



**HAL**  
open science

# Asymptotic Error in Euler's Method with a Constant Step Size

Jawher Jerray, Adnane Saoud, Laurent Fribourg

► **To cite this version:**

Jawher Jerray, Adnane Saoud, Laurent Fribourg. Asymptotic Error in Euler's Method with a Constant Step Size. *European Journal of Control*, 2022, 68, pp.100694. 10.1016/j.ejcon.2022.100694 . hal-03816598

**HAL Id: hal-03816598**

**<https://hal.science/hal-03816598>**

Submitted on 17 Oct 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

# Asymptotic Error in Euler's Method with a Constant Step Size

Jawher Jerray<sup>1</sup>, Adnane Saoud<sup>2</sup>, Laurent Fribourg<sup>3</sup>

---

## Abstract

In the gradient descent method, one often focus on the convergence of the sequence generated by the algorithm, but less often on the deviation of these points from the solutions of the original continuous-time differential equation (gradient flow). This also happens when discretizing other ordinary differential equations. In the case of a discretization by explicit Euler's method with a constant step  $h$ , we provide here sufficient conditions, in terms of strong monotonicity and co-coercivity, for the deviation between discrete and continuous solutions to tend asymptotically towards zero. This analysis could shed new light on some applications of the gradient descent algorithm.

*Keywords:* Euler's method, optimization

---

## 1. Introduction

Different techniques have been proposed in the literature to provide a deeper understanding of different optimization algorithms through the use of continuous-time differential equations. Indeed, the use of differential equations makes it possible to exploit the plethora of tools developed within dynamical systems analysis and control community in order to analyse stability and convergence properties. In this context, the authors in [1] derived a continuous-time ver-

---

<sup>1</sup>Jawher Jerray is with University Sorbonne Paris Nord, LIPN, CNRS, UMR 7030, F-93430, Villetaneuse, France [jerray@lipn.univ-paris13.fr](mailto:jerray@lipn.univ-paris13.fr)

<sup>2</sup>Adnane Saoud is with Laboratoire des Signaux et Systèmes, CentraleSupélec, Université Paris Saclay, Gif-sur-Yvette, France. [adnane.saoud@centralesupelec.fr](mailto:adnane.saoud@centralesupelec.fr)

<sup>3</sup>Laurent Fribourg is with University Paris-Saclay, CNRS, ENS Paris-Saclay, LMF, F-91190 Gif-Sur-Yvette, France [fribourg@lsv.fr](mailto:fribourg@lsv.fr)

sion of the Nesterov accelerated gradient and used a Lyapunov formulation to estimate the convergence rate. Finally, they transformed the continuous time Lyapunov function into a discrete-time version and provided a new proof of the convergence rate of the Nesterov accelerated gradient method. A similar Lyapunov-based approach has been used to provide proofs of the convergence rates for the Nesterov accelerated algorithm in [2] and the accelerated mirror descent method in [3]. However, all the aforementioned approaches fail to provide a general discretization procedure that generate a provably convergent optimization algorithm, except of the work in [4], where the authors used a Runge-Kutta scheme for a particular case of an accelerated Nesterov algorithm. Indeed, the main difficulty comes from the fact that designing a Lyapunov function of a continuous-time ordinary differential equation (ODE) is the “art of the designer”.

In this paper, we explore another strategy that does not require an a priori knowledge of a Lyapunov function, but rather relies on the *contractivity* properties of the continuous-time differential equation. More precisely, we give here conditions, as general as possible, on the flow of ODEs in order to guarantee the convergence to 0 of the error in Euler’s method, making use of an analytic formula  $\delta(t)$  bounding this error at time  $t$  [5].

**Related work.** Let us mention that the concept of contractivity has been previously used in the literature, where the authors in [6] were able to generate a contractive continuous-time version of the accelerated Nesterov algorithm.

Let us also note that an important part of the literature studies the convergence of the discretization error as the step size  $h \rightarrow 0$ . In contrast here, the size  $h$  is constant while the number of steps  $k \rightarrow \infty$ . Such a convergence at constant  $h$  was the subject of pioneering work by [7] who developed the notion of  $A$ -stability and studied the disk around the origin within which the error remains bounded. However, these studies concern a particular type of linear differential equation of the type  $\dot{x} = -\alpha x$  with  $\alpha > 0$  (cf. [8] for extension to “homogeneous systems”), and not arbitrary differential equations as here. [9]

provides an, linear in time, upper bound on the discretization error under some stability-type conditions on the solutions to the differential equation, for general numerical methods, without showing the convergence of the discretization error to zero. The authors in [10] provides an upper bound on the discretization error while using an explicit Runge-Kutta method with an adaptive and variable step size. Similarly, [11] shows the convergence of the error dynamics while using Runge-Kutta method and for a fixed step size under the restrictive *circle condition* [12]. In comparison with existing approaches in the literature, in this paper, we provide sufficient conditions in terms of *strong monotonicity* and *co-coercivity* to show the *convergence of the discretization error* to 0 for the explicit Euler’s method, while using a constant step size.

Regarding the analysis of the gradient descent algorithm, it is well-known that the error (between the sequence generated by the algorithm and the optimal point) is *non-decreasing* when the step size satisfies  $h < 2/L$ , where  $L$  is the Lipschitz constant of the gradient [7, 13, 14]. The same constraint is used here for the gradient descent (see Corollary 1), but, when coupled with an assumption of *strong monotonicity*, we show that the error tends to 0. We also make use of the property of *co-coercivity* (Theorem 3) whose role has already been observed by Zhu and Marcotte [15] in the different context of iterative schemes for solving variational inequalities.

The remainder of this paper is organized as follows. Section 2 recalls some classical results on convexity, strong monotonicity and co-coercivity. Section 3 gives some sufficient conditions on the flow of the differential equation and the discretization time-step to make the error in Euler’s method converge to 0. We give some final remarks in Section 4.

