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OUTER AUTOMORPHISMS OF CLASSICAL ALGEBRAIC GROUPS

ANNE QUÉGUINER-MATHIEU AND JEAN-PIERRE TIGNOL

ABSTRACT. The so-called Tits class, associated to an adjoint absolutely almost simple algebraic group, provides a cohomological obstruction for this group to admit an outer automorphism. If the group has inner type, this obstruction is the only one. In this paper, we prove this is not the case for classical groups of outer type, except for groups of type ${}^2\mathbf{A}_n$ with n even, or $n = 5$. More precisely, we prove a descent theorem for exponent 2 and degree 6 algebras with unitary involution, which shows that their automorphism groups have outer automorphisms. In all other relevant classical types, namely ${}^2\mathbf{A}_n$ with n odd, $n \geq 3$ and ${}^2\mathbf{D}_n$, we provide explicit examples where the Tits class obstruction is satisfied, and yet the group does not have outer automorphism. As a crucial tool, we use “generic” sums of algebras with involution.

1. INTRODUCTION

Every automorphism of an absolutely almost simple algebraic group scheme G of adjoint type over an arbitrary field F induces an automorphism of its Dynkin diagram Δ . Inner automorphisms of G act trivially on Δ , and there is an exact sequence of algebraic group schemes

$$(1) \quad 1 \rightarrow G \rightarrow \mathbf{Aut}(G) \rightarrow \mathbf{Aut}(\Delta) \rightarrow 1,$$

see [2, Exp. XXIV, 1.3, 3.6]. If G is split, the corresponding sequence of groups of rational points is exact and split, see [6, (25.16)], [11, §16.3]. Therefore, a split adjoint group G admits outer automorphisms if and only if its Dynkin diagram admits automorphisms, i.e., if G has type \mathbf{A}_n with $n \geq 2$, \mathbf{D}_n with $n \geq 3$ or \mathbf{E}_6 . Moreover, in all three cases, $\mathbf{Aut}(\Delta)(F)$ lifts to an isomorphic subgroup in $\mathbf{Aut}(G)(F)$. This property does not hold generally for nonsplit groups. For instance, if G is the connected component of the identity in the group scheme of automorphisms of a central simple F -algebra with quadratic pair (A, σ, f) , then G has no outer automorphisms if A is not split by the quadratic étale F -algebra defined by the discriminant of the quadratic pair, see §2.2 below. More generally, Garibaldi identified in [4, §2] a cohomological obstruction to the existence of outer automorphisms of an arbitrary absolutely almost simple algebraic group scheme G : the group $\mathbf{Aut}(\Delta)(F)$ acts

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on $H^2(F, C)$, where C is the center of the simply connected group scheme isogenous to G , and the Tits class $t_G \in H^2(F, C)$ is invariant under the action of the image of $\mathbf{Aut}(G)(F)$ in $\mathbf{Aut}(\Delta)(F)$. Therefore, automorphisms of Δ that do not leave t_G invariant do not lift to outer automorphisms of G . For adjoint or simply connected groups of inner type, Garibaldi showed in [4, §2] that this is the only obstruction to the lifting of automorphisms of Δ . As he explains in [4, Thm 11] this has interesting consequences in Galois cohomology. In a subsequent paper, Garibaldi–Petersson [5, Conjecture 1.1.2] conjectured that this Tits class obstruction is the only obstruction, also for adjoint or simply connected groups of outer type.

In this paper, we provide a complete answer to the question raised by Garibaldi and Petersson for groups of outer type A and D, leaving aside trialitarian groups (see the Appendix). Thus, in all the cases we consider, $\mathbf{Aut}(\Delta)(F)$ has order 2. Our main goal is to compare the following three conditions, listed from weaker to stronger:

- (Out 1): The Tits class t_G is fixed under $\mathbf{Aut}(\Delta)(F)$;
- (Out 2): G admits an outer automorphism defined over F ;
- (Out 3): G admits an outer automorphism of order 2 defined over F .

Under condition (Out 2), the sequence

$$1 \rightarrow G(F) \rightarrow \mathbf{Aut}(G)(F) \rightarrow \mathbf{Aut}(\Delta)(F) \rightarrow 1$$

is exact, and under condition (Out 3), it is split. In [4], Garibaldi proves that all three conditions are equivalent if G has inner type A or D (see Remarks 2.3 and 2.7). This is not the case for groups of outer type, and our main result is the following:

Theorem 1.1. *Let G be an absolutely almost simple adjoint or simply connected algebraic group scheme of type ${}^2\mathbf{A}_n$, with $n \geq 2$, or ${}^2\mathbf{D}_n$, with $n \geq 3$.*

- (1) *If G has type ${}^2\mathbf{A}_n$, with n even, or ${}^2\mathbf{A}_5$, then conditions (Out 1), (Out 2) and (Out 3) are equivalent.*
- (2) *In all the other types, there are examples of groups for which (Out 1) holds and (Out 2) does not hold, and examples of groups for which (Out 2) holds and (Out 3) does not hold.*

In other words, assertion (2) says there are examples where the condition on the Tits class is satisfied, and yet G does not have any outer automorphism, and examples where G has an outer automorphism, but no outer automorphism of order 2. In particular, this disproves Conjecture 1.1.2 in [5], and provides examples of simply connected absolutely simple algebraic group schemes G for which the Galois cohomology sequence

$$H^1(F, C) \rightarrow H^1(F, G) \rightarrow H^1(F, \mathbf{Aut}(G))$$

from [4, Thm 11(b)] (where C is the center of G) is not exact.

Every absolutely almost simple algebraic group scheme of adjoint type ${}^2\mathbf{A}_n$ over F is isomorphic to $\mathbf{PGU}(B, \tau) = \mathbf{Aut}_K(B, \tau)$ for some central simple algebra B of degree $n+1$ over a separable quadratic field extension K of F with a K/F -unitary involution τ . As explained below in §2.1, condition (Out 1) holds for the group $\mathbf{PGU}(B, \tau)$ if and only if B has exponent at most 2, and condition (Out 3) holds if and only if (B, τ) has a descent, i.e., $(B, \tau) = (B_0, \tau_0) \otimes_F (K, \iota)$ for some central

simple F -algebra with F -linear involution (B_0, τ_0) . For n even, Theorem 1.1(1) can be reformulated in a more precise form:

Theorem 1.2. *Let (B, τ) be a central simple algebra with unitary involution. If $\deg B$ is odd, then conditions (Out 1), (Out 2), and (Out 3) for $\mathbf{PGU}(B, \tau)$ are equivalent and hold if and only if B is split.*

The proof is easy: see Corollary 2.4.

Now, assume $G = \mathbf{PGU}(B, \tau)$ has type ${}^2\mathbf{A}_5$, i.e., B has degree 6. If the exponent of B is at most 2, then its index is at most 2. Therefore, Theorem 1.1(1) for such groups follows from the following descent theorem for algebras with unitary involution, proved in §4.1:

Theorem 1.3. *Let (B, τ) be a central simple algebra of degree at most 6 and index at most 2, with a K/F -unitary involution. There exists a central simple algebra with orthogonal involution (B_0, τ_0) over F , of the same index as B , such that $(B, \tau) = (B_0, \tau_0) \otimes (K, \iota)$, where ι is the unique nontrivial F -automorphism of K .*

It also follows from this theorem that assertion (1) does hold for groups of type ${}^2\mathbf{A}_3$ when the underlying algebra B has index at most 2; but this does not apply to all groups of type ${}^2\mathbf{A}_3$, since a degree 4 central simple algebra of exponent 2 can be of index 4. An example of a degree 4 and exponent 2 algebra with unitary involution that does not have a descent will be provided in §3.3.2 below (see Remark 3.14).

As usual for classical groups, we use as a crucial tool their explicit description in terms of algebras with involution or quadratic pair. How conditions (Out 1), (Out 2) and (Out 3) translate into conditions on these algebraic structures is explained in §2. Section 3 studies in more details the ${}^2\mathbf{D}_n$ case. In particular, we introduce our main tool for proving assertion (2) of Theorem 1.1, namely “generic” orthogonal sums of hermitian forms or involutions. In §4, using the same kind of strategy, we prove Theorem 1.3, and complete the proof of Theorem 1.1 by producing examples of outer type ${}^2\mathbf{A}_n$.

We refer the reader to [6] for definitions and basic facts on central simple algebras, involutions, and quadratic pairs. Recall that if $\text{char } F \neq 2$, then for any quadratic pair (σ, f) , σ is an involution of orthogonal type, and f is the map defined on the set $\text{Sym}(A, \sigma)$ of σ -symmetric elements by $f(s) = \frac{1}{2} \text{Trd}_A(s)$. Hence the quadratic pair is uniquely determined by the involution, and we usually write (A, σ) for (A, σ, f) in this case.

