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# Submodular functions in additive combinatorics problems for group actions and representations

Vincent Beck and Cédric Lecouvey

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## Abstract

We establish analogues in the context of group actions or group representations of some classical problems and results in additive combinatorics of groups. We also study the notion of left invariant submodular function defined on power sets which plays a central role in our proofs.

## 1 Introduction

Consider a multiplicative group  $G$  acting on the left on a non empty set  $X$ . When  $A$  and  $Y$  are respectively finite nonempty subsets of  $G$  and  $X$  what can be said about the cardinality  $|A \cdot Y|$  of the set  $A \cdot Y = \{a \cdot y \mid (a, y) \in A \times Y\}$ ? Here  $a \cdot y$  means the image of  $y$  under the action of  $a$ . When  $X = G$  and the action considered is the action by left multiplication (thus  $a \cdot y = ay$  the product of the two elements in the group  $G$ ), this question relates to additive (or here multiplicative) combinatorics on groups and there exists in the literature numerous results yielding lower and upper bounds for the cardinality of the Kronecker product set  $AY$  (see for example [16] and [19]). Among them, Kneser's theorem is a corner stone claiming that in any abelian group

$$|G_{AY}| + |AY| \geq |A| + |Y|$$

where  $G_{AY} = \{g \in G \mid gAY = AY\}$  is the stabilizer of the product set  $AY$ . This theorem does not remain true for non abelian groups even it is not immediate to find a simple counterexample. Therefore, if we consider  $G_{A \cdot Y} = \{g \in G \mid g \cdot (A \cdot Y) = A \cdot Y\}$ , the inequality

$$|G_{A \cdot Y}| + |A \cdot Y| \geq |A| + |Y| \tag{1}$$

*does not hold* in the general left action context. In contrast, it is very easy to find a counterexample by considering the action of the symmetric group  $\mathfrak{S}_n$  on the set  $\{1, \dots, n\}$  (see Example 2.1).

Although Kneser's theorem does not have an immediate generalization in the group action context, we shall see in this paper that it is nevertheless possible to obtain interesting analogues of various other results in this setting, most of them being inspired by results or tools coming from additive combinatorics for non abelian groups. Among them is the notion of submodular function defined on subsets of  $G$  or subsets of  $X$ . In fact, we will often obtain two different families of statements by fixing  $Y$  and letting  $A$  running on  $\mathcal{P}(G)$  (the power set of  $G$ ) or fixing  $A$  and letting  $Y$  running on  $\mathcal{P}(X)$  (the power set of  $X$ ). This is for example the case for Theorems 5.6 and 5.9 which both are declinations of the same theorem proved by Tao in [20] for product sets in general groups. Even if the context of group action studied in this paper presents some analogies with the combinatorics of groups (i.e. the case of an action by multiplication), it is worth mentioning that there are important differences. Maybe the most important comes from the fact that the multiplication in a group can be performed on the left and on the right whereas a group action is only one-hand sided. This makes many classical tools like the Dyson or Diderrich transforms on subsets of groups (see for example [16] and [3]) irrelevant for group actions.

The present paper can also be regarded as a contribution of the general project to extend methods developed in additive combinatorics of groups to more general contexts. In the linear context, where the cardinalities of sets are replaced by the dimensions of vectors spaces, this was initiated in [8] for field extensions and developed in particular in [4],[12], [1] (for fields and division rings) and in [2] and [15] (for associative algebras). As far as we are aware, the group action setting presented in this paper was first considered very more recently in [13] and [14] in connection with the notion of approximate groups. Our approach here, based on tools coming from group theory and on the notion of submodular function, is different. Most often, we are also able to state linear analogues of our results where group actions on finite sets are replaced by finite-dimensional group representations.

Let us now describe more precisely the content of the paper. Section 2 is devoted to a general overview of additive combinatorics type problems in the context of group actions and group representations. In particular, we explain how the problem of determining lower and upper bounds for the previous cardinality  $|A \cdot Y|$  can theoretically be reduced to the classical group setting when sufficiently many information on the orbit decompositions and the stabilizers of the elements is available. This is for example the case for free actions. We also consider the particular case of a faithful action which gives a direct counterexample to the inequality (1). Nevertheless in general, this reduction is not easy to perform and does not yield statements so simple and elegant as in the group setting. The further sections give examples of such results. Section 3 focuses on the notion of submodular function with various defining sets relevant for our group action and group representation context. We define in particular a natural analogue of the classical graph cut submodular function. Section 4 relies on Hamidoune's notions of fragments and atoms (see [7]) also developed by Tao in [20]. In particular Proposition 4.3 gives informations on the structure of atoms associated to a left regular submodular function defined on  $\mathcal{P}(X)$ . In Section 5, we state and prove the analogues of theorems by Hamidoune, Tao and Petridis in our group action and group representation setting which are at the heart of this paper. In Section 6, we establish complementary results more in the spirit of the paper [13] by Murphy. Finally in Section 7, we give an analogue of a theorem by Ruzsa for the action of a product set  $AB$  in the group  $G$  on a subset  $Y$  of  $X$ .

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## 2 Additive combinatorics problems in the context of group actions and representations

### 2.1 Group actions and representations

In the sequel we consider  $G$  a discrete group and  $X$  a set on which  $G$  acts. As usual, for any  $(g, x) \in G \times X$ , we shall denote by  $g \cdot x$  the element of  $X$  corresponding to the action of  $g$  on  $x$ . Let us write

$$G_x = \{g \in G \mid g \cdot x = x\}$$

for the stabilizer of  $x$  in  $G$ . For any subset  $Y \subset X$  and any  $g \in G$ , set  $g \cdot Y = \{g \cdot y \mid y \in Y\}$ . Let

$$G_Y = \{g \in G \mid g \cdot Y = Y\}$$

be the stabilizer of  $Y$  in  $G$ . Observe that for any fixed  $g \in G$ , the map

$$\begin{cases} X \rightarrow X \\ x \mapsto g \cdot x \end{cases} \quad (2)$$

is bijective. In particular, for any finite subset  $Y \subset X$ , we have  $|g \cdot Y| = |Y|$ , that is the sets  $g \cdot Y$  and  $Y$  have the same cardinality.

For any subset  $A \subset G$  and  $Y \subset X$ , define

$$A \cdot Y = \{a \cdot y \mid (a, y) \in A \times Y\}.$$

In the sequel, we will study lower and upper bounds for the cardinality  $|A \cdot Y|$  when  $A$  and  $Y$  are supposed finite. In the particular case  $X = G$  and  $G$  acts on itself by left translation, we recover the classical problem in additive combinatorics of determining lower and upper bounds for Minkowski products of finite subsets of an ambient group.

It will also be interesting to replace the set  $X$  by its linear analogue, that is to consider a representation  $(\rho, V)$  of the group  $G$  instead of an action of  $G$  on  $X$ . Recall that a representation  $(\rho, V)$  is a morphism of groups

$$\rho : G \rightarrow GL(V)$$

where  $V$  is a finite-dimensional vector space over a given field  $k$ . This can essentially be thought as a linear action of  $G$  on the vector space  $V$  and we will write  $g \cdot v$  the action of any element  $g \in G$  on any vector  $v \in V$ . We thus have for any  $(\lambda_1, \lambda_2) \in k^2$  and any  $(v_1, v_2) \in V^2$

$$g \cdot (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (g \cdot v_1) + \lambda_2 (g \cdot v_2).$$

For any subset  $Z$  in  $V$ , we denote by  $\langle Z \rangle$  the  $k$ -subspace of  $V$  generated by the vectors in  $Z$ . We then write for short  $\dim(Z)$  instead of  $\dim(\langle Z \rangle)$ . Given any  $k$ -subspace  $W$  of  $V$  and any subset  $A$  of  $G$ , the set

$$A \cdot W = \langle a \cdot v \mid (a, v) \in A \times W \rangle.$$

We will study its dimension  $\dim(A \cdot W)$  in terms of  $\dim(W)$  and  $|A|$ .

## 2.2 Orbit decomposition method for a group action

In this paragraph, we will assume that the set  $X$  is finite. Given an element  $x$  in  $X$ , we denote by  $O_x = \{g \cdot x \mid g \in G\}$  its orbit. Let us fix  $x_1, \dots, x_r$  in  $X$  so that

$$X = \bigsqcup_{i=1}^r O_{x_i}$$

is the disjoint union of the orbits  $O_{x_i}, i = 1, \dots, r$ . It is classical that for any  $i = 1, \dots, r$  the map

$$\phi_i : \begin{cases} G/G_{x_i} \rightarrow O_{x_i} \\ gG_{x_i} \mapsto g \cdot x_i \end{cases}$$

is well-defined and bijective. Assume now that we have fixed a representative  $g[i]$  in each left coset  $gG_{x_i}$  of  $G/G_{x_i}$ . Also for any subset  $S$  in  $G$ , write  $|S|_i$  for the cardinality of the set of cosets  $\{gG_{x_i} \mid g \in S\}$  which is the same as the cardinality of the set  $S[i] = \{g[i] \mid g[i] \in S\}$ .

For any subset  $Y \subset X$ , write  $Y_i = Y \cap O_{x_i}$ . Then, we have for any subset  $A \subset G$

$$A \cdot Y = \bigsqcup_{i=1}^r A \cdot Y_i.$$

Finally, for any  $i = 1, \dots, r$ , we get by setting  $B_i = \{g[i] \mid gG_{x_i} \in \phi_i^{-1}(Y_i)\}$  the equalities  $|Y_i| = |B_i|$ ,  $|A \cdot Y_i| = |AB_i|_i$  and

$$|A \cdot Y| = \sum_{i=1}^r |AB_i|_i.$$

Therefore, the problem of studying the cardinality of  $|A \cdot Y|$  can be formally reduced to the problem of studying first each product sets  $AB_i$  in the group  $G$  and next the number  $|AB_i|_i$  of left cosets attained by the elements of  $AB_i$ . Since we have

$$\frac{|AB_i|}{|G_{x_i}|} \leq |AB_i|_i \leq |AB_i|, i = 1, \dots, r$$

we get

$$\sum_{i=1}^r \frac{|AB_i|}{|G_{x_i}|} \leq |A \cdot Y| \leq \sum_{i=1}^r |AB_i|$$

which theoretically reduces the question to classical estimations of product sets in groups which is largely addressed in the literature. In particular, when the action is simply transitive (that is when there is only one orbit and each stabilizer is trivial), both problems are equivalent. When the action is free (each stabilizer is trivial) we just get

$$|A \cdot Y| = \sum_{i=1}^r |AB_i|$$

so that the study of  $|A \cdot Y|$  can be performed from the group setting up to the determination of the orbits of the action of  $G$  on  $X$ . Nevertheless, in the general case, in addition to the orbit decomposition, this method imposes to have many information on the different stabilizers, their associated left cosets and the maps  $\phi_i, i = 1, \dots, r$  and do not yield statements so simple and elegant as in the group setting. In the sequel, we will see that it is in fact possible to overcome the orbit decomposition method to get in the group action setting results of comparable complexity to their group setting analogues.