## 2. Preliminaries

*Notation.* The symbols  $\mathbb{N}$ ,  $\mathbb{N}_{>0}$ , and  $\mathbb{R}$  and  $\mathbb{R}_{>0}$  denote the set of positive integers, non-negative integers, real and non-negative real numbers, respectively. The Euclidean norm is denoted by  $\|\cdot\|$ . For a differentiable map  $x \mapsto f(x) \in \mathbb{R}$ ,

$\nabla f$  denotes the gradient of  $f$  with respect to  $x$ . For an invertible matrix  $A \in \mathbb{R}^{m_x \times m_x}$ ,  $A^\top$  denotes its transpose. For  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $\mathbb{B}(x, r) = \{z \in \mathbb{R}^n : \|x - z\| \leq r\}$ .

We recall some definitions and classical results on convexity, monotonicity and co-coercivity (see, e.g., [14, 16, 17]) that will be used through the paper.

**Definition 1.** Consider a continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . The function  $f$  is *convex* if, for all  $x, y \in \mathbb{R}^n$ :

$$f(y) - f(x) \geq (\nabla f(x))^\top (y - x).$$

If  $f$  is convex and bounded from below, then a minimum of  $f$  exists in some  $x^* \in \mathbb{R}^n$ , and  $\nabla f(x^*) = 0$ .

**Definition 2.** A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is

- *L-Lipschitz continuous* if for all  $x, y \in \mathbb{R}^n$ :

$$\|g(x) - g(y)\| \leq L\|x - y\|.$$

- *strongly monotone* if there exists  $m > 0$  such that, for all  $x, y \in \mathbb{R}^n$ :

$$(g(x) - g(y))^\top (x - y) \geq m\|x - y\|^2. \quad (1)$$

- *co-coercive* if there exists a positive constant  $a$  such that for all  $x, y \in \mathbb{R}^n$ :

$$(g(y) - g(x))^\top (y - x) \geq a\|g(y) - g(x)\|^2.$$

*Remark 1.* Property (1) of strong monotonicity is equivalent to saying that  $g$  has a negative *one-sided Lipschitz constant (OSL)*  $\lambda$  (see [7, 18]; cf Definition 3 below), equal to  $-m$ .

We consider a differential system of the form

$$\dot{x}(t) = g(x(t)) \quad (2)$$

with initial condition  $x_0 \in \mathbb{R}^n$ , where  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a differentiable Lipschitzian function of constant  $L > 0$ . Without loss of understanding, we will denote by  $x(t; x_0)$ , or more simply by  $x(t)$ , the solution of this system at time  $t$ .

We denote by  $y_k$  the (explicit) Euler discretization of  $\dot{x}(t) = g(x(t))$  at time  $t = kh$ , for  $k \in \mathbb{N}$ , where  $h > 0$  is a constant step size. Given an initial point  $y_0 \in \mathbb{R}^n$ ,  $y_k$  is defined, for  $k \geq 1$ , by:

$$y_k = y_{k-1} + hg(y_{k-1}) \tag{3}$$

**Proposition 1.** *On a domain where  $-g$  is strongly monotone (i.e.,  $g$  has an OSL constant  $\lambda < 0$ ), any two solutions  $x(t; x_0)$  and  $x(t; y_0)$  of (2) with initial conditions  $x_0$  and  $y_0$  respectively, converge to each other exponentially in time:*

$$\|x(t; x_0) - x(t; y_0)\| \leq \|x_0 - y_0\|e^{\lambda t}.$$

*It follows that there is at most one stationary point  $x^*$  (with  $g(x^*) = 0$ ).*

This result states that strong monotonicity leads to a form of *contraction* (as studied in [6, 19]).

The following result can be found in [20] and allows to relate the concepts of convexity, Lipschitzness and co-coercivity.

**Proposition 2.** *Consider a continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . If  $f$  is convex and  $\nabla f$   $L$ -Lipschitz, then  $\nabla f$  is co-coercive of constant  $1/L$ .*

**Proposition 3.** *Consider a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . If the following conditions are satisfied:*

1.  $-g$  is co-coercitive,
2. the step size  $h < 2/L$ ,
3.  $g(x^*) = 0$  for some  $x^* \in \mathbb{R}^n$  (existence of a stationary point).

*Then  $\|g(y_k)\| \rightarrow 0$  with rate  $O(1/k)$  for the averaged iterates, where  $\{y_k\}_{k \in \mathbb{N}}$  is defined as in (3).*

A proof (adapted) from [21] is given in Appendix in order to be self-contained.

In the following, we show how the result of Proposition 3 applies to the case of the gradient descent algorithm. We first recall the link between the gradient flow and gradient descent algorithm.

Consider a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the *gradient descent* algorithm generates a sequence  $\{x_k\}_{k \in \mathbb{N}}$  described as:

$$x_{k+1} = x_k - h \nabla f(x_k) \quad (4)$$

where  $h > 0$  is a constant step size. This algorithm is generally used to resolve optimization problems of the form  $\min_{x \in \mathbb{R}^n} f(x)$  for a function  $f$ . The algorithm in (4) can be seen as the (explicit) Euler discretization of the gradient flow differential equation described by:

$$\dot{x} = -\nabla f(x) \quad (5)$$

for a continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

From Proposition 2 and Proposition 3, it follows:

**Proposition 4.** *Consider a continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . If the following conditions are satisfied*

1.  $f$  convex of minimizer  $x^*$ ,
2.  $\nabla f$   $L$ -Lipschitzian,
3. the step size satisfies  $h < 2/L$ .

*Then  $\|\nabla f(x_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ , with rate  $O(1/k)$  for the averaged iterates, where  $\{x_k\}_{k \in \mathbb{N}}$  is defined as in (4).*

### 3. Asymptotic Error in Euler's Method

#### 3.1. Error bound in Euler's method

Our objective is to analyze the error between the solutions of the differential equation (2) and their Euler approximations in (3). For this reason, we define a function  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  in order to upperbound  $\|y_k - x(t)\|$  for  $t = kh$  (see [5] and Theorem 1 below).