Notation. If \mathfrak{A} is a structure (such as an algebra with involution or an algebraic group scheme) defined over a field F , we write $\mathbf{Aut}(\mathfrak{A})$ for the algebraic group scheme of automorphisms of \mathfrak{A} and $\text{Aut}(\mathfrak{A})$ for its (abstract) group of rational points:

$$\text{Aut}(\mathfrak{A}) = \mathbf{Aut}(\mathfrak{A})(F).$$

We use a similar convention for classical groups; thus for instance if (B, τ) is a central simple algebra with unitary involution over a separable quadratic field extension K of F , then

$$\mathbf{PGU}(B, \tau) = \mathbf{Aut}_K(B, \tau) \quad \text{and} \quad \text{PGU}(B, \tau) = \mathbf{PGU}(B, \tau)(F).$$

Note that an absolutely almost simple simply connected algebraic group scheme and its isogenous adjoint group have the same automorphism group, hence it is

enough to consider adjoint groups. For isogenous groups that are neither adjoint nor simply connected, obstruction to the existence of an outer automorphism can arise from the fundamental group.

2. GROUPS OF TYPE A AND D, AND ASSOCIATED ALGEBRAS WITH INVOLUTION

The main purpose of this section is to point out how conditions (Out 1), (Out 2) and (Out 3) can be translated in terms of the corresponding algebra with involution or quadratic pair. Part of Theorem 1.1 follows immediately, as we will show. Throughout this section, F is an arbitrary field.

2.1. Type A. Let K be an étale quadratic F -algebra, and ι be the nontrivial F -automorphism of K . Consider a central simple K -algebra with K/F -unitary involution (B, τ) . We denote by $({}^tB, {}^t\tau)$ the conjugate algebra with involution defined by ${}^tB = \{{}^tx \mid x \in B\}$ with the operations

$${}^tx + {}^ty = {}^t(x + y), \quad {}^tx {}^ty = {}^t(xy), \quad \lambda {}^tx = {}^t(\iota(\lambda)x) \quad \text{and} \quad {}^t\tau({}^tx) = {}^t(\tau(x))$$

for $x, y \in B$ and $\lambda \in K$.

The following propositions were proven by Garibaldi–Petersson [5]:

Proposition 2.1. *Let $G = \mathbf{PGU}(B, \tau)$, with $\deg B \geq 3$.*

- (1) *Condition (Out 1) holds for G if and only if B has exponent at most 2;*
- (2) *Condition (Out 2) holds for G if and only if (B, τ) admits a ι -semilinear automorphism, i.e., (B, τ) is isomorphic to $({}^tB, {}^t\tau)$;*
- (3) *Condition (Out 3) holds for G if and only if (B, τ) admits a ι -semilinear automorphism of order 2.*

Proposition 2.2. *Condition (Out 3) holds for $\mathbf{PGU}(B, \tau)$ if and only if (B, τ) has a descent, i.e., there exists a central simple F -algebra with F -linear involution (B_0, τ_0) such that $(B, \tau) \simeq (B_0, \tau_0) \otimes (K, \iota)$.*

Proof of Proposition 2.1. Those assertions are taken from [5, §9]; for the reader's convenience, we briefly sketch an argument. One may understand the action of $\mathbf{Aut}(\Delta)(F)$ on the Tits class by looking at the action on the Tits algebras. For groups of type A, the symmetry of the diagram, together with the description of the Tits algebras given in [6, §27.B], shows that $t_{\mathbf{PGU}(B, \tau)}$ is invariant under the action of $\mathbf{Aut}(\Delta)(F)$ if and only if B is invariant under the action of the Galois group of K/F , i.e., if B is isomorphic to its conjugate tB . Since τ is a semilinear involution, it induces an anti-automorphism between B and tB . Therefore, B and tB are isomorphic if and only if B is isomorphic to its opposite algebra, i.e., B has exponent at most 2.

Recall from [6, (26.9)] that there is an equivalence of categories between the groupoid $\mathbf{A}_n(F)$ of central simple algebras of degree $n+1$ with a unitary involution over some étale quadratic F -algebra and the groupoid $\overline{\mathbf{A}}^n(F)$ of adjoint absolutely almost simple linear algebraic groups of type \mathbf{A}_n defined over F , under which (B, τ) maps to the adjoint group $\mathbf{PGU}(B, \tau)$. Hence, $\mathbf{PGU}(B, \tau)$ and (B, τ) have the same automorphisms. More precisely, the automorphisms of $\mathbf{PGU}(B, \tau)$ defined over F coincide with the F -automorphisms of (B, τ) , see [6, (26.10)]. Among those, the inner automorphisms are the K -linear automorphisms of (B, τ) , while outer automorphisms coincide with ι -semilinear automorphisms of (B, τ) . Therefore, $\mathbf{PGU}(B, \tau)$ admits an outer automorphism if and only if (B, τ) is isomorphic

to $({}^tB, {}^t\tau)$. Note that the condition $\deg B \geq 3$ is crucial here. Indeed, if $B = Q$ is a quaternion algebra, $\mathbf{PGU}(Q, \tau)$ has no outer automorphism, while (Q, τ) does admit semilinear automorphisms. \square

Proof of Proposition 2.2. If $(B, \tau) \simeq (B_0, \tau_0) \otimes (K, \iota)$, then $\text{Id}_{B_0} \otimes \iota$ is a semilinear automorphism of B which commutes with $\tau = \tau_0 \otimes \iota$, and has order 2. Therefore, it induces an outer automorphism of $\mathbf{PGU}(B, \tau)$ of order 2. Conversely, assume (B, τ) has a ι -semilinear automorphism φ of order 2. The F -algebra of fixed points $B_0 = B^\varphi$ is a central simple F -algebra of the same degree as B , hence

$$B = B_0 \otimes_F K.$$

Moreover, since φ commutes with τ , the restriction of τ induces an F -linear involution τ_0 of B_0 , and we have $(B, \tau) = (B_0, \tau_0) \otimes_F (K, \iota)$ as required. \square

Remark 2.3. If G has inner type ${}^1\mathbf{A}_n$, then $K \simeq F \times F$ and the corresponding algebra with involution (B, τ) is isomorphic to $(E \times E^{\text{op}}, \varepsilon)$ for some central simple F -algebra E , with ε the exchange involution (see [6, (2.14)]). If condition (Out 1) holds, then E has exponent at most 2, hence by a theorem of Albert (see [6, (3.1)]) E carries an F -linear involution γ . Identifying $E \otimes_F (F \times F)$ with $E \times E$, one may check that the map $(x, y) \in E \times E \mapsto (x, \gamma(y)^{\text{op}}) \in E \times E^{\text{op}}$ induces an isomorphism between $(E, \gamma) \otimes_F (F \times F, \iota)$ and $(E \times E^{\text{op}}, \varepsilon)$. Therefore $(E \times E^{\text{op}}, \varepsilon)$ has a descent, provided E has exponent at most 2. This shows that conditions (Out 1), (Out 2) and (Out 3) are equivalent for groups of inner type ${}^1\mathbf{A}_n$, as observed by Garibaldi [4, Ex. 17(i)]. Moreover, these conditions hold if and only if $G = \mathbf{PGU}(E \times E^{\text{op}}, \varepsilon) = \mathbf{PGL}(E)$ with E of exponent at most 2. If n is even, then E has odd degree $n + 1$, and the conditions hold if and only if E is split.

Combining Proposition 2.1 and 2.2 we already get Theorem 1.2. More precisely, we have

Corollary 2.4. *Let $G = \mathbf{PGU}(B, \tau)$ with $\deg B \geq 3$.*

- (1) *If B is split, then G admits outer automorphisms of order 2.*
- (2) *If G has type ${}^2\mathbf{A}_n$, with n even, conditions (Out 1), (Out 2) and (Out 3) are equivalent, and hold if and only if B is split.*

Proof. If B is split, we may assume $B = \text{End}_K V$ for some K -vector space V . Then τ is the adjoint involution with respect to some nondegenerate hermitian form $h: V \times V \rightarrow K$. Pick a diagonalization of h , corresponding to a K -basis $(e_i)_{1 \leq i \leq n}$ of V . For all i , we have $h(e_i, e_i) \in F^\times$, hence h restricts to a nondegenerate symmetric bilinear form b on the F -vector space $V_0 = e_1F + \cdots + e_nF$. Therefore, $(B, \tau) = (\text{End}_F V_0, \text{ad}_b) \otimes_F (K, \iota)$ has a descent, so (Out 3) holds for $\mathbf{PGU}(B, \tau)$.

Now, assume that G has type ${}^2\mathbf{A}_n$ for some $n \geq 3$, with n even. Then $G = \mathbf{PGU}(B, \tau)$, where B has odd degree $n + 1$. Hence, under condition (Out 1), B is split, so (Out 3) holds by the first assertion, and this concludes the proof. \square

Corollary 2.4 was proved by Garibaldi–Petersson, see [5, Cor 9.1.2].

To prove Theorem 1.1(2), we will give in § 3 and § 4 examples of algebras with unitary involutions (B, τ) such that either B has exponent 2 and (B, τ) is not isomorphic to its conjugate $({}^tB, {}^t\tau)$, or (B, τ) and $({}^tB, {}^t\tau)$ are isomorphic, yet (B, τ) does not have a descent. We provide examples of degree 4 and index 4, and examples of degree $n + 1$ and index 2 for all odd $n \geq 7$; see Remark 3.14 and § 4.3.

