\eta} \text{dist}(T_{A_0}(\eta)y^* + S_A(\eta)F(t^*, y^*), C(t^* + \eta)) < \frac{\epsilon}{4} \\ \sup_{0 \leq t \leq 2\eta} \|T_{A_0}(t)z - z\| \leq \epsilon, z \in K \end{cases} \right\} \quad (45)$$

is non empty and we set  $\eta^* = \frac{\sup(\mathcal{I}^*)}{4}$ . This ensures that  $[\eta^*, 2\eta^*] \subset \mathcal{I}^*$ . Using the convergence of the sequences  $(\eta_k)$  and  $(t_k, y_k)$ , respectively to 0 and  $(t^*, y^*)$  it follows that there exists  $k_0 \geq 1$  large enough such that

$$\begin{cases} \|F(t_k, y_k) - F(t^*, y^*)\| \leq \frac{\epsilon}{2}, \forall k \geq k_0 \\ \|y^* - y_k\| \leq \eta^*, \forall k \geq k_0 \\ 0 < t^* - t_k \leq \eta^*, \forall k \geq k_0 \\ \hat{\eta}_k < \eta^*, \forall k \geq k_0. \end{cases} \quad (46)$$

Next, we introduce the sequence

$$s_k := t^* - t_k + \eta^* > \eta^*, \forall k \geq 0 \Leftrightarrow s_k + t_k = t^* + \eta^*, \forall k \geq 0 \quad (47)$$

and observe that

$$s_k \in [\eta^*, 2\eta^*] \subset \mathcal{I}^*, \forall k \geq k_0 \text{ and } \hat{\eta}_k < \eta^* < s_k, \forall k \geq k_0. \quad (48)$$

Note that (48) implies that  $s_k \notin \mathcal{I}_k$  for all  $k \geq k_0$ . Moreover, we have for all  $k \geq k_0$

$$(|t - t_k| \leq s_k \Rightarrow |t - t^*| \leq 3\eta^*) \text{ and } (\|y - y_k\| \leq s_k \Rightarrow \|y - y^*\| \leq 3\eta^*). \quad (49)$$

Thus, using (45) and (49) it holds that if  $|t - t_k| \leq s_k$  and  $\|y - y_k\| \leq s_k$  then for all  $k \geq k_0$

$$\begin{cases} \|F(t, y) - F(t_k, y_k)\| \leq \|F(t, y) - F(t^*, y^*)\| + \|F(t^*, y^*) - F(t_k, y_k)\| \leq \epsilon \\ \sup_{0 \leq t \leq s_k} \|T_{A_0}(t)y_k - y_k\| \leq \sup_{z \in K} \sup_{0 \leq t \leq 2\eta^*} \|T_{A_0}(t)z - z\| \leq \epsilon. \end{cases} \quad (50)$$

Hence, the above inequalities (48) and (50) ensure that we must have  $s_k \notin \mathcal{I}_k$  for all  $k \geq k_0$  with

$$\frac{1}{s_k} \text{dist}(T_{A_0}(s_k)y_k + S_A(s_k)F(t_k, y_k), C(t_k + s_k)) \geq \frac{\epsilon}{2}. \quad (51)$$

Recalling that from (47) we have  $t_k + s_k = t^* + \eta^*$  and by letting  $k \rightarrow +\infty$  in (51) we obtain

$$\frac{1}{\eta^*} \text{dist}(T_{A_0}(\eta^*)y^* + S_A(\eta^*)F(t^*, y^*), C(t^* + \eta^*)) \geq \frac{\epsilon}{2}, \quad (52)$$

which contradict the fact that  $\eta^* \in \mathcal{I}^*$ . This proves that there exists  $n_\epsilon \geq 0$  such that  $t_k = t_{n_\epsilon} = \tau$  for all  $k \geq n_\epsilon$ . To conclude to the stationarity of  $(y_k)$ , we use the relationship

$$y_{k+1} = T_{A_0}(t_{k+1} - t_k)y_k + S_A(t_{k+1} - t_k)F(t_k, y_k) + (t_{k+1} - t_k)H_k, \quad k \geq 0$$

together with  $S_A(0) = 0$  to obtain

$$y_k = y_{n_\epsilon}, \quad \forall k \geq n_\epsilon.$$

■



## 5.2 Construction of the approximate solution

Let us now proceed to the construction of the approximate mild solutions. To this end, we define for  $k = 0, \dots, n_\epsilon - 1$  the map  $w_k^\epsilon : [t_k, t_{k+1}] \rightarrow X$  by

$$w_k^\epsilon(t) := \frac{t_{k+1} - t}{t_{k+1} - t_k} y_k + \frac{t - t_k}{t_{k+1} - t_k} y_{k+1}, \quad \forall t \in [t_k, t_{k+1}]. \quad (53)$$

