



HAL
open science

The War of Attrition under Uncertainty: Theory and Robust Testable Implications

Jean-Paul Décamps, Fabien Gensbittel, Thomas Mariotti

► **To cite this version:**

Jean-Paul Décamps, Fabien Gensbittel, Thomas Mariotti. The War of Attrition under Uncertainty: Theory and Robust Testable Implications. 2022. hal-03816389v3

HAL Id: hal-03816389

<https://hal.science/hal-03816389v3>

Preprint submitted on 12 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The War of Attrition under Uncertainty: Theory and Robust Testable Implications*

Jean-Paul Décamps[†] Fabien Gensbittel[‡] Thomas Mariotti[§]

June 30, 2024

Abstract

We study a symmetric-information war of attrition in which the players' payoffs depend on exogenous market conditions that evolve according to a homogeneous linear diffusion. Using that a Markov strategy can be represented as a stopping region along with an intensity measure of stopping, we fully characterize mixed-strategy Markov-perfect equilibria through a variational system for the players' value functions. When players are asymmetric, in any such equilibrium each player randomizes at a discrete set of thresholds for market conditions. As a result, players may alternatively find themselves in a position of strength or weakness on the equilibrium path. Delayed concessions occur because a player currently in a position of weakness can hope for market conditions to reverse in his favor. In the standard duopoly model of exit under uncertainty, the firms' stock prices and their return volatilities comove negatively over the attrition region and exhibit patterns documented by technical analysis.

Keywords: War of Attrition, Mixed-Strategy Equilibrium, Uncertainty.

JEL Classification: C61, D25, D83.

*This paper was first circulated by the same authors under the title “Mixed-Strategy Equilibria in the War of Attrition under Uncertainty” (arXiv:2210.08848). We thank René Aïd, Olivier Compte, Tiziano De Angelis, Erik Ekström, Bertrand Gobillard, Aaron Kolb, Pauli Murto, Frank Riedel, Jean-Charles Rochet, Julia Salmi, Sara Shahanaghi, Jan-Henrik Steg, Jean Tirole, Juuso Välimäki, Nikhil Vellodi, and Stéphane Villeneuve for very valuable feedback. We also thank seminar audiences at Collegio Carlo Alberto, HEC Paris, Helsinki Graduate School of Economics, London Mathematical Finance Seminar, Paris School of Economics, Séminaire Bachelier, and Toulouse School of Economics, as well as conference participants at the 2021 Quimper Workshop on Dynamic Games and the 2024 Universität Bielefeld Conference on Strategic Timing Problems in Economics and Mathematical Finance for many useful discussions. This research has benefited from financial support of the ANR (Programmes d'Investissements d'Avenir CHES ANR-17-EURE-0010 and ANITI ANR-19-PI3A-0004) and the research foundation TSE-Partnership (Chaire Marchés des Risques et Création de Valeur, Fondation du Risque/SCOR).

[†]Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: jean-paul.decamps@tse-fr.eu.

[‡]Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: fabien.gensbittel@tse-fr.eu.

[§]Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France, CEPR, and CESifo. Email: thomas.mariotti@tse-fr.eu.

1 Introduction

The war of attrition (WoA) is a workhorse model of situations in which each player has to decide when to concede and forfeit a prize to his opponent. Examples include animal conflict (Maynard Smith (1974)), public good provision (Bliss and Nalebuff (1984)), exit from a declining industry (Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986)), labor strikes (Kennan and Wilson (1989)), macroeconomic stabilizations (Alesina and Drazen (1991), Drazen and Grilli (1993)), competing standards (Bulow and Klemperer (1999)), bargaining (Abreu and Gul (2000)), investment under learning externalities (Décamps and Mariotti (2004)), and boycotts (Egorov and Harstad (2017)). A growing literature attempts to test the predictions of these models and to estimate the welfare cost of delayed concessions (Hendricks and Porter (1996), Ghemawat (1997), Padovano and Venturi (2001), Geraghty and Wiseman (2008), Wang (2009), Takahashi (2015)).

Because attrition generates costs for all players, a natural question is why it may occur in the first place.¹ Under complete information, if a player believes that his opponent is stubborn, so that she will never concede, then he is better off avoiding conflict altogether by conceding immediately. If his opponent correctly anticipates this, then she in turn has every reason to be stubborn. Hence, attrition can occur in equilibrium only if players do not know each other's intentions, that is, if each player believes that his opponent concedes in a random manner. A common alternative explanation for attrition is asymmetric information, which arises when some player does not know how strong—how powerful, patient, enduring, or committed—his opponent is. Yet, this explanation is debatable, because conflicts in practice can last for a considerable amount of time despite large observable differences in strength between the parties involved.² Besides, even when information is asymmetric, immediate concession can occur in equilibrium if players have different distributions of waiting costs (Martinelli and Escorza (2007), Myatt (2024)).

This paper argues that delayed concessions in the WoA may be caused by unpredictable changes in the environment rather than by players' asymmetric information about each others' characteristics. Our approach is motivated by the simple observation that uncertainty is a pervasive feature of the WoAs that are deemed to take place in practice. For instance, firms fighting to be the last to exit from a declining industry may still be uncertain about the future evolution of demand (Dixit and Pindyck (1994)). Similarly, political groups

¹Fearon (1995) raises this general point in the context of wars: given that wars are costly for all the parties involved, how can one rationalize the fact that they nonetheless recur and persist?

²Wars, labor strikes, and boycotts are obvious cases in point, as well as, to some extent, conflicts over macroeconomic stabilizations (Labán and Sturzenegger (1994)).

fighting about how to share the tax increase necessary to stabilize the economy may face random fluctuations in the interest rate as well as random shifts in the level of aid or foreign intervention (Alesina and Drazen (1991)). The new mechanism we emphasize is that, under these circumstances, a player currently finding himself in a position of weakness vis-à-vis his opponent may want to delay concession simply because he hopes for events to turn again in his favor. The challenge is to show that this simple mechanism can cause delays over and above those due to the mere presence of uncertainty, even though changes in market conditions do not a priori favor either player—so that being in a position of strength or weakness is not a built-in feature of players’ preferences, but rather an endogenous feature of equilibrium play. In so doing, we identify a new class of mixed-strategy equilibria that have robust and novel testable implications.

To this end, we study a general two-player symmetric-information WoA in which players’ payoffs depend on an exogenous state variable, hereafter generically referred to as market conditions. Both players continuously observe the evolution of market conditions, which follow a homogeneous linear diffusion. Based on this information, each player then decides whether to remain in the market or to irreversibly exit. There is a second-mover advantage in the sense that, if and when a player exits, his payoff is lower than the payoff he would have obtained if his opponent had exited under the same market conditions. Players may be asymmetric, allowing us for instance to capture observable differences in waiting costs. In our running example, two firms face uncertainty about future demand. A WoA arises because each firm would like to liquidate its assets if demand were to deteriorate enough, but would meanwhile individually fare better as a monopolist than as a duopolist. Firms may be asymmetric in that one firm’s assets may have a higher liquidation value than its opponent’s, so that it is a priori more willing to exit the market.

Special cases of our running example have been studied in the literature, with a natural focus on Markov-perfect equilibria (MPEs) in which firms’ exit decisions at any point in time only depend on current market conditions (Maskin and Tirole (2001)). Murto (2004) characterizes pure-strategy MPEs in which each firm exits with probability 1 over some region of the state space. In the case of symmetric firms, Steg (2015) and Georgiadis, Kim, and Kwon (2022) construct a symmetric mixed-strategy MPE in which both firms exit at a stochastic rate over an interval of market conditions. Attrition in this MPE is maximal, in that each firm obtains the stand-alone value it would obtain if its opponent were stubborn. Importantly, Georgiadis, Kim, and Kwon (2022) show that no such MPE exists as soon as there is the slightest asymmetry between the firms.

As we argue in this paper, however, this negative result does not imply that attrition cannot occur in equilibrium when players are asymmetric. The reason is that the set of mixed Markov strategies is not exhausted by strategies defined by stochastic exit rates. The latter, for instance, do not allow one to capture the behavior of a player who would wish to exit the market, with positive but finite intensity, each time the market conditions hit a single point of the state space.³ Yet, this behavior naturally emerges as the limit of mixed Markov strategies with stochastic exit rates defined over a sequence of intervals degenerating to that point, or, given appropriate normalizations, of mixed Markov strategies defined over discretized state spaces and time grids with increasingly finer mesh.

Our first contribution is thus to provide a full characterization of mixed Markov strategies, which we model as randomized stopping times. Our first main result, Theorem 1, states that a randomized stopping time is Markovian if and only if it can be represented by a pair (μ, S) , where S is a subset of the state space over which the player exits with probability 1 and μ is a measure over the complement of S representing the player's exit intensity at states at which he randomizes. Special cases of this representation include pure strategies, in which the intensity measure μ is degenerate, mixed strategies with a stochastic exit rate, in which μ is absolutely continuous with respect to Lebesgue measure, and mixed strategies in which μ is discrete, capturing the behavior of a player who exits the market with finite intensity each time the market conditions hit a countable set of states. More generally, one can conceive strategies in which the intensity measure μ has an arbitrary component that is singular with respect to Lebesgue measure.

Our second main result, Theorem 2, states that, if players are asymmetric, then all mixed-strategy MPEs involve strategies with discrete intensity measures, whose supports form two intertwined sequences of randomization thresholds, one for each player. At any such threshold, the corresponding player is indifferent between remaining in the market and exiting. This implies that the state space is endogenously partitioned into intervals that can all be reached on the equilibrium path, and over which each player is alternatively in a position of strength or weakness depending on how far current market conditions are from one of his randomization thresholds. The reason why a player currently in a position of weakness has an incentive to wait and to randomize at specific thresholds for market conditions is that he can hope for a reversal of situation in his favor—that is, for market conditions to transit to a neighboring interval of the state space over which he will be again

³Importantly, exiting with finite *intensity* at a point should be distinguished from exiting with positive *probability* each time the market conditions hit that point: given the infinite oscillations of Brownian motion, the latter is indistinguishable from exiting with probability 1 at that point.

in a position of strength. As a result, the balance of power between the players randomly fluctuates as market conditions vary over time. This new finding contrasts with the outcomes of pure-strategy MPEs, in which one of the players always remains in a position of strength until his opponent eventually exits. It also contrasts, when players are symmetric, with the outcome of the regular mixed-strategy MPE in which the players exit at a stochastic rate over an interval of the state space.

Finally, our third main result, Theorem 3, characterizes these singular mixed-strategy MPEs by a pair of variational systems for the players' equilibrium value functions. These variational systems are linked by the intensity with which each player exits at each of his randomization thresholds. At any such point, the other player's equilibrium value function reaches a peak and exhibits a kink whose size is proportional to the intensity with which the randomizing player exits the market; this kink reflects that exit by the latter is unpredictable given the current market conditions. Importantly, this characterization also applies to the case of symmetric players.

Admittedly, these results do not contribute to solving the multiplicity problem that plagues WoA models (Riley (1980), Hendricks, Weiss, and Wilson (1988)): if anything, we exhibit new equilibria that have been disregarded in the literature, both in the cases of symmetric and asymmetric players. However, the key point is that the mixed-strategy MPEs that survive when players are asymmetric share a common structure, and lead to qualitatively similar testable implications. In that sense, we provide a robust characterization of MPEs in which attrition takes place in the WoA under uncertainty.

We illustrate these findings in our running example by providing sufficient conditions under which there exists an MPE in which one firm randomizes at a single point while the other firm plays a pure strategy. This MPE exists when firms have the same liquidation values and is robust to some asymmetry in the firms' liquidation values, as long as it is not too large. In equilibrium, the firm with the lowest liquidation value randomizes between remaining in the market and exiting at the exit threshold for market conditions that would be optimal if its opponent were stubborn. By contrast, the firm with the highest liquidation value exits with probability 1 if market conditions fall below a lower threshold, the value of which is determined precisely so as to meet its opponent's indifference condition. The intensity with which the firm with the lowest liquidation value exits the market at its randomization threshold in turn makes it optimal for its opponent to exit at its lower threshold. This illustrates the general point that incentives in singular mixed-strategy MPEs are nonlocal due to their alternating structure.

In this MPE, the total value of the randomizing firm goes down to its liquidation value at its randomization threshold, while the total value of its opponent simultaneously reaches the peak of its total value function. A similar pattern more generally emerges in any MPE with multiple randomization thresholds. Novel asset-pricing implications ensue when these firms are publicly traded on a frictionless financial market.

First, along any path of the diffusion process modeling the evolution of market conditions, the firms' stock prices and the volatilities of their returns fluctuate randomly over the attrition region, moving in opposite directions as long as no firm exits the market. These negative comovements of the firms' stock prices and of the volatilities of their returns stand in sharp contrast with the predictions of the regular mixed-strategy MPE that arises when firms have identical liquidation values, in which firms' stock prices stay constant and equal to their liquidation value over the attrition region—a very strong prediction that is unlikely to be validated empirically.

Second, when the stock price of a firm reaches a peak of its total value function, two events may occur. Either its opponent does not exit the market, causing the firm's stock price to bounce downward. Or its opponent exits the market, causing the firm's stock price to jump upwards to its monopoly value. Because exit by the opponent is unpredictable, these downward bounces exactly compensate for this upward jump. As a result, rational investors have no means to arbitrage away the profits associated to these downward bounces by short-selling the firm's stock at its peak without incurring the risk of a sudden upward jump in its price. We argue that this pattern is consistent with what technical analysts describe as a resistance level in stock prices, for which our analysis provides an illustration in a setting where stock prices are only driven by fundamentals. Needless to say, we do not claim that the patterns documented by technical analysis can only be rationalized by our model, as many other rational or behavioral factors may be at play. Still, our model seems to capture reasonably well the intuitive idea that resistance levels in stock prices can be discontinuously broken by unpredictable changes in the environment above investors' expectations—in this instance, the exit of a competitor.

Our findings pave the way to many other applications. For instance, WoA models of macroeconomic stabilizations typically do not account for exogenous random changes in the environment. Our analysis suggests that such changes may considerably delay the needed reforms even if the parties involved—say, workers and capitalists—are well aware of each other's waiting costs. The singular mixed-strategy MPEs that we construct, in which concession by one player is stochastic and tied to the hitting of critical levels of the exogenous

state variable, may also help explain why actual stabilizations need not follow significant observable changes in the macroeconomic environment (Alesina and Drazen (1991)).

Related Literature This paper belongs to the large literature on the continuous-time WoA, starting with the seminal contribution of Maynard Smith (1974) on animal conflict. Ghemawat and Nalebuff (1985) study a WoA between duopolists that must decide when to exit from a declining industry. Hendricks, Weiss, and Wilson (1988) offer an exhaustive characterization of pure- and mixed-strategy equilibria in the symmetric-information WoA when players have potentially asymmetric payoffs that are deterministic functions of time. Riley (1980), Bliss and Nalebuff (1984), and Fudenberg and Tirole (1986) extend the analysis to asymmetric-information settings where, for instance, a player is uncertain about his opponent's waiting cost. Myatt (2024) studies the impact of players' perceived strengths on equilibrium concession times when there is an exogenous deadline. Décamps and Mariotti (2004) study an investment game that has the structure of a WoA because a firm's investment generates a public signal about the return of a common-value project.

These papers except the last one focus on deterministic settings. By contrast, starting with Lambrecht (2001) and Murto (2004), a small literature examines the case where players in a WoA have symmetric information but are uncertain about their future payoffs, which are driven by a diffusion process. Lambrecht (2001) analyzes how the order in which firms go bankrupt in an industry is influenced by aggregate factors and firm-specific factors such as their financial structure. Murto (2004) studies a stochastic version of Ghemawat and Nalebuff (1985)'s exit model, and shows that the firm with the lowest liquidation value may end up exiting the market first in equilibrium, despite being a priori more enduring than its opponent. These papers allow for asymmetries between players but restrict attention to pure-strategy MPEs, as in Fine and Li (1989)'s discrete-time model of exit from a stochastically declining industry. By contrast, Steg (2015) characterizes the regular mixed-strategy MPE of the symmetric game. Kwon and Palczewski (2022) extend this construction to a WoA with asymmetric information and a continuum of types; they show that the symmetric-equilibrium pure strategy, seen as a randomized strategy using each player's type as a randomization device, has an absolutely continuous intensity that depends on the exogenous diffusion process and on an endogenous belief process.

Closest to the present paper in this literature is Georgiadis, Kim, and Kwon (2022). In a setting that extends Murto (2004), they show that, as soon as firms have different liquidation values, there exists no mixed-strategy MPE in which firms exit the market at a stochastic rate, that is, according to absolutely continuous intensity measures. This shows

that the regular mixed-strategy MPE characterized by Steg (2015) is not robust to even small asymmetries between firms. They conclude that, when firms are asymmetric, only pure-strategy MPEs exist, and, therefore, that attrition cannot actually take place on the equilibrium path. Our analysis shows that this conclusion is unwarranted once the possibility for firms to exit the market according to Markovian randomized stopping times with singular intensity measures is accounted for.

Gieczewski (2024) studies a symmetric-information WoA that has the structure of a supermodular game because changes in the underlying state variable affect the players' waiting costs in opposite ways. He shows that there exists a unique pure-strategy MPE in which one player exits when the state hits a lower threshold from above, while his opponent exits when the state hits an upper threshold from below. In the intermediate region, each player is willing to wait because he hopes that the state will hit his opponent's exit threshold first. Our model differs in that, although players may be asymmetric, changes in market conditions affect them in similar ways. As a result, market conditions do not per se determine whether a player is in a position of strength or weakness; instead, this is endogenously determined in equilibrium.

We have borrowed from Touzi and Vieille (2002) our concept of a randomized stopping time, which they use to show that any continuous-time zero-sum Dynkin game admits a value. Singular mixed strategies have been emphasized in the recent literature on dynamic games of incomplete information where, in equilibrium, an informed player uses a randomized stopping time to control the belief process of an uninformed player using a pure strategy (Daley and Green (2012), Kolb (2019), De Angelis, Ekström, and Glover (2022)). The resulting singular cumulative distribution function is related to the local time of the uninformed player's belief process at some critical threshold and, at this threshold, can generate a kink in the value function of the uninformed player (Kolb (2019)).

In our WoA model, the players have symmetric information, and randomized stopping times with singular cumulative distribution functions play a different role. Specifically, in equilibrium, each player randomizes at some critical thresholds at which he is indifferent between exiting and remaining in the market. The intensity with which he exits at each of these thresholds makes his opponent willing to remain in the market in intervals of states around these thresholds and indifferent between exiting and remaining in the market at the ends of such intervals. This gives rise to the alternating structure of singular mixed-strategy MPEs. The resulting singular cumulative distribution function is related to the local times of the exogenous diffusion process of market conditions at these critical thresholds; such local

times, in our running example, play a key role in the dynamics of firms' stock prices. Finally, the fact that the cumulative distribution function of any Markovian randomized stopping time can be represented via an integral of such local-time processes is precisely what enables us to obtain a full characterization of mixed-strategy MPEs. This representation result is of independent interest and may prove useful for the study of general stochastic timing games in which the state variable is driven by a Brownian motion.

Keller, Rady, and Cripps (2005) study a strategic experimentation model with two-armed bandits where the risky arm might distribute lump-sum payoffs according to a Poisson process. They show that the unique symmetric MPE, in which, over some range of the belief space, each player devotes an interior amount of resource to experimentation, is Pareto-dominated over this range by asymmetric MPEs in which players take turns in experimenting and playing the safe action. This alternating structure of asymmetric MPEs, as well as the idea that the latter may lead to Pareto improvements over the symmetric one, is reminiscent of that which arises in the singular mixed-strategy MPEs of our model. Keller and Rady (2010) generalize this insight to a situation in which a single success on the risky arm does not fully reveal its type. They show that this generates an encouragement effect and that players may alternatively find themselves in a position of strength or weakness. Despite these similarities with our results, the strategic interaction between the players is entirely different as the experimentation model does not involve irreversible decisions. Another difference is that our analysis is cast in a Brownian rather than in a Poisson setup.

The paper is organized as follows. Section 2 describes the model. Section 3 defines our strategy and equilibrium concepts and provides preliminary properties of MPEs. Section 4 heuristically shows how to construct a mixed-strategy MPE involving a singular intensity measure for one player. Section 5 states our main characterization results. The main Appendix provides the proofs of Theorems 1–3. The Online Supplement collects detailed proofs of technical lemmas and claims used in the derivation of these theorems.

2 The Model

2.1 A General Model of the WoA under Uncertainty

Two players, 1 and 2, face uncertainty about future market conditions. In what follows, i (he) refers to an arbitrary player and j (she) to his opponent. Time is continuous and indexed by $t \geq 0$. The evolution of market conditions is modeled as a one-dimensional time-homogeneous diffusion process $X \equiv (X_t)_{t \geq 0}$ defined over the canonical space $(\Omega, \mathcal{F}, \mathbf{P}_x)$

of continuous trajectories with $X_0 = x$ under \mathbf{P}_x , which is solution in law to the SDE

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad t \geq 0, \quad (1)$$

driven by some Brownian motion $W \equiv (W_t)_{t \geq 0}$. The state space for X is $\mathcal{I} \equiv (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$, and the functions b and σ are continuous, with $\sigma > 0$ over \mathcal{I} . We assume that α and β are inaccessible (natural) endpoints for the diffusion. Therefore, X is regular over \mathcal{I} and the SDE (1) admits a weak solution that is unique in law.

Each player observes the evolution of market conditions, and decides at each instant whether to hold fast or to irreversibly concede to his opponent. Denoting by τ^1 and τ^2 the random times at the players choose to concede, every player i 's expected payoff is⁴

$$J^i(x, \tau^1, \tau^2) = \mathbf{E}_x \left[1_{\{\tau^i \leq \tau^j\}} e^{-r\tau^i} R^i(X_{\tau^i}) + 1_{\{\tau^i > \tau^j\}} e^{-r\tau^j} G^i(X_{\tau^j}) \right], \quad (2)$$

where $r > 0$ is the players' common discount rate. The payoff functions R^i and G^i in (2) satisfy $G^i \geq R^i$, with $G^i(x) > R^i(x)$ for all x above some threshold $\alpha^i \in [\alpha, \beta)$. Therefore, if $\tau^i \leq \tau^j$, then player i obtains a payoff $R^i(X_{\tau^i})$, whereas, if $\tau^j < \tau^i$ and $X_{\tau^j} > \alpha^i$, then player i obtains a payoff $G^i(X_{\tau^j})$ strictly higher than the payoff $R^i(X_{\tau^j})$ he would have obtained by conceding at τ^j . The functions b , σ , R^i , and G^i are common knowledge.

2.2 Assumptions

We now detail our assumptions on the payoff functions R^i and G^i and emphasize useful properties of the *stand-alone exit problem*

$$V_{R^i}(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [e^{-r\tau} R^i(X_\tau)] \quad (3)$$

faced by player i when player j is *stubborn*, that is, plays $\tau^j = \infty$; here \mathcal{T} is the set of all stopping times of the usual augmentation $(\mathcal{F}_t)_{t \geq 0}$ of the natural filtration generated by X .

Discount Factors The infinitesimal generator of X is defined by $\mathcal{L}u \equiv bu' + \frac{1}{2}\sigma^2u''$ for all $u \in \mathcal{C}^2(\mathcal{I})$. Because $\sigma > 0$ over \mathcal{I} , the ODE $\mathcal{L}u - ru = 0$ admits a two-dimensional vector space of solutions in $\mathcal{C}^2(\mathcal{I})$, spanned by two positive fundamental solutions ψ and ϕ , respectively increasing and decreasing, and uniquely defined up to a linear transformation. Because the boundaries α and β of \mathcal{I} are natural,

$$\lim_{x \rightarrow \alpha^+} \psi(x) = 0, \quad \lim_{x \rightarrow \beta^-} \psi(x) = \infty, \quad \lim_{x \rightarrow \alpha^+} \phi(x) = \infty, \quad \lim_{x \rightarrow \beta^-} \phi(x) = 0. \quad (4)$$

⁴By convention, we let $f(X_\tau) \equiv 0$ over $\{\tau = \infty\}$ for any Borel function f and any random time τ .

Letting $\tau_y \equiv \inf \{t \geq 0 : X_t = y\}$ be the hitting time by X of $y \in \mathcal{I}$ from $X_0 = x$, we then obtain the following formula for the expected discount factor associated to x and τ_y :

$$\mathbf{E}_x[e^{-r\tau_y}] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{if } x \leq y \\ \frac{\phi(x)}{\phi(y)} & \text{if } x > y \end{cases}. \quad (5)$$

Assumptions on the Payoff Functions For each $i = 1, 2$, R^i is \mathcal{C}^2 over \mathcal{I} and satisfies

A1 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t \geq 0} e^{-rt} |R^i(X_t)|] < \infty$.

A2 For each $x \in \mathcal{I}$, $\lim_{t \rightarrow \infty} e^{-rt} R^i(X_t) = 0$ \mathbf{P}_x -a.s.

A3 For some $x_0^i \in \mathcal{I}$, $\mathcal{L}R^i - rR^i < 0$ over (α, x_0^i) and $\mathcal{L}R^i - rR^i > 0$ over (x_0^i, β) .

A1 guarantees that the family $(e^{-r\tau} R^i(X_\tau))_{\tau \in \mathcal{T}}$ is uniformly integrable. A1–A2 imply

$$\lim_{x \rightarrow \alpha^+} \frac{R^i(x)}{\phi(x)} = \lim_{x \rightarrow \beta^-} \frac{R^i(x)}{\psi(x)} = 0. \quad (6)$$

A3 states that the gains from holding fast increase only as long as market conditions remain in (x_0^i, β) . As a result, the optimal stopping region $\{x \in \mathcal{I} : V_{R^i}(x) = R^i(x)\}$ for the stand-alone problem (3) is of the form $(\alpha, x_{R^i}]$ for some threshold $x_{R^i} < x_0^i$, and

$$V_{R^i}(x) = \begin{cases} R^i(x) & \text{if } x \leq x_{R^i} \\ \frac{\phi(x)}{\phi(x_{R^i})} R^i(x_{R^i}) & \text{if } x > x_{R^i} \end{cases}. \quad (7)$$

The smooth-fit property applies at x_{R^i} , that is, $R^{i'}(x_{R^i}) = \frac{\phi'(x_{R^i})}{\phi(x_{R^i})} R^i(x_{R^i})$ (Dayanik and Karatzas (2003, Corollary 7.1)). It follows from standard optimal stopping theory that $(e^{-rt} V_{R^i}(X_t))_{t \geq 0}$ is a supermartingale and that $\mathcal{L}V_{R^i} - rV_{R^i} \leq 0$ over $\mathcal{I} \setminus \{x_{R^i}\}$. The following lemma is standard (Décamps, Gensbittel, and Mariotti (2024)).

Lemma 1 $V_{R^i} > 0$ over \mathcal{I} and $R^i > 0$ over $(\alpha, x_{R^i}]$.

For each $i = 1, 2$, G^i is \mathcal{C}^1 and piecewise \mathcal{C}^2 over \mathcal{I} and satisfies

A4 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t \geq 0} e^{-rt} G^i(X_t)] < \infty$.

A5 For each $x \in \mathcal{I}$, $\lim_{t \rightarrow \infty} e^{-rt} G^i(X_t) = 0$ \mathbf{P}_x -a.s.

A6 $G^i \geq V_{R^i}$ over \mathcal{I} and $G^i(x) > V_{R^i}(x)$ if and only if $x > \alpha^i$ for some $\alpha^i < x_{R^i}$.

A7 $\mathcal{L}G^i - rG^i \leq 0$ everywhere $G^{i'}$ is defined.

It should be noted that A7 together with $G^i \geq R^i$ implies the first statement in A6. A7

implies that player i prefers to obtain the payoff $G^i(X_t)$ sooner than later, as is the case when G^i is itself the value function of an optimal stopping problem. From A6 and Lemma 1, we have $G^i > 0$ over \mathcal{I} ; hence A4 guarantees that the family $(e^{-r\tau}G^i(X_\tau))_{\tau \in \mathcal{T}}$ is uniformly integrable. A4–A5 imply

$$\lim_{x \rightarrow \alpha^+} \frac{G^i(x)}{\phi(x)} = \lim_{x \rightarrow \beta^-} \frac{G^i(x)}{\psi(x)} = 0. \quad (8)$$

2.3 A Running Example: Exit in Duopoly

Two firms are initially present on the market. As long as they both remain in the market, each firm earns a flow duopoly profit X_t , where X follows a geometric Brownian motion with drift $b < r$ and volatility σ over $\mathcal{I} \equiv (0, \infty)$. If firm i exits the market at $\tau^i \leq \tau^j$, then its assets are liquidated for a value $l^i > 0$. If firm j exits the market at $\tau^j < \tau^i$, then firm i thereafter earns a flow monopoly profit mX_t for $m > 1$ until it eventually decides to liquidate its assets. Given (τ^1, τ^2) , the total value of every firm i is thus

$$F^i(x, \tau^1, \tau^2) \equiv \mathbf{E}_x \left[\int_0^{\tau^1 \wedge \tau^2} e^{-rt} X_t dt + 1_{\{\tau^i \leq \tau^j\}} e^{-r\tau^i} l^i + 1_{\{\tau^i > \tau^j\}} e^{-r\tau^j} V_m^i(X_{\tau^j}) \right],$$

where $V_m^i(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x \left[\int_0^\tau e^{-rt} mX_t dt + e^{-r\tau} l^i \right]$ is firm i 's monopoly value. Letting $E(x) \equiv \frac{x}{r-b}$, $R^i \equiv l^i - E$, and $G^i \equiv V_m^i - E$, we obtain that $J^i(\cdot, \tau^1, \tau^2) \equiv F^i(\cdot, \tau^1, \tau^2) - E$ satisfies (2). From standard computations (Dixit and Pindyck (1994)), $x_{R^i} = \frac{\rho^-}{\rho^- - 1} (r - b) l^i$ for $\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$, $\alpha^i = \frac{x_{R^i}}{m}$, and $G^i = R^i$ over $(\alpha, \alpha^i]$. That A1–A7 are satisfied can be checked along the lines of Décamps, Gensbittel, and Mariotti (2024).

3 Mixed Strategies and Equilibrium Concept

In this section, we first recall the definition and basic properties of randomized stopping times. Imposing a Markov restriction leads to our first main result, which is a representation theorem for Markovian randomized stopping times. We then define the concept of Markov-perfect equilibrium and give some important properties of best replies.

3.1 Randomized Stopping Times

One classical definition of a randomized stopping time consists, following Aumann (1964), in enlarging the probability space; this compensates for the absence of a natural measurable structure over the space of stopping times. For every player $i = 1, 2$, the corresponding enlarged probability space is $(\Omega^i, \mathcal{F}^i) \equiv (\Omega \times [0, 1], \mathcal{F} \otimes \mathcal{B}([0, 1]))$, endowed with the product

probability $\mathbf{P}_x^i \equiv \mathbf{P}_x \otimes \text{Leb}$, where $\mathcal{B}([0, 1])$ is the Borel σ -field over the sampling space $[0, 1]$ and Leb is Lebesgue measure. Hence the following definition (Touzi and Vieille (2002)).

Definition 1 *A randomized stopping time for player $i = 1, 2$ is an $\mathcal{F} \otimes \mathcal{B}([0, 1])$ -measurable function $\gamma^i : \Omega^i \rightarrow \mathbb{R}_+$ such that, for Leb -a.e. $u^i \in [0, 1]$, $\gamma^i(\cdot, u^i) \in \mathcal{T}$. The process $\Gamma^i \equiv (\Gamma_t^i)_{t \geq 0}$ defined by*

$$\Gamma_t^i(\omega) \equiv \int_{[0,1]} \mathbf{1}_{\{\gamma^i(\omega, u^i) \leq t\}} du^i, \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \quad (9)$$

is the conditional cumulative distribution function (ccdf) of the randomized stopping time γ^i . The process $\Lambda^i \equiv (\Lambda_t^i)_{t \geq 0}$ defined by $\Lambda_t^i \equiv 1 - \Gamma_t^i$ is the conditional survival function (csf) of the randomized stopping time γ^i .