2.2. Type D. Let A be a central simple F -algebra of even degree, and let (σ, f) be a quadratic pair on A . We write $\mathrm{GO}(A, \sigma, f)$ for the (abstract) group of similitudes of (A, σ, f) , defined as

$$\mathrm{GO}(A, \sigma, f) = \{g \in A^\times \mid \sigma(g)g \in F^\times \text{ and } f \circ \mathrm{Int}(g) = f\}.$$

The scalar $\mu(g) = \sigma(g)g$ is called the *multiplier* of g . Mapping $g \in \mathrm{GO}(A, \sigma, f)$ to $\mathrm{Int}(g)$ yields an identification of $\mathrm{GO}(A, \sigma, f)/F^\times$ with the group of rational points $\mathrm{PGO}(A, \sigma, f) = \mathrm{Aut}(A, \sigma, f)$. Every automorphism of (A, σ, f) induces an automorphism of the Clifford algebra $C(A, \sigma, f)$. A similitude is said to be *proper* if the induced automorphism of $C(A, \sigma, f)$ is the identity on the center Z ; otherwise it is said to be *improper*. The proper similitudes form a subgroup $\mathrm{GO}^+(A, \sigma, f)$ which satisfies $\mathrm{GO}^+(A, \sigma, f)/F^\times = \mathrm{PGO}^+(A, \sigma, f)$ for $\mathbf{PGO}^+(A, \sigma, f)$ the connected component of the identity in $\mathbf{PGO}(A, \sigma, f) = \mathbf{Aut}(A, \sigma, f)$.

If $A = \mathrm{End}_F V$ for some F -vector space V , then every quadratic pair (σ, f) on A is adjoint to some nonsingular quadratic form q on V , see [6, (5.11)]. In that case, we write simply $\mathrm{GO}(V, q)$, $\mathbf{PGO}(V, q)$, etc. for $\mathrm{GO}(A, \sigma, f)$, $\mathbf{PGO}(A, \sigma, f)$, etc.

Proposition 2.5. *Let $G = \mathbf{PGO}^+(A, \sigma, f)$, with $\deg A = 2n \geq 4$, and let Z be the discriminant quadratic F -algebra of (σ, f) , i.e., Z is the center of the Clifford algebra $C(A, \sigma, f)$. Assume Z is a field.*

- (1) *Condition (Out 1) holds for G if and only if A is split by Z ;*
- (2) *Condition (Out 2) holds for G if and only if (A, σ, f) admits improper similitudes;*
- (3) *Condition (Out 3) holds for G if and only if (A, σ, f) admits square-central improper similitudes.*

In particular, condition (Out 1) holds if and only if the algebra A is Brauer-equivalent to a quaternion algebra split by Z . This condition is necessary for the existence of an improper similitude by the generalization of Dieudonné's theorem on multipliers of similitudes given in [6, (13.38)].

Proof. (1): Let ι denote the nontrivial F -automorphism of Z , and let $C = C(A, \sigma, f)$. The Tits class t_G is invariant under the action of $\mathrm{Aut}(\Delta)$ if and only if C is isomorphic to its conjugate algebra ${}^\iota C$, or equivalently $C \otimes_Z {}^\iota C^{\mathrm{op}}$ is split. Recall from [6, (9.12)] the fundamental relations between A and C : if n is even, then $C \otimes_Z C$ is split and the corestriction $\mathrm{Cor}_{Z/F} C$ is Brauer-equivalent to A . After scalar extension to Z , it follows from the latter relation that the Z -algebra A_Z is Brauer-equivalent to $C \otimes_Z {}^\iota C$. If n is odd, then $C \otimes_Z C$ is Brauer-equivalent to A_Z , while $\mathrm{Cor}_{Z/F} C$ is split, hence $C \otimes_Z {}^\iota C$ is split. Thus, in each case A_Z is Brauer-equivalent to $C \otimes_Z {}^\iota C^{\mathrm{op}}$, and we get that (Out 1) holds for G if and only if A_Z is split.

(2) and (3): If $\deg A \neq 8$, we may argue along the same lines as for Proposition 2.1, using the equivalence of categories between the groupoid $\mathbf{D}_n(F)$ of central simple F -algebras of degree $2n$ with quadratic pair and the groupoid $\overline{\mathbf{D}}^n(F)$ of adjoint absolutely almost simple groups of type \mathbf{D}_n , which maps the algebra A with quadratic pair (σ, f) to $\mathbf{PGO}^+(A, \sigma, f)$, see [6, (26.15)]. This line of argument does not apply to the case where $\deg A = 8$, however, because the description of $\mathbf{D}_4(F)$ is different (see [6]). Therefore, we give a different proof, which applies in all cases where $\deg A = 2n \geq 4$.

We will need the following lemma, which is probably well-known:

Lemma 2.6. *Let (V, q) be a hyperbolic space of dimension $2n \geq 4$ over an infinite field E . The map $\mathrm{PGO}(V, q) \rightarrow \mathrm{Aut}(\mathbf{PGO}^+(V, q))$ which carries $g \in E^\times$ to $\mathrm{Int}(g)$ is injective.*

Proof. Let b be the polar bilinear form of q , and let $e_1, f_1, \dots, e_n, f_n$ be a symplectic base of (V, q) , i.e., a base such that $q(e_i) = q(f_i) = 0$ and

$$b(e_i, f_i) = 1, \quad b(e_i, e_j) = b(e_i, f_j) = b(f_i, f_j) = 0 \text{ for all } i, j = 1, \dots, n \text{ with } i \neq j.$$

Since E is infinite, we may find $\alpha_1, \dots, \alpha_n \in E^\times$ such that $\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}$ are pairwise distinct and moreover, if $\mathrm{char} E \neq 2$,

$$\{\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}\} \neq \{-\alpha_1, -\alpha_1^{-1}, \dots, -\alpha_n, -\alpha_n^{-1}\}.$$

Consider the proper isometry $a \in \mathrm{GO}^+(V, q)$ defined by

$$a(e_i) = \alpha_i e_i \quad \text{and} \quad a(f_i) = \alpha_i^{-1} f_i \quad \text{for } i = 1, \dots, n.$$

Let $g \in \mathrm{GO}(V, q)$ be such that $\mathrm{Int}(g)$ is the identity on $\mathrm{PGO}^+(V, q)$. Then $g^{-1}ag = \lambda a$ for some $\lambda \in E^\times$. Because $g^{-1}ag$ and a are isometries, we must have $\lambda = \pm 1$. Moreover, by evaluating $ag = \lambda ga$ on e_1, \dots, f_n , we obtain

$$ag(e_i) = \lambda \alpha_i g(e_i) \quad \text{and} \quad ag(f_i) = \lambda \alpha_i^{-1} g(f_i) \quad \text{for } i = 1, \dots, n.$$

Thus, $g(e_i)$ (resp. $g(f_i)$) is an eigenvector of a with eigenvalue $\lambda \alpha_i$ (resp. $\lambda \alpha_i^{-1}$). But the eigenvalues of a are $\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}$, hence

$$\{\lambda \alpha_1, \lambda \alpha_1^{-1}, \dots, \lambda \alpha_n, \lambda \alpha_n^{-1}\} = \{\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}\}$$

with $\lambda = \pm 1$. By the choice of $\alpha_1, \dots, \alpha_n$ we must have $\lambda = 1$, hence $g(e_i)$ must be a scalar multiple of e_i and $g(f_i)$ a scalar multiple of f_i . Therefore, there exist $\gamma_1, \dots, \gamma_n \in E^\times$ such that, letting $\mu = \mu(g)$ be the multiplier of g ,

$$g(e_i) = \gamma_i e_i \quad \text{and} \quad g(f_i) = \mu \gamma_i^{-1} f_i.$$

Thus, the matrix of g with respect to the base e_1, \dots, f_n is diagonal. Using [6, (12.24)] if $\mathrm{char} E \neq 2$ and [6, (12.12)] if $\mathrm{char} E = 2$, it is then easy to check that g is a proper similitude. Since the map of algebraic group schemes $\mathbf{PGO}^+(V, q) \rightarrow \mathbf{Aut}(\mathbf{PGO}^+(V, q))$ is injective (cf. (1)), it follows that the homomorphism $\mathrm{PGO}^+(V, q) \rightarrow \mathrm{Aut}(\mathbf{PGO}^+(V, q))$ is injective, hence $g \in F^\times$. \square

Proof of Proposition 2.5(2) and (3). The map $g \mapsto \mathrm{Int}(g)$ induces a map of algebraic group schemes Φ which fits in the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{PGO}^+(A, \sigma, f) & \longrightarrow & \mathbf{PGO}(A, \sigma, f) & \longrightarrow & \mathbf{Aut}_F(Z) \longrightarrow 1 \\ & & \parallel & & \downarrow \Phi & & \downarrow \Psi \\ 1 & \longrightarrow & \mathbf{PGO}^+(A, \sigma, f) & \longrightarrow & \mathbf{Aut}(\mathbf{PGO}^+(A, \sigma, f)) & \longrightarrow & \mathbf{Aut}(\Delta) \longrightarrow 1 \end{array}$$