Since  $w_k^\epsilon$  is constructed by a convex combination of  $y_k$  and  $y_{k+1}$ , it follows from Lemma 5.2 that

$$\begin{cases} w_k^\epsilon(t) \in C_0 \text{ and } \|w_k^\epsilon(t)\| \leq r_0, \quad \forall t \in [t_k, t_{k+1}] \\ w_k^\epsilon(t_k) = y_k \text{ and } w_k^\epsilon(t_{k+1}) = y_{k+1}. \end{cases} \quad (54)$$

Moreover using (37) and (53), we also have

$$\|w_k^\epsilon(t) - y_k\| \leq M_0 \epsilon, \quad \forall t \in [t_k, t_{k+1}] \quad (55)$$

with  $M_0$  the constant defined in (38). From (53), (54) and (55) it is now easy to see that the map  $w_\epsilon : [0, \tau] \rightarrow X$  defined by

$$w_\epsilon(t) := w_k^\epsilon(t), \quad \forall t \in [t_k, t_{k+1}] \quad (56)$$

is continuous in  $[0, \tau]$  and satisfies

$$\begin{cases} w_\epsilon(t) \in C_0 \text{ and } \|w_\epsilon(t)\| \leq r_0, \quad \forall t \in [0, \tau] \\ w_\epsilon(t_k) = y_k, \quad k = 0, \dots, n_\epsilon \\ \|w_\epsilon(t) - y_k\| \leq M_0 \epsilon, \quad \forall t \in [t_k, t_{k+1}], \quad k = 0, \dots, n_\epsilon - 1. \end{cases} \quad (57)$$

The next, step is to show that the map  $w_\epsilon$  defined in (57) is an approximate solution to (2) that is

$$\omega_\epsilon(t) - T_{A_0}(t)x_0 - \lim_{\lambda \rightarrow +\infty} \int_0^t T_{A_0}(t-s) \lambda R_\lambda(A) F(s, \omega_\epsilon(s)) ds = \mathcal{O}(\epsilon)$$

and that it converges to a unique solution to (2).

## 5.3 Convergence of the approximate solution

We proceed into three steps. In the first step, we show that that map  $v_\epsilon : [0, \tau] \rightarrow C_0$  defined by

$$v_\epsilon(t) := \omega_\epsilon(t) - T_{A_0}(t)x_0 - \lim_{\lambda \rightarrow +\infty} \int_0^t T_{A_0}(t-s) \lambda R_\lambda(A) F(s, \omega_\epsilon(s)) ds, \quad \forall t \in [0, \tau] \quad (58)$$

is of order  $\epsilon$  uniformly in  $[0, \tau]$ . This means that  $w_\epsilon : [0, \tau] \rightarrow C_0$  is an  $\epsilon$ -approximate mild solution to (2). In the second step, we prove that  $w_\epsilon : [0, \tau] \rightarrow C_0$  converges to  $w \in C([0, \tau], X)$ . In the third step we prove that  $w(t) \in C(t)$  for all  $t \in [0, \tau]$ .

**Step 1:** In this step, we do estimate  $v_\epsilon$  on  $[0, \tau]$ . To this end, note that for  $t \in [t_k, t_{k+1}]$  with  $k = 0, \dots, n_\epsilon - 1$  we have

$$\begin{aligned} v_\epsilon(t) &= \omega_\epsilon(t) - T_{A_0}(t-t_k) \left[ T_{A_0}(t_k)x_0 + \lim_{\lambda \rightarrow +\infty} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k-s) \lambda R_\lambda(A) F(t_i, y_i) \right] \\ &\quad - \lim_{\lambda \rightarrow +\infty} T_{A_0}(t-t_k) \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k-s) \lambda R_\lambda(A) [F(s, \omega_\epsilon(s)) - F(t_i, y_i)] ds \\ &\quad - \lim_{\lambda \rightarrow +\infty} \int_{t_k}^t T_{A_0}(t-s) \lambda R_\lambda(A) F(s, \omega_\epsilon(s)) ds \end{aligned}$$

and recalling that

$$y_k = T_{A_0}(t_k)x_0 + \sum_{i=0}^{k-1} \lim_{\lambda \rightarrow +\infty} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - s) \lambda R_\lambda(A) F(t_i, y_i) ds \\ + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t_k - t_{i+1}) H_i ds$$

we obtain

$$v_\epsilon(t) = \omega_\epsilon(t) - T_{A_0}(t - t_k)y_k \\ - \lim_{\lambda \rightarrow +\infty} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t - s) \lambda R_\lambda(A) [F(s, \omega_\epsilon(s)) - F(t_i, y_i)] ds \\ - \lim_{\lambda \rightarrow +\infty} \int_{t_k}^t T_{A_0}(t - s) \lambda R_\lambda(A) F(s, \omega_\epsilon(s)) ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} T_{A_0}(t - t_{i+1}) H_i ds \quad (59)$$

