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Anne-Marie Aubert, Yujie Xu

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ON HECKE ALGEBRAS FOR p-ADIC REDUCTIVE GROUPS: THE EXAMPLE OF G_2

ANNE-MARIE AUBERT AND YUJIE XU

Abstract. We study the endomorphism algebras attached to Bernstein components of reductive p-adic groups. By using recent results of Solleveld, we prove a reduction to depth zero case result for the components attached to regular supercuspidal representations of Levi subgroups, and construct a correspondence with the appropriate set of enhanced L-parameters.

In particular, for Levi subgroups of maximal parabolic of the split exceptional group of type G_2, we compute the explicit parameters for the corresponding Hecke algebras, and show that they satisfy a conjecture of Lusztig's. We also give examples for a generalized version of Yu's conjecture using type theory for G_2.

1. Introduction

Let G be the group of F-points of a reductive connected group $\mathbf{G}$, defined over a local non-archimedean field F of residual characteristic p.

Let M be the Levi subgroup of a parabolic subgroup P of G. We denote by M_1 the subgroup of M generated by all its compact subgroups. We recall that a character of M is said to be unramified if it is trivial on M_1, and denote by $\mathfrak{X}_{\mathrm{nr}}(M)$ the group of unramified characters of M.

Let $\mathfrak{s} = [M, \sigma]_G$ be the inertial class of a supercuspidal irreducible representation σ of G (that is, $\mathfrak{s}$ is the G-conjugacy class of $(M, \mathfrak{X}_{\mathrm{nr}}(M) \cdot \sigma)$ where $\mathfrak{X}_{\mathrm{nr}}(M) \cdot \sigma$ is the orbit of σ under $\mathfrak{X}_{\mathrm{nr}}(M)$). We denote by $\mathrm{Irr}^{\mathfrak{s}}(G)$ the Bernstein series of irreducible representations of G attached to $\mathfrak{s}$.

Let $W_G^{\mathfrak{s}}$ denote the extended finite Weyl group $\mathrm{N}_G(\mathfrak{s}_M)/M$, where

$$\mathrm{N}_G(\mathfrak{s}_M) := \{g \in M \,:\, {}^gM = M \text{ and } {}^g\sigma \simeq \chi \otimes \sigma, \text{ for some } \chi \in \mathfrak{X}_{\mathrm{nr}}(M)\}\,, \tag{1.0.1}$$

and let $W_G^{\mathfrak{s},x}$ be the the stabilizer in $W_G^{\mathfrak{s}}$ of $x \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$. Solleveld has constructed in [Sol21a] a collection of 2-cocycles

$$\natural_x \colon W_G^{\mathfrak{s},x} \times W_G^{\mathfrak{s},x} \longrightarrow \mathbb{C}^\times,$$

for $x \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$ and a bijection

$$\xi_G^{\mathfrak{s}} \colon \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural, \tag{1.0.2}$$

where $\mathfrak{s}_M = [M, \sigma]_M$, and $(\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural$ is the twisted extended quotient in the sense of [ABPS17a, §2.1], the definition of which is recalled in (1.0.7).

A parallel picture to (1.0.2) exists on the Galois side. We denote by W_F the absolute Weil group of F and by I_F its inertia subgroup. Let $M^\vee$ be the Langlands

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dual group of M (a complex Lie group with root system dual to that of M) equipped with an action of W_F, and we write ${}^LM := M^\rtimes W_F$. The group $M^\vee$ acts on the set of cuspidal M-relevant enhanced L-parameters for M (a terminology which is based on Lusztig's notion of cuspidal pairs, see Definition 3.1.5) and we denote by $\Phi_{\mathrm{e}}^{\mathrm{c}}(M)$ the set of $M^\vee$-conjugacy classes of these enhanced parameters for M.

Let $\mathrm{Z}_{M^\vee\rtimes I_F}$ be the center of $M^\vee\rtimes I_F$. The group $\mathfrak{X}_{\mathrm{nr}}({}^LM) := (\mathrm{Z}_{M^\vee\rtimes I_F})^\circ_{W_F}$, which is naturally isomorphic to the group $\mathfrak{X}_{\mathrm{nr}}(M)$ (see [Hai14, §3.3.1]), acts naturally on the set of cuspidal M-relevant enhanced L-parameters for M. We denote by $\mathfrak{s}^\vee = [{}^LM, \varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}]_{G^\vee}$ the $G^\vee$-conjugacy class of the orbit of $(\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}) \in \Phi_{\mathrm{e}}^{\mathrm{c}}(M)$ under the action of $\mathfrak{X}_{\mathrm{nr}}({}^LM)$ and by $\mathfrak{B}^\vee(G)$ be the set of these $\mathfrak{s}^\vee$.

In [AMS18], a partition à la Bernstein of the set of G-relevant enhanced Langlands parameters was constructed:

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mathfrak{s}^\vee\in\mathfrak{B}^\vee(G)} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G). \tag{1.0.3}$$

We write $\mathfrak{s}_M^\vee := [{}^LM, \varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}]_{M^\vee}$, and, analogously to (1.0.1), denote by $W_{G^\vee}^{\mathfrak{s}^\vee}$ the stabilizer in $\mathrm{N}_{G^\vee}(M^\vee)/M^\vee$ of $\mathfrak{s}_M^\vee$ and by $W_{G^\vee}^{\mathfrak{s}^\vee,y}$ the stabilizer in $W_{G^\vee}^{\mathfrak{s}^\vee}$ of $y \in \Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)$. Then [AMS18, Theorem 9.3] provides a bijection

$$\xi_{G^\vee}^{\mathfrak{s}^\vee} \colon \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \longrightarrow (\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee})_{{}^L\natural}, \tag{1.0.4}$$

where $(\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee})_{{}^L\natural}$ is a twisted extended quotient with respect to a collection of 2-cocycles

$${}^L\natural_y \colon W_{G^\vee}^{\mathfrak{s}^\vee,y} \times W_{G^\vee}^{\mathfrak{s}^\vee,y} \to \mathbb{C}^\times.$$

Assuming the existence of a map

$$\begin{array}{rccc} \mathfrak{L}^{\mathfrak{s}_M}\colon & \mathrm{Irr}^{\mathfrak{s}_M}(M) & \longrightarrow & \Phi_{\mathrm{e}}^{\mathrm{c}}(M) \\ & \sigma & \mapsto & (\varphi_\sigma, \varrho_\sigma) \end{array}$$

such that the following properties are satisfied for any $\sigma \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$:

(1) For any $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$, we have

$$(\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) = \chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma),$$

where $\chi \mapsto \chi^\vee$ is the canonical isomorphism $\mathfrak{X}_{\mathrm{nr}}(M) \xrightarrow{\sim} \mathfrak{X}_{\mathrm{nr}}({}^LM)$.

(2) For any $w \in W(M)$, we have

$${}^{w^\vee}(\varphi_\sigma, \varrho_\sigma) \simeq (\varphi_{{}^w\sigma}, \varrho_{{}^w\sigma}),$$

where $w \mapsto w^\vee$ is the canonical isomorphism $W(M) \xrightarrow{\sim} W(M^\vee)$.

we establish the following result.

Theorem 1. (See Theorem 3.1.15)
We have a natural isomorphism

$$\mathfrak{e}\colon \mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}} \xrightarrow{\sim} \Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee}.$$

We suppose in the rest of this Introduction that the group $\mathbf{G}$ splits over a tamely ramified extension of F and that p does not divide the order of the Weyl group of G. Then there exists a compact mod center subgroup $\widetilde{K}_M$ of M and an irreducible representation ρ_M^d of it such that $\sigma = \mathrm{ind}_{\widetilde{K}_M}^M \rho_M^d$, see [Yu01, Fin21]. Kim and Yu constructed in [KY17] an $\mathfrak{s}$-type in the sense of Bushnell-Kutzko [BK98] $(K_{\mathcal{D}}, \rho_{\mathcal{D}})$, formed of an open compact subgroup $K_{\mathcal{D}}$ of G and irreducible representation of $\rho_{\mathcal{D}}$ of $K_{\mathcal{D}}$. Here $\mathcal{D}$, as defined in (2.3.2), is a 5-tuple which includes a finite sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \cdots, \mathbf{G}^d)$ of twisted E-Levi subgroups of $\mathbf{G}$, with E a tamely ramified finite extension of F. Attached to $\mathcal{D}$ there is also a depth zero supercuspidal irreducible representation σ^0 of a Levi subgroup M^0 of G^0. We denote by $\mathfrak{s}^0 = [M^0, \sigma^0]_{G^0}$ the inertial class of σ^0.

Let $\mathcal{H}^{\mathfrak{s}}(G)$ denote the endomorphism algebra of the Bernstein progenerator of $\mathfrak{s}$ (see (2.1.3)) and let $\mathcal{H}(G, \rho_{\mathcal{D}})$ be the intertwining algebra of $(K_{\mathcal{D}}, \rho_{\mathcal{D}})$. We prove in Proposition 2.3.15 that the algebras $\mathcal{H}^{\mathfrak{s}}(G)$ and $\mathcal{H}(G, \rho_{\mathcal{D}})$ are isomorphic.

We suppose that σ is regular in the terminology of [Kal19a]. Then Kaletha has attached to σ a supercuspidal Langlands parameter $\varphi_\sigma \colon W_F \to {}^L M$.

By applying Theorem 1 to the map $\mathfrak{L}^{\mathfrak{s}_M} \colon \sigma \mapsto (\varphi_\sigma, 1)$, we obtain the following result.

Theorem 2. (See Theorem 3.1.19)
We suppose that the collections of 2-cocycles $\natural$ and ${}^L\natural$ are both trivial and the characteristic of F is 0 then the map

$$\mathfrak{L} := (\xi_{G^\vee}^{\mathfrak{s}^\vee})^{-1} \circ \mathfrak{e} \circ \xi_G^{\mathfrak{s}} \colon \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \tag{1.0.5}$$

is a bijection.

We expect that the bijection $\mathfrak{L}$ from (1.0.5) satisfies the desired properties of the local Langlands correspondence, which include the properties described in [AP22, Definition 3.25] and in [Kal22], and hence provides new cases of the latter.

Theorem 3. (See Theorem 2.4.9)
We suppose that p is good for G (in the sense of [Car93]) and does not divide the order of the fundamental group of $\mathbf{G}_{\mathrm{der}}$, and that the representation σ is regular in the terminology of [Kal19a]. Then there is a bijection

$$\varrho_{\mathfrak{s}^0}^{\mathfrak{s}} \colon \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow \mathrm{Irr}(G^0)^{\mathfrak{s}^0}.$$

It induces a bijection

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}}(G)) \longrightarrow \mathrm{Irr}(\mathcal{H}^{\mathfrak{s}^0}(G^0))$$

between the sets of equivalence classes of simple modules of the algebras $\mathcal{H}^{\mathfrak{s}}(G)$ and $\mathcal{H}^{\mathfrak{s}^0}(G^0)$.

Theorem 3 proves the validity of [AM21, Conjecture 1.1], under the above assumption on p, for all regular supercuspidal representations of M. The bijection $\varrho_{\mathfrak{s}^0}^{\mathfrak{s}}$ is defined as

$$\varrho_{\mathfrak{s}^0}^{\mathfrak{s}} := (\xi_{G^0}^{\mathfrak{s}^0})^{-1} \circ \mathfrak{l}_\sigma \circ \xi_G^{\mathfrak{s}}, \tag{1.0.6}$$

where $(\mathrm{Irr}^{\mathfrak{s}_{M^0}}(M^0)//W_{G^0}^{\mathfrak{s}^0})_{\natural^0}$ is the twisted extended quotient with respect to a certain collection $\natural^0$ of 2-cocycles, the definition of which is recalled in (2.4.8), and

$$\natural_\sigma \colon (\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_{M^0}}(M^0)//W_{G^0}^{\mathfrak{s}^0})_{\natural^0}$$

is the isomorphism constructed by Adler and Mishra in [AM21].

In Section 4, we study in more details the case when G is the exceptional group of type G_2. The case where M is a torus being known thanks to the work of Roche for principal series of split p-adic groups [Roc98], it suffices to consider the cases where $M \simeq \mathrm{GL}_2(F)$ is a maximal Levi subgroup. The G_2-covers of the supercuspidaL types in M were computed explicitly in [Blo99] when M corresponds to the long simple root, and in [Des21] when M corresponds to the short simple root, but the intertwining algebras of these types were still unknown. We compute these latter in §4.5, and in particular, by computing their parameters explicitly, we show that they satisfy the expectation made by Lusztig in [Lus20, 1.(a)].

Twisted extended quotients. Let Γ be a group acting on a topological space X and let Γ_x denote the stabilizer in Γ of $x \in X$. Let $\natural$ be a collection of 2-cocycles

$$\natural_x \colon \Gamma_x \times \Gamma_x \to \mathbb{C}^\times,$$

such that $\natural_{\gamma x}$ and $\gamma_* \natural_x$ define the same class in $H^2(\Gamma_{\gamma x}, \mathbb{C}^\times)$, where $\gamma_* \colon \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma\alpha\gamma^{-1}$. Let $\mathbb{C}[\Gamma_x, \natural_x]$ be the group algebra of Γ_x twisted by $\natural_x$. We set

$$\widetilde{X}_\natural := \{(x, \tau) \,:\, x \in X,\, \tau \in \mathrm{Irr}\ \mathbb{C}[\Gamma_x, \natural_x]\},$$

and topologize $\widetilde{X}_\natural$ by decreeing that a subset of $\widetilde{X}_\natural$ open if its projection to the first coordinate is open in X.

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} \colon \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

satisfying the conditions

(a) if $\gamma x = x$, then $\phi_{\gamma,x}$ is conjugation by an element of $\mathbb{C}[\Gamma_x, \natural_x]^\times$;
(b) $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma$ and $x \in X$.

Define a Γ-action on $\widetilde{X}_\natural$ by $\gamma \cdot (x, \tau) := (\gamma x, \tau \circ \phi_{\gamma,x}^{-1})$. The *twisted extended quotient* of X by Γ with respect to $\natural$ is defined to be

$$(X//\Gamma)_\natural := \widetilde{X}_\natural / \Gamma. \tag{1.0.7}$$

In the case when the 2-cocycles $\natural_x$ are trivial, we write simply $X//\Gamma$ for $(X//\Gamma)_\natural$ and refer to it as the (spectral) *extended quotient* of X by Γ.

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Notations. Let F be a local non-archimedean field. Let $\mathfrak{o}_F$ denote the ring of integers of F, $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$ and $k_F := \mathfrak{o}_F/\mathfrak{p}_F$ the residue field of F. We assume that k_F is finite and denote by $q = q_F$ its cardinality. Let $\mathrm{v}_F \colon F \to \mathbb{Z} \cup \{\infty\}$ be a valuation of F.

We fix a separable closure F_{sep} of F and denote by $W_F \subset \mathrm{Gal}(F_{\mathrm{sep}}/F)$ the absolute Weil group of F and let $I_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ be its inertia subgroup. We denote by F_{nr} the maximal unramified extension of F inside F_{sep} and by Fr_F the element of $\mathrm{Gal}(F_{\mathrm{nr}}/F)$ that induces the automorphism $a \mapsto a^q$ on the residue field $\overline{k}_F$ of F_{nr}. Then $W_F = I_F \rtimes \langle \mathrm{Fr}_F \rangle$. We take $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ for the Weil-Deligne group of F. Let I_F^+ denote the wild inertia group of F (i.e., the maximal pro-p open normal subgroup of I_F). We have $I_F^{\mathrm{t}} = \mathrm{Gal}(F_{\mathrm{sep}}/F_{\mathrm{t}}) \simeq I_F/I_F^+$, where F_{t} is the tame closure of F in F_{sep}. The group I_F^{t} is pro-cyclic and we denote by ζ_F a generator of it.

Let G be the F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F. We denote by $\mathbf{G}_{\mathrm{der}}$ the derived group of $\mathbf{G}$, and by $\mathbf{G}_{\mathrm{sc}}$ and $\mathbf{G}_{\mathrm{ad}}$ the simply connected cover and adjoint quotient of $\mathbf{G}_{\mathrm{der}}$. Let $\mathrm{Z}_{\mathbf{G}}$ be the center of $\mathbf{G}$ and $\mathbf{A}_{\mathbf{G}}$ the maximal F-split torus contained in $\mathrm{Z}_{\mathbf{G}}$.

Suppose that H is a group, H_1 a subgroup of H and h an element of H. We set ${}^hH_1 := hH_1h^{-1}$. If π is a representation of H_1, we denote by ${}^h\pi$ the representation $h_1 \mapsto \pi(h^{-1}h_1h)$ of ${}^hH_1$. We denote by $\mathrm{Irr}(H)$ the set of of equivalence classes of irreducible representations of H.

The category of right modules over an algebra $\mathcal{A}$ is denoted $\mathcal{A}-\mathrm{Mod}$. We write $\mathrm{Irr}(\mathcal{A})$ for the set of equivalence classes of simple modules of $\mathcal{A}$.

2. Hecke algebras and Bernstein Center

2.1. **General framework.** Let $\mathfrak{R}(G)$ denote the category of all smooth complex representations of G. This is an abelian category admitting arbitrary coproducts. Let M be a Levi subgroup of a parabolic subgroup P of G, and let $\mathfrak{X}_{\mathrm{nr}}(M)$ be the group of unramified characters of M. Let σ be an irreducible supercuspidal smooth representation of M. We write $\mathfrak{s} := [M, \sigma]_G$ for the G-conjugacy class of the pair $(M, \mathfrak{X}_{\mathrm{nr}}(M) \cdot \sigma)$ and $\mathfrak{B}(G)$ for the set of such classes $\mathfrak{s}$. We set $\mathfrak{s}_M := [M, \sigma]_M$.

We denote by $\mathfrak{R}^{\mathfrak{s}}(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every G-subquotient of π is equivalent to a subquotient of a parabolically induced representation $i_P^G(\sigma')$, where i_P^G is the functor of normalized parabolic induction and $\sigma' \in \mathfrak{X}_{\mathrm{nr}}(M) \cdot \sigma$. We write $\mathrm{Irr}^{\mathfrak{s}}(G)$ for the class of irreducible objects in $\mathfrak{R}^{\mathfrak{s}}(G)$.