The differential $d\Phi$ is injective, since the restriction of Φ to the connected component of the identity $\mathbf{PGO}^+(A, \sigma, f)$ is the identity map. Moreover, Lemma 2.6 shows that over an algebraic closure F_{alg} the map

$$\Phi_{\mathrm{alg}}: \mathbf{PGO}(A, \sigma, f)(F_{\mathrm{alg}}) \rightarrow \mathbf{Aut}(\mathbf{PGO}^+(A, \sigma, f))(F_{\mathrm{alg}})$$

is injective. It follows by [6, (22.2)] that Φ is injective, and likewise Ψ is injective. We have $\mathbf{Aut}(\Delta) \simeq \mathbf{Aut}_F(Z)$ if $n \neq 4$, and $\mathbf{Aut}(\Delta) \simeq \mathbf{Aut}_F(F \times Z)$ if $n = 4$. Since Z is assumed to be a field, in each case the group of F -rational points is

$$\mathbf{Aut}(\Delta) \simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathbf{Aut}_F(Z).$$

Therefore, the diagram above yields the following diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{PGO}^+(A, \sigma, f) & \longrightarrow & \mathbf{PGO}(A, \sigma, f) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & & \parallel & & \downarrow \Phi_F & & \parallel \\ 1 & \longrightarrow & \mathbf{PGO}^+(A, \sigma, f) & \longrightarrow & \mathbf{Aut}(\mathbf{PGO}^+(A, \sigma, f)) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

It follows that Φ_F is an isomorphism, which proves (2) and (3) of Proposition 2.5. \square

Remark 2.7. (i) If the algebra A is split, which means that $\mathbf{PGO}^+(A, \sigma, f) = \mathbf{PGO}^+(V, q)$ for some quadratic space (V, q) , then (Out 3) holds, since each quadratic space admits improper isometries of order 2.

(ii) The arguments in the proof of Proposition 2.5 also apply in the case where $Z \simeq F \times F$. It follows that in this case (Out 1), (Out 2), and (Out 3) are equivalent, and hold if and only if A is split. Thus, adjoint groups of inner type D_n admit outer automorphisms of order 2 whenever the Tits class obstruction vanishes, as pointed out by Garibaldi [4].

In the outer case, condition (Out 3) induces additional restrictions on the algebra A when its degree is divisible by 4, as we now proceed to show:

Lemma 2.8. *Let $G = \mathbf{PGO}^+(A, \sigma, f)$ for some F -algebra with quadratic pair (A, σ, f) , such that $\deg A \equiv 0 \pmod{4}$, so G has type D_n with n even. If G admits an outer automorphism of order 2, then A is split.*

Proof. In view of Remark 2.7, it suffices to consider the case where the center Z of the Clifford algebra $C = C(A, \sigma, f)$ is a field. By Proposition 2.5, if G admits an outer automorphism of order 2, then (A, σ, f) admits a square-central improper similitude g . As explained in [6, §13.A], g induces an automorphism $C(g)$ of order 2 of C , which commutes with the canonical involution $\underline{\sigma}$. Moreover, since g is improper, $C(g)$ acts non trivially on Z . Therefore, the fixed points $C^{C(g)}$ form an F -algebra C_0 of the same degree as C , and we have $C \simeq C_0 \otimes_F Z$. Since $C(g)$ commutes with the canonical involution $\underline{\sigma}$ of the Clifford algebra, $\underline{\sigma}$ restricts to an F -linear involution on C_0 , so C_0 has exponent at most 2. In view of the fundamental relations [6, (9.12)], we get that A is Brauer-equivalent to $\mathrm{Cor}_{Z/F}(C_0 \otimes_F Z) \simeq C_0 \otimes C_0 \sim 0$, hence A is split, as required. \square

To prove Theorem 1.1, we will construct in § 3.3 below examples of algebras with quadratic pairs such that either A is split by the discriminant quadratic extension, yet (A, σ, f) does not admit improper similitudes, or (A, σ, f) admits improper similitudes, but no improper similitudes of order 2. We provide examples of degree $2n$ for arbitrary $n \geq 3$. The index of A is 2, as required by condition (Out 1).

3. OUTER AUTOMORPHISMS AND SIMILITUDES: THE ORTHOGONAL CASE

Throughout this section, we assume that the base field F has characteristic different from 2. Hence, we consider orthogonal involutions instead of quadratic pairs. Our goal is to produce examples of groups of type ${}^2\mathbf{D}_n$, for all $n \geq 3$, for which (Out 1) holds and (Out 2) fails, or (Out 2) holds and (Out 3) fails. Before describing the explicit examples, we first recall a few well-known facts on similitudes of hermitian forms, and we introduce our main tool in this section, namely “generic” sums of hermitian forms.

By Proposition 2.5(1), if $\mathbf{PGO}^+(A, \sigma)$ satisfies (Out 1), then A is split by the discriminant quadratic algebra Z . In particular, A has index at most 2. Moreover, Remark 2.7 shows that we may assume A is not split. Hence, our main case of interest is when $A = M_n(Q)$ for some quaternion division algebra Q over F . However, our discussion of generic sums is more general, because we think this tool could be useful in various other contexts.

3.1. Similitudes of hermitian forms. Let D be a central division F -algebra. Assume D carries an F -linear involution ρ , and let $\delta = \pm 1$. Let (V, h) be a δ -hermitian space over (D, ρ) . By definition, an element $g \in \text{End}_D V$ is a similitude of (V, h) with multiplier $\mu(g) = \mu$ if

$$h(g(x), g(y)) = \mu h(x, y) \quad \text{for all } x, y \in V.$$

We write $\text{Sim}(V, h)$ or $\text{Sim}(h)$ for the group of similitudes of (V, h) , which is also the group of similitudes of $\text{End}_D V$ for the adjoint involution ad_h . Depending on δ and the type of the reference involution ρ , this group is a form of an orthogonal or a symplectic group:

$$\text{Sim}(V, h) = \begin{cases} \text{GO}(\text{End}_D V, \text{ad}_h) & \text{if } \text{ad}_h \text{ is orthogonal,} \\ \text{GSp}(\text{End}_D V, \text{ad}_h) & \text{if } \text{ad}_h \text{ is symplectic.} \end{cases}$$

For the rest of this subsection, let $A = \text{End}_D V$ and $\deg A = 2n$, and suppose ad_h is orthogonal; this case occurs if and only if $\delta = 1$ and ρ is orthogonal, or $\delta = -1$ and ρ is symplectic, see [6, (4.2)]. Since $\text{char } F \neq 2$, we may distinguish as follows between proper and improper similitudes: for $g \in \text{Sim}(V, h)$, taking the reduced norm of each side of the equation $\mu(g) = \sigma(g)g$, we see that $\mu(g)^{2n} = \text{Nrd}_A(g)^2$, hence $\text{Nrd}_A(g) = \pm \mu(g)^n$. The similitude g is proper if $\text{Nrd}_A(g) = \mu(g)^n$, and improper if $\text{Nrd}_A(g) = -\mu(g)^n$ (see [6, (12.24)]).

Suppose now $V = V_1 \perp \dots \perp V_r$ for some subspaces $V_1, \dots, V_r \subset V$, hence h restricts to a nonsingular δ -hermitian form h_i on each V_i . For $i = 1, \dots, r$, let $A_i = \text{End}_D V_i$, pick $g_i \in A_i$, and let $g = g_1 \oplus \dots \oplus g_r \in A$ be the map defined by

$$g(x_1 + \dots + x_r) = g_1(x_1) + \dots + g_r(x_r) \quad \text{for } x_1 \in V_1, \dots, x_r \in V_r.$$

Lemma 3.1. *With the notation above, g is a similitude of h with multiplier μ if and only if each g_i is a similitude of h_i with multiplier μ . When this condition holds, the similitude g is proper if and only if the number of improper similitudes among g_1, \dots, g_r is even.*

Proof. The first part is clear since $h(g(x), g(y)) = \mu h(x, y)$ for all $x, y \in V$ if and only if $h_i(g_i(x), g_i(y)) = \mu h_i(x, y)$ for all i , and all $x, y \in V_i$. To prove the second part, let $\deg A_i = 2n_i$ for $i = 1, \dots, r$, hence $n = n_1 + \dots + n_r$, and suppose

$\text{Nrd}_{A_i}(g_i) = \varepsilon_i \mu^{n_i}$ with $\varepsilon_i = \pm 1$. We then have

$$\text{Nrd}_A(g) = \prod_{i=1}^r \text{Nrd}_{A_i}(g_i) = \left(\prod_{i=1}^r \varepsilon_i \right) \mu^{n_1 + \dots + n_r}. \quad \square$$

We next consider the particular case where D is a quaternion division algebra Q and ρ is the canonical involution $-$, hence $\delta = -1$. The generalization of Dieudonné's theorem on multipliers of similitudes [6, (13.38)] then allows to distinguish between proper and improper similitudes as follows: a similitude g of (V, h) is proper if the quaternion algebra $(Z, \mu(g))_F$ is split (we write simply $(Z, \mu(g))_F = 0$ in this case), and improper if it is isomorphic to Q . For 1-dimensional skew-hermitian forms, we have the following more precise result:

Lemma 3.2. *Let q be a nonzero pure quaternion in a quaternion division algebra Q , and let $a = q^2 \in F^\times$. Define*

$$G_+(a) = \{\mu \in F^\times \mid (a, \mu)_F = 0\} \quad \text{and} \quad G_-(a) = \{\mu \in F^\times \mid (a, \mu)_F = Q\}.$$

Then $G_+(a)$ is the group of multipliers of proper similitudes of the skew-hermitian form $\langle q \rangle$, and $G_-(a)$ is the coset of multipliers of improper similitudes of $\langle q \rangle$. Moreover, the improper similitudes of $\langle q \rangle$ are all square-central.