### 2.3 Dual problem

Assume that the group  $G$  acts on the set  $X$  and consider a finite subset  $Z$  in  $X$ . Let

$$\mathcal{D}_Z = \{(A, Y) \in \mathcal{P}(G) \times \mathcal{P}(Y) \mid A \cdot Y = Z\}.$$

Observe first that  $(\{1\} \times Z)$  belongs to  $\mathcal{D}_Z$  which is thus nonempty. For the left action of an abelian group  $G$  on itself, Kneser's theorem (1) can be interpreted as an upper bound for the sum  $|A| + |Y|$  when  $(A, Y)$  runs over  $\mathcal{D}_Z$

$$\max_{(A, Y) \in \mathcal{D}_Z} (|A| + |Y|) \leq |G_Z| + |Z|.$$

In the general case, it is thus natural to ask for a subset  $U_Z$  in  $G$  depending only on  $Z$  and such that

$$\max_{(A, Y) \in \mathcal{D}_Z} (|A| + |Y|) \leq |U_Z| + |Z|.$$

In the sequel, we will study the more tractable following situation. Fix the subsets  $Y \subset X, A_0 \subset G$  and  $Z = A_0 \cdot Y$ . Next consider the set

$$\kappa_Y(Z) = \{g \in G \mid g \cdot Y \subset Z\} \supset A_0.$$

Clearly, for any  $g \in \kappa_Y(Z)$ , we have  $G_Z g \subset \kappa_Y(Z)$ . Hence, the set  $\kappa_Y(Z)$  is a disjoint union of right cosets of  $G_Z \setminus G$ . Then we get

$$\max_{A \subset G \mid A \cdot Y = Z} |A| + |Y| \leq |\kappa_Y(Z)| + |Y| \leq |\kappa_Y(Z)| + |Z|$$

since  $|Z| = |A \cdot Y| \geq |Y|$  and  $A \subset \kappa_Y(Z)$ . The first bound is sharp because  $\kappa_Y(Z) \cdot Y = Z$ . In general, the set  $\kappa_Y(Z)$  is not a subgroup (or a translated of a subgroup of  $G$ ). We will see in Section 5 how the study of affine maps of the form  $|A| \mapsto \lambda|A| + |Y|$  (instead of  $|A| \mapsto |A| + |Y|$ ) where  $\lambda$  is a positive real permits to make appear bounds related to subgroups of  $G$  more in the spirit of Kneser's theorem.

### 2.4 Faithful action

Recall that the action of  $G$  on  $X$  is said faithful when the group homomorphism

$$\left\{ \begin{array}{l} G \rightarrow \mathfrak{S}(X) \\ g \mapsto \left\{ \begin{array}{l} X \rightarrow X \\ x \mapsto g \cdot x \end{array} \right. \end{array} \right.$$

is injective. Here  $\mathfrak{S}(X)$  is the group of bijections of  $X$ . When  $X$  is finite, we can then assume that  $X = \{1, \dots, n\}$  and  $G$  is then a subgroup of  $\mathfrak{S}_n$  the symmetric group on  $\{1, \dots, n\}$ . Thus, if we consider  $Y \subset \{1, \dots, n\}$  and  $A \subset G \subset \mathfrak{S}_n$ , we get

$$A \cdot Y = \bigcup_{\sigma \in A} \sigma(Y).$$

**Example 2.1** Assume  $Y = \{1, \dots, k\}$ ,  $G = \mathfrak{S}_n$  and consider  $n > \ell \geq k$ . Let  $A_0$  be the set of permutations  $\sigma \in \mathfrak{S}_n$  such that  $\sigma(\{1, \dots, k\}) \subset \{1, \dots, \ell\}$ . One easily checks that

$$|A_0| = \frac{\ell!}{(\ell - k)!} (n - k)!.$$

Now, observe that  $A_0 \cdot Y = \{1, \dots, \ell\}$  and the stabilizer  $\mathfrak{S}_{A_0 \cdot Y}$  of  $A_0 \cdot Y$  thus has cardinality

$$|\mathfrak{S}_{A_0 \cdot Y}| = \ell! \times (n - \ell)!.$$

We so get

$$|A_0 \cdot Y| + |\mathfrak{S}_{A_0 \cdot Y}| \geq |Y| + |A_0| \iff \ell + \ell! \times (n - \ell)! \geq k + \frac{\ell!}{(\ell - k)!} (n - k)! \iff \frac{\ell - k}{\ell!(n - \ell)!} \geq \binom{n - k}{n - \ell} - 1$$

which can only hold when  $\ell = k$  for otherwise

$$\frac{\ell - k}{\ell!(n - \ell)!} < 1 \text{ and } \binom{n - k}{n - \ell} - 1 \geq 1.$$

In particular, when  $\ell > k$ , the inequality (1) does not hold. When  $k = \ell$ , the set  $A_0$  is a group isomorphic to the direct product  $\mathfrak{S}_k \times \mathfrak{S}_{n - k}$  and we get  $A_0 \cdot Y = Y$  with  $\mathfrak{S}_{A_0 \cdot Y} = A_0$  in which case (1) becomes an equality.

## 3 Submodular functions

### 3.1 Background

Consider a set  $S$  (in the sequel  $S$  could be a group  $G$  or the set  $X$  on which  $G$  acts). Let  $\mathcal{P}(S)$  be the power set of  $S$ .

**Definition 3.1** The map  $f : \mathcal{P}(S) \rightarrow \mathbb{R}$  is said submodular when

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B) \tag{3}$$

for any subsets  $A$  and  $B$  in  $\mathcal{P}(S)$ .

The submodular function  $f$  is said

- increasing when  $f(A) \leq f(B)$  for any subsets  $A \subset B \subset S$ ,
- symmetric when  $f(S \setminus A) = f(A)$  for any subset  $A \subset S$ .

Very often, we shall consider submodular functions defined on the set  $\mathcal{P}_{\text{fin}}(S)$  of finite subsets in  $S$  rather than on  $\mathcal{P}(S)$ . When  $S$  is finite, one can check that  $f$  is submodular if and only if for any subsets  $A_1 \subset A_2$  of  $\mathcal{P}(S)$  and any  $s \in S \setminus A_2$ , we have

$$f(A_1 \cup \{s\}) - f(A_1) \geq f(A_2 \cup \{s\}) - f(A_2). \tag{4}$$

Now, consider  $V$  a vector space over the field  $k$ . For any  $S \subset V$ , write  $\langle S \rangle$  for the  $k$ -vector space generated by  $S$  in  $V$ .

**Definition 3.2** The map  $f : \mathcal{P}(V) \rightarrow \mathbb{R}$  is said  $k$ -submodular when it is submodular and for any subset  $S$  of  $V$ , we have

$$f(S) = f(\langle S \rangle),$$

that is,  $f$  is constant on all the generating subsets of a given  $k$ -subspace of  $V$ .

Let us now introduce examples of submodular functions relevant for our purposes.

### 3.2 Group action and graph cut type submodular function

Let  $G$  be a finite group acting on the finite set  $X$ . For any subset  $Y \subset X$ , set

$$E_Y = \{(g, y) \in G \times Y \mid g \cdot y \notin Y\}.$$

Consider the cut function

$$f : \begin{cases} \mathcal{P}(X) \rightarrow \mathbb{Z}_{\geq 0} \\ Y \mapsto |E_Y| \end{cases} \quad (5)$$

**Proposition 3.3** The previous function  $f$  is submodular and nonnegative. Moreover, for any  $g \in G$  and any  $Y \in \mathcal{P}(X)$ , we have  $f(g \cdot Y) = f(Y)$ .

**Proof.** Consider  $Y_1$  and  $Y_2$  subsets of  $X$  such that  $Y_1 \subset Y_2$  and  $y_0 \in X \setminus Y_2$ . Then

$$E_{Y_1 \cup \{y_0\}} = \{(g, y) \in G \times Y_1 \mid g \cdot y \notin Y_1\} \setminus \{(g, y) \in G \times Y_1 \mid g \cdot y = y_0\} \sqcup \{(g, y_0) \mid g \cdot y_0 \notin Y_1 \cup \{y_0\}\}.$$

This gives

$$f(E_{Y_1 \cup \{y_0\}}) = f(E_{Y_1}) + |\{(g, y_0) \mid g \cdot y_0 \notin Y_1 \cup \{y_0\}\}| - |G_{y_0}| |\mathcal{O}_{y_0} \cap Y_1|.$$

Similarly, we have

$$f(E_{Y_2 \cup \{y_0\}}) = f(E_{Y_2}) + |\{(g, y_0) \mid g \cdot y_0 \notin Y_2 \cup \{y_0\}\}| - |G_{y_0}| |\mathcal{O}_{y_0} \cap Y_2|.$$

Now, the assumption  $Y_1 \subset Y_2$  implies the set inclusions

$$\{(g, y_0) \mid g \cdot y_0 \notin Y_2 \cup \{y_0\}\} \subset \{(g, y_0) \mid g \cdot y_0 \notin Y_1 \cup \{y_0\}\} \text{ and } \mathcal{O}_{y_0} \cap Y_1 \subset \mathcal{O}_{y_0} \cap Y_2.$$

This gives

$$f(E_{Y_1 \cup \{y_0\}}) - f(E_{Y_1}) \geq f(E_{Y_2 \cup \{y_0\}}) - f(E_{Y_2})$$

and  $f$  is submodular by (4). Moreover, the function  $f$  is clearly nonnegative. Finally, for any  $Y \subset X$  and any  $g_0 \in G$  the map

$$\chi_{g_0} : \begin{cases} E_Y \rightarrow E_{g_0 \cdot Y} \\ (g, y) \mapsto (g_0 g g_0^{-1}, g_0 \cdot y) \end{cases}$$

is a bijection which implies that  $f(g_0 \cdot Y) = f(Y)$  as desired. ■

**Remark 3.4** When the action of  $G$  on  $X$  is free, it can be represented by an oriented graph  $\Gamma = (X, E)$  with set of vertices  $X$  and set of arrows  $x \rightarrow x'$  when there exists  $g \in G$  such that  $x' = g \cdot x$ . Observe that such an element  $g$  is then unique by assumption. Then the previous function  $f$  becomes the cut function of  $\Gamma$  which is classical in graph theory and known to be submodular.