The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ in a direct product (see [Ber84]):

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

If $\Pi^{\mathfrak{s}}$ is a progenerator of $\mathfrak{R}^{\mathfrak{s}}(G)$, then the functor $V \mapsto \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, V)$ is an equivalence from $\mathfrak{R}^{\mathfrak{s}}(G)$ to the algebra $\mathrm{End}_G(\Pi^{\mathfrak{s}})$ (see for instance [Roc02, § 1.1]).

Let $\mathfrak{s} = [M, \sigma]_G \in \mathfrak{B}(G)$ and let M_1 denote the subgroup of M generated by all compacts subgroups, M_1 is the intersection of kernels of unramified characters of M. Let V be the underlying vector space for the supercuspidal representation σ of M and σ_1 an irreducible component of the restriction of σ to M_1. We denote by $\mathrm{ind}_{M_1}^M$ the functor of compact induction. As noticed in [Roc02, § 1.2], the isomorphism class of

$$\Pi_M^{\mathfrak{s}_M} := \mathrm{ind}_{M_1}^M(\sigma_1) \tag{2.1.1}$$

is independent of the choice of σ_1. It was shown by Bernstein that

$$\Pi_G^{\mathfrak{s}} := i_P^G(\Pi_G^{\mathfrak{s}_M}) \tag{2.1.2}$$

is a progenerator of $\mathfrak{R}^{\mathfrak{s}}(G)$ (see [Roc02, §1.6]). We write

$$\mathcal{H}^{\mathfrak{s}}(G) := \mathrm{End}_G(\Pi_G^{\mathfrak{s}}). \tag{2.1.3}$$

Hence we have an equivalence of categories of right modules

$$\mathfrak{R}^{\mathfrak{s}}(G) \simeq \mathcal{H}^{\mathfrak{s}}(G) - \mathrm{Mod}\,. \tag{2.1.4}$$

We set $B := \mathbb{C}[M/M_1]$ and $V_B := V \otimes_{\mathbb{C}} B$. Then $i_P^G(V_B)$ is also a progenerator of $\mathfrak{R}^{\mathfrak{s}}(G)$. Thus we have the equivalence of categories of right modules

$$\begin{array}{rccc} \mathcal{E}: & \mathfrak{R}^{\mathfrak{s}}(G) & \longrightarrow & \mathrm{End}_G(i_P^G(V_B)) - \mathrm{Mod} \\ & \mathcal{V} & \mapsto & \mathrm{Hom}_G(i_P^G(V_B), \mathcal{V}) \end{array}. \tag{2.1.5}$$

We write

$$\mathfrak{X}_{\mathrm{nr}}(M, \sigma) := \{\chi \in \mathfrak{X}_{\mathrm{nr}}(M) \,:\, \chi \otimes \sigma \simeq \sigma\}. \tag{2.1.6}$$

It is a finite subgroup of $\mathfrak{X}_{\mathrm{nr}}(M)$.

Remark 2.1.7. In the case where $M = \mathrm{GL}_n(F)$ with n a positive integer, the supercuspidal representation σ contains a simple type (J, λ) (as defined in [BK93, (5.5.10)]) and the order of $\mathfrak{X}_{\mathrm{nr}}(M, \sigma)$ is $n/e(L|F)$, where $e(L|F)$ is the ramification index of the extension L/F involved in the definition of (J, λ) (see [BK93, (6.0.1) and (6.2.5)].

We denote by $\mathcal{O}$ the orbit of σ under the action of $\mathfrak{X}_{\mathrm{nr}}(M)$. The map $\chi \mapsto \chi \otimes \sigma$ defines a bijection

$$\mathfrak{X}_{\mathrm{nr}}(M)/\mathfrak{X}_{\mathrm{nr}}(M, \sigma) \to \mathcal{O} = \{\chi \otimes \sigma \,:\, \chi \in \mathfrak{X}_{\mathrm{nr}}(M)\} = \mathrm{Irr}^{\mathfrak{s}_M}(M). \tag{2.1.8}$$

We set $W(M) := \mathrm{N}_G(M)/M$ and define

$$W_G^{\mathfrak{s}} := W(M, \mathcal{O}) := \{n \in \mathrm{N}_G(M) \,:\, {}^n\mathcal{O} \simeq \mathcal{O}\}/M. \tag{2.1.9}$$

Recall that $\mathbf{A}_M$ is the maximal split torus contained in the center of $\mathbf{M}$. We denote by $\Sigma(A_M) \subset X^*(A_M)$ the set of nonzero weights occurring in the adjoint representation of A_M on the Lie algebra of G, and by $\Sigma_{\mathrm{red}}(A_M)$ be the set of indivisible elements therein.

For every $\gamma \in \Sigma_{\mathrm{red}}(A_M)$, let $M_\gamma \supset M$ denote the centralizer of $\ker \gamma$ in G (it is a Levi subgroup of G whose semisimple rank is one higher than that of M). Let μ^G be the Harish-Chandra μ-function for G (see [Sil79, §1] or [Wal03, §V.2]). The restriction of μ^G to $\mathcal{O}$ is a rational $W(M, \mathcal{O})$-invariant function on $\mathcal{O}$ [Wal03, Lemma V.2.1]. The set

$$\Sigma_{\mathcal{O},\mu} := \left\{\gamma \in \Sigma_{\mathrm{red}}(A_M) \,:\, \mu^{M_\gamma} \text{ has a zero on } \mathcal{O}\right\}, \tag{2.1.10}$$

is a root system (see [Hei11, Proposition1.3]). Let $W_{\mathcal{O}}$ denote the Weyl group of $\Sigma_{\mathcal{O},\mu}$.

Let $P = MN$ be a parabolic subgroup of G with Levi factor M. The group $W(M, \mathcal{O})$ decomposes as

$$W(M, \mathcal{O}) = W_{\mathcal{O}} \rtimes R(\mathcal{O}), \tag{2.1.11}$$

where

$$R(\mathcal{O}) := \{w \in W(M, \mathcal{O}) \,:\, w(\Sigma_{\mathcal{O},\mu}(P)) = \Sigma_{\mathcal{O},\mu}(P)\}, \tag{2.1.12}$$

with $\Sigma_{\mathcal{O},\mu}(P)$ the set of positive roots of $\Sigma_{\mathcal{O},\mu}$ determined by P (see [Hei11, 1.12]).

The action of every element w of $W_G^{\mathfrak{s}}$ can be lifted to a transformation $\widetilde{w}$ of $\mathfrak{X}_{\mathrm{nr}}(M)$. Let $W(M, \sigma, \mathfrak{X}_{\mathrm{nr}}(M))$ be the group of permutations of $\mathfrak{X}_{\mathrm{nr}}(M)$ generated by $\mathfrak{X}_{\mathrm{nr}}(M, \sigma)$ and the $\widetilde{w}$'s. It satisfies

$$W(M, \sigma, \mathfrak{X}_{\mathrm{nr}}(M))/\mathfrak{X}_{\mathrm{nr}}(M, \sigma) \simeq W_G^{\mathfrak{s}}.$$

Let $K(B) := \mathbb{C}(\mathfrak{X}_{\mathrm{nr}}(M))$ denote the quotient field of $B := \mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(M)]$. Solleveld showed in [Sol21a, Corollary 5.8] the existence of a 2-cocycle

$$\kappa \colon W(M, \sigma, \mathfrak{X}_{\mathrm{nr}}(M))^2 \to \mathbb{C}^\times \tag{2.1.13}$$

(defined in [Sol21a, Lemma 5.7], where it is denoted by $\natural$) and of an algebra isomorphism

$$K(B) \otimes_B \mathrm{End}_G(i_P^G(V_B) \simeq K(B) \rtimes \mathbb{C}[W(M, \sigma, \mathfrak{X}_{\mathrm{nr}}(M)), \kappa],$$

where $\mathbb{C}[W(M,\sigma,\mathfrak{X}_{\mathrm{nr}}(M)),\kappa]$ is the twisted group algebra of $W(M,\sigma,\mathfrak{X}_{\mathrm{nr}}(M))$ (it has basis elements t_w that multiply as $t_w t_{w'} = \kappa(w,w')t_{ww'}$) and the symbol $\rtimes$ denotes a crossed product: as vector space it just means the tensor product, and the multiplication rules on that are determined by the action of $W(M,\sigma,\mathfrak{X}_{\mathrm{nr}}(M))$ on $K(B)$. The cocycle κ is trivial on $W_{\mathcal{O}}$.

Remark 2.1.14. If $R(\mathcal{O})$ has order at most 2, then the intertwining operators can be normalized such that the cocycle κ is trivial (see [Sol21a, Proposition 5.2 & above Lemma 5.7]).

For any $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$, let $W_G^{\mathfrak{s},\chi\otimes\sigma}$ denote the stabilizer in $W_G^{\mathfrak{s}}$ of $\chi\otimes\sigma$ and $\natural_\chi$ the restriction to $W_G^{\mathfrak{s},\chi\otimes\sigma}$ of the 2-cocycle $\natural_u$ defined in [Sol21a, (6.18)].

Let $(\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural$ be the twisted extended quotient with respect to the collection $\natural$ of the 2-cocycles $\natural_\chi$.

Proposition 2.1.15. *There is a bijection*

$$\xi_G^{\mathfrak{s}}\colon \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural. \tag{2.1.16}$$

Proof. First [Sol21a, Theorem 9.7], provides bijections

$$\mathrm{Irr}^{\mathfrak{s}}(G) \xrightarrow{\mathcal{E}} \mathrm{Irr}(\mathrm{End}_G(i_P^G(V_B)) \xrightarrow{\zeta} \mathrm{Irr}(\mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(M)] \rtimes \mathbb{C}[W(M,\sigma,\mathfrak{X}_{\mathrm{nr}}(M)),\kappa],$$

where $\mathcal{E}$ is induced by the equivalence of categories defined in (2.1.5). On the other hand, [Sol21a, Lemma 9.8] shows that $\mathrm{Irr}(\mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(M)] \rtimes \mathbb{C}[W(M,\sigma,\mathfrak{X}_{\mathrm{nr}}(M)),\kappa]$ is canonically isomorphic to $(\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural$, where $\mathfrak{s}_M := [M,\sigma]_M$. □

Corollary 2.1.17. *Let $\mathfrak{s} = [M,\sigma]_G \in \mathfrak{B}(G)$. There is a bijection*

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}}(G)) \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural.$$

Proof. The result follows from the proof of Proposition 2.1.15 by using (2.1.4). □

Remark 2.1.18. As observed in [Sol21a, (10.12)], if the restriction of σ to M_1 is multiplicity free, we have

$$\Pi_G^{\mathfrak{s}} = \big(i_P^G(V_B)\big)^{\mathfrak{X}_{\mathrm{nr}}(M,\sigma)} \quad \text{and} \quad \mathrm{End}_G(i_P^G(V_B)) \simeq \mathcal{H}^{\mathfrak{s}}(G) \otimes_{\mathbb{C}} \mathrm{M}_{[M:M_\sigma]}(\mathbb{C}).$$

Note that if σ is generic, then its restriction to M_1 is multiplicity free (see [Roc09, Remark 1.6.1.3]). In particular, if σ is a supercuspidal irreducible representation of a proper Levi subgroup M of G_2, then its restriction to M_1 is multiplicity free.

2.2. **Theory of types.** We fix a Haar measure on G, write $\mathcal{H}(G)$ for the space of locally constant, compactly supported functions $f\colon G\to\mathbb{C}$ and view $\mathcal{H}(G)$ as a $\mathbb{C}$-algebra via convolution relative to the Haar measure.

Let (ρ,V_ρ) be a smooth representation of a compact open subgroup K of G, and let $(\widetilde{\rho},V_{\widetilde{\rho}})$ denote its contragredient. We define $\mathcal{H}(G,\rho)$ to be the space of compactly supported functions $f\colon G\to \mathrm{End}_G(V_{\widetilde{\rho}})$ such that

$$f(kgk') = \widetilde{\rho}(k)f(g)\widetilde{\rho}(k'), \quad \text{where } k,k'\in K \text{ and } g\in G. \tag{2.2.1}$$

The convolution product gives $\mathcal{H}(G,\rho)$ the structure of a unitary associative $\mathbb{C}$-algebra.

Let $e_\rho \in \mathcal{H}(G)$ be the function defined by

$$e_\rho(g) := \begin{cases} \frac{\dim \rho}{\mathrm{meas}(K)} \operatorname{tr}(\rho(g^{-1})) & \text{if } g \in K, \\ 0 & \text{if } g \in G,\ g \notin K. \end{cases}$$

Then e_ρ is idempotent, and $e_\rho \star \mathcal{H}(G) \star e_\rho$ is a sub-algebra of $\mathcal{H}(G)$ with unit e_ρ.

Bushnell and Kutzko defined in [BK98, (2.12)] a canonical isomorphism:

$$\mathcal{H}(G,\rho) \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(V_\rho) \ \to\ e_\rho \star \mathcal{H}(G) \star e_\rho.$$

The algebras $\mathcal{H}(G,\rho)$ and $e_\rho \star \mathcal{H}(G) \star e_\rho$ are therefore canonically Morita equivalent. Hence, we get an equivalence of categories:

$$\mathcal{H}(G,\rho) - \mathrm{Mod} \ \simeq\ e_\rho \star \mathcal{H}(G) \star e_\rho - \mathrm{Mod}\,. \tag{2.2.2}$$

We write $\mathfrak{R}_\rho(G)$ for the full sub-category of $\mathfrak{R}(G)$ whose objects are those V satisfying $V = \mathcal{H}(G) \star e_\rho \star V$, that is, $\mathfrak{R}_\rho(G)$ is generated over G by the subspace $e_\rho \star V$.

Definition 2.2.3. The pair (K,ρ) is called an $\mathfrak{s}$-type for G if the category $\mathfrak{R}_\rho(G)$ is closed by subquotients. A supercuspidal type for G is an $\mathfrak{s}$-type such that is $\mathfrak{s} = [G,\sigma]_G$.

If (K,ρ) is an $\mathfrak{s}$-type for G, then we have $\mathfrak{R}_\rho(G) = \mathfrak{R}^{\mathfrak{s}}(G)$ (see [BK98, (4.1)–(4.2)]) and the latter is equivalent to the category of modules of $\mathcal{H}(G,\rho)$ (see [BK98, Theorem 3.5]):

$$\mathfrak{R}^{\mathfrak{s}}(G) \ \simeq\ \mathcal{H}(G,\rho) -\ \mathrm{Mod}\,. \tag{2.2.4}$$

By combining (2.2.4) and (2.1.4), we obtain an equivalence

$$\mathcal{H}^{\mathfrak{s}}(G) - \mathrm{Mod} \ \simeq\ \mathcal{H}(G,\rho)\ -\ \mathrm{Mod}\,. \tag{2.2.5}$$

Let (K_M,ρ_M) be a $\mathfrak{s}_M$-type for $\mathfrak{s}_M \in \mathfrak{B}(M)$. If the pair (K,ρ) is a G-cover of (K_M,ρ_M) as defined in [BK98, Definition 8.1], then K decomposes with respect to M in the sense of [BK98, Definition 6.1] (in particular, $K_M = K \cap M$ and $\rho_M = \rho|_{K_M}$) and the equivalence of categories (2.2.4) commutes with parabolic induction and parabolic restriction in the appropriate sense (see [BK98, Corollary 8.4]).

Proposition 2.2.6. *Let (K_M,ρ_M) be a $\mathfrak{s}_M$-type for $\mathfrak{s}_M \in \mathfrak{B}(M)$, which has the property that $\Pi_M^{\mathfrak{s}_M} \simeq \text{c-Ind}_{K_M}^M(\rho_M, V_{\rho_M})$, and let (K,ρ) be a G-cover of (K_M,ρ_M).*

Then we have

$$\Pi_G^{\mathfrak{s}} \simeq \text{c-Ind}_K^G(\rho, V_\rho),$$

and, as a consequence,

$$\mathcal{H}^{\mathfrak{s}}(G) := \operatorname{End}_G(\Pi_G^{\mathfrak{s}}) \simeq \mathcal{H}(G,\rho). \tag{2.2.7}$$

Proof. See [BS21, Lemma B.3]. □

2.3. **The Kim-Yu types.**

In this section we suppose that $\mathbf{G}$ splits over a tamely ramified extension of F and that p does not divide the order of the Weyl group of G. By a Levi subgroup of $\mathbf{G}$, we mean an F-subgroup of $\mathbf{G}$ which is a Levi factor of a parabolic F-subgroup of $\mathbf{G}$. Let L/F be be a finite extension. By twisted L-Levi subgroup of $\mathbf{G}$ we mean an F-subgroup $\mathbf{G}'$ of $\mathbf{G}$ such that $\mathbf{G}' \otimes_F L$ is a Levi subgroup of $\mathbf{G} \otimes_F L$. If L/F is tamely ramified, the $\mathbf{G}'$ is called a tamely ramified twisted Levi subgroup of $\mathbf{G}$. A tamely ramified twisted Levi sequence in $\mathbf{G}$ is a finite sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \cdots, \mathbf{G}^d)$ of twisted E-Levi subgroups of $\mathbf{G}$, with E/F tamely ramified [Yu01, p 586].

Let $\mathcal{B}(\mathbf{G}, F)$ denote the (enlarged) building of G:

$$\mathcal{B}(\mathbf{G}, F) = \mathcal{B}(\mathbf{G}/\mathrm{Z}_{\mathbf{G}}, F) \times X_*(\mathrm{Z}_G) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{2.3.1}$$

where $X_*(\mathrm{Z}_G)$ is the set of F-algebraic cocharacters of Z_G. We recall that if $\mathbf{G}'$ is a ramified twisted Levi subgroup of $\mathbf{G}$, then there is a family of natural embeddings of $\mathcal{B}(\mathbf{G}', F)$ into $\mathcal{B}(\mathbf{G}, F)$.

If x is a point in $\mathcal{B}(\mathbf{G}, F)$, we denote by $G_{x,0}$ the associate parahoric subgroup, and by $G_{x,0^+}$ the pro-unipotent radical of the latter. In general, for r a positive real number, $G_{x,r}$ is the corresponding Moy-Prasad filtration subgroup of $G_{x,0}$.