Notice that the ccdf process Γ^i defined by (9) takes values in $[0, 1]$ and has nondecreasing and right-continuous trajectories. The following representation result is useful.

Lemma 2 *The ccdf process Γ^i is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Gamma_t^i = \mathbf{P}_x^i[\gamma^i \leq t | \mathcal{F}_t] \quad \mathbf{P}_x\text{-a.s.} \quad (10)$$

By convention, we let $\Gamma_{0-}^i \equiv 0$. This allows us in what follows to interpret integrals of the form $\int_{[0, \tau)} \cdot d\Gamma_t^i$ in the Stieltjes sense for any ccdf Γ^i .

If the players use randomized stopping times γ^1 and γ^2 , then their expected payoffs are defined over the product probability space $\Omega \times [0, 1] \times [0, 1]$ with canonical element (ω, u^1, u^2) , endowed with the product probability $\bar{\mathbf{P}}_x \equiv \mathbf{P}_x \otimes \text{Leb} \otimes \text{Leb}$. Specifically,

$$J^i(x, \gamma^1, \gamma^2) \equiv \bar{\mathbf{E}}_x \left[\mathbf{1}_{\{\gamma^i \leq \gamma^j\}} e^{-r\gamma^i} R^i(X_{\gamma^i}) + \mathbf{1}_{\{\gamma^i > \gamma^j\}} e^{-r\gamma^j} G^i(X_{\gamma^j}) \right], \quad (11)$$

where $\gamma^1 \equiv \gamma^1(\omega, u^1)$ and $\gamma^2 \equiv \gamma^2(\omega, u^2)$, reflecting that players 1 and 2 use the independent randomization devices u^1 and u^2 , respectively.

In line with Touzi and Vieille (2002) and Riedel and Steg (2017), the following lemma shows that we may equivalently work with the family of ccdf processes Γ^i .

Lemma 3 *If the players use randomized stopping times with ccdfs Γ^1 and Γ^2 , then their expected payoffs write as*

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x \left[\int_{[0, \infty)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{[0, \infty)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j \right]. \quad (12)$$

Moreover, any nondecreasing, right-continuous, \mathcal{F}_t -adapted, $[0, 1]$ -valued process Γ^i is the ccdf of the randomized stopping time $\hat{\gamma}^i$ defined by

$$\hat{\gamma}^i(u^i) \equiv \inf \{t \geq 0 : \Gamma_t^i > u^i\}. \quad (13)$$

3.2 Markovian Randomized Stopping Times

Our goal in this paper is to characterize equilibria in which players concede according to mixed Markov strategies that only depend on current market conditions. Notice that such strategies have to be defined for any initial market conditions $x \in \mathcal{I}$. The following definition is standard (Revuz and Yor (1999, Chapter I, §3)).

Definition 2 Let $Y \equiv (Y_t)_{t \geq 0}$ be the coordinate process over the canonical space Ω , defined by $Y_t(\omega) \equiv \omega_t$ for all $\omega \in \Omega$ and $t \geq 0$. Then, for each $t \geq 0$, the shift operator $\theta_t : \Omega \rightarrow \Omega$ is defined by $Y_s \circ \theta_t \equiv Y_{s+t}$ for all $s \geq 0$.

In words, the effect of θ_t on a trajectory ω is to forget the part of the trajectory prior to time t and to shift back the remaining part by t units of time. We are now ready to define our notion of a Markovian randomized stopping time.

Definition 3 A randomized stopping time for player $i = 1, 2$ with csf $\Lambda^i : \Omega \times \mathbb{R}_+ \rightarrow [0, 1]$ is Markovian if, for all $x \in \mathcal{I}$, $\tau \in \mathcal{T}$, and $s \geq 0$,

$$\Lambda_{\tau+s}^i = \Lambda_\tau^i(\Lambda_s^i \circ \theta_\tau) \text{ over } \{\tau < \infty\} \mathbf{P}_x\text{-a.s.} \quad (14)$$

Definition 3 can be interpreted as follows. According to Definition 1 and Lemma 2, $\Lambda_{\tau+s}^i$ is the probability that player i concedes after time $\tau + s$ conditionally on $\mathcal{F}_{\tau+s}$. The Markov restriction then states that, conditionally on player i not conceding by τ , the probability that he holds fast for at least s additional units of time should not depend on the trajectory of X prior to time τ . This probability is thus given by $\Lambda_s^i \circ \theta_\tau$, that is, the probability induced by the randomized strategy applied to the shifted trajectory. Formula (14) then follows from the standard formula for conditional probabilities.

Processes satisfying (14) are known as multiplicative functionals of the Markov process X (Blumenthal and Gettoor (1968)). Combining a result by Sharpe (1971) with the classical representation result of additive functionals of regular diffusions (Borodin and Salminen (2002, Part I, Chapter II, Section 4, §23)), we obtain the following representation result.

Theorem 1 For each $i = 1, 2$, $\Lambda^i : \Omega \times \mathbb{R}_+ \rightarrow [0, 1]$ is the csf of a Markovian randomized stopping time for player i if and only if there exists a closed set $S^i \subset \mathcal{I}$ and a Radon measure⁵ μ^i over $\mathcal{I} \setminus S^i$ such that, for all $x \in \mathcal{I}$ and $t \geq 0$,

$$\Lambda_t^i = 1_{\{t < \tau_{S^i}\}} e^{-\int_{\mathcal{I} \setminus S^i} L_t^y \mu^i(dy)} \mathbf{P}_x\text{-a.s.}, \quad (15)$$

⁵Recall that a Radon measure over an open set $U \subset \mathbb{R}$ is a nonnegative Borel measure that is locally finite in the sense that every point of U has a neighborhood having finite measure.

where $L_t^y \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(y-\varepsilon, y+\varepsilon)}(X_s) \sigma^2(X_s) ds$ is the local time of X at (y, t) , and $\tau_{S^i} \equiv \inf \{t \geq 0 : X_t \in S^i\}$ is the hitting time by X of S^i . In particular, the mapping $t \mapsto \Lambda_t^i$ is continuous over $[0, \tau_{S^i})$ \mathbf{P}_x -a.s.

Theorem 1 allows us to interchangeably refer to a Markov strategy as a pair (μ^i, S^i) , a cdf Γ^i , or a csf Λ^i . The interpretation of (15) is that player i concedes with probability 1 over S^i and with finite intensity over $\text{supp } \mu^i$. Three special cases are worth mentioning.

The Pure Stopping Case If $\mu^i \equiv 0$, then the Markov strategy $(0, S^i)$ is just the standard stopping time τ_{S^i} .

The Absolutely Continuous Case If $\mu^i \equiv g^i \cdot \text{Leb}$ is absolutely continuous, then, from the occupation time formula (Revuz and Yor (1999, Chapter VI, §1, Corollary 1.6)),

$$\Lambda_t^i = 1_{\{t < \tau_{S^i}\}} e^{-\int_{\mathcal{I}} L_t^y g^i(y) dy} = 1_{\{t < \tau_{S^i}\}} e^{-\int_0^t g^i(X_s) \sigma^2(X_s) ds}. \quad (16)$$

Outside S^i , this amounts for player i to concede with intensity $\lambda^i(X_t) \equiv g^i(X_t) \sigma^2(X_t)$, that is, during a short time interval of length dt , with probability $\lambda^i(X_t) dt$.

The Singular Case If $\mu^i \equiv a^i \delta_{x^i}$, where $a^i > 0$ and δ_{x^i} is the Dirac mass at $x^i \in \mathcal{I} \setminus S$, then the corresponding csf writes as

$$\Lambda_t^i = 1_{\{t < \tau_{S^i}\}} e^{-a^i L_t^{x^i}}. \quad (17)$$

In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is singular over $[0, \tau_{S^i}(\omega))$ for \mathbf{P}_x -a.e. $\omega \in \Omega$ such that X crosses x^i ; that is, its derivative is zero for Leb -a.e. $t \in [0, \tau_{S^i}(\omega))$, though Λ_t^i decreases each time X crosses x^i . Heuristically, (17) means that, when $X_t = x^i$, player i concedes with instantaneous probability $a^i dL_t^{x^i}$. Thus a^i truly represents an *intensity* of exit; this should not be confused with a positive *probability* of exit at x^i , which would, because X \mathbf{P}_{x^i} -a.s. crosses x^i infinitely often in any interval $[t, t + dt]$, be undistinguishable from exiting the market with probability 1 at x^i .

Whereas strategies such as (17) have not been considered in the WoA literature, they naturally emerge as limits of more familiar ones:

- (i) Discretizing the state space and the time dimension, suppose that player i concedes with positive probability only when the current state is x^i . Then, using the appropriate normalization—which consists, for a time period of duration dt , to concede at x^i with a probability of order \sqrt{dt} (Feller (1968, Chapter III, §5, Theorem 1))—the limit of such Markov strategies when the mesh of the discretization goes to 0 corresponds to a distribution with hazard rate proportional to the local time of the diffusion at x^i .

- (ii) Consider the Markov strategy that, outside S^i , consists in conceding with intensity $\lambda_\varepsilon^i(X_t) \equiv \frac{a^i}{2\varepsilon} \sigma^2(X_t) 1_{(x^i-\varepsilon, x^i+\varepsilon)}$ for $a^i > 0$ and some small $\varepsilon > 0$. By (16), the corresponding csf writes as $\Lambda_{\varepsilon,t}^i = 1_{\{t < \tau_{S^i}\}} e^{-\int_0^t \lambda_\varepsilon^i(X_s) ds}$. Then, by definition of the local time, for each $t \geq 0$, $\Lambda_{\varepsilon,t}^i$ converges \mathbf{P}_x -a.s. to Λ_t^i in (17) as ε goes to 0.

The second example suggests that the space of Markov strategies allowed for by Theorem 1 is, in a topological sense, a natural completion of the space of Markov strategies with absolutely continuous intensity measures. This is indeed the case, provided an adequate weak topology is defined over the space of intensity measures, though a formal proof of this fact is beyond the scope of this paper.

An important property of a Markov strategy, such as (17), associated to a singular intensity measure with an atom at x^i , is that the total probability of conceding before time dt starting from x^i is itself of order \sqrt{dt} ,⁶ whereas the same quantity is of order dt for a Markov strategy, such as (16), associated to an absolutely continuous intensity measure. As we will explain in Section 4.3, this generates a kink in player j ' equilibrium value function.

3.3 Markov-Perfect Equilibrium and Properties of Best Replies

We are now ready to define our equilibrium concept and to provide some basic properties of best replies. Our first result is standard and reflects the fact that a player, given the behavior of his opponent, cannot improve his payoff by merely randomizing over pure strategies, that is, over standard stopping times.

Lemma 4 *For each $x \in \mathcal{I}$ and for any pair of randomized stopping times with cdfs (Γ^1, Γ^2) , $J^i(x, \Gamma^i, \Gamma^j) \leq \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, \Gamma^j)$.*

This motivates the following definition.

Definition 4 *A Markov-perfect equilibrium (MPE) is a profile $((\mu^1, S^1), (\mu^2, S^2))$ of Markov strategies such that, for all $x \in \mathcal{I}$ and $i = 1, 2$,*

$$J^i(x, (\mu^i, S^i), (\mu^j, S^j)) = \bar{J}^i(x, (\mu^j, S^j)) \equiv \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j)).$$

That is, for each $i = 1, 2$, (μ^i, S^i) is a perfect best reply (pbr) for player i to (μ^j, S^j) , and $\bar{J}^i(\cdot, (\mu^j, S^j))$ is player i 's best-reply value function (brvf) to (μ^j, S^j) .

When no confusion can arise as to the strategy of player j , we write \bar{J}^i instead of $\bar{J}^i(\cdot, (\mu^j, S^j))$. The next proposition provides useful general properties of pbrs and brvfs.

⁶This follows from the properties of the local time, see Peskir (2019, Lemma 15).

Proposition 1 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf \bar{J}^i , then $V_{R^i} \leq \bar{J}^i \leq G^i$. Furthermore,*

- (i) $S^1 \cap S^2 \cap (\alpha^i, \beta) = \emptyset$;
- (ii) $S^i \subset C^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = R^i(x)\}$;
- (iii) $\text{supp } \mu^i \setminus S^j \subset C^i$ and $\text{supp } \mu^i \cap S^j \subset D^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = G^i(x)\}$;
- (iv) $S^i \cup (\text{supp } \mu^i \setminus S^j) \subset (\alpha, x_{R^i}]$;
- (v) $(0, S^i)$ is also a pbr to (μ^j, S^j) ; more generally, $(\tilde{\mu}^i, S^i)$ is a pbr to (μ^j, S^j) for any $\tilde{\mu}^i$ such that $\text{supp } \tilde{\mu}^i \subset C^i \cup S^j$.

Property (i) intuitively states that player i should never concede when market conditions are such that player j concedes with probability 1 and player i 's payoff from conceding is strictly less than the payoff from letting player j concede, that is, $x \in S^j$ and $G^i(x) > V_{R^i}(x)$. Property (ii) simply expresses the fact that player i 's brvf coincides with R^i over the portion S^i of the state space over which he concedes with probability 1. Property (iii) states that player i 's payoff is R^i when he concedes with positive intensity outside of player j 's stopping region S^j . Property (iv) reflects that player i should never concede when market conditions are above the optimal threshold x_{R^i} for his stand-alone exit problem; intuitively, this is because waiting for X to drop down to x_{R^i} before conceding is player i 's optimal strategy even in the worst-case scenario in which player j is stubborn, that is, $(\mu^j, S^j) = (0, \emptyset)$. Finally, property (v) states that, when conceding with positive intensity outside of S^i , player i should be indifferent between holding fast and conceding.

Remark Murto (2004) requires in his definition of an MPE that $(\alpha, \alpha^i] \subset S^i$ for $i = 1, 2$. The rationale is that, because $G^i = V_{R^i} = R^i$ over $(\alpha, \alpha^i]$, holding fast over (α, α^i) is weakly dominated for player i by conceding with probability 1 over this interval. For instance, being stubborn is a best reply for player i over (α, α^i) only if player j concedes with probability 1 over this interval, except perhaps over a set of Lebesgue measure 0. This behavior is not per se inconsistent with an MPE, but, as pointed out by Ghemawat and Nalebuff (1985) in a deterministic context, it is not consistent with trembling-hand perfection in the spirit of Selten (1975). Hereafter, we do not systematically impose this refinement, especially in Section 5 where this allows us to simplify notation; however, we indicate which MPEs can be modified so as to satisfy it.

We close this section with an important global regularity result.

Proposition 2 *If $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE, then, for each $i = 1, 2$, player i 's bruf \bar{J}^i is continuous over \mathcal{I} .*

4 Derivation and Implications of A Singular Mixed-Strategy MPE

We first recall within our general framework two standard MPEs, respectively in pure and mixed strategies, that have been emphasized in the literature. We next describe a novel type of MPE involving a singular strategy for one player. Our heuristic derivation leads to a variational system that fully characterizes this candidate MPE. We provide sufficient conditions for the existence of a solution to this variational system in the context of our running example described in Section 2.3. We finally compare the resulting singular mixed-strategy MPE with the two standard MPEs and discuss its asset-pricing implications.

4.1 A Pure-Strategy MPE

We say that player 1 is *as least as enduring* as player 2 if $\alpha^1 \leq \alpha^2$ and $x_{R^1} \leq x_{R^2}$; intuitively, player 1 is at least as willing to hold fast as player 2. Suppose then that player 1 threatens to hold fast maximally by conceding only at $\tau^1 = \inf\{t \geq 0 : X_t \leq \alpha^1\}$. Then, because $\alpha^1 \leq \alpha^2$, we have $G^2(X_{\tau^1}) = R^2(X_{\tau^1})$ by definition of α^2 . In light of (2)–(3), this implies that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$, $J^2(x, \tau^1, \tau^2) = \mathbf{E}_x[e^{-r\tau^1 \wedge \tau^2} R^2(X_{\tau^1 \wedge \tau^2})] \leq V_{R^2}(x)$. Thus a pbr for player 2 to τ^1 is to concede at $\tau^2 = \inf\{t \geq 0 : X_t \leq x_{R^2}\}$. As for player 1, if player 2 concedes at τ^2 , then, for each $x \in \mathcal{I}$, $\mathbf{E}_x[e^{-r\tau^2} G^1(X_{\tau^2})] \geq R^1(x)$. For $x \leq x_{R^2}$, this follows from the fact that $G^1(x) \geq R^1(x)$ by A6, with a strict inequality if $x > \alpha^1$. For $x > x_{R^2}$, this follows from A6 again along with the fact that the process $(e^{-rt} V_{R^1}(X_t))_{t \geq 0}$ is a martingale up to $\tau_{x_{R^1}}$, the hitting time by X of x_{R^1} , which is no less than τ^2 because $x_{R^1} \leq x_{R^2}$ by assumption. Thus a pbr for player 1 to τ^2 is to concede at τ^1 . Hence the following standard result (Murto (2004), Georgiadis, Kim, and Kwon (2022)).

Proposition 3 *If player 1 is at least as enduring as player 2, then $((0, (\alpha, \alpha^1)), (0, (\alpha, x_{R^2})))$ is a pure-strategy MPE that satisfies Murto (2004)'s refinement.*

In the case of a small asymmetry between the players, $((0, (\alpha, x_{R^1})), (0, \emptyset))$ is also an MPE in which the more enduring player 1 follows his stand-alone optimal strategy because the less enduring player 2 is stubborn (Georgiadis, Kim, and Kwon (2022)). However, this MPE does not satisfy Murto (2004)'s refinement, because, for $x \in (\alpha^1, \alpha^2)$, player 2's strategy is

no longer a best response when player 1 does not concede with probability 1 in any small enough neighborhood of x . Interestingly, Murto (2004) shows that, when we allow player 1's stopping set S^1 to exhibit a gap, there may exist an MPE satisfying his refinement in which, when $x > x_{R^1}$, player 1 exits first when X reaches x_{R^1} .

4.2 A Regular Mixed-Strategy MPE in the Symmetric Case

The following result is standard (Steg (2015), Georgiadis, Kim, and Kwon (2022)).

Proposition 4 *If the players are as enduring as each other, then the strategy profile $((\lambda^1(x)\sigma^{-2}(x) \cdot Leb, (\alpha, \alpha^*]), (\lambda^2(x)\sigma^{-2}(x) \cdot Leb, (\alpha, \alpha^*]))$ defined, for each $i = 1, 2$, by*

$$\lambda^i(x) \equiv \frac{rR^j(x) - \mathcal{L}R^j(x)}{G^j(x) - R^j(x)} 1_{\{\alpha^* < x \leq x^*\}} \quad (18)$$

for $\alpha^* \equiv \alpha^1 = \alpha^2$ and $x^* \equiv x_{R^1} = x_{R^2}$, is a mixed-strategy MPE.

In this MPE, each player exits the market with an intensity measure over $(\alpha^*, x^*]$ that makes his opponent indifferent between holding fast and conceding; for instance, in our running example, each firm obtains its liquidation value over $(\alpha^*, x^*]$. Each player's equilibrium value function coincides with the value function of his stand-alone exit problem. Thus, in expectation, all rents are dissipated in equilibrium.

4.3 A Singular Mixed-Strategy MPE

When there is no uncertainty about future payoffs, the WoA admits mixed-strategy equilibria in which players' strategies are described by absolutely continuous distributions over some interval of exit times (Hendricks, Weiss, and Wilson (1988)). However, under Brownian uncertainty, when $x_{R^1} \neq x_{R^2}$, there exists no mixed-strategy MPE in which the players concede with absolutely continuous intensities (Georgiadis, Kim, and Kwon (2022)). For all that, it is incorrect to conclude that only pure-strategy MPEs exist in this case. This section argues for this claim by describing an MPE involving a singular strategy for one player. For the sake of simplicity, the analysis in this section remains heuristic; a full justification of our arguments is provided in Section 5.

As in Section 4.1, let us suppose that player 1 is at least as enduring as player 2, so that players may be asymmetric or symmetric, and consider the following equation in x :

$$R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)} G^1(x). \quad (19)$$

Lemma S.5 in the Online Supplement shows that (19) admits a unique solution $\underline{x}^2 \in$

(α^1, x_{R^1}) . The threshold \underline{x}^2 is such that, if player 2 threatens to concede only at $\tau^2 = \inf \{t \geq 0 : X_t \leq \underline{x}^2\}$, then, at x_{R^1} , player 1 is indifferent between conceding and obtaining $R^1(x_{R^1})$ immediately and waiting for player 2 to concede at τ^2 and obtaining $G^1(\underline{x}^2)$ only then. Our goal is to construct an MPE $((a^1 \delta_{x_{R^1}}, (\alpha, \alpha^1)), (0, (\alpha, \underline{x}^2)))$ in which player 1 concedes with intensity a^1 at x_{R^1} and player 2 concedes with probability 1 at τ^2 .

4.3.1 Necessary Conditions

To this end, we first suppose that such an MPE exists, and we derive necessary conditions for the brvfs \bar{J}^1 and \bar{J}^2 . Notice that $\bar{J}^1 \geq R^1$ and $\bar{J}^2 \geq R^2$, because every player i can guarantee himself R^i by conceding immediately.

Player 1 Player 1, whose strategy involves randomization at x_{R^1} , should be indifferent at x_{R^1} between conceding and holding fast until τ^2 . This implies that his brfv \bar{J}^1 must be \mathcal{C}^2 over (\underline{x}^2, β) , with $\bar{J}^1(x_{R^1}) = R^1(x_{R^1})$ (value matching). Because $\bar{J}^1 \geq R^1$, it follows in turn that $\bar{J}^{1'}(x_{R^1}) = R^{1'}(x_{R^1})$ as well (smooth pasting). Moreover, by standard dynamic-programming arguments, $\mathcal{L}\bar{J}^1 - r\bar{J}^1 = 0$ over (\underline{x}^2, β) . Thus

$$\bar{J}^1(x) = \frac{\phi(x)}{\phi(x_{R^1})} R^1(x_{R^1}), \quad x \in (\underline{x}^2, \beta). \quad (20)$$

In particular, $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$: player 1 does not benefit from the presence of player 2 over $[x_{R^1}, \beta)$. By contrast, $\bar{J}^1 > V_{R^1}$ over $[\underline{x}^2, x_{R^1})$ because player 1 can hope for player 2 to concede at \underline{x}^2 before he himself concedes at x_{R^1} .

Player 2 Player 2 plays a pure strategy and hopes for player 1 to concede at x_{R^1} . We guess that \bar{J}^2 is \mathcal{C}^2 over $(\underline{x}^2, \beta) \setminus \{x_{R^1}\}$, with $\bar{J}^2(\underline{x}^2) = R^2(\underline{x}^2)$ (value-matching) and $\bar{J}^{2'}(\underline{x}^2) = R^{2'}(\underline{x}^2)$ (smooth pasting), and that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over that region. There remains to characterize the behavior of \bar{J}^2 at x_{R^1} . Because player 1 randomizes between holding fast and conceding at x_{R^1} , we expect that $G^2(x_{R^1}) > \bar{J}^2(x_{R^1}) > R^2(x_{R^1})$. This implies that \bar{J}^2 is not differentiable at x_{R^1} . Indeed, by the properties of the local time, starting from x_{R^1} , player 1 concedes during a short time interval of length dt with probability $\mathbf{E}_{x_{R^1}}[\Gamma_{dt}] = a^1 c \sqrt{dt} + o(\sqrt{dt})$, where $\Gamma_{dt} = 1 - e^{-a^1 L_{dt}^{x_{R^1}}}$ and c is a positive constant. If player 1 concedes, then player 2 benefits from the follower payoff $G^2(x_{R^1})$, whereas if player 1 holds fast, then player 2 achieves the value $\bar{J}^2(X_{dt})$. Thus

$$\bar{J}^2(x_{R^1}) = a^1 c \sqrt{dt} G^2(x_{R^1}) + (1 - a^1 c \sqrt{dt}) \mathbf{E}_{x_{R^1}}[e^{-r dt} \bar{J}^2(X_{dt})] + o(\sqrt{dt}). \quad (21)$$

Now, suppose, by way of contradiction, that J^2 is \mathcal{C}^2 in a neighborhood of x_{R^1} . Then, from

Itô's formula, we have

$$\mathbf{E}_{x_{R^1}}[e^{-rdt}\bar{J}^2(X_{dt})] = \bar{J}^2(x_{R^1}) + (\mathcal{L}\bar{J}^2 - r\bar{J}^2)(x_{R^1}) dt + o(dt). \quad (22)$$

Plugging (22) into (21) yields $a^1 c[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})]\sqrt{dt} + o(\sqrt{dt}) = 0$, a contradiction as $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$ and a^1 and c are positive constants. This is an indication that \bar{J}^2 is not differentiable at x_{R^1} , so let us denote by $\Delta\bar{J}^{2'}(x_{R^1}) \equiv \bar{J}^{2'+}(x_{R^1}) - \bar{J}^{2'-}(x_{R^1})$ the corresponding derivative jump. From the Itô–Tanaka–Meyer formula, which generalizes Itô's formula to functions, such as \bar{J}^2 , that can be written as the difference of two convex functions (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.5)), we have

$$\begin{aligned} \mathbf{E}_{x_{R^1}}[e^{-rdt}\bar{J}^2(X_{dt})] &= \bar{J}^2(x_{R^1}) + \mathbf{E}_{x_{R^1}} \left[\int_0^{dt} e^{-rs} (\mathcal{L}\bar{J}^2 - r\bar{J}^2)(X_s) ds \right. \\ &\quad \left. + \int_0^{dt} e^{-rs} \bar{J}^{2'-}(X_s) \sigma(X_s) dW_s + \frac{1}{2} \Delta\bar{J}^{2'}(x_{R^1}) L_{dt}^{x_{R^1}} \right] \\ &= \bar{J}^2(x_{R^1}) + \frac{1}{2} \Delta\bar{J}^{2'}(x_{R^1}) c \sqrt{dt} + o(\sqrt{dt}), \end{aligned} \quad (23)$$

where the second equality follows from the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over $(\underline{x}^2, \beta) \setminus \{x_{R^1}\}$ and from the properties of local time. Plugging (23) into (21) yields

$$a^1[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})] + \frac{1}{2} \Delta\bar{J}^{2'}(x_{R^1}) = 0.$$

From this and $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$, we obtain $\Delta\bar{J}^{2'}(x_{R^1}) < 0$; intuitively, player 2 gets increasingly optimistic as X approaches x_{R^1} , but is disappointed if X crosses x_{R^1} but player 1 holds fast at x_{R^1} .

The Variational System Our discussion so far leads to the following variational system: find a constant $a^1 > 0$, and two functions $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2\})$ and $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2, x_{R^1}\})$ such that

$$w^1 \geq R^1 \text{ over } \mathcal{I}, \quad (24)$$

$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (\underline{x}^2, \beta), \quad (25)$$

$$w^1 = G^1 \text{ over } (\alpha, \underline{x}^2], \quad (26)$$

$$w^1(x_{R^1}) = R^1(x_{R^1}), \quad (27)$$

$$w^1(\beta-) = 0, \quad (28)$$

$$w^2 \geq R^2 \text{ over } \mathcal{I}, \quad (29)$$

$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (\underline{x}^2, \beta) \setminus \{x_{R^1}\}, \quad (30)$$

$$w^2 = R^2 \text{ over } (\alpha, \underline{x}^2], \quad (31)$$

$$w^{2'}(\underline{x}^2) = R^{2'}(\underline{x}^2), \quad (32)$$

$$a^1[G^2(x_{R^1}) - w^2(x_{R^1})] + \frac{1}{2}\Delta w^{2'}(x_{R^1}) = 0, \quad (33)$$

$$w^2(\beta-) = 0. \quad (34)$$

4.3.2 Sufficient Conditions

Our main characterization result, Theorem 3, implies that, if $(a^1, \bar{J}^1, \bar{J}^2)$ is a solution to the variational system (24)–(34), then \bar{J}^1 is the brfv to $(0, (\alpha, \underline{x}^2])$ and \bar{J}^2 is the brfv to $(a^1\delta_{x_{R^1}}, (\alpha, \alpha^1])$, so that $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1]), (0, (\alpha, \underline{x}^2])$ is indeed an MPE strategy profile. As for \bar{J}^1 , we have already seen that (24)–(25) and (27) pin down a unique solution, given by (20), which satisfies $\bar{J}^1(\underline{x}^2) = G^1(\underline{x}^2)$ by definition of \underline{x}^2 . As for \bar{J}^2 , the analysis is a bit more delicate due to the presence of the derivative jump $\Delta\bar{J}^{2'}(x_{R^1})$ at x_{R^1} , which, by (33), is itself determined by the intensity a^1 with which player 1 exits at x_{R^1} . The following result provides sufficient conditions for the existence of a singular mixed-strategy MPE in our running example.

Proposition 5 *In the running example, if the firms' liquidation values $l^1 \leq l^2$ are close enough to each other, and if m is sufficiently large and $b > 0$, then there exists a mixed-strategy MPE $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1]), (0, (\alpha, \underline{x}^2])$ in which the more enduring firm 1 randomizes between holding fast and conceding at x_{R^1} while the less enduring firm 2 exits with probability 1 as soon as market conditions fall below $\underline{x}^2 < x_{R^1}$.⁷ This MPE satisfies Murto (2004)'s refinement.*

Remark It should be noted that the above analysis is wholly conducted in terms of thresholds for the state variable. This suggests that similar arguments would apply to a model in which the only observable difference in players' characteristics is that they have unequal discount factors.

4.3.3 Comparisons with the Standard MPEs

The MPE constructed in Proposition 5 differs from the pure-strategy MPE of Proposition 3 in that, for $x \geq x_{R^1}$, the more enduring firm 1 does not benefit from the presence of firm 2 as $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$, whereas $\bar{J}^2 > V_{R^2}$ over $[x_{R^1}, \beta)$. The reason is that firm 2 adopts a tougher stance by threatening to exit the market only at $\underline{x}^2 < x_{R^1} \leq x_{R^2}$, which makes firm 1 indifferent between holding fast and conceding at x_{R^1} . In our singular mixed-strategy

⁷Numerical simulations suggest that, when firms' liquidation values $l^1 \leq l^2$ are close enough to each other, the variational system (24)–(34) admits a solution whatever the parameter values of the model if $b > 0$, and, if $b < 0$, as long as $m \in [1, C]$ for some constant C that increases with σ .