### 3.3 Action on a fixed set or subspace

As before assume  $G$  acts on  $X$  and fix  $Y$  a finite subset of  $X$ . Let  $\mathcal{P}_{\text{fin}}(G)$  be the subset of finite subsets in  $G$ . Then the map

$$f_Y : \begin{cases} \mathcal{P}_{\text{fin}}(G) \rightarrow \mathbb{Z}_{\geq 0} \\ A \mapsto |A \cdot Y| \end{cases}$$

is increasing submodular. Indeed, we have for any finite subsets  $A$  and  $B$

$$(A \cap B) \cdot Y \subset (A \cdot Y) \cap (B \cdot Y) \text{ and } (A \cup B) \cdot Y = (A \cdot Y) \cup (B \cdot Y).$$

Similarly, when  $(\rho, V)$  is a linear representation of  $G$  and  $W$  a fixed subspace of  $V$ , the map

$$\phi_W : \begin{cases} \mathcal{P}_{\text{fin}}(G) \rightarrow \mathbb{Z}_{\geq 0} \\ A \mapsto \dim(A \cdot W) \end{cases}$$

is increasing submodular because

$$\langle (A \cap B) \cdot W \rangle \subset \langle A \cdot W \rangle \cap \langle B \cdot W \rangle \text{ and } \langle (A \cup B) \cdot W \rangle = \langle A \cdot W \rangle + \langle B \cdot W \rangle.$$

### 3.4 Action of a fixed subset in a group

When  $G$  acts on  $X$  and  $A$  is a fixed finite subset of  $G$ , we can alternatively consider the map

$$f_A : \begin{cases} \mathcal{P}_{\text{fin}}(X) \rightarrow \mathbb{Z}_{\geq 0} \\ Y \mapsto |A \cdot Y| \end{cases}$$

defined on the set  $\mathcal{P}_{\text{fin}}(X)$  of finite subsets of  $X$ . This gives yet a submodular function since for any  $Y, Z$  in  $\mathcal{P}_{\text{fin}}(X)$ , we have

$$A \cdot (Y \cap Z) \subset (A \cdot Y) \cap (A \cdot Z) \text{ and } A \cdot (Y \cup Z) = (A \cdot Y) \cup (A \cdot Z).$$

Also, if we have a representation  $(\rho, V)$  of  $G$  we can define similarly the map

$$\phi_A : \begin{cases} \mathcal{P}_{k\text{fin}}(X) \rightarrow \mathbb{Z}_{\geq 0} \\ Y \mapsto \dim(A \cdot Y) \end{cases}$$

where  $\langle A \cdot Y \rangle$  is the  $k$ -vector space generated by  $A \cdot Y$  and  $\mathcal{P}_{k\text{fin}}(V)$  is the set of subsets  $Y$  of  $V$  contained in a finite-dimensional subspace. This gives a  $k$ -submodular function since we have  $\dim(A \cdot Y_1) = \dim(A \cdot Y_2)$  for any two subsets  $Y_1$  and  $Y_2$  in  $\mathcal{P}_{k\text{fin}}(X)$  such that  $\langle Y_1 \rangle = \langle Y_2 \rangle$ .

### 3.5 Combinations of submodular functions

The following property is easy to check.

**Proposition 3.5** *The set of nonnegative submodular functions defined from a set  $S$  is a cone: given  $f$  and  $g$  nonnegative submodular on  $\mathcal{P}(S)$  and  $(\lambda, \mu) \in \mathbb{R}_{\geq 0}$ , the map*

$$\lambda f + \mu g$$

*is yet submodular nonnegative.*

Now assume  $f$  is submodular ( $f$  is not assumed nonnegative here) and  $u$  is a modular map defined on  $\mathcal{P}(S)$ , that is satisfying

$$u(A \cup B) + u(A \cap B) = u(A) + u(B).$$

Then, we also get the following proposition.



**Proposition 3.6** For any real  $\lambda \in \mathbb{R}$ , the map  $f - \lambda u$  is submodular on  $\mathcal{P}(S)$ .

**Proof.** Consider  $A$  and  $B$  subsets of  $S$ . We have

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \quad (6)$$

because  $f$  is submodular and

$$\lambda u(A \cup B) + \lambda u(A \cap B) = \lambda u(A) + \lambda u(B)$$

by definition of the map  $u$ . By subtracting the last inequality to (6), we get the submodularity of  $f - \lambda u$ . ■

**Remark 3.7** The two previous propositions remain true if the notion of submodular functions defined on  $\mathcal{P}(S)$  is replaced by that of  $k$ -submodular functions defined on  $\mathcal{P}(V)$  where  $V$  is a  $k$ -vector space.

**Example 3.8** We resume the notation of the previous section.

1. When  $G$  acts on  $X$  and  $Y \subset X$  is a fixed finite subset of  $X$  or  $(\rho, V)$  is a representation of  $G$  and  $Y \subset V$  is a fixed subset of  $V$  such that  $\dim(Y)$  is finite, the maps

$$c_Y : \begin{cases} \mathcal{P}_{\text{fin}}(G) \rightarrow \mathbb{R} \\ A \mapsto |A \cdot Y| - \lambda |A| \end{cases} \quad \text{and} \quad \gamma_Y : \begin{cases} \mathcal{P}_{\text{fin}}(G) \rightarrow \mathbb{R} \\ A \mapsto \dim(A \cdot Y) - \lambda |A| \end{cases}$$

are submodular for any real  $\lambda$ .

2. When  $G$  acts on  $X$  or  $(\rho, V)$  is a representation of  $G$  and  $A$  is a fixed finite subset of  $G$ , the maps

$$d_A : \begin{cases} \mathcal{P}_{\text{fin}}(X) \rightarrow \mathbb{R} \\ Y \mapsto |A \cdot Y| - \lambda |Y| \end{cases} \quad \text{and} \quad \delta_A : \begin{cases} \mathcal{P}_{k\text{fin}}(V) \rightarrow \mathbb{R} \\ Y \mapsto \dim(A \cdot Y) - \lambda \dim(Y) \end{cases}$$

are respectively submodular and  $k$ -submodular for any real  $\lambda$  (recall here that  $\mathcal{P}_{k\text{fin}}(V)$  is the set of subsets  $Y$  of  $V$  contained in a finite-dimensional subspace).

## 4 Fragments and atoms

### 4.1 Definitions and general properties

In this paragraph, we fix a submodular function  $f$  defined on  $\mathcal{P}(S)$  such that  $m = \min_{Y \neq \emptyset \in \mathcal{P}_{\text{fin}}(S)} f(Y)$  exists. Then a *fragment* for  $f$  is a nonempty finite subset  $Y$  of  $S$  such that  $f(Y) = m$ . An *atom* for  $f$  is a fragment of minimal cardinality. Observe that there exists at least one fragment and one atom by the hypotheses on  $f$ . Moreover, by definition, *all the atoms have the same finite cardinality*.

**Lemma 4.1** Assume  $A_1$  and  $A_2$  are two atoms of the submodular function  $f$ . Then  $A_1 = A_2$  or  $A_1 \cap A_2 = \emptyset$ .

**Proof.** Assume  $A_1 \cap A_2$  is not empty. Since  $f$  is a submodular function on  $\mathcal{P}_{\text{fin}}(S)$ , we can write

$$f(A_1 \cap A_2) + f(A_1 \cup A_2) \leq f(A_1) + f(A_2) = 2m$$

by using that  $A_1$  and  $A_2$  are atoms. We have  $f(A_1 \cap A_2) \geq m$  and  $f(A_1 \cup A_2) \geq m$  since  $m = \min_{Y \neq \emptyset \in \mathcal{P}_{\text{fin}}(S)} f(Y)$ , we get  $f(A_1 \cap A_2) = f(A_1 \cup A_2) = m$ . Hence both  $A_1 \cup A_2$  and  $A_1 \cap A_2$  are fragments for  $f$ . Now, observe that  $A_1 \cap A_2 \subset A_1$ . Thus by minimality of the cardinality of an atom, we have  $|A_1 \cap A_2| = |A_1|$  and therefore  $A_1 \cap A_2 = A_1$  which means that  $A_1 \subset A_2$ . But  $A_1$  and  $A_2$  have same cardinality since they are atoms. So  $A_1 = A_2$ . ■

## 4.2 Left invariant submodular functions on groups

Let  $G$  be a group and  $f : \mathcal{P}_{\text{fin}}(G) \rightarrow \mathbb{R}$  a submodular function. It is said left invariant when  $f(gA) = f(A)$  for any  $g \in G$  and any finite subset  $A \subset G$ .