As in [KY17, § 7.1], a *depth-zero G-datum* is a triple $((\mathbf{G}, \mathbf{M}), (y, \iota), (K_M, \rho_M))$ such that

- $\mathbf{G}$ is a connected reductive group over F and $\mathbf{M}$ a Levi subgroup of $\mathbf{G}$;
- $y \in \mathcal{B}(M)$ is such that $M_{y,0}$ is a maximal parahoric subgroup of M, and $\iota\colon \mathcal{B}(M) \hookrightarrow \mathcal{B}(G)$ is a 0-generic embedding relative to y (see [KY17, Definition 3.2]);
- K_M is a compact open subgroup of M containing $M_{y,0}$, and ρ_M is an irreducible smooth representation of K_M such that $\rho_M | M_{y,0}$ contains a cuspidal representation of $M_{y,0}/M_{y,0^+}$.

Let $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \cdots, \mathbf{G}^d)$ be a tamely ramified twisted Levi sequence in $\mathbf{G}$. To $\vec{\mathbf{G}}$, we associate a sequence of Levi subgroups $\vec{\mathbf{M}} = (\mathbf{M}^0, \cdots, \mathbf{M}^d)$, where $\mathbf{M}^i$ is a Levi subgroup of $\mathbf{G}^i$ given as the centralizer of $A_{\mathbf{M}^0}$ in $\mathbf{G}^i$, with $A_{\mathbf{M}^0}$ the maximal F-split torus of the center $\mathrm{Z}_{\mathbf{M}^0}$ of M^0.

Then a $\mathbf{G}$-*datum* is a 5-tuple

$$\mathcal{D} = ((\vec{\mathbf{G}}, \mathbf{M}^0), (y, \{\iota\}), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}) \tag{2.3.2}$$

satisfying the following:

D1. $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \cdots, \mathbf{G}^d)$ is a tamely ramified twisted Levi sequence in $\mathbf{G}$, and $\mathbf{M}^0$ a Levi subgroup of $\mathbf{G}^0$. Let $\vec{\mathbf{M}}$ be associated to $\vec{\mathbf{G}}$ as above;

D2. y is a point in $\mathcal{B}(M^0)$ and $\{\iota\}$ is a commutative diagram of $\vec{s}$-generic embeddings of buildings relative to y (see [KY17, Definition 3.5]), where $\vec{s} = (0, r_0/2, \cdots, r_{d-1}/2)$;

D3. $\vec{r} = (r_0, r_1, \cdots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \cdots < r_{d-1} \leq r_d$ if $d > 0$, and $0 \leq r_0$ if $d = 0$;

D4. (K_{M^0}, ρ_{M^0}) is such that $\mathcal{D}^0 := ((\mathbf{G}^0, \mathbf{M}^0), (y, \iota), (K_{M^0}, \rho_{M^0}))$ is a depth zero datum;

D5. $\vec{\phi} = (\phi_0, \phi_1, \cdots, \phi_d)$ is a sequence of quasi-characters, where ϕ_i is a quasi-character of G^i such that ϕ_i is G^{i+1}-generic of depth r_i relative to x for all $x \in \mathcal{B}(G^i)$ (in the terminology of [Yu01, § 9]).

The construction. For a given $\mathbf{G}$-datum $\mathcal{D}$ as above, we write

$$K_{\mathcal{D}^0} := K_{M^0} G^0_{\iota(y),0}. \tag{2.3.3}$$

We have

$$K_{\mathcal{D}^0}/G^0_{\iota(y),0+} \simeq K_{M^0}/M^0_{y,0+}, \tag{2.3.4}$$

and we define $\rho_{\mathcal{D}^0}$ to be the representation of $K_{\mathcal{D}^0}$ obtained by composing the isomorphism (2.3.4) with ρ_{M^0}.

Following the recipe in [Yu01], Kim and Yu constructed in [KY17, §7.4] a pair $(K_{\mathcal{D}}, \rho_{\mathcal{D}})$ formed by an open compact subgroup

$$K_{\mathcal{D}} := K_{\mathcal{D}^0} G^1_{\iota(y),s_0} \cdots G^d_{\iota(y),s_{d-1}} \tag{2.3.5}$$

and an irreducible representation $\rho_{\mathcal{D}}$ of K^d.

To $\vec{\mathbf{G}}$, we associate a tamely ramified twisted Levi sequence $\vec{\mathbf{M}} = (\mathbf{M}^0, \ldots, \mathbf{M}^d)$ of $\mathbf{M}$, where $\mathbf{M}^i$ is the centralizer of $A_{\mathbf{M}}$ in $\mathbf{G}^i$. Then we write

$$\mathcal{D}_M := (\vec{\mathbf{M}}, y, \vec{r}, \rho_{M^0}, \vec{\phi}). \tag{2.3.6}$$

It is a datum for constructing a supercuspidal type in M.

Let $K^d_M := K_{\mathcal{D}} \cap M$ and let $\widetilde{K}^d_M$ denote the normalizer in M of K^d_M, this is a compact mod center subgroup of M. We write $\rho^d_M := \rho_{\mathcal{D}}|_{K^d_M}$ and

$$\sigma_{\mathcal{D}_M} := \mathrm{ind}^M_{\widetilde{K}_M} \rho^d_M. \tag{2.3.7}$$

Theorem 2.3.8. ([KY17, Fin21]) *Suppose that $K_{M^0} = M^0_y$. Then*

1. *(K^d_M, ρ^d_M) is a supercuspidal type on M^d (as defined in Definition 2.2.3) and $\sigma_{\mathcal{D}_M}$ is an irreducible supercuspidal representation of M;*
2. *$(K_{\mathcal{D}}, \rho_{\mathcal{D}})$ is a G-cover of (K^d_M, ρ^d_M) and hence it is an $\mathfrak{s}$-type, where $\mathfrak{s} = [M, \sigma_{\mathcal{D}_M}]_G$.*

Proposition 2.3.9. *Let $\mathcal{D}$ and $\dot{\mathcal{D}}$ be two G-data*

$$\mathcal{D} = ((\vec{\mathbf{G}}, \mathbf{M}^0), (y, \iota), \vec{r}, (K_{M^0}, \rho_{M^0}), \vec{\phi}) \text{ and } \dot{\mathcal{D}} = ((\dot{\vec{\mathbf{G}}}, \dot{\mathbf{M}}^0), (\dot{y}, i), \dot{\vec{r}}, (K_{\dot{M}^0}, \rho_{\dot{M}^0}), \dot{\vec{\phi}})$$

such that $K_{M^0} = M^0_y$ and $K_{\dot{M}^0} = \dot{M}^0_{\dot{y},0}$. Let $\mathfrak{s} := [M, \sigma_{\mathcal{D}_M}]_G$, and $\dot{\mathfrak{s}} = [M, \sigma_{\dot{\mathcal{D}}_M}]_G$.

Then we have $\mathfrak{s} = \dot{\mathfrak{s}}$ if and only if there exists $g \in G$ such that

$${}^g K_{M^0} = K_{\dot{M}^0} \quad \text{and} \quad {}^g(\rho_{M^0} \otimes \phi) \simeq \rho_{\dot{M}^0} \otimes \dot{\phi}, \tag{2.3.10}$$

where $\phi := \prod_{i=0}^d (\phi_i|_{M^0})$ and $\dot{\phi} := \prod_{i=0}^d (\dot{\phi}_i|_{M^0})$.

Proof. It is a reformulation of [KY17, Theorem 10.3]. Indeed, due to the conditions $K_{M^0} = M^0_y$ and $K_{\dot{M}^0} = \dot{M}^0_{\dot{y},0}$, we have $\mathfrak{s} = \dot{\mathfrak{s}}$ if and only the types $(K_{\mathcal{D}}, \rho_{\mathcal{D}})$ and $(K_{\dot{\mathcal{D}}}, \rho_{\dot{\mathcal{D}}})$ are equivalent in the sense of [KY17, Definition 10.1].

Note that [KY17, Theorems 10.2 and 10.3] still hold without assuming the hypothesis $C(\vec{\mathbf{G}})$ of [HM08, Remark 2.49 & above], since [Kal19a, §3.5] shows that [HM08, Theorems 6.6 and 6.7] are valid without assuming $C(\vec{\mathbf{G}})$. □

Remark 2.3.11. If $G = M$, it follows from [HM08, Theorems 6.6 and 6.7] that $\mathfrak{s}_M = \dot{\mathfrak{s}}_M$ if and only the data $\mathcal{D}_M = (\vec{\mathbf{M}}, y, \vec{r}, \rho_{M^0}, \vec{\phi})$ and $\dot{\mathcal{D}}_M = (\vec{\mathbf{M}}, \dot{y}, \vec{\dot{r}}, \rho_{\dot{M}^0}, \vec{\dot{\phi}})$ are equivalent in the sense of [HM08, Definition 5.3].

For any $\mathfrak{s} = [M, \sigma] \in \mathfrak{B}(G)$, we define

$$\mathrm{N}_G^{\mathfrak{s}} := \{n \in \mathrm{N}_{\mathbf{G}}(\mathbf{M})(F) \,:\, {}^n\sigma \simeq \chi \otimes \sigma \ \text{ for some } \chi \in \mathfrak{X}_{\mathrm{nr}}(M)\}\,. \tag{2.3.12}$$

Corollary 2.3.13. *For every $n \in \mathrm{N}_G^{\mathfrak{s}}$ there exists $m \in M$ such that*

$$ {}^{mn}K_{M^0} = K_{M^0} \quad \textit{and} \quad {}^{mn}(\rho_{M^0} \otimes \phi) \simeq \rho_{M^0} \otimes \phi,$$

where $\phi := \prod_{i=0}^{d}(\phi_i|_{M^0})$.

Proof. We have $\sigma = \sigma_{\mathcal{D}_M}$, for an M-datum $\mathcal{D}_M = (\vec{\mathbf{M}}, y, \vec{r}, \rho_{M^0}, \vec{\phi})$. Let $n \in \mathrm{N}_G^{\mathfrak{s}}$. Thus ${}^n\sigma \simeq \chi \otimes \sigma$ for some $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$. We write $\dot{\mathcal{D}}_M := (\vec{\mathbf{M}}, y, \vec{r}, \rho_{M^0} \otimes \chi|_{K_{M^0}}, \vec{\phi})$. From (2.3.7), we have

$$\chi \otimes \sigma = (\mathrm{ind}_{\widetilde{K}_M}^{M} \rho_M^d) \otimes \chi \simeq \mathrm{ind}_{\widetilde{K}_M}^{M}(\rho_M^d \otimes \chi|_{\widetilde{K}_M}).$$

Since χ is unramified, we have $\mathrm{ind}_{\widetilde{K}_M}^{M}(\rho_M^d \otimes \chi|_{\widetilde{K}_M}) = \sigma_{\dot{\mathcal{D}}_M}$, and hence $\chi \otimes \sigma \simeq \sigma_{\dot{\mathcal{D}}_M}$. By applying Remark 2.3.11 to the M-data ${}^n\mathcal{D}_M$ and $\dot{\mathcal{D}}_M$, we see that these data are equivalent. Hence there exists $m \in M$ such that

$$ {}^{mn}K_{M^0} = K_{M^0} \quad \text{and} \quad {}^{mn}(\rho_{M^0} \otimes \phi) \simeq \rho_{M^0} \otimes \chi|_{K_{M^0}} \otimes \phi = \rho_{M^0} \otimes \phi,$$

since χ is trivial on K_{M^0}, where $\phi := \prod_{i=0}^{d}(\phi_i|_{M^0})$. □

As shown in [HM08], it follows from Yu's construction [Yu01] that ρ_M^d is of the form $\rho_M^d = \rho_{M^0} \otimes \kappa$, where the representation $\kappa = \kappa_G$ depends only on $\vec{\phi}$.

Recall that we have assumed that K_{M^0} is the fixator of y in M^0. We denote by $\widetilde{K}_{M^0}$ the normalizer in M^0 of K_{M^0}, and define

$$\sigma^0 := \mathrm{ind}_{\widetilde{K}_{M^0}}^{M^0} \rho_{M^0}. \tag{2.3.14}$$

The representation σ^0 is a depth-zero irreducible supercuspidal representation of M^0.

We write $\mathfrak{s}^0 = [M^0, \sigma^0]_{G^0}$ and define $\Pi_{G^0}^{\mathfrak{s}^0}$ as in (2.1.2): it a progenerator of the category $\mathfrak{R}^{\mathfrak{s}^0}(G^0)$. We set $\mathcal{H}^{\mathfrak{s}^0}(G^0) := \mathrm{End}(\Pi_{G^0}^{\mathfrak{s}^0})$.

Proposition 2.3.15.

(1) *The algebras $\mathcal{H}^{\mathfrak{s}^0}(G^0)$ and $\mathcal{H}(G^0, \rho_{\mathcal{D}^0})$ are isomorphic.*
(2) *The algebras $\mathcal{H}^{\mathfrak{s}}(G)$ and $\mathcal{H}(G, \rho_{\mathcal{D}})$ are isomorphic.*

Proof. We will check that the assumptions in Proposition 2.2.6 are satisfied. Firstly, $(K_{\mathcal{D}^0}, \rho_{\mathcal{D}^0})$ is a G^0-cover of (K_{M^0}, ρ_{M^0}) (see [KY17, §7.1]) and $(K_{\mathcal{D}}, \rho_{\mathcal{D}})$ is a G-cover of (K_M^d, ρ_M^d) (see [KY17, Theorem 7.5]).

Secondly, since σ^0 and σ are supercuspidal irreducible representations, an element m^0 of M^0 intertwines ρ_{M^0} if and only if $m^0 \in \widetilde{K}_{M^0}$, and an element m of M intertwines ρ_M if and only if $m \in \widetilde{K}_M$. Then the proof of [BS21, Lemma B.4] applies, and shows that $\Pi_M^{\mathfrak{s}_M} \simeq \text{c-Ind}_{K_M}^M(\rho_M, V_{\rho_M})$. Then the result follows from Proposition 2.2.6. □

Proposition 2.3.16. *If $W_G^{\mathfrak{s}} = \{1\}$, then there is an algebra isomorphism*

$$\mathcal{H}(G, \rho_{\mathcal{D}}) \simeq \mathcal{H}(M, \rho_M^d),$$

which preserves support of functions, and the algebra $\mathcal{H}(G, \rho_{\mathcal{D}})$ is commutative.

Proof. Since $W_G^{\mathfrak{s}} = \{1\}$, we have $\mathrm{N}_G(\mathfrak{s}) \subset M$. Then the first assertion follows from [BK98, (12.1)]. On the other hand, the algebra $\mathcal{H}(M, \rho_M^d)$ is commutative (see for instance [BK98, (5.6)]). □

Remark 2.3.17. Applying Proposition 2.3.16 to the group G^0 shows that, if $W_{G^0}^{\mathfrak{s}^0} = \{1\}$, then there is an algebra isomorphism

$$\mathcal{H}(G^0, \rho_{\mathcal{D}^0}) \simeq \mathcal{H}(M^0, \rho_{M^0}^0),$$

which preserves support of functions, and the algebra $\mathcal{H}(G^0, \rho_{\mathcal{D}^0})$ is commutative.

2.4. **Regular Bernstein blocks.** We assume that p is odd, good for G and does not divide the order of the fundamental groups of $\mathbf{G}_{\mathrm{der}}$. Let $\mathbf{M}$ be an F-rational Levi subgroup of an F-rational parabolic subgroup of $\mathbf{G}$. Then p satisfies the same assumptions relative to $\mathbf{M}$.

Let $(\mathbf{S}, \theta)$ be a pair consisting of a tamely ramified torus $\mathbf{S}$ in $\mathbf{M}$ and a character $\theta\colon S \to \mathbb{C}^\times$. For any positive real number r, we set

$$\Sigma_r^{\mathbf{S},\theta} := \left\{\gamma \in \Sigma(\mathbf{M}, \mathbf{S}) \,:\, (\theta \circ \mathrm{N}_{E/F})(\gamma^\vee(E_r^\times)) = 1\right\}. \tag{2.4.1}$$

We have $\Sigma_s^{\mathbf{S},\theta} \subset \Sigma_r^{\mathbf{S},\theta}$ for $s < r$, and we set $\Sigma_{r+}^{\mathbf{S},\theta} := \bigcap_{s>r} \Sigma_s^{\mathbf{S},\theta}$. Then $r \mapsto \Sigma_r^{\mathbf{S},\theta}$ defines a $\mathrm{Gal}(F_{\mathrm{sep}}/F)$-invariant filtration. Let $r_{d-1} > r_{d-2} > \cdots > r_0 > 0$ denote the breaks of the filtration, that is, the r such that $\Sigma_r^{\mathbf{S},\theta} \neq \Sigma_{r+}^{\mathbf{S},\theta}$. We set $r_{-1} := 0$ and write r_d for the depth of θ. We have $r_d \geq r_{d-1}$. For each i such that $0 \leq i \leq d$, we denote by $\mathbf{M}^i$ the connected reductive subgroup of $\mathbf{M}$ with maximal torus $\mathbf{S}$ and root system $\Sigma_{r_{i-1}+}^{\mathbf{S},\theta}$. By definition the root system of $\mathbf{M}^d$ is $\Sigma(\mathbf{M}, \mathbf{S})$, so $\mathbf{M}^d = \mathbf{M}$. The $\mathbf{M}^i$'s are tame twisted Levi subgroups of $\mathbf{M}$ by [Kal19a, Lemma 3.6.1]. We write $M^i := \mathbf{M}^i(F)$. Moreover, the root system of $\mathbf{M}^0$ is $\Sigma_{0+}^{\mathbf{S},\theta}$, and, if the latter is empty, then we have $\mathbf{M}^0 = \mathbf{S}$.

As proved by Kaletha in [Kal19a, Proposition 3.6.7], the pair $(\mathbf{S}, \theta)$ has a Howe factorization with respect of a sequence $(\phi_{-1}, \phi_0, \ldots, \phi_d)$ of characters $\phi_{-1}\colon S \to \mathbb{C}^\times$ and $\phi_i\colon M^i \to \mathbb{C}^\times$ for $0 \leq i \leq d$. It means that

$$\theta = \prod_{i=-1}^{d} \phi_i, \tag{2.4.2}$$

where the character ϕ_i is trivial on $(\mathbf{M}^i)_{\mathrm{sc}}$ for any $i \in \{0, \ldots, d\}$, has depth r_i and is $\mathbf{M}^{i+1}$-generic for any $i \neq d$, and ϕ_d if trivial if $r_d = r_{d-1}$ and has depth r_d otherwise.

