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GRAVITATIONAL MONOPOLES

GOURAB BHATTACHARYA

ABSTRACT. We introduce a new equation for the self-dual Weyl tensor W^+ and a harmonic spinor ψ on a 4-manifold. The equation is,

$$(0.1) \quad \begin{aligned} \nabla\!\!\!/ \psi &= (d + d^*)\psi = 0, \\ c(W_g^+) &= \frac{1}{4}(e_i \cdot e_j \psi, \psi) e^i \wedge e^j, \end{aligned}$$

c is the Clifford map.

1. PHYSICAL MOTIVATIONS

In this research announcement, it will be described that a gravitational analogue of the usual monopole equation of Seiberg-Witten [6] is feasible in 4-dimensions, and will be described in somehow informally.

It is well-known that the $U(1)$ electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 0, 1, 2, 3$ (on a 4-dimensional Minkowski background) has a decomposition in terms of $SL(2, \mathbb{C})$ -spinors, more specifically, one has,

$$(1.1) \quad F_{AA'BB'} = F_{AB'AB'} = \Phi_{(AB)}\epsilon_{A'B'} + \Psi_{(A'B')}\epsilon_{AB}.$$

One usually identifies $\Phi_{(AB)}\epsilon_{A'B'}$ with the self-dual part $F_{AB'AB'}^+$ of $F_{AB'AB'}$. Similar for $\Psi_{(A'B')}\epsilon_{AB}$. It is interesting to note that in [6] Witten assumed, the self-dual part of $F_{\mu\nu}$, that is $\Psi_{(A'B')}$ is equal to $\psi \frac{1}{2}[\gamma^\mu, \gamma^\nu]\bar{\psi}$ coupled to the Dirac equation $\gamma^\mu \partial_{A_\mu} \psi = 0$.

A similar idea was floating for many decades for the whole Riemann-Christoffel curvature tensor, however, the gigantic nature of the curvature tensor and various difficulties associated with the complexification of Lorentzian manifold (as in Euclidean Quantum Gravity), one may not able to proceed very far. The new idea is to exploit the King-Nekrasov equation [1] and consider it as a quantization condition for Hermitian manifolds. In a separate article, the ideas of the Hermitian Quantization are described in [4].

The Gravitational analogue of the electromagnetism is encoded in the Weyl tensor and corresponding 2-form. In this article, we shall describe it in a somewhat sketchy way, the details will be written elsewhere.

So, we shall start with the decomposition of the Riemann-Christoffel curvature tensor for a Lorentzian metric in terms of $SL(2, \mathbb{C})$ -spinors, and hope for a similar generalization in terms of Weyl tensor. A decomposition of the Riemann-Christoffel tensor R_{abcd} in terms of 2-spinors is the following,

$$(1.2) \quad \begin{aligned} R_{abcd} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \\ &+ \frac{R}{24}(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}) \end{aligned}$$

The conformally invariant part of the Weyl-tensor then has the following decomposition,

$$(1.3) \quad W_{abcd} = W_{abcd}^+ + W_{abcd}^-$$

with,

$$(1.4) \quad W_{abcd}^- = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}$$

and,

$$(1.5) \quad W_{abcd}^+ = \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$$

Assume,

$$(1.6) \quad \begin{aligned} \partial_{AA'}\phi_A &= 0 \\ \Psi_{ABCD} &= \frac{i}{4!}(\phi_A\bar{\phi}_B\phi_C\bar{\phi}_D + \phi_B\bar{\phi}_C\phi_D\bar{\phi}_A + \phi_C\bar{\phi}_D\phi_A\bar{\phi}_B + \phi_D\bar{\phi}_A\phi_B\bar{\phi}_C) \end{aligned}$$

with the following convention with the Weyl tensor,

$$(1.7) \quad \Psi_{ABCD} = W_{ijkl}\Gamma_{ABCD}^{ijkl}$$

where Γ_{ABCD}^{ijkl} is totally antisymmetric in γ^i -matrices $\forall i = 1, 2, 3, 4$.