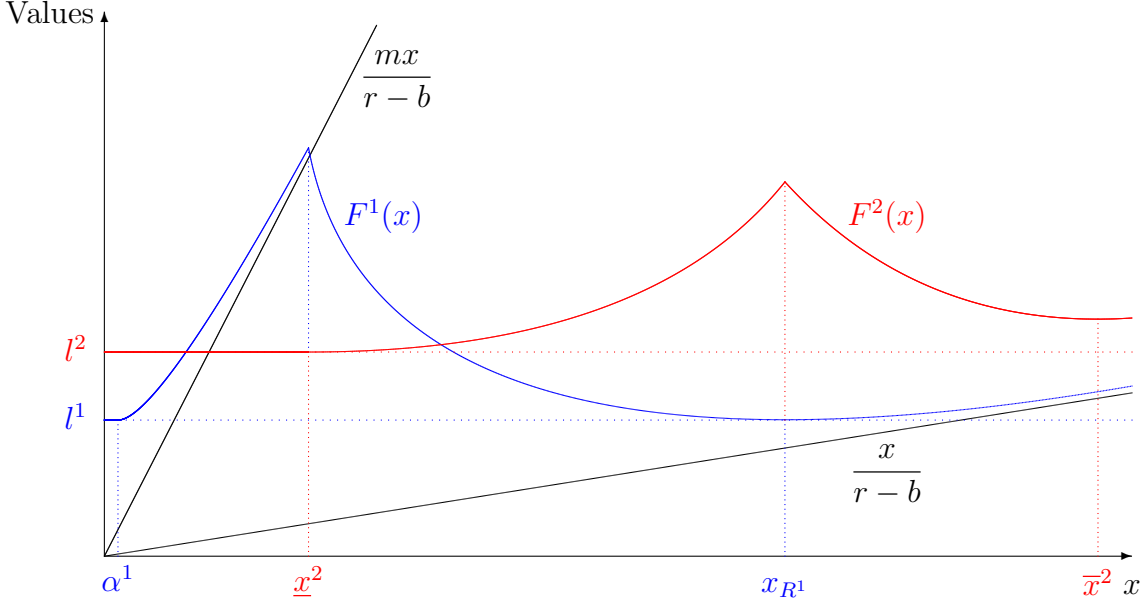


Figure 1: The total values of never exiting the market in a duopoly and in a monopoly (in black), firm 1's total value (in blue), and firm 2's total value (in red) in the singular mixed-strategy MPE of Proposition 5.

MPE, $\max S^1 \vee \max S^2 = \underline{x}^2 < x_{R^1} \leq x_{R^2}$, whereas $\max S^1 \vee \max S^2 \in \{x_{R^1}, x_{R^2}\}$ in any pure-strategy MPE. Thus mixing by firm 1 delays the time at which some firm must exit the market. This leads to a richer dynamics, whereby firms can alternatively find themselves in a position of strength or weakness depending on the random fluctuations of market conditions. As illustrated on Figure 1, firm 1 is in a position of strength when X is close to \underline{x}^2 and in a position of weakness when X is close to x_{R^1} , whereas the reverse holds for firm 2. The general results of Section 5 confirm that such alternation in the balance of power is a robust feature of singular mixed-strategy MPEs, reflecting that a player in a position of weakness can hope for a reversal of market conditions in its favor. It should be noted that whether a player is currently in a position of strength or weakness is an endogenous feature of the equilibrium under consideration.

In the limiting case of symmetric firms, in which $l^1 = l^2 \equiv l$, $\alpha^1 = \alpha^2 \equiv \alpha^*$, and $x_{R^1} = x_{R^2} \equiv x^*$, one may also compare our singular mixed-strategy MPE with the regular mixed-strategy MPE of Proposition 4. In the latter, over $(\alpha^*, x^*]$, the probability of any firm exiting the market during a short time interval of length dt is of order dt ; moreover, the two firms' total values are constant and equal to their common liquidation value l . Thus attrition leads to a complete dissipation of rents. By contrast, in our singular mixed-strategy MPE, the probability that firm 1, starting at $x_{R^1} = x^*$, exits the market during a short time interval of length dt is now of order \sqrt{dt} . It also follows from Figure 1 that our singular mixed-strategy

MPE Pareto-dominates the regular mixed-strategy MPE for any initial market condition $x > \alpha^*$ and that the firms' total values are not monotonic in market conditions.

When firms are asymmetric, with $l^1 < l^2$, a regular mixed-strategy MPE does not exist (Georgiadis, Kim, and Kwon (2022)) and another benchmark is the pure-strategy MPE characterized by Murto (2004). Because firm 1 does not exit the market with probability 1 at x_{R^1} , firm 2's total value at x_{R^1} , $F^2(x_{R^1})$, must be less than its monopoly value $V_m^2(x_{R^1})$. Because firm 1's total value coincides with his stand-alone total value over $[x_{R^1}, \infty)$, for any initial market condition $x > x_{R^1}$, our singular mixed-strategy MPE is Pareto dominated by any pure-strategy MPE in which player 1 exits the market at x_{R^1} , reflecting that wasteful attrition can take place on the equilibrium path.

Finally, it should be noted that, whereas the regular mixed-strategy MPE of Proposition 4 has no counterpart when there is even the slightest degree of asymmetry in the firms' liquidation values, our singular mixed-strategy MPE also exists in the case of symmetric liquidation values and is robust to asymmetry.

4.3.4 Asset-Pricing Implications

We now draw the asset-pricing implications of our singular mixed-strategy MPE.

Assets and Investors Suppose that both firms are all-equity firms whose stocks are traded on a frictionless financial market. At any time t , each firm's stock distributes a payout flow X_t if neither firm has conceded, and a 0 or mX_t payout flow otherwise, depending on whether or not the firm has exited the market. Shareholders are risk-neutral and observe market conditions and firms' exit decisions. Thus their information set at any time t is

$$\hat{\mathcal{F}}_t \equiv \mathcal{F}_t \vee \sigma(1_{\{\gamma^1 \leq s\}}, 0 \leq s \leq t). \quad (35)$$

Notice that (35) reflects that, as $\tau_{\underline{x}^2}$ is $(\mathcal{F}_t)_{t \geq 0}$ -adapted, the information that firm 2 has conceded by time t is already included in $\hat{\mathcal{F}}_t$.

Stock Prices Because shareholders are risk-neutral, the firms' stock prices up to the first time $\tau^c \equiv \gamma^1 \wedge \tau_{\underline{x}^2}$ at which one of them concedes are given, for each $t \geq 0$, by

$$\begin{aligned} V_t^{1, \tau^c} &\equiv F^1(X_{t \wedge \tau^c}), \\ V_t^{2, \tau^c} &\equiv F^2(X_{t \wedge \tau^c}) + [V_m^2(x_{R^1}) - F^2(x_{R^1})]1_{\{t \wedge \tau_{\underline{x}^2} \geq \gamma^1\}}, \end{aligned}$$

where the second term in the definition of V_t^{2, τ^c} reflects that firm 1 concedes over $\{\tau_{\underline{x}^2} > \gamma^1\}$, so that firm 2's market value jumps upwards to its monopoly value at γ^1 ; there is no analogous

term in the definition of V_t^{1,τ^c} as $F^1(\underline{x}^2) = V_m^1(\underline{x}^2)$. Applying Itô's formula to F^1 and the Itô–Tanaka–Meyer formula to F^2 yields

$$V_t^{1,\tau^c} = F_1(x) + \int_0^{t \wedge \tau^c} [rF^1(X_s) - X_s] ds + \int_0^{t \wedge \tau^c} \sigma X_s F^{1'}(X_s) dW_s, \quad (36)$$

$$\begin{aligned} V_t^{2,\tau^c} = & F_2(x) + \int_0^{t \wedge \tau^c} [rF^2(X_s) - X_s] ds + \int_0^{t \wedge \tau^c} \sigma X_s F^{2'}(X_s) dW_s \\ & + [V_m^2(x_{R^1}) - F^2(x_{R^1})](1_{\{t \wedge \tau_{\underline{x}^2} \geq \gamma^1\}} - a^1 L_{t \wedge \tau^c}^{x_{R^1}}). \end{aligned} \quad (37)$$

The Martingale Property The absence of arbitrage opportunities requires that each firm i 's discounted cum-dividend stock-price process $(e^{-rt \wedge \tau^c} V_t^{i,\tau^c} + \int_0^{t \wedge \tau^c} e^{-rs} X_s ds)_{t \geq 0}$ be a martingale with respect to the shareholders' filtration $(\hat{\mathcal{F}}_t)_{t \geq 0}$. For firm 1, this readily follows from (36). The analysis of firm 1's stock price is then the same as in the corporate-finance models of Merton (1974), Leland (1994), and Goldstein, Ju, and Leland (2001), except that it is not stopped with probability 1 at $\tau_{x_{R^1}}$. In particular, firm 1's stock price is not a function of its current payout level. For firm 2, the martingale property is more subtle. At first sight, the presence of the local-time term $L_{t \wedge \tau^c}^{x_{R^1}}$ in (37) seems to create an arbitrage opportunity that consists to sell firm 2's stock each time $X_t = x_{R^1}$ at price $F^2(x_{R^1})$ and then to repurchase firm 2's stock at price $F^2(X_{t+dt}) < F^2(x_{R^1})$ at $t + dt$; to a naive investor, this strategy seems to yield a gain of order $dL_t^{x_{R^1}}$ each time $X_t = x_{R^1}$.⁸ However, this does not account for the possibility that firm 1 may exit when $X_t = x_{R^1}$, causing firm 2's stock price to jump upwards to $V_m^2(x_{R^1})$. Once this risk is taken into account, the expected gain of this strategy is exactly zero, reflecting that the term $1_{\{t \wedge \tau_{\underline{x}^2} \geq \gamma^1\}} - a^1 L_{t \wedge \tau^c}^{x_{R^1}}$ in (37) is an $(\hat{\mathcal{F}}_t)_{t \geq 0}$ -martingale.

Comovements of Stock Prices and their Volatilities A takeaway from Figure 1 is that, as long as no firm exits the market, firms' stock prices comove negatively in bad times, when $X_t \in (\underline{x}^2, \bar{x}^2)$, while they comove positively in good times, when $X_t > \bar{x}^2$. The general results of Section 5 confirm that this is a robust feature of singular mixed-strategy MPEs in our running example. Figure 2 illustrates sample paths of stock prices before any firm exits the market.

As predicted by (37), each time $X_t = x_{R^1}$ without firm 1 exiting the market, firm 2's stock price is continuously reflected downward. Moreover, as Lemma S.8 in the Online Supplement

⁸This strategy is in the spirit of Karatzas and Shreve (1998, Appendix B) and Jarrow and Protter (2005, Theorem 4.3), who show that a singular term in the dynamics of a cum-dividend stock prices leads to arbitrage opportunities. Of course, this is not the case for an ex-dividend stock-price process, as in dynamic security-design models (DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007)) or cash-management models (Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011)).

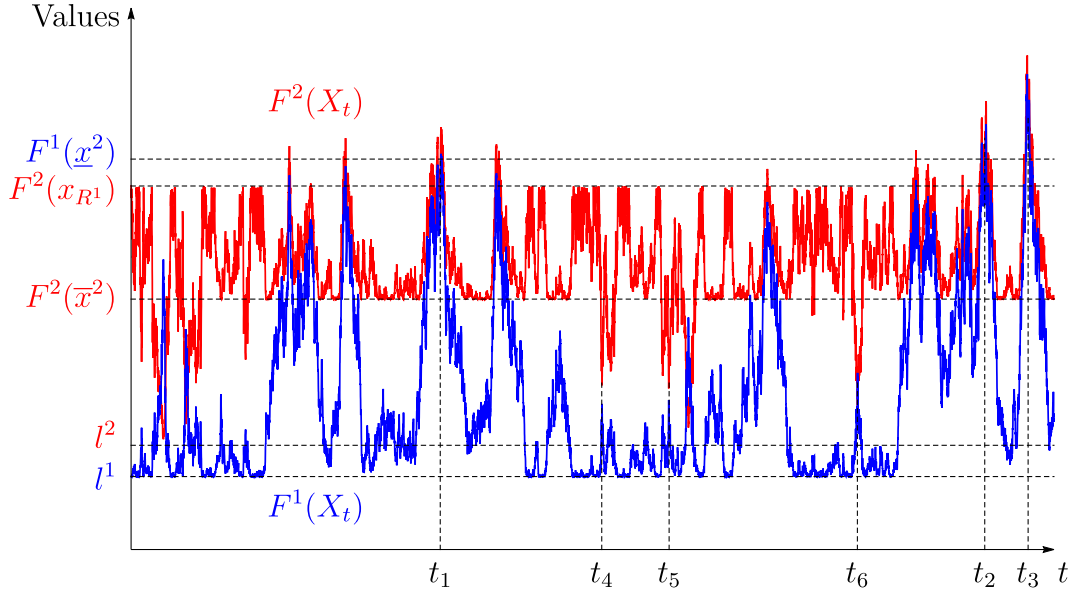


Figure 2: Sample paths of firm 1's stock price (in blue) and firm 2's stock price (in red) before any firm exits the market. The selected parameter values are $l^1 = 0.97$, $l^2 = 1$, $m = 2$, $\mu = 0.15$, $r = 0.8$, and $\sigma = 0.3$.

shows, (37) predicts that the volatility $\left| \frac{\sigma X_t F^{2'}(X_t)}{F^2(X_t)} \right|$ of firm 2's stock returns peaks when firm 2's stock price approaches the reflecting boundary $F^2(x_{R^1})$, and drops to zero when firm 2's stock price approaches its liquidation value l^2 . (As long as $X_t > x_{R^1}$, the volatility of firm 2's stock returns also drops to zero when firm 2's stock price approaches $F^2(\bar{x}^2)$.) Similarly, (36) predicts that the volatility $\left| \frac{\sigma X_t F^{1'}(X_t)}{F^1(X_t)} \right|$ of firm 1's stock returns peaks when firm 1's stock price approaches its monopoly value $V_m^1(\underline{x}^2)$, and drops to zero when firm 1's stock price approaches its liquidation value l^1 . Thus the volatilities of firms' stock returns comove negatively as long as no firm exits the market and the market conditions remain in the attrition region $(\underline{x}^2, \bar{x}^2)$. Again, this is a robust feature of singular mixed-strategy MPEs in our running example.

A Rationale for Resistance and Support Levels Technical analysts claim that they can predict financial price movements using limited information sets, including past prices (Edwards, Magee, and Bassetti (2013)). Faced with a chart such as Figure 2, a technical analyst unaware of the fundamental relationship between market conditions and stock prices would interpret $F^2(x_{R^1})$ as a predictable *resistance level* for firm 2's stock price, at which upward trends tend to be reversed. Similarly, he may interpret l^1 and $F^2(\bar{x}^2)$ as predictable *support levels* for firm 1's and firm 2's stock prices. Our analysis provides a rationale for these well-documented stylized facts while maintaining the assumption that stock prices are

only driven by fundamentals.⁹

A *breakup* of the resistance level $F^2(x_{R^1})$ for firm 2 can in turn occur in two cases. (1) In good times, when a large improvement in market conditions leads firm 2's stock price to break, in a continuous way, the resistance level $F^2(x_{R^1})$. This happens at times t_1 , t_2 , and t_3 in Figure 2. These events are preceded and followed by episodes of positive comovements of stock prices, reflecting that market conditions have left the attrition region $(\underline{x}_2, \bar{x}_2)$. (2) In bad times, if firm 1 concedes at x_{R^1} , causing an upward jump in firm 2's stock price from $F^2(x_{R^1})$ to its monopoly value $V_m^2(x_{R^1})$. This second case is in line with the observation often made in technical analysis that, when prices rise above their resistance levels, they tend to do so decisively. In contrast with continuous breakups, such a discontinuous breakup is preceded by an episode of negative comovements of stock prices. A *breakdown* of the support level $F^2(\bar{x}^2)$ can only happen in a continuous way, and only after firm 2's stock price has reached its resistance level $F^2(x_{R^1})$. This happens at times t_4 , t_5 , and t_6 in Figure 2.

Technical analysts often explain decisive breakups of resistance levels by unpredictable changes in earnings, management, or competition above investors' expectations. This is exactly what happens in our model. Where we differ from technical analysis is that the downward bounces in firm 2's stock price at the resistance level $F^2(x_{R^1})$ are no more predictable from past prices than the upward jump in firm 2's stock price that occurs when firm 1 exits the market at x_{R^1} , and thus cannot be arbitrated away by rational investors.

5 Main Results

We first provide a necessary condition for mixed-strategy MPEs, establishing that any such MPE is either singular and exhibits an alternating threshold structure, or—and only if $x_{R^1} = x_{R^2}$ —is regular, involving absolutely continuous intensity measures. We then characterize singular MPEs by a variational system satisfied by the players' value functions. The proofs of our main results make use of an additional regularity assumption.

A8 The functions b , σ , and $R^{i''}$ are locally Lipschitz.

5.1 The Alternating Structure of Singular Mixed-Strategy MPEs

By convention, we let $\max \emptyset \equiv \alpha$ and, for any MPE $((\mu^1, S^1), (\mu^2, S^2))$, we let $s^i \equiv \max S^i$.

⁹The interpretation of l^1 and $F^2(\bar{x}^2)$ as support levels for firms 1 and 2, respectively, is a little less clear-cut than that of $F^2(x_{R^1})$ as a resistance level for firm 2. Indeed, the volatilities of firm 1's and firm 2's stock prices drop to zero at x_{R^1} and \bar{x}^2 , respectively, making it less likely to detect a trend reversal at l^1 and $F^2(\bar{x}^2)$ than at $F^2(x_{R^1})$, where the volatility of firm 2's stock price reaches a peak, see Figure 2.

The following result then holds.

Theorem 2 *For any mixed-strategy MPE $((\mu^1, S^1), (\mu^2, S^2))$,*

- (i) *if $x_{R^1} \neq x_{R^2}$, then the restrictions of the intensity measures μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$ are purely atomic;*
- (ii) *if $x_{R^1} = x_{R^2}$, either the restrictions of the intensity measures μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$ are purely atomic, or they are absolutely continuous, with densities characterized by (18) with α^* replaced by $\alpha^1 \vee \alpha^2$.*

Theorem 2 first confirms the basic insight of Georgiadis, Kim, and Kwon (2022), according to which there exists no mixed-strategy MPE with absolutely continuous intensity measures when $x_{R^1} \neq x_{R^2}$. Thus, if a mixed-strategy MPE exists at all in this case, it must feature intensity measures that are singular with respect to Lebesgue measure. The key information provided by Theorem 2 is that these measures must be discrete.

The proof can be sketched as follows.

Let us consider a mixed-strategy MPE $((\mu^1, S^1), (\mu^2, S^2))$, supposing one exists. First, Proposition 1(iv) implies $\max \text{supp } \mu^i \cap (s^1 \vee s^2, \beta) \leq x_{R^i}$ for every player i ; we show that this must in fact be an equality for the largest maximum of the supports. Next, Proposition 1(v) and dynamic-programming arguments imply that, for each $i = 1, 2$, $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over any interval (q, q') where player j does not concede; it also follows from Proposition 1 that $\bar{J}^i \geq V_{R^i}$ and that $\bar{J}^i(q^i) = R^i(q^i)$ for all $q^i \in \text{supp } \mu^i$. Fixing such an interval (q, q') , and assuming that $q, q' \in \text{supp } \mu^j$, we deduce from this that there must exist a single point $q^i \in (q, q') \cap \text{supp } \mu^i$ at which player i is indifferent between conceding or holding fast. The reason why such a point q^i must exist is that, otherwise, player j would expect, starting from any initial market condition $x \in (q, q')$, to obtain either $R^j(q)$ or $R^j(q')$ when leaving this interval. However, because $\mathcal{L}R^j - rR^j < 0$ over (q, q') as $q' \leq x_{R^j} < x_0^j$, player j would be strictly better off conceding and obtaining $R^j(x)$ at x , a contradiction. It follows that \bar{J}^i coincides with the solution to $\mathcal{L}u - ru = 0$ that is tangent to R^i at q^i . This, together with $\mathcal{L}R^i - rR^i < 0$, implies that q^i is unique. As a result, the set of accumulation points of the supports of μ^1 and μ^2 in $(s^1 \vee s^2, \beta)$ must coincide.

Consider first the asymmetric case $x_{R^1} \neq x_{R^2}$, and, to fix ideas, assume that $q_1^1 = x_{R^1}$ and $q_1^1 \geq q_1^2$ for $q_1^i \equiv \max \text{supp } \mu^i$. We verify that it is not optimal for player 2 to concede at q_1^1 . Therefore, q_1^1 must be an isolated point of $\text{supp } \mu^1$ and $q_1^1 > q_1^2$. Iterating this argument and using the preceding remarks, we show that, for each $i = 1, 2$, and for any two consecutive

points $q_n^i > q_{n+1}^i > s^1 \vee s^2$ in the support of μ^i , there must exist a single point $q_n^j \in (q_{n+1}^i, q_n^i)$ in the support of μ^j at which player j is indifferent between conceding or holding fast. We thus obtain two decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$, with either $N^1 = N^2 = \infty$ or $0 \leq N^1 - N^2 \leq 1$, which are intertwined in the sense that $q_1^1 > q_1^2 > q_2^1 > q_2^2 > \dots$ as long as these thresholds are defined. We also show that if $N^1 = N^2 = \infty$, any such intertwined sequences must converge to α . These two sequences characterize the restrictions of μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$. As a result, when $x_{R^1} \neq x_{R^2}$, any mixed-strategy MPE must fall into one of three categories, which are delineated in Corollary 1 below.

In the symmetric case $x_{R^1} = x_{R^2}$, analogous arguments show that the common set of accumulation points of the supports of μ^1 and μ^2 is either empty or equal to $(s^1 \vee s^2, x_{R^1}]$. In the latter case, analytic arguments imply that the measures μ^i are absolutely continuous, with densities characterized by (18) for $\alpha^* \equiv s^1 \vee s^2$.

Corollary 1 *Let $((\mu^1, S^1), (\mu^2, S^2))$ be a singular mixed-strategy MPE. Then, for every player i , $\text{supp } \mu^i \cap (s^1 \vee s^2, \beta) = \{q_n^i : n = 1, \dots, N^i\}$ for intertwined decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$ satisfying, with no loss of generality, $q_1^1 > q_1^2$. Moreover, $q_1^1 = x_{R^1}$ and one of the following three conditions holds:*

1. $N^1 = N^2 \equiv N \in \mathbb{N} \setminus \{0\}$ and $q_N^1 > q_N^2 > s^1 > s^2$;
2. $N^1 = N^2 + 1 \equiv N \in \mathbb{N} \setminus \{0\}$ and $q_{N-1}^2 > q_N^1 > s^2 > s^1$, with $q_0^2 \equiv \beta$ by convention;
3. $N^1 = N^2 = \infty$ and $\lim_{n \rightarrow \infty} q_n^1 = \lim_{n \rightarrow \infty} q_n^2 = s^1 = s^2 = \alpha$, so that $S^1 = S^2 = \emptyset$.

In an MPE of type 1, player 1 exits the market with probability 1 at s^1 , and player 2 has the lowest randomization threshold. In an MPE of type 2, player 1 has the lowest randomization threshold, and player 2 exits the market with probability 1 at s^2 —the example of Section 4.3 is a case in point, with $N^1 = 1$ and $N^2 = 0$. In an MPE of type 3, neither player exits the market with probability 1 at any point of the state space, and players keep randomizing all the way down to α . It should be noted that an MPE of type 3 can exist only if $\alpha^1 = \alpha^2 = \alpha$; indeed, every player i such that $\alpha^i > \alpha$ would not be willing to delay exiting the market over (α, α^i) if his opponent were to do the same.

The upshot from Theorem 2 and Corollary 1 is that, when players have different stand-alone optimal exit thresholds, alternation is a robust feature of singular mixed-strategy MPEs. In the attrition region, players randomize between conceding and holding fast at isolated thresholds. Thus, generalizing the MPE constructed in Proposition 5, players can

alternatively find themselves in a position of strength or weakness; the difference is that both players may now randomize, leading to a richer set of equilibrium outcomes. In an MPE of type 1 and type 2, this process may persist until one player eventually reaches his stopping region and exits the market with probability 1. By contrast, in an MPE of type 3, exit must take place at a randomization threshold.

Corollary 1 fully characterizes equilibrium outcomes for an MPE of type 3, because any market condition in \mathcal{I} can be reached with positive probability from any initial market conditions $x \in \mathcal{I}$. The same holds true for MPEs of types 1 and 2, provided $x > x_{R^1}$, with $q_1^1 > q_1^2$ by convention. Indeed, for any such MPE $((\mu^1, S^1), (\mu^2, S^2))$ and for each $x > x_{R^1}$, there exists an outcome-equivalent MPE $((\tilde{\mu}^1, \tilde{S}^1), (\tilde{\mu}^2, \tilde{S}^2))$ such that $\text{supp } \tilde{\mu}^i = \text{supp } \mu^i \cap (s^1 \vee s^2, \beta)$ for every player i and $\tilde{S}^1 = (\alpha, s^1)$ and $\tilde{S}^2 = \emptyset$ (for an MPE of type 1), or $\tilde{S}^1 = \emptyset$ and $\tilde{S}^2 = (\alpha, s^2)$ (for an MPE of type 2). By contrast, Corollary 1 does not pin down equilibrium outcomes of MPEs of types 1 and 2 for lower initial market conditions. Indeed, as in Murto (2004), it is possible to construct MPEs in which the stopping regions S^1 and S^2 exhibit gaps; moreover, these gaps may themselves include randomization thresholds.

5.2 The Characterization Result

Our final theorem provides a necessary and sufficient condition for the existence of an MPE of type 2. Analogous results hold for MPEs of types 1 and 3; their statements and proofs proceed along similar lines, and are omitted for the sake of brevity.

Theorem 3 *Let $N \in \mathbb{N} \setminus \{0\}$ and let be given*

- *two finite sequences $(q_n^1)_{n=1}^N$ and $(q_n^2)_{n=0}^{N-1}$ of numbers in \mathcal{I} , with $q_0^2 \equiv \beta$ by convention, and a number $s^2 \in \mathcal{I}$ such that $q_1^1 = x_{R^1} > q_1^2 > q_2^1 > \dots > q_{N-1}^1 > q_{N-1}^2 > q_N^1 > s^2$;*
- *two finite sequences $(a_n)_{n=1}^N$ and $(b_n)_{n=0}^{N-1}$ of positive real numbers.*

Then the strategy profile $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2)))$, with $\sum_{n=1}^0 \equiv 0$ by convention, is an MPE of type 2 if and only if $s^2 > \alpha^2$ and there exists two functions $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q_n^2 : 1 \leq n \leq N-1\} \cup \{s^2\}))$ and $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q_n^1 : 1 \leq n \leq N\} \cup \{s_2\}))$ that satisfy the variational system

$$w^1 \geq R^1 \text{ over } \mathcal{I}, \quad (38)$$

$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (s^2, \beta) \setminus \{q_n^2 : 1 \leq n \leq N-1\}, \quad (39)$$

$$w^1 = G^1 \text{ over } (\alpha, s^2], \quad (40)$$

$$w^1(q_n^1) = R^1(q_n^1), \quad 1 \leq n \leq N, \quad (41)$$

$$b_n[G^1(q_n^2) - w^1(q_n^2)] + \frac{1}{2}\Delta w^1(q_n^2) = 0, \quad 1 \leq n \leq N-1, \quad (42)$$

$$w^1(\beta-) = 0, \quad (43)$$

$$w^2 \geq R^2 \text{ over } \mathcal{I}, \quad (44)$$

$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (s^2, \beta) \setminus \{q_n^1 : 1 \leq n \leq N\}, \quad (45)$$

$$w^2 = R^2 \text{ over } (\alpha, s^2], \quad (46)$$

$$w^2(q_n^2) = R^2(q_n^2), \quad 1 \leq n \leq N-1, \quad (47)$$

$$w^{2'}(s^2) = R^{2'}(s^2), \quad (48)$$

$$a_n[G^2(q_n^1) - w^2(q_n^1)] + \frac{1}{2}\Delta w^{2'}(q_n^1) = 0, \quad i = 1 \leq n \leq N, \quad (49)$$

$$w^2(\beta-) = 0. \quad (50)$$

Moreover, whenever $\alpha^1 \leq \alpha^2$, $((\sum_{n=1}^N a_n \delta_{q_n^1}, (\alpha, \alpha^1], (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2]))$) is an outcome-equivalent MPE that satisfies Murto (2004)'s refinement.

The proof of Theorem 3 is based on the properties obtained in the proof of Theorem 2, together with classical methods employed in verification theorems for optimal-stopping and stopping-game theory. In particular, conditions (42) and (49) are obtained by applying the Itô–Tanaka–Meyer formula. A key insight from this characterization is that incentives are nonlocal: for instance, when $N \geq 2$, player 2 randomizes at the threshold q_n^2 , $1 \leq n \leq N-1$, with appropriate intensity b_n , to make player 1 willing to randomize at the thresholds q_{n+1}^1 and q_n^1 , and vice versa. This contrasts with the regular mixed-strategy equilibrium in the symmetric case $x_{R^1} = x_{R^2}$, in which, at any point of the attrition region, each player randomizes so as to make his opponent indifferent between holding fast and conceding at the very same point.

The ultimate justification for Proposition 5 follows from applying Theorem 3 for $N = 1$, which yields the variational system (24)–(34). In our running example, MPEs of type 2 with $N > 1$ have similar implications as the MPE of Proposition 5. In particular, in the attrition region, the firms' stock prices and the volatilities of their returns comove negatively. The difference is that firms' stock prices now exhibit several resistance levels— $F^1(q_n^2)$, $1 \leq n \leq N-1$, for firm 1, and $F^2(q_n^1)$, $1 \leq n \leq N$, for firm 2—resulting in a richer price dynamics. MPEs of types 1 and 3 have similar robust implications.

Importantly, the variational characterization in Theorem 3 and the analogous results for MPEs of types 1 and 3 hold for both symmetric and asymmetric players. Thus our results provide a characterization of mixed-strategy MPE outcomes in the WoA under uncertainty

that are robust to even the slightest degree of heterogeneity between players.

6 Concluding Remarks

This paper has offered a complete study of mixed-strategy MPE outcomes in the symmetric-information WoA when the players' payoffs are driven by a homogenous linear diffusion. Our contribution is threefold.

First, we have provided a characterization result for Markov strategies in terms of an intensity measure over the state space together with a subset of the state space over which the player concedes with probability 1. This covers the standard cases of pure strategies and of mixed strategies in which intensity measures are absolutely continuous over the state space. In addition, this representation allows for mixed Markov strategies with singular intensity measures, a possibility that has been disregarded in the literature.

Second, we have argued that, far from being artificial or exotic, such singular strategies are key to the identification of robust mixed-strategy MPE outcomes, both in the cases of symmetric and asymmetric players. We have provided a variational characterization of singular mixed-strategy MPEs and we have shown that they are characterized by intertwined sequences of randomization thresholds for the players. As a result, each player on the equilibrium path can alternately be in a position of strength or weakness, reflecting that a weak player can hope for a reversal of situation, a novel insight in the literature.

Third, we have seen that, in the standard model of exit in a duopoly, this characterization leads to new testable asset-pricing implications when firms are publicly traded. Namely, the firms' stock prices and the volatilities of their returns comove negatively over the attrition region and exhibit patterns documented by technical analysis. This contrasts with the predictions of the standard regular mixed-strategy MPE that only exists when firms are symmetric, in which firms' stock prices are perfectly aligned and are constant and equal to their common liquidation value over the attrition region.

Taken together, our results show that mixed-strategy MPEs that are robust to even slight asymmetries between players' payoffs share a common structure, and lead to qualitatively similar empirical implications. This yields rich and robust predictions for the WoA under uncertainty—something that is precluded by focusing on pure-strategy MPEs, or regular mixed-strategy MPEs of symmetric games, whose implications are too stark to fruitfully lend themselves to applied analysis. Our hope is that these insights may pave new avenues for empirical work.

Appendix

Notation A property is satisfied a.s. if, for each $x \in \mathcal{I}$, it is satisfied for \mathbf{P}_x -a.e. $\omega \in \Omega$.