**Proposition 4.2** *Assume  $f$  is submodular and left invariant such that  $m = \min_{A \neq \emptyset \in \mathcal{P}_{\text{fin}}(G)} f(A)$  exists. Then, there exists a unique atom  $H$  for  $f$  containing 1. Moreover  $H$  is a finite subgroup of  $G$ , the atoms of  $G$  are the left cosets  $gH$  with  $g \in G$  and they yield a partition of  $G$ .*

**Proof.** The existence of an atom is obtained as in the previous paragraph. Now, if  $A$  is an atom, since it is nonempty, we get that  $a^{-1}A$  is also an atom for any  $a \in A$  because  $f(a^{-1}A) = m$ . Then  $H = a^{-1}A$  is an atom containing 1. Let  $H'$  be another atom containing 1. Then  $H \cap H'$  is nonempty, thus by Lemma 4.1, we must have  $H = H'$  which proves that there exists indeed a unique atom  $H$  containing 1. Given  $h \in H$ , we show similarly that  $h^{-1}H$  is an atom containing 1 so that  $h^{-1}H = H$ . Therefore, for any  $h, h' \in H$  we get that  $h^{-1}h'$  belongs to  $H$  which shows that  $H$  is a subgroup of  $G$  (finite by definition of  $f$ ). Let  $A$  be an atom for  $H$ . Then, for any  $a \in A$ , the atom  $a^{-1}A$  coincide with  $H$  because it contains 1. Thus,  $A = aH$  is a left coset of  $H$ . It is then well-known that the left cosets of  $H$  yields a partition of  $G$ . ■

## 4.3 Left invariant submodular functions for group actions

Assume that  $G$  acts on the set  $X$  and consider  $f : \mathcal{P}_{\text{fin}}(X) \rightarrow \mathbb{R}$  a submodular function such that  $m = \inf_{Y \neq \emptyset \in \mathcal{P}_{\text{fin}}(X)} f(Y)$  exists. The function  $f$  is said left invariant if for any  $g \in G$  and any  $Y \subset X$ , we have  $f(g \cdot Y) = f(Y)$ . In this case, we get by Lemma 4.1 that for any atom  $Y_0$  and any  $g \in G$

$$g \cdot Y_0 \text{ is an atom such that } g \cdot Y_0 = Y_0 \text{ or } g \cdot Y_0 \cap Y_0 = \emptyset.$$

Let  $\mathcal{A}$  be the set of atoms for  $f$ . We thus get an action of the group  $G$  on the set of atoms  $\mathcal{A}$ . Now given any element  $y_0$  in the atom  $Y_0$ , we obtain the inclusion  $G_{y_0} \subset G_{Y_0}$  of the stabilizers of  $y_0$  and  $Y_0$  for the action of  $G$  on  $X$ . Indeed, for any  $g \in G_{y_0}$ , we have  $y_0 = g \cdot y_0 \in g \cdot Y_0$  and also  $y_0 \in Y_0$ . Therefore  $g \cdot Y_0 \cap Y_0 \neq \emptyset$  and  $g \cdot Y_0 = Y_0$  which means that  $g$  belongs to  $G_{Y_0}$ . Observe also that if  $y_0$  belongs to the atom  $Y_0$ , then any element  $g \cdot y_0$  also belongs to an atom (because  $g \cdot y_0$  belongs to  $g \cdot Y_0$ ). We will call the set

$$\mathcal{C}(X) = \coprod_{Y_0 \in \mathcal{A}} Y_0$$

the *core* of  $X$ . The action of  $G$  on  $X$  restricts to an action on  $\mathcal{C}(X)$  and thus, the set  $\mathcal{C}(X)$  is a disjoint union of orbits for the action of  $G$  on  $X$ . Moreover, for any such orbit  $\mathcal{O}$  and any atom  $Y_0$ , we have

$$\mathcal{O} \cap Y_0 = \emptyset \text{ or } \mathcal{O} \cap Y_0 = \{g \cdot y_0 \mid \bar{g} \in G_{Y_0}/G_{y_0}\} \text{ with } y_0 \in \mathcal{O} \cap Y_0$$

that is  $\mathcal{O} \cap Y_0$  is empty or parametrized by the elements of the coset  $G_{Y_0}/G_{y_0}$  with  $y_0 \in \mathcal{O} \cap Y_0$  since  $G_{y_0}$  is then a subgroup of  $G_{Y_0}$ . In particular, if the action of  $G$  on  $X$  is assumed to be transitive, we have a unique orbit,  $\mathcal{C}(X) = X$  and the atoms form a partition of  $X$ . Let us summarize the previous observations.

**Proposition 4.3** *Assume that  $G$  acts on the set  $X$  and  $f : \mathcal{P}_{\text{fin}}(X) \rightarrow \mathbb{R}$  is a left invariant submodular function such that  $m = \inf_{Y \neq \emptyset \in \mathcal{P}_{\text{fin}}(X)} f(Y)$  exists. We have the following properties.*

1. *The group  $G$  acts on the set  $\mathcal{A}$  of atoms for  $f$ .*
2. *The action of  $G$  restricts to the core  $\mathcal{C}(X)$  of  $X$ , defined as the disjoint union of the atoms for  $f$  which is thus also a disjoint union of orbits for the action of  $G$  on  $X$ .*
3. *For any atom  $Y_0$  and any orbit  $\mathcal{O}$ , the intersection  $\mathcal{O} \cap Y_0$  is empty or parametrized by the elements of the coset  $G_{Y_0}/G_{y_0}$  with  $y_0 \in \mathcal{O} \cap Y_0$ .*

4. When the action of  $G$  on  $X$  is transitive,  $\mathcal{C}(X) = X$ , each element of  $X$  belongs to one atom.

**Example 4.4** For each action of a finite group  $G$  on the finite set  $X$ , one can consider the cut function  $f$  as defined in (5). By Proposition 3.3, it is nonnegative submodular and left invariant. Also the minimum of  $f$  is equal to zero and is attained in any subset  $Y$  such that  $g \cdot y \in Y$  for any  $g \in G$  and any  $y \in Y$ . This means that the fragments of  $f$  are the disjoint union of orbits and the atoms are the orbits of minimal cardinality. The core is the disjoint union of the orbits with minimal cardinality.

**Example 4.5** Here is another example in which atoms are the orbits with minimal cardinality; the submodular function considered is the function  $d_A$  of Example 3.8 and Subsection 4.4. Fix  $\sigma \in \mathfrak{S}_n$ , consider  $X = \{1, \dots, n\}$  and  $A = \langle \sigma \rangle$ . As suggested in Section 2.2,  $X$  can be written as  $X = X_1 \sqcup \dots \sqcup X_r$  where the  $X_i$  are the orbits of  $X$  under the action of  $A$ . In this case,  $d_A(Y) = \sum_{j, X_j \cap Y \neq \emptyset} |X_j| - \lambda|Y|$ . Among the subset  $Z$  of  $X$  meeting non trivially exactly the same  $X_i$  as  $Y$ ,  $d_A(Z)$  is minimal precisely when  $Z = \cup_{j, X_j \cap Y \neq \emptyset} X_j$ . In this case,  $d_A(Z) = (1 - \lambda)|Z|$ . Thus, for  $\lambda < 1$ , the fragments and atoms coincide and are the  $X_j$  with minimal cardinality. When  $\lambda = 1$ , every union of orbits is a fragment and the atoms are the  $X_j$  with minimal cardinality.

#### 4.4 Behavior of the atoms for the submodular function $d_A$

The function  $d_A$  defined in Example 3.8 on  $\mathcal{P}_{\text{fin}}(X)$  by  $d_A(Y) = |A \cdot Y| - \lambda|Y|$  is submodular nonnegative for any  $\lambda \in [0, 1]$  and left invariant when  $G$  is abelian (see § 5.3). In contrast to the previous examples, the corresponding atoms and cores depend on  $\lambda$  and on the definition of the action. Our goal in this paragraph is to show that, roughly speaking, the cardinality of fragments is bounded by  $|A|$  for small values of  $\lambda$  whereas, for values of  $\lambda$  close to 1 and when the action is free, the cardinality of fragments become larger than  $|A|$ .

More precisely, assume that  $\lambda < 1/|A|$ . Then every fragment  $Y$  for  $d_A$  verifies  $|Y| \leq |A|$ . Indeed if we assume  $|Y| \geq |A| + 1$ , we get for any  $y \in Y$

$$d_A(Y) = |A \cdot Y| - \lambda|Y| \geq (1 - \lambda)|Y| \geq (1 - \lambda)(|A| + 1) = |A| - \lambda + 1 - \lambda|A| > |A| - \lambda \geq d_A(\{y\})$$

because  $|A \cdot \{y\}| \leq |A|$ . This thus gives the contradiction  $d_A(\{y\}) < d_A(Y)$ .

Now assume that the action of  $G$  on  $X$  is free and  $A \subset G$  is such that  $|X| \geq |A|$ . Consider  $\mu \leq 1$  and any subset  $Y$  of  $X$  such that  $|Y| < \mu|A|$ . For any  $\lambda \in [0, 1]$  such that

$$\frac{|X| - |A|}{|X| - \mu|A|} \leq \lambda \leq 1$$

we get for the function  $d_A$  corresponding to  $\lambda$

$$d_A(Y) = |A \cdot Y| - \lambda|Y| \geq |A| - \lambda|Y| > |A| - \mu\lambda|A|.$$

Here we use that our action is free to insure that  $|A \cdot Y| \geq |A|$ . By observing that

$$\lambda \geq \frac{|X| - |A|}{|X| - \mu|A|} \iff |A| - \mu\lambda|A| \geq (1 - \lambda)|X|$$

and  $d_A(X) = (1 - \lambda)|X|$ , we get that  $Y$  cannot be a fragment. In particular, when  $\mu > 1 - \frac{1}{|A|}$ , we obtain that the fragments are of cardinality at least  $|A|$  for any function  $d_A$  such that

$$\lambda \geq \frac{|X| - |A|}{|X| - \mu|A|} \geq \frac{|X| - |A|}{|X| - |A| + 1}.$$

Thus, atoms and fragments indeed strongly depend on  $\lambda$  and are in general not easy to determine explicitly.

## 4.5 Left invariant $k$ -submodular functions and group representations

When  $(\rho, V)$  is a linear representation of  $G$ , we can proceed similarly for any left invariant  $k$ -submodular function  $f : \mathcal{P}_{\text{kfin}}(V) \rightarrow \mathbb{R}$  such that  $m = \inf_{0 < \dim(Y) < +\infty} f(Y)$  exists. We define a fragment as a  $k$ -vector subspace  $W$  of  $V$  which is not reduced to  $\{0\}$  and such that  $f(W) = m$ . By definition of the  $k$ -submodular function  $f$ , we can replace  $W$  by any of its generating set  $Y$  since  $f(Y) = f(W)$ . An atom for  $f$  will be a fragment with minimal dimension. The function  $f$  is said left invariant if for any  $g \in G$  and any subspace  $W$  of  $V$ , we have  $f(g \cdot W) = f(W)$ . all the atoms have the same dimension and we have a linear analogue of Lemma 4.1 whose proof is similar.

**Lemma 4.6** *Assume  $W_1$  and  $W_2$  are two atoms of  $f$ . Then  $W_1 = W_2$  or  $W_1 \cap W_2 = \{0\}$ .*

## 5 Generalizations of results in additive group theory

We resume the definition of the submodular functions  $c_Y, \gamma_Y, d_A, \delta_A$  of Example 3.8. In any cases, recall that we consider an action of the discrete group  $G$  on a set  $X$  (or a linear action on a vector space  $V$ ). The functions  $c_Y, \gamma_Y, Y \subset X$  are defined on  $\mathcal{P}_{\text{fin}}(G)$  from a fixed finite subset of  $X$  or  $V$  whereas the functions  $d_A$  and  $\gamma_A$  are defined on  $\mathcal{P}_{\text{fin}}(X)$  or  $\mathcal{P}_{\text{kfin}}(V)$  from a finite fixed subset  $A \subset G$ . Also, all these functions attain their minimum on their restrictions to nonempty subsets as soon as they are nonnegative because their images are discrete subsets of  $\mathbb{R}_{\geq 0}$ .