Now one can try the following sets of equations also,

$$(1.8) \quad \Psi_{ABCD} = \frac{i}{4!}\phi_{(A}\phi_B\phi_C\phi_{D)}, \quad \partial\phi = 0.$$

The Anti-Self-Duality condition on the Yang-Mills curvature is

$$(1.9) \quad F_A^+ = 0.$$

One similarly can consider the Gravitational instanton condition on the Weyl tensor, and there is a huge amount of literature dedicated to the following equation

$$(1.10) \quad W^+ = 0.$$

The choice of the *Gravitational Monopole Equation*

$$(1.11) \quad \begin{aligned} \nabla\psi &= 0, \\ W^+ &= q(\psi) \end{aligned}$$

seems new. The name Gravitational monopole was chosen with a similar terminology to [6]. In the next section, we formalised the above physical motivation. In the section [6] the author mentioned a curious finding of his [3], namely a relationship between Perelman's $\bar{\lambda}$ -functional and the signature $\sigma(M)$ of a self-dual manifold M that allows the Gravitational monopole equations (3.2).

Remark 1.1. In a 4-dimensional curved space with a boundary and a Yang-Mills background field [9] the index is,

$$(1.12) \quad n_+ - n_- = -\frac{1}{384\pi^2} \int_X \text{Tr}(W \wedge W^*) + \frac{2}{3} \frac{T(T+1)(2T+1)}{16\pi^2} \int_X \text{Tr}(F \wedge F^*) - \frac{1}{2} \eta(0) + \int_{\partial X} Q$$

n_+ and n_- are the numbers of positive and negative helicity solutions of the zero-rest mass Dirac equation, $\eta(s) := \sum_{\mu} (\text{sign } \mu) |\mu|^{-s}$, η 's are eigenvalues of the Dirac operator at the boundary ∂X in question, T is the parameter of the $(2T+1)$ -dimensional representation space of the $SU(2)$, the integrand Q contains terms involving the curvature and the second fundamental form of the boundary in X and Yang-Mills connections. If the manifold X is not *Spin*, then the index $n_+ - n_-$ can be a rational number contrary to the definition of index, as $n_+ - n_- = \dim_{\mathbb{R}} \ker \not{D} - \dim_{\mathbb{R}} \ker \not{D}^* \in \mathbb{Z}$. So, a representation of a new structure resolves the ambiguity, known as *Spin^C*-structure. This can be crucial to our definition of the equation in the section [3].

2. FOUR-DIMENSIONAL RIEMANNIAN GEOMETRY.

The notations and facts of this section is taken from [7]. Let X be an oriented Riemannian manifold of even dimension $2l$ and we also assume X is a spin manifold, that is the first and second Stiefel-Whitney classes vanish. We denote by Λ^p the bundle of exterior p -forms with $A^p = \Gamma(\Lambda^p)$ its space of smooth sections. The Hodge star operator $*\Lambda^p \rightarrow \Lambda^{2l-p}$ is defined by,

$$(2.1) \quad \alpha \wedge *\beta = (\alpha, \beta)\omega \in \bigwedge^{2l}$$

where $\alpha, \beta \in \Lambda^p$, (α, β) is the induced inner product on p -forms and ω is the volume form.

From now everything will be 4-dimensional unless otherwise stated. We start with the symmetry of the equations, namely the Lie algebras. The Lie algebra $\mathfrak{so}(4)$ of the special orthogonal group $SO(4)$ is not simple. It can be decomposed into the direct sum of two copies the Lie algebra $\mathfrak{so}(3)$ of the group $SO(3)$:

$$(2.2) \quad \mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

In terms of the group theory, one understands the above decomposition corresponds to the fact that the universal covering group of $SO(4)$ is the product of the two copies of $SU(2)$. This fact in quantum mechanics corresponds to $\pm \frac{1}{2}$ spins of an electron for each factor $SU(2)$.

In terms of the geometry of the vector bundles, the decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ induces the following decomposition (for a choice of g on X^4) for the vector bundle $\bigwedge^2 T^*X \rightarrow X$,

$$(2.3) \quad \bigwedge^2 T^*X \cong \bigwedge^+ \oplus \bigwedge^-,$$

as a Whitney sum of two oriented 3-plane bundles. One can choose an oriented orthonormal frame for $T_U X$ for an open set $U \subset X$. One therefore has,

$$(2.4) \quad \bigwedge^\pm = \text{Span}\{(e^1 \wedge e^2 \pm e^3 \wedge e^4), (e^2 \wedge e^3 \pm e^1 \wedge e^4), (e^3 \wedge e^1 \pm e^2 \wedge e^4)\}.$$