From now one we assume that $\mathbf{S}$ is an elliptic maximal torus of $\mathbf{M}$, the splitting extension of $\mathbf{S}$ is tamely ramified, $\mathbf{S}$ is maximally unramified inside $\mathbf{M}^0$ (that is, S coincides with its maximal unramified subtorus, see [Kal19a, Definition 3.4.2]), and θ is k_F-regular with respect to M^0 in the sense of [Kal19b, Definition 3.1.1].

For any point x in the building of M, we denote by $[x]$ the projection of x on the reduced building $\mathcal{B}_{\mathrm{red}}(M)$, and by M_x (resp. $M_{[x]}$) the subgroup of M fixing x (resp. $[x]$). Recall that $M_{[x]} = \mathrm{N}_M(M_{x,0})$, [Yu01, Lemma 3.3].

Then we can associate to S a vertex $[y]$ of $\mathcal{B}_{\mathrm{red}}(M)$ (see [Kal19a, Lemma 3.4.3]), which is the unique $\mathrm{Gal}(F^{\mathrm{nr}}/F)$-fixed point in the apartment $\mathcal{A}_{\mathrm{red}}(S, F^{\mathrm{nr}})$ of the reduced Bruhat-Tits building of M.

Let S_{b} be the unique maximal bounded subgroup of S (which is also the unique maximal compact subgroup of S). Denoting by $\mathfrak{S}^\circ$ the connected Neron model of $\mathbf{S}$, we write $S_0 := \mathfrak{S}^\circ(\mathfrak{o}_f) \subset S_{\mathrm{b}}$ (see [Kal19a, §3.1] for more details).

We write $\mathbf{G}^0 := \mathbf{G}^0$ and $\mathbf{M}^0 := \mathbf{M}^0$, and we denote by $\mathbb{M}^0_y$ the connected reductive k_F-group such that

$$\mathbb{M}^0_{y,0} := \mathbb{M}^0_{y,0}(k_F) = M^0_{y,0}/M^0_{y,0+}. \tag{2.4.3}$$

There exists an elliptic maximal k_F-torus $\mathbb{S}$ of $\mathbb{M}^0_{y,0}$ such that for every unramified extension F' of F, the image of $\mathbf{S}(F')_0$ in $\mathbf{M}(F')_{y,0}/\mathbf{M}(F')_{y,0+}$ is equal to $\mathbb{S}(k_{F'})$, see [Kal19a, Lemma 3.4.4].

By [Kal19a, Lemma 3.4.14], the character $\phi_{-1}|_{S_0}$ factors through a *regular* character $\overline{\phi_{-1}}$ of $\mathbb{S} := \mathbb{S}(k_F)$ as defined in [Kal19a, Definition 3.4.16] (in particular, $\overline{\phi_{-1}}$ is in general position in [DL76, Definition 5.15] terminology). Then it follows from [DL76, Proposition 7.4,Theorem 8.3] that the Deligne-Lusztig character

$$(-1)^{\mathrm{r}(\mathbb{M}^0_{y,0})-\mathrm{r}(\mathbb{S})} R_{\mathbb{S}}^{\mathbb{M}^0_{y,0}}(\overline{\phi_{-1}}) \tag{2.4.4}$$

can be represented by a cuspidal $\mathbb{M}^0_{y,0}$-module $\boldsymbol{\kappa}_{S,\phi_{-1}}$, where $\mathrm{r}(?)$ denotes the k_F-rank of ?. Then $\boldsymbol{\kappa}_{S,\phi_{-1}}$ is irreducible (see [DL76, Definition 5.15]), and its pull-back to $M^0_{y,0}$ extends uniquely to a representation $\kappa_{S,\phi_{-1}}$ of $SM^0_{y,0}$, and we define

$$\rho_{\mathbf{S},\theta} := \mathrm{Ind}_{SM^0_{y,0}}^{M^0_{[y]}} \kappa_{S,\phi_{-1}} \quad \text{and} \quad \sigma^0 := \text{c-Ind}_{M^0_{[y]}}^{M^0} \rho_{\mathbf{S},\theta}. \tag{2.4.5}$$

Then σ^0 is a depth-zero irreducible regular supercuspidal representation of M^0 (see [Kal19a, Definition 3.4.19 & Proposition 3.4.20]). We set $\mathfrak{s}_{M^0} := [M^0, \sigma^0]_{M^0}$.

More generally, we define an irreducible supercuspidal representation σ of M by using the twisted Yu construction of [FKS21]. As observed in [Kal19b, §3.4], it has the same effect as using the original Yu construction from [Yu01] applied to the character $\theta \cdot \epsilon$, where $\epsilon\colon S \to \{\pm 1\}$ is the product of the characters $\epsilon_y^{M^i/M^{i-1}}$ of [FKS21, Theorem 3.4]. The representation σ is regular (i.e., satisfies [Kal19a, Definition 3.7.13]). Then $\chi \otimes \sigma$ is regular for any $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$, and we say that $\mathfrak{s} = [M, \sigma]_G$ is regular.

For $\mathfrak{s}_M = [M, \sigma]_M$ and $\mathfrak{s}_{M^0} = [M^0, \sigma^0]_{M^0}$, the map

$$\mathfrak{f}\colon \begin{array}{ccc} \mathrm{Irr}^{\mathfrak{s}_M}(M) & \longrightarrow & \mathrm{Irr}^{\mathfrak{s}_{M^0}}(M^0) \\ \chi \otimes \sigma & \mapsto & \sigma^0 \otimes \chi|_{M^0} \end{array}, \quad \chi \in \mathfrak{X}_{\mathrm{nr}}(M), \tag{2.4.6}$$

is an isomorphism of varieties (see for [Mis19, Theorem 6.1]). We denote by $\mathcal{O}^0$ the orbit of σ^0 under the action of $\mathfrak{X}_{\mathrm{nr}}(M^0)$ and write $W_{G^0}^{\mathfrak{s}^0} = W_{\mathcal{O}^0} \rtimes R(\mathcal{O}^0)$ for the decomposition analogous to (2.1.11). Then (2.4.6) and (2.1.8) (applied to both $\mathfrak{s}_M$ and $\mathfrak{s}_{M_0}$) show that the orbits $\mathcal{O}$ and $\mathcal{O}^0$ are isomorphic.

The following result is a combination of [AM21, Theorem 7.3] and [AM21, Theorem 9.3].

Theorem 2.4.7. (Adler-Mishra) *We suppose that that p is good for G and does not divide the order of the fundamental group of $\mathbf{G}_{\mathrm{der}}$.*

Let $\mathfrak{s} = [M, \sigma]_G \in \mathfrak{B}(G)$ be a regular inertial class.

Then

(1) *there is a group isomorphism*

$$\mathfrak{w}_\sigma \colon W_G^{\mathfrak{s}} \longrightarrow W_{G^0}^{\mathfrak{s}^0},$$

where $\mathfrak{s}^0 = [M^0, \sigma^0]_{G^0}$, and $\mathfrak{f}$ is equivariant with respect to $\mathfrak{w}_\sigma$.

(2) *there is an isomorphism*

$$\mathfrak{l}_\sigma \colon (\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}})_\natural \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_{M^0}}(M^0)//W_{G^0}^{\mathfrak{s}^0})_{\natural^0}.$$

The collection $\natural^0$ of 2-cocycles is defined as follows. For $x \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$, let $W_G^{\mathfrak{s},x}$ denote the stabilizer of x in $W^{\mathfrak{s}}$. Since $\mathfrak{f}$ is equivariant with respect to $\mathfrak{w}_\sigma$, the latter induces an isomorphism

$$\mathfrak{w}_\sigma|_{W_G^{\mathfrak{s},x}} \colon W_G^{\mathfrak{s},x} \longrightarrow W_{G^0}^{\mathfrak{s}^0,\mathfrak{f}(x)},$$

and every 2-cocycle

$$\natural_x \colon W_G^{\mathfrak{s},x} \times W_G^{\mathfrak{s},x} \longrightarrow \mathbb{C}^\times$$

defines a 2-cocycle

$$\natural^0_{\mathfrak{f}(x)} \colon W_{G^0}^{\mathfrak{s}^0,\mathfrak{f}(x)} \times W_{G^0}^{\mathfrak{s}^0,\mathfrak{f}(x)} \longrightarrow \mathbb{C}^\times. \tag{2.4.8}$$

As a consequence, we prove in Theorem 2.4.9(2) new instances of [AM21, Conjecture 1.1].

Theorem 2.4.9. *We suppose that p is good for G and does not divide the order of the fundamental group of $\mathbf{G}_{\mathrm{der}}$.*

Let $\mathfrak{s} = [M, \sigma]_G \in \mathfrak{B}(G)$ be a regular inertial class.

(1) *Then*

$$(\xi_{G^0}^{\mathfrak{s}^0})^{-1} \circ \mathfrak{l}_\sigma \circ \xi_G^{\mathfrak{s}} \colon \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow \mathrm{Irr}(G^0)^{\mathfrak{s}^0}$$

is a bijection.

(2) *We have a bijection*

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}}(G)) \longrightarrow \mathrm{Irr}(\mathcal{H}^{\mathfrak{s}^0}(G^0)).$$

Proof. (1) It follows from the fact that the map ξ_G defined in (2.1.16) and the analogous map

$$\xi_{G^0}^{\mathfrak{s}^0} \colon \mathrm{Irr}(G)^{\mathfrak{s}^0} \longrightarrow (\mathrm{Irr}^{\mathfrak{s}_{M^0}}(M^0)//W_{G^0}^{\mathfrak{s}^0})_\natural.$$

are isomorphisms.

(2) By [Sol21a, Theorem 9.7] applied to G and G^0, we have

$$\mathrm{Irr}^{\mathfrak{s}}(G) \cong \mathrm{Irr}(\mathrm{End}(i_P^G(V_B)) \quad \text{and} \quad \mathrm{Irr}^{\mathfrak{s}^0}(G^0) \cong \mathrm{Irr}(\mathrm{End}(i_{P_0}^{G^0}(V_{B^0})).$$

Thus by (1), we have

$$\mathrm{Irr}(\mathrm{End}(i_P^G(V_B)) \cong \mathrm{Irr}^{\mathfrak{s}}(G) \cong \mathrm{Irr}^{\mathfrak{s}^0}(G^0) \cong \mathrm{Irr}(\mathrm{End}(i_{P^0}^{G^0}(V_{B^0})).$$

Then the result follows by applying Corollary 2.1.17 to both $\mathfrak{s}$ and $\mathfrak{s}^0$. □

Remark 2.4.10. The algebras $\mathcal{H}^{\mathfrak{s}}(G)$ and $\mathcal{H}^{\mathfrak{s}^0}(G^0)$ are not always isomorphic as [GR05, Example 11.8] shows for $G = \mathrm{SL}_n(F)$. However, we will show in Theorem 4.5.2 that they are isomorphic in the case $G = G_2$ and M is a maximal Levi subgroup.

3. A Langlands correspondence for non-supercuspidal Bernstein blocks

3.1. The construction of the correspondence.

Let $G^\vee$ denote the Langlands dual group of G, that is, the complex Lie group with root system dual to that of G. Let $\mathrm{Z}_{G^\vee}$ be the center of $G^\vee$ and $G^\vee_{\mathrm{ad}}$ the quotient $G^\vee/\mathrm{Z}_{G^\vee}$. The L-group of G is defined to be ${}^LG := G^\vee \rtimes W_F$.

Similarly, $M^\vee$ denotes the Langlands dual group of M. We write $\mathrm{Z}_{M^\vee \rtimes I_F}$ for the center of $M^\vee \rtimes I_F$, and define

$$\mathfrak{X}_{\mathrm{nr}}({}^LM) := (\mathrm{Z}_{M^\vee \rtimes I_F})^\circ_{W_F}. \tag{3.1.1}$$

The group $\mathfrak{X}_{\mathrm{nr}}({}^LM)$ is naturally isomorphic to the group $\mathfrak{X}_{\mathrm{nr}}(M)$. We will denote the isomorphism $\mathfrak{X}_{\mathrm{nr}}(M) \to \mathfrak{X}_{\mathrm{nr}}({}^LM)$ by $\chi \mapsto \chi^\vee$.

An L-parameter is a continuous morphism $\varphi\colon W_F' \to {}^LG$ such that $\varphi(w)$ is semisimple for each $w \in W_F$, and the restriction $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is a morphism of complex algebraic groups. An L-parameter φ is said to be discrete if $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$. The group $G^\vee$ acts on the set of L-parameters and we denote by $\Phi(G)$ the set of $G^\vee$-classes of G-relevant L-parameters.

We attach to each L-parameter φ for G, several (possibly disconnected) complex reductive groups. We write $\mathrm{Z}_{G^\vee}(\varphi) := \mathrm{Z}_{G^\vee}(\varphi(W_F'))$ and denote by $\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\phi)$ the inverse image under the quotient map $G^\vee_{\mathrm{sc}} \to G^\vee_{\mathrm{ad}}$ of $\mathrm{Z}_{G^\vee}(\phi)/\mathrm{Z}_{G^\vee}(\phi) \cap \mathrm{Z}_{G^\vee}$ (viewed as a subgroup of $G^\vee_{\mathrm{ad}}$). Then we set

$$\mathcal{G}_\varphi := \mathrm{Z}^1_{G^\vee}(\varphi|_{W_F}). \tag{3.1.2}$$

We define the following component group

$$\mathcal{S}_\varphi := \mathrm{Z}^1_{G^\vee}(\varphi)/\mathrm{Z}^1_{G^\vee}(\varphi)^\circ. \tag{3.1.3}$$

An enhancement of φ is an irreducible representation ϱ of S_φ. The pairs (φ, ϱ) are called enhanced L-parameters (for G and its inner forms).

We let $G^\vee$ act on the set of enhanced L-parameters by

$$g \cdot (\varphi, \varrho) = (g\varphi g^{-1}, g \cdot \varrho). \tag{3.1.4}$$

We define an action of $\mathfrak{X}_{\mathrm{nr}}({}^L M)$ on $\Phi_{\mathrm{e}}(M)$ as follows. Given $(\varphi, \varrho) \in \Phi_{\mathrm{e}}(M)$, and $\xi \in \mathfrak{X}_{\mathrm{nr}}({}^L M)$, we define $(\xi\varphi, \varrho) \in \Phi_{\mathrm{e}}(M)$ by $\xi\varphi := \varphi$ on $I_F \times \mathrm{SL}_2(\mathbb{C})$ and $(\xi\varphi)(\mathrm{Fr}_F) := \widetilde{\xi}\varphi(\mathrm{Fr}_F)$, where $\widetilde{\xi} \in \mathrm{Z}^{\circ}_{M^\vee \rtimes I_F}$ represents z.

For φ an L-parameter for G, we denote by u_φ the image under φ of $(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. Then we have $u_\varphi \in \mathcal{G}^{\circ}_{\varphi}$ and $\mathcal{S}_\varphi \simeq \mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)/\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)^\circ$ (see [AMS18, (92)]).

Let ϱ be an irreducible representation of $A_{\mathcal{G}^{\circ}_{\varphi}}(u_\varphi)$. The pair (u_φ, ϱ) is called cuspidal if it determines a $\mathcal{G}^\circ$-equivariant cuspidal local system on the $\mathcal{G}^{\circ}_{\varphi}$-conjugacy class of u_φ as defined by Lusztig in [Lus84].

Definition 3.1.5. An enhanced L-parameter (φ, ϱ) is called cuspidal if φ is discrete and (u_φ, ϱ) is a cuspidal pair in $\mathcal{G}_\phi$.

Let $(\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}})$ be a cuspidal enhanced L-parameter for M, we write

$$\mathfrak{s}^\vee := [{}^L M, \varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}]_{G^\vee} \tag{3.1.6}$$

for the $G^\vee$-conjugacy class of $({}^L M, \mathcal{O}^\vee)$, where $\mathcal{O}^\vee$ is the orbit of $(\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}})$ under the action of $\mathfrak{X}_{\mathrm{nr}}({}^L M)$. We denote by $\mathfrak{B}^\vee(G)$ the set of such $\mathfrak{s}^\vee$. We write $\mathfrak{s}^\vee_M := [{}^L M, \varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}]_{M^\vee}$.

We define

$$\mathrm{N}^{\mathfrak{s}^\vee}_{G^\vee} := \left\{ n \in \mathrm{N}_{G^\vee}(M^\vee) \,:\, {}^n(\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}) \simeq (\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}) \otimes \chi^\vee \ \text{ for some } \chi^\vee \in \mathfrak{X}_{\mathrm{nr}}({}^L M) \right\},$$

and we write

$$W^{\mathfrak{s}^\vee}_{G^\vee} = W(M^\vee, \mathcal{O}^\vee) = \mathrm{N}^{\mathfrak{s}^\vee}_{G^\vee} / M^\vee. \tag{3.1.7}$$

The group $W^{\mathfrak{s}^\vee}_{G^\vee}$ is a finite extended Weyl group, i.e., it decomposes as

$$W^{\mathfrak{s}^\vee}_{G^\vee} = W_{\mathcal{O}^\vee} \rtimes R(\mathcal{O}^\vee), \tag{3.1.8}$$

where $W_{\mathcal{O}^\vee}$ is a finite Weyl group and $R(\mathcal{O}^\vee)$ is a finite abelian group.

Let $\Phi^{\mathrm{c}}_{\mathrm{e}}(M)$ denote the set of $M^\vee$-conjugacy classes of cuspidal enhanced L-parameters for M. From [AMS18, (115)], we have a partition of $\Phi_{\mathrm{e}}(G)$ into series of enhanced L-parameters indexed by the set $\mathfrak{B}^\vee(G)$:

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi^{\mathfrak{s}^\vee}_{\mathrm{e}}(G), \tag{3.1.9}$$

such that, for any $(\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}) \in \Phi^{\mathrm{c}}_{\mathrm{e}}(M)$, we have

$$\Phi^{\mathfrak{s}^\vee_M}_{\mathrm{e}}(M) = \mathfrak{X}_{\mathrm{nr}}({}^L M) \cdot (\varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}). \tag{3.1.10}$$

For $\mathfrak{s}^\vee = [{}^L M, \varphi_{\mathrm{c}}, \varrho_{\mathrm{c}}]_{G^\vee} \in \mathfrak{B}^\vee(G)$, [AMS18, Theorem 9.3] provides a bijection

$$\xi^{\mathfrak{s}^\vee}_{G^\vee} \colon \Phi^{\mathfrak{s}^\vee}_{\mathrm{e}}(G) \longrightarrow (\Phi^{\mathfrak{s}^\vee_M}_{\mathrm{e}}(M) // W^{\mathfrak{s}^\vee}_{G^\vee})_{{}^L\natural}, \tag{3.1.11}$$

where ${}^L\natural \colon W^{\mathfrak{s}^\vee}_{G^\vee} \times W^{\mathfrak{s}^\vee}_{G^\vee} \to \mathbb{C}^\times$ is a 2-cocycle, that is is trivial on $W_{\mathcal{O}^\vee}$.