### 5.1 Group action context and submodular functions $c_Y$

Let us start with an observation which is not relevant in the context of additive group theory but crucial in our group action context. Consider the map

$$q_Y : \begin{cases} \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\} \rightarrow \mathbb{Q}_{>0} \\ A \mapsto \frac{|A \cdot Y|}{|A|} \end{cases}$$

Then it might happen that

$$\mu = \inf_{A \in \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\}} q_Y(A) = 0. \quad (7)$$

This will be in particular the case if  $G_Y$  is an infinite subgroup of  $G$  since subsets  $A$  in  $G_Y$  may have arbitrary large cardinalities whereas  $|AY| = |Y|$  is then fixed. In the opposite direction, we will always have  $\mu > 0$  when

1. there exists an element  $y_0 \in Y$  such that  $G_{y_0} = \{1\}$  and then  $\mu \geq 1$  (this is in particular true if we consider the action by left translation of  $G$  on itself),
2. or the group  $G$  is finite and then  $\mu \geq \frac{|Y|}{|G|}$  because we always have  $|A \cdot Y| \geq |Y|$  and  $Y$  is fixed.

To overcome this difficulty, we need in general the assumption

$$\mu = \inf_{A \in \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\}} \frac{|A \cdot Y|}{|A|} > 0. \quad (8)$$

#### Example 5.1

1. *When the action is free (for example in the case of the left translation of  $G$  on itself), we have  $|A \cdot Y| \geq |A|$  so that  $\mu \geq 1$ .*

2. For the action of the symmetric group  $\mathfrak{S}_n$  on  $\{1, \dots, n\}$ , when  $|A \cdot Y| = \ell$ , we get with the notation of Example 2.1

$$\inf_{A \neq \emptyset \in \mathcal{P}_{\text{fin}}(\mathfrak{S}_n) \mid |A \cdot Y| = \ell} \frac{|A \cdot Y|}{|A|} = \frac{|A_0 \cdot Y|}{|A_0|} = \frac{\ell}{\frac{\ell!}{(\ell-k)!} (n-k)!}$$

which is minimal for  $\ell = n$  and then

$$\mu = \frac{n}{n!} = \frac{1}{(n-1)!}.$$

3. Assume  $G$  is finite and acts on itself by conjugation. If we consider  $Y$  a subset of  $Z(G)$  the center of  $G$ , we get  $A \cdot Y = Y$  for any subset  $A \subset G$ . Then

$$\mu = \inf_{A \in \mathcal{P}(G) \setminus \{\emptyset\}} \frac{|A \cdot Y|}{|A|} = \frac{|Y|}{|G|}.$$

4. We get similarly  $\mu = \frac{|Y|}{|G|}$  as soon as  $Y$  is a set of fixed elements under the action of  $G$ .

**Remark 5.2** Assume  $G$  is finite and the infimum  $\mu$  in (8) is attained for the subset  $A_0 \subset G$ , that is  $\mu = \frac{|A_0 \cdot Y|}{|A_0|} > 0$ . Since we have  $G_Y \cdot Y = Y$  for the stabilizer  $G_Y$  of  $Y$ , the set  $A_0$  is a disjoint union of left  $G_Y$ -cosets. In particular,  $G_Y$  is finite.

Under the assumption  $\mu > 0$ , for any  $\lambda \in [0, \mu]$ , the function  $c_Y$  defined on  $\mathcal{P}_{\text{fin}}(G)$  by  $c_Y(A) = |A \cdot Y| - \lambda |A|$  is submodular. Moreover, we have

$$c_Y(A) = |A \cdot Y| - \lambda |A| \geq |A \cdot Y| - \mu |A| \geq 0$$

and  $c_Y$  is nonnegative. Observe that

$$c_Y(A) \geq (\mu - \lambda) |A|.$$

Also for any  $g \in G$

$$c_Y(gA) = |gA \cdot Y| - \lambda |gA| = c_Y(A)$$

thus the map  $c_Y$  is also left invariant. We get the following theorem.

**Theorem 5.3** Consider a subset  $Y \subset X$  and set

$$\mu = \inf_{A \in \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\}} \frac{|A \cdot Y|}{|A|}.$$

Then

- either  $\mu = 0$ ,
- or for any  $\lambda \in [0, \mu]$ , there exists a finite subgroup  $H$  of  $G$  containing  $G_Y$  such that

$$|A \cdot Y| \geq \lambda |A| + |H \cdot Y| - \lambda |H| \geq \lambda |A| + |Y| - \lambda |H| \quad (9)$$

for any subset  $A$  in  $G$ .

**Proof.** Assume  $\mu > 0$  and set as usual  $m = \min_{A \neq \emptyset \in \mathcal{P}_{\text{fin}}(G)} c_Y(A)$ . The case  $\lambda = 0$  is trivial (take  $H = G_Y$  and remark that  $G_Y$  is finite by a previous remark). Consider  $\lambda \in ]0, \mu]$  and  $A_0 \in \mathcal{P}_{\text{fin}}(G)$  such that  $\mu = \frac{|A_0 \cdot Y|}{|A_0|}$ . Then, for any  $A \in \mathcal{P}_{\text{fin}}(G)$ , we have

$$c_Y(A) = |A \cdot Y| - \lambda |A| \geq |A_0 \cdot Y| - \lambda |A_0| \geq (\mu - \lambda) |A_0| \geq 0$$

so that  $c_Y$  is a nonnegative submodular function. By Proposition 4.2, there thus exists a unique atom  $H$  for  $c_Y$  containing 1 which is a subgroup of  $G$ . Assume there exists  $g \in G_Y$  such  $g \notin H$ . Then

$$c_Y(H \cup \{g\}) = |(H \cup \{g\}) \cdot Y| - \lambda |H \cup \{g\}| = c_Y(H) - \lambda < c_Y(H)$$

and  $H \cup \{g\}$  is nonempty. This contradicts the fact that  $H$  is an atom. Thus, we must have  $G_Y \subset H$ . Also since  $H$  is an atom, we have for any finite subset  $A$  in  $G$

$$c_Y(A) = |A \cdot Y| - \lambda |A| \geq |H \cdot Y| - \lambda |H| = c_Y(H)$$

which gives

$$|A \cdot Y| \geq \lambda |A| + |H \cdot Y| - \lambda |H|.$$

■

**Remark 5.4**

1. Observe that when  $\mu = 0$ , the inequality (9) still holds since it reduces to  $|A \cdot Y| \geq |Y|$ .
2. When  $Y$  contains an element with trivial stabilizer, we have  $\mu \geq 1$  and the theorem generalizes Hamidoune's one when  $G$  acts on itself by left translation.
3. Note that we must have  $H = \{1\}$  when  $G$  is torsion free because  $H$  is a finite subgroup of  $G$ .

The theorem also has a dual interpretation in the spirit of § 2.3. Consider a finite subset  $Y$  in  $X$  such that  $\mu > 0$ .

**Corollary 5.5** For any  $\lambda \in ]0, \mu]$  and any finite subset  $A_0$  in  $G$  there exists a subgroup  $H$  of  $G$  containing  $G_Y$  such that

$$\lambda \max_{A \subset G | A \cdot Y = A_0 \cdot Y} |A| + |Y| \leq \lambda |H| + |A_0 \cdot Y|.$$

In another direction, we can also get the following analogue of a Theorem by Tao and Petridis (see [19]) in our group action context.

**Theorem 5.6** Consider  $A$  a finite subset of  $G$  and  $Y$  a finite subset of  $X$ . Assume that

$$|A \cdot Y| \leq \alpha |A|$$

with  $\alpha \in \mathbb{R}_{\geq 0}$ . Then, there exists a nonempty subset  $B$  in  $A$  such that

$$|CB \cdot Y| \leq \alpha |CB|$$

for any finite subset  $C$  of  $G$ .

**Proof.** Define the map  $q_{A,Y}$  such that

$$q_{A,Y} : \begin{cases} \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow \mathbb{Q}_{>0} \\ C \mapsto \frac{|C \cdot Y|}{|C|} \end{cases}$$

and its minimum  $\mu$  (which indeed exists since  $\mathcal{P}(A)$  is finite). Let  $B \subset A$  such that  $\mu = \frac{|B \cdot Y|}{|B|}$ . Now let us consider the function  $c_Y$  defined on  $\mathcal{P}_{\text{fin}}(G)$  by  $c_Y(C) = |C \cdot Y| - \mu |C|$ . We have seen that the function  $c_Y$  is submodular and left invariant. We also have here  $c_Y(B) = 0$  and for any  $C \subset A$  we get  $c_Y(C) \geq c_Y(B) = 0$ . Nevertheless,  $c_Y$  is not nonnegative on  $\mathcal{P}_{\text{fin}}(G)$  in general. Now for any nonempty finite subset  $S$  of  $G$  and any  $g \in G$ , we can write

$$c_Y(B \cup g^{-1}S) + c_Y(B \cap g^{-1}S) \leq c_Y(B) + c_Y(g^{-1}S) \leq c_Y(S)$$

because  $c_Y(B) = 0$  and  $c_Y(g^{-1}S) = c_Y(S)$ . We also have  $c_Y(B \cap g^{-1}S) \geq 0$  because  $B \cap g^{-1}S \subset B \subset A$  which implies that  $c_Y(B \cup g^{-1}S) \leq c_Y(S)$  for any  $g \in G$  and any  $S \in \mathcal{P}_{\text{fin}}(G)$ . By left invariance, this gives

$$c_Y(gB \cup S) \leq c_Y(S) \quad (10)$$

for any  $g \in G$  and any  $S \in \mathcal{P}_{\text{fin}}(G)$ .