**Definition 3.** Given a sequence  $\{\mu_k\}_{k \geq 0}$  of nonnegative reals, we define  $\delta_{\mu_k} : [0, h] \rightarrow \mathbb{R}_{\geq 0}$  as follows: for all  $t \in [0, h]$ ,

if  $\lambda < 0$  :

$$\delta_{\mu_k}(t) = \left( \mu_k^2 e^{\lambda t} + \frac{C_k^2}{\lambda^2} \left( t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} (1 - e^{\lambda t}) \right) \right)^{\frac{1}{2}}$$

if  $\lambda = 0$  :

$$\delta_{\mu_k}(t) = \left( \mu_k^2 e^t + C_k^2 (-t^2 - 2t + 2(e^t - 1)) \right)^{\frac{1}{2}}$$

if  $\lambda > 0$  :

$$\delta_{\mu_k}(t) = \left( \mu_k^2 e^{3\lambda t} + \frac{C_k^2}{3\lambda^2} \left( -t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2} (e^{3\lambda t} - 1) \right) \right)^{\frac{1}{2}}$$

where

- $L$  denotes the Lipschitz constant for  $g$ ,
- $C_k := L\|g(y_k)\|$ ,
- $\lambda$  is the “one-sided Lipschitz constant (OSL)” (or “logarithmic Lipschitz constant” [22]) associated to  $g$ , i.e., the minimal constant such that, for all  $x_1, x_2 \in \mathbb{R}^n$ :

$$(g(x_1) - g(x_2))^\top (x_1 - x_2) \leq \lambda \|x_1 - x_2\|^2, \quad (6)$$

Note that the constant  $C_k$  depends on the value of  $y(t)$  at time  $t = kh^4$ . The constant  $\lambda$  can also be calculated “locally” for the zone occupied by  $x(t)$  at time  $t \in [kh, (k+1)h]$ , using a nonlinear optimization solver (see, e.g., [23] for details).

Let us now consider the sequence  $\{\mu_k\}_{k \geq 0}$  where  $\mu_k$  is defined recursively, for  $k \geq 1$  as

$$\mu_k = \delta_{\mu_{k-1}}(h).$$

We can now define  $\delta_{\mu_0}(2h)$  as  $\delta_{\mu_1}(h)$  with  $\mu_1 = \delta_{\mu_0}(h)$ , and more generally, we can extend the definition of  $\delta_{\mu_0}(\cdot)$  on  $[0, \infty)$  as follows: for all  $k \geq 0$  and  $t \in [0, h]$ ,

$$\delta_{\mu_0}(kh + t) = \delta_{\mu_k}(t).$$

In particular, it is easy to see that:  $\delta_{\mu_0}(kh) = \delta_{\mu_k}(0) = \mu_k$  and  $\delta_{\mu_0}((k+1)h) = \delta_{\mu_k}(h) = \mu_{k+1}$ .

---

<sup>4</sup>This slight modification of the definition of  $C$ , as originally given in [5], is justified by a simple inspection of the proof of Theorem 1 in [5].



**Theorem 1.** [5] Consider the system  $\dot{x}(t) = g(x(t))$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Let  $x(t)$  and  $(y_k)$  be defined as in (2) and (3) respectively, and  $\mu_0 := \|y_0 - x_0\|$ . Then, for all  $t = kh$  with  $k \geq 0$ :

$$\|y_k - x(t)\| \leq \delta_{\mu_0}(t).$$

In the following, we give sufficient conditions on  $g$  and  $h$  ensuring that  $\delta_{\mu_0}(kh)$  (hence  $\|y_k - x(kh)\|$ ) converge to 0 as  $k \rightarrow \infty$ . In the rest of the paper, we will always assume that  $g$  is Lipschitz continuous with constant  $L > 0$ .

### 3.2. Strong monotonicity

**Lemma 1.** Consider the system  $\dot{x}(t) = g(x(t))$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Let  $x(t)$  and  $(y_k)$  be defined as in (2) and (3) respectively, and  $\mu_0 := \|y_0 - x_0\|$ . If the following conditions are satisfied:

1.  $\|g(y_k)\| \rightarrow 0$  as  $k \rightarrow \infty$  with a convergence rate  $r_k$ ,
2.  $g$  of OSL constant  $\lambda < 0$  (i.e.,  $-g$  strongly monotone).

Then  $\delta_{\mu_0}(kh) \rightarrow 0$  as  $k \rightarrow \infty$  with a convergence rate  $r_k$ .

*Proof.* By Taylor-Lagrange's theorem, we find:

$$\delta_{\mu_0}^2((k+1)h) \leq \delta_{\mu_0}^2(kh)e^{\lambda h} + C_k^2 h^3 / (3|\lambda|). \quad (7)$$

Since  $C_k = L\|g(y_k)\| \rightarrow 0$  as  $k \rightarrow \infty$  with rate  $r_k$ , and  $e^{\lambda h} < 1$ , it follows that there exists  $\ell \in \mathbb{N}$  such that:  $\delta_{\mu_0}^2((k+1)h) < \delta_{\mu_0}^2(kh)$  for all  $k \geq \ell$ . Hence, the sequence  $\{\delta_{\mu_0}(kh)\}_{k \geq \ell}$  is decreasing, and converges to a limit  $\delta^* \geq 0$  with rate  $r_k$ . This limit  $\delta^*$  must satisfy (7), i.e.:

$$(\delta^*)^2 \leq (\delta^*)^2 e^{\lambda h} + C_k^2 h^3 / (3|\lambda|).$$

Since  $C_k = L\|g(y_k)\| \rightarrow 0$  as  $k \rightarrow \infty$  and  $e^{\lambda h} < 1$ , one must have  $\delta^* = 0$ . Therefore  $\delta_{\mu_0}(kh) \rightarrow 0$  with rate  $r_k$ .  $\square$

*Remark 2.* Let us mention that the result in Lemma 1 can be relaxed to show that the sequence  $\delta_{\mu_0}(kh)$ ,  $k \in \mathbb{N}$ , is bounded if the sequence  $\|g(y_k)\|$ ,  $k \in \mathbb{N}$  is bounded.

From Lemma 1 and Theorem 1, it follows:

**Theorem 2.** *Consider the system  $\dot{x}(t) = g(x(t))$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$   $L$ -Lipschitz continuous. Let  $x(t)$  and  $(y_k)$  as defined in (2) and (3) respectively. Suppose:*

1.  $\|g(y_k)\| \rightarrow 0$  with a convergence rate  $r_k$ ,
2.  $g$  of OSL constant  $\lambda < 0$  (i.e.,  $-g$  strongly monotone).