Let $\mathrm{Irr}^{\mathrm{sc}}(M) \subset \mathrm{Irr}(M)$ denote the set of equivalence classes of irreducible supercuspidal representations of M. Let $g \in \mathrm{N}_{\mathbf{G}}(\mathbf{M})(F)$. We have ${}^g\sigma \in \mathrm{Irr}^{\mathrm{sc}}(M)$ for any $\sigma \in \mathrm{Irr}^{\mathrm{sc}}(M)$. We denote by c_g the isomorphism $c_g \colon (M, \sigma) \xrightarrow{\sim} (M, {}^g\sigma)$. It induces a

map $c_g^\vee\colon M^\vee \to M^\vee$ as in [Spr79, §1-2] and hence an L-isomorphism ${}^L c_g\colon {}^L M \to {}^L M$ defined by ${}^L c_g(m^\vee, v) := (c_g^\vee(m^\vee), v)$ for any $m^\vee \in M^\vee$ and $v \in W_F$.

We have $c_{n_w}^\vee(m^\vee) = {}^{n_{w^\vee}} m^\vee$, where $w \mapsto w^\vee$ is the canonical isomorphism from $W(M) := \mathrm{N}_{\mathbf{G}}(\mathbf{M})(F)/M$ to $W(M^\vee) := \mathrm{N}_{G^\vee}(M^\vee)/M^\vee$ (see [ABPS17a, Proposition 3.1]), and n_w, $n_{w^\vee}$ are representatives of w and $w^\vee$ in $\mathrm{N}_{\mathbf{G}}(\mathbf{M})(F)$ and $\mathrm{N}_{G^\vee}(M^\vee)$, respectively. Since for $\sigma \in \mathrm{Irr}(M)$, the equivalence class of ${}^{n_w}\sigma$ does not depend on the choice of the representative n_w, we will write it simply as ${}^w\sigma$. Similarly denote $w^\vee$ instead of $n_{w^\vee}$for the action of $\mathrm{N}_{G^\vee}(M^\vee)$ on $\Phi_{\mathrm{e}}(M)$. We state the following property.

Property 3.1.12. *Let M be a Levi subgroup of G and $\mathfrak{s}_M := [M, \sigma]_M \in \mathfrak{B}(M)$. Then there exists a map*

$$\begin{array}{rccc} \mathfrak{L}^{\mathfrak{s}_M}\colon & \mathrm{Irr}^{\mathfrak{s}_M}(M) & \longrightarrow & \Phi_{\mathrm{e}}^{\mathrm{c}}(M) \\ & \sigma & \mapsto & (\varphi_\sigma, \varrho_\sigma) \end{array}$$

such that the following properties are satisfied for any $\sigma \in \mathrm{Irr}^{\mathfrak{s}_M}(M)$:

(1) *For any $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$, we have*

$$(\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) = \chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma),$$

where $\chi \mapsto \chi^\vee$ is the canonical isomorphism $\mathfrak{X}_{\mathrm{nr}}(M) \xrightarrow{\sim} \mathfrak{X}_{\mathrm{nr}}({}^L M)$.

(2) *For any $w \in W(M)$, we have*

$${}^{w^\vee}(\varphi_\sigma, \varrho_\sigma) \simeq (\varphi_{{}^w\sigma}, \varrho_{{}^w\sigma}),$$

where $w \mapsto w^\vee$ is the canonical isomorphism $W(M) \xrightarrow{\sim} W(M^\vee)$.

Remark 3.1.13. Property 3.1.12(1) is closely related to Borel's *desiderata* [Bor79, 10.3.(2)]. Property 3.1.12(2) implies [Bor79, 10.3.(5)] for $\eta = c_g$, and can be viewed as an analogue for enhanced L-parameters of [Hai14, Conjecture 5.2.4] for supercuspidal representations.

Lemma 3.1.14. *Let $\mathfrak{s} = [M, \sigma]_M \in \mathfrak{B}(G)$ satisfy* Property *3.1.12. Then there is a group isomorphism*

$$\mathfrak{r}\colon W_G^{\mathfrak{s}} \xrightarrow{\sim} W_{G^\vee}^{\mathfrak{s}^\vee}, \quad \textit{where } \mathfrak{s}^\vee := [{}^L M, \mathfrak{L}^{\mathfrak{s}_M}(\sigma)]_{G^\vee},$$

and $\mathfrak{L}^{\mathfrak{s}_M}$ is equivariant with respect to $\mathfrak{r}$.

Proof. Let $w \in W_G^{\mathfrak{s}} \subset W(M)$. By definition of $W_G^{\mathfrak{s}}$, we have ${}^w\sigma \simeq \chi \otimes \sigma$ for some $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$. By Property 3.1.12, we get

$${}^{w^\vee}(\varphi_\sigma, \varrho_\sigma) \simeq (\varphi_{{}^w\sigma}, \varrho_{{}^w\sigma}) \simeq (\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) \simeq \chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma).$$

Thus $w^\vee \in W_{G^\vee}^{\mathfrak{s}^\vee}$, and $w \mapsto w^\vee$ defines a group morphism from $W_G^{\mathfrak{s}}$ to $W_{G^\vee}^{\mathfrak{s}^\vee}$. By reversing the argument, we see that it is a isomorphism. □

Theorem 3.1.15. *Suppose that Property 3.1.12 holds. Then we have a canonical isomorphism*

$$\mathfrak{e}\colon \mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}} \xrightarrow{\sim} \Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee}.$$

Proof. Let $\mathfrak{s} = [M, \sigma]_G$. We write $(\varphi_\sigma, \varrho_\sigma) := \mathfrak{L}^{\mathfrak{s}_M}(\sigma)$. Then, the isomorphism $\mathfrak{X}_{\mathrm{nr}}(M) \simeq \mathfrak{X}_{\mathrm{nr}}({}^L M)$ combined with (3.1.10) shows that

$$\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M) = \left\{\chi^\vee \otimes (\varphi_\sigma, \varrho_\sigma) \,:\, \chi \in \mathfrak{X}_{\mathrm{nr}}(M)\right\}.$$

By using Property 3.1.12(1), we obtain

$$\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M) = \{(\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) \,:\, \chi \in \mathfrak{X}_{\mathrm{nr}}(M)\} \simeq \mathrm{Irr}^{\mathfrak{s}_M}(M),$$

thanks to (2.1.8).

Recall that $W_G^{\mathfrak{s},\chi\otimes\sigma}$ denotes the stabilizer of $\chi \otimes \sigma$ in $W_G^{\mathfrak{s}}$. We have

$$\mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}} = \bigsqcup_{\chi\in\mathfrak{X}_{\mathrm{nr}}(M)} \mathrm{Irr}(W_G^{\mathfrak{s},\chi\otimes\sigma})/W_G^{\mathfrak{s}}.$$

For $\chi \in \mathfrak{X}_{\mathrm{nr}}(X)$, we denote by $W_{G^\vee}^{\mathfrak{s}^\vee,\chi^\vee\cdot(\varphi_\sigma,\varrho_\sigma)}$ the stabilizer of $\chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma)$ in $W_{G^\vee}^{\mathfrak{s}^\vee}$. Then we get

$$\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee} = \bigsqcup_{\chi\in\mathfrak{X}_{\mathrm{nr}}(M)} \mathrm{Irr}(W_{G^\vee}^{\mathfrak{s}^\vee,\chi^\vee\cdot(\varphi_\sigma,\varrho_\sigma)})/W_{G^\vee}^{\mathfrak{s}^\vee}.$$

Let $w \in W_G^{\mathfrak{s},\chi\otimes\sigma}$. Thanks to Property 3.1.12 we have

$${}^{w^\vee}(\chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma)) = {}^{w^\vee}(\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) = (\varphi_{w(\chi\otimes\sigma)}, \varrho_{w(\chi\otimes\sigma)} = (\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}).$$

Thus, the morphism $\mathfrak{r}$ restricts to an isomorphism between $W_G^{\mathfrak{s},\chi\otimes\sigma}$ and $W_{G^\vee}^{\mathfrak{s}^\vee,\chi^\vee\cdot(\varphi_\sigma,\varrho_\sigma)}$, and we obtain an isomorphism,

$$\mathfrak{e}\colon\ \mathrm{Irr}^{\mathfrak{s}_M}(M)//W_G^{\mathfrak{s}} \longrightarrow \Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M)//W_{G^\vee}^{\mathfrak{s}^\vee}.$$

□

Remark 3.1.16. When G is a split classical group, F has characteristic zero and $\mathfrak{L}^{\mathfrak{s}_M}$ is the LLC defined by Arthur in [Art13], then Lemma 3.1.14 was proved by Moussaoui in [Mou17a, Theorem 4.1] and Theorem 3.1.15 follows from [Mou17b, §3.2 & 3.3].

Corollary 3.1.17. *We assume that the collections of* 2*-cocycles* $\natural$ *and* ${}^L\natural$ *are both trivial and that Property 3.1.12 holds. Then we have a bijection*

$$\mathfrak{L} := (\xi_{G^\vee}^{\mathfrak{s}^\vee})^{-1} \circ \mathfrak{e} \circ \xi_G^{\mathfrak{s}}\colon\ \mathrm{Irr}^{\mathfrak{s}}(G) \longrightarrow \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G).$$

Proof. It follows directly from the combination of Theorem 3.1.15, Proposition 2.1.15 and (3.2.11). □

Remark 3.1.18. The 2-cocycles involved in $\natural$ and ${}^L\natural$ are expected to be often trivial. In particular, they are trivial if the groups $R(\mathcal{O})$ and $R(\mathcal{O}^\vee)$ have cardinality at most 2, and hence when M is a Levi subgroup of a maximal parabolic subgroup of G. They are trivial also when G is a symplectic group or a special orthogonal group [Hei11] and [Mou17b], and for principal series representatons of split groups [Roc98] and [ABPS17c]. However, there exist cases when they are not trivial (see [ABPS17b, Example 5.5] for $\natural$ and [AMS18, Example 9.4] for ${}^L\natural$).

Theorem 3.1.19. *Let σ be a regular supercuspidal irreducible representation of M and $\varphi_\sigma\colon W_F \to {}^LM$ it L-parameter of σ, as defined in [Kal19b]. Let $\mathfrak{s} = [M,\sigma]_G$ and $\mathfrak{s}^\vee = [M^\vee, (\varphi_\sigma, 1)]_{G^\vee}$*

If the collections of 2-cocycles $\natural$ and ${}^L\natural$ are both trivial and the characteristic of F is 0, then the map

$$\mathfrak{L}\colon \operatorname{Irr}^{\mathfrak{s}}(G) \longrightarrow \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$$

is a bijection.

Proof. Let $\mathfrak{L}^{\mathfrak{s}_M}$ be the map

$$\mathfrak{L}^{\mathfrak{s}_M} : \sigma \mapsto (\varphi_\sigma, 1).$$

Property 3.1.12 (1) follows from the construction of σ in [Kal19a]. Since σ is regular, its L-packet for G is a singleton, and hence Property 3.1.12 (2) follows from the functoriality result established in [BM21]. □

3.2. **Matching of simple modules of extended affine Hecke algebras.**

The goal of this section is to use Corollary 3.1.17 in order to get a bijection between simple modules of extended affine Hecke algebras. We suppose that Property 3.1.12 is satisfied. A (possibly twisted) extended affine Hecke algebra $\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee)$ was constructed in [AMS17]. We recall the definition below. We write

$$\mathcal{M}_{\varphi_{\mathrm{c}}} := \mathrm{Z}_{M^\vee}(\varphi_{\mathrm{c}}(W_F)), \tag{3.2.1}$$

and denote by $\mathcal{A}_{\varphi_{\mathrm{c}}}$ the identity component of the center of $\mathcal{M}_{\varphi_{\mathrm{c}}}$. We set

$$\mathcal{J}_\varphi = \mathcal{J}_\varphi^{G^\vee} := \mathrm{Z}_{G^\vee}(\varphi(I_F)), \tag{3.2.2}$$

and define $\Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})$ as the set of $\alpha \in X^*(\mathcal{A}_{\varphi_{\mathrm{c}}}) \setminus \{0\}$ which appear in the adjoint action of $\mathcal{A}_{\varphi_{\mathrm{c}}}$ on the Lie algebra of $\mathcal{J}_\varphi^\circ$. It is a root system (see [AMS17, Proposition 3.9]). We denote by $\Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})^+$ the positive system defined by an F-rational Borel subgroup of $\mathcal{J}_\varphi^\circ$.

Let Δ be a basis of the reduced root system $\Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})_{\mathrm{red}}$, and let $a \in \mathcal{A}_{\varphi_{\mathrm{c}}}$ such that $\alpha(a^{-1})$ is an eigenvalue of $\mathrm{Ad}(\varphi(\mathrm{Fr}))$ for any $\alpha \in \Delta$. Then we define $\varphi_a \in \Phi(M)$ by

$$\varphi_a|_{I_F \times \mathrm{SL}_2(\mathbb{C})} := \varphi_{\mathrm{c}}|_{I_F \times \mathrm{SL}_2(\mathbb{C})} \quad \text{and} \quad \varphi_a(\mathrm{Fr}_F) := a \cdot \varphi_{\mathrm{c}}(\mathrm{Fr}_F). \tag{3.2.3}$$

We have $\Sigma(\mathcal{G}_{\varphi_a}, \mathcal{A}_{\varphi_{\mathrm{c}}})_{\mathrm{red}} = \Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})_{\mathrm{red}}$, where $\mathcal{G}_{\varphi_a} = \mathrm{Z}_{G^\vee}(\varphi_a(W_F))$ (see [AMS17, Proposition 3.9]).

We write

$$\mathfrak{X}_{\mathrm{nr}}(M^\vee, \varphi_a) := \left\{ x \in \mathfrak{X}_{\mathrm{nr}}(M^\vee) \,:\, (z\varphi_a)_{M^\vee} = (\varphi_a)_{M^\vee} \right\} \tag{3.2.4}$$

and $T_{\mathcal{O}^\vee} := \mathfrak{X}_{\mathrm{nr}}(M^\vee)/\mathfrak{X}_{\mathrm{nr}}(M^\vee, \varphi_a)$. For each $\alpha \in \Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})_{\mathrm{red}}$, letting $m_\alpha \in \mathbb{Z}_{>0}$ be the smallest integer such that

$$\ker(m_\alpha \alpha) \supset \left\{ a' \in \mathcal{A}_{\varphi_{\mathrm{c}}} \,:\, (a'\varphi_a)_{M^\vee} = (\varphi_a)_{M^\vee} \right\},$$

we set

$$\Sigma_{\mathcal{O}^\vee} := \left\{ m_\alpha \,:\, \alpha \in \Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_{\mathrm{c}}})_{\mathrm{red}} \right\} \subset X^*(T_{\mathcal{O}^\vee}). \tag{3.2.5}$$

Then $W_{\mathcal{O}^\vee}$ is the finite Weyl group of $\Sigma_{\mathcal{O}^\vee}$, and

$$R(\mathcal{O}^\vee) := \left\{ w \in W(M^\vee, \mathcal{O}^\vee) \,:\, w \cdot \Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_c})^+ = \Sigma(\mathcal{J}_\varphi^\circ, \mathcal{A}_{\varphi_c})^+ \right\}. \tag{3.2.6}$$

Let

$$\lambda^\vee \colon \Sigma_{\mathcal{O}^\vee} \to \mathbb{Z}_{\geq 0} \quad \text{and} \quad \lambda^{*\vee} \colon \left\{ m_\alpha \alpha \in \Sigma_{\mathcal{O}^\vee} : (m_\alpha \alpha)^\vee \in 2X_*(T_{\mathfrak{s}^\vee}) \right\} \to \mathbb{Z}_{\geq 0}$$

be the two parameter functions defined as in the proof of Lemma 3.12 in [AMS17]. We recall from [AMS17, (28)] that $\lambda^{*\vee}(\alpha) = \lambda^\vee(\alpha)$ unless α is a short root in a type B root subsystem of $R_{\mathcal{O}^\vee}$.

Then the algebra $\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee)$ is defined to be

$$\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee) := \mathcal{H}_{\mathrm{aff}}(\mathcal{O}^\vee, \Sigma_{\mathcal{O}^\vee}, \lambda^\vee, \lambda^{\vee *}, z) \rtimes \mathbb{C}[R(\mathcal{O}^\vee), \kappa^\vee], \tag{3.2.7}$$

where z is a positive real number, $\mathcal{H}_{\mathrm{aff}}(G^\vee, \mathfrak{s}^\vee) := \mathcal{H}_{\mathrm{aff}}(\mathcal{O}^\vee, \Sigma_{\mathcal{O}^\vee}, \lambda^\vee, \lambda^{*\vee}, z)$ is the corresponding affine Hecke algebra with affine Weyl group $W_{\mathcal{O}^\vee} \ltimes X^*(T_{\mathcal{O}^\vee})$, and $\kappa^\vee$ a 2-cocycle on $R(\mathcal{O}^\vee)$.

Theorem 3.2.8. *Let $\mathfrak{s} = [M, \sigma]_G$. If $\mathfrak{s}^\vee = [{}^L M, \varphi_\sigma, 1]_{G^\vee}$, then there is a bijection*

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}}(G)) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee)).$$

Proof. From Corollary 2.1.17, we have a bijection

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}}(G)) \xrightarrow{\sim} (\mathrm{Irr}^{\mathfrak{s}_M}(M) // W_G^{\mathfrak{s}})_\natural. \tag{3.2.9}$$

By Lemma 3.1.14, we have

$$(\mathrm{Irr}^{\mathfrak{s}_M}(M) // W_G^{\mathfrak{s}})_\natural \simeq (\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M) // W_{G^\vee}^{\mathfrak{s}^\vee})_{{}^L\natural}. \tag{3.2.10}$$

On the other hand, from (3.1.11), we get a bijection

$$(\Phi_{\mathrm{e}}^{\mathfrak{s}_M^\vee}(M) // W_{G^\vee}^{\mathfrak{s}^\vee})_{{}^L\natural} \xrightarrow{\sim} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G). \tag{3.2.11}$$

Finally, [AMS17, Theorem 3.18] shows that there is a bijection

$$\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee)). \tag{3.2.12}$$

Combining equations 3.2.9, 3.2.10, 3.2.11 and 3.2.12, we obtain the desired bijection. □

Corollary 3.2.13. *With the same assumption as in Theorem 3.2.8.*

$$\mathrm{Irr}(\mathcal{H}^{\mathfrak{s}^\vee}(G^\vee)) \longrightarrow \mathrm{Irr}(\mathcal{H}^{(\mathfrak{s}^0)^\vee}((G^0)^\vee)).$$

Proof. This follows from combining Theorems 3.2.8 and 2.4.9. □

4. The G_2 case

4.1. Some background.

The aim of the sequel this section is to introduce notation and background in view of studying the G_2 case.