Now, let us consider a subset  $C$  of  $G$  such that  $C = \{g_1, g_2, \dots, g_m\}$  and  $C^b = \{g_1, g_2, \dots, g_{m-1}\}$ . We get for any  $S' \in \mathcal{P}_{\text{fin}}(G)$

$$c_Y(CB \cup S') = c_Y(g_m B \cup (C^b B \cup S')) \leq c_Y(C^b B \cup S')$$

by applying (10) with  $g = g_m$  and  $S = C^b B \cup S'$ . By an easy induction on  $m$  we finally obtain

$$c_Y(CB \cup S') \leq c_Y(S')$$

for any  $S' \in \mathcal{P}_{\text{fin}}(G)$ . In particular for  $S' = \emptyset$ , we get

$$c_Y(CB) \leq 0 \iff |CB \cdot Y| - \mu |CB| \leq 0 \iff |CB \cdot Y| \leq \mu |CB|$$

since  $c_Y(\emptyset) = 0$ . We conclude by observing that  $\mu = \min_{C \subset A, C \neq \emptyset} \frac{|C \cdot Y|}{|C|} \leq \frac{|A \cdot Y|}{|A|} \leq \alpha$ . ■

## 5.2 Group representation context and submodular functions $\gamma_Y$

If we consider a representation  $(\rho, V)$  of  $G$  and a finite-dimensional  $k$ -subspace  $W = k\langle Y \rangle$  in  $V$ , we can get an analogue of Theorem 5.3. The proof uses the same arguments and is thus omitted here.

**Theorem 5.7** *Consider a subset  $Y \subset X$  and set*

$$\mu = \inf_{A \neq \emptyset \in \mathcal{P}_{\text{fin}}(G)} \frac{\dim(A \cdot Y)}{|A|}.$$

*Then*

- *either  $\mu = 0$*
- *or for any  $\lambda \in [0, \mu]$ , there exists a finite subgroup  $H$  of  $G$  containing  $G_Y$  such that*

$$\dim(A \cdot Y) \geq \lambda |A| + \dim(H \cdot Y) - \lambda |H| \geq \lambda |A| + \dim(Y) - \lambda |H|$$

*for any subset  $A$  in  $G$ .*

We also have an analogue of Theorem 5.6.

**Theorem 5.8** *Consider  $A$  a finite subset of  $G$  and  $W$  a finite-dimensional  $k$ -subspace of  $V$ . Assume that*

$$\dim\langle A \cdot Y \rangle \leq \alpha |A|$$

*with  $\alpha \in \mathbb{R}_{\geq 0}$ . Then, there exists a nonempty subset  $B$  in  $A$  such that*

$$\dim\langle CB \cdot Y \rangle \leq \alpha |CB|$$

*for any finite subset  $C$  of  $G$ .*

### 5.3 Group action context and submodular functions $d_A$

Recall that for any fixed finite subset  $A$  in  $G$  and any  $\lambda \geq 0$ , the submodular function  $d_A$  is defined on  $\mathcal{P}_{\text{fin}}(X)$  by  $d_A(Y) = |A \cdot Y| - \lambda|Y|$ . Observe, the function  $d_A$  is not left invariant in general as defined in § 4.3 but this is nevertheless the case when  $G$  is assumed abelian. Indeed, we then have for any  $g \in G$

$$d_A(g \cdot Y) = |A \cdot (g \cdot Y)| - \lambda|g \cdot Y| = |gA \cdot Y| - \lambda|g \cdot Y| = |A \cdot Y| - \lambda|Y| = d_A(Y)$$

because we have  $|gA \cdot Y| = |A \cdot Y|$  and  $|g \cdot Y| = |Y|$  by (2). The function  $d_A$  is not nonnegative for any  $\lambda \geq 0$  but it is the case when  $\lambda \in [0, 1]$  because we have for any  $A \subset G$  and any  $Y \subset X$  the inequalities  $|A \cdot Y| \geq |Y| \geq \lambda|Y|$ .

We get the following theorem.

**Theorem 5.9** *Assume  $G$  is abelian. Consider  $A$  a finite subset of  $G$  and  $Y$  a finite subset of  $X$ . Assume that*

$$|A \cdot Y| \leq \alpha|Y|$$

with  $\alpha \in \mathbb{R}_{\geq 0}$ . Then, there exists a nonempty subset  $Z$  in  $Y$  such that

$$|AC \cdot Z| \leq \alpha|C \cdot Z|$$

for any finite subset  $C$  of  $G$ .

**Proof.** Define the map  $q_{Y,A}$  such that

$$q_{Y,A} : \begin{cases} \mathcal{P}_{\text{fin}}(Y) \setminus \{\emptyset\} \rightarrow \mathbb{Q}_{>0} \\ S \mapsto \frac{|A \cdot S|}{|S|} \end{cases}$$

and its minimum  $\mu$ . Let  $Z \subset Y$  such that  $\mu = \frac{|A \cdot Z|}{|Z|}$ . Now let us consider the function  $d_A$  defined on  $\mathcal{P}_{\text{fin}}(X)$  by  $d_A(S) = |A \cdot S| - \mu|S|$ . The function  $d_A$  is submodular and left invariant because  $G$  is abelian. We have  $d_A(Z) = 0$  and for any  $S \subset Y$  we get  $d_A(S) \geq 0$ . Here again, the function  $d_A$  is not nonnegative on  $\mathcal{P}_{\text{fin}}(X)$  in general. Now for any nonempty finite subset  $S$  of  $X$  and any  $g \in G$ , we can write

$$d_A(Z \cup g^{-1}S) + d_A(Z \cap g^{-1}S) \leq d_A(Z) + d_A(g^{-1}S) \leq d_A(S)$$

because  $d_A(Z) = 0$  and  $d_A(g^{-1}S) = d_A(S)$ . We also have  $d_A(Z \cap g^{-1}S) \geq 0$  because  $Z \cap g^{-1}S \subset Z \subset Y$  which implies that  $d_A(Z \cup g^{-1}S) \leq d_A(S)$  for any  $g \in G$  and any  $S \in \mathcal{P}_{\text{fin}}(X)$ . By left invariance, this gives

$$d_A(gZ \cup S) \leq d_A(S) \tag{11}$$

for any  $g \in G$  and any  $S \in \mathcal{P}_{\text{fin}}(X)$ .

Now, let us consider a subset  $C$  of  $G$  such that  $C = \{g_1, g_2, \dots, g_m\}$  and  $C^b = \{g_1, g_2, \dots, g_{m-1}\}$ . We get for any  $S' \in \mathcal{P}_{\text{fin}}(X)$

$$d_A((C \cdot Z) \cup S') = d_A((g_m \cdot Z) \cup ((C^b \cdot Z) \cup S')) \leq d_A((C^b \cdot Z) \cup S')$$

by applying (11) with  $g = g_m$  and  $S = (C^b \cdot Z) \cup S'$ . By induction on  $m$  we finally obtain

$$d_A((C \cdot Z) \cup S') \leq d_A(S')$$

for any  $S' \in \mathcal{P}_{\text{fin}}(X)$ . In particular for  $S' = \emptyset$ , we get since  $d_A(\emptyset) = 0$

$$d_A(C \cdot Z) \leq 0 \iff |A \cdot (C \cdot Z)| - \mu|C \cdot Z| \leq 0 \iff |AC \cdot Z| \leq \mu|C \cdot Z|.$$

We conclude by observing that  $\mu = \min_{S \subset Y, S \neq \emptyset} \frac{|A \cdot S|}{|S|} \leq \frac{|A \cdot Y|}{|Y|} \leq \alpha$ . ■

Under the same hypotheses as in the previous theorem, we get the following interesting corollary.



**Corollary 5.10** *Assume  $G$  is abelian and  $|A \cdot Y| \leq \alpha |Y|$ . Then, there exists a nonempty subset  $Z$  in  $Y$  such that for any integer  $n \geq 1$  we have*

$$|A^n \cdot Z| \leq \alpha^n |Z|.$$

**Proof.** By applying the theorem, we get a subset  $Z$  of  $Y$  such that  $|AC \cdot Z| \leq \alpha |C \cdot Z|$  for any finite subset  $C$  of  $G$ . In particular, with  $C = \{1\}$ , this gives  $|A \cdot Z| \leq \alpha |Z|$ , that is the corollary for  $n = 1$ . Now consider an integer  $n \geq 2$  and assume by induction that we have  $|A^{n-1} \cdot Z| \leq \alpha^{n-1} |Z|$ . We then get

$$|A^n \cdot Z| = |A \cdot A^{n-1} \cdot Z| \leq \alpha |A^{n-1} \cdot Z| \leq \alpha^n |Z|$$

where the first inequality is obtained by using the theorem with  $C = A^{n-1}$  and the second one is the induction hypothesis. ■

## 5.4 Group representation context and submodular functions $\delta_A$

Here again, we have a group representation version of the results in § 5.3 when  $(\rho, V)$  is a representation of  $G$ .

**Theorem 5.11** *Assume  $G$  is abelian. Consider  $A$  a finite subset of  $G$  and  $W = k\langle Y \rangle$  a finite-dimensional  $k$ -subspace of  $V$ . Assume that*

$$\dim(A \cdot Y) \leq \alpha \dim(Y)$$

with  $\alpha \in \mathbb{R}_{\geq 0}$ . Then, there exists a  $k$ -subspace  $Z \neq \{0\}$  in  $W$  such that

$$\dim(AC \cdot Z) \leq \alpha \dim(C \cdot Y)$$

for any finite subset  $C$  of  $G$ .