Then  $\delta_{\mu_0}(kh) \rightarrow 0$  and  $\|y_k - x(kh)\| \rightarrow 0$  with a convergence rate  $r_k$  as  $k \rightarrow \infty$ .

Note that Theorem 2 applies even if  $\|y_k\| \rightarrow \infty$ , as shown in Example 1.

*Remark 3.* For the sake of simplicity of the statement, we assume in Theorem 2 (as well as Theorem 3 and Corollary 3 below), that the property of Lipschitz continuity and strong monotonicity (as well as, later on, co-coercivity and existence of stationary point) hold over all  $\mathbb{R}^n$ . In fact, it is enough that these properties hold on a *subdomain*  $\mathbb{D} \subset \mathbb{R}^n$  only, provided that  $\mathbb{D}$  be “invariant” in the following sense:

$$\exists \ell \in \mathbb{N} \forall k \geq \ell : B(y_k, \delta_{\mu_0}(kh)) \subseteq \mathbb{D}^5.$$

**Example 1.** Let  $g(x) = \frac{1}{2\sqrt{x}}$ ,  $y_0 = 1$ ,  $h = 0.5$ ,  $\mu_0 = 0.5$ , and let  $\mathbb{D} = [1, +\infty)$ . We have  $y_{k+1} = y_k + h \frac{1}{2\sqrt{y_k}} > y_k$ , therefore  $\{y_k\}$  is an increasing sequence which converges to  $+\infty$ , and  $g(y_k) = \frac{1}{2\sqrt{y_k}}$  converges to 0. Using the software ORBITADOR [24], one calculates that  $L \leq 0.5$  and  $\lambda < 0$  on  $\mathbb{D}$  (see Figure 1). Since  $\|g(y_k)\| \rightarrow 0$  (Figure 2), it follows by Theorem 2:  $\delta_{\mu_0}(kh) \rightarrow 0$  and  $\|y_k - x(kh)\| \rightarrow 0$  as  $k \rightarrow \infty$  (see Figures 3 and 4).

### 3.3. Application to gradient descent

An immediate consequence of Theorem 2, using Lemma 1 and Proposition 4 with  $g = -\nabla f$ , is the following result (which avoids mentioning  $\|g(y_k)\| \rightarrow 0$  as an explicit assumption).

**Corollary 1.** *Consider a continuously differentiable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Suppose:*

---

<sup>5</sup> $B(x, r)$  denotes the ball of centre  $x \in \mathbb{R}^n$  and radius  $r \geq 0$ .

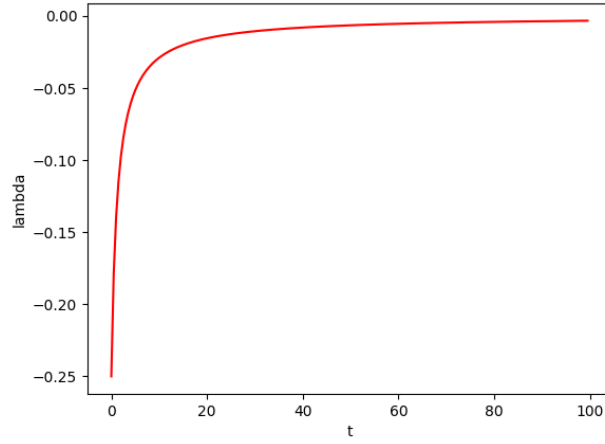


Figure 1: Evolution of  $\lambda$

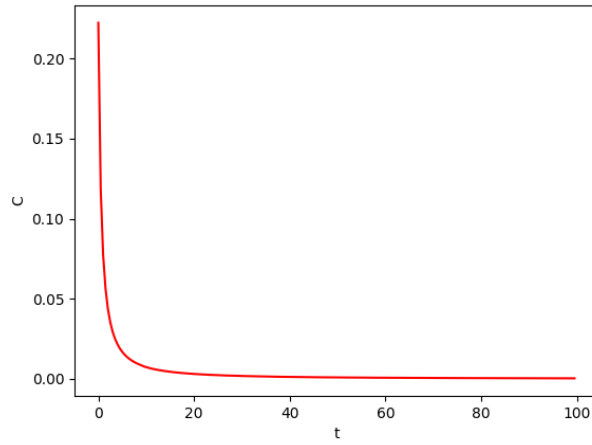


Figure 2: Evolution of  $C_k = L\|g(y_k)\|$

1.  $f$  convex of minimizer  $x^*$ ,
2.  $\nabla f$   $L$ -Lipschitzian,
3.  $-\nabla f$  OSL  $\lambda < 0$  (i.e.,  $\nabla f$  strongly monotone),
4.  $h < 2/L$

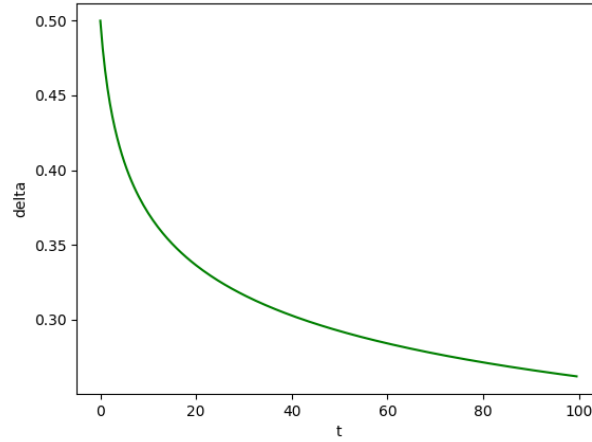


Figure 3: Evolution of  $\delta_{\mu_0}$

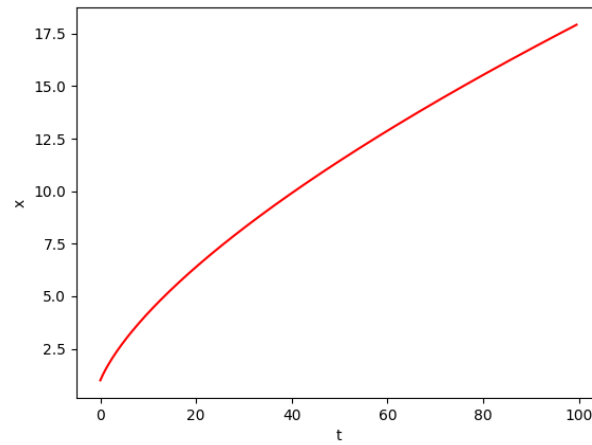


Figure 4: Evolution of  $y_k$

Then  $y_k \rightarrow x^*$  and  $x(kh) \rightarrow x^*$  as  $k \rightarrow \infty$  with rate  $O(1/k)$  for the averaged iterates.