PROOF OF THEOREM 1: (Necessity) We hereafter omit the index i for the sake of clarity. If Λ is the csf of a Markovian randomized stopping time, then, for all $t, s \geq 0$, $\Lambda_{t+s} = \Lambda_t(\Lambda_s \circ \theta_t)$ a.s. In particular, applying this property at $t = s = 0$ yields $\Lambda_0 = (\Lambda_0)^2$ and, hence, $\Lambda_0 \in \{0, 1\}$ a.s. In the terminology of Blumenthal and Gettoor (1968, Definition III.1.1), Λ is a right-continuous multiplicative functional of X adapted to $(\mathcal{F}_t)_{t \geq 0}$. The set $E_\Lambda \equiv \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 1] = 1\}$ is called the set of permanent points for Λ . Using Blumenthal's 0–1 law (Blumenthal and Gettoor (1968, Proposition I.5.17)) and the fact that $\Lambda_0 \in \{0, 1\}$ a.s., we have $\mathcal{I} \setminus E_\Lambda = \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 0] = 1\}$. The stopping time $\tau \equiv \inf\{t > 0 : \Lambda_t = 0\} \in \mathcal{T}$ is called the lifetime of Λ . The proof consists of three steps.

Step 1 In order to apply the main result of Sharpe (1971), we need to check that Λ is an exact multiplicative functional in the sense of Blumenthal and Gettoor (1968, Definition III.4.13). According to Blumenthal and Gettoor (1968, Proposition III.5.9) it is sufficient to prove that, for all $x \in \mathcal{I} \setminus E_\Lambda$ and $t > 0$,

$$\lim_{u \downarrow 0} \mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] = 0. \quad (\text{A.1})$$

To this end, we first claim that, for any such x and t , and for each $u \in (0, t)$, we have $1_{\{t-u \geq \tau_x \circ \theta_u\}}(\Lambda_{t-u} \circ \theta_u) = 0$ \mathbf{P}_x -a.s. Indeed, if $t - u \geq \tau_x \circ \theta_u(\omega)$ for some $\omega \in \Omega$, then the trajectory $\theta_u(\omega)$ crosses x over the interval $[0, t - u]$. Because, by (14), $\Lambda_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega)) = \Lambda_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega))\Lambda_0(\theta_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega))) = 0$ \mathbf{P}_x -a.s. as $x \in \mathcal{I} \setminus E_\Lambda$, this implies that $\Lambda_{t-u}(\theta_u(\omega)) = 0$ as the mapping $s \mapsto \Lambda_s(\theta_u(\omega))$ is nonincreasing and nonnegative. The claim follows. This implies in particular that, for $u < \frac{t}{2}$,

$$\mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] \leq \mathbf{P}_x[t - u < \tau_x \circ \theta_u] = \mathbf{E}_x[\mathbf{P}_{X_u}[t - u < \tau_x]] \leq \mathbf{E}_x\left[\mathbf{P}_{X_u}\left[\frac{t}{2} < \tau_x\right]\right].$$

The mapping $y \mapsto \mathbf{P}_y[\frac{t}{2} < \tau_x]$ is bounded and $\lim_{y \rightarrow x} \mathbf{P}_y[\frac{t}{2} < \tau_x] = 0$ as X is a regular diffusion. Hence (A.1) follows by bounded convergence along with the fact that $\lim_{u \downarrow 0} X_u = x$ \mathbf{P}_x -a.s. Exactness of Λ implies that E_Λ is open and thus that $\mathcal{I} \setminus E_\Lambda$ is closed, see Blumenthal and Gettoor (1968, page 126, last paragraph) together with the fact that the fine topology over \mathcal{I} associated to X coincides with the usual topology, see Blumenthal and Gettoor (1968, Definition II.4.1 and Exercise II.4.16).

Step 2 We are now in a position to apply Sharpe (1971, Theorem 7.1, Formula (7.1)), which expresses Λ_t as the product of three factors.

1. The first factor is equal to 1 because X has continuous trajectories, so that the terms $F(X_{s-}, X_s)$ vanish as $F = 0$ over the diagonal of \mathcal{I} , see Sharpe (1971, Theorem 5.1 and proof of Theorem 7.1).

2. The second factor can be written as $1_{\{t < \tau_B\}}$, where τ_B is the hitting time by X of a Borel subset B of \mathcal{I} ; this is because the lifetime of X is infinite and X has continuous trajectories. In turn, because X is a diffusion process and $\sigma > 0$ over \mathcal{I} , this term is a.s. equal to $1_{\{t < \tau_S\}}$, where S is the closure of B .

3. The third factor is of the form $e^{-\int_0^t f(X_s) dA_s}$, where $f : \mathcal{I} \rightarrow \mathbb{R}_+$ is Borel-measurable and A is a continuous additive functional of X (Revuz and Yor (1999, Chapter X, §1, Definition 1.1)) such that the mapping $x \mapsto \mathbf{E}_x[\int_0^\infty e^{-t} dA_t]$ is bounded.

Thus, for each $t \geq 0$, we have the representation

$$\Lambda_t = 1_{\{t < \tau_S\}} e^{-\int_0^t f(X_s) dA_s} \text{ a.s.} \quad (\text{A.2})$$

Moreover, the integral $\int_0^t f(X_s) dA_s$ is \mathbf{P}_x -a.s. finite for all $t < \tau_S$ except maybe for x in an M -polar set, where M is the multiplicative functional defined by $M_t \equiv 1_{\{t < \tau_S\}}$ for all $t \geq 0$ (Blumenthal and Gettoor (1968, II.2.18 and III.1.4)). According to Sharpe (1971, Definition, page 29), $B \subset \mathcal{I}$ is an M -polar set if there exists a nearly Borel subset (Blumenthal and Gettoor (1968, Definition I.10.21)) $C \supset B$ of \mathcal{I} such that the hitting time by X of C is a.s. greater than or equal to the lifetime of M , that is, τ_S . Hence, because the trajectories of X are continuous and S is closed, an M -polar set must be a subset of S , and it follows that $\int_0^t f(X_s) dA_s$ is \mathbf{P}_x -a.s. finite for all $t < \tau_S$ and $x \in \mathcal{I} \setminus S$. Finally, observe that we can with no loss of generality assume that $f = 0$ over S , as replacing f by $f1_{\mathcal{I} \setminus S}$ does not alter the right-hand side of (A.2).

Step 3 Using the classical representation result for additive functionals of X (Borodin and Salminen (2002, Part I, Chapter I, Section 4, §23)), there exists a Radon measure ν over $\mathcal{I} \setminus S$ such that $A_t = \int_{\mathcal{I} \setminus S} L_t^y \nu(dy)$ a.s. Therefore, for each $t < \tau_S$,

$$\tilde{A}_t \equiv \int_0^t f(X_s) dA_s = \int_0^t \int_{\mathcal{I} \setminus S} f(X_s) dL_s^y \nu(dy) = \int_{\mathcal{I} \setminus S} L_t^y f(y) \nu(dy) \text{ a.s.}$$

We claim that $\mu \equiv f \cdot \nu$ is a Radon measure, which concludes the first part of the proof. To this end, we only need to prove that μ is locally finite. Indeed, if it were not so, then there would exist $x \in \mathcal{I} \setminus S$ such that $\int_{[x-\varepsilon, x+\varepsilon]} f(y) \nu(dy) = \infty$ for all $\varepsilon > 0$ such that $[x - \varepsilon, x + \varepsilon] \subset \mathcal{I} \setminus S$. For each $t > 0$, $L_t^x(\omega) > 0$ for all ω in a set of \mathbf{P}_x -probability 1. Therefore, as the local time of X is a.s. jointly continuous (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), we have that, for any such ω , there exists $\varepsilon(\omega) > 0$ such that

$[x - \varepsilon(\omega), x + \varepsilon(\omega)] \subset \mathcal{I} \setminus S$ and $L_t^y(\omega) > 0$ for all $y \in [x - \varepsilon(\omega), x + \varepsilon(\omega)]$. This implies that, if $0 < t < \tau_S(\omega)$, then

$$\tilde{A}_t(\omega) = \int_{\mathcal{I} \setminus S} L_t^y(\omega) f(y) \nu(dy) \geq \min_{y \in [x - \varepsilon(\omega), x + \varepsilon(\omega)]} L_t^y(\omega) \int_{[x - \varepsilon(\omega), x + \varepsilon(\omega)]} f(y) \nu(dx),$$

which is infinite by assumption. Because $\mathbf{P}_x[\tau_S > 0] = 1$ as $x \in \mathcal{I} \setminus S$, this contradicts the fact that, for each $t < \tau_S$, $\tilde{A}_t = \int_0^t f(X_s) dA_s$ is \mathbf{P}_x -a.s. finite. The claim follows.

(Sufficiency) Reciprocally, if S is a closed subset of \mathcal{I} and μ is a Radon measure over $\mathcal{I} \setminus S$, then the process defined by $\Lambda_t = 1_{\{t < \tau_S\}} e^{-\int_{\mathcal{I} \setminus S} L_t^y \mu(dy)}$ is well-defined and, as the local time of X is a strong additive functional of X (Revuz and Yor (1999, Chapter X, §1, Proposition 1.2)), is a right-continuous multiplicative functional that satisfies (14). In particular, $\Gamma \equiv 1 - \Lambda$ satisfies the assumptions of Lemma 3 and thus is the cdf of a randomized stopping time. Hence the result. \blacksquare

PROOF OF THEOREM 2: Let $s \equiv s^1 \vee s^2$ and $E^i \equiv \text{supp } \mu^i \cap (s, \beta)$ for $i = 1, 2$. E^i is a relatively closed subset of (s, β) that can be written as a disjoint union $E^i = A^i \cup K^i$, where A^i is the set of accumulation points of E^i in (s, β) , which is relatively closed in (s, β) , and K^i is the (countable) set of isolated points of E^i . Observe that $E^i \subset (s, x_{R^i}]$ by Proposition 1(iv) as $E^i \cap S^j = \emptyset$. If $E^1 = E^2 = \emptyset$, there is nothing to prove and the MPE under consideration is outcome-equivalent to a pure-strategy MPE. Let us otherwise denote by \bar{J}^i player i 's equilibrium value function. The proof then consists of four steps and repeatedly uses assertions (i)–(iii) of Lemma A.1 below.

Lemma A.1 *Let u be a \mathcal{C}^2 function defined over an open interval $(a, b) \subset \mathcal{I}$ and such that $\mathcal{L}u - ru = 0$ over (a, b) . Then, the following holds:*

- (i) *if $b = \beta$, $u(\beta-) = 0$, $u(a+) = R^i(a)$, and $u \geq V_{R^i}$ over (a, β) , then $a = x_{R^i}$;*
- (ii) *if $u \geq V_{R^i}$ over (a, b) , then $\{x \in (a, b) : u(x) = R^i(x)\}$ contains at most one point;*
- (iii) *if $b \leq x_{R^i}$, $u(b-) = R^i(b)$, and either $a > \alpha$ and $u(a+) = R^i(a)$ or $a = \alpha$ and $u(a+) = 0$, then $u < R^i$ over (a, b) ;*
- (iv) *if $\alpha < a \leq x_{R^i}$, $u \geq R^i$ over (a, b) , $u(a) = R^i(a)$, and $u'(a+) > R^{i'}(a)$, then, for every sufficiently small $\varepsilon > 0$, the function f_ε solution to $\mathcal{L}f - rf = 0$ over $(a - \varepsilon, a + \varepsilon)$ with $f_\varepsilon(a - \varepsilon) = R^i(a - \varepsilon)$ and $f_\varepsilon(a + \varepsilon) = u(a + \varepsilon)$ satisfies $f_\varepsilon(a) > u(a)$.*

Step 1 We first claim that every connected component (a, b) of $(s, \beta) \setminus E^i$ such that (a)

$a > s$ or $a = s = s^i$ or $a = s = \alpha$, and (b) $b \leq x_{R^i}$, contains exactly one point of E^j .

Suppose first, by way of contradiction, that $E^j \cap (a, b) = \emptyset$. By Proposition 1(v), the strategy $(0, S^i)$ is a pbr to the strategy (μ^j, S^j) . Therefore, τ_{S^i} is a solution to the optimal-stopping problem $\bar{J}^i(x) = \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j))$. Letting τ be the first exit time of X from (a, b) , we have $\tau_{S^i} \geq \tau$ \mathbf{P}_x -a.s. for all $x \in (a, b)$. We deduce from this that the brvf \bar{J}^i satisfies, for each $x \in (a, b)$,

$$\bar{J}^i(x) = J^i(x, (0, S^i), (\mu^j, S^j)) = \mathbf{E}_x[e^{-r\tau} \bar{J}^i(X_\tau)], \quad (\text{A.3})$$

where the last inequality follows from the strong Markov property (S.7). As $E^j \cap (a, b) = \emptyset$, it then follows from standard arguments that \bar{J}^i is \mathcal{C}^2 and $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (a, b) . Now, consider the conditions in the claim. First, if $a > s$, then $a \in \text{supp } \mu^i$ by definition of a connected component of $(s, \beta) \setminus E^i$, and thus $\bar{J}^i(a) = R^i(a)$ by Proposition 1(iii); the same reasoning shows that $\bar{J}^i(b) = R^i(b)$. Next, if $a = s = s^i$, then $\bar{J}^i(a) = R^i(a)$ by Proposition 1(iii). Finally, if $a = s = \alpha$, then τ coincides with the hitting time of b , and thus (A.3) and (4)–(5) together imply, letting x go to $\alpha+$, that $\bar{J}^i(a+) = 0$. Thanks to $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (a, b) and $b \leq x_{R^i}$, we are thus in a position to apply Lemma A.1(iii); we obtain $\bar{J}^i < R^i$ over (a, b) , a contradiction as $\bar{J}^i \geq V_{R^i}$ over \mathcal{I} . Therefore, $E^j \cap (a, b) \neq \emptyset$. Finally, using again standard arguments, it must be that $\mathcal{L}\bar{J}^j - r\bar{J}^j = 0$ over (a, b) . Because $\bar{J}^j \geq V_{R^j}$, Lemma A.1(ii) implies that $E^j \cap (a, b)$ contains exactly one point. The claim follows. It should be noted that the same arguments show that every interval $(a, b) \subset (s, \beta)$ such that (a) $a > s$ or $a = s = s^i$ or $a = s = \alpha$, (b) $b \leq x_{R^i}$, and (c) $\bar{J}^i(a+) = R^i(a+)$ and $\bar{J}^i(b) = R^i(b)$ contains at least one point of E^j .

Step 2 We next claim that $A^1 = A^2$.

Let $x \in A^i$. Suppose first, by way of contradiction, that $x \notin E^j$. Then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap E^j = \emptyset$, where ε can be chosen sufficiently small so that $x - \varepsilon > s$. As x is an accumulation point of E^i and E^i is relatively closed in (s, β) , one of the two following conditions must hold:

- (i) $(x - \varepsilon, x + \varepsilon)$ includes a connected component (a, b) of $(s, \beta) \setminus E^i$ such that $a > s$ and $b \leq x_{R^i}$;
- (ii) E^i includes a nondegenerate interval $\mathcal{I}_0 \subset (x - \varepsilon, x + \varepsilon)$ that contains x .

In case (i), the connected component (a, b) must contain one point of E^j by Step 1, a contradiction. In case (ii), notice that $\mathcal{I}_0 \cap S^j = \emptyset$ by definition of s , E^i , and E^j . Thus, by Proposition 1(ii), it must be that $\bar{J}^i = R^i$ over \mathcal{I}_0 . On the other hand, because $(0, S^i)$

is also a pbr to (μ^j, S^j) and $E^j \cap \mathcal{I}_0 = \emptyset$, we obtain as in Step 1 that \bar{J}^i must be \mathcal{C}^2 and satisfy $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over the interior of \mathcal{I}_0 . But then $\mathcal{L}R^i - rR^i = 0$ over a nondegenerate interval, a contradiction by A3. We conclude that $x \in E^j$ and in turn that $A^i \subset E^j$. Let us now prove that $A^i \subset A^j$. If $x \in A^i$ belongs to the relative closure of $A^i \setminus \{x\}$ in (s, β) , then $x \in A^j$ as $A^i \setminus \{x\} \subset E^j$. If not, then x must be the limit of a sequence of points $(x_n)_{n \geq 1}$ in K^i , which we can assume to be strictly monotone. By Step 1, for every sufficiently large n , the interval formed by two consecutive elements x_n and x_{n+1} of this sequence contains exactly one point y_n of E^j , and thus $x = \lim_{n \rightarrow \infty} y_n \in A^j$ as it is an accumulation point of E^j . We conclude that $A^i \subset A^j$ and in turn that $A^i = A^j$ by exchanging the role of the players. The claim follows.

Step 3 We then claim that, if $A^1 = A^2 = \emptyset$ and $K^1 \cup K^2 \neq \emptyset$, then the measures μ^1 and μ^2 are discrete or degenerate, with at least a nondegenerate one, and their supports are described by one of the cases in Corollary 1.

By assumption, μ^1 and μ^2 are discrete measures and their supports have no accumulation points in $(s, x_{R^i}]$. Therefore, either their supports are finite, or they are infinite, with s as a unique accumulation point. In both cases, for each $i = 1, 2$, $E^i = K^i \equiv \{q_n^i : 1 \leq n \leq N^i\}$ for some decreasing sequence $(q_n^i)_{n=1}^{N^i}$ in $(s, x_{R^i}]$, with N^i finite or infinite, and possibly equal to 0 for some i , in which case μ^i is degenerate. We now establish three key properties of the sequences $(q_n^i)_{n=1}^{N^i}$, $i = 1, 2$, which together imply the claim.

First, it must be that $q_1^i = x_{R^i}$ for some i . Indeed, suppose that $K^i \neq \emptyset$ and $\max E^j \leq q_1^i$, where $\max \emptyset = \alpha$. We first have $\bar{J}^i(q_1^i) = R^i(q_1^i)$ by Proposition 1(ii)–(iii) as $q_1^i > s \geq s^j$. Next, because $E^j \cap (q_1^i, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that \bar{J}^i is \mathcal{C}^2 and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (q_1^i, β) . As a result, $\bar{J}^i = A\phi + B\psi$ over this interval for some constants A and B . From this, it follows in turn that $\bar{J}^i(\beta^-) = 0$. Indeed, by Lemma 1 and Proposition 1, we have $0 \leq \bar{J}^i \leq G^i$, which, together with (8), implies $B = 0$. That $\bar{J}^i(\beta^-) = 0$ follows then from (4). Finally, $J^i \geq V_{R^i}$ by Proposition 1. Thus \bar{J}^i satisfies all the conditions of Lemma A.1(i), from which we conclude that $q_1^i = x_{R^i}$.

Next, it must be that the sequences $(q_n^i)_{n=1}^{N^i}$, $i = 1, 2$, are intertwined. Indeed, Step 1 implies that, if at least one of these sequence has at least two elements, then, between two consecutive elements of each sequence, there must be exactly one element of the other sequence. Similarly, if $1 \leq N^i < \infty$ and $s = s^i$ or $s = \alpha$, then $s < q_{N^i}^i$ and there must be one element of K^j in $(s, q_{N^i}^i)$. These properties have two main implications. (a) First, the sequences $(q_n^i)_{n=1}^{N^i}$, $i = 1, 2$, have no common element. Indeed, suppose, by way of contradiction, that $q^1 = q^2 = q$ for two components of these two sequences. We distinguish

two cases. If at least one of the sets K^1 and K^2 is not a singleton, then, because K^1 and K^2 have s as their only possible accumulation point, there exists some $i = 1, 2$ for which the distance $\inf_{q' \in K^i \setminus \{q\}} |q' - q| > 0$ is minimized, with $\inf_{q' \in \emptyset} |q' - q| \equiv \infty$ for all $q \in \mathcal{I}$ by convention. Let this minimal distance be reached at q' . But then, as argued above, there must exist $q'' \in K^j$ in between q and q' , so that $|q'' - q| < |q' - q|$, in contradiction with the definition of q' . If both K^1 and K^2 are singletons, then it must be that $K^1 = K^2 = \{x_{R^1}\} = \{x_{R^2}\}$ by the first property above. Applying Step 1 to the connected component (s^i, x_{R^i}) of $(s, \beta) \setminus E^i$ for a player i such that $s = s^i$, we obtain that (s^i, x_{R^i}) contains exactly one point of E^j , a contradiction. (b) Second, and as a result, if $\max S^j \cup E^j < q_1^i$, then either N^i is finite and $N^j \in \{N^i - 1, N^i\}$, or $N^i = N^j = \infty$.

Finally, if $N^1 = N^2 = \infty$, the sequences $(q_n^i)_{n \geq 1}$, $i = 1, 2$, must converge to α , so that $s = \alpha$ and $S^1 = S^2 = \emptyset$. This is a consequence of the following general lemma.

Lemma A.2 *Let $((\mu^1, S^1), (\mu^2, S^2))$ be a mixed-strategy MPE for which there exists two intertwined decreasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ in $\text{supp } \mu^1 \cap (s, \beta)$ and $\text{supp } \mu^2 \cap (s, \beta)$, respectively, such that, for each $i = 1, 2$, $\text{supp } \mu^i \cap (\inf_{n \geq 1} \chi_n^i, \chi_1^i) = \{\chi_n^i : n \geq 1\}$. Then these two sequences converge to α . Similarly, there are no intertwined increasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ in $\text{supp } \mu^1 \cap (s, \beta)$ and $\text{supp } \mu^2 \cap (s, \beta)$, respectively, such that, for each $i = 1, 2$, $\text{supp } \mu^i \cap [\chi_0^i, \sup_{n \geq 1} \chi_n^i) = \{\chi_n^i : n \geq 1\}$.*

The claim follows.

Step 4 We finally claim that, if $A \equiv A^1 = A^2 \neq \emptyset$, then $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, $s = \alpha^1 \vee \alpha^2$, and, for each $i = 1, 2$, the restriction of μ^i to $(s, x_R]$ is absolutely continuous with density $\sigma^{-2} \lambda^i$, where λ^i is given by (18) with s instead of α^* .

We first show that $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, and $s = \alpha^1 \vee \alpha^2$. The argument is fourfold.

We first claim that $A \subset (s, x_{R^1} \wedge x_{R^2}]$ is an interval. Indeed, suppose, by way of contradiction, that this is not so. Then there exists an interval $(a, b) \subset (s, \beta) \setminus A$ such that $a > s$ and $a, b \in A$. Because (a, b) cannot be a connected component of both $(s, \beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (a, b) \neq \emptyset$ for some i . Fix some $\chi_1^i \in K^i \cap (a, b)$. Then $\bar{J}^i(a) = R^i(a)$ and $\bar{J}^i(\chi_1^i) = R^i(\chi_1^i)$ by Proposition 1(ii)–(iii) as $a > s \geq s^j$, so that $K^j \cap (a, \chi_1^i) \neq \emptyset$ by the final remark of Step 1. Because $\chi_1^i \in (a, b)$ is not an accumulation point of E^j , we have $\chi_1^i > \chi_1^j \equiv \sup K^j \cap (a, \chi_1^i) \in K^j$. Applying this argument recursively, we obtain two infinite intertwined decreasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ in $K^1 \cap (a, b)$ and $K^2 \cap (a, b)$, respectively. Because these sequences are bounded below by $a > s$ and (a, b)

is a connected component of $(s, \beta) \setminus A$, they both converge to a . Moreover, arguing as in Step 3, it is easy to check that $\text{supp } \mu^1 \cap (\inf_{n \geq 1} \chi_n^1, \chi_1^1] = \{\chi_n^1 : n \geq 1\}$, and similarly for player 2. Thus, by Lemma A.2, it must be the case that $a = \alpha$, a contradiction as $a > s$. The claim follows. As A is relatively closed in (s, β) , $\sup A = \max A \in A$.

We next claim that $\max A = x_{R^1} = x_{R^2}$. Indeed, suppose first, by way of contradiction, that $\max A < x_{R^1} \wedge x_{R^2}$. Arguing as in Step 3, we obtain that $x_{R^i} \in K^i$ for some $i = 1, 2$. Hence $\bar{J}^i(\max A) = R^i(\max A)$ and $\bar{J}^i(x_{R^i}) = R^i(x_{R^i})$ by Proposition 1(ii)–(iii) as $\max A > s \geq s^j$, so that $K^j \cap (\max A, x_{R^1} \wedge x_{R^2}] \neq \emptyset$ by the final remark of Step 1 along with the fact that $K^j \subset (s, x_{R^j}]$. We can then repeat the above argument, leading again to a contradiction. We conclude that $\max A = x_{R^1} \wedge x_{R^2} = x_{R^j}$ for some $j = 1, 2$, so that $\max E^i \geq x_{R^j}$. Now, $\bar{J}^i(x_{R^j}) = R^i(x_{R^j})$ by Proposition 1(ii)–(iii) as $x_{R^j} > s \geq s^j$. Because $E^j \cap (x_{R^j}, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that \bar{J}^i is \mathcal{C}^2 and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (x_{R^j}, β) . Finally, we can use similar arguments as in Step 3 to show that $\bar{J}^i(\beta-) = 0$. As $\bar{J}^i \geq V_{R^i}$ by Proposition 1, \bar{J}^i satisfies all the conditions of Lemma A.1(i), from which we conclude that $x_{R^j} = x_{R^i} \equiv x_R$. The claim follows.

We then claim that $\inf A = s$. Indeed, suppose, by way of contradiction, that $\inf A > s$. Because $(s, \inf A)$ cannot be a connected component of both $(s, \beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (s, \inf A) \neq \emptyset$ for some i . Fixing some $\chi_1^i \in K^i \cap (s, \inf A)$, we can then mirror the above argument to obtain two infinite intertwined *increasing* sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ in K^1 and K^2 , respectively, converging to $\inf A$, and such that for $i = 1, 2$, $\text{supp } \mu^i \cap [\chi_1^i, \sup_{n \geq 1} \chi_n^i) = \{\chi_n^i : n \geq 1\}$, a contradiction by Lemma A.2. We conclude that $\inf A = s$ and thus that $A = (s, x_R]$. The claim follows.

We finally claim that $s = \alpha^1 \vee \alpha^2$. Notice first that $s \geq \alpha^1 \vee \alpha^2$ by Lemma S.4(ii) in the Online Supplement. Now, suppose, by way of contradiction that $s > \alpha^1 \vee \alpha^2$ and $s \in S^i$. Then, by Proposition 1(i), $s \notin S^j$, so that $\bar{J}^j(s) = G^j(s)$. But $\bar{J}^j(s+) = R^j(s) < G^j(s)$ as $(s, x_{R^j}] \subset \text{supp } \mu^j$ and $s > \alpha^j$, a contradiction as \bar{J}^j is continuous by Proposition 2. The claim follows.

We have thus shown that, if $A \neq \emptyset$, then $A = (s, x_R]$, with $s = \alpha^1 \vee \alpha^2$ and $x_R = x_{R^1} = x_{R^2}$. By Proposition 1(iii), it follows that, for each $i = 1, 2$, $\bar{J}^i = R^i$ over $(s, x_R]$. Therefore, by Lemma 3, Proposition 1(v), and the strong Markov property, we have, letting $\Gamma_t^j \equiv 1 - e^{-\int_{(s, x_R]} L_t^j \mu^j(dy)}$,

$$\bar{J}^i(x) = f^i(x, \mu^j) \equiv \mathbf{E}_x \left[\int_{[0, \tau_s)} e^{-rt} G^i(X_t) d\Gamma_t^j + e^{-r\tau_s} R^i(s) (1 - \Gamma_{\tau_s}^j) \right] = R^i(x) \quad (\text{A.4})$$

for all $x \in (s, x_R]$. Notice that the right-hand side of (A.4) does not depend on μ^j , so

that neither does $f^i(x, \mu^j)$ in equilibrium for all $x \in (s, x_R]$. Defining then the measure $\bar{\mu}^j \equiv \sigma^{-2} \lambda^j \cdot \text{Leb}$ over (α^i, β) , where $\lambda^j(x) \equiv \frac{rR^i(x) - \mathcal{L}R^i(x)}{G^i(x) - R^i(x)} 1_{\{\alpha^i < x \leq x_R\}}$, it can be verified as in Steg (2015) and Georgiadis, Kim, and Kwon (2022) that, as in Proposition 4, the pair $((\alpha, \alpha^2], \bar{\mu}^1), ((\alpha, \alpha^1], \bar{\mu}^2)$ is an MPE with equilibrium value functions (V_{R^1}, V_{R^2}) . In particular, Proposition 1 implies that player j 's strategy makes player i indifferent between holding fast and conceding over $(\alpha^i, x_{R^i}]$, which, together with the Markov property, implies that $\bar{\mu}^j$ is solution to (A.4).

To conclude, we show that $\mu^j = \bar{\mu}^j|_{\mathcal{B}((s, \beta))}$, that is, (A.4) has a unique solution over the Borel σ -field $\mathcal{B}((s, \beta))$. The strong Markov property implies that, for each $x \in (s, x_R]$ and for every stopping time $\tau < \tau_s$,

$$\begin{aligned} f^i(x, \mu^j) &= \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^i(X_t) d\Gamma_t^j + e^{-r\tau} f^i(X_\tau, \mu^j) \right] \\ &= \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^i(X_t) d\Gamma_t^j + e^{-r\tau} R^i(X_\tau) \right]. \end{aligned}$$

Because $f^i(x, \bar{\mu}^j) = f^i(x, \mu^j) = R^i(x)$ and similarly for $f^i(x, \bar{\mu}^j)$, it follows that

$$\mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^i(X_t) d(\Gamma_t^j - \bar{\Gamma}_t^j) \right] = 0,$$

where $\bar{\Gamma}_t^j \equiv 1 - e^{-\int_{(s, x_R]} L_u^y \bar{\mu}^j(dy)}$. Because this equality holds for any stopping time $\tau < \tau_s$, the process $u \mapsto M_u \equiv \int_{[0, u]} e^{-rt} G^i(X_t) d(\Gamma_t^j - \bar{\Gamma}_t^j)$ is a martingale over $[0, \tau_s)$ (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Therefore, being a continuous process of bounded variation, it is indistinguishable from 0 over $[0, \tau_s)$. As $G^i > V_{R^i} > 0$ by Lemma 1, it follows that the process $u \mapsto \Gamma_u^j - \bar{\Gamma}_u^j = \int_{[0, u]} \frac{e^{rt}}{G^i(X_t)} dM_t$ is indistinguishable from 0 over $[0, \tau_s)$, so that the processes $u \mapsto \int_{(s, x_{R^i}]} L_u^y \mu^j(dy)$ and $u \mapsto \int_{(s, x_{R^i}]} L_u^y \bar{\mu}^j(dy)$ are indistinguishable from each other over $[0, \tau_s)$. In turn, these two processes can be seen as additive functionals of the diffusion X over (s, β) , where s is modified into a killing boundary. This implies that $\mu^j = \bar{\mu}^j$, because both the measure associated to an additive functional of a diffusion and the killing measure of a diffusion are unique (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §4, and Section 4, §23)). Hence the result. \blacksquare

PROOF OF THEOREM 3: (Necessity) Let $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2]))$ be an MPE of type 2, and consider the brvf \bar{J}^2 to (μ^1, S^1) . Our goal is to show that \bar{J}^2 satisfies the variational system (44)–(50).