We now use the unique Levi-Civita connection ∇ on $\bigwedge^2 T^*X$ to find a suitable decomposition of the curvature tensor under the action of $O(4)$. The first step towards it is to note that $\nabla g = 0$, this however means that ∇ is covariantly constant, that is ∇ maps sections of \bigwedge^\pm into $\bigwedge^\pm \otimes T^*X$; there is no mixed term mapping \bigwedge^+ into $\bigwedge^- \otimes T^*X$. The curvature of the Levi-Civita connection defines a section of $\bigwedge^2 T^*X \otimes \bigwedge^2 T^*X$, correspondingly a decomposition of $\bigwedge^2 T^*X \otimes \bigwedge^2 T^*X$ into four matrix-blocks of size 3×3 , more precisely, the Riemann curvature tensor defines, in general, a self-adjoint linear transformation $\mathcal{R} : \bigwedge^2 \rightarrow \bigwedge^2$ such that,

$$(2.5) \quad \mathcal{R}(e^i \wedge e^j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e^k \wedge e^l,$$

relative to the decomposition $\bigwedge^2 = \bigwedge^+ \otimes \bigwedge^-$, the operator \mathcal{R} has the following form,

$$(2.6) \quad \mathcal{R} = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$$

where, $B \in \text{Hom}(\bigwedge^-, \bigwedge^+)$ (is the traceless Ricci curvature) $\overset{0}{Ric}$, and $A \in \text{End}(\bigwedge^+)$, that is A is symmetric about its diagonal, that is $A^t = A$, similarly for $C \in \text{End}(\bigwedge^-)$ we have $C^t = C$.

Fix a Riemannian metric g on X and consider the $SO(3)$ bundle \bigwedge^+ with the induced Riemannian connection. The adjoint bundle \mathfrak{g} is in this case \bigwedge^+ itself and the curvature of the induced connection is that part of the Riemann tensor that belongs to $\bigwedge^+ \otimes \bigwedge^2$, that is,

$$(2.7) \quad F_\nabla = A + B^t \in A^2(\bigwedge^+)$$

Now since

$$(2.8) \quad A^2(\bigwedge^+) = A_+^2(\bigwedge^+) \oplus A_-^2(\bigwedge^+),$$

therefore,

$$(2.9) \quad F_\nabla = A + B^t \in A^2(\bigwedge^+) \implies F_\nabla = A + B^t \in A_+^2(\bigwedge^+) \oplus A_-^2(\bigwedge^+)$$

where ∇ is the induced Levi-Civita connection on Λ^+ and F_∇ the corresponding curvature, therefore,

$$(2.10) \quad A \in A_+^2(\underset{+}{\bigwedge}), \quad \text{and} \quad B^t \in A_-^2(\underset{+}{\bigwedge})$$

therefore the induced connection on Λ_+^2 is self-dual if and only if $B^t = 0$, in other words, if and only if the metric is Einstein. Therefore we get the following theorem,

Theorem 2.1. *A 4-manifold X is Einstein if and only if the Levi-Civita connection on Λ^+ is self-dual.*

So, we can write the curvature tensor as,

$$(2.11) \quad \mathcal{R} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} W_g^+ + \frac{s}{12} \cdot \text{Id} & 0 \\ 0 & W_g^- - \frac{s}{12} \cdot \text{Id} \end{bmatrix} = \begin{bmatrix} F_\nabla^+ & 0 \\ 0 & F_\nabla^- \end{bmatrix}$$

If we denote the projection operator by,

$$(2.12) \quad P_\pm := \frac{1}{2}(1 \pm *) : \Lambda \rightarrow \Lambda^\pm,$$

then,

$$(2.13) \quad W^\pm = P_\pm \circ Rm \circ P_\pm - \frac{s}{12} Id_\pm$$

Definition 2.2. The metric g is called self-dual (anti-self-dual) if $W^\mp = 0$ or, which is the same as $*W = \pm W$.

It is worth mentioning that reversing the orientation of X changes the self-dual part of curvature to the anti-self-dual part. The conformal flat metrics will be anti-self-dual. We have a large number of anti-self-dual metrics that are Kähler metrics with zero scalar curvature, since the self-dual part of the Weyl tensor is given by,

$$(2.14) \quad W^+ = \left(\frac{s}{6}, -\frac{s}{12}, -\frac{s}{12} \right).$$

In particular, a $K3$ surface with a Calabi–Yau metric is anti-self-dual. This also follows from the Gauss–Bonnet formula and the Hirzebruch signature formula, since with the canonical orientation the index and the Euler characteristics are respectively $\tau = -16$, $\chi = 24$.