**Example 2.** <sup>6</sup> Let us consider the following function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined for

---

<sup>6</sup>adapted from

$t \geq 0$ , by:

$$f(w) = \frac{1}{2} \sum_{i=1}^2 (b_i - (w_1 \cdot a_i + w_2))^2,$$

with  $w = (w_1, w_2)$ ,  $a_1 = 1, a_2 = 2, b_1 = 5$  and  $b_2 = 7$ . Let us compute its minimum  $w^* = (w_1^*, w_2^*)$  by gradient descent, with initial point  $w^0 = (w_1^0, w_2^0) = (9, 10)$ . Using the software ORBITADOR [24], we find  $L < 7$  and  $\lambda \leq -0.187$  (see Figure 5). For  $h = 0.2$  (which satisfies:  $h < 2/L$ ), it follows by Proposition 4 that  $\|\nabla f(w(kh))\| \rightarrow 0$  and  $C = L\|\nabla f(w(kh))\| \rightarrow 0$  respectively, as  $k \rightarrow \infty$  (see Figures 6 and 7 respectively). Letting  $\mu_0 = 0.1$ , it follows from Corollary 1 that  $\delta_{\mu_0}(t) \rightarrow 0$  (see Figure 8 for  $t = kh \leq 95$  and Figure 9, with a change of scale, for  $95 \leq t = kh \leq 120$ ),  $y_k \rightarrow w^*$  and  $x(kh) \rightarrow w^*$ , with  $w^* = (2, 3)$  (which is the stationary point of  $\nabla f$  and the minimizer of  $f$ ).

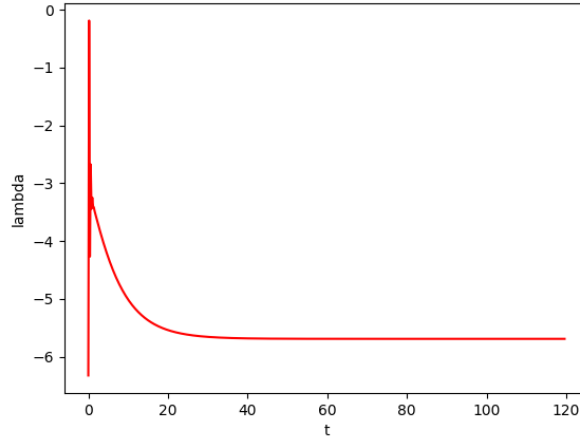


Figure 5: Evolution of  $\lambda$  computed locally ( $\lambda(t) \leq -0.187$ )

### 3.4. Co-coercivity

The following result is a generalization of Corollary 1.

---

<https://codingvision.net/gradient-descent-simply-explained-with-example>

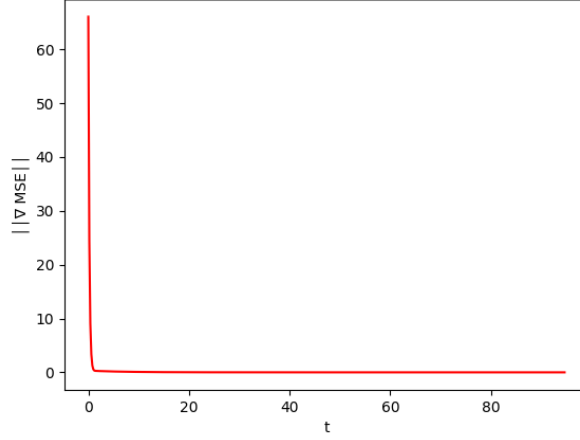


Figure 6: Evolution of  $\|\nabla f(w(t))\|$

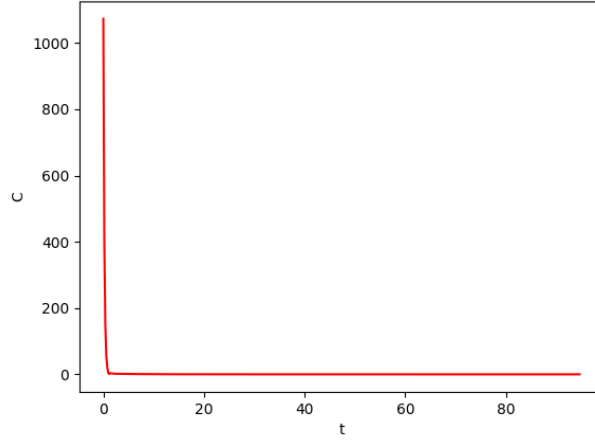


Figure 7: Evolution of  $C(t) = L\|\nabla f(t)\|$  varying from 918 (for  $t = 0$ ) to 0.0272 (for  $t = 30$ )

**Theorem 3.** Consider the system  $\dot{x}(t) = g(x(t))$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$   $L$ -Lipschitz continuous. Let  $x(t)$  and  $(y_k)$  as defined in (2) and (3) respectively. Suppose

1.  $h < 2/L$ ,
2.  $-g$  co-coercive with constant  $1/L$ ,
3.  $g$  of OSL constant  $\lambda < 0$  (i.e.,  $-g$  strongly monotone),