Proof. The lemma follows from the explicit description of similitudes of $\langle q \rangle$ given in [6, (12.18)]: the proper similitudes form the multiplicative group $F(q)^\times \subset Q^\times$, while the improper similitudes are the elements $u \in Q^\times$ such that $uq = -qu$. \square

In the case where each V_i is 1-dimensional, Lemma 3.1 yields:

Lemma 3.3. *Let q_1, \dots, q_n be pure quaternions in Q , consider the skew-hermitian form h over $(Q, -)$ defined by $h = \langle q_1, \dots, q_n \rangle$, and let $a_i = q_i^2 \in F^\times$. If $\mu \in F^\times$ satisfies $(\mu, a_i)_F \in \{0, Q\}$ for all i , then (V, h) admits a similitude with multiplier μ . Moreover, this similitude is proper if and only if the number of pure quaternions among q_1, \dots, q_n satisfying $(\mu, a_i)_F = Q$ is even.*

Proof. From the condition on μ , it follows by Lemma 3.2 that each $\langle q_i \rangle$ admits a similitude g_i with multiplier μ . Then $g = g_1 \oplus \dots \oplus g_n$ is a similitude of h with multiplier μ . Lemma 3.1 shows that this similitude is proper if and only if the number of indices i such that $(\mu, a_i)_F = Q$ is even. \square

Of course, most similitudes do not act diagonally, and the multipliers of similitudes of (V, h) need not satisfy the condition given in the lemma; nevertheless, as we explain in the next section, this condition actually characterizes multipliers of similitudes for some particular involutions, which we call “generic sums of orthogonal involutions.”

3.2. Generic sums. Let D be a central division algebra over an arbitrary field F of characteristic different from 2. Assume D carries an involution ρ of the first kind, let $\delta = \pm 1$, and let $(V_1, h_1), \dots, (V_n, h_n)$ be δ -hermitian spaces over (D, ρ) . Consider the field of iterated Laurent series in n indeterminates

$$\widehat{F} = F((t_1)) \dots ((t_n)),$$

and let

$$\widehat{D} = D \otimes_F \widehat{F} \quad \text{and} \quad \widehat{V}_i = V_i \otimes_F \widehat{F} \quad \text{for } i = 1, \dots, n.$$

The involution ρ extends to an involution $\widehat{\rho} = \rho \otimes \text{Id}_{\widehat{F}}$ on \widehat{D} . We also extend h_i to a δ -hermitian form \widehat{h}_i on \widehat{V}_i , and we let

$$(\widehat{V}, \widehat{h}) = (\widehat{V}_1 \oplus \cdots \oplus \widehat{V}_n, \langle t_1 \rangle \widehat{h}_1 \perp \cdots \perp \langle t_n \rangle \widehat{h}_n).$$

The adjoint involution $\text{ad}_{\widehat{h}}$ is an orthogonal sum, in the sense of Dejaiffe [1], of the involutions $\text{ad}_{\widehat{h}_i}$; we call it a ‘‘generic orthogonal sum’’ since each \widehat{h}_i is extended from an involution h_i defined over F , and scaled by some indeterminate t_i . We assume throughout that h_1, \dots, h_n are anisotropic, hence \widehat{h} is anisotropic. Our goal is to relate the multipliers of similitudes of $(\widehat{V}, \widehat{h})$ to the multipliers of similitudes of $(V_1, h_1), \dots, (V_n, h_n)$, with the help of a norm on the vector space \widehat{V} , i.e., a valuation-like map for which \widehat{V} contains a splitting base (see [9, §2]). More precisely, we prove:

Theorem 3.4. *Let $(\widehat{V}, \widehat{h})$ be a ‘‘generic sum’’ of δ -hermitian spaces (V_i, h_i) for $1 \leq i \leq n$, as defined above.*

- (1) *If $n \geq 3$, every similitude $g \in \text{Sim}(\widehat{V}, \widehat{h})$ has the form $g = \lambda g'$ for some $\lambda \in \widehat{F}^\times$ and some similitude g' with multiplier in F^\times .*
- (2) *For every similitude $g \in \text{Sim}(\widehat{V}, \widehat{h})$ such that $\mu(g) \in F^\times \subset \widehat{F}^\times$, there exist similitudes $g_i \in \text{Sim}(V_i, h_i)$ for $i = 1, \dots, n$ with $\mu(g) = \mu(g_1) = \cdots = \mu(g_n)$.*

Proof. The field \widehat{F} carries the (t_1, \dots, t_n) -adic valuation v with value group \mathbb{Z}^n ordered lexicographically from right to left. This valuation is Henselian; it extends in a unique way to a valuation on \widehat{D} with value group \mathbb{Z}^n . We write again v for this valuation on \widehat{D} . Because \widehat{h} is anisotropic and v is Henselian, we may define a norm ν on \widehat{V} by the following formula (see [10, Prop. 3.1], [9, Cor. 3.6, Th. 4.6, Prop. 4.2]):

$$\nu(x) = \frac{1}{2}v(\widehat{h}(x, x)) \quad \text{for } x \in \widehat{V}.$$

To describe the value set of this norm, let $\varepsilon_i = (0, \dots, 1, \dots, 0) = v(t_i)$ be the i -th element in the standard base of \mathbb{Z}^n . For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ we write $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \in \widehat{F}$. Every nonzero vector $x_i \in \widehat{V}_i$ can be written as a series $x_i = \sum_{\alpha} x_{i\alpha} t^\alpha$ with $x_{i\alpha} \in V_i$, where the support $\{\alpha \mid x_{i\alpha} \neq 0\}$ is a well-ordered subset of \mathbb{Z}^n . If α_0 is the minimal element in this support, then since $\widehat{h}(x_{i\alpha_0}, x_{i\alpha_0}) = t_i \widehat{h}_i(x_{i\alpha_0}, x_{i\alpha_0}) \neq 0$ we have $v(\widehat{h}(x_i, x_i)) = 2\alpha_0 + \varepsilon_i$. Thus,

$$\nu(\widehat{V}_i \setminus \{0\}) = \frac{1}{2}\varepsilon_i + \mathbb{Z}^n \subset (\frac{1}{2}\mathbb{Z})^n.$$

It follows that $\nu(\widehat{V}_i) \cap \nu(\widehat{V}_j) = \{\infty\}$ for $i \neq j$. Therefore, for $x = x_1 + \cdots + x_n$ with $x_i \in \widehat{V}_i$ for all i , we have

$$(2) \quad \nu(x) = \min(\nu(x_1), \dots, \nu(x_n)).$$

Thus, the value set of \widehat{V} , for which we use the notation $\Gamma_{\widehat{V}}$, is

$$\Gamma_{\widehat{V}} = \{\nu(x) \mid x \in \widehat{V} \setminus \{0\}\} = \bigcup_{i=1}^n (\frac{1}{2}\varepsilon_i + \mathbb{Z}^n) \subset (\frac{1}{2}\mathbb{Z})^n.$$

We also need to consider the graded structures associated to norms and valuations. For $\alpha \in \mathbb{Z}^n$ we let

$$\widehat{F}_{\geq \alpha} = \{a \in F \mid v(a) \geq \alpha\}, \quad \widehat{F}_{> \alpha} = \{a \in F \mid v(a) > \alpha\}, \quad \text{and } \widehat{F}_\alpha = \widehat{F}_{\geq \alpha} / \widehat{F}_{> \alpha}.$$

Thus, \widehat{F}_α is a 1-dimensional vector space over F , spanned by the image of t^α . We let

$$\mathbf{gr}(\widehat{F}) = \bigoplus_{\alpha \in \mathbb{Z}^n} \widehat{F}_\alpha.$$

For each nonzero $a \in \widehat{F}$, let $\tilde{a} = a + \widehat{F}_{>v(a)} \in \mathbf{gr}(\widehat{F})$. We also let $\tilde{0} = 0 \in \mathbf{gr}(\widehat{F})$, and note that the multiplication in \widehat{F} induces a multiplication on $\mathbf{gr}(\widehat{F})$, which turns this F -vector space into a commutative graded ring in which every nonzero homogeneous element is invertible:

$$\mathbf{gr}(\widehat{F}) = F[\tilde{t}_1, \tilde{t}_1^{-1}, \dots, \tilde{t}_n, \tilde{t}_n^{-1}].$$

The same construction can be applied to \widehat{D} , yielding the graded ring $\mathbf{gr}(\widehat{D}) = D \otimes_F \mathbf{gr}(\widehat{F})$, and also to \widehat{V} , yielding the graded module $\mathbf{gr}(\widehat{V})$ over $\mathbf{gr}(\widehat{D})$. From (2) it follows that

$$\mathbf{gr}(\widehat{V}) = \mathbf{gr}(\widehat{V}_1) \oplus \dots \oplus \mathbf{gr}(\widehat{V}_n),$$

see [9, Remark 2.6]. For each i , we have $\mathbf{gr}(\widehat{V}_i) = V_i \otimes_F \mathbf{gr}(\widehat{F})$, with a grading shifted by $\frac{1}{2}\varepsilon_i$. Note that the grade sets of $\mathbf{gr}(\widehat{V}_1), \dots, \mathbf{gr}(\widehat{V}_n)$, which are the value sets of $\widehat{V}_1, \dots, \widehat{V}_n$, are pairwise disjoint, hence every homogeneous component of $\mathbf{gr}(\widehat{V})$ lies in exactly one $\mathbf{gr}(\widehat{V}_i)$.