**Corollary 5.12** *Assume  $G$  is abelian and  $\dim\langle A \cdot Y \rangle \leq \alpha \dim\langle Y \rangle$ . Then, there exists a  $k$ -subspace  $Z \neq \{0\}$  in  $\langle Y \rangle$  such that for any integer  $n \geq 1$  we have*

$$\dim(A^n \cdot Z) \leq \alpha^n \dim\langle Z \rangle.$$

## 6 Symmetry sets and simple upper bounds

Assume that the group  $G$  acts on the set  $X$  and consider a finite nonempty subset  $Y$  of  $X$ . The notion of symmetry set was introduced by Murphy in [13]. Consider a real  $\alpha \in ]0, 1]$ , the *symmetry set* of  $Y$  in  $G$  for  $\alpha$  is defined as

$$\text{Sym}_\alpha(Y) = \{g \in G \mid |g \cdot Y \cap Y| \geq \alpha |Y|\}.$$

We also introduce the *weak stabilizer* of  $Y$  as

$$\Gamma_Y = \{g \in G \mid g \cdot Y \cap Y \neq \emptyset\} = \bigcup_{\alpha \in ]0, 1]} \text{Sym}_\alpha(Y).$$

One immediately checks that  $1 \in \Gamma_Y$  and  $g \in \text{Sym}_\alpha(Y)$  if and only if  $g^{-1} \in \text{Sym}_\alpha(Y)$ . Also, if  $G$  acts on itself by left translation and  $A \subset G$ , we have  $\Gamma_A = AA^{-1}$ . Observe also that  $\text{Sym}_1(Y) = G_Y$  is the stabilizer of  $Y$  in  $G$ . In general we always have  $G_Y \subset \text{Sym}_\alpha(Y)$  for any  $\alpha \in ]0, 1]$  and more generally  $\text{Sym}_\alpha(Y) \subset \text{Sym}_{\alpha'}(Y)$  for  $\alpha' \leq \alpha$ . Therefore the sequence of subsets  $(\text{Sym}_\alpha(Y))_{\alpha \in ]0, 1]}$  decreases from  $G$  to  $G_Y$  when  $\alpha$  increases in  $]0, 1]$ . The set

$$\left\{ \frac{|g \cdot Y \cap Y|}{|Y|} \mid g \in \Gamma_Y \right\} \subset \mathbb{Q}_{>0}$$

is discrete and not empty. Thus it admits a minimum  $\alpha_0$  and we then have  $\Gamma_Y = \text{Sym}_{\alpha_0}(Y)$ .

When  $(\rho, V)$  is a linear representation of  $G$  such that  $V \neq \{0\}$ , we define similarly for any  $k$ -subspace  $W \neq \{0\}$  of  $V$

$$\begin{aligned}\text{Sym}_\alpha(W) &= \{g \in G \mid \dim g \cdot W \cap W \geq \alpha \dim W\}. \\ \Gamma_W &= \{g \in G \mid g \cdot W \cap W \neq \{0\}\}.\end{aligned}$$

We yet have  $1 \in \text{Sym}_\alpha(W)$  and  $g \in \text{Sym}_\alpha(W)$  if and only if  $g^{-1} \in \text{Sym}_\alpha(W)$ .

In this section, we examine what information can be extracted when assumption are done on the cardinality ratio  $\frac{|A \cdot Y|}{|Y|}$  (or the dimension ratio  $\frac{\dim(A \cdot Y)}{\dim(Y)}$ ). This problem was addressed in detail by Murphy in [13] for group actions setting. Let us start by recalling the first result of [13] and state its linear version.

**Proposition 6.1**

1. Assume that  $|A \cdot Y| = |Y|$ . Then  $H = \langle A^{-1}A \rangle^1$  is a subgroup of  $G_Y$  and  $Y$  decomposes into  $H$ -orbits.
2. Assume that  $\dim(A \cdot W) = \dim W$ . Then  $H = \langle A^{-1}A \rangle$  is a subgroup of  $G_W$ . When  $k$  has characteristic zero and  $H$  is finite, the  $k$ -space  $W$  decomposes into irreducible representations for the group  $H$

**Proof.** 1: For any  $a \in A$ , we have  $1 \in a^{-1}A$  and  $a^{-1}A \cdot Y = Y$  because  $Y \subset a^{-1}A \cdot Y$  and  $|a^{-1}A \cdot Y| = |A \cdot Y| = |Y|$ . This shows that  $A^{-1}A \cdot Y = Y$  and thus the desired inclusion  $\langle A^{-1}A \rangle \subset G_Y$ . Since  $H$  is a subgroup of  $G_Y$ , it acts on  $Y$  which yields the decomposition in  $H$ -orbits.

2: We get similarly  $A^{-1}A \cdot W = W$  and the decomposition of  $W$  in irreducible representations for the finite group  $H$  follows from the semisimplicity of its representation theory in characteristic zero. ■

We now examine simple cases where the hypotheses of the previous proposition are relaxed. In the following  $\alpha$  is a fixed real in  $]0, 1[$ .

**Lemma 6.2** 1. Assume  $A \subset G$  and  $Y \subset X$  are finite and nonempty and such that  $|A \cdot Y| \leq (2-\alpha)|Y|$ . Then  $A^{-1}A \subset \text{Sym}_\alpha(Y)$ .

2. Assume  $A \subset G$  and  $W$  is a finite dimensional  $k$ -subspace of  $V$  such that  $\dim(A \cdot W) \leq (2-\alpha)\dim W$ . Then  $A^{-1}A \subset \text{Sym}_\alpha(W)$

**Proof.** 1: Consider  $a, b$  in  $A$ . Since we have  $|a \cdot Y| = |b \cdot Y| = |Y|$ ,  $a \cdot Y \subset A \cdot Y$ ,  $b \cdot Y \subset A \cdot Y$  and  $|A \cdot Y| \leq (2-\alpha)|Y|$ , we must have  $|(a \cdot Y) \cap (b \cdot Y)| \geq \alpha|Y|$ . We thus obtain  $|(b^{-1}a \cdot Y) \cap Y| \geq \alpha|Y|$  and the desired inclusion  $A^{-1}A \subset \text{Sym}_\alpha(Y)$ .

2: This works similarly: given  $a, b$  in  $A$ , we must have  $(a \cdot W) \cap (b \cdot W) \neq \{0\}$  and thus  $a^{-1}b$  belongs to  $\text{Sym}_\alpha(W)$ . ■

Given a subset  $S$  of  $G$ ,  $\langle S \rangle$  be the subgroup of  $G$  generated by the elements in  $S$ .

**Proposition 6.3**

1. Assume  $A \subset G$  and  $Y \subset X$  are nonempty and such that  $|A^{-1} \cdot Y| \leq \frac{3-\alpha}{2}|Y|$ . Then  $(AA^{-1})^2$  is contained in  $\text{Sym}_\alpha(Y)$ .
2. Assume  $A \subset G$  and  $W$  is a  $k$ -subspace of  $V$  such that  $\dim(A^{-1} \cdot W) \leq \frac{3-\alpha}{2}\dim W$ . Then  $(AA^{-1})^2$  is contained in  $\text{Sym}_\alpha(W)$ .

---

<sup>1</sup>Here  $\langle A^{-1}A \rangle$  means the subgroup of  $G$  generated by  $A^{-1}A$ .

**Proof. 1:** Consider  $u = ab^{-1}$  in  $AA^{-1}$  with  $a, b$  in  $A$ . We have

$$\begin{aligned} |(a^{-1} \cdot Y) \cap (b^{-1} \cdot Y)| &= |a^{-1} \cdot Y| + |b^{-1} \cdot Y| - |(a^{-1} \cdot Y) \cup (b^{-1} \cdot Y)| \geq \\ &2|Y| - |A^{-1} \cdot Y| \geq \frac{1+\alpha}{2}|Y| \end{aligned}$$

where the second inequality follows from the inclusions  $a^{-1} \cdot Y \subset A^{-1} \cdot Y$  and  $b^{-1} \cdot Y \subset A^{-1} \cdot Y$  together with the hypothesis  $|A^{-1} \cdot Y| \leq \frac{3-\alpha}{2}|Y|$ . We thus get

$$|Y \cap u \cdot Y| \geq \frac{1+\alpha}{2}|Y|.$$

For any  $v \in AA^{-1}$ , we get similarly

$$|v^{-1} \cdot Y \cap Y| = |Y \cap v \cdot Y| \geq \frac{1+\alpha}{2}|Y|.$$

This implies that both sets  $Y \cap u \cdot Y$  and  $v^{-1} \cdot Y \cap Y$  intersect non trivially in  $Y$  and

$$|u \cdot Y \cap v^{-1} \cdot Y \cap Y| \geq |Y \cap u \cdot Y| + |v^{-1} \cdot Y \cap Y| - |Y| \geq \alpha|Y|.$$

Therefore we obtain that  $|vu \cdot Y \cap Y| \geq \alpha|Y|$  and the product  $vu$  of any two elements  $u, v$  in  $AA^{-1}$  belongs to  $\text{Sym}_\alpha(Y)$ . In particular, by taking  $v = 1 \in AA^{-1}$ , we get that  $AA^{-1}$  is contained in  $\text{Sym}_\alpha(Y)$ .

2: The proof can be easily adapted to the context of a the linear representation  $V$  of  $G$ . ■

#### Remark 6.4

1. When  $G$  acts on itself by left translation and  $Y = A$ , we have  $\Gamma_A = AA^{-1}$  and the hypothesis  $|A^{-1} \cdot Y| < \frac{3}{2}|Y|$  implies that  $\langle AA^{-1} \rangle_G \subset AA^{-1}$ , that is  $AA^{-1}$  is itself a subgroup of  $G$ . Indeed, for some  $\alpha$ ,  $(AA^{-1})^2 \subset \text{Sym}_\alpha(Y) \subset \Gamma_A = AA^{-1}$ .
2. If we assume  $|A \cdot Y| < \frac{3}{2}|Y|$ , we get similarly that  $(A^{-1}A)^2$  is contained  $\Gamma_Y$ .

Assertion 1 of the previous remark suggests the following corollary of Proposition 6.3.

**Corollary 6.5** Assume that  $A \subset G$  and  $Y \subset X$  are nonempty and that there exists  $\alpha \in ]0, 1[$  such that

$$\text{Sym}_\alpha(Y) \subset AA^{-1} \text{ and } |A^{-1} \cdot Y| \leq \frac{3-\alpha}{2}|Y|.$$

Then  $AA^{-1}$  is a subgroup of  $G$ .

**Proof.** By Proposition 6.3, we get  $(AA^{-1})^2 \subset \text{Sym}_\alpha(Y) \subset AA^{-1}$ . Therefore,  $AA^{-1}$  is a finite subgroup of  $G$ . ■

We can also use Theorem 5.3 to generalize the previous results.