We start with some simple observations. First, $\bar{J}^2 \in \mathcal{C}^0(\mathcal{I})$ by Proposition 2, as requested. Second, we know from Proposition 1 that $\bar{J}^2 \geq V_{R^2}$ over \mathcal{I} and from (3) that $V_{R^2} \geq R^2$ over \mathcal{I} . Hence \bar{J}^2 satisfies (44). Third, $\bar{J}^2 = R^2$ over $S^2 = (\alpha, s^2]$ by Proposition 1(ii). Hence

\bar{J}^2 satisfies (46). Fourth, $\text{supp } \mu^2 = \{q_n^2 : 1 \leq n \leq N-1\} \subset \{x \in \mathcal{I} : \bar{J}^2(x) = R^2(x)\}$ by Proposition 1(iii). Hence \bar{J}^2 satisfies (47). Fifth, as in Steps 1 and 3 of the proof of Theorem 2, it can be verified that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q_1^1, β) and (s^2, q_N^1) , and that $\bar{J}^2 = T_{q_n^2}^2$ over (q_{n+1}^1, q_n^1) for $1 \leq n \leq N-1$, where T_q^2 is the solution to $\mathcal{L}u - ru = 0$ that is tangent to R^2 at q . Hence \bar{J}^2 satisfies (45). Sixth, as in Step 3 of the proof of Theorem 2, the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q_1^1, β) implies that $\bar{J}^2 = A\phi + B\psi$ over this interval for some constants A, B , and the fact that $0 \leq \bar{J}^2 \leq G^2$ together with (8) implies that $B = 0$ and thus $\bar{J}^2(\beta-) = 0$. Hence \bar{J}^2 satisfies (50). Seventh, and as a result, $\bar{J}^2 \in \mathcal{C}^2(\mathcal{I} \setminus (\{q_n^1 : 1 \leq n \leq N\} \cup \{s_2\}))$ and $|\bar{J}^{2'-}(x)| \vee |\bar{J}^{2'+}(x)| < \infty$ for all $x \in \{q_n^1 : 1 \leq n \leq N\} \cup \{s_2\}$, as requested.

Let us now check that \bar{J}^2 satisfies (48). Because $\bar{J}^2 \geq R^2$, with equality at s^2 , it must be that $\bar{J}^{2'+}(s^2) \geq R^{2'+}(s^2)$. Suppose, by way of contradiction, that this inequality is strict. Consider the stopping time $\tau_\varepsilon \equiv \inf\{t \geq 0 : X_t \notin (s^2 - \varepsilon, s^2 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $\alpha < s^2 - \varepsilon < s^2 + \varepsilon < q_N^1$. Define $f_\varepsilon(x) \equiv \mathbf{E}_x[e^{-r\tau_\varepsilon} \bar{J}^2(X_{\tau_\varepsilon})]$ for $x \in (s^2 - \varepsilon, s^2 + \varepsilon)$. Recalling that τ_{S^2} is a best reply to (μ^1, S^1) by Proposition 1(v) and invoking the strong Markov property, we obtain that $f_\varepsilon(x)$ is the payoff of player 2 against (μ^1, S^1) when using the non-Markovian stopping time $\tau_\varepsilon + \tau_{S^2} \circ \theta_{\tau_\varepsilon}$ that consists in holding fast up to τ_ε and then conceding the first time X hits S^2 in the continuation game. By construction, $\mathcal{L}f_\varepsilon - rf_\varepsilon = 0$ over $(s^2 - \varepsilon, s^2 + \varepsilon)$. Applying Lemma A.1(iv) with $i = 2$, $a = s^2$, $b = q_N^1$, and $u = \bar{J}^2$, we deduce that $f_\varepsilon(s^2) > \bar{J}^2(s^2)$ for ε sufficiently small, a contradiction as (μ^2, S^2) is a pbr to (μ^1, S^1) . Hence \bar{J}^2 satisfies (48).

Let us finally check that \bar{J}^2 satisfies (49). The following lemma provides two expressions for \bar{J}^2 that result from the Markov property and the Itô–Tanaka–Meyer formula.

Lemma A.3 *Let $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2)))$ be an MPE of type 2. Then, for all $x \in \mathcal{I}$ and $\tau \in \mathcal{T}$,*

$$\begin{aligned} \bar{J}^2(x) = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} e^{-rt} G^2(q_n^1) \Lambda_t^1 a_n dL_t^{q_n^1} \right. \\ \left. + 1_{\{\tau_{S^2} < \tau\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 + 1_{\{\tau_{S^2} \geq \tau\}} e^{-r\tau} \bar{J}^2(X_\tau) \Lambda_\tau^1 \right] \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \bar{J}^2(x) = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right. \\ \left. + 1_{\{\tau_{S^2} < \tau\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 + 1_{\{\tau_{S^2} \geq \tau\}} e^{-r\tau} \bar{J}^2(X_\tau) \Lambda_\tau^1 \right]. \end{aligned} \quad (\text{A.6})$$

An immediate implication of (A.5)–(A.6) is that, for each $\tau \in \mathcal{T}$,

$$\begin{aligned} \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0,\tau]} 1_{\{\tau_{S^2} > t\}} e^{-rt} G^2(q_n^1) \Lambda_t^1 a_n dL_t^{q_n^1} \right] \\ = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0,\tau]} 1_{\{\tau_{S^2} > t\}} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right]. \end{aligned}$$

Equivalently, for each $\tau \in \mathcal{T}$, $\mathbf{E}_x[M_\tau] = \mathbf{E}_x[M_0] = 0$, where

$$M_t \equiv \sum_{n=1}^N \int_{[0,t]} 1_{\{\tau_{S^2} > s\}} e^{-rs} \left\{ a_n [G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right\} \Lambda_s^1 dL_s^{q_n^1} \quad (\text{A.7})$$

for all $t \geq 0$. It follows that the process $(M_t)_{t \geq 0}$ is a martingale (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Because it is a continuous process of bounded variation, it must then be that, for each $\tau \in \mathcal{T}$, $M_\tau = M_0 = 0$ \mathbf{P}_x -a.s. Now, suppose, by way of contradiction, that $a_n [G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \neq 0$ for some n such that $1 \leq n \leq N$. Let $x \equiv q_n^1$ and $\tau_\varepsilon \equiv \inf \{t \geq 0 : X_t \notin (q_n^1 - \varepsilon, q_n^1 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $q_{n+1}^1 < q_n^1 - \varepsilon < q_n^1 + \varepsilon < q_{n-1}^1$, with $q_0^1 \equiv \beta$ and $q_{N+1}^1 \equiv s^2$ by convention. From the properties of local time, we have that, for each $t > 0$, $L_t^{q_n^1} > 0$ $\mathbf{P}_{q_n^1}$ -a.s. (see, for instance, Revuz and Yor (1999, Chapter VI, §2, Proof of Proposition 2.5)). It then follows from (A.7) that $M_{\tau_\varepsilon} \neq 0$ $\mathbf{P}_{q_n^1}$ -a.s., a contradiction. Hence \bar{J}^2 satisfies (49). This completes the proof that \bar{J}^2 satisfies the variational system (44)–(50). The proof that \bar{J}^1 satisfies the variational system (38)–(43) is similar, and is omitted for the sake of brevity.

(Sufficiency) That the variational system (38)–(50) characterizes players' value functions in MPEs of type 2 is an immediate consequence of the following verification lemma.

Lemma A.4 *Let w^1 and w^2 be solutions to the systems (38)–(43) and (44)–(50), respectively, for some $N \in \mathbb{N} \setminus \{0\}$, four sequences $(q_n^1)_{n=1}^N$, $(q_n^2)_{n=0}^{N-1}$, $(a_n)_{n=1}^N$, $(b_n)_{n=0}^{N-1}$, and a number s^2 as in the statement of Theorem 3. Then, for each $i = 1, 2$,*

$$w^i(x) \geq \sup_{\tau \in \mathcal{T}} J^i(x, \tau, (\mu^j, S^j)), \quad (\text{A.8})$$

$$w^i(x) = J^i(x, (\mu^1, S^1), (\mu^2, S^2)), \quad (\text{A.9})$$

where $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2)))$.

(Refinement) On the one hand, we have $\alpha^1 \leq \alpha^2 \leq s^2$, where the second inequality follows from Lemma S.4(ii) in the Online Supplement, and thus it can be easily checked that, for each $x \in \mathcal{I}$, $J^1(x, (\mu^1, (\alpha, \alpha^1)), (\mu^2, S^2)) = J^1(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^1(x)$, which implies that $(\mu^1, (\alpha, \alpha^1))$ is a pbr to (μ^2, S^2) . On the other hand, using (11) along with the

fact that $G^2 = R^2$ over $(\alpha, \alpha^1]$ as $\alpha^1 \leq \alpha^2$, it is easily checked that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$, $J^2(x, (\mu^1, (\alpha, \alpha^1]), \tau^2) = J^2(x, (\mu^1, \emptyset), \tau^2 \wedge \tau_{(\alpha, \alpha^1]}) \leq \bar{J}^2(x)$ and $J^2(x, (\mu^1, (\alpha, \alpha^1]), (\mu^2, S^2)) = J^2(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^2(x)$, which implies that (μ^2, S^2) is a pbr to $(\mu^1, (\alpha, \alpha^1])$. Hence the result. ■

References

- ABREU, DILIP, AND FARUK GUL (2000): “Bargaining and Reputation,” *Econometrica*, 68 (1), 85–117.
- ALESINA, ALBERTO, AND ALLAN DRAZEN (1991): “Why are Stabilizations Delayed?” *American Economic Review*, 81 (5), 1170–1188.
- AUMANN, ROBERT J. (1964): “Mixed and Behavior Strategies in Infinite Extensive Games,” in *Advances in Game Theory*, Annals of Mathematics Study, Vol. 52, ed. by M. Dresher, L.S. Shapley, and A.W. Tucker. Princeton: Princeton University Press, 627–650.
- BIAIS, BRUNO, THOMAS MARIOTTI, GUILLAUME PLANTIN, AND JEAN-CHARLES ROCHET (2007): “Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications,” *Review of Economic Studies*, 74 (2), 345–390.
- BLISS, CHRISTOPHER, AND BARRY NALEBUFF (1984): “Dragon-Slaying and Ballroom Dancing: The Private Supply of a Public Good,” *Journal of Public Economics*, 25 (1–2), 1–12.
- BLUMENTHAL, ROBERT M., AND RONALD K. GETOOR (1968): *Markov Processes and Potential Theory*. New York: Academic Press.
- BOLTON, PATRICK, HUI CHEN, AND NENG WANG (2011): “A Unified Theory of Tobin’s q , Corporate Investment, Financing, and Risk Management,” *Journal of Finance*, 66 (5), 1545–1578.
- BORODIN, ANDREI N., AND PAAVO SALMINEN (2002): *Handbook of Brownian Motion—Facts and Formulae*. Basel, Boston, Berlin: Birkhäuser Verlag.
- BULOW, JEREMY, AND PAUL KLEMPERER (1999): “The Generalized War of Attrition,” *American Economic Review*, 89 (1), 175–189.
- DALEY, BRENDAN, AND BRETT GREEN (2012): “Waiting for News in the Market for Lemons,” *Econometrica*, 80 (4), 1433–1504.

- DAYANIK, SAVAS, AND IOANNIS KARATZAS (2003): “On the Optimal Stopping Problem for One-Dimensional Diffusions,” *Stochastic Processes and their Applications*, 107 (2), 173–212.
- DE ANGELIS, TIZIANO, ERIK EKSTRÖM, AND KRISTOFFER GLOVER (2022): “Dynkin Games with Incomplete and Asymmetric Information,” *Mathematics of Operations Research*, 47 (1), 560–586.
- DÉCAMPS, JEAN-PAUL, FABIEN GENSBITTEL, AND THOMAS MARIOTTI (2024): “Investment Timing and Technological Breakthroughs,” *Mathematics of Operations Research*, forthcoming.
- DÉCAMPS, JEAN-PAUL, AND THOMAS MARIOTTI (2004): “Investment Timing and Learning Externalities,” *Journal of Economic Theory*, 118 (1), 80–102.
- DÉCAMPS, JEAN-PAUL, THOMAS MARIOTTI, JEAN-CHARLES ROCHET, AND STÉPHANE VILLENEUVE (2011): “Free Cash Flow, Issuance Costs, and Stock Prices,” *Journal of Finance*, 66 (5), 1501–1544.
- DEMARZO, PETER M., AND YULIY SANNIKOV (2006): “Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model,” *Journal of Finance*, 61 (6), 2681–2724.
- DIXIT, AVINASH K., AND ROBERT S. PINDYCK (1994): *Investment under Uncertainty*. Princeton: Princeton University Press.
- DRAZEN, ALLAN, AND VITTORIO GRILLI (1993): “The Benefits of Crises for Economic Reform,” *American Economic Review*, 83 (3), 598–607.
- EDWARDS, ROBERT D., JOHN MAGEE, AND W.H.C. BASSETTI (2013): *Technical Analysis of Stock Trends*. Boca Raton: CRC Press.
- EGOROV, GEORGY, AND BÅRD HARSTAD (2017): “Private Politics and Public Regulation,” *Review of Economic Studies*, 84 (4), 1652–1682.
- FEARON, JAMES D. (1995): “Rationalist Explanations for War,” *International Organization*, 49 (3), 379–414.
- FELLER, WILLIAM (1968): *An Introduction to Probability Theory and Its Applications*. New York, London, Sidney: John Wiley & Sons, Inc.

- FINE, CHARLES H., AND LODE LI (1989): “Equilibrium Exit in Stochastically Declining Industries,” *Games and Economic Behavior*, 1 (1), 40–59.
- FUDENBERG, DREW, AND JEAN TIROLE (1986): “A Theory of Exit in Duopoly,” *Econometrica*, 54 (4), 943–960.
- GEORGIADIS, GEORGE, YOUNGSOO KIM, AND H. DHARMA KWON (2022): “The Absence of Attrition in a War of Attrition under Complete Information,” *Games and Economic Behavior*, 131, 171–185.
- GERAGHTY, THOMAS M., AND THOMAS WISEMAN (2008): “Wage Strikes in 1880s America: A Test of the War of Attrition Model,” *Explorations in Economic History*, 45 (4), 303–326.
- GHEMAWAT, PANKAJ (1997): *Games Businesses Play: Cases and Models*. Cambridge, MA: Cambridge University Press.
- GHEMAWAT, PANKAJ, AND BARRY NALEBUFF (1985): “Exit,” *RAND Journal of Economics*, 16 (2), 184–194.
- GIECZEWSKI, GERMÁN (2024): “Evolving Wars of Attrition,” Unpublished Manuscript, Princeton University.
- GOLDSTEIN, ROBERT, NENGJIU JU, AND HAYNE E. LELAND (2001): “An EBIT-Based Model of Dynamic Capital Structure,” *Journal of Business*, 74 (4), 483–512.
- HENDRICKS, KENNETH, AND ROBERT H. PORTER (1996): “The Timing and Incidence of Exploratory Drilling on Offshore Wildcat Tracts,” *American Economic Review*, 86 (3), 388–407.
- HENDRICKS, KENNETH, ANDREW WEISS, AND CHARLES WILSON (1988): “The War of Attrition in Continuous Time with Complete Information,” *International Economic Review*, 29 (4), 663–680.
- JARROW, ROBERT, AND PHILIP PROTTER (2005): “Large Traders, Hidden Arbitrage, and Complete Markets,” *Journal of Banking and Finance*, 29 (11), 2803–2820.
- KARATZAS, IOANNIS, AND STEVEN E. SHREVE (1998): *Methods of Mathematical Finance*. New York: Springer-Verlag.

- KELLER, GODFREY, AND SVEN RADY (2010): “Strategic Experimentation with Poisson Bandits,” *Theoretical Economics*, 5 (2), 275–311.
- KELLER, GODFREY, SVEN RADY, AND MARTIN CRIPPS (2005): “Strategic Experimentation with Exponential Bandits,” *Econometrica*, 73 (1), 39–68.
- KENNAN, JOHN, AND ROBERT WILSON (1989): “Strategic Bargaining Models and Interpretation of Strike Data,” *Journal of Applied Econometrics*, 4 (S1), S87–S130.
- KOLB, AARON (2019): “Strategic Real Options,” *Journal of Economic Theory*, 183, 344–383.
- KWON, H. DHARMA, AND JAN PALCZEWSKI (2022): “Exit Game with Private Information,” Unpublished Manuscript, arXiv:2210.01610.
- LABÁN, RAÚL, AND FEDERICO STURZENEGGER (2007): “Distributional Conflict, Financial Adaptation, and Delayed Stabilizations,” *Economics and Politics*, 6 (3), 257–276.
- LAMBRECHT, BART M. (2001): “The Impact of Debt Financing on Entry and Exit in a Duopoly,” *Review of Financial Studies*, 14 (3), 765–804.
- LELAND, HAYNE E. (1994): “Corporate Debt Value, Bond Covenants, and Optimal Capital Structure,” *Journal of Finance*, 49 (4), 1213–1252.
- MASKIN, ERIC, AND JEAN TIROLE (2001): “Markov Perfect Equilibrium: I. Observable Actions,” *Journal of Economic Theory*, 100 (2), 191–219.
- MAYNARD SMITH, JOHN (1974): “The Theory of Games and the Evolution of Animal Conflicts,” *Journal of Theoretical Biology*, 47 (1), 209–221.
- MARTINELLI, CÉSAR, AND RAÚL ESCORZA (2007): “When Are Stabilizations Delayed: Alesina–Drazen Revisited,” *European Economic Review*, 51 (5), 1223–1245.
- MERTON, ROBERT C. (1974): “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” *Journal of Finance*, 29 (2), 449–470.
- MURTO, PAULI (2004): “Exit in Duopoly under Uncertainty,” *RAND Journal of Economics*, 35 (1), 111–127.
- MYATT, DAVID P. (2024): “The Impact of Perceived Strength in the War of Attrition,” Unpublished Manuscript, London Business School.

- PADOVANO, FABIO, AND LARISSA VENTURI (2001): “Wars of Attrition in Italian Government Coalitions and Fiscal Performance: 1948–1994,” *Public Choice*, 109 (1), 15–54.
- PESKIR, GORAN (2019): “Continuity of the Optimal Stopping Boundary for Two-Dimensional Diffusions,” *Annals of Applied Probability*, 29 (1), 505–530.
- REVUZ, DANIEL, AND MARC YOR (1999): *Continuous Martingales and Brownian Motion*. Berlin, Heidelberg, New York: Springer-Verlag.
- RIEDEL, FRANK, AND JAN-HENRIK STEG (2017): “Subgame-Perfect Equilibria in Stochastic Timing Games,” *Journal of Mathematical Economics*, 72, 36–50.
- RILEY, JOHN G. (1980): “Strong Evolutionary Equilibrium and the War of Attrition,” *Journal of Theoretical Biology*, 82 (3), 383–400.
- SELTEN, REINHARD (1975): “Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games,” *International Journal of Game Theory*, 4 (1), 25–55.
- SHARPE, MICHAEL J. (1971): “Exact Multiplicative Functionals in Duality,” *Indiana University Mathematics Journal*, 21 (1), 27–60.
- STEG, JAN-HENRIK (2015): “Symmetric Equilibria in Stochastic Timing Games,” Center for Mathematical Economics Working Paper No. 543, Universität Bielefeld.
- TAKAHASHI, YUYA (2015): “Estimating a War of Attrition: The Case of the US Movie Theater Industry,” *American Economic Review*, 105 (7), 2204–2241.
- TOUZI, NIZAR, AND NICOLAS VIEILLE (2002): “Continuous-Time Dynkin Games with Mixed Strategies,” *SIAM Journal on Control and Optimization*, 41 (4), 1073–1088.
- WANG, ZHONGMIN (2009): “(Mixed) Strategy in Oligopoly Pricing: Evidence from Gasoline Price Cycles Before and Under a Timing Regulation,” *Journal of Political Economy*, 117 (6), 987–1030.

Supplement to “The War of Attrition under Uncertainty: Theory and Robust Testable Implications”: Additional Proofs

Jean-Paul Décamps* Fabien Gensbittel[†] Thomas Mariotti[‡]

June 30, 2024

Abstract

Section S.1 provides useful preliminary results. Section S.2 gathers proofs of lemmas that appear elsewhere in the literature or follow directly from existing results. Section S.3 provides the proofs of Propositions 1 and 2. Section S.4 gathers proofs of results in Section 4. Section S.5 proves two key lemmas for Theorem 2. Section S.6 proves two key lemmas for Theorem 3.

*Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: jean-paul.decamps@tse-fr.eu.

[†]Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: fabien.gensbittel@tse-fr.eu.

[‡]Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France, CEPR, and CESifo. Email: thomas.mariotti@tse-fr.eu.

S.1 Preliminaries

S.1.1 The Fundamental Filtration

We start with some definitions (Revuz and Yor (1999, Chapter I, §4)). The process X is defined over the canonical space (Ω, \mathcal{F}) of continuous trajectories, and \mathbf{P}_μ denotes the law of the process X given an initial distribution $\mu \in \Delta(\mathcal{I})$, where $\Delta(\mathcal{I})$ is the space of Borel probability measures over \mathcal{I} . We denote by $(\mathcal{F}_t^0)_{t \geq 0}$ the natural filtration $(\sigma(X_s; s \leq t))_{t \geq 0}$ generated by X , and we let $\mathcal{F}_\infty^0 \equiv \sigma(\bigcup_{t \geq 0} \mathcal{F}_t^0)$. For each $\mu \in \Delta(\mathcal{I})$, we denote by \mathcal{F}_∞^μ the completion of \mathcal{F}_∞^0 with respect to \mathbf{P}_μ , and, for each $t \geq 0$, we let \mathcal{F}_t^μ be the augmentation of \mathcal{F}_t^0 by the \mathbf{P}_μ -null, \mathcal{F}_∞^μ -measurable sets. The usual augmented filtration $(\mathcal{F}_t)_{t \geq 0}$ is then defined by $\mathcal{F}_t \equiv \bigcap_{\mu \in \Delta(\mathcal{I})} \mathcal{F}_t^\mu$ for all $t \geq 0$. Because the process X is a Feller process in the sense of Revuz and Yor (1999, Chapter III, §2, Definition 2.5) and a standard process in the sense of Blumenthal and Gettoor (1968, Chapter I, Definition 9.2), the filtration $(\mathcal{F}_t)_{t \geq 0}$ is actually right-continuous. As usual in this literature, we say that a property of the trajectories $\omega \in \Omega$ is satisfied *almost surely* if it is satisfied \mathbf{P}_μ -almost surely for all $\mu \in \mathcal{I}$ or, equivalently, \mathbf{P}_x -almost surely for all $x \in \mathcal{I}$.

S.1.2 A Useful Change of Variables

Dayanik and Karatzas (2003) introduced an elegant change of variables that we use in several proofs. Specifically, for each $x \in \mathcal{I}$, define $\zeta(x) \equiv \frac{\phi(x)}{\psi(x)}$, which is strictly decreasing in x and maps \mathcal{I} onto $(0, \infty)$. Then, for any function $g : \mathcal{I} \rightarrow \mathbb{R}$, define the function \hat{g} by

$$\hat{g}(y) \equiv \frac{g}{\psi} \circ \zeta^{-1}(y), \quad y \in (0, \infty). \quad (\text{S.1})$$

Observe that $\hat{\phi}(y) = y$ and $\hat{\psi}(y) = 1$ for all $y \in (0, \infty)$. A direct computation shows that, if $g \in \mathcal{C}^2(\mathcal{I})$, then

$$\hat{g}''(\zeta(x)) = \frac{2\phi(x)^3}{[\varrho\sigma(x)p'(x)]^2} (\mathcal{L}g - rg)(x), \quad x \in \mathcal{I}, \quad (\text{S.2})$$

where p is the scale function of the diffusion X , which is uniquely defined up to an affine transformation by

$$p(x) \equiv \int_c^x \exp\left(-\int_c^y \frac{2\mu(z)}{\sigma^2(z)} dz\right) dy, \quad x \in \mathcal{I}, \quad (\text{S.3})$$

for some fixed $c \in \mathcal{I}$ (Karatzas and Shreve (1998, Chapter 5, Section 5, §B)), and

$$\varrho \equiv \frac{\psi'(x)\phi(x) - \psi(x)\phi'(x)}{p'(x)} > 0, \quad (\text{S.4})$$

the ratio of the Wronskian of ψ and ϕ and of the derivative of the scale function, is a constant independent of x by Abel's theorem. From A3 and (S.2), we deduce that $\hat{R}'''(\zeta(x)) < 0$ for all $x \in (\alpha, x_0^i)$ or, equivalently, that $\hat{R}'''(y) < 0$ for all $y \in (\zeta(x_0^i), \infty)$ and thus, in particular, for all $y \in (\zeta(x_{R^i}), \infty)$ as $x_{R^i} < x_0^i$. From A7 and (S.2), we deduce that $\hat{G}''' \leq 0$ everywhere \hat{G}''' is defined. Another useful remark is that, from Lemma 1 and A6, we have $G^i > 0$ over \mathcal{I} . Thus, $\hat{G}^i > 0$ over $(0, \infty)$, and (8) implies

$$\lim_{y \rightarrow 0} \hat{G}^i(y) = \lim_{y \rightarrow \infty} \frac{\hat{G}^i(y)}{y} = 0. \quad (\text{S.5})$$

S.2 Basic Lemmas

PROOF OF LEMMA 1: The proof proceeds along the same lines as in Décamps, Gensbittel, and Mariotti (2024, Lemma 1). The result follows. \blacksquare

PROOF OF LEMMA 2: For each $\mu \in \Delta(\mathcal{I})$, ω and u^i are independent under $\mathbf{P}_\mu^i \equiv \mathbf{P}_\mu \otimes \text{Leb}$, and hence, for each $t \geq 0$,

$$\Gamma_t^i(\omega) = \mathbf{P}_\mu^i[\gamma^i \leq t | \mathcal{F}](\omega)$$

for \mathbf{P}_μ -almost every $\omega \in \Omega$. We may assume that $\gamma(\cdot, u^i) \in \mathcal{T}$ for all u^i , as we can replace γ^i by the constant stopping time 0 for all u^i in a Borel set of Lebesgue measure zero without modifying the process Γ^i . Therefore, for all $u^i \in [0, 1]$ and $t \geq 0$, we have $\{\omega \in \Omega : \gamma^i(\omega, u^i) \leq t\} \in \mathcal{F}_t$ as $\gamma(\cdot, u^i) \in \mathcal{T}$. It follows from Solan, Tsirelson, and Vieille (2012, Corollary 2) that this implies that Γ_t^i is measurable with respect to the augmentation of \mathcal{F}_t by the \mathbf{P}_μ -null, \mathcal{F}_∞^μ -measurable sets, which coincides with \mathcal{F}_t^μ . As this is true for all $\mu \in \Delta(\mathcal{I})$, we deduce that Γ^i is adapted with respect to \mathcal{F}_t . In particular, letting $\mu \equiv \delta_x$ yields

$$\Gamma_t^i(\omega) = \mathbf{P}_x^i[\gamma^i \leq t | \mathcal{F}_t](\omega)$$

for \mathbf{P}_x -almost every $\omega \in \Omega$ by the law of iterated expectations. The result follows. \blacksquare

PROOF OF LEMMA 3: Suppose that, for each $i = 1, 2$, γ^i is a randomized stopping time with ccdf Γ^i . We have

$$\begin{aligned} \bar{\mathbf{E}}_x \left[1_{\{\gamma^i \leq \gamma^j\}} e^{-r\gamma^i} R^i(X_{\gamma^i}) \right] &= \int_0^1 \int_0^1 \mathbf{E}_x \left[1_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}} e^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \right] du^j du^i \\ &= \int_0^1 \mathbf{E}_x \left[e^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \int_0^1 1_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}} du^j \right] du^i \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \mathbf{E}_x \left[e^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda_{\gamma^i(u^i)-}^j \right] du^i \\
&= \mathbf{E}_x \left[\int_0^1 e^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda_{\gamma^i(u^i)-}^j du^i \right] \\
&= \mathbf{E}_x \left[\int_{[0,\infty)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i \right],
\end{aligned}$$

where the second and fourth equalities follow from Fubini's theorem, and the third equality follows from the definition of Λ^j . The last equality follows from observing that, for \mathbf{P}_x -almost every $\omega \in \Omega$, $t \mapsto \Gamma_t^i(\omega)$ is the cdf of the random variable $\gamma^i(\omega, \cdot)$ defined on the probability space $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$ and taking values in $[0, \infty]$, where $\Gamma_\infty^i(\omega) \equiv 1$ by convention; Fubini's theorem then implies that the random variable $u^i \mapsto e^{-r\gamma^i(\omega, u^i)} R^i(X_{\gamma^i(\omega, u^i)}) \Lambda_{\gamma^i(\omega, u^i)-}^j$ is Lebesgue integrable over $[0, 1]$ for \mathbf{P}_x -almost every $\omega \in \Omega$,¹ and we can thus apply the usual formula for the expectation. The proof for the second term appearing in (11) and (12) is similar and thus omitted.

Let us then verify that (13) defines a randomized stopping time in the sense of Definition 1. That $\hat{\gamma}^i(u^i) \in \mathcal{T}$ for *Leb*-almost every $u^i \in [0, 1]$ is standard (Jacod and Shiryaev (1994, Proposition I.1.28)). The random variable $(\omega, u^i) \mapsto \hat{\gamma}^i(u^i)(\omega)$ is $\mathcal{F}_\infty \otimes \mathcal{B}([0, 1])$ -measurable as it is nondecreasing and right-continuous with respect to u^i . That the ccdf associated to $\hat{\gamma}^i$ is Γ^i is proven in De Angelis, Ferrari, and Moriarty (2018, Lemma 4.1), who use this representation as the definition of a randomized stopping time. The result follows. \blacksquare

PROOF OF LEMMA 4: We focus on player 1, the proof for player 2 being symmetrical. Observe from (12) that, for each $\tau^1 \in \mathcal{T}$, player 1's payoff from playing τ^1 against Γ^2 is

$$J^1(x, \tau^1, \Gamma^2) = \mathbf{E}_x \left[e^{-r\tau^1} R^1(X_{\tau^1}) \Lambda_{\tau^1-}^2 + \int_{[0, \tau^1)} e^{-rt} G^1(X_t) d\Gamma_t^2 \right]. \quad (\text{S.6})$$

Letting $\hat{\gamma}^1$ be the randomized stopping time associated to the ccdf Γ^1 by (13), we have

$$\begin{aligned}
J^1(x, \Gamma^1, \Gamma^2) &= \int_0^1 \mathbf{E}_x \left[e^{-r\hat{\gamma}^1(u^1)} R^1(X_{\hat{\gamma}^1(u^1)}) \Lambda_{\hat{\gamma}^1(u^1)-}^2 + \int_{[0, \hat{\gamma}^1(u^1))} e^{-rt} G^1(X_t) d\Gamma_t^2 \right] du^1 \\
&= \int_0^1 J^1(x, \hat{\gamma}^1(u^1), \Gamma^2) du^1 \\
&\leq \sup_{u^1 \in [0, 1]} J^1(x, \hat{\gamma}^1(u^1), \Gamma^2) \\
&\leq \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, \Gamma^2).
\end{aligned}$$

where the first equality follows along the same steps as in the proof of Lemma 3, and the second equality follows from (S.6). The result follows. \blacksquare

¹Recall that, by convention, this random variable is equal to 0 if $\gamma^i(\omega, u^i) = \infty$.

The following consequence of the strong Markov property will be used several times throughout this Online Supplement.