4.1.1. *General background.*

We denote by $\mathfrak{a}_M$ the real Lie algebra of $A_{\mathbf{M}}$, by $\mathfrak{a}_M^*$ its dual, by $\mathfrak{a}_{M,\mathbb{C}}^*$ the complexification of the latter, by $|\cdot|_F$ the modulus of F, and by H_M the map $M \to \mathfrak{a}_M$ which satisfies $q^{-\langle H_M(m),\alpha\rangle} = |\alpha(m)|_F$ for every rational character α of M and every $m \in M$. The kernel of H_M is equal to M_1.

If s is a complex number, we denote by χ_s the character defined by

$$\chi_s(m) := |\det(m)|_F^s \quad \text{for any } m \in M. \tag{4.1.1}$$

We have $\chi_s \in \mathfrak{X}_{\mathrm{nr}}(M)$. The group

$$M_\sigma := \bigcap_{\chi \in \mathfrak{X}_{\mathrm{nr}}(M,\sigma)} \ker \chi \tag{4.1.2}$$

has finite index in M_1. We have

$$\mathrm{Irr}(M_\sigma/M_1) \simeq \mathfrak{X}_{\mathrm{nr}}(M)/\mathfrak{X}_{\mathrm{nr}}(M,\sigma) \quad \text{and} \quad \mathbb{C}[M_\sigma/M_1] \simeq \mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(M)/\mathfrak{X}_{\mathrm{nr}}(M,\sigma)].$$

We write $(M_\sigma/M_1)^\vee := \mathrm{Hom}_{\mathbb{Z}}(M_\sigma/M_1, \mathbb{Z})$. Composition with H_M and $\mathbb{R}$-linear extension of maps $H_M(M_\sigma/M_1) \to \mathbb{Z}$ determines an embedding

$$H_M^\vee \colon (M_\sigma/M_1)^\vee \to \mathfrak{a}_M^*.$$

For $m \in M$, let b_m be the element of $\mathbb{C}[\mathfrak{X}_{\mathrm{nr}}(M)]$ defined by $b_m(\chi) := \chi(m)$ for any $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)$. Let h_α be the unique generator of M_σ/M_1 such that $\mathrm{v}_F(\alpha(h_\alpha)) > 0$. We define $X_\alpha \in \mathbb{C}(\mathfrak{X}_{\mathrm{nr}}(M)/\mathfrak{X}_{\mathrm{nr}}(M,\sigma)$ by

$$X_\alpha(\chi) := \chi(h_\alpha), \tag{4.1.3}$$

where $\chi \in \mathfrak{X}_{\mathrm{nr}}(M)/\mathfrak{X}_{\mathrm{nr}}(M,\sigma)$. In particular, we have

$$X_\alpha(\chi_s) = |\det(h_\alpha)|_F^s, \tag{4.1.4}$$

where χ_s is the unramified character defined in (4.1.1). Let $\widetilde{\alpha}$ be the element of $\mathfrak{a}_M^*$ defined as $\widetilde{\alpha} := \langle \rho_P, \alpha^\vee\rangle^{-1}\rho_P$, where ρ_P is half the sum of the roots of A_M in $\mathrm{Lie}\, N$, with $P = MN$. Then $s\widetilde{\alpha} \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$.

We recall the description of the Plancherel measure from [Sil79] (see also [Sol21b] or [Hei11] for the notations used here): for $\alpha \in \Sigma_{\mathcal{O},\mu}$, where is $\Sigma_{\mathcal{O},\mu}$ is the root system defined in (2.1.10), there exists $q_\alpha, q_{\alpha^*} \in \mathbb{R}_{\geq 1}, c'_{s_\alpha} \in \mathbb{R}_{>0}$ for $\alpha \in \Sigma_{\mathcal{O},\mu}$, such that

$$\mu^{M_\alpha}(\sigma \otimes \cdot) = c'_{s_\alpha} \frac{(1-X_\alpha)(1-X_\alpha^{-1})}{(1-q_\alpha^{-1}X_\alpha)(1-q_\alpha^{-1}X_\alpha^{-1})} \cdot \frac{(1+X_\alpha)(1+X_\alpha^{-1})}{(1+q_{\alpha^*}^{-1}X_\alpha)(1+q_{\alpha^*}^{-1}X_\alpha^{-1})} \tag{4.1.5}$$

Let $\alpha \in \Sigma_{\mathcal{O},\mu}$. Then there is a unique $\alpha^\sharp \in (M_\sigma/M_1)^\vee$ such that $H_M^\vee(\alpha^\sharp) \in \mathbb{R}\alpha$ and $\langle h_\alpha, \alpha^\sharp\rangle = 2$ (see [Sol21a, Proposition 3.1]). We set

$$\Sigma_{\mathcal{O}} := \left\{\alpha^\sharp \,:\, \alpha \in \Sigma_{\mathcal{O},\mu}\right\} \quad \text{and} \quad \Sigma_{\mathcal{O}}^\vee := \left\{\alpha^\sharp \,:\, h_\alpha \in \Sigma_{\mathcal{O},\mu}\right\}.$$

The quadruple $(\Sigma_{\mathcal{O}}^\vee, M_\sigma/M_1, \Sigma_{\mathcal{O}}, (M_\sigma/M_1)^\vee)$ is a root datum with Weyl group $W_{\mathcal{O}}$, the group $W(M,\mathcal{O})$ acts naturally on it, and $R(\mathcal{O})$ is the stabilizer of the basis

determined by P (see [Sol21a, Proposition 3.1]). We endow this based root datum with the parameter $q = q_F$ and the labels

$$\lambda(\alpha) := \log(q_\alpha q_{\alpha^*})/\log(q_F) \quad \text{and} \quad \lambda^*(\alpha) := \log(q_\alpha q_{\alpha^*}^{-1})/\log(q_F). \tag{4.1.6}$$

To the above data we associate the affine Hecke algebra

$$\mathcal{H}^{\mathfrak{s}}_{\mathrm{aff}}(\lambda, \lambda^*, q_F) := \mathcal{H}_{\mathrm{aff}}(\Sigma_{\mathcal{O}}^\vee, M_\sigma/M_1, \Sigma_{\mathcal{O}}, (M_\sigma/M_1)^\vee, \lambda, \lambda^*, q_F). \tag{4.1.7}$$

By definition it is the vector space $\mathbb{C}[W_{\mathcal{O}}] \otimes_{\mathbb{C}} \mathbb{C}[M_\sigma/M_1]$ with multiplication given by the following rules:

- $\mathbb{C}[W_{\mathcal{O}}] = \mathrm{span}\{T_w : w \in W_{\mathcal{O}}\}$ is embedded as the Iwahori-Hecke algebra $\mathcal{H}(W_{\mathcal{O}}, q_F^\lambda)$, that is,
$$T_w T_v = T_{wv} \quad \text{if } \ell(w) + \ell(v) = \ell(ww),$$
$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) = 0 \quad \text{if } \alpha \in \Delta_{\mathcal{O},\mu},$$
where $\ell(w)$ is the word length of w[1];
- $\mathbb{C}[M_\sigma/M_1] \simeq \mathbb{C}[\mathcal{O}]$ is embedded as a subalgebra,
- for $\alpha \in \Delta_{\mathcal{O},\mu}$ and $x \in M_\sigma/M_1$ (corresponding to $\theta_x \in \mathbb{C}[M_\sigma/M_1]$):

$$\theta_x T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(x)} = \left(q_F^{\lambda(\alpha)} - 1 + X_\alpha^{-1}(q^{\frac{\lambda(\alpha)+\lambda^*(\alpha)}{2}} - q^{\frac{\lambda(\alpha)-\lambda^*(\alpha)}{2}})\right)\frac{\theta_x - \theta_{s_\alpha(x)}}{1 - X_\alpha^{-2}}.$$

We set

$$W^{\mathfrak{s}}_{\mathrm{aff}} := W_{\mathcal{O}} \ltimes \mathbb{Z}\Sigma_{\mathcal{O}}^\vee. \tag{4.1.8}$$

From now on we assume that the parabolic subgroup P is maximal. Then we have $M_\alpha = G$, and $W(M)$ is either trivial or of order 2.

Remark 4.1.9.

(1) The groups $W(M, \mathcal{O})$, $W_{\mathcal{O}}$, and $R(\mathcal{O})$ are either trivial or of order 2. In particular, $\Sigma_{\mathcal{O},\mu}$ is either empty or $\{\alpha, -\alpha\}$.

(2) For $G = G_2$, if $\sigma \not\simeq \sigma^\vee$, then $W(M, \mathcal{O}) = 1$. It suffices to only check the case where $\sigma \simeq \sigma^\vee$.

In general, if $W(M, \mathcal{O}) = 1$, then the parabolically induced representation is irreducible, so we do not need to work with the case. In the case of G_2, the condition $W(M, \mathcal{O}) \neq 1$ happens to be characterized by the condition that σ is self-dual. See [Sha89] for more details.

If $\Sigma_{\mathcal{O},\mu} \neq \emptyset$, then $W(M) \neq \{1\}$ and the group $W_{\mathcal{O}}$ is generated by the unique non-trivial element of $W(M)$, say s_M, and then we get $W_{\mathcal{O}} = W(M, \mathcal{O}) = W(M)$. In particular, if $\Sigma_{\mathcal{O},\mu} \neq \emptyset$, we have $R(\mathcal{O}) = \{1\}$.

The condition $\Sigma_{\mathcal{O},\mu} = \emptyset$ is equivalent to the following

$$\mu^G(\chi \otimes \sigma) \neq 0 \quad \text{for any } \chi \in X_{\mathrm{nr}}(M). \tag{4.1.10}$$

We recall the following Harish-Chandra theorem.

[1] i.e., $\ell(w)$ is the smallest integer $\ell \geq 0$ such that w is a product of ℓ generators s_α.

Theorem 4.1.11. (Harish-Chandra) [Sil79, 5.4.2.2 and 5.4.2.3] *Let M be a Levi subgroup of a maximal parabolic subgroup of G and let σ be a supercuspidal irreducible representation of M.*
(a) *If $\mu^G(\sigma) = 0$, then $W(M) = \{1, s_M\} \neq \{1\}$ and $s_M\sigma \simeq \sigma$.*
(b) *Suppose that $W(M) \neq \{1\}$. Then, in order that $\mu^G(\sigma) \neq 0$, it is necessary and sufficient that the representation $i_P^G(\sigma)$ is reducible. The representation $i_P^G(\sigma)$ is then the direct sum of two non-isomorphic irreducible representations.*

Corollary 4.1.12. *Suppose that $W(M) \neq \{1\}$. Then $W_{\mathcal{O}} = \{1\}$ if and only if, for any $\chi \in X_{\mathrm{nr}}(M)$, the representation $i_P^G(\chi \otimes \sigma)$ is reducible.*

4.1.2. *Some background on the group G_2.* In our specific case where G is the split G_2, we can obtain more precise results. Denote by T a maximal split torus in G, and by R the set of roots of G with respect to T. Let $(\epsilon_1, \epsilon_2, \epsilon_3)$ be the canonical basis of $\mathbb{R}^3$, equipped with the scalar product $(\ |\ )$ for which this basis is orthonormal. Then $\alpha := \varepsilon_1 - \varepsilon_2$, $\beta := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ defines a basis of R and

$$R^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

is a subset of positive roots in R. We have

$$(\alpha|\alpha) = 2, \quad (\beta|\beta) = 6 \quad \text{and} \quad (\alpha|\beta) = -3. \tag{4.1.13}$$

Hence, α is short root, while β is long root.

Following [Mui97], we fix an isomorphism $\eta_\alpha \colon T \xrightarrow{\sim} F^\times \times F^\times$:

$$t \longmapsto ((2\alpha+\beta)(t), (\alpha+\beta)(t)). \tag{4.1.14}$$

Under this identification we have

$$\alpha^\vee(a) = \eta_\alpha^{-1}(a, a^{-1}) \quad \text{and} \quad \beta^\vee(a) = \eta_\alpha^{-1}(1, a) \quad \text{for any } a \in F^\times. \tag{4.1.15}$$

Let $G^\vee$ be the dual group of G over $\mathbb{C}$ obtained via an identification of the roots of $G^\vee$ with the coroots of G_2 and vice versa. Then $G^\vee$ is a complex reductive group of type G_2, with simple roots $\alpha^\vee$ and $\beta^\vee$. Notice that $\alpha^\vee$ is the long root of $G^\vee$ and $\beta^\vee$ the short root. The torus $T^\vee$ dual to T is a maximal torus of $G^\vee$. We fix an isomorphism $\eta_{\beta^\vee} \colon T \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$:

$$t \longmapsto ((\alpha^\vee + 2\beta^\vee)(t), (\alpha^\vee + \beta^\vee)(t)). \tag{4.1.16}$$

We have that

$$\alpha^\vee(a) = \eta_{\beta^\vee}^{-1}(1, a) \quad \text{and} \quad \beta^\vee(a) = \eta_{\beta^\vee}^{-1}(a, a^{-1}) \quad \text{for any } a \in F^\times. \tag{4.1.17}$$

For each root $\gamma \in R(G)$, we fix root groups homomorphisms $x_\gamma \colon F \to G$ and $\mathbb{Z}$-homomorphisms $\zeta_\gamma \colon \mathrm{SL}_2(F) \to G$ for $\gamma \in R$ as in [BT72, (6.1.3) (b)]. We have

$$x_\gamma(u) = \zeta_\gamma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad x_{-\gamma}(u) = \zeta_{-\gamma} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad \gamma^\vee(t) = \zeta_\gamma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

For $\gamma \in \{\alpha, \beta\}$, let P_γ be the maximal standard parabolic subgroup of G generated by γ, and M_γ be the centralizer of the image of $(\gamma')^\vee$ in G, where γ' is the unique

positive root orthogonal to γ, i.e.

$$\gamma' := \begin{cases} 3\alpha+\beta & \text{if } \gamma=\alpha, \\ 3\alpha+2\beta & \text{if } \gamma=\beta. \end{cases}$$

Then M_γ is a Levi factor for P_γ, and M_α, M_β are representatives of the two conjugacy classes of maximal Levi subgroups of G.

We extend $\zeta_\gamma\colon \mathrm{SL}_2(F)\to M_\gamma$ to an isomorphism $\zeta_\gamma\colon \mathrm{GL}_2(F)\to M_\gamma$ by

$$\zeta_\gamma\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\right) := \zeta_{\gamma'}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\right), \quad \text{for } t\in\ F^\times.$$

Then the restriction to T of the inverse map of ζ_γ coincides with the isomorphism

$$\eta_\gamma\colon T\to F^\times\times F^\times,$$

where η_α has been defined in (4.1.14), while

$$\eta_\beta\colon t\mapsto ((\alpha+\beta)(t),\alpha(t)).$$

4.2. **Explicit parameters for G_2.**

4.2.1. *The long root case.* Let ψ be a fixed nontrivial additive character of F, and $\overline{\psi}$ be the dual of ψ. Assume the Levi factor M of $P=MN$ is generated by the long root of G. Let σ be an irreducible unitary supercuspidal representation of M. We denote by $\omega=\omega_\sigma$ the central character of σ. Let L/F be a quadratic extension and χ be a character of $L^\times$. Let χ' be the conjugate of χ, i.e. $\chi'(a)=\chi(\overline{a})$. Let $\Pi(\sigma)$ denote the Gelbart-Jacquet lift of σ as defined in [GJ78]. Our notations follow [Sha89]. The Plancherel measure $\mu(s\widetilde{\alpha},\sigma)$ has the following four possibilities ([AEF+21]).

4.2.2. *Case I.* If ω is unramified, and if $\sigma=\sigma(\tau)$ with $\tau=\mathrm{Ind}_{W_L}^{W_F}\chi$, with $\chi^2\chi'$ unramified, then

$$\mu(s\widetilde{\alpha},\sigma)=\gamma(G/P)^2 q_F^{n(\omega)+n(\sigma\times\Pi(\sigma))-n(\sigma)}\frac{(1-\omega(\varpi)q_F^{-2s})(1-\omega^{-1}(\varpi)q_F^{2s})}{(1-\omega^{-1}(\varpi)q_F^{-1+2s})(1-\omega(\varpi)q_F^{-1-2s})} \cdot\frac{(1-\chi^2\chi'^{-1}(\varpi_L)q_L^{-s})(1-\chi^{-2}\chi'(\varpi_L)q_L^{s})}{(1-\chi^2\chi'^{-1}(\varpi_L)q_L^{-1-s})(1-\chi^{-2}\chi'(\varpi_L)q_L^{-1+s})}$$

Comparing to the Plancherel formula in (4.1.5), we have

$$\begin{cases} X_\alpha(s) & =\omega(\varpi_F)q_F^{-2s} \\ X_\alpha(s) & =-\chi^2\chi'^{-1}(\varpi_L)q_L^{-s} \end{cases}$$

which implies that

$$\omega(\varpi_F)q_F^{-2s}+\chi^2\chi'^{-1}(\varpi_L)q_L^{-s}=0 \tag{4.2.1}$$

Since $q_L=q^{f(L/F)}$, the above only has a solution when $f(L/F)=2$ and

$$\omega(\varpi_F)+\chi^2\chi'^{-1}(\varpi_L)=0$$

which is satisfied in our case. In particular, we have

$$q_\alpha=q_F,\quad q_{\alpha^*}=q_L=q_F^{f(L/F)} \tag{4.2.2}$$

Thus

$$\lambda(\alpha) = \log(q_\alpha q_{\alpha^*})/\log(q_F) = 1+f(L/F), \quad \lambda^*(\alpha) = |\log(q_\alpha q_{\alpha^*}^{-1})/\log(q_F)| = |1-f(L/F)|.$$

Hence the parameters for the Hecke algebra in this case are $q_F^{1+f(L/F)}$ and $q_F^{|1-f(L/F)|}$.