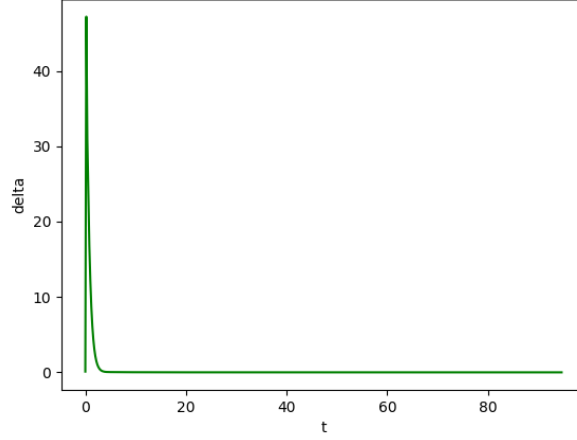


Figure 8: Evolution of  $\delta(t)$  for  $t \in [0, 95]$

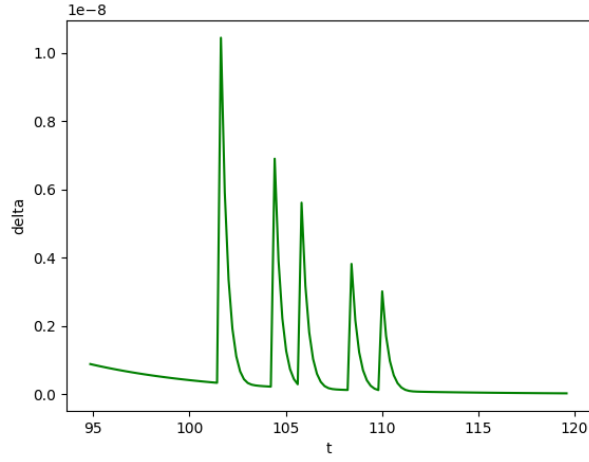


Figure 9: Evolution of  $\delta(t)$  for  $t \in [95, 120]$

4.  $g(x^*) = 0$  for some  $x^* \in \mathbb{R}^n$  (existence of a stationary point).

Then we have:

- $x^*$  is the unique stationary point of  $\mathbb{R}^n$ ,
- $y_k \rightarrow x^*$  and  $x(kh) \rightarrow x^*$  as  $k \rightarrow \infty$  with rate  $O(1/k)$  for the averaged

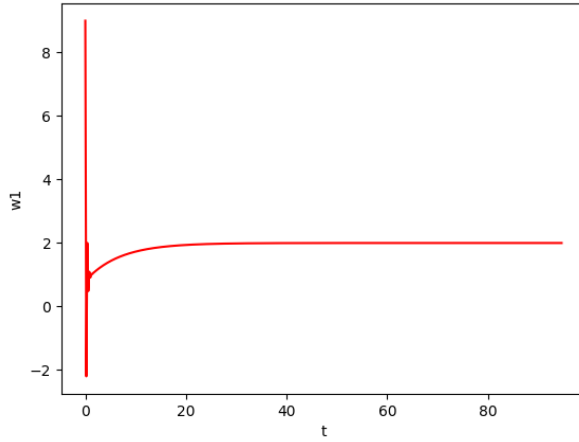


Figure 10: Evolution of  $w_1(t)$  (which converges to  $w_1^* = 2$ )

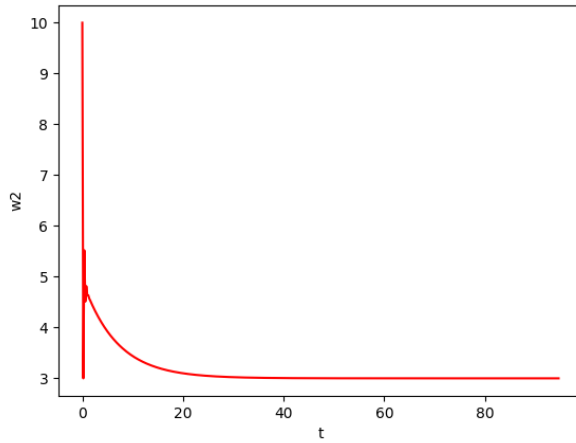


Figure 11: Evolution of  $w_2(t)$  (which converges to  $w_3^* = 3$ )

*iterates.*

*Proof.* By Proposition 1, Lemma 1, Proposition 3 and Theorem 1. □

**Example 3.** Consider the differential equation  $\dot{x} = g(x)$  with  $g(x) = -4x^3 + 6x^2$ , and its Euler discretization with  $y_0 = 0.25$  and  $h = 0.12$ . Using OR-



BITADOR, one calculates  $L \leq 12$ , where  $L$  is the Lipschitz constant of  $g$ . Let  $\mathbb{D} = [1.25, 1.75]$ . For  $\mu_0 = 0.1$  and  $h = 0.12 < 2/L$ , one can show, using ORBITADOR:

- $\lambda < 0$  on  $\mathbb{D}$  (see Figure 12),
- $B(y_k, \delta_{\mu_0}(kh)) \subseteq \mathbb{D}$  for all  $k \geq 12$ , and
- $-g$  co-coercive of constant  $1/L$  on  $\mathbb{D}$ .

Besides,  $x^* = 1.5 \in \mathbb{D}$  is a stationary point ( $g(y^*) = 0$ ). From Theorem 3, it follows:  $\delta_{\mu_0}(kh) \rightarrow 0$ , (see Figure 13),  $x^*$  is the unique stationary point of  $\mathbb{D}$ ,  $y_k \rightarrow x^*$  and  $x(kh) \rightarrow x^*$  as  $k \rightarrow \infty$  (see Figure 14). We can also check  $C = L\|g(y_k)\| \rightarrow 0$  (see Figure 15). Note that  $x^*$  is here the minimizer of the non-convex function  $f(x) = x^4 - 2x^3 + 2$  (with  $-\nabla f(x) = g(x)$ ). See Figure 16.