Note that for each  $t \in [t_k, t_{k+1}]$

$$\|\omega_\epsilon(t) - T_{A_0}(t - t_k)y_k\| \leq \|\omega_\epsilon(t) - y_k\| + \|y_k - T_{A_0}(t - t_k)y_k\| \\ \leq \|\omega_\epsilon(t) - y_k\| + \sup_{0 \leq s \leq t_{k+1} - t_k} \|y_k - T_{A_0}(s)y_k\|$$

and we infer from (57) and the definition of the sequence  $(t_k, y_k)$  in Proposition 5.1 together with  $t_{k+1} - t_k \leq \eta_k$  to obtain

$$\|\omega_\epsilon(t) - T_{A_0}(t - t_k)y_k\| \leq M_0\epsilon + \epsilon, \quad \forall t \in [t_k, t_{k+1}], \quad k = 0, \dots, n_\epsilon - 1. \quad (60)$$

Furthermore, for  $s \in [t_k, t_{k+1}]$ ,  $k = 0, \dots, n_\epsilon - 1$  we also have

$$\|F(s, \omega_\epsilon(s)) - F(t_k, y_k)\| \leq \|F(s, \omega_\epsilon(s)) - F(s, y_k)\| + \|F(s, y_k) - F(t_k, y_k)\| \\ \leq \kappa \|y_k - \omega_\epsilon(s)\| + \|F(t_k, y_k) - F(s, y_k)\|$$

hence using (57) and the definition of the sequence  $(t_k, y_k)$  in Proposition 5.1 combined with  $s - t_k \leq t_{k+1} - t_k \leq \eta_k$  we obtain

$$\|F(t_k, y_k) - F(s, \omega_\epsilon(s))\| \leq \kappa M_0\epsilon + \epsilon, \quad \forall s \in [t_k, t_{k+1}], \quad k = 0, \dots, n_\epsilon - 1. \quad (61)$$

Since  $s \in [0, \tau]$  and  $\omega_\epsilon(s) \in C_0$  with  $\|\omega_\epsilon(s)\| \leq r_0$ , the inequality (29) implies that

$$\|F(s, \omega_\epsilon(s))\| \leq M, \quad \forall s \in [0, \tau]. \quad (62)$$

Now combining (60) (61) and (62), it is easy to obtain from (59) that for all  $t \in [t_k, t_{k+1}]$

$$\begin{aligned}
\|v_\epsilon(t)\| &\leq M_0\epsilon + \epsilon + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M_A^2 e^{\omega_A(t-s)} (\kappa M_0\epsilon + \epsilon) ds \\
&\quad + \int_{t_k}^t M_A^2 e^{\omega_A^+(t-s)} M ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M_A e^{\omega_A(t-t_{i+1})} \epsilon ds \\
&\leq M_0\epsilon + \epsilon + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M_A^2 e^{\omega_A^+\tau} (\kappa M_0\epsilon + \epsilon) ds \\
&\quad + \int_{t_k}^t M_A^2 e^{\omega_A^+\tau} M ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M_A e^{\omega_A^+\tau} \epsilon ds \\
&\quad + \int_{t_k}^t M_A^2 e^{\omega_A^+(t-s)} M ds + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} M_A e^{\omega_A(t-t_{i+1})} \epsilon ds \tag{63} \\
&\leq M_0\epsilon + \epsilon + t_k M_A^2 e^{\omega_A^+\tau} (\kappa M_0\epsilon + \epsilon) ds \\
&\quad + (t - t_k) \int_{t_k}^t M_A^2 e^{\omega_A^+\tau} M ds + t_k M_A e^{\omega_A^+\tau} \epsilon ds \\
&\leq M_0\epsilon + \epsilon + \tau M_A^2 e^{\omega_A^+\tau} (\kappa M_0\epsilon + \epsilon) \\
&\quad + (t_{k+1} - t_k) M_A^2 e^{\omega_A^+\tau} M + \tau M_A e^{\omega_A^+\tau} \epsilon
\end{aligned}$$

and since by construction  $t_{k+1} - t_k \leq \eta_k \leq \epsilon$  we obtain

$$\|v_\epsilon(t)\| \leq \epsilon M_1, \quad \forall t \in [t_k, t_{k+1}], \quad k = 0, \dots, n_\epsilon - 1 \tag{64}$$

with

$$M_1 := M_0 + 1 + \tau M_A^2 e^{\omega_A^+\tau} (\kappa M_0 + 1) + M_A^2 e^{\omega_A^+\tau} M + \tau M_A e^{\omega_A^+\tau} \epsilon. \tag{65}$$