Lemma S.1 *If the players use Markovian randomized stopping times with cdfs (Γ^1, Γ^2) , then, for all $x \in \mathcal{I}$ and $\tau \in \mathcal{T}$, their expected payoffs write as*

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{[0, \tau)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j + e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^j \Lambda_{\tau-}^i \right]. \quad (\text{S.7})$$

PROOF: It follows from Lemma 3 that

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{[0, \tau)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j + e^{-r\tau} R^i(X_\tau) \Lambda_{\tau-}^j (\Gamma_\tau^i - \Gamma_{\tau-}^i) + e^{-r\tau} G^i(X_\tau) \Lambda_\tau^i (\Gamma_\tau^j - \Gamma_{\tau-}^j) + \int_{(\tau, \infty)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{(\tau, \infty)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j \right]. \quad (\text{S.8})$$

Notice from (15) that the only jump of Λ^i occurs at τ_{S^i} , at which time Λ^i jumps down to 0 and remains there forever after, and similarly for Λ_j . Hence

$$\begin{aligned} & e^{-r\tau} R^i(X_\tau) \Lambda_{\tau-}^j (\Gamma_\tau^i - \Gamma_{\tau-}^i) + e^{-r\tau} G^i(X_\tau) \Lambda_\tau^i (\Gamma_\tau^j - \Gamma_{\tau-}^j) \\ &= 1_{\{\tau_{S^j} \geq \tau = \tau_{S^i}\}} e^{-r\tau} R^i(X_\tau) \Lambda_{\tau-}^j \Lambda_{\tau-}^i + 1_{\{\tau_{S^i} > \tau = \tau_{S^j}\}} e^{-r\tau} G^i(X_\tau) \Lambda_{\tau-}^i \Lambda_{\tau-}^j \\ &= 1_{\{\tau_{S^j} \geq \tau = \tau_{S^i}\}} e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^j \Lambda_{\tau-}^i + 1_{\{\tau_{S^i} > \tau = \tau_{S^j}\}} e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^i \Lambda_{\tau-}^j \\ &= 1_{\{\tau \geq \tau_{S^i} \wedge \tau_{S^j}\}} e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^j \Lambda_{\tau-}^i, \end{aligned} \quad (\text{S.9})$$

where the last equality follows from the fact that $e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^i \Lambda_{\tau-}^j$ vanishes over $\{\tau > \tau_{S^i} \wedge \tau_{S^j}\}$. On the other hand, we have

$$\begin{aligned} & \int_{(\tau, \infty)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{(\tau, \infty)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j \\ &= 1_{\{\tau < \tau_{S^i} \wedge \tau_{S^j}\}} e^{-r\tau} \left[\int_{(0, \infty)} e^{-rt} R^i(X_{\tau+t}) \Lambda_{(\tau+t)-}^j d\Gamma_{\tau+t}^i + \int_{(0, \infty)} e^{-rt} G^i(X_{\tau+t}) \Lambda_{\tau+t}^i d\Gamma_{\tau+t}^j \right] \\ &= 1_{\{\tau < \tau_{S^i} \wedge \tau_{S^j}\}} e^{-r\tau} \Lambda_\tau^j \Lambda_\tau^i \left[\int_{(0, \infty)} e^{-rt} R^i(X_t \circ \theta_\tau) (\Lambda_{t-}^j \circ \theta_\tau) d(\Gamma_t^i \circ \theta_\tau) + \int_{(0, \infty)} e^{-rt} G^i(X_t \circ \theta_\tau) (\Lambda_t^i \circ \theta_\tau) d(\Gamma_t^j \circ \theta_\tau) \right] \\ &= 1_{\{\tau < \tau_{S^i} \wedge \tau_{S^j}\}} e^{-r\tau} \Lambda_{\tau-}^j \Lambda_{\tau-}^i \left[\int_{[0, \infty)} e^{-rt} R^i(X_t \circ \theta_\tau) (\Lambda_{t-}^j \circ \theta_\tau) d(\Gamma_t^i \circ \theta_\tau) + \int_{[0, \infty)} e^{-rt} G^i(X_t \circ \theta_\tau) (\Lambda_t^i \circ \theta_\tau) d(\Gamma_t^j \circ \theta_\tau) \right], \end{aligned}$$

where the second equality follows from (14). Taking expectations and applying the strong Markov property at τ yields

$$\begin{aligned} \mathbf{E}_x \left[\int_{(\tau, \infty)} e^{-rt} R^i(X_t) \Lambda_{t-}^j d\Gamma_t^i + \int_{(\tau, \infty)} e^{-rt} G^i(X_t) \Lambda_t^i d\Gamma_t^j \right] \\ = \mathbf{E}_x \left[1_{\{\tau < \tau_{S^i} \wedge \tau_{S^j}\}} e^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda_{\tau-}^j \Lambda_{\tau-}^i \right]. \end{aligned} \quad (\text{S.10})$$

Inserting (S.9) and (S.10) into (S.8) yields (S.7). The result follows. \blacksquare

S.3 Proofs of Propositions 1 and 2

PROOF OF PROPOSITION 1: Suppose, with no loss of generality, that $i = 1$. We first prove that $V_{R^1} \leq \bar{J}^1 \leq G^1$. For the first inequality, let $\tau^1 \equiv \tau_{(\alpha, x_{R^1}]}$, the hitting time by X of $(\alpha, x_{R^1}]$, and let $\hat{\gamma}^2(u)$ be defined by (13). Using Lemma 3 and $G^1 \geq V_{R^1}$ by A6, we obtain

$$\begin{aligned} \bar{J}^1(x) &\geq J^1(x, \tau^1, \Gamma^2) \\ &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u)) du \\ &\geq \int_0^1 \mathbf{E}_x \left[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}} e^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}} e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) \right] du \end{aligned}$$

for all $x \in \mathcal{I}$. For each $u \in [0, 1]$, we have

$$e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x \left[e^{-r\tau^1} R^1(X_{\tau^1}) \mid \mathcal{F}_{\hat{\gamma}^2(u)} \right]$$

\mathbf{P}_x -almost surely over $\{\tau^1 > \hat{\gamma}^2(u)\}$. Thus, by the tower property of conditional expectation,

$$\mathbf{E}_x \left[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}} e^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}} e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) \right] = \mathbf{E}_x \left[e^{-r\tau^1} R^1(X_{\tau^1}) \right] = V_{R^1}(x),$$

and we conclude that, for each $x \in \mathcal{I}$,

$$\bar{J}^1(x) \geq \int_0^1 V_{R^1}(x) du = V_{R^1}(x).$$

For the second inequality, we have $R^1 \leq V_{R^1} \leq G^1$ by A6. Hence, for each $\tau^1 \in \mathcal{T}$,

$$\begin{aligned} J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u)) du \\ &\leq \int_0^1 \mathbf{E}_x \left[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}} e^{-r\tau^1} G^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}} e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \right] du \\ &= \int_0^1 \mathbf{E}_x \left[e^{-r(\tau^1 \wedge \hat{\gamma}^2(u))} G^1(X_{\tau^1 \wedge \hat{\gamma}^2(u)}) \right] du \\ &\leq \int_0^1 G^1(x) du \\ &= G^1(x) \end{aligned}$$

for all $x \in \mathcal{I}$, where the second inequality follows from the fact that $(e^{-rt}G^1(X_t))_{t \geq 0}$ is a supermartingale by A7. We now prove properties (i)–(v) in turn.

(i) It is not optimal for player 1 to concede at $x \in S^2$ if $R^1(x) < G^1(x)$, that is, if $x > \alpha^1$. Therefore, if (μ^1, S^1) is a pbr to (μ^2, S^2) , then $S^1 \cap S^2 \cap (\alpha^1, \beta) = \emptyset$.

(ii) This directly follows from the definition (11) of players' payoffs.

(iii) By Lemma 3, for each $x \in \text{supp } \mu^1$, we have

$$\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2) du,$$

where $\hat{\gamma}^1(u) = \inf \{t \geq 0 : \Gamma_t^1 > u\}$. Thus the inequality $J^1(x, \hat{\gamma}^1(u), \Gamma^2) \leq \bar{J}^1(x)$, which holds for all $u \in [0, 1]$, must be an equality for all u in a set U of Lebesgue measure 1. By definition of Γ^1 , $\hat{\gamma}^1(u) = \inf \{t \geq 0 : 1 - e^{-\int_{\mathcal{I} \setminus S^1} L_t^y \mu^1(dy)} > u\} \wedge \tau_{S^1}$ for all $u \in [0, 1]$. Notice that $\hat{\gamma}^1(u) > 0$ \mathbf{P}_x -almost surely for all $u \in (0, 1)$ as the mapping $t \mapsto 1_{\{t < \tau_{S^1}\}} e^{-\int_{\mathcal{I} \setminus S^1} L_t^y \mu^1(dy)}$ is continuous over $[0, \tau_{S^1})$ by Theorem 1. We claim that, because $x \in \text{supp } \mu^1$, we also have $\lim_{u \rightarrow 0} \hat{\gamma}^1(u) = 0$ \mathbf{P}_x -almost surely. Indeed, $\hat{\gamma}^1(u, \omega)$ is nondecreasing in u for all ω and converges to $\hat{\gamma}^1(0, \omega) = \inf \{t \geq 0 : \int_{\mathcal{I} \setminus S^1} L_t^y(\omega) \mu^1(dy) > 0\} \wedge \tau_{S^1}(\omega)$. Let us fix a continuous version $(t, y) \mapsto L_t^y$ of the local time of X (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), and observe that $L_t^x > 0$ \mathbf{P}_x -almost surely for all $t > 0$. Thus there exist a sequence $(t_n)_{n \geq 1}$ converging to 0 and, for each $n \geq 1$, a set $\Omega_{t_n} \in \mathcal{F}$ of \mathbf{P}_x -probability 1 such that $L_{t_n}^x(\omega) > 0$ and $y \mapsto L_{t_n}^y(\omega)$ is continuous at x for all $\omega \in \Omega_{t_n}$. Now, $x \in \text{supp } \mu^1$ and $\text{supp } \mu^1$ being closed jointly imply that any open interval of \mathcal{I} containing x has positive μ^1 -measure. From these observations, it follows that, for each $n \geq 1$, $\int_{\mathcal{I} \setminus S^1} L_{t_n}^y(\omega) \mu^1(dy) > 0$ for all $\omega \in \Omega_{t_n}$, so that $\hat{\gamma}^1(0, \omega) = 0$ for all $\omega \in \bigcap_{n \geq 1} \Omega_{t_n}$ and thus \mathbf{P}_x -almost surely, as claimed. Finally, for each $u \in U$,

$$\bar{J}^1(x) = J^1(x, \hat{\gamma}^1(u), \Gamma^2) = \mathbf{E}_x \left[\int_{[0, \hat{\gamma}^1(u))} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\hat{\gamma}^1(u)} R^1(X_{\hat{\gamma}^1(u)}) \Lambda_{\hat{\gamma}^1(u)-}^2 \right]. \quad (\text{S.11})$$

Using bounded convergence to take the limit as $u \in U$ goes to 0, two cases must be distinguished. If $x \notin S^2$, then Γ_t^2 is continuous at $t = 0$, from which it follows that $\bar{J}^1(x) = R^1(x)$. If $x \in S^2$, then $\Gamma_{0-}^2 = 0$, $\Gamma_0^2 = 1$, and $\Lambda_{\hat{\gamma}^1(u)-}^2 = 0$ for all $u \in (0, 1)$, from which it follows that $\bar{J}^1(x) = G^1(x)$.

(iv) We claim that, for each $x \in \mathcal{I}$,

$$\bar{J}^1(x) \geq J^1(x, (0, (\alpha, x_{R^1})), (\mu^2, S^2)) \geq J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)). \quad (\text{S.12})$$

The first inequality in (S.12) directly follows from the fact that (μ^1, S^1) is a pbr to (μ^2, S^2) . For the second one, recall that, by A6,

$$G^1(x) \geq V_{R^1}(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [e^{-r\tau} R^1(X_\tau)] = \mathbf{E}_x [e^{-r\tau^1} R^1(X_{\tau^1})],$$

where $\tau^1 \equiv \tau_{(\alpha, x_{R^1}]}$. We have

$$\begin{aligned} J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u)) \, du \\ &= \int_0^1 \mathbf{E}_x \left[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}} e^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}} e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \right] \, du. \end{aligned}$$

Over $\{\tau^1 > \hat{\gamma}^2(u)\}$, we have

$$e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \geq e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x \left[e^{-r\tau^1} R^1(X_{\tau^1}) \mid \mathcal{F}_{\hat{\gamma}^2(u)} \right].$$

\mathbf{P}_x -almost surely by A6. Therefore, using the tower property of conditional expectation,

$$J^1(x, \tau^1, \hat{\gamma}^2(u)) \geq \mathbf{E}_x \left[e^{-r\tau^1} R^1(X_{\tau^1}) \right],$$

which implies the second inequality of (S.12) upon integrating with respect to u . The conclusion follows from noticing that $J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)) = V_{R^1}(x) > R^1(x)$ for all $x > x_{R^1}$ and applying (ii) and the first assertion in (iii).

(v) Arguing as in (iii) yields that $\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2) \, du$ for all u in a set U of Lebesgue measure 1. Moreover, using the explicit expression for $\hat{\gamma}^1(u)$ given in (iii), it is easy to check that $\lim_{u \rightarrow 1} \hat{\gamma}^1(u) = \tau_{S^1}$. Therefore, taking the limit in (S.11) as $u \in U$ goes to 1, we deduce that

$$\bar{J}^1(x) = \mathbf{E}_x \left[\int_{[0, \tau_{S^1})} e^{-rt} G^1(X_t) \, d\Gamma_t^2 + e^{-r\tau_{S^1}} R^1(X_{\tau_{S^1}}) \Lambda_{\tau_{S^1}-}^2 \right] = J^1(x, \tau_{S^1}, \Gamma^2)$$

by bounded convergence, from which the first assertion follows. For the second assertion, let $\tilde{\Gamma}^1$ be the cdf associated to $(\tilde{\mu}^1, S^1)$ and $\tilde{\gamma}^1(u) = \inf\{t \geq 0 : \tilde{\Gamma}_t^1 > u\}$. By assumption,

$$\bar{J}^1(X_{\tilde{\gamma}^1(u)}) = R^1(X_{\tilde{\gamma}^1(u)}). \quad (\text{S.13})$$

for all $u \in [0, 1]$. On the one hand,

$$J^1(x, \tilde{\Gamma}^1, \Gamma^2) = \int_0^1 \mathbf{E}_x \left[\int_{[0, \tilde{\gamma}^1(u))} e^{-rt} G^1(X_t) \, d\Gamma_t^2 + e^{-r\tilde{\gamma}^1(u)} R^1(X_{\tilde{\gamma}^1(u)}) \Lambda_{\tilde{\gamma}^1(u)-}^2 \right] \, du. \quad (\text{S.14})$$

On the other hand, using that $\bar{J}^1 = J^1(\cdot, \tau_{S^1}, \Gamma^2)$ and applying the strong Markov property at $\tilde{\gamma}^1(u)$ in (12) yields

$$\begin{aligned} \bar{J}^1(x) &= \mathbf{E}_x \left[\int_{[0, \tilde{\gamma}^1(u))} e^{-rt} G^1(X_t) \, d\Gamma_t^2 + e^{-r\tilde{\gamma}^1(u)} \bar{J}^1(X_{\tilde{\gamma}^1(u)}) \Lambda_{\tilde{\gamma}^1(u)-}^2 \right] \\ &= \mathbf{E}_x \left[\int_{[0, \tilde{\gamma}^1(u))} e^{-rt} G^1(X_t) \, d\Gamma_t^2 + e^{-r\tilde{\gamma}^1(u)} R^1(X_{\tilde{\gamma}^1(u)}) \Lambda_{\tilde{\gamma}^1(u)-}^2 \right], \end{aligned} \quad (\text{S.15})$$

where the second equality follows from (S.13). Integrating (S.15) with respect to u yields (S.14), from which the second assertion follows. Hence the result. \blacksquare

PROOF OF PROPOSITION 2: Our argument requires some technical results on processes $A \equiv (A_t)_{t \geq 0}$ of the form $A_t \equiv \int_{\mathcal{I} \setminus S} L_t^x \mu(dx)$, where $S \subset \mathcal{I}$ is a closed set and μ is a Radon measure over $\mathcal{I} \setminus S$. Precisely, if τ is the first exit time of $(a, b) \subset \mathcal{I} \setminus S$, with $[a, b] \subset \mathcal{I}$, then

$$\mathbf{E}_x[A_\tau] = \int_{\mathcal{I} \setminus S} \mathbf{E}_x[L_\tau^y] \mu(dy) = \int_{(a,b)} \mathbf{E}_x[L_\tau^y] \mu(dy) = \int_{(a,b)} 2[p'(y)]^{-1} \Phi_{a,b}(x, y) \mu(dy), \quad (\text{S.16})$$

where p' is the derivative of the scale function (S.3) of the diffusion X , and

$$\Phi_{a,b}(x, y) \equiv \frac{[p(x \wedge y) - p(a)][p(b) - p(x \vee y)]}{p(b) - p(a)}$$

is the Green function of the diffusion X killed at the boundaries a and b (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §11, and Section 2, §13)). It is easy to check that $\mathbf{E}_x[A_\tau]$ is finite if and only if, for some $x \in (a, b)$,

$$\int_a^x [p(y) - p(a)] \mu(dy) < \infty \quad \text{and} \quad \int_x^b [p(b) - p(y)] \mu(dy) < \infty.$$

A more precise result can be stated as follows (Çetin (2018, Theorem 2.1)):

$$A_{\tau_a} 1_{\{\tau_a < \tau_b\}} = \infty \text{ a.s. if } \int_a^x [p(y) - p(a)] \mu(dy) = \infty \text{ for some } x \in (a, b), \quad (\text{S.17})$$

$$A_{\tau_a} 1_{\{\tau_a < \tau_b\}} < \infty \text{ a.s. otherwise.} \quad (\text{S.18})$$

A symmetric result holds for b . The following lemma is key to our continuity result.

Lemma S.2 *For each $t \geq 0$, let $A_t \equiv \int_{(a,b)} L_t^y \mu(dy)$ for some Radon measure μ over $(a, b) \subset \mathcal{I}$. Then the function h defined, for nonnegative constants C_a and C_b , by*

$$h(x) = \mathbf{E}_x \left[C_a 1_{\{\tau_a < \tau_b\}} e^{-A_{\tau_a}} + C_b 1_{\{\tau_b < \tau_a\}} e^{-A_{\tau_b}} \right], \quad x \in (a, b),$$

is nonnegative, p -convex,² and continuous over (a, b) . Moreover, the limits $h(a+)$ and $h(b-)$ exist and are given by

$$h(a+) = \begin{cases} 0 & \text{if } \int_a^x [p(y) - p(a)] \mu(dy) = \infty \text{ for some } x \in (a, b), \\ C_a & \text{otherwise} \end{cases}, \quad (\text{S.19})$$

$$h(b-) = \begin{cases} 0 & \text{if } \int_x^b [p(b) - p(y)] \mu(dy) = \infty \text{ for some } x \in (a, b). \\ C_b & \text{otherwise} \end{cases}. \quad (\text{S.20})$$

²That is,

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq h(x_1) \frac{p(x_2) - p(\lambda x_1 + (1 - \lambda)x_2)}{p(x_2) - p(x_1)} + h(x_2) \frac{p(\lambda x_1 + (1 - \lambda)x_2) - p(x_1)}{p(x_2) - p(x_1)}$$

for all $x_1, x_2 \in (a, b)$ and $\lambda \in [0, 1]$.

PROOF: First, h is clearly nonnegative. Next, applying the strong Markov property to $h(\lambda x_1 + (1 - \lambda)x_2)$ at $\tau_{x_1} \wedge \tau_{x_2}$ yields

$$h(\lambda x_1 + (1 - \lambda)x_2) = \mathbf{E}_{\lambda x_1 + (1 - \lambda)x_2} \left[h(x_1) 1_{\{\tau_{x_1} < \tau_{x_2}\}} e^{-A\tau_{x_1}} + h(x_2) 1_{\{\tau_{x_2} < \tau_{x_1}\}} e^{-A\tau_{x_2}} \right].$$

Using that $e^{-At} \leq 1$, we then obtain from standard computations (Karatzas and Shreve (1998, Chapter 5, Section 5, §C)) that h is p -convex. Finally, that h is continuous follows from its being p -convex (Revuz and Yor (1999, Appendix, §3)).

Consider now (S.19). If $\int_a^x [p(y) - p(a)] \mu(dy) = \infty$ for some $x \in (a, b)$, then by (S.17) $h(x) = \mathbf{E}_x [C_b 1_{\{\tau_b < \tau_a\}} e^{-A\tau_b}]$ and thus $0 \leq h(x) \leq C_b \mathbf{P}_x[\tau_b < \tau_a]$, which goes to 0 as x goes to a . Hence $h(a+) = 0$. If $\int_a^x [p(y) - p(a)] \mu(dy) < \infty$ for some $x \in (a, b)$, then by (S.18) $e^{-A\tau_a} > 0$ \mathbf{P}_x -almost surely. If $(a_n)_{n \geq 1}$ is a decreasing sequence converging to a and strictly bounded above by x , then, applying the strong Markov property to $h(x)$ at τ_{a_n} , we have

$$h(x) = \mathbf{E}_x \left[h(a_n) 1_{\{\tau_{a_n} < \tau_b\}} e^{-A\tau_{a_n}} + C_b 1_{\{\tau_b < \tau_{a_n}\}} e^{-A\tau_b} \right].$$

Using bounded convergence to take the limit along any subsequence $(h(a_{n_k}))_{k \geq 1}$ converging to some $z < \infty$, we obtain that

$$h(x) = \mathbf{E}_x \left[z 1_{\{\tau_a < \tau_b\}} e^{-A\tau_a} + C_b 1_{\{\tau_b < \tau_{a_n}\}} e^{-A\tau_b} \right],$$

and thus $z = C_a$ as $\mathbf{E}_x [1_{\{\tau_a < \tau_b\}} e^{-A\tau_a}] > 0$. It follows that $\lim_{n \rightarrow \infty} h(a_n) = C_a$. Because this is true for any decreasing sequence $(a_n)_{n \geq 0}$ converging to a , this implies that $h(a+)$ exists and is equal to C_a . This concludes the proof of (S.19). The argument for (S.20) proceeds along similar lines, using (S.18). The result follows. \blacksquare

The proof of Proposition 2 relies on two preliminary lemmas.

Lemma S.3 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf \bar{J}^i , then the restriction of \bar{J}^i to $[a, b]$ is continuous for any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$.*

PROOF: Suppose, with no loss of generality, that $i = 1$. Given $x \notin S^1 \cup S^2$, and for each integer $n \geq 1$, let $\tilde{\tau}_n \equiv \tau_{x-\eta} \wedge \tau_{x+\varepsilon_n}$, where $\eta > 0$, $(\varepsilon_n)_{n \geq 1}$ is a decreasing sequence converging to 0, and $[x - \eta, x + \varepsilon_n] \subset \mathcal{I} \setminus (S^1 \cup S^2)$. Applying Lemma S.1 with $\tau \equiv \tilde{\tau}_n$ yields

$$\begin{aligned} & \bar{J}^1(x) \\ &= \mathbf{E}_x \left[\int_{[0, \tilde{\tau}_n)} e^{-rt} R^1(X_t) \Lambda_{t-}^2 d\Gamma_t^1 + \int_{[0, \tilde{\tau}_n)} e^{-rt} G^1(X_t) \Lambda_t^1 d\Gamma_t^2 + e^{-r\tilde{\tau}_n} \bar{J}^1(X_{\tilde{\tau}_n}) \Lambda_{\tilde{\tau}_n-}^2 \Lambda_{\tilde{\tau}_n-}^1 \right] \\ &= \mathbf{E}_x \left[\int_{[0, \tilde{\tau}_n)} e^{-rt} R^1(X_t) \Lambda_t^2 d\Gamma_t^1 + \int_{[0, \tilde{\tau}_n)} e^{-rt} G^1(X_t) \Lambda_t^1 d\Gamma_t^2 + e^{-r\tilde{\tau}_n} \bar{J}^1(X_{\tilde{\tau}_n}) \Lambda_{\tilde{\tau}_n}^2 \Lambda_{\tilde{\tau}_n}^1 \right], \quad (\text{S.21}) \end{aligned}$$

where the second equality follows from the fact that $\Lambda_{t-}^i = \Lambda_t^i$ over $\{t \leq \tilde{\tau}_n\}$. Consider a subsequence $(\bar{J}^1(x + \varepsilon_{n_k}))_{k \geq 1}$ converging to some z . Because η is fixed, $\tilde{\tau}_{n_k}$ goes to 0 \mathbf{P}_x -almost surely as k goes to ∞ , and $\mathbf{P}_x[\tilde{\tau}_{n_k} = \tau_{x+\varepsilon_{n_k}}]$ goes to 1. The equality $X_{\tilde{\tau}_{n_k}} = 1_{\{\tilde{\tau}_{n_k} = \tau_{x-\eta}\}}(x - \eta) + 1_{\{\tilde{\tau}_{n_k} = \tau_{x+\varepsilon_{n_k}}\}}(x + \varepsilon_{n_k})$ then implies that $\bar{J}^1(X_{\tilde{\tau}_{n_k}})$ goes to z \mathbf{P}_x -almost surely as k goes to infinity. Using bounded convergence to take the limit in (S.21), and taking advantage of the fact that both Γ_t^1 and Γ_t^2 are continuous at $t = 0$ as $x \notin S^1 \cup S^2$, we obtain that $\bar{J}^1(x) = z$, from which it follows as in the proof of Lemma S.2 that \bar{J}^1 is right-continuous at x . The proof that \bar{J}^1 is left-continuous at x is similar.

Now, let us consider an interval $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$. That \bar{J}^1 is continuous over (a, b) follows from the preceding argument; but we need to check that \bar{J}^1 is right-continuous at a and left-continuous at b . We focus on a , the arguments for b being symmetrical. Because $S^1 \cup S^2$ is closed, the only difficulty arises when $a \in S^1 \cup S^2$. We distinguish two cases.

Case 1 Suppose first that $a \in S^1$, so that $\bar{J}^1(a) = R^1(a)$ by Proposition 1(ii). By Proposition 1(v), $\bar{J}^1 = J^1(\cdot, (0, S^1), (\mu^2, S^2))$. Applying Lemma S.1 with $\tau \equiv \tau_a \wedge \tau_b$ yields

$$\bar{J}^1(x) = \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau) \right]$$

for all $x \in (a, b)$. Moreover,

$$0 \leq \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 \right] \leq C \mathbf{E}_x [1 - e^{-A_\tau^2}],$$

where C is an upper bound for G^1 over $[a, b]$. Because $a \notin S^2$, μ^2 is locally finite at a . Applying Lemma S.2 with $C_a = C_b \equiv 1$ and $\mu \equiv \mu^2$ then yields that $\mathbf{E}_x [1 - e^{-A_\tau^2}]$ goes to 0 as $x > a$ goes to a . Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.2 also yields that $\mathbf{E}_x [e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]$ goes to $\bar{J}^1(a) = R^1(a)$ as $x > a$ goes to a . Thus \bar{J}^1 is right-continuous at a .

Case 2 Suppose next that $a \in S^2$, so that $\bar{J}^1(a) = G^1(a)$ by Proposition 1(i) and (iii). Fix some $\varepsilon \in (0, b - a)$. As in Case 1 with $\tau \equiv \tau_a \wedge \tau_{a+\varepsilon}$, we have

$$\bar{J}^1(x) = \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau) \right]$$

for all $x \in (a, a + \varepsilon)$. If $\int_a^x [p(y) - p(a)] \mu^2(dy) < \infty$, the proof proceeds along the same lines as in Case 1. Thus let us assume that $\int_a^x [p(y) - p(a)] \mu^2(dy) = \infty$. Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.2 yields that $\mathbf{E}_x [e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]$ goes to 0 as $x > a$ goes to a . Moreover,

$$\mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 \right] \geq \min_{y \in [a, a+\varepsilon]} G^1(y) \mathbf{E}_x [e^{-r\tau} - e^{-r\tau} e^{-A_\tau^2}].$$

By (5) and Lemma S.2, the last expectation goes to 1 as $x > 0$ goes to a . We deduce that $\liminf_{x \rightarrow a^+} \bar{J}^1(x) \geq \min_{y \in [a, a+\varepsilon]} G^1(y)$ and thus that $\liminf_{x \rightarrow a^+} \bar{J}^1(x) \geq G^1(a)$ by letting ε go to zero. Finally, we also have $\limsup_{x \rightarrow a^+} \bar{J}^1(x) \leq G^1(a)$ as $\bar{J}^1 \leq G^1$ by Proposition 1, and this concludes the proof that \bar{J}^1 is right-continuous at a . The result follows. \blacksquare

Lemma S.4 *The following holds:*

(i) *If (μ^i, S^i) is a pbr to (μ^j, S^j) , then $(\alpha, \alpha^i) \subset S^1 \cup S^2$;*

(ii) *If $\alpha^1 < \alpha^2$ and $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE, then S^1 and S^2 cannot both intersect $(\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2]$, so that either $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^1$ or $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^2$.*

PROOF: (i) Suppose, with no loss of generality, that $i = 1$, and recall that $\bar{J}^1 = R^1 = V_{R^1} = G^1$ over $(\alpha, \alpha^1]$. Suppose, by way of contradiction, that $x \in (\alpha, \alpha^1) \setminus (S^1 \cup S^2)$. Let $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$, with $b < \alpha^1$ and $x \in (a, b)$. Because $(0, S^1)$ is also a pbr to (μ^2, S^2) by Proposition 1(v), applying Lemma S.1 with $\tau \equiv \tau_{\mathcal{I} \setminus (a, b)}$ yields

$$\begin{aligned}
\bar{J}^1(x) &= \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau} \bar{J}^1(X_\tau) \Lambda_{\tau-}^2 \right] \\
&= \mathbf{E}_x \left[\int_{[0, \tau)} e^{-rt} R^1(X_t) d\Gamma_t^2 + e^{-r\tau} R^1(X_\tau) \Lambda_{\tau-}^2 \right] \\
&= \int_0^1 \mathbf{E}_x \left[1_{\{\hat{\gamma}^2(u, \cdot) < \tau\}} e^{-r\hat{\gamma}^2(u, \cdot)} R^1(X_{\hat{\gamma}^2(u, \cdot)}) + 1_{\{\hat{\gamma}^2(u, \cdot) \geq \tau\}} e^{-r\tau} R^1(X_\tau) \right] du \\
&= \int_0^1 \mathbf{E}_x \left[e^{-r(\hat{\gamma}^2(u, \cdot) \wedge \tau)} R^1(X_{\hat{\gamma}^2(u, \cdot) \wedge \tau}) \right] du \\
&< R^1(x),
\end{aligned} \tag{S.22}$$

where the third equality follows along the same lines as in Lemma 3, and the inequality follows from A3 together with the fact that, for each $u > 0$, $\tau \wedge \hat{\gamma}^2(u, \cdot) > 0$ \mathbf{P}_x -almost surely as Γ^2 is continuous over $[0, \tau_{S^2})$ and $\tau_{S^2} > 0$ \mathbf{P}_x -almost surely. By (S.22), $J(x) < R^1(x)$, in contradiction with Proposition 1. Therefore, $(\alpha, \alpha^1) \subset S^1 \cup S^2$, from which (i) follows as $S^1 \cup S^2$ is closed.