Let $\tilde{\rho} = \rho \otimes \text{Id}_{\mathbf{gr}(F)}$ be the involution of the first kind on $\mathbf{gr}(\widehat{D})$ extending ρ . By [9, Th. 4.6, Prop. 4.2], we have $v(\widehat{h}(x, y)) \geq \nu(x) + \nu(y)$ for all $x, y \in \widehat{V}$. Therefore, the δ -hermitian form \widehat{h} induces a δ -hermitian form \tilde{h} on $\mathbf{gr}(\widehat{V})$, defined on homogeneous elements by

$$\tilde{h}(\tilde{x}, \tilde{y}) = \begin{cases} \widetilde{\widehat{h}(x, y)} & \text{if } v(\widehat{h}(x, y)) = \nu(x) + \nu(y), \\ 0 & \text{if } v(\widehat{h}(x, y)) > \nu(x) + \nu(y), \end{cases}$$

and extended by bilinearity to $\mathbf{gr}(\widehat{V})$. Letting \tilde{h}_i denote the restriction of \tilde{h} to $\mathbf{gr}(\widehat{V}_i)$, we have

$$(\mathbf{gr}(\widehat{V}), \tilde{h}) = (\mathbf{gr}(\widehat{V}_1), \tilde{h}_1) \perp \dots \perp (\mathbf{gr}(\widehat{V}_n), \tilde{h}_n).$$

As observed above, we have $\mathbf{gr}(\widehat{V}_i) = V_i \otimes_F \mathbf{gr}(\widehat{F})$. The δ -hermitian form \tilde{h}_i is given by

$$\tilde{h}_i(x, y) = h_i(x, y) \otimes \tilde{t}_i \quad \text{for } x, y \in V_i.$$

Now, suppose $g: \widehat{V} \rightarrow \widehat{V}$ is a similitude of $(\widehat{V}, \widehat{h})$, with multiplier $\mu(g) \in \widehat{F}^\times$. For $x \in \widehat{V}$ we have $\widehat{h}(g(x), g(x)) = \mu(g)\widehat{h}(x, x)$, hence

$$\nu(g(x)) = \nu(x) + \frac{1}{2}v(\mu(g)) \quad \text{for all } x \in \widehat{V}.$$

As a result, g induces a homomorphism of $\mathbf{gr}(\widehat{D})$ -modules $\tilde{g}: \mathbf{gr}(\widehat{V}) \rightarrow \mathbf{gr}(\widehat{V})$, defined on homogeneous elements by

$$\tilde{g}(\tilde{x}) = \widetilde{g(x)} \quad \text{for } x \in \widehat{V}.$$

This homomorphism is a similitude of $(\mathbf{gr}(\widehat{V}), \tilde{h})$ with multiplier $\widetilde{\mu(g)}$, and it shifts the grading by $\frac{1}{2}v(\mu(g))$. It follows that the value set $\Gamma_{\widehat{V}}$, which is the grade set of $\mathbf{gr}(\widehat{V})$, is stable under translation by $\frac{1}{2}v(\mu(g)) \in (\frac{1}{2}\mathbb{Z})^n$. We must therefore have for all $i = 1, \dots, n$

$$\frac{1}{2}\varepsilon_i + \frac{1}{2}v(\mu(g)) \in \bigcup_{\ell=1}^n (\frac{1}{2}\varepsilon_\ell + \mathbb{Z}^n).$$

Suppose i, j are such that $\frac{1}{2}\varepsilon_i + \frac{1}{2}v(\mu(g)) \in \frac{1}{2}\varepsilon_j + \mathbb{Z}^n$, and $i \neq j$. For $k \neq i, j$ we then have

$$\frac{1}{2}\varepsilon_k + \frac{1}{2}v(\mu(g)) \in \frac{1}{2}\varepsilon_k + \frac{1}{2}\varepsilon_j - \frac{1}{2}\varepsilon_i + \mathbb{Z}^n \not\subset \bigcup_{\ell=1}^n (\frac{1}{2}\varepsilon_\ell + \mathbb{Z}^n).$$

This contradiction implies that $\frac{1}{2}\varepsilon_i + \frac{1}{2}v(\mu(g)) \in \frac{1}{2}\varepsilon_i + \mathbb{Z}^n$ for all i , hence $v(\mu(g)) \in 2\mathbb{Z}^n$. Let $v(\mu(g)) = 2v(\lambda_0)$ for some $\lambda_0 \in \widehat{F}^\times$, hence $v(\mu(\lambda_0^{-1}g)) = 0$. Consider the residue $\mu(\lambda_0^{-1}g) = a \in F^\times$. We have $\mu(\lambda_0^{-1}g) = a(1+m)$ for some $m \in \widehat{F}$ with $v(m) > 0$. Since \widehat{F} is Henselian and the characteristic of the residue field F is different from 2, we may find $\lambda_1 \in \widehat{F}$ with $\lambda_1^2 = 1+m$. Then $\mu(\lambda_1^{-1}\lambda_0^{-1}g) = a$, so we may write $g = \lambda g'$ with $\lambda = \lambda_0\lambda_1 \in \widehat{F}^\times$ and $g' = \lambda^{-1}g$. Then $g' \in \text{Sim}(\widehat{V}, \widehat{h})$ and $\mu(g') = a \in F^\times$. The first assertion of the theorem is thus proved.

Now, we prove the second assertion. Consider a similitude $g \in \text{Sim}(\widehat{V}, \widehat{h})$ and assume its multiplier $\mu(g)$ is in $F^\times \subset \widehat{F}^\times$. Since $v(\mu(g)) = 0$, the similitude $\tilde{g} \in \text{Sim}(\text{gr}(\widehat{V}), \tilde{h})$ preserves the grading. We may therefore consider its restriction g_i to the homogeneous component of degree $\frac{1}{2}\varepsilon_i$, which is V_i . Because \tilde{g} is a similitude with multiplier $\widetilde{\mu(g)} = \mu(g)$ and

$$\tilde{h}(x, y) = h_i(x, y)\tilde{t}_i \quad \text{for } x, y \in V_i,$$

it follows that g_i is a similitude of (V_i, h_i) with multiplier $\mu(g)$. \square

The last part of the proof above establishes the following result:

Lemma 3.5. *For every similitude $g \in \text{Sim}(\widehat{V}, \widehat{h})$ such that $\mu(g) \in F^\times \subset \widehat{F}^\times$, the similitude $\tilde{g} \in \text{Sim}(\text{gr}(\widehat{V}), \tilde{h})$ has the form*

$$\tilde{g} = (g_1 \otimes \text{Id}_{\text{gr}(\widehat{F})}) \oplus \cdots \oplus (g_n \otimes \text{Id}_{\text{gr}(\widehat{F})})$$

for some similitudes $g_i \in \text{Sim}(V_i, h_i)$ with $\mu(g) = \mu(g_1) = \cdots = \mu(g_n)$.

Abusing notation, we write $g_1 \oplus \cdots \oplus g_n$ for $(g_1 \otimes \text{Id}_{\text{gr}(\widehat{F})}) \oplus \cdots \oplus (g_n \otimes \text{Id}_{\text{gr}(\widehat{F})})$. Note that conversely, given similitudes $g_i \in \text{Sim}(V_i, h_i)$ for $i = 1, \dots, n$ such that $\mu(g_1) = \cdots = \mu(g_n)$, we may define a similitude $g \in \text{Sim}(\widehat{V}, \widehat{h})$ such that $\tilde{g} = g_1 \oplus \cdots \oplus g_n$ and $\mu(g) = \mu(g_1) \in F^\times$ by

$$g = (g_1 \otimes \text{Id}_{\widehat{F}}) \oplus \cdots \oplus (g_n \otimes \text{Id}_{\widehat{F}}).$$

Now, let us apply these results to the setting of a generic orthogonal sum of 1-dimensional skew-hermitian forms over a quaternion division algebra Q over F . The following proposition is a key tool for the examples we produce below.