4.2.3. *Case II.* If ω_σ is ramified and $\sigma = \sigma(\tau)$ with $\tau = \mathrm{Ind}_{W_L}^{W_F}\chi$, and $\chi^2\chi'$ unramified,

(4.2.3)

$$\mu(s\widetilde{\alpha},\sigma) = \gamma(G/P)^2 q_F^{n(\sigma\times\Pi(\sigma))-n(\sigma)} \frac{(1-\chi^2\chi'^{-1}(\varpi_L)q_L^{-s})(1-\chi^{-2}\chi'(\varpi_L)q_L^{s})}{(1-\chi^2\chi'^{-1}(\varpi_L)q_L^{-1-s})(1-\chi^{-2}\chi'(\varpi_L)q_L^{-1+s})}$$

We compare (4.2.3) to the Plancherel formula (4.1.5) and get

$$q_{\alpha^*} = 1, \quad q_\alpha = q_L = q_F^{f(L/F)} \tag{4.2.4}$$

Recall the definition of X_α as

$$X_\alpha(\chi) := \chi(h_\alpha^\vee) \tag{4.2.5}$$

where $\chi \in \mathfrak{X}^{\mathrm{nr}}(M)/\mathfrak{X}^{\mathrm{nr}}(M,\sigma)$. Since the map $\psi_s\colon m \mapsto |\det(m)|_F^s$ is an unramified character of M, we have

$$X_\alpha(\psi_s) = (\chi^2\chi'^{-1})(\varpi_L)q_L^{-s} \tag{4.2.6}$$

Recall from (4.1.6) that

$$q_F^{\lambda(\alpha)} = q_\alpha q_{\alpha^*} \in \mathbb{R}_{>1}. \tag{4.2.7}$$

Thus by (4.2.4), we have $q_F^{\lambda(\alpha)} = q_L = q_F^{f(L/F)}$, where $f(L/F)$ is the residue degree and is thus 1 if L/F is ramified, and 2 if L/F is unramified. In particular

$$\lambda(\alpha) = f(L/F), \quad \lambda^*(\alpha) = f(L/F). \tag{4.2.8}$$

Note that for $w \in W(M,\mathcal{O})$, one may check that

$$w(X_\alpha) = X_{w(\alpha)} \tag{4.2.9}$$

Since $w(\alpha) = \alpha$ for $w \in W(M,\mathcal{O})$ in the $G = G_2$ case, (4.2.9) is simply $w(X_\alpha) = X_\alpha$. On the other hand, by [Sol21b, Prop 1.1] we have

$$wX_\alpha(\chi) = w(X_\alpha(\chi)) = w(\chi(h_\alpha^\vee)) = \chi(w(h_\alpha^\vee)) = \chi(h_{w(\alpha)}^\vee) = \chi(h_\alpha^\vee) = X_\alpha(\chi)$$

Thus $wX_\alpha = X_\alpha = X_{w(\alpha)}$. This reduces to check, in the long root case, that

$$s_{2\alpha+\beta}(\chi^2\chi'^{-1}(\varpi_L)q_L^{-s}) = \chi^2\chi'^{-1}(\varpi_L)q_L^{-s} \tag{4.2.10}$$

Since $\Sigma_{\mathcal{O}}^\vee = \{1, 2\alpha+\beta\}$ in the long root $M = M^\beta$ case, we have

$$W(\Sigma_{\mathcal{O}}^\vee) = \{1, s_{2\alpha+\beta}\}. \tag{4.2.11}$$

Thus the spherical Hecke algebra is given by

$$\mathcal{H}(W(\Sigma_{\mathcal{O}}^\vee), q_F^\lambda) = \begin{cases} \mathcal{H}(\{1, s_{2\alpha+\beta}\}, q_F), & L/F \text{ is ramified} \\ \mathcal{H}(\{1, s_{2\alpha+\beta}\}, q_F^2), & L/F \text{ is unramified} \end{cases} \tag{4.2.12}$$

Therefore, the affine Hecke algebra in this case is given by

$$\mathcal{H}_{\mathrm{aff}}(M^\beta) = \mathcal{H}(\{1, s_{2\alpha+\beta}\}, q_F^{f(L/F)}) \ltimes \mathbb{C}[\mathcal{O}] \tag{4.2.13}$$

4.2.4. *Case III.* If ω is unramified and $\sigma \neq \sigma(\tau)$ or $\chi^2\chi'$ is ramified,

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q_F^{n(\omega)+n(\sigma\times\Pi(\sigma))-n(\sigma)} \frac{(1-\omega(\varpi)q_F^{-2s})(1-\omega^{-1}(\varpi)q_F^{2s})}{(1-\omega^{-1}(\varpi)q_F^{-1+2s})(1-\omega(\varpi)q_F^{-1-2s})} \tag{4.2.14}$$

In this case, we have

$$X_\alpha(s) = \omega(\varpi)q_F^{-2s} \tag{4.2.15}$$

and

$$q_{\alpha^*} = 1, \quad q_\alpha = q_F \tag{4.2.16}$$

Thus $\lambda(\alpha) = 1$ and $\lambda^*(\alpha) = 1$. The parameters in this case are simply q_F.

4.2.5. *Case IV.* If ω ramified and $\sigma \neq \sigma(\tau)$ or $\chi^2\chi'$ is ramified,

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q_F^{n(\sigma\times\Pi(\sigma)))-n(\sigma)} \tag{4.2.17}$$

In this case, we have

$$q_\alpha = 1, \quad q_{\alpha^*} = 1 \tag{4.2.18}$$

Thus $\lambda(\alpha) = 0$ and $\lambda^*(\alpha) = 0$. Thus the parameters in this case are trivial.

4.2.6. *The short root case.*

As an example, we also give an explicit computation in the short root case. Assume the Levi factor M of $P = MN$ is generated by the short root of split G_2. Let σ be an irreducible unitary supercuspidal representation of M. Let $\omega = \omega_\sigma$ be its central character. Then by [Sha91, Proposition 6.2] the Plancherel measure $\mu(s\widetilde{\alpha}, \sigma)$ is given by the formula

$$\mu(s\widetilde{\alpha}, \sigma) = \begin{cases} \gamma(G/P)^2 q_F^{n(\sigma)+n(\sigma\otimes\omega)} \frac{(1-\omega(\varpi)q_F^{-2s})(1-\omega(\varpi)^{-1}q_F^{2s})}{(1-\omega(\varpi)q^{-1-2s})(1-\omega(\varpi)^{-1}q_F^{-1+2s})} & \text{if } \omega \text{ is unramified} \\ \gamma(G/P)^2 q_F^{n(\sigma)+n(\omega)+n(\sigma\otimes\omega)} & \text{otherwise} \end{cases} \tag{4.2.19}$$

Here $n(\sigma)$, $n(\omega)$ and $n(\sigma \otimes \omega)$ are the corresponding conductors.

4.2.7. *Case I.* If ω is unramified,

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q_F^{n(\sigma)+n(\sigma\otimes\omega)} \frac{(1-\omega(\varpi)q_F^{-2s})(1-\omega(\varpi)^{-1}q_F^{2s})}{(1-\omega(\varpi)q_F^{-1-2s})(1-\omega(\varpi)^{-1}q_F^{-1+2s})} \tag{4.2.20}$$

We compare 4.2.20 to (4.1.5). This implies that

$$q_\alpha = q_F, \quad q_{\alpha^*} = 1 \tag{4.2.21}$$

Since χ_s is an unramified character of M, we have

$$X_\alpha(\chi_s) = \omega(\varpi)q_F^{-2s}. \tag{4.2.22}$$

Recall from (4.1.6) that

$$q_F^{\lambda(\alpha)} = q_\alpha q_{\alpha^*} \in \mathbb{R}_{>1}. \tag{4.2.23}$$

Thus by 4.2.21, we have $q_F^{\lambda(\alpha)} = q_F$ and thus $\lambda(\alpha) = 1$ and $\lambda^*(\alpha) = 1$.

Note that for $w \in W(M, \mathcal{O})$, one may check that

$$w(X_\alpha) = X_{w(\alpha)} \tag{4.2.24}$$

Since $w(\alpha) = \alpha$ for $w \in W(M, \mathcal{O})$ in the $G = G_2$ case, (4.2.24) is simply $w(X_\alpha) = X_\alpha$. On the other hand, by [Sol21b, Prop 1.1] we have

$$wX_\alpha(\chi) = w(X_\alpha(\chi)) = w(\chi(h_\alpha)) = \chi(w(h_\alpha)) = \chi(h_{w(\alpha)}) = \chi(h_\alpha) = X_\alpha(\chi)$$

Thus $wX_\alpha = X_\alpha = X_{w(\alpha)}$. Since $\Sigma_{\mathcal{O}}^\vee = \{2\alpha + \beta\}$ in the short root $M = M_\alpha$ case,

$$W(\Sigma_{\mathcal{O}}^\vee) = \{1, s_{2\alpha+\beta}\} \tag{4.2.25}$$

Therefore we have the affine Hecke algebra

$$\mathcal{H}_{\mathrm{aff}}(M_\alpha) = \mathcal{H}(\{1, s_{2\alpha+\beta}\}, q_F) \ltimes \mathbb{C}[\mathcal{O}]. \tag{4.2.26}$$

4.2.8. *Case II.* In the other case,

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q^{n(\sigma)+n(\omega)+n(\sigma\otimes\omega)} \tag{4.2.27}$$

Comparing (4.2.27) and (4.1.5) gives us

$$q_\alpha = 1, \quad q_{\alpha^*} = 1. \tag{4.2.28}$$

Therefore we have $q_F^\lambda = 1$ and thus $\lambda(\alpha) = 0$ and $\lambda^*(\alpha) = 0$. Therefore the affine Hecke algebra in this case is given by

$$\mathcal{H}_{\mathrm{aff}}(M_\alpha) = \mathcal{H}(\{1, s_{2\alpha+\beta}\}, 1) \ltimes \mathbb{C}[\mathcal{O}]. \tag{4.2.29}$$

Remark 4.2.30. The computations of Hecke algebras with explicit parameters in this section will be collected into tables in the next section.

4.3. **Intertwining algebras of Kim-Yu types for G_2.**

For $b \in F^\times/F^{\times 2}$, let $\mathrm{U}_b(1,1)$ be the quasi split unitary group and $\mathrm{U}_b(2)$ be the compact unitary group in two variables in $F(\sqrt{b})$. Writing as $F^\times/F^{\times 2} = \{1, \varepsilon, \varpi, \varepsilon\varpi\}$, the possible unitary groups in 2 variables are:

$$\mathrm{U}_\varepsilon(1,1),\ \mathrm{U}_\varepsilon(2),\ \mathrm{U}_\varpi(1,1),\ \mathrm{U}_\varpi(2),\ \mathrm{U}_{\varepsilon\varpi}(1,1),\ \mathrm{U}_{\varepsilon\varpi}(2).$$

The group $\mathrm{U}_\epsilon(1,1)$ is an unramified group, and $\mathrm{U}_{\varpi'}(1,1)$ is ramified where $\varpi' \in \{\varpi, \varepsilon\varpi\}$.

We now enumerate the twisted Levi sequences in G_2 (up to conjugacy) such that $\mathbf{M} = \mathbf{M}_\gamma$ with $\gamma \in \{\alpha, \beta\}$:

(1) Essentially depth zero case [AEF+21]: If ρ_{Σ_M} is an essentially depth zero supercuspidal type on M, then Σ_M is of the form (M, y, ϕ, r, ρ_M) (hence in particular $M^0 = M$), where $K_{\Sigma_M} = M_{y,0} \simeq \mathrm{GL}_2(\mathfrak{o}_F)$ is a maximal compact subgroup of $\mathrm{GL}_2(F)$ and $r = \mathrm{depth}(\rho_{\Sigma_M})$ is an integer. If $r = 0$, we may assume that $\phi = 1$ without loss of generality.

(a) $\vec{\mathbf{G}} = (\mathbf{G})$ (here $\mathbf{G}^0 = \mathbf{G}$, it is a depth zero case: $r = 0$),
(b) $\vec{\mathbf{G}} = (\mathbf{M}^0, \mathbf{G})$ (here $\mathbf{G}^0 = \mathbf{M} = \mathbf{M}^0$ and $r \neq 0$).

(2) Positive depth cases [AEF+21]:
(a) $\vec{\mathbf{G}} = (\mathrm{U}_\varepsilon(1,1), \mathbf{G})$,
(b) $\vec{\mathbf{G}} = (\mathrm{U}_{\varpi'}(1,1), \mathbf{G})$, with $\varpi' \in \{\varpi, \varepsilon\varpi\}$,
(c) $\vec{\mathbf{G}} = (\mathbf{M}^0, \mathbf{G})$,
(d) $\vec{\mathbf{G}} = (\mathbf{M}^0, \mathbf{M}, \mathbf{G})$,
where $\mathbf{M}^0$ is a torus in $\mathbf{G}^0$.

When $\mathbf{M} = \mathbf{M}_\gamma$, we have three possibilities for $\mathbf{M}^0$, denoted $T_{\gamma,\varepsilon}$, $T_{\gamma,\varpi}$ and $T_{\gamma,\varpi}$. If ϕ_0 has trivial restriction to Z_M°, the it can be extended to to a character of U_b and we use the same notation ϕ_0 for the extended character.

Let $\mathbb{G}_y^0$ denote the reductive quotient of G_y^0. Let $\varpi' \in \{\varpi, \varepsilon\varpi\}$. We have

$$\mathbb{G}_y^0 = \begin{cases} \mathrm{U}(1,1) & \text{if } G^0 = \mathrm{U}_\varepsilon(1,1), \\ \mathrm{SO}(2) & \text{if } G^0 = \mathrm{U}_{\varpi'}(1,1). \end{cases}$$

Remark 4.3.1. The central character ω_σ of σ can be either ramified or unramified. It is unramified if and only if ω_{σ^0} is trivial. In the case when ω_σ is ramified, ω_{σ^0} is quadratic.

Lemma 4.3.2. *We have*

$$W_G^{\mathfrak{s}} \simeq W_{G^0}^{\mathfrak{s}^0}.$$

Proof. The representation σ is regular and p is good for G (that is, $p \neq 2, 3$) and does not divide the order ($= 1$) of the fundamental group of $\mathbf{G}_{\mathrm{der}}$. Hence Theorem 2.4.7 applies. □

4.4. **The intertwining algebras of types attached to G^0.**

4.4.1. *The case $G^0 = M^0$.* It occurs in both the essentially depth-zero case with $r \neq 0$ and in the positive depth cases. We have two possibilities for M^0: either $M^0 \simeq \mathrm{GL}_2(F)$ or M^0 is a torus. In both cases, the algebra $\mathcal{H}(G^0, \rho_{\mathcal{D}^0}) = \mathcal{H}(M^0, \rho_{\mathcal{D}^0})$ is commutative by [BK98, 5.5,5.6].

4.4.2. *The case $G^0 = \mathrm{U}_\varepsilon(1,1)$.* If $W_{G^0}^{\mathfrak{s}^0} = \{1\}$, then $\mathcal{H}(G^0, \rho_{\mathcal{D}^0})$ as seen in Remark 2.3.17 is commutative.

We suppose from now on that $W_{G^0}^{\mathfrak{s}^0} \neq \{1\}$. We write $a \mapsto \overline{a}$ for the non-trivial element of $\mathrm{Gal}(L/F)$, and set

$$w^0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w_1 := \begin{pmatrix} 0 & \overline{\varpi}_L^{-1} \\ \varpi_L & 0 \end{pmatrix}, \quad \textbf{and} \quad \mathfrak{P} := \begin{pmatrix} \mathfrak{o}_L^\times & \mathfrak{o}_L \\ \mathfrak{p}_L & \mathfrak{o}_L^\times \end{pmatrix} \cap G^0.$$

Recall that $\widetilde{\rho_{\mathcal{D}^0}}$ denotes the contragredient representation of ρ^0. Then the Iwahori-Matsumoto presentation of $\mathcal{H}(\mathrm{U}_\varepsilon(1,1), \rho_{\mathcal{D}^0})$ is the following (see [Bad20, §3.1]): $\mathcal{H}(\mathrm{U}_\varepsilon(1,1), \rho_{\mathcal{D}^0})$ is the space spanned by the functions

$$T_{w_i} \colon G^0 \to \mathrm{End}_G(V_{\widetilde{\rho_{\mathcal{D}^0}}}), \quad \text{for } i \in \{0, 1\},$$

such that

$$T_{w_i}(pgp') = \widetilde{\rho_{\mathcal{D}^0}}(p)T_{w_i}(g)\widetilde{\rho_{\mathcal{D}^0}}(p'), \quad \text{where } p, p' \in \mathfrak{P} \text{ and } g \in G^0, \tag{4.4.1}$$

where T_{w_i} is supported on $\mathfrak{P}w_i\mathfrak{P}$ and satisfies the quadratic relation

$$(T_{w_i} - q_F)(T_{w_i} + 1) = 0. \tag{4.4.2}$$

We can deduce the Bernstein presentation of $\mathcal{H}(\mathrm{U}_\varepsilon(1,1), \rho_{\mathcal{D}^0})$ using [Lus89, §3]. In particular, we have $q_F^{\lambda(\alpha)} = q_F$.

4.4.3. *The case* $G^0 = \mathrm{U}_{\varpi'}(1,1)$. For $\varpi' \in \{\varpi, \varepsilon\varpi\}$, since $\mathrm{U}_{\varpi'}(1,1)$ is ramified, it follows from [Bad20, §5.1.1] that the algebra $\mathcal{H}(\mathrm{U}_{\varpi'}(1,1), \rho_{\mathcal{D}^0})$ has trivial parameters with $R(\mathcal{O}^0) \neq 1$ and $W_{\mathcal{O}^0} = 1$ if $\omega_\sigma|_{\mathfrak{o}_F^\times} \neq 1$, and the Hecke algebra has parameter q_F otherwise (in which case $W_{\mathcal{O}^0} \neq 1$ and $R(\mathcal{O}^0) = 1$).