(ii) Suppose, with no loss of generality, that $\alpha^1 < \alpha^2$. By Proposition 1(i), $S^1 \cap S^2 \cap (\alpha^1, \alpha^2] = \emptyset$, and, as shown in (i), $(\alpha^1, \alpha^2] \subset S^1 \cup S^2$. It follows that $S^1 \cap (\alpha^1, \alpha^2]$ and $S^2 \cap (\alpha^1, \alpha^2]$, which are both relatively closed sets in $(\alpha^1, \alpha^2]$, are complementary sets in $(\alpha^1, \alpha^2]$, and thus are both relatively open in $(\alpha^1, \alpha^2]$. As their union $(\alpha^1, \alpha^2]$ is a connected set, either one or the other must be empty. Thus either $(\alpha^1, \alpha^2] \subset S^1$ or $(\alpha^1, \alpha^2] \subset S^2$, from which (ii) follows as both S^1 and S^2 are closed sets. The result follows. \blacksquare

We are now ready to complete the proof of Proposition 2. We focus on the right-continuity of the functions \bar{J}^i , $i = 1, 2$, the arguments for their left-continuity being symmetrical. For any function $J : \mathcal{I} \rightarrow \mathbb{R}$ and for each $S \subset \mathcal{I}$, we denote by $J|_S$ the restriction of J to S . Suppose, with no loss of generality, that $\alpha^1 \leq \alpha^2$. For each $i = 1, 2$, $R^i = G^i$ over $(\alpha, \alpha^i]$ and $R^i \leq \bar{J}^i \leq G^i$ by Proposition 1. Thus \bar{J}^i is continuous over $(\alpha, \alpha^i]$ and, in particular, over $(\alpha, \alpha^1]$. Moreover, by Lemma S.4(ii), J^1 coincides with R^1 or G^1 over $(\alpha^1, \alpha^2]$. We conclude that, for each $i = 1, 2$, $\bar{J}^i_{|(\alpha, \alpha^2]}$ is continuous. Notice that \bar{J}^2 is right-continuous at α^2 and that the same is true for \bar{J}^1 if $\alpha^1 = \alpha^2$. By Lemma S.3, for each $i = 1, 2$, $\bar{J}^i_{|[a, b]}$ is continuous for any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$; moreover, $\bar{J}^i_{|S^i} = R^i_{|S^i}$ and $\bar{J}^i_{|S^j} = G^i_{|S^j}$ are also continuous. Therefore, if \bar{J}^1 or \bar{J}^2 is not right-continuous at x , it must be that $x \geq \alpha^2$, that $x \in S^1 \cup S^2$, and that, for each $\varepsilon > 0$, $[x, x + \varepsilon)$ intersects both $S^1 \cup S^2$ and $\mathcal{I} \setminus (S^1 \cup S^2)$; we refer to this last property as Property P. We distinguish two cases.

Case 1 Let us first consider the case where $x \in S^2$ and $x > \alpha^2$, and suppose, by way of contradiction, that \bar{J}^1 or \bar{J}^2 is not right-continuous at x , so that Property P is satisfied. As $(\alpha^2, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^1$. Hence, because S^1 is closed, there exists $\varepsilon > 0$ such that $[x, x + \varepsilon) \cap S^1 = \emptyset$. If (a, b) is a connected component of the open set $[x, x + \varepsilon) \setminus S^2$, so that $a, b \in S^2$, then it must be that $\mu^1[(a, b)] > 0$. Indeed, suppose, by way of contradiction, that this is not the case. Then, for each $y \in (a, b)$, we have

$$\bar{J}^2(y) = J^2(y, (0, S^1), (\mu^2, S^2)) = \mathbf{E}_y[e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})] < R^2(y)$$

by A3 as $b \leq x_{R^2}$ by Proposition 1(iv), in contradiction with Proposition 1. Thus $\mu^1[(a, b)] > 0$ and, by Proposition 1(iii), there exists some $y \in (a, b)$ such that $\bar{J}^1(y) = R^1(y)$. As this is true for every connected component of $[x, x + \varepsilon) \setminus S^2$, Property P implies that there exists a decreasing sequence $(y_n)_{n \geq 1}$ converging to x such that $\bar{J}^1(y_n) = R^1(y_n)$, as well as a sequence of connected components $((a_n, b_n))_{n \geq 1}$ of $[x, x + \varepsilon) \setminus S^2$ such that $y_n \in (a_n, b_n)$ for all $n \geq 1$ and whose length goes to zero as n goes to ∞ . By Proposition 1, $\bar{J}^1(a_n) = G^1(a_n)$ and $\bar{J}^1(b_n) = G^1(b_n)$. Because $x > \alpha^1$, $G^1(x) > R^1(x)$. For each $n \geq 1$, because $(0, S^2)$ is a best reply to (μ^1, S^1) by Proposition 1(v), applying Lemma S.1 to $\tau_n \equiv \tau_{a_n} \wedge \tau_{b_n}$ yields

$$\begin{aligned} \bar{J}^1(y_n) &= \mathbf{E}_{y_n} \left[\int_{[0, \tau_n)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau_n} \bar{J}^1(X_{\tau_n}) \Lambda_{\tau_n}^2 \right] \\ &= \mathbf{E}_{y_n} \left[\int_{[0, \tau_n)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau_n} G^1(X_{\tau_n}) \Lambda_{\tau_n}^2 \right]. \end{aligned}$$

G^1 and R^1 being locally Lipschitz, there exists $\varepsilon > 0$ such that, for any sufficiently large n ,

$$G^1(y) > R^1(y) + \varepsilon, \quad y \in (a_n, b_n).$$

Hence, for any such n ,

$$\bar{J}^1(y_n) \geq [R^1(y_n) + \varepsilon] \mathbf{E}_{y_n} \left[\int_{[0, \tau_n)} e^{-rt} d\Gamma_s^2 + e^{-r\tau_n} \Lambda_{\tau_n-}^2 \right] \geq [R^1(y_n) + \varepsilon] \mathbf{E}_{y_n} [e^{-r\tau_n}].$$

We have $\mathbf{E}_{y_n} [e^{-r\tau_n}] = A_n \phi(y_n) + B_n \psi(y_n)$, where the coefficients A_n and B_n are such that

$$A_n \phi(a_n) + B_n \psi(a_n) = A_n \phi(b_n) + B_n \psi(b_n) = 1.$$

It follows that these coefficients are bounded, and, therefore, as ϕ and ψ are locally Lipschitz, that $\mathbf{E}_{y_n} [e^{-r\tau_n}]$ goes to 1 as n goes to ∞ . This, for n sufficiently large, contradicts the fact that $\bar{J}^1(y_n) = R^1(y_n)$. Thus \bar{J}^1 and \bar{J}^2 are right-continuous at x . The right-continuity of \bar{J}^1 and \bar{J}^2 at x in case $x \in S^1$ and $x > \alpha^2$ and the right-continuity of \bar{J}^1 at x in case $x \in S^2$ and $x = \alpha^2 > \alpha^1$ can be proven in a similar way.

Case 2 It remains only to prove that \bar{J}^1 is right-continuous at x in case $x \in S^1$ and $x = \alpha^2 > \alpha^1$. Suppose that Property P is satisfied so that \bar{J}^1 may not be right-continuous at x . As $(\alpha^1, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^2$. Hence, because S^2 is closed, there exists $\varepsilon > 0$ such that $[x, x + \varepsilon) \cap S^2 = \emptyset$. Notice that $\mu^2([x, x + \varepsilon)) < \infty$ as μ^2 is locally finite on $\mathcal{I} \setminus S^2$. If (a, b) is a connected component of the open set $(x, x + \varepsilon) \setminus S^1$, so that $a, b \in S^1$, then, for $y \in (a, b)$ and $\tau \equiv \tau_a \wedge \tau_b$, we have

$$\begin{aligned} \bar{J}^1(y) - R^1(y) &= J^1(y, (0, S^1), (\mu^2, S^2)) - R^1(y) \\ &= \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau} R^1(X_\tau) \Lambda_{\tau-}^2 \right] - R^1(y) \\ &\geq 0, \end{aligned} \tag{S.23}$$

where the first equality follows from Proposition 1(v). We also have

$$\begin{aligned} \bar{J}^1(y) - R^1(y) &= \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} G^1(X_t) d\Gamma_t^2 + e^{-r\tau} R^1(X_\tau) \Lambda_{\tau-}^2 \right] - R^1(y) \\ &= \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} R^1(X_t) d\Gamma_t^2 + e^{-r\tau} R^1(X_\tau) \Lambda_{\tau-}^2 - R^1(y) + \int_{[0, \tau)} e^{-rt} [G^1(X_t) - R^1(X_t)] d\Gamma_t^2 \right] \\ &= \int_0^1 \mathbf{E}_y \left[e^{-r[\tau \wedge \hat{\gamma}^2(u, \cdot)]} R^1(X_{\tau \wedge \hat{\gamma}^2(u, \cdot)}) - R^1(y) \right] du + \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} (G^1(X_t) - R^1(X_t)) d\Gamma_t^2 \right] \\ &= \int_0^1 \mathbf{E}_y \left[\int_0^{\tau \wedge \hat{\gamma}^2(u, \cdot)} (\mathcal{L}R^1 - rR^1)(X_t) dt \right] du + \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} [G^1(X_t) - R^1(X_t)] d\Gamma_t^2 \right] \\ &\leq \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} [G^1(X_t) - R^1(X_t)] d\Gamma_t^2 \right], \end{aligned} \tag{S.24}$$

where the third equality follows along the same lines as in the proof of Lemma 3, the fourth equality follows from Itô's formula, and the inequality follows from A3 and Proposition 1(iv). Letting $C > 0$ be an upper bound for $G^1 - R^1$ over $[x, x + \varepsilon)$, we then have

$$\begin{aligned} \mathbf{E}_y \left[\int_{[0, \tau)} e^{-rt} [G^1(X_t) - R^1(X_t)] d\Gamma_t^2 \right] &\leq C \mathbf{E}_y [\Gamma_\tau^2] \\ &= C \mathbf{E}_y [1 - \Lambda_\tau^2] \\ &= C \mathbf{E}_y [1 - e^{-A_\tau^2}] \\ &\leq C \mathbf{E}_y [A_\tau^2]. \end{aligned}$$

From (S.16), we have, for some positive constant C' ,

$$\mathbf{E}_y [A_\tau^2] = \int_a^b 2[p'(z)]^{-1} \Phi_{a,b}(y, z) \mu^2(dz) \leq C' \mu^2[(a, b)],$$

as the mapping $z \mapsto 2[p'(z)]^{-1} \Phi_{a,b}(y, z)$ is uniformly bounded over $[x, x + \varepsilon)$. Property P implies that there exists a sequence $((a_n, b_n))_{n \in \mathbb{N}}$ of connected components of $[x, x + \varepsilon) \setminus S^1$ whose length goes to zero as n goes to ∞ . Because μ^2 is locally bounded at x , it must be that $\mu^2[(a_n, b_n)]$ goes to 0 as n goes to ∞ , and the inequalities $0 \leq \bar{J}^1(y) - R^1(y) \leq C C' \mu^2[(a_n, b_n)]$ along with the fact that the constants C and C' are independent of n imply that \bar{J}^1 is right-continuous at x . Hence the result. \blacksquare

S.4 Proofs for Section 4

Lemma S.5 *The equation*

$$R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)} G^1(x) \tag{S.25}$$

has a unique solution $\underline{x}^2 \in (\alpha^1, x_{R^1})$ and $R_1(x_{R^1}) < \frac{\phi(x_{R^1})}{\phi(x)} G^1(x)$ over (\underline{x}^2, β) .

PROOF: For each $x \in \mathcal{I}$, let $f(x) \equiv \frac{\phi(x)}{\phi(x_{R^1})} R^1(x_{R^1})$. Notice that $f = V_{R^1} \geq R^1$ over $[x_{R^1}, \beta)$ and that $f'(x_{R^1}) = R^{1'}(x_{R^1})$ by the smooth-fit property. Applying the change-of-variables formula (S.1) to f , a direct computation shows that \underline{x}^2 is a solution to (S.25) if and only if $\zeta(\underline{x}^2)$ is a solution to

$$\hat{f}(y) = \hat{G}^1(y), \text{ that is, } \frac{R^1(x_{R^1})}{\phi(x_{R^1})} y = \hat{G}^1(y).$$

Because $f = V_{R^1}$ over $[x_{R^1}, \beta)$, it follows from A6 that $\hat{f} < \hat{G}^1$ over $(0, \zeta(x_{R^1}))$. Because \hat{G}^1 is positive, concave, and satisfies (S.5), it follows in turn that (S.25) admits a unique solution

$\underline{x}^2 < x_{R^1}$, and that $\frac{\phi}{\phi(x_{R^1})} R^1(x_{R^1}) > G^1$ over $(\alpha, \underline{x}^2)$ and $\frac{\phi}{\phi(x_{R^1})} R^1(x_{R^1}) < G^1$ over (\underline{x}^2, β) . Finally, recall from A3 and A6 that $\alpha^1 < x_{R^1} < x_0^1$ and that \hat{R}^1 is strictly concave over $(\zeta(x_0^1), \infty)$. Therefore, $\hat{f} > \hat{R}^1$ over $(\zeta(x_{R^1}), \infty)$ as \hat{f} is linear and tangent to \hat{R}^1 at $\zeta(x_{R^1})$. Hence, if $\alpha^1 > \alpha$, it must be that $\alpha^1 < \underline{x}^2$ as $G^1 = R^1$ over $(\alpha, \alpha^1]$. The result follows. ■

PROOF OF PROPOSITION 5: The next lemma provides sufficient conditions for the variational system (24)–(34) to admit a solution.

Lemma S.6 *In the running example, if the firms' liquidation values $l^1 \leq l^2$ are close enough to each other, and if m is sufficiently large and $b > 0$, then there exists a constant $a^1 > 0$ and two functions $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2\})$ and $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2, x_{R^1}\})$ solution to the variational system (24)–(34).*

PROOF: We shall use the standard fact (see, for instance, Dixit and Pindyck (1994)) that, in the running example, $\phi(x) = x^{\rho^-}$ and $\psi(x) = x^{\rho^+}$ for all $x \in (0, \infty)$, where

$$\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \quad \text{and} \quad \rho^+ \equiv \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \quad (\text{S.26})$$

The proof then consists of two parts. We first characterize a candidate solution to (24)–(34) and provide sufficient conditions for its existence. We then show that these conditions are met under our parameter restrictions.

A Candidate Solution Using the notation of Section 2.3, we have

$$V_{R^i}(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [e^{-r\tau} R^i(X_\tau)] = \begin{cases} \frac{\phi(x)}{\phi(x_{R^i})} (l^i - \frac{1}{r-b} x_{R^i}) & \text{if } x > x_{R^i} \\ l^i - \frac{1}{r-b} x & \text{if } x \leq x_{R^i} \end{cases},$$

where $x_{R^i} = \frac{\rho^-}{\rho^- - 1} (r - b) l^i$. Similarly,

$$V_m^i(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x \left[\int_0^\tau e^{-rt} m X_t dt + e^{-r\tau} l^i \right] = \begin{cases} \frac{m}{r-b} x + \frac{\phi(x)}{\phi(\alpha^i)} (l^i - \frac{m}{r-b} \alpha^i) & \text{if } x > \alpha^i \\ l^i & \text{if } x \leq \alpha^i \end{cases},$$

where $\alpha^i = \frac{x_{R^i}}{m} < x_{R^i}$. Thus

$$G^i(x) = (V_m^i - E)(x) = \begin{cases} \frac{m-1}{r-b} x + \frac{\phi(x)}{(1-\rho^-)\phi(\alpha^i)} l^i & \text{if } x > \alpha^i \\ l^i - \frac{1}{r-b} x & \text{if } x \leq \alpha^i \end{cases}.$$

This allows us to rewrite (S.25) as

$$\frac{x^{\rho^-}}{(1-\rho^-) \left[\frac{\rho^-}{\rho^- - 1} (r-b) l^1 \right]^{\rho^-}} l^1 = \frac{m-1}{r-b} x + \frac{x^{\rho^-} m^{\rho^-}}{(1-\rho^-) \left[\frac{\rho^-}{\rho^- - 1} (r-b) l^1 \right]^{\rho^-}} l^1.$$

Solving this equation yields $\underline{x}^2 = \xi x_{R^1}$, where

$$\xi \equiv \left[\frac{1 - m^{\rho^-}}{\rho^- (1 - m)} \right]^{\frac{1}{1 - \rho^-}} \in \left(\frac{1}{m}, 1 \right). \quad (\text{S.27})$$

It follows that the function w^1 defined by

$$w^1(x) \equiv \begin{cases} \frac{\phi(x)}{\phi(x_{R^1})} (l^1 - \frac{1}{r-b} x_{R^1}) & \text{if } x > \underline{x}^2 \\ G^1(x) & \text{if } x \leq \underline{x}^2 \end{cases}$$

is, by construction, solution to the variational system (24)–(28).

If a solution (w^2, a^1) to the variational system (29)–(34) exists, then, letting T_x^2 denote the unique solution to $\mathcal{L}u - ru = 0$ that is tangent to R^2 at x , it must be that $w^2 = T_{\underline{x}^2}^2$ over $(\underline{x}^2, x_{R^1})$. Specifically, we have $T_{\underline{x}^2}^2 = B\psi + C\phi$ with positive coefficients B and C given by³

$$B = \frac{-\phi'(\underline{x}^2)(l^2 - \frac{1}{r-b} \underline{x}^2) - \frac{1}{r-b} \phi(\underline{x}^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)} \quad \text{and} \quad C = \frac{\psi'(\underline{x}^2)(l^2 - \frac{1}{r-b} \underline{x}^2) + \frac{1}{r-b} \psi(\underline{x}^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)}.$$

Similarly, we have $w^2 = A\phi$ over (x_{R^1}, ∞) for

$$A \equiv B \frac{\psi(x_{R^1})}{\phi(x_{R^1})} + C,$$

as required by the continuity of w^2 at x_{R^1} . It follows that

$$\Delta w^{2'}(x_{R^1}) = B \left[\frac{\psi(x_{R^1})}{\phi(x_{R^1})} \phi'(x_{R^1}) - \psi'(x_{R^1}) \right] < 0$$

by (S.4). We deduce that, if

$$G^2(x_{R^1}) > T_{\underline{x}^2}^2(x_{R^1}) > T_{x_{R^2}}^2(x_{R^1}), \quad (\text{S.28})$$

then

$$w^2 = 1_{(0, \underline{x}^2]} R^2 + 1_{(\underline{x}^2, x_{R^1}]} T_{\underline{x}^2}^2 + 1_{(x_{R^1}, \infty)} A\phi \quad \text{and} \quad a^1 = -\frac{\Delta w^{2'}(x_{R^1})}{G^2(x_{R^1}) - w^2(x_{R^1})}$$

is solution to the variational system (29)–(34). In (S.28), the first inequality ensures that $a^1 > 0$, while the second inequality ensures that $\underline{x}^2 < x_{R^2}$ and that $w^2 \geq R^2$ over (x_{R^1}, ∞) . The convexity of $T_{\underline{x}^2}^2$ and the linearity of R^2 imply that $w^2 \geq R^2$ over $(\underline{x}^2, x_{R^1}]$.

Checking the Sufficient Conditions We now show that, if $l^1 \leq l^2$ are close enough to

³That B and C are positive can be seen as follows. First, the denominator of B and C is positive by (S.4). Second, because $\phi' < 0$, $\phi'' > 0$, $\underline{x}^2 < x_{R^1}$, and $l^2 \geq l^1$, the numerator of B is greater than or equal to $-\phi'(x_{R^1})(l^1 - \frac{1}{r-b} x_{R^1}) - \frac{1}{r-b} \phi(x_{R^1})$, which is equal to 0 by the smooth-pasting condition for (3) with $i = 1$. Third, because $\psi' > 0$, $\underline{x}^2 < x_{R^1}$, and $l^2 \geq l^1$, the numerator of C is positive.

each other, and if m is sufficiently large and $b > 0$, then (S.28) holds. Letting $\Delta\rho \equiv \rho^+ - \rho^-$, direct computations lead to

$$B = \frac{\rho^-(\underline{x}^2)^{-\rho^+}}{\Delta\rho} (\xi l^1 - l^2) \quad \text{and} \quad C = \frac{\rho^+(\underline{x}^2)^{-\rho^-}}{\Delta\rho} \left(l^2 - \frac{\rho^+ - 1}{\rho^+} \frac{\rho^-}{\rho^- - 1} \xi l^1 \right).$$

Using that $\underline{x}^2 = \xi x_{R^1}$, we deduce from this that

$$T_{\underline{x}^2}^2(x_{R^1}) = Bx_{R^1}^{\rho^+} + Cx_{R^1}^{\rho^-} = \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho} (\xi l^1 - l^2) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho} \left(l^2 - \frac{\rho^+ - 1}{\rho^+} \frac{\rho^-}{\rho^- - 1} \xi l^1 \right).$$

Now, we have $T_{x_{R^2}}^2 = \frac{l^2}{(1-\rho^-)\phi(x_{R^2})} \phi$, so that

$$T_{x_{R^2}}^2(x_{R^1}) = \frac{l^2}{1-\rho^-} \left(\frac{l^1}{l^2} \right)^{\rho^-}.$$

If l^1 and l^2 are close enough to each other so that $l^1 \geq \frac{l^2}{m}$, then $x_{R^1} \geq \alpha^2$ and thus

$$G^2(x_{R^1}) = \frac{m-1}{r-\mu} x_{R^1} + \frac{\phi(x_{R^1})}{(1-\rho^-)\phi(\alpha^2)} l^2 = \frac{\rho^-}{\rho^- - 1} (m-1)l^1 + \frac{l^2}{1-\rho^-} \left(\frac{l^1}{l^2} \right)^{\rho^-} m^{\rho^-}.$$

Therefore, if $l^1 \geq \frac{l^2}{m}$, then (S.28) holds if and only if

$$\begin{aligned} & \frac{\rho^-}{\rho^- - 1} (m-1)l^1 + \frac{l^2}{1-\rho^-} \left(\frac{l^1}{l^2} \right)^{\rho^-} m^{\rho^-} \\ & > \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho} (\xi l^1 - l^2) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho} \left(l^2 - \frac{\rho^+ - 1}{\rho^+} \frac{\rho^-}{\rho^- - 1} \xi l^1 \right) \\ & > \frac{l^2}{1-\rho^-} \left(\frac{l^1}{l^2} \right)^{\rho^-}. \end{aligned}$$

This is true for any close enough values of l^1 and l^2 if

$$\begin{aligned} & \frac{\rho^-}{\rho^- - 1} (m-1) + \frac{1}{1-\rho^-} m^{\rho^-} \\ & > \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho} (\xi - 1) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho} \left(1 - \frac{\rho^+ - 1}{\rho^+} \frac{\rho^-}{\rho^- - 1} \xi \right) \\ & > \frac{1}{1-\rho^-}. \end{aligned} \tag{S.29}$$

As for the second inequality in (S.29), notice from (S.27) that, as $\rho^- < 0$, ξ goes to 0 as m goes to ∞ , and thus, as $\rho^+ > 0$ and $\rho^-(\xi - 1) > 0$, that the left-hand side goes to ∞ as m goes to ∞ . Therefore, the second inequality in (S.29) is satisfied if m is sufficiently large. As for the first inequality in (S.29), notice from (S.27) that the right-hand side is of the order $m^{\frac{\rho^+}{1-\rho^-}}$ as m goes to ∞ , while the left-hand side is of the order m . Therefore, the

first inequality in (S.29) is satisfied if m is sufficiently large and $\rho^+ + \rho^- < 1$, that is, from (S.26), if $b > 0$. The result follows. \blacksquare

Given Lemma S.6, Proposition 5 is then a direct consequence of Theorem 3 as explained in the main text. Hence the result. \blacksquare

The martingale property of firm 2's discounted cum-dividend stock-price process is a direct consequence of the following lemma.

Lemma S.7 *Let γ^1 be a randomized stopping time of player 1 associated to $\Lambda^1 \equiv (\Lambda_t^1)_{t \geq 0} \equiv (e^{-a^1 L_t^{x_{R^1}}})_{t \geq 0}$ of the form*

$$\gamma^1 \equiv \inf\{t \geq 0 : \Gamma_t^1 > U^1\},$$

where U^1 is uniformly distributed over $[0, 1]$ and independent of X . Then γ^1 is an $(\hat{\mathcal{F}}_t)_{t \geq 0}$ -stopping time and its $(\hat{\mathcal{F}}_t)_{t \geq 1}$ -predictable compensator is $(a^1 L_{t \wedge \gamma^1}^{x_{R^1}})_{t \geq 0}$, where $(\hat{\mathcal{F}}_t)_{t \geq 0}$ is the shareholders' filtration defined by (35). In particular, the processes $(1_{\{t \wedge \tau_{x^2} \geq \gamma^1\}} - a^1 L_{t \wedge \tau^c}^{x_{R^1}})_{t \geq 0}$ and $(e^{-rt \wedge \tau^c} V_t^{2, \tau^c} + \int_0^{t \wedge \tau^c} e^{-rs} X_s ds)_{t \geq 0}$ are $(\hat{\mathcal{F}}_t)_{t \geq 0}$ -martingales.

PROOF: We only need to check that $Z \equiv (Z_t)_{t \geq 0} \equiv (1_{\{t \geq \gamma^1\}} - a^1 L_{t \wedge \gamma^1}^{x_{R^1}})_{t \geq 0}$ is an $(\hat{\mathcal{F}}_t)_{t \geq 0}$ -martingale for all \mathbf{P}_x , $x \in \mathcal{I}$. Let $s \leq t$, and consider the random variable $U_s^1(\omega, u^1) \equiv u^1 1_{\{\Gamma_s^1(\omega) \geq u^1\}} + 1_{\{\Gamma_s^1(\omega) < u^1\}}$ over the probability space $\Omega^1 \equiv \Omega \times [0, 1]$. It is easy to check that

$$\hat{\mathcal{F}}_s = \mathcal{F}_s \vee \sigma(U_s^1) \subset \mathcal{F}_\infty \vee \sigma(U_s^1).$$

From the definition of \mathcal{F}_∞ , we have

$$\mathbf{E}_x[Z_t - Z_s | \mathcal{F}_\infty \vee \sigma(U_s^1)] = \mathbf{E}_x[Z_t - Z_s | \omega, U_s^1].$$

A version of the conditional law of U^1 given $(\omega, U_s^1(\omega, U^1))$ is

$$1_{\{U_s^1 < 1\}} \delta_{U_s^1} + 1_{\{U_s^1 = 1\}} \mathcal{U}_{[\Gamma_s^1, 1]},$$

where $\mathcal{U}_{[a, b]}$ denotes the uniform distribution over $[a, b]$. Hence

$$1_{\{U_s^1 = 1\}} \mathbf{P}_x[\gamma^1 \leq t | \omega, U_s^1] = 1_{\{U_s^1 = 1\}} \frac{\Gamma_t^1 - \Gamma_s^1}{1 - \Gamma_s^1}.$$

We deduce that

$$\begin{aligned} & \mathbf{E}_x[Z_t - Z_s | \omega, U_s^1] \\ &= \mathbf{E}_x \left[1_{\{s < \gamma^1 \leq t\}} - a^1 (L_{t \wedge \gamma^1}^{x_{R^1}} - L_s^{x_{R^1}}) | \omega, U_s^1 \right] \\ &= \mathbf{E}_x \left[1_{\{\Gamma_s^1 < U^1 \leq \Gamma_t^1\}} - a^1 (L_{t \wedge \gamma^1}^{x_{R^1}} - L_s^{x_{R^1}}) | \omega, U_s^1 \right] \\ &= \frac{1_{U_s^1 = 1}}{1 - \Gamma_s^1} \left[\Gamma_t^1 - \Gamma_s^1 - a^1 \int_s^t (L_u^{x_{R^1}} - L_s^{x_{R^1}}) d\Gamma_u^1 - a^1 (1 - \Gamma_t^1) (L_t^{x_{R^1}} - L_s^{x_{R^1}}) \right] \\ &= 0, \end{aligned}$$

where the fourth equality follows from the integration by parts formula and the fact that

$$\Gamma_t^1 - \Gamma_s^1 = \int_s^t a^1(1 - \Gamma_u^1) dL_u^{x_{R^1}}.$$

We conclude that $\mathbf{E}_x[Z_t - Z_s | \hat{\mathcal{F}}_s] = 0$ by using the law of iterated conditional expectations. The result follows. \blacksquare

The fact that the volatilities of firms' stock returns comove negatively over the attrition zone is a direct consequence of the following lemma.⁴

Lemma S.8 *For each $x \in (0, \infty)$, let $F(x) \equiv Bx^{\rho^+} + Cx^{\rho^-} + \frac{x}{r-b}$ for two nonnegative numbers B and C such that $(B, C) \neq (0, 0)$. Then $\frac{x F'(x)}{F(x)}$ is strictly increasing in x .*

PROOF: For concision, let us write $J(x) \equiv Bx^{\rho^+} + Cx^{\rho^-}$ for all $x \in (0, \infty)$. Then

$$\begin{aligned} \frac{d}{dx} \left[\frac{x F'(x)}{F(x)} \right] &= \frac{[x F''(x) + F'(x)] F(x) - x [F'(x)]^2}{[F(x)]^2} \\ &\propto \left[x J''(x) + J'(x) + \frac{1}{r-b} \right] \left[J(x) + \frac{x}{r-b} \right] - x \left[J'(x) + \frac{1}{r-b} \right]^2 \\ &= x J''(x) \left[J(x) + \frac{x}{r-b} \right] + J(x) J'(x) - \frac{x J'(x)}{r-b} + \frac{J(x)}{r-b} - x [J'(x)]^2 \\ &= x \left[B \rho^+ (\rho^+ - 1) x^{\rho^+ - 2} + C \rho^- (\rho^- - 1) x^{\rho^- - 2} \right] \left[B x^{\rho^+} + C x^{\rho^-} + \frac{x}{r-b} \right] \\ &\quad + (B x^{\rho^+} + C x^{\rho^-}) (B \rho^+ x^{\rho^+ - 1} + C \rho^- x^{\rho^- - 1}) - x \frac{B \rho^+ x^{\rho^+ - 1} + C \rho^- x^{\rho^- - 1}}{r-b} \\ &\quad + \frac{B x^{\rho^+} + C x^{\rho^-}}{r-b} - x (B \rho^+ x^{\rho^+ - 1} + C \rho^- x^{\rho^- - 1})^2 \\ &= \frac{B(\rho^+ - 1)^2}{r-b} x^{\rho^+} + \frac{C(\rho^- - 1)^2}{r-b} x^{\rho^-} + BC(\rho^+ - \rho^-)^2 x^{\rho^+ + \rho^- - 1} \\ &> 0. \end{aligned}$$

The result follows. \blacksquare

S.5 Proofs of Lemmas for Theorem 2

PROOF OF LEMMA A.1: Recall that any solution $u \in \mathcal{C}^2((a, b))$ to the ODE $\mathcal{L}u - ru = 0$ is of the form $u = A\phi + B\psi$ for some constants A and B . Whenever needed, we use the change of variables (S.1) to reexpress the assumptions and the conclusions of (i)–(iv). For instance, $u(x) \geq V_{R^i}(x)$ for all $x \in (a, b)$ if and only if $\hat{u}(z) = Az + B \geq \hat{V}_{R^i}(z)$ for all $z \in (\zeta(b), \zeta(a))$.

⁴It follows from Footnote 5 that Lemma S.8 can be used to reach the same conclusion for any singular mixed-strategy MPE in our running example.

Recall also that $\hat{V}_{R^i} \in \mathcal{C}^1((0, \infty))$, that, for some $C^i > 0$, $\hat{V}_{R^i}(z) = C^i z > \hat{R}^i(z)$ for all $z \in (0, \zeta(x_{R^i}))$, and that $\hat{V}_{R^i} = \hat{R}^i$ is \mathcal{C}^2 and strictly concave over $[\zeta(x_{R^i}), \infty)$.

(i) The assumption $u(\beta-) = 0$ implies $B = 0$, and thus $\hat{u}(0+) = 0$. The assumption that $\hat{u} \geq \hat{V}_{R^i}$ over $(0, \zeta(a))$ implies $A \geq C^i$. If this inequality were strict, then we would have $Az > C^i z \geq \hat{R}^i(z)$ for all $z > 0$ as \hat{V}_{R^i} is concave, in contradiction to the assumption $\hat{u}(\zeta(a)-) = A\zeta(a) = \hat{R}^i(\zeta(a))$. We conclude that $A = C^i$ and, from the properties of \hat{V}_{R^i} , that the unique solution to $Az = \hat{R}^i(z)$ is $\zeta(x_{R^i})$.