Proposition 3.6. *Let Q be a quaternion division algebra over F , and consider pure quaternions q_1, \dots, q_n , with respective squares $a_1, \dots, a_n \in F^\times$. Let \widehat{F} be the field of iterated Laurent series in n indeterminates t_1, \dots, t_n over F , let $\widehat{Q} = Q \otimes_F \widehat{F}$, and consider the involution σ on $A = M_n(\widehat{Q})$ adjoint to the skew-hermitian form $\widehat{h} = \langle t_1q_1, \dots, t_nq_n \rangle$. If $n \geq 3$, then*

- (1) *The involution σ has discriminant $\text{disc } \sigma = a_1 \dots a_n \cdot \widehat{F}^{\times 2}$;*
- (2) *The involution σ admits improper similitudes if and only if there exist $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ such that $\varepsilon_1 \dots \varepsilon_n = -1$ and*

$$G_{\varepsilon_1}(a_1) \cap \cdots \cap G_{\varepsilon_n}(a_n) \neq \emptyset.$$

- (3) *The involution σ admits square-central improper similitudes if and only if n is odd and*

$$G_-(a_1) \cap \dots \cap G_-(a_n) \neq \emptyset.$$

Proof. The discriminant of σ is the product of the discriminants of the involutions adjoint to $\langle t_i q_i \rangle$ for all i . Since the discriminant of the adjoint involution of $\langle q \rangle$, for any nonzero pure quaternion q , is the square class of q^2 , we get assertion (1).

Suppose that the hermitian form \widehat{h} admits improper similitudes. Since it is a generic orthogonal sum, as defined above, of the 1-dimensional skew-hermitian forms $h_i = \langle q_i \rangle$, we may apply Theorem 3.4 and Lemma 3.5. Therefore, since $n \geq 3$, we may find an improper similitude g of \widehat{h} with multiplier $\mu = \mu(g) \in F^\times \subset \widehat{F}^\times$. By Lemma 3.5 we have $\widetilde{g} = g_1 \oplus \dots \oplus g_n$ with $g_i \in \text{Sim}(h_i)$ and $\mu(g_i) = \mu$ for $i = 1, \dots, n$. Because g and \widetilde{g} are improper, the same computation as in Lemma 3.1 shows that the number of improper similitudes among g_1, \dots, g_n is odd. Letting $\varepsilon_i = +1$ if g_i is proper and $\varepsilon_i = -1$ if g_i is improper, we thus have

$$\mu \in G_{\varepsilon_1}(a_1) \cap \dots \cap G_{\varepsilon_n}(a_n) \quad \text{and} \quad \varepsilon_1 \dots \varepsilon_n = -1.$$

Assume in addition g is square-central. From $\sigma(g)g = \mu$, we get $g^2 = \varepsilon\mu$ for some $\varepsilon \in \{\pm 1\}$. Hence we also have $\widetilde{g}^2 = \varepsilon\widetilde{\mu} = \varepsilon\mu$. By Lemma 3.5, this occurs if and only if $g_i^2 = \varepsilon\mu$ for $i = 1, \dots, n$. Since g is improper, there is at least one i for which g_i is improper. From the description of similitudes recalled in the proof of Lemma 3.2, we get that g_i is a pure quaternion that anticommutes with q_i . Therefore

$$\mu = \mu(g_i) = \sigma(g_i)g_i = q_i^{-1}\overline{g_i}q_i = g_i^2.$$

It follows that $\varepsilon = 1$. Now assume for the sake of contradiction that g_j is proper for some j . Then g_j is a quaternion that commutes with q_j , i.e., $g_j \in F(q_j)$, and it is square-central, hence it belongs to $F^\times \cup F^\times q_j$. The first case leads to $\mu = \mu(g_j) \in F^{\times 2}$, which is impossible since $Q = (\mu, a_i)_F$ is a division algebra. The second case leads to $\mu = \mu(g_j) = -g_j^2$, which is impossible since $\varepsilon = 1$. Therefore, g_j is improper for all j , that is $\varepsilon_1 = \dots = \varepsilon_n = -1$. Since g is improper, this implies n is odd.

We have thus proved the “only if” parts of (2) and (3). The converse statements are easy consequences of Lemma 3.1 and Lemma 3.3. \square

3.3. Examples of groups of type 2D_n . With Proposition 3.6 in hand, we can now produce explicit examples of groups of type D_n , proving that conditions (Out 1), (Out 2), and (Out 3) are not equivalent.

In our examples, the algebra has the form $A = M_n(Q)$ for some integer $n \geq 3$, and some quaternion division algebra Q over F . As a preliminary observation concerning condition (Out 1), note that the set of discriminants of orthogonal involutions on A is $(-1)^n \text{Nrd}_Q(Q^\times)$. This follows easily from the fact that any quaternion can be written as a product of two pure quaternions. On the other hand, a quadratic extension $F(\sqrt{\delta})$ of F is a splitting field of Q if and only if Q contains a pure quaternion q such that $\delta = q^2 = -\text{Nrd}_Q(q)$. Hence, if n is odd, for any splitting field $F(\sqrt{\delta})$, A does admit orthogonal involutions σ with discriminant δ , and (Out 1) holds for the corresponding group. As opposed to this, it is not always true that A admits an involution σ for which (Out 1) holds if n is even, as we now proceed to show.

3.3.1. *Type 2D_n with n even.* In this subsection, we assume $A = M_n(Q)$ with $n = 2m$ even, $m \geq 2$. We first prove:

Proposition 3.7. *Assume $A = M_n(Q)$ with n even. The algebra A admits an orthogonal involution σ such that A is split by the discriminant quadratic algebra Z of σ if and only if $-1 \in \text{Nrd}_Q(Q^\times)$.*

Proof. If A is split by the discriminant algebra $Z = F(\sqrt{\delta})$ of some orthogonal involution σ , then $\delta = q^2 = -\text{Nrd}_Q(q)$ for some pure quaternion q , and $\delta = \text{Nrd}_A(x)$ for some σ skew-symmetric $x \in A$, so that $\delta \in \text{Nrd}_Q(Q^\times)$. Hence, δ and $-\delta$ are reduced norms, and we get $-1 \in \text{Nrd}_Q(Q^\times)$.

Assume conversely that $-1 \in \text{Nrd}_Q(Q^\times)$, and pick an arbitrary quadratic field $Z = F(\sqrt{\delta})$ that splits Q . There exists a pure quaternion $q \in Q^0$ such that $\delta = q^2 = -\text{Nrd}_Q(q)$. Since $-1 \in \text{Nrd}_Q(Q^\times)$, we get $\delta \in \text{Nrd}_Q(Q^\times)$, and since n is even, it follows that there exists an orthogonal involution σ of discriminant δ . \square

In view of Proposition 2.5, the following result provides examples of groups $\text{PGO}^+(A, \sigma)$ of type 2D_n , with n even and $n \geq 3$, which admit outer automorphisms but no outer automorphisms of order 2.

Proposition 3.8. *Let Q be a quaternion division algebra such that $-1 \in \text{Nrd}_Q(Q^\times)$, and let Z be a quadratic splitting field for Q . For every even integer $n \geq 2$ there exists an orthogonal involution σ of $M_n(Q)$ with discriminant Z such that (A, σ) admits improper similitudes. Moreover, (A, σ) does not have square-central improper similitudes.*

Proof. Since Z is a quadratic splitting field for Q , there exists $\delta, \nu \in F^\times$ such that $Z = F(\sqrt{\delta})$ and $Q = (\delta, \nu)_F$. Moreover, since the norm form of Q represents -1 , the quadratic form $\langle 1, -\nu, -\delta, \nu\delta, 1 \rangle$ is isotropic. After scaling, we get that $\langle -\nu, 1, \delta\nu, -\delta, -\nu \rangle$ also is isotropic, hence $\langle 1, -\nu \rangle$ and $\langle \delta, \nu, -\delta\nu \rangle$ represent a common value. This means there exists a pure quaternion $q \in Q^0$ such that $a = q^2$ is a norm for the quadratic field extension $F(\sqrt{\nu})/F$, or equivalently $(a, \nu)_F = 0$. So we have $Q = (\delta, \nu)_F = (a\delta, \nu)_F$. Let q' be a pure quaternion with square $a\delta$, and let σ be the adjoint involution with respect to the skew-hermitian form $h = \langle q', q, q, \dots, q \rangle$. Since n is even, σ has discriminant δ . Moreover, by Lemma 3.3, σ admits an improper similitude with multiplier ν . Since Q is a division algebra, the last assertion follows from Lemma 2.8. \square

To produce examples of groups satisfying (Out 1) but with no outer automorphisms, we use the ‘‘orthogonal generic sums’’ defined above. More precisely, we consider the following:

Proposition 3.9. *Let Q be a quaternion division algebra. Assume Q contains pure quaternions q_1, q_2, q_3 with respective squares a_1, a_2 , and a_3 such that*

- (1) Q is split by $F(\sqrt{a_1 a_2})$;
- (2) Q is not split by $F(\sqrt{a_1 a_3})$ nor by $F(\sqrt{a_2 a_3})$.

Then the involution σ on $A = M_n(\widehat{Q})$, with n even, $n \geq 3$, defined as in Proposition 3.6 with q_1, q_2, q_3 as above and $q_4 = \dots = q_n = q_3$, admits no improper similitudes, yet Q is split by the discriminant quadratic extension Z/F .