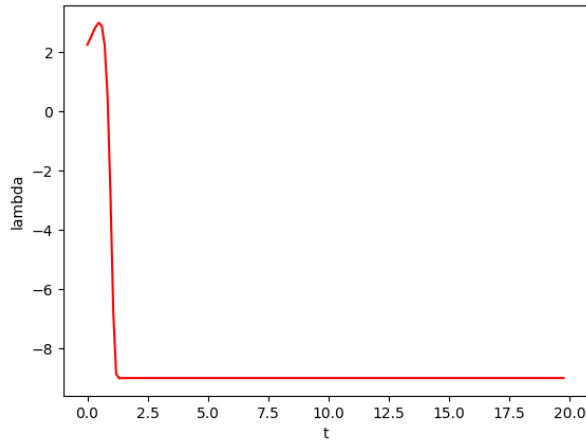


Figure 12: Evolution of  $\lambda$

#### 4. Conclusion

In this paper, we have focused on the following problem: given an ODE  $\dot{x} = g(x)$  and the sequence  $(y_k)$  generated by its discretization with explicit Euler's

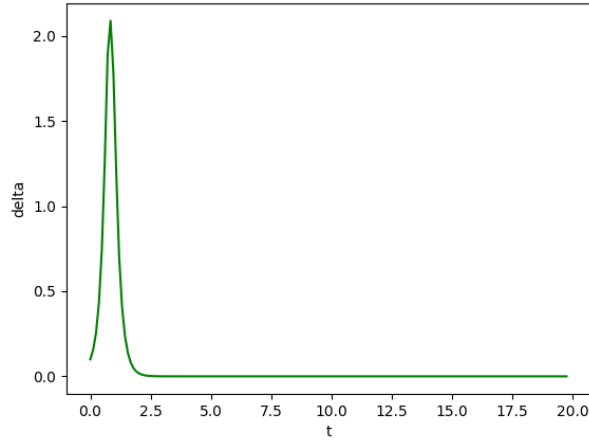


Figure 13: Evolution of  $\delta_{\mu_0}$  (which converges to 0)

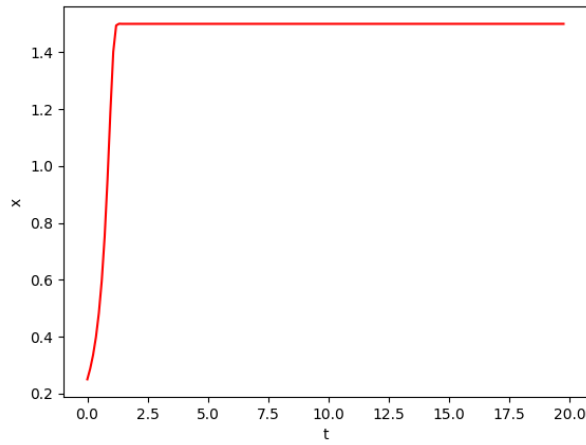


Figure 14: Evolution of  $y_k$  (which converges to  $x^* = 1.5$ )

method, what are the sufficient conditions on  $g$  to ensure that the discretization error  $\|x(kh) - y_k\|$  converges to 0 as  $k \rightarrow \infty$ . In particular, under certain properties of  $g$  called “strong monotonicity” and “co-coercivity”, we have shown the convergence of the discretization error to 0 (Theorem 3). This can shed new light on the relationship between the convergence of continuous differential

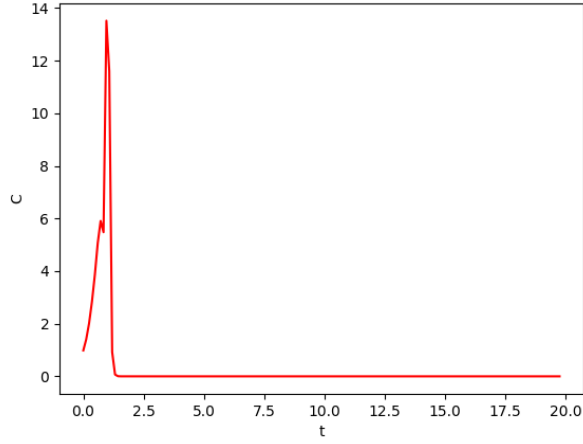


Figure 15: Evolution of  $C = L\|g(y)\|$

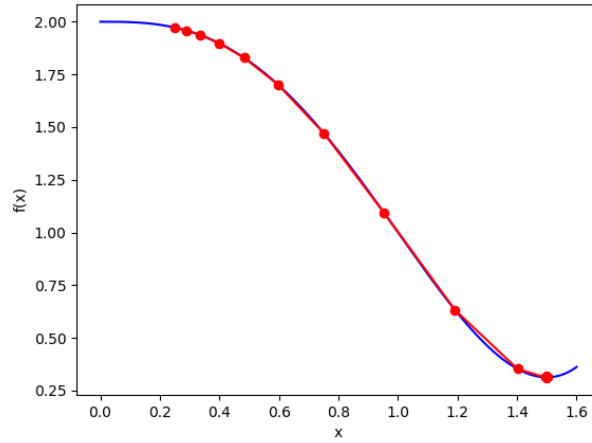


Figure 16: Graph  $(y_k, f(y_k))$

equations (such as the gradient flow) and their discretization (gradient descent algorithm), especially in the context of the gradient descent algorithms for neural networks. Indeed, in a recent work, [25] has shown that for a shallow fully connected NN with ReLu activation functions, if the number  $m$  of hidden nodes is sufficiently large and if the initial values of the weights of the NN are chosen

according to a Gaussian distribution, then the gradient descent converges to a globally optimal solution. In future work, we will use the approach developed in this paper to show the convergence of the gradient descent algorithm for the class of neural networks considered in [25].

## References

- [1] W. Su, S. Boyd, E. Candes, A differential equation for modeling nesterov’s accelerated gradient method: Theory and insights, *Advances in neural information processing systems* 27 (2014) 2510–2518.
- [2] B. Shi, S. S. Du, M. I. Jordan, W. J. Su, Understanding the acceleration phenomenon via high-resolution differential equations, *Mathematical Programming* (2021) 1–70.
- [3] W. Krichene, A. Bayen, P. Bartlett, Accelerated mirror descent in continuous and discrete time, *Advances in neural information processing systems* 28 (2015) 2845–2853.
- [4] J. Zhang, A. Mokhtari, S. Sra, A. Jadbabaie, Direct Runge-Kutta discretization achieves acceleration, *arXiv preprint arXiv:1805.00521*.
- [5] A. Le Coënt, F. De Vuyst, L. Chamoin, L. Fribourg, Control synthesis of nonlinear sampled switched systems using Euler’s method, in: E. Ábrahám, S. Bogomolov (Eds.), *SNR*, Vol. 247 of *EPTCS*, 2017, pp. 18–33. doi: 10.4204/EPTCS.247.2.
- [6] P. Cisneros-Velarde, F. Bullo, A contraction theory approach to optimization algorithms from acceleration flows, *arXiv preprint arXiv:2105.08832*.
- [7] G. Dahlquist, Error analysis for a class of methods for stiff non-linear initial value problems, in: G. A. Watson (Ed.), *Numerical Analysis*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1976, pp. 60–72.