(ii) Notice that $V_{R^i} > R^i$ over (x_{R^i}, β) , so that $\hat{V}_{R^i} > \hat{R}^i$ over $(0, \zeta(x_{R^i}))$. Hence, if there exists $z_0 \in (\zeta(b), \zeta(a))$ such that $\hat{u}(z_0) = \hat{R}^i(z_0)$, then it must be that $z_0 \geq \zeta(x_{R^i})$. In this case, \hat{u} is tangent to the concave \mathcal{C}^1 function \hat{V}_{R^i} at z_0 . Over $[\zeta(x_{R^i}), \infty)$, $\hat{V}_{R^i} = \hat{R}^i$ is strictly concave. As a result, $\hat{u}(z) > \hat{R}^i(z)$ for all $z \neq z_0$ in $[\zeta(x_{R^i}), \infty) \cap (\zeta(b), \zeta(a))$, and thus for all $z \neq z_0$ in $(\zeta(b), \zeta(a))$ by the preceding remark.

(iii) If $a > \alpha$, then \hat{u} is an affine function over $(\zeta(b), \zeta(a))$ that coincides with \hat{R}^i at both boundaries of this interval. The fact that \hat{R}^i is strictly concave over $[\zeta(x_{R^i}), \infty)$ together with $\zeta(b) \geq \zeta(x_{R^i})$ then implies that $\hat{u} < \hat{R}^i$ over $(\zeta(b), \zeta(a))$. If $a = \alpha$, then $u(a+) = 0$ implies that $u = B\psi$ for some constant B by (4), and thus that $\hat{R}^i(\zeta(b)) = \hat{u}(\zeta(b)) = B$. The function \hat{R}^i is strictly concave and, by Lemma 1, positive over $[\zeta(x_{R^i}), \infty)$. It is thus increasing over this interval, which implies that $\hat{u} = B < \hat{R}^i$ over $(\zeta(b), \infty)$.

(iv) The function \hat{u} satisfies $\hat{u}(z) = Az + B$ for all $z \in (\zeta(b), \zeta(a))$ for some constants A and B . A direct computation yields

$$A = \hat{u}'(\zeta(a)-) = \frac{\psi(a)u'(a+) - \psi'(a)u(a)}{\psi(a)^2\zeta'(a)} \quad \text{and} \quad \hat{R}^{i'}(\zeta(a)) = \frac{\psi(a)R^{i'}(a) - \psi'(a)R^i(a)}{\psi(a)^2\zeta'(a)},$$

so that, as $u(a) = R^i(a)$, $u'(a+) > R^{i'}(a)$, and $\zeta'(a) < 0$,

$$\hat{R}^{i'}(\zeta(a)) - A = \frac{R^{i'}(a) - u'(a+)}{\psi(a)\zeta'(a)} > 0.$$

Hence $\hat{R}^i(\zeta(a - \varepsilon)) > A\zeta(a - \varepsilon) + B$ for $\varepsilon > 0$ small enough. Similarly, the function \hat{f}_ε satisfies $\hat{f}_\varepsilon(z) = A'z + B'$ for all $z \in (0, \infty)$ for some constants A' and B' . Moreover, $\hat{f}_\varepsilon(\zeta(a - \varepsilon)) = \hat{R}^i(\zeta(a - \varepsilon))$ and $\hat{f}_\varepsilon(\zeta(a + \varepsilon)) = \hat{u}(\zeta(a + \varepsilon))$. Hence

$$A'\zeta(a + \varepsilon) + B' = A\zeta(a + \varepsilon) + B \quad \text{and} \quad A'\zeta(a - \varepsilon) + B' > A\zeta(a - \varepsilon) + B,$$

so that $A'\zeta(a) + B' > A\zeta(a) + B$ as $\zeta(a) \in (\zeta(a + \varepsilon), \zeta(a - \varepsilon))$. The result follows. \blacksquare

PROOF OF LEMMA A.2: As in the proof of Lemma S.6, let T_x^i denote, for each $x < x_{R^i}$, the unique solution to $\mathcal{L}u - ru = 0$ that is tangent to R^i at x . Then $T_x^i \geq R^i$ over (x_{R^i}, β)

and $T_x^i \equiv A_x \phi + B_x \psi$ for some positive coefficients A_x and B_x .⁵ For each $z \geq \zeta(x_{R^i})$, let $\hat{T}_z^i \equiv \widehat{T_{\zeta^{-1}(z)}^i}$ be the affine function tangent to \hat{R}^i at z , which is given by

$$\hat{T}_z^i(y) = A_{\zeta^{-1}(z)}y + B_{\zeta^{-1}(z)} = \hat{R}^i(z) + \hat{R}'^i(z)(y - z), \quad y \in (0, \infty). \quad (\text{S.30})$$

Now, suppose, by way of contradiction, that $\chi_\infty \equiv \lim_{n \rightarrow \infty} \chi_n^1 = \lim_{n \rightarrow \infty} \chi_n^2 > \alpha$. Also suppose, with no loss of generality, that $\chi_1^1 > \chi_1^2$, and let $y_{2n-1} \equiv \zeta(\chi_n^1)$ and $y_{2n} \equiv \zeta(\chi_n^2)$ for all $n \geq 1$. Because $(\chi_n^i)_{n \geq 1}$ is a sequence in $\text{supp } \mu^i \cap (s, \beta)$ and, hence, in $(\alpha, x_{R^i}]$ by Proposition 1(iv), $(y_n)_{n \geq 1}$ is a sequence in $[\zeta(x_{R^i}), \infty)$. As in Step 3 of the proof of Theorem 2, that player 1 does not stop over the interval (χ_{n+1}^1, χ_n^1) and that $\chi_n^2 \in (\chi_{n+1}^1, \chi_n^1)$ belongs to the support of μ^2 implies that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (χ_{n+1}^1, χ_n^1) and that $\bar{J}^2 \geq V_{R^2}$ and $\bar{J}^2(\chi_n^2) = R^2(\chi_n^2)$. Moreover, as \bar{J}^2 is continuous, it coincides with $T_{\chi_n^2}^j$ on $[\chi_{n+1}^1, \chi_n^1]$. It follows that, for each $n \geq 1$, $\bar{J}^2(\chi_{n+1}^1) = T_{\chi_n^2}^j(\chi_{n+1}^1) = T_{\chi_{n+1}^2}^j(\chi_{n+1}^1)$, and a similar property holds for \bar{J}^1 . Using (S.30) to rewrite these equalities yields, for each $n \geq 1$,

$$\begin{aligned} \hat{R}^1(y_{2n-1}) + \hat{R}'^1(y_{2n-1})(y_{2n} - y_{2n-1}) &= \hat{R}^1(y_{2n+1}) + \hat{R}'^1(y_{2n+1})(y_{2n} - y_{2n+1}), \\ \hat{R}^2(y_{2n}) + \hat{R}'^2(y_{2n})(y_{2n+1} - y_{2n}) &= \hat{R}^2(y_{2n+2}) + \hat{R}'^2(y_{2n+2})(y_{2n+1} - y_{2n+2}). \end{aligned}$$

With $y < y' < y''$ three appropriate consecutive terms of the sequence $(y_n)_{n \geq 1}$, these equalities can be compactly rewritten for $i = 1, 2$ as

$$\hat{R}^i(y) + \hat{R}'^i(y)(y' - y) - \hat{R}^i(y') = \hat{R}^i(y'') + \hat{R}'^i(y'')(y' - y'') - \hat{R}^i(y'). \quad (\text{S.31})$$

Using Taylor's theorem with integral remainder, (S.31) is equivalent to

$$-\int_y^{y'} (y' - z) \hat{R}^{i''}(z) dz = -\int_{y'}^{y''} (z - y') \hat{R}^{i''}(z) dz. \quad (\text{S.32})$$

Because $\hat{R}^{i''} < 0$ over $[y_1, \infty) \subset [\zeta(x_{R^i}), \infty)$, the right-hand side of (S.32) is increasing in y'' . Therefore, given $y' > y \geq y_1$, if a solution $y'' > y'$ to (S.32) exists, it is unique. By assumption, $\lim_{n \rightarrow \infty} y_n = y_\infty \equiv \zeta(\chi_\infty) < \infty$. Moreover, because $\hat{R}^{i''}$ is locally Lipschitz by A8, there exists $K > 0$ such that $|\hat{R}^{i''}(z) - \hat{R}^{i''}(y')| \leq K|z - y'|$ for all $z, y' \in [y_1, y_\infty]$. Thus

$$-\int_y^{y'} (y' - z) \hat{R}^{i''}(z) dz \geq -R^{i''}(y') \frac{(y' - y)^2}{2} - K \frac{(y' - y)^3}{3}, \quad (\text{S.33})$$

$$-\int_{y'}^{y''} (z - y') \hat{R}^{i''}(z) dz \leq -R^{i''}(y') \frac{(y'' - y')^2}{2} + K \frac{(y'' - y')^3}{3}. \quad (\text{S.34})$$

By (S.32), we have

$$(y'' - y')^2 + \frac{2K}{3|\hat{R}^{i''}(y')|} (y'' - y')^3 \geq (y' - y)^2 - \frac{2K}{3|\hat{R}^{i''}(y')|} (y' - y)^3. \quad (\text{S.35})$$

⁵That A_x and B_x are positive follows from $x < x_{R^i}$ along the same lines as in Footnote 3.

Let C such that, for each $y' \in [y_1, y_\infty]$,

$$\frac{2K}{3|\hat{R}'''(y')|} \leq C.$$

Then, by (S.35), we have

$$(y'' - y')^2 + C(y'' - y')^3 \geq (y' - y)^2 - C(y' - y)^3.$$

Letting $u_n \equiv y_{n+1} - y_n$ for all $n \geq 1$, the upshot of the above analysis is that $h(u_{n+1}) \geq g(u_n)$, where $g(u) \equiv u^2 - Cu^3$ and $h(u) \equiv u^2 + Cu^3$. By assumption, $y_1 + \sum_{n \geq 1} u_n = y_\infty < \infty$, which implies that $\lim_{n \rightarrow \infty} u_n = 0$. Therefore, for n sufficiently large, $g(u_n) > 0$ and $u_{n+1} \geq h^{-1}(g(u_n))$, where h^{-1} denotes the inverse of h restricted to $[0, \infty)$. Because $h^{-1}(z) = \sqrt{z} - \frac{C}{2}z + o(z)$, we have $h^{-1}(g(u)) = u - Cu^2 + o(u^2)$. Hence

$$u_{n+1} \geq u_n - Cu_n^2 + o(u_n^2)$$

and, as a result,

$$\frac{1}{u_{n+1}} - \frac{1}{u_n} \leq \frac{1}{u_n} \left[\frac{1}{1 - Cu_n + o(u_n)} - 1 \right] = C + o(1).$$

We obtain

$$\frac{1}{u_n} = \frac{1}{u_1} + \sum_{k=1}^{n-1} \left(\frac{1}{u_{k+1}} - \frac{1}{u_k} \right) \leq nC + o(n)$$

and thus

$$u_n \geq \frac{1}{nC} + o\left(\frac{1}{n}\right),$$

so that $\sum_{n \geq 1} u_n = \infty$, a contradiction. The case of increasing sequences, whose limit must be in $(\alpha, x_{R^i}]$, can be dealt in a similar way by replacing the inequalities (S.33) and (S.34) by an upper bound and a lower bound of the same type, respectively. The result follows. ■

S.6 Proofs of Lemmas for Theorem 3

PROOF OF LEMMA A.3: From Proposition 1(v), if $((\mu^1, S^1), (\mu^2, S^2))$ is a MPE, then $(0, S^2)$ is a pbr to (μ^1, S^1) . Applying the strong Markov property (S.7) to the value function of player 2 associated to the pair of Markov strategies $((\mu^1, S^1), (0, S^2))$ yields, for all $x \in \mathcal{I}$ and $\tau \in \mathcal{T}$,

$$\begin{aligned} \bar{J}^2(x) = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} e^{-rt} G^2(q_n^1) \Lambda_t^1 a_n dL_t^{q_n^1} \right. \\ \left. + 1_{\{\tau_{S^2} < \tau\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 + 1_{\{\tau_{S^2} \geq \tau\}} e^{-r\tau} \bar{J}^2(X_\tau) \Lambda_\tau^1 \right], \end{aligned}$$

where we used that $d\Gamma_t^1 = \sum_{n=1}^N a_n \Lambda_t^1 dL_t^{q_n^1}$. This proves (A.5).

To prove (A.6), we apply the Itô–Tanaka–Meyer formula to $e^{-r(\tau \wedge \tau_{S^2} \wedge \tau_k)} \bar{J}^2(X_{\tau \wedge \tau_{S^2} \wedge \tau_k}) \Lambda_{\tau \wedge \tau_{S^2} \wedge \tau_k}^1$, where, for each $k \in \mathbb{N}$, $\tau_k \equiv \inf\{t \geq 0 : X_t \notin [\alpha_k, \beta_k]\}$ for some increasing sequence $([\alpha_k, \beta_k])_{k \in \mathbb{N}}$ of compact intervals of \mathcal{I} such that $\bigcup_{k \in \mathbb{N}} [\alpha_k, \beta_k] = \mathcal{I}$. Observe that $\mathbf{E}_x[\tau_k] < \infty$ (Karatzas and Shreve (1998, Chapter 5, Section 5, §C)) and that $X_t \in [\alpha_k, \beta_k]$ over $\{t \leq \tau_k\}$ \mathbf{P}_x -almost surely for all $x \in [\alpha_k, \beta_k]$. Moreover, because X does not explode in finite time, $\lim_{k \rightarrow \infty} \tau_k = \infty$ and, hence, $\lim_{k \rightarrow \infty} \tau \wedge \tau_k = \tau$ for all $\tau \in \mathcal{T}$. We obtain

$$\begin{aligned} \bar{J}^2(x) &= e^{-r(\tau \wedge \tau_{S^2} \wedge \tau_k)} \bar{J}^2(X_{\tau \wedge \tau_{S^2} \wedge \tau_k}) \Lambda_{\tau \wedge \tau_{S^2} \wedge \tau_k}^1 - \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \bar{J}^2(X_t) d\Lambda_t^1 \\ &\quad - \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} [\mathcal{L} \bar{J}^2(X_t) - r \bar{J}^2(X_t)] \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dt \\ &\quad - \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \sigma(X_t) \bar{J}^{2'}(X_t) \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dW_t \\ &\quad - \frac{1}{2} \sum_{n=1}^N \Delta \bar{J}^{2'}(q_n^1) \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1}. \end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned} \bar{J}^2(x) &= \mathbf{E}_x \left[e^{-r\tau \wedge \tau_{S^2} \wedge \tau_k} \bar{J}^2(X_{\tau \wedge \tau_{S^2} \wedge \tau_k}) \Lambda_{\tau \wedge \tau_{S^2} \wedge \tau_k}^1 - \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \bar{J}^2(X_t) d\Lambda_t^1 \right. \\ &\quad \left. - \frac{1}{2} \sum_{n=1}^N \Delta \bar{J}^{2'}(q_n^1) \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1} \right], \end{aligned}$$

where we have used the fact that \bar{J}^2 satisfies (45) and that

$$\mathbf{E}_x \left[\int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \sigma(X_t) \bar{J}^{2'}(X_t) \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dW_t \right] = 0. \quad (\text{S.36})$$

Indeed, notice that σ is continuous on I , and that $\bar{J}^2 \in \mathcal{C}^1(\mathcal{I} \setminus \{(q_n^1)_{1 \leq n \leq N}\})$ with $|\bar{J}^{2'}(x+)| < \infty$ and $|\bar{J}^{2'}(x-)| < \infty$ for $x \in \{q_n^1 : 1 \leq n \leq N\}$. Thus there exists $C_k > 0$ such that $|\sigma(X_t) \bar{J}^{2'}(X_t)| \leq C_k$ over $\{t \leq \tau_{S^2} \wedge \tau_k\}$ \mathbf{P}_x -almost surely, which implies (S.36). Hence

$$\begin{aligned} \bar{J}^2(x) &= \mathbf{E}_x [1_{\{\tau_{S^2} \geq \tau \wedge \tau_k\}} e^{-r\tau \wedge \tau_k} \bar{J}^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1] + \mathbf{E}_x [1_{\{\tau_{S^2} < \tau \wedge \tau_k\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1] \\ &\quad + \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \bar{J}^2(X_t) \Lambda_t^1 a_n dL_t^{q_n^1} \right] \\ &\quad - \mathbf{E}_x \left[\frac{1}{2} \sum_{n=1}^N \Delta \bar{J}^{2'}(q_n^1) \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1} \right]. \end{aligned}$$

Using that the measure $dL_t^{q_n^1}$ only charges the set $\{t \geq 0 : X_t = q_n^1\}$, we obtain

$$\bar{J}^2(x) = \mathbf{E}_x [1_{\{\tau_{S^2} \geq \tau \wedge \tau_k\}} e^{-r\tau \wedge \tau_k} \bar{J}^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1] + \mathbf{E}_x [1_{\{\tau_{S^2} < \tau \wedge \tau_k\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1]$$

$$+ \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right]. \quad (\text{S.37})$$

By the monotone convergence theorem,

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbf{E}_x \left[\int_{[0, \tau \wedge \tau_{S^2} \wedge \tau_k)} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right] \\ = \mathbf{E}_x \left[\int_{[0, \tau \wedge \tau_{S^2})} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right] \end{aligned}$$

for all n , and

$$\lim_{k \rightarrow \infty} \mathbf{E}_x \left[1_{\{\tau_{S^2} < \tau \wedge \tau_k\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 \right] = \mathbf{E}_x \left[1_{\{\tau_{S^2} < \tau\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 \right].$$

Because $0 \leq \bar{J}^2 \leq G^2$ by Proposition 1, it follows from A4 that the sequence $(1_{\{\tau \wedge \tau_k \leq \tau_{S^2}\}} e^{-r\tau \wedge \tau_k} \bar{J}^2(X_{\tau \wedge \tau_k}))_{k \in \mathbb{N}}$ is uniformly integrable. Therefore, by Vitali's convergence theorem,

$$\lim_{k \rightarrow \infty} \mathbf{E}_x \left[1_{\{\tau_{S^2} \geq \tau \wedge \tau_k\}} e^{-r(\tau \wedge \tau_k)} \bar{J}^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1 \right] = \mathbf{E}_x \left[1_{\{\tau_{S^2} \geq \tau\}} e^{-r\tau} \bar{J}^2(X_\tau) \Lambda_\tau^1 \right].$$

Finally, $1_{\{\tau_{S^2} \geq \tau \wedge \tau_k\}} e^{-r(\tau \wedge \tau_k)} \bar{J}^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1 = 1_{\{\tau_{S^2} \geq \tau_k\}} e^{-r\tau_k} \bar{J}^2(X_{\tau_k}) \Lambda_{\tau_k}^1$ over $\{\tau = \infty\}$. For k large enough, $x \in (\alpha_k, \beta_k)$. Hence

$$\begin{aligned} \mathbf{E}_x \left[1_{\{\tau_{S^2} \geq \tau_k\}} e^{-r\tau_k} \bar{J}^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \right] \\ \leq \mathbf{E}_x \left[1_{\{X_{\tau_k} = \alpha_k\}} e^{-r\tau_k} \bar{J}^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \right] + \mathbf{E}_x \left[1_{\{X_{\tau_k} = \beta_k\}} e^{-r\tau_k} \bar{J}^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \right] \\ \leq \mathbf{E}_x \left[e^{-r\tau_k} \bar{J}^2(\alpha_k) \Lambda_{\tau_{\alpha_k}}^1 \right] + \mathbf{E}_x \left[e^{-r\tau_k} \bar{J}^2(\beta_k) \Lambda_{\tau_{\beta_k}}^1 \right] \\ \leq \frac{\phi(x)}{\phi(\alpha_k)} G^2(\alpha_k) + \frac{\psi(x)}{\psi(\beta_k)} G^2(\beta_k). \end{aligned}$$

Because $\bar{J}^2 \geq 0$, it then follows from the growth properties (8) that

$$\lim_{k \rightarrow \infty} \mathbf{E}_x \left[1_{\{\tau_{S^2} \geq \tau_k\}} e^{-r\tau_k} \bar{J}^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \right] = 0.$$

Thus, letting k go to ∞ in (S.37) yields

$$\begin{aligned} \bar{J}^2(x) = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} e^{-rt} \left[\bar{J}^2(q_n^1) a_n - \frac{1}{2} \Delta \bar{J}^{2'}(q_n^1) \right] \Lambda_t^1 dL_t^{q_n^1} \right. \\ \left. + 1_{\{\tau_{S^2} < \tau\}} e^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda_{\tau_{S^2}}^1 + 1_{\{\tau_{S^2} \geq \tau\}} e^{-r\tau} \bar{J}^2(X_\tau) \Lambda_\tau^1 \right]. \end{aligned}$$

This shows (A.6). The result follows. \blacksquare

PROOF OF LEMMA A.4: Suppose, with no loss of generality, that $i = 2$ and $j = 1$. First,

let us observe that (S.7) leads to

$$J^2(x, (\mu^1, S^1), \tau) = \mathbf{E}_x \left[e^{-r\tau} R^2(X_\tau) \Lambda_\tau^1 + \sum_{n=1}^N \int_{[0, \tau)} e^{-rt} G^2(X_t) \Lambda_s^1 a_n dL_t^{q_n^1} \right].$$

Let w^2 be a solution to (44)–(50). Applying the Itô–Tanaka–Meyer formula to $e^{-r(\tau \wedge \tau_k)} w^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1$, with τ_k defined as in the proof of Lemma A.3, we obtain

$$\begin{aligned} w^2(x) &= e^{-r(\tau \wedge \tau_k)} w^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1 - \int_{[0, \tau \wedge \tau_k)} e^{-rt} w^2(X_t) d\Lambda_t^1 \\ &\quad - \int_{[0, \tau \wedge \tau_k)} e^{-rt} [\mathcal{L}w^2(X_t) - rw^2(X_t)] \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dt \\ &\quad - \int_{[0, \tau \wedge \tau_k)} e^{-rt} \sigma(X_t) w^{2'}(X_t) \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dW_t \\ &\quad - \frac{1}{2} \sum_{n=1}^N \Delta w^{2'}(q_n^1) \int_{[0, \tau \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1}. \end{aligned} \quad (\text{S.38})$$

From (46) and A3, we have $\mathcal{L}w^2 - rw^2 = \mathcal{L}R^2 - rR^2 \leq 0$ over $(\alpha, s^2) \subset (\alpha, x_{R^2}]$. It then follows from (45) that

$$\mathbf{E}_x \left[- \int_{[0, \tau \wedge \tau_k)} e^{-rt} [\mathcal{L}w^2(X_t) - rw^2(X_t)] \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 dt \right] \geq 0. \quad (\text{S.39})$$

Next, we have

$$\begin{aligned} &\mathbf{E}_x \left[- \frac{1}{2} \sum_{n=1}^N \Delta w^{2'}(q_n^1) \int_{[0, \tau \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1} \right] \\ &= \mathbf{E}_x \left[\sum_{n=1}^N a_n [G^2(q_n^1) - w^2(q_n^1)] \int_{[0, \tau \wedge \tau_k)} e^{-rt} \Lambda_t^1 dL_t^{q_n^1} \right] \\ &= \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_k)} e^{-rt} G^2(X_t) \Lambda_t^1 a_n dL_t^{q_n^1} - \sum_{n=1}^N \int_{[0, \tau \wedge \tau_k)} e^{-rt} w^2(X_t) \Lambda_t^1 a_n dL_t^{q_n^1} \right] \\ &= \mathbf{E}_x \left[\int_{[0, \tau \wedge \tau_k)} e^{-rt} G^2(X_t) d\Gamma_t^1 + \int_{[0, \tau \wedge \tau_k)} e^{-rt} w^2(X_t) d\Lambda_t^1 \right], \end{aligned} \quad (\text{S.40})$$

where the first equality follows from (49), the second equality follows from the fact that the measure $dL_t^{q_n^1}$ only charges the set $\{t \geq 0 : X_t = q_n^1\}$, and the third equality follows from the representation (15). We obtain from (S.38)–(S.40) that

$$\begin{aligned} w^2(x) &\geq \mathbf{E}_x \left[e^{-r(\tau \wedge \tau_k)} w^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1 + \int_{[0, \tau \wedge \tau_k)} e^{-rt} G^2(X_t) d\Gamma_t^1 \right] \\ &\geq \mathbf{E}_x \left[e^{-r(\tau \wedge \tau_k)} R^2(X_{\tau \wedge \tau_k}) \Lambda_{\tau \wedge \tau_k}^1 + \int_{[0, \tau \wedge \tau_k)} e^{-rt} G^2(X_t) d\Gamma_t^1 \right], \end{aligned}$$

where the first inequality follows from the fact that the stochastic integral in (S.38) is a centered square-integrable random variable as shown in the proof of Lemma A.3, and the second inequality follows from (44). Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x) \geq \mathbf{E}_x \left[e^{-r\tau} R^2(X_\tau) \Lambda_\tau^1 + \int_{[0,\tau)} e^{-rt} G^2(X_t) d\Gamma_t^1 \right] = J^2(x, (\mu^1, S^1), \tau),$$

where the equality follows from (12). Taking the supremum over $\tau \in \mathcal{T}$ yields (A.8).

To establish (A.9), we apply the Itô–Tanaka–Meyer formula to $e^{-r\tau_k} w^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \Lambda_{\tau_k-}^2$. Taking expectations, we obtain

$$\begin{aligned} w^2(x) = \mathbf{E}_x \left[e^{-r\tau_k} w^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \Lambda_{\tau_k-}^2 - \int_{[0,\tau_k)} e^{-rt} w^2(X_t) \Lambda_{t-}^2 d\Lambda_t^1 \right. \\ \left. - \int_{[0,\tau_k)} e^{-rt} w^2(X_t) \Lambda_t^1 d\Lambda_t^2 - \frac{1}{2} \sum_{n=1}^N \Delta w^{2'}(q_n^1) \int_{[0,\tau_k)} e^{-rt} \Lambda_t^1 \Lambda_{t-}^2 dL_t^{q_n^1} \right], \end{aligned} \quad (\text{S.41})$$

where, as in the proof of Lemma A.3, we have used that

$$\mathbf{E}_x \left[\int_{[0,\tau_k)} e^{-rt} \sigma(X_t) w^{2'}(X_t) \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 \Lambda_{t-}^2 dW_s \right] = 0$$

and that

$$\mathbf{E}_x \left[\int_{[0,\tau \wedge \tau_k)} e^{-rt} [\mathcal{L} \bar{w}^2(X_t) - r w^2(X_t)] \prod_{n=1}^N 1_{\{X_t \neq q_n^1\}} \Lambda_t^1 \Lambda_{t-}^2 dt \right] = 0,$$

which follows from (45) and from the fact that $\Lambda_{t-}^2 = 1_{\{t \leq \tau_{S^2}\}} e^{-\int_X L_t^y \mu^2(dy)}$ vanishes over $\{X_t < s^2\}$. Now, using that the measure $d\Gamma_t^2$ only charges the set $\{t \geq 0 : w^2(X_t) = R^2(X_t)\}$, we have

$$\begin{aligned} \mathbf{E}_x \left[- \int_{[0,\tau_k)} e^{-rt} w^2(X_t) \Lambda_t^1 d\Lambda_t^2 \right] &= \mathbf{E}_x \left[\int_{[0,\tau_k)} e^{-rt} w^2(X_t) \Lambda_t^1 d\Gamma_t^2 \right] \\ &= \mathbf{E}_x \left[\int_{[0,\tau_k)} e^{-rt} R^2(X_t) \Lambda_t^1 d\Gamma_t^2 \right]. \end{aligned} \quad (\text{S.42})$$

Next, using (43), and following the same steps as for (S.40), we have

$$\begin{aligned} \mathbf{E}_x \left[- \frac{1}{2} \sum_{n=1}^N \Delta w^{2'}(q_n^1) \int_{[0,\tau_k)} e^{-rt} \Lambda_t^1 \Lambda_{t-}^2 dL_t^{q_n^1} \right] \\ = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0,\tau_k)} e^{-rt} G^2(q_n^1) \Lambda_t^1 \Lambda_{t-}^2 a_n dL_t^{q_n^1} - \sum_{n=1}^N \int_{[0,\tau_k)} e^{-rt} w^2(q_n^1) \Lambda_t^1 \Lambda_{t-}^2 a_n dL_t^{q_n^1} \right] \\ = \mathbf{E}_x \left[\int_{[0,\tau_k)} e^{-rt} G^2(X_t) \Lambda_{t-}^2 d\Gamma_t^1 + \int_{[0,\tau_k)} e^{-rt} w^2(X_t) \Lambda_{t-}^2 d\Lambda_t^1 \right]. \end{aligned} \quad (\text{S.43})$$

We obtain from (S.41)–(S.43) that

$$w^2(x) = \mathbf{E}_x \left[e^{-r\tau_k} w^2(X_{\tau_k}) \Lambda_{\tau_k}^1 \Lambda_{\tau_k-}^2 \right. \quad (\text{S.44})$$

$$\left. + \int_{[0, \tau_k)} e^{-rt} R^2(X_t) \Lambda_t^1 d\Gamma_t^2 + \int_{[0, \tau_k)} e^{-rt} G^2(X_t) \Lambda_{t-}^2 d\Gamma_t^1 \right]. \quad (\text{S.45})$$

Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x) = \mathbf{E}_x \left[\int_{[0, \infty)} e^{-rt} R^2(X_t) \Lambda_t^1 d\Gamma_t^2 + \int_{[0, \infty)} e^{-rt} G^2(X_t) \Lambda_{t-}^2 d\Gamma_t^1 \right] = J^2(x, (\mu^1, S^1), (\mu^2, S^2)),$$

where the equality follows from (12). The result follows. \blacksquare

References

- BLUMENTHAL, ROBERT M., AND RONALD K. GETTOOR (1968): *Markov Processes and Potential Theory*. New York: Academic Press.
- BORODIN, ANDREI N., AND PAAVO SALMINEN (2002): *Handbook of Brownian Motion—Facts and Formulae*. Basel, Boston, Berlin: Birkhäuser Verlag.
- ÇETIN, UMUT (2003): “Integral Representation of Subharmonic Functions and Optimal Stopping with Random Discounting,” arXiv:1809.08029.
- DAYANIK, SAVAS, AND IOANNIS KARATZAS (2003): “On the Optimal Stopping Problem for One-Dimensional Diffusions,” *Stochastic Processes and their Applications*, 107 (2), 173–212.
- DE ANGELIS, TIZIANO, GIORGIO FERRARI, AND JOHN MORIARTY (2018): “Nash Equilibria of Threshold Type for Two-Player Nonzero-Sum Games of Stopping,” *Annals of Applied Probability*, 28 (1), 112–147.
- DÉCAMPS, JEAN-PAUL, FABIEN GENSBITTEL, AND THOMAS MARIOTTI (2024): “Investment Timing and Technological Breakthroughs,” *Mathematics of Operations Research*, forthcoming.
- DIXIT, AVINASH K., AND ROBERT S. PINDYCK (1994): *Investment under Uncertainty*. Princeton: Princeton University Press.
- JACOD, JEAN, AND ALBERT N. SHIRYAEV (2003): *Limit Theorems for Stochastic Processes*. Berlin, Heidelberg, New York: Springer-Verlag.

KARATZAS, IOANNIS, AND STEVEN E. SHREVE (1991): *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.

REVUZ, DANIEL, AND MARC YOR (1999): *Continuous Martingales and Brownian Motion*. Berlin, Heidelberg, New York: Springer-Verlag.

SOLAN, EILON, BORIS TSIRELSON, AND NICOLAS VIEILLE (2012): “Random Stopping Times in Stopping Problems and Stopping Games,” arXiv:1211.5802.