3. THE GRAVITATIONAL MONOPOLE EQUATION

Let (X^4, g) be a Riemannian spin 4-manifold. Then the Clifford algebra bundle $\mathcal{Cl}(X^4)$ is a vector bundle over X^4 with fibre at $x \in X^4$ is the Clifford algebra $\mathcal{Cl}(T_x X)$. With respect to the metric g , one identifies (isomorphism) $\mathcal{Cl}(T_x X)$ with $\mathcal{Cl}(T_x^* X)$. Therefore, as a vector space, this is isomorphic to $\wedge T_x^* X$. Let us also assume $E \rightarrow X$ is a Clifford module bundle with a covariant derivative ∇^E . Then for each $x \in X$ there is a Clifford action $c : T_x^* X \otimes E_x \rightarrow E_x$ via $c(\alpha \otimes s) = c(\alpha)s$.

Definition 3.1. The twisted Dirac operator associated to (E, ∇^E) is the operator,

$$(3.1) \quad \not{D} := c \circ \nabla^E : C^\infty(X, E) \rightarrow C^\infty(X, E).$$

The equations we wish to consider are (sometimes we omit the mapping c and the dimension 4 for the convenience of computations),

$$(3.2) \quad \begin{aligned} \not{D}\psi &= (d + d^*)\psi = 0, \\ c(W_g^+) &= \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j, \\ \text{or, } c\left((W_g^+)_{ijkl}\right) e^i \wedge e^j &= \frac{1}{4} \langle e_k \cdot e_l \psi, \psi \rangle. \end{aligned}$$

The Weitzenböck's formula [8]: the decomposition of the Laplace-Beltami operator as a generalized Laplacian is, with the left-Clifford multiplication $\epsilon_l = ext_l - int_l$, and corresponding right Clifford action c_r ,

$$(3.3) \quad (d + d^*)^2 = \Delta^{\wedge T^*X} - \sum_{ijkl} R_{ijkl} \epsilon_l^i \epsilon_l^j \epsilon_r^k \epsilon_r^l + \frac{1}{4} s.$$

4. THE SEIBERG-WITTEN EQUATIONS AS A REDUCTION

4.1. The Seiberg-Witten Equations. We here review the notions of the Seiberg-Witten equations. As we saw in the remark (1.1), if the manifold is not *Spin*, then due to index theoretic calculations, we are obliged to bring *Spin*^C-structure. More about the *Spin*^C-structures can be found in the book [10].

The Seiberg-Witten equations of a 4-manifold X are given in terms of a pair (A, ψ) where A is a spin-connection and ψ is a section of \mathbb{S}^+ or a section of $\mathbb{S}^+ \otimes \sqrt{L}$, in case the manifold is not *Spin* one twists the spin-bundle \mathbb{S}^+ by a square-root of a non-trivial line bundle L (that is $\sqrt{L} \otimes \sqrt{L} \cong L$). L -becomes trivial in case the manifold is *spin*. In the physics literature one usually twists by $L^2 = \mathcal{O}(2s)$, otherwise, the sections of $\sqrt{L} = \mathcal{O}(s/2)$ come with weight $s/2$ giving half-integral spins to bosons! but we will keep the mathematician's notation by giving a "wrong" spin to bosons keeping the tradition. Let \not{D}_A be the Dirac operator acting on the representation space of *Spin*^C group, and ∇ the induced connection on L , the equations are,

$$(4.1) \quad \begin{aligned} \not{D}_A \psi &= 0, \\ F_{\nabla}^+ &= \frac{1}{4} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j. \end{aligned}$$

$\{e_i\}$ is a local basis of TX , it acts on the spinor ψ by Clifford action $c(v) = ext(v) - int_v$.

4.2. The Reduction. One notes that, as a special case, from (2.11) and the new equation (3.2), one recovers the Seiberg-Witten equations, namely,

$$(4.2) \quad \begin{aligned} \not{D}_A \psi &= 0, \\ F_{\nabla}^+ &= q(\psi). \end{aligned}$$

where \not{D}_A is a Dirac operator on L (can be trivial) over X^4 (otherwise one twists the bundle by \sqrt{L}), and F_{∇} is the curvature of \wedge^+ .