It is now clear from (64) and (65) that

$$\|v_\epsilon(t)\| \leq \epsilon M_1, \quad \forall t \in [0, \tau]. \tag{66}$$

**Step 2:** In this step, we prove that the map  $w_\epsilon \in C([0, \tau], C_0)$  converges when  $\epsilon \rightarrow 0^+$ . Let  $\epsilon, \delta \in (0, 1)$ . Then using (58) we obtain

$$w_\epsilon(t) - w_\delta(t) = v_\epsilon(t) - v_\delta(t) + \lim_{\lambda \rightarrow +\infty} \int_0^t T_{A_0}(t-s) \lambda R_\lambda(A) [F(s, \omega_\epsilon(s)) - F(s, \omega_\delta(s))] ds, \quad \forall t \in [0, \tau].$$

Recalling that by construction we have

$$\|w_\epsilon(t)\| \leq r_0, \quad \forall t \in [0, \tau]$$

it follows from (28), (58) and (64) that

$$\begin{aligned}
\|w_\epsilon(t) - w_\delta(t)\| &\leq \|v_\epsilon(t) - v_\delta(t)\| + \int_0^t M_A^2 e^{\omega_A(t-s)} \kappa \|w_\epsilon(s) - w_\delta(s)\| ds \\
&\leq 2M_1(\epsilon + \delta) + \int_0^t M_A^2 e^{\omega_A(t-s)} \kappa \|w_\epsilon(s) - w_\delta(s)\| ds \\
&\leq 2M_1(\epsilon + \delta) + \int_0^t M_A^2 e^{\omega_A^+\tau} \kappa \|w_\epsilon(s) - w_\delta(s)\| ds, \quad \forall t \in [0, \tau]
\end{aligned}$$

and by Gronwall's lemma, we get

$$\|w_\epsilon(t) - w_\delta(t)\| \leq 2M_1(\epsilon + \delta)e^{\kappa_0 t}, \quad \forall t \in [0, \tau], \quad \text{with } \kappa_0 := M_A^2 e^{\omega_A^+ \tau} \kappa.$$

This shows that  $w_\epsilon : [0, \tau] \rightarrow C_0$ ,  $\epsilon \in (0, 1)$  is a Cauchy sequence in  $C([0, \tau], X)$ . Since  $C_0$  is closed, there exists  $w \in C([0, \tau], C_0)$  such that

$$\lim_{\epsilon \rightarrow 0^+} w_\epsilon(t) = w(t) \quad \text{in } C([0, \tau], C_0).$$

Therefore, using (58) and (64) and letting  $\epsilon \rightarrow 0^+$  we obtain

$$w(t) = T_{A_0}(t)x_0 + \lim_{\lambda \rightarrow +\infty} \int_0^t T_{A_0}(t-s)\lambda R_\lambda(A)F(s, w(s))ds, \quad t \in [0, \tau].$$

**Step 3:** Let  $t \in [0, \tau]$  be fixed. Recall that, by construction, for each  $\epsilon \in (0, 1)$  there exists a sequence  $(t_k, y_k)_{k \in \{0, \dots, n_\epsilon\}}$  such that  $t_{k+1} - t_k \leq \epsilon$  and  $t_{n_\epsilon} = \tau$ . Since  $t \in [0, \tau]$  there exists  $k_\epsilon \in \{0, \dots, n_\epsilon - 1\}$  such that  $t \in [t_{k_\epsilon}, t_{k_\epsilon+1}]$ . This means that for each  $\epsilon \in (0, 1)$  there exists  $k_\epsilon \in \mathbb{N}$  such that

$$\begin{cases} 0 \leq t_{k_\epsilon} \leq t \leq t_{k_\epsilon+1} \\ t_{k_\epsilon+1} - t_{k_\epsilon} \leq \epsilon \\ t_{k_\epsilon} \in [0, \tau] \end{cases}$$

so that  $t_{k_\epsilon} \rightarrow t$  when  $\epsilon \rightarrow 0^+$ . Next, we note that

$$\|w(t) - y_{k_\epsilon}\| \leq \sup_{s \in [0, \tau]} \|w(t) - w_\epsilon(t)\| + \|w_\epsilon(t) - y_{k_\epsilon}\|$$

and by using (57) we obtain

$$\|w(t) - y_{k_\epsilon}\| \leq \sup_{s \in [0, \tau]} \|w(t) - w_\epsilon(t)\| + M_0\epsilon.$$

Therefore

$$\lim_{\epsilon \rightarrow 0^+} t_{k_\epsilon} = t \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} y_{k_\epsilon} = w(t)$$

which implies by hypothesis that  $w(t) \in C(t)$ .

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