4.5. **The intertwining algebras of types attached to G.**

4.5.1. *Long root essentially depth zero case.*
(a) $r = 0$, $\chi^3 = 1$ case and $\sigma = \sigma(\tau)$ for $\tau = \mathrm{Ind}_{W_L}^{W_F}\chi$:

We have ρ_M self-dual, σ and τ correspond via LLC for $\mathrm{GL}_2(F)$. Since σ has depth zero, L/F is unramified (so $e(L/F) = 1$ and $f(L/F) = 2$). We have the following four cases:

- The central character $\omega_\sigma = 1$ and $\chi^2\chi'^{-1}$ unramified. This corresponds to Case 4.2.2, in which case the Plancherel formula has a zero, and the Hecke algebra is affine non-commutative, with parameters q_F^3 and q_F. We have $W_{\mathcal{O}} \neq 1$ and $R(\mathcal{O}) = 1$. Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.
- The central character $\omega_\sigma \neq 1$ is ramified, and $\chi^2\chi'^{-1}$ is unramified. This corresponds to Case 4.2.3, in which case the Plancherel formula has a zero, and the Hecke algebra is affine non-commutative, with parameters q_F^2. We have $W_{\mathcal{O}} \neq 1$ and $R(\mathcal{O}) = 1$. Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.
- The central character $\omega_\sigma = 1$ and $\chi^2\chi'^{-1}$ ramified. This corresponds to Case 4.2.4, in which case the Plancherel formula has a zero, and the Hecke algebra is affine non-commutative, with parameter q_F. We have $W_{\mathcal{O}} \neq 1$ and $R(\mathcal{O}) = 1$. Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.
- The central character $\omega_\sigma \neq 1$ is ramified, and $\chi^2\chi'^{-1}$ is ramified. This corresponds to Case 4.2.5, in which case the Plancherel formula has no zero, and the Hecke algebra is affine commutative of the form $\mathbb{C}[R(\mathcal{O})]$ plus the translation part $\mathbb{C}[\mathcal{O}]$. We have $W_{\mathcal{O}} = 1$ (and we don't know what $R(\mathcal{O})$ is in this case). Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.

(b) $r = 0$ and $\sigma \neq \sigma(\tau)$: We have $\sigma = \sigma(\tau')$ where $\tau' = \mathrm{Ind}_{W_L}^{W_F}\zeta$ for ζ such that $\zeta^{-1} = \overline{\zeta}$ (the Galois conjugate). Since σ is still depth zero, we still have L/F unramified.

- The central character $\omega_\sigma = 1$. This corresponds to Case 4.2.4, in which case the Plancherel formula has a zero, and the Hecke algebra is affine non-commutative, with parameters q_F. We have $W_{\mathcal{O}} \neq \{1\}$ and $R(\mathcal{O}) = \{1\}$. Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.
- The central character $\omega_\sigma \neq 1$ ramified. This corresponds to Case 4.2.5, in which case the Plancherel formula has no zero, and the Hecke algebra is affine commutative of the form $\mathbb{C}[R(\mathcal{O})]$ plus the translation part $\mathbb{C}[\mathcal{O}]$. we have $W_{\mathcal{O}} = \{1\}$ (and we don't know what $R(\mathcal{O})$ is in this case). Since $G = G^0$ in this case, $W_{\mathcal{O}} = W_{\mathcal{O}^0}$ and $R(\mathcal{O}) = R(\mathcal{O}^0)$.

(c) $r \neq 0$ essentially depth zero case: Recall from §4.4.1 that $G^0 = M = M^0$. Thus we have

$$W_{G^0}^{\mathfrak{s}^0} \subset \mathrm{N}_{G^0}(M^0)/M^0 = \mathrm{N}_M(M)/M = \{1\}.$$

By Lemma 4.3.2, we get $W_G^{\mathfrak{s}} = \{1\}$. In this case, $W(M, \mathcal{O}) = W(M^0, \mathcal{O}^0) = 1$. Thus the algebras $\mathcal{H}(G, \rho)$ and $\mathcal{H}(G^0, \rho^0)$ are both of the form $\mathbb{C}[\mathcal{O}]$, and they are isomorphic.

4.5.2. *Table for long root essentially depth zero cases.*

r	$\mathcal{D}$	ω_σ	$\chi^2\chi'^{-1}$	$R(\mathcal{O})$	$R(\mathcal{O}^0)$	L/F	$\#X_{\mathrm{nr}}(M,\sigma)$	$W_{\mathcal{O}}$	$W_{\mathcal{O}^0}$	$\mathcal{H}(G,\rho)$	$\mathcal{H}(G^0,\rho^0)$
$r=0$	$((G,M),(y,\iota),(M_{y,0},\rho_M))$	$=1$	unramified χ cubic	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm, q_F^3, q_F	non-comm, q_F^3, q_F
		$\neq 1$	unramified χ cubic	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm, q_F^2	non-comm, q_F^2
		$=1$	ramified χ cubic	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm, q_F	non-comm, q_F
		$\neq 1$	ramified χ cubic	$*$	$*$	unramified	2	$=1$	$=1$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$
		$=1$	χ not cubic	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm,q_F	non-comm,q_F
		$\neq 1$	N/A	$*$	$*$	unramified	2	$=1$	$=1$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$
$r \neq 0$	$(((M,G),M),(y,\iota),(r,0),(\phi,1),(M_{y,0},\rho_M))$	$\neq 1$	N/A	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}]$

TABLE 4.5.2.

4.5.3. *Long root positive depth case.*
(a) $\mathrm{U}_{\varpi'}(1,1)$ case: $\sigma = \sigma(\tau') \neq \sigma(\tau)$, where τ' is induction of some quadratic character. (Note that the cubic character only occurs in depth zero, because we are assuming $p \neq 2, 3$. There are two possibilities, $\phi_0|_{\mathrm{Z}_M^0}$ could be either trivial or non-trivial:

- When $\phi_0|_{\mathrm{Z}_M^0} = 1$ unramified, since $\sigma = \sigma(\tau') \neq \sigma(\tau)$, this corresponds to Case 4.2.4, in which case the Plancherel formula has a zero, and the Hecke algebra is affine non-commutative, with parameters q_F. We have $W_{\mathcal{O}} \neq \{1\}$ and $R(\mathcal{O}) = 1$. By 4.4.3, the Hecke algebra for G^0 also has q_F parameter. Thus we have $W_{\mathcal{O}^0} \neq \{1\}$ and $R(\mathcal{O}^0) = 1$.
- When $\phi_0|_{\mathrm{Z}_M^0}$ = sign character ramified, since $\sigma = \sigma(\tau') \neq \sigma(\tau)$, this corresponds to Case 4.2.5, in which case the Plancherel formula has no zero, and the Hecke algebra is of the form $\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$. We have $W_{\mathcal{O}} = \{1\}$. By 4.4.3, we have $W_{\mathcal{O}^0} = \{1\}$ and $R(\mathcal{O}^0) \neq 1$. Thus $R(\mathcal{O}) \cong R(\mathcal{O}^0) \neq 1$ by 2.4.7.

(b) $\mathrm{U}_\epsilon(1,1)$ case: $\sigma = \sigma(\tau') \neq \sigma(\tau)$, where τ' is the induction of some quadratic character.

- When $\phi_0|_{Z_M^0} = 1$ unramified, since $\sigma = \sigma(\tau') \neq \sigma(\tau)$, this corresponds to 4.2.4, in which case the Plancherel formula has a zero, and the Hecke algebra for G is affine non-commutative, with parameters q. We have $W_{\mathcal{O}} \neq \{1\}$ and $R(\mathcal{O}) = \{1\}$. From 4.4.2, we have $W_{\mathcal{O}^0} \neq \{1\}$ and $R(\mathcal{O}^0) = \{1\}$. Note that the cardinality of $\mathfrak{X}_{\mathrm{nr}}(M, \sigma(\tau'))$ is 2 (see Remark 2.1.7).

4.5.4. *Table for long root positive depth cases.*

We summarize the above in the following table:

M^0	$\phi_0\vert_{Z_M^0}$	ϕ_1	$\vec{G}$	$R(\mathcal{O})$	$R(\mathcal{O}^0)$	L/F	$\#X_{\mathrm{nr}}(M,\mathcal{D})$	$W_{\mathcal{O}}$	$W_{\mathcal{O}^0}$	$\mathcal{H}(G,\rho)$	$\mathcal{H}(G^0,\rho^0)$
$T_{\beta,\varpi'}$	$=1$	$=1$	$(U_{\varpi'}(1,1),G)$	$=1$	$=1$	ramified	1	$\neq 1$	$\neq 1$	non-comm, q_F	non-comm, q_F
	$=$ sign character	$\neq 1$		$\neq 1$	$\neq 1$	ramified	1	$=1$	$=1$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$
	$\neq 1$ $\neq$ sign character	$=1$	(M^0,G)	$=1$	$=1$	ramified	1	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
	both	$\neq 1$	(M^0,M,G)	$=1$	$=1$	ramified	1	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
$T_{\beta,\varepsilon}$	$=1$	$=1$	$(U_\varepsilon(1,1),G)$	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm. q_F	non-comm. q_F
	$\neq 1$	$=1$	(M^0,G)	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
	both	$\neq 1$	(M^0,M,G)	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$

TABLE 4.5.4.

4.5.5. *Short root essentially depth zero case.*

(a) $r = 0$**,** there are only two cases:

- When $\rho_M|_{Z_M^\circ} = 1$, this corresponds to the central character being unramified case, and in this case the Plancherel formula in 4.2.7 has a zero. Thus thus $W_{\mathcal{O}} \neq 1$ and thus $R(\mathcal{O}) = 1$. In this case the Hecke algebra is non-commutative, and the q-parameter is just $q = q_F$. The case for G^0 again follows from 4.4.3.
- When $\rho_M|_{Z_M^\circ} \neq 1$, this corresponds to the central character being ramified case, and in this case the Plancherel formula in 4.2.8 has no zero, and thus $W_{\mathcal{O}} = \{1\}$. In this case the Hecke algebra is commutative, and the q-parameter is trivial.

(b) $r \neq 0$ essentially depth zero case. The same argument as in §4.5.1(c) applies here.

4.5.6. *Table for short root essentially depth zero cases.*

r	$\mathcal{D}$	ω_σ	$R(\mathcal{O})$	$R(\mathcal{O}^0)$	L/F	$\#X_{\mathrm{nr}}(M,\sigma)$	$W_{\mathcal{O}}$	$W_{\mathcal{O}^0}$	$\mathcal{H}(G,\rho)$	$\mathcal{H}(G^0,\rho^0)$
$r=0$	$((G,M),(y,\iota),(M_{y,0},\rho_M))$	$=1$	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm, q_F, q_F	non-comm, q_F, q_F
		$\neq 1$	$*$	$*$	unramified	2	$=1$	$=1$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$
$r \neq 0$	$(((M,G),M),(y,\iota),(r,0),(\phi,1),(M_{y,0},\rho_M))$	$=1$ / $\neq 1$	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$

TABLE 4.5.6.

4.5.7. *Short root positive depth case.*
(a) $G^0 = \mathrm{U}_{\varpi'}(1,1)$ case:

- When $\phi_0|_{\mathrm{Z}^0_M} = 1$, the Plancherel formula on the G_2 side in 4.2.7 has a zero, and thus $W_{\mathcal{O}} \neq \{1\}$ and thus $R(\mathcal{O}) = \{1\}$. In this case the Hecke algebra $\mathcal{H}(G,\rho)$ is non-commutative, and the q-parameter is just $q = q_F$. By 2.4.7, we have $W(M^0, \mathcal{O}^0) \neq 1$, and since $W_{\mathcal{O}^0} \neq 1$ by 4.4.3, we have $R(\mathcal{O}^0) = 1$. Moreover, the Hecke algebra $\mathcal{H}(G^0, \rho^0)$ has parameter q_F by 4.4.3.
- When the central character (of $\mathrm{GL}_2^{\mathrm{short}}$) $\phi_0|_{Z^0_M}$ = sign character $\neq 1$ is ramified, the Plancherel formula on the G2 side in 4.2.8 has no zero, and thus $W_{\mathcal{O}} = \{1\}$. In this case the Hecke algebra $\mathcal{H}(G,\rho) = \mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$ has trivial q-parameter. On the other hand, since $I(\sigma)$ is reducible by [Sha91, Proposition 6.2], we have $R_\sigma \neq 1$. Since $W_\sigma \rtimes R_\sigma = W(M,\sigma) \subseteq W(M,\mathcal{O}) = W_{\mathcal{O}} \rtimes R(\mathcal{O})$ and $W_{\mathcal{O}} = 1$, we have $R_\sigma \subseteq R(\mathcal{O})$ and thus $R(\mathcal{O}) \neq 1$. By 2.4.7, we also have $R(\mathcal{O}^0) \neq 1$ since $W_{\mathcal{O}^0} = 1$ by 4.4.3.

(b) $G^0 = \mathrm{U}_\epsilon(1,1)$ case: When $\phi_0|_{\mathrm{Z}^0_M} = 1$ the Plancherel formula on the G_2 side in 4.2.7 has a zero, and thus $W_{\mathcal{O}} \neq \{1\}$ and thus $R(\mathcal{O}) = \{1\}$. In this case the Hecke algebra $\mathcal{H}(G,\rho)$ is non-commutative, and the q-parameter is just $q = q_F$. From 4.4.2, we have $W_{\mathcal{O}^0} \neq \{1\}$ and $R(\mathcal{O}^0) = \{1\}$.

4.5.8. *Table for short root positive depth cases.* We summarize the above in the following table:

M^0	$\phi_0\vert_{\mathrm{Z}^0_M}$	ϕ_1	$\vec{G}$	$R(\mathcal{O})$	$R(\mathcal{O}^0)$	L/F	$\mathfrak{X}_{\mathrm{nr}}(M,\sigma)$	$W_{\mathcal{O}}$	$W_{\mathcal{O}^0}$	$\mathcal{H}(G,\rho)$	$\mathcal{H}(G^0,\rho^0)$
$T_{\alpha,\varpi'}$	$=1$	$=1$	$(U_{\varpi'}(1,1),G)$	$=1$	$=1$	ramified	1	$\neq 1$	$\neq 1$	non-comm, q_F	non-comm, q_F
	$=$ sign character	$\neq 1$		$\neq 1$	$\neq 1$	ramified	1	$=1$	$=1$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}]$	$\mathbb{C}[R(\mathcal{O})] \ltimes \mathbb{C}[\mathcal{O}^0]$
	$\neq 1$ $\neq$ sign character	$=1$	(M^0,G)	$=1$	$=1$	ramified	1	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
	both	$\neq 1$	(M^0,M,G)	$=1$	$=1$	ramified	1	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
$T_{\alpha,\varepsilon}$	$=1$	$=1$	$(U_\varepsilon(1,1),G)$	$=1$	$=1$	unramified	2	$\neq 1$	$\neq 1$	non-comm. q_F	non-comm. q_F
	$\neq 1$	$=1$	(M^0,G)	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$
	both	$\neq 1$	(M^0,M,G)	$=1$	$=1$	unramified	2	$=1$	$=1$	$\mathbb{C}[\mathcal{O}]$	$\mathbb{C}[\mathcal{O}^0]$

TABLE 4.5.8.

4.5.1. We keep the notations of §2.1. The following Theorem establishes the validity for G_2 of an extension of a conjecture of Yu's [Yu01, Conjecture 0.2] for supercuspidal types, which was proved by Ohara in [Oha21].

The following result shows that a stronger version of Theorem 2.4.9(2) holds for the group G_2.

Theorem 4.5.2. *We assume $p \neq 2,3$. Then the algebras $\mathcal{H}^{\mathfrak{s}}(G) = \mathrm{End}_G(\Pi^{\mathfrak{s}}_G)$ and $\mathcal{H}^{\mathfrak{s}^0}(G^0) = \mathrm{End}_{G^0}(\Pi^{\mathfrak{s}^0}_{G^0})$ are isomorphic*

Proof. From Proposition 2.3.15, it equivalent to show that the algebras $\mathcal{H}(G, \rho_{\mathcal{D}})$ and $\mathcal{H}(G^0, \rho_{\mathcal{D}^0})$ are isomorphic. The latter can be read directly from the tables 4.5.2, 4.5.4, 4.5.6 and 4.5.8. □

The following corollary is a stronger version of Lemma 4.3.2 (i.e. [AM21, Theorem 7.3]) for $G = G_2$.

Corollary 4.5.3. *The groups $R(\mathcal{O}) \simeq R(\mathcal{O}^0)$ and $W_{\mathcal{O}} \simeq W_{\mathcal{O}^0}$.*

Proof. This can be read directly from our tables 4.5.2, 4.5.4, 4.5.6 and 4.5.8, with explanations given in the sections before them. □

4.5.9. *On Lusztig's conjecture.*

Let $L^{\mathfrak{s}} \colon W^{\mathfrak{s}}_{\mathrm{aff}} \to \mathbb{N}$ be the weight function[2] on $W^{\mathfrak{s}}_{\mathrm{aff}}$ defined by

$$L^{\mathfrak{s}}(s_\alpha) := \lambda(\alpha) \quad \text{and} \quad L^{\mathfrak{s}}(s'_\alpha) := \lambda^*(\alpha). \tag{4.5.4}$$

In [Lus20, §1.a], Lusztig made the following conjecture.

Conjecture 4.5.5. (Lusztig) The function $L^{\mathfrak{s}}$ on the affine Weyl group $W^{\mathfrak{s}}_{\mathrm{aff}}$ is in the collection of weight functions described in [Lus91, Lus95, Lus02].

Solleveld has proved the validity of Conjecture 4.5.5 in many cases, including the representation in the principal series of G, in [Sol21b].

Theorem 4.5.6. *Conjecture 4.5.5 holds for the group G_2.*

Proof. It follows from Tables 4.5.2, 4.5.4, 4.5.6 and 4.5.8. □

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[2] i.e., $L^{\mathfrak{s}}(w) > 0$ for all $w \in W^{\mathfrak{s}}_{\mathrm{aff}} - \{1\}$, and $L^{\mathfrak{s}}(ww') = L^{\mathfrak{s}}(w) + L^{\mathfrak{s}}(w')$ for any $w, w' \in W^{\mathfrak{s}}_{\mathrm{aff}}$ such that $\ell(ww') = \ell(w) + \ell(w')$.

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Sorbonne Université and Université de Paris, CNRS, IMJ-PRG, F-75005 Paris, France
Email address: anne-marie.aubert@imj-prg.fr

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, Massachusetts 02138, USA
Email address: yujiex@math.harvard.edu