**Theorem 6.6** Consider a discrete group  $G$  acting on  $X$ . Let  $A, Y$  be nonempty finite subsets respectively of  $G$  and  $X$  such that  $|A| \geq |Y|$ . Assume that

$$\mu = \inf_{S \neq \emptyset \in \mathcal{P}_{\text{fin}}(G)} \frac{|S \cdot Y|}{|S|} > 0 \text{ and there exists } \varepsilon > 0 \text{ such that } |A \cdot Y| \leq (2 - \varepsilon)\mu|Y|.$$

Then, there exists a finite subgroup  $H$  of  $G$  such that  $Y$  is contained in the disjoint union  $H \cdot Y$  of  $H$ -orbits with

$$|H| \leq \left(\frac{2}{\varepsilon} - 1\right)|Y| \text{ and } |H \cdot Y| \leq \mu \left(\frac{2}{\varepsilon} - 1\right)|Y|.$$

**Proof.** Set  $\lambda = \mu(1 - \frac{\varepsilon}{2})$ . By definition of  $\mu$  and by considering the submodular nonnegative left invariant function  $c_Y$  defined in Example 3.8, we must have

$$c_Y(S) = |S \cdot Y| - \lambda |S| \geq (\mu - \lambda) |S| \geq \mu \frac{\varepsilon}{2} |S| \geq 0 \quad (12)$$

for any finite subset  $S \subset G$ . Now, we can write by using the hypotheses  $|A \cdot Y| \leq (2 - \varepsilon)\mu |Y|$  and  $|A| \geq |Y|$

$$c_Y(A) = |A \cdot Y| - \mu(1 - \frac{\varepsilon}{2}) |A| \leq (2 - \varepsilon)\mu |Y| - \mu(1 - \frac{\varepsilon}{2}) |Y| = \mu(1 - \frac{\varepsilon}{2}) |Y|. \quad (13)$$

Let  $H$  be the unique atom for  $c_Y$  containing 1. By Theorem 5.3, we know that  $H$  is a finite subgroup of  $G$  and  $c_Y(H) \leq c_Y(A)$ . We must have by (12) and (13)

$$|H| \leq \frac{2}{\varepsilon\mu} c_Y(H) \leq \frac{2}{\varepsilon\mu} c_Y(A) \leq (\frac{2}{\varepsilon} - 1) |Y|$$

as desired.

We also get

$$c_Y(H) = |H \cdot Y| - \mu(1 - \frac{\varepsilon}{2}) |H| \leq c_Y(A) \leq \mu(1 - \frac{\varepsilon}{2}) |Y|.$$

Therefore

$$|H \cdot Y| \leq \mu(1 - \frac{\varepsilon}{2}) |H| + \mu(1 - \frac{\varepsilon}{2}) |Y|.$$

Since  $\mu = \inf_{S \in \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\}} \frac{|S \cdot Y|}{|S|}$  and  $H \in \mathcal{P}_{\text{fin}}(G) \setminus \{\emptyset\}$ , we should have  $\mu |H| \leq |H \cdot Y|$  which gives

$$|H \cdot Y| \leq (1 - \frac{\varepsilon}{2}) |H \cdot Y| + \mu(1 - \frac{\varepsilon}{2}) |Y|.$$

By gathering the occurrences of  $|H \cdot Y|$ , we finally obtain and the announced upper bound for  $|H \cdot Y|$

$$|H \cdot Y| \leq \mu(\frac{2}{\varepsilon} - 1) |Y|.$$

■

## 7 Action of a product subset of $G$ on a subset of $X$

Assume that  $G$  acts on the set  $X$ . We now address the question of determining an upper bound of  $AB \cdot Y$  when  $A, B$  are nonempty finite subsets of  $G$  and  $Y$  a finite subset of  $X$ . This is an action group version of Theorem 9.2 in [18].

**Theorem 7.1** *With the previous notation we have*

$$|AB \cdot Y|^2 \leq |AB| |B \cdot Y| \max_{b \in B} \{|Ab \cdot Y|\}. \quad (14)$$

*In particular, when the elements of  $A$  commute with those of  $B$  we have*

$$|AB \cdot Y|^2 \leq |AB| |B \cdot Y| |A \cdot Y|.$$

**Proof.** We proceed by induction on  $|B|$ . When  $B = \{b\}$ , we obtain

$$|Ab \cdot Y|^2 \leq |Ab| |b \cdot Y| \max_{b \in B} \{|Ab \cdot Y|\}$$

by observing that  $|Ab \cdot Y| \leq |Ab| |Y|$  and  $|b \cdot Y| = |Y|$ . Now assume  $|B| > 1$ , set  $m = \max_{u \in B} \{|Au \cdot Y|\}$  and fix  $b \in B$  such that  $m = |Ab \cdot Y|$ . Write  $B = B' \cup \{b\}$ . Set  $A = \{a_1, \dots, a_r\}$  and  $Y = \{y_1, \dots, y_s\}$ . We have  $AB = AB' \cup Ab$ . There exists a subset  $A^b$  of  $A$  such that

$$AB = AB' \bigsqcup_{a \in A^b} ab.$$

Similarly, there exists a subset  $Y^b$  of  $Y$  such that

$$B \cdot Y = (B' \cdot Y) \bigsqcup_{y \in Y^b} b \cdot y.$$

We get

$$AB \cdot Y = (AB' \cdot Y) \cup_{a \in A^b} (ab \cdot Y) = (AB' \cdot Y) \cup_{a \in A^b} (aB \cdot Y) = (AB' \cdot Y) \cup_{a \in A^b} (aB' \cdot Y) \cup_{a \in A^b} \bigcup_{y \in Y^b} (ab \cdot y).$$

Since we have  $\bigcup_{a \in A^b} (aB' \cdot Y) \subset AB' \cdot Y$ , we have in fact

$$AB \cdot Y = (AB' \cdot Y) \cup_{a \in A^b} \bigcup_{y \in Y^b} (ab \cdot y).$$

By the previous decomposition, there exists  $X \subset A^b \times Y^b$  such that

$$AB \cdot Y = (AB' \cdot Y) \bigsqcup_{(a,y) \in X} (ab \cdot y).$$

Set  $\alpha = |X|$ ,  $\beta = |A^b|$  and  $\gamma = |Y^b|$ . Since  $|AB \cdot Y| = |AB' \cdot Y| + \alpha$ , the desired inequality (14) is equivalent to

$$(|AB' \cdot Y| + \alpha)^2 \leq (|AB'| + \beta)(|B' \cdot Y| + \gamma)m. \quad (15)$$

By the induction hypothesis, we have

$$|AB' \cdot Y|^2 \leq |AB'| |B' \cdot Y| m. \quad (16)$$

because  $\max_{u \in B'} \{|Au \cdot Y|\} \leq \max_{u \in B} \{|Au \cdot Y|\} = m$ . We have  $\bigcup_{(a,y) \in X} (ab \cdot y) \subset Ab \cdot Y$  and therefore  $\alpha \leq m$ . Since  $X \subset A^b \times Y^b$ , we have also  $\alpha \leq \beta\gamma$ . We get  $\alpha^2 \leq m\beta\gamma$ . By multiplying in (16), this gives

$$\alpha^2 |AB' \cdot Y|^2 \leq |AB'| |B' \cdot Y| m^2 \beta \gamma.$$

Therefore

$$\alpha |AB' \cdot Y| \leq m \sqrt{\gamma |AB'| \times \beta |B' \cdot Y|} \leq m \frac{\gamma |AB'| + \beta |B' \cdot Y|}{2}.$$

So

$$2\alpha |AB' \cdot Y| \leq m\gamma |AB'| + m\beta |B' \cdot Y|.$$

Combining this last inequality with  $\alpha^2 \leq m\beta\gamma$  and (16), we finally get

$$\begin{aligned} (|AB' \cdot Y| + \alpha)^2 &= |AB' \cdot Y|^2 + 2\alpha |AB' \cdot Y| + \alpha^2 \leq \\ & m |AB'| |B' \cdot Y| + m\gamma |AB'| + m\beta |B' \cdot Y| + m\beta\gamma = (|AB'| + \beta)(|B' \cdot Y| + \gamma)m \end{aligned}$$

as desired. ■

Vincent Beck  
 Institut Denis Poisson (UMR CNRS 7013)  
 Université d'Orléans

Cédric Lecouvey  
 Institut Denis Poisson (UMR CNRS 7013)  
 Université de Tours

## References

- [1] C. BACHOC, A. COUVREUR AND G. ZÉMOR, Towards a function field version of Freiman's Theorem, *Algebraic Combinatorics*, **4** (2018), 501-521.
- [2] V. BECK and C. LECOUCVEY, Additive combinatorics methods in associative algebras, *Confluentes Math.*, **9** (2017), 3-27.
- [3] G. T. DIDERRICH, On Kneser's addition theorem in groups, *Proc. Ams.* **38** (1973), 443-451.
- [4] S. ELIAHOU. and C. LECOUCVEY, On linear versions of some addition theorems, *Linear Algebra and multilinear algebra*, **57** (2009), 759-775.
- [5] D. GRYNKIEWICZ. Structural additive theory, *Developments in Mathematics*, vol 30 (2013).
- [6] X. D. HOU, K. H. LEUNG AND XIANG. Q, A generalization of an addition theorem of Kneser, *Journal of Number Theory* **97** (2002), 1-9.
- [7] Y. O. HAMIDOUNE, On the connectivity of Cayley digraphs, *Europ. J. Comb.*, **5** (1984), 309-312.
- [8] X. D. HOU, On a vector space analogue of Kneser's theorem, *Linear Algebra and its Applications* **426** (2007) 214-227.
- [9] F. KAINRATH, On local half-factorial orders, in *Arithmetical Properties of Commutative Rings and Monoids*, Chapman & Hall/CRC, Lect. Notes. Pure Appl. Math. 241-316 (2005).
- [10] J. H. B. KEMPERMAN, On complexes in a semigroup, *Indag. Math.* **18** (1956), 247-254.
- [11] S. LANG, *Algebra*, Graduate Texts in Mathematics, Springer-Verlag New York Inc (2005).
- [12] C. LECOUCVEY, Plünnecke and Kneser type theorems for dimension estimates, *Combinatorica*, **34**, (2014) 331-358.
- [13] B. MURPHY, Group action combinatorics, preprint arXiv: 190713569.
- [14] B. MURPHY, Group action in arithmetic combinatorics, Ph.D. Thesis, University of Rochester (2016).
- [15] D. MIRANDOLA, G. ZEMOR, Critical pairs for the product singleton bound, preprint 2015 arXiv: 150106419.
- [16] M. B. NATHANSON, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Graduate Text in Mathematics **165**, Springer-Verlag New York (1996).
- [17] J. E. OLSON, On the sum of two sets in a group, *J. Number Theory* **18** (1984), 110-120.
- [18] I. Z. RUZSA, *Sumsets and structure*, Combinatorial Number Theory and additive group theory, Springer New York (2009).
- [19] T. TAO, Non commutative sets of small doublings, *European Journal of Combinatorics* **34** (2013), 1459-1465.
- [20] T. TAO, Product set estimates for non-commutative groups, *Combinatorica* **28** (2009), 547-594.