- [8] D. Efimov, A. Polyakov, A. Levant, W. Perruquetti, Realization and discretization of asymptotically stable homogeneous systems, *IEEE Transactions on Automatic Control* 62 (11) (2017) 5962–5969. doi:10.1109/TAC.2017.2699284.
- [9] D. Viswanath, Global errors of numerical ode solvers and Lyapunov’s theory of stability, *IMA journal of numerical analysis* 21 (1) (2001) 387–406.
- [10] M. Calvo, D. Higham, J. Montijano, L. Randaz, Global error estimation with adaptive explicit Runge-Kutta methods, *IMA journal of numerical analysis* 16 (1) (1996) 47–63.
- [11] J. F. B. M. Kraaijevanger, Contractivity of Runge-Kutta methods, *BIT Numerical Mathematics* 31 (3) (1991) 482–528.
- [12] G. Dahlquist, Generalized disks of contractivity for explicit and implicit Runge-Kutta methods, Tech. rep., CM-P00069451 (1979).
- [13] J. M. Sanz-Serna, K. C. Zygalakis, Contractivity of Runge-Kutta methods for convex gradient systems, *SIAM J. Numer. Anal.* 58 (4) (2020) 2079–2092. doi:10.1137/19M1299256.  
URL <https://doi.org/10.1137/19M1299256>
- [14] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, Springer Publishing Company, 2014.
- [15] D. L. Zhu, P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM Journal on Optimization* 6 (3) (1996) 714–726. doi:10.1137/S1052623494250415.  
URL <https://epubs.siam.org/doi/10.1137/S1052623494250415>
- [16] S. P. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2014. doi:10.1017/CB09780511804441.  
URL <https://web.stanford.edu/%7Eboyd/cvxbook/>

- [17] E. K. Ryu, S. Boyd, A primer on monotone operator methods, *Applied Computational Mathematics* 15 (1) (2016) 3–43.
- [18] E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II*, Springer-Verlag, 1996.
- [19] P. M. Wensing, J. E. Slotine, Cooperative adaptive control for cloud-based robotics, in: *2018 IEEE International Conference on Robotics and Automation, ICRA 2018, Brisbane, Australia, May 21-25, 2018*, IEEE, 2018, pp. 6401–6408. doi:10.1109/ICRA.2018.8460856.  
URL <https://doi.org/10.1109/ICRA.2018.8460856>
- [20] A. Chambolle, *Continuous optimization, an introduction*, Lecture Notes, 2016.
- [21] R. M. Gower, *Convergence theorems for gradient descent* (2020).
- [22] Z. Aminzare, E. D. Sontag, Contraction methods for nonlinear systems: A brief introduction and some open problems, in: *53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014*, 2014, pp. 3835–3847.
- [23] J. Jerray, L. Fribourg, É. André, Robust optimal periodic control using guaranteed Euler’s method, in: *2021 American Control Conference, ACC 2021, New Orleans, LA, USA, May 25-28, 2021*, IEEE, 2021, pp. 986–991. doi:10.23919/ACC50511.2021.9482621.  
URL <https://doi.org/10.23919/ACC50511.2021.9482621>
- [24] J. Jerray, Orbitador: A tool to analyze the stability of periodical dynamical systems, in: *ARCH21. 8th International Workshop on Applied Verification of Continuous and Hybrid Systems (ARCH21)*, Berlin, Germany, July, 2021, EPiC Series in Computing, EasyChair, 2021.
- [25] S. S. Du, X. Zhai, B. Póczos, A. Singh, Gradient descent provably optimizes over-parameterized neural networks, *CoRR* abs/1810.02054. arXiv:1810.02054.

### Appendix: Proof of Proposition 3

*Proof.* This proof Let  $\alpha := Lh > 0$ . We have:

$$\begin{aligned} \|y_{k+1} - x^*\|^2 &= \|y_k - x^* + \frac{\alpha}{L}g(y_k)\|^2 \\ &= \|y_k - x^*\|^2 - 2\frac{\alpha}{L}(y_k - x^*)^\top(-g(y_k)) + \frac{\alpha^2}{L^2}\|g(y_k)\|^2 \end{aligned} \quad (8)$$

Now, from the co-coercivity of  $-g$  and  $g(x^*) = 0$ , we have:

$$(y_k - x^*)^\top(-g(y_k)) \geq \frac{1}{L}\|g(y_k)\|^2.$$

Therefore, from (8), we derive:

$$\|y_{k+1} - x^*\|^2 \leq \|y_k - x^*\|^2 + \frac{-2\alpha + \alpha^2}{L^2}\|g(y_k)\|^2$$

Hence,

$$\frac{\alpha(2 - \alpha)}{L^2}\|g(y_k)\|^2 \leq \|y_k - x^*\|^2 - \|y_{k+1} - x^*\|^2.$$

Therefore, by dividing by  $\alpha(2 - \alpha)$  (which is positive since  $2 - \alpha = 2 - hL > 0$ ):

$$\|g(y_k)\|^2 \leq \frac{L^2}{\beta}\|y_k - x^*\|^2 - \frac{L^2}{\beta}\|y_{k+1} - x^*\|^2$$

with  $\beta = \alpha(2 - \alpha) > 0$ .

Summing up from  $k = 1, \dots, K$  and dividing by  $K$  gives

$$\begin{aligned} \frac{1}{K}\sum_{i=1}^K\|g(y_k)\|^2 &\leq \frac{L^2}{\beta K}(\|y_1 - x^*\|^2 - \|y_{K+1} - x^*\|^2) \\ &\leq \frac{L^2}{\beta K}\|y_1 - x^*\|^2. \end{aligned}$$

Consequently

$$\min_{k=1, \dots, K}\|g(y_k)\|^2 \leq \frac{1}{K}\sum_{k=1}^K\|g(y_k)\|^2 \leq \frac{L^2}{\beta K}\|y_1 - x^*\|^2.$$

It follows that  $\|g(y_k)\| \rightarrow 0$  as  $k \rightarrow \infty$  with rate  $O(1/k)$  for the averaged iterates.  $\square$