Remark 4.1. One notes that the Seiberg-Witten equation (4.2) can be defined for \wedge^+ without the Einstein condition (2.11).

Proposition 1. For $k \neq l$, $\langle e_k e_l \psi, \psi \rangle$ is purely imaginary.

Proposition 2.

$$(4.3) \quad |\langle e_k e_l \psi, \psi \rangle|^2 = 2|\psi|^4.$$

We keep in mind the Seiberg-Witten analysis, and analogously define and get the following definition of *Gravitational-Monopole functional*,

Definition 4.2. The Gravitational-Monopole functional of a pair (ψ, g) is given by,

$$(4.4) \quad \begin{aligned} S(g, \psi) &= \int_X \left(|\nabla \psi|^2 + |W^+ - \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j|^2 \right) d(vol)_g \\ i.e., S(g, \psi) &= \int_X \left(|(d + d^*)\psi|^2 + |W^+ - \frac{1}{4} \langle e_i \cdot e_j \psi, \psi \rangle e^i \wedge e^j|^2 \right) d(vol)_g. \end{aligned}$$

Proposition 3.

$$(4.5) \quad S(g, \psi) = \int_X \left(|\nabla \psi|^2 + |W^+|^2 + \frac{s}{4} |\psi|^2 + \frac{1}{8} |\psi|^4 \right) d(vol)_g$$

Proposition 4. As a direct consequence of (3), if the scalar curvature of X is non-negative, all solutions of (3.2) have $\psi \equiv 0$.

5. COMPARISON WITH A KÄHLER SITUATION

Suppose now that (X, g, ω) is a Kähler manifold of real dimension four (see [11]). We fix the canonical orientation, ω is then a section of Λ^+ . For all $x \in X$, ω_x can be completed to an orthogonal basis ω_x, η, θ of Λ_x^+ with $|\eta| = |\theta| = \sqrt{2}$. Then the tensor $C_x = \eta \otimes \eta \oplus \theta \otimes \theta$ does not depend on the choice of η and θ , and one calculates that the tensor field C on X is parallel. One defines the non-trivial parallel tensor field

$$(5.1) \quad A = \frac{1}{12}\omega \otimes \omega - \frac{1}{24}C.$$

This is an endomorphism of $\Lambda^2 X$, and A is given by

$$(5.2) \quad \zeta \mapsto \langle \zeta^+, \omega \rangle \frac{\omega}{8} - \frac{\zeta^+}{12}$$

ζ^+ being the Λ^+ component of $\zeta \in \Lambda^2 X$. We have the following proposition,

Proposition 5. *(X, g, ω) is a Kähler manifold of real dimension four, oriented in the natural way. Considering W^+ as an endomorphism of $\Lambda^2 X$, trivial on $\Lambda^- X$, we have*

$$(5.3) \quad W^+ = s \cdot A,$$

where s is the scalar curvature and A denotes the non-trivial parallel tensor field which is given by (5.1).

Remark 5.1. One can not ignore the formal resemblance of the equation (5.3) of the proposition (5) and our new equation (3.2). One might hope, the new equation will provide the Kähler geometry of non-Kählerian geometry. In terms of a table,

Kähler Case	The New Equation
$W^+ = s \cdot A$	$W^+ = q(\psi)$
$\nabla A = 0$	$\nabla \psi = 0.$

6. FUTURE DIRECTIONS

The author is currently investigating different topological implications of the equation (3.2) on algebraic surfaces, hermitian manifolds, etc. Also, the author is investigating the following scenario: since Einstein's equation on a self-dual manifold reduces to Ashtekar equations, it is worth studying Einstein's equation on a manifold that allows the new equation (3.2) and sees if one needs a modification of the Ashtekar equations and corresponding variables.

In an article [3] the author proved the following theorem that states a relationship between the signature (as well as the first Pontryagin number of the manifold) of a 4-dimensional self-dual manifold and Perelman's $\bar{\lambda}$ -functional

Theorem 6.1. *For a self-dual manifold M admitting the gravitational monopole equations, the following relationship between the signature, first Pontryagin number, and Perelman's $\bar{\lambda}(M)$ -functional holds:*

$$(6.1) \quad \bar{\lambda}(M) \leq -8\pi \left(\frac{1}{3} p_1(M) \right)^{\frac{1}{2}} = -8\pi \left(\sigma(M) \right)^{\frac{1}{2}}.$$

In a future paper, we will discuss various aspects of the theorem (6.1).

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