Proof. Since n is even, σ has discriminant $a_1 a_2$, hence the first condition guarantees that Q is split by Z . It remains to prove that Q has no improper similitudes. By

Proposition 3.6, this means we have to prove

$$G_{\varepsilon_1}(a_1) \cap \dots \cap G_{\varepsilon_n}(a_n) = \emptyset,$$

for all $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ such that $\varepsilon_1 \dots \varepsilon_n = -1$. Recall that $\mu \in G_+(a_i)$ (respectively $G_-(a_i)$) if and only if $(\mu, a_i)_F = 0$ (respectively $(\mu, a_i)_F = Q$). Since Q is a division algebra, it follows that $G_+(a_3) \cap G_-(a_3) = \emptyset$. Thus, if the intersection above is nonempty, then $\varepsilon_3 = \dots = \varepsilon_n$. Since n is even, we have $\varepsilon_3 \dots \varepsilon_n = \varepsilon_3^{n-2} = 1$. Therefore, it is enough to prove that the following intersections are empty:

- (i) $G_+(a_1) \cap G_-(a_2) \cap G_+(a_3) = \emptyset$, (ii) $G_-(a_1) \cap G_+(a_2) \cap G_+(a_3) = \emptyset$,
 (iii) $G_+(a_1) \cap G_-(a_2) \cap G_-(a_3) = \emptyset$, (iv) $G_-(a_1) \cap G_+(a_2) \cap G_-(a_3) = \emptyset$.

Assume that some $\mu \in F^\times$ belongs to the intersection (i) (respectively (iv)). The two quaternion algebras $(\mu, a_1)_F = (\mu, a_3)_F$ are split (respectively equal to Q), while the third one is $(\mu, a_2)_F = Q$ (respectively is split). In each case, we get that $Q = (\mu, a_2 a_3)_F$. This is impossible, since we assumed that $F(\sqrt{a_2 a_3})$ does not split Q . Similarly, if μ belongs to the intersection (ii) or (iii), we get $Q = (\mu, a_1 a_3)_F$, which again is impossible. \square

The following example provides an explicit quaternion algebra Q satisfying the conditions of Proposition 3.9, hence examples of groups $\mathbf{PGO}^+(A, \sigma)$ of type ${}^2\mathbf{D}_n$ with n even, $n \geq 3$, for which (Out 1) holds but not (Out 2).

Example 3.10. Consider a field k of characteristic $\neq 2$ such that $-1 \in k^{\times 2}$. Assume k is the center of a quaternion division algebra $(a_1, a_2)_k$, and let $F = k(r, s, t)$ where r, s , and t are independent indeterminates. Let $Q = (a_1, a_2)_F$ and $a_3 = a_1 r^2 + a_2 s^2 + a_1 a_2 t^2 \in F^\times$. Clearly, Q is a quaternion division algebra containing pure quaternions q_1, q_2, q_3 with $q_i^2 = a_i$ for $i = 1, 2, 3$. Since $-1 \in F^{\times 2}$, the algebra $Q = (a_1, a_2)_F = (a_1, a_1 a_2)_F$ is split by $F(\sqrt{a_1 a_2})$. If Q is split by $F(\sqrt{a_1 a_3})$, then $a_1 a_3$ is represented over F by the quadratic form $\langle a_1, a_2, a_1 a_2 \rangle$, hence (after scaling by a_1) a_3 is represented by $\langle 1, a_2, a_1 a_2 \rangle$ over F . Because r, s, t are indeterminates, Pfister's subform theorem [8, Th. IX.2.8] shows that this condition implies that $\langle a_1, a_2, a_1 a_2 \rangle \simeq \langle 1, a_2, a_1 a_2 \rangle$ over k , hence (by Witt's cancellation theorem or by comparing discriminants) $a_1 \in k^{\times 2}$. This is impossible since $(a_1, a_2)_k$ is a division algebra. Similarly, if Q is split by $F(\sqrt{a_2 a_3})$, then $a_2 a_3$ is represented by $\langle a_1, a_2, a_1 a_2 \rangle$ over F , hence a_3 is represented by $\langle 1, a_1, a_1 a_2 \rangle$ over F , and $\langle a_1, a_2, a_1 a_2 \rangle \simeq \langle 1, a_1, a_1 a_2 \rangle$ over k , a contradiction since $a_2 \notin k^{\times 2}$. Hence, the quaternion algebra Q satisfies the conditions of Proposition 3.9.

3.3.2. *Type ${}^2\mathbf{D}_n$, with n odd.* We again use the orthogonal generic sums defined in §3.2. More precisely, we have the following:

Proposition 3.11. *Let Q be a quaternion division algebra. Assume Q contains pure quaternions q_1, q_2, q_3 with respective squares a_1, a_2 , and a_3 such that*

- (1) Q is split by $F(\sqrt{a_1 a_2 a_3})$;
 (2) There is no $\mu \in F^\times$ such that $Q = (a_1, \mu)_F = (a_2, \mu)_F = (a_3, \mu)_F$.

Consider the involution σ of $A = M_n(\widehat{Q})$, with n odd, $n \geq 3$, defined as in Proposition 3.6, with q_1, q_2, q_3 as above and $q_4 = \dots = q_n = q_3$. This involution admits no square-central improper similitudes, yet Q is split by the discriminant quadratic extension Z/F . Moreover, if in addition $-1 \notin \mathrm{Nrd}_Q(Q^\times)$, then σ has no improper similitudes.

Proof. Since n is odd, σ has discriminant $a_1 a_2 a_3$. Therefore condition (1) guarantees that Q is split by the discriminant quadratic algebra Z . Moreover, arguing as in the proof of Proposition 3.9, and taking into account the fact that n is now odd, we get that σ has improper similitudes if and only if one of the following intersections is nonempty:

$$\begin{aligned} \text{(i)} \quad & G_+(a_1) \cap G_+(a_2) \cap G_-(a_3), & \text{(ii)} \quad & G_+(a_1) \cap G_-(a_2) \cap G_+(a_3), \\ \text{(iii)} \quad & G_-(a_1) \cap G_+(a_2) \cap G_+(a_3), & \text{(iv)} \quad & G_-(a_1) \cap G_-(a_2) \cap G_-(a_3). \end{aligned}$$

In addition, we know by Proposition 3.6 that σ has a square-central improper similitude if and only if the fourth intersection is nonempty, or equivalently, if there exists $\mu \in F^\times$ such that $Q = (\mu, a_i)_F$ for $i = 1, 2, 3$. This is impossible by condition (2).

If the involution σ has an improper similitude, then one of the intersections (i), (ii) or (iii) is nonempty. So assume for instance there exists $\mu \in F^\times$ such that $Q = (\mu, a_3)_F$ and $(\mu, a_1)_F = (\mu, a_2)_F = 0$. The first equation shows that there exists a pure quaternion z such that $\mu = z^2 = -\text{Nrd}_Q(z)$. On the other hand, since $(\mu, a_1)_F = 0$, there exists a quaternion $z' \in F(q_1)$ such that $\mu = N_{F(q_1)/F}(z') = \text{Nrd}_Q(z')$. Therefore, both μ and $-\mu$ are reduced norms, and it follows -1 also is a reduced norm. This concludes the proof of the proposition. \square

Adapting a construction from [3] (see also [12, §10.2.2]), we now describe an explicit example of a quaternion algebra satisfying the conditions of Proposition 3.11, and we use it to give examples of groups of type 2D_n , with n odd, satisfying (Out 1) and not (Out 2), or (Out 2) and not (Out 3).

Example 3.12. Let k be an arbitrary field of characteristic 0, and let $F = k(a_1, a_2)$, where a_1 and a_2 are independent indeterminates. Consider the quaternion division algebra $Q = (a_1, a_2)_F$, and let

$$(3) \quad a_3 = a_1((1 - a_1)^2(1 + a_2)^2 - 4(1 - a_1)a_2).$$

The algebra Q satisfies the conditions (1) and (2) of Proposition 3.11.

Proof. It is clear that Q contains pure quaternions q_1, q_2 with $q_1^2 = a_1$ and $q_2^2 = a_2$. Computation yields

$$(1 - a_1)^2(1 + a_2)^2 - a_1^{-1}a_3 = 4(1 - a_1)a_2,$$

hence the quaternion algebra $(a_1^{-1}a_3, (1 - a_1)a_2)_F$ is split. Therefore,

$$(a_3, (1 - a_1)a_2)_F \simeq (a_1, (1 - a_1)a_2)_F \simeq Q,$$

and it follows that Q contains a pure quaternion q_3 with $q_3^2 = a_3$.

Another computation yields

$$(1 - a_1)^2(1 - a_2)^2 - a_1^{-1}a_3 = 4a_1(1 - a_1)a_2,$$

hence the quaternion algebra $(a_1^{-1}a_3, a_1(1 - a_1)a_2)_F$ is split. Since we already observed that $(a_1^{-1}a_3, (1 - a_1)a_2)_F$ is split, it follows that $(a_1^{-1}a_3, a_1)_F$ is split, hence

$$(4) \quad (a_1, a_3)_F \simeq (a_1, a_1)_F \simeq (a_1, -1)_F.$$

We thus see that $(a_1, -a_3)_F$ is split, hence

$$Q \simeq (a_1, -a_2a_3)_F \simeq (a_1, a_1a_2a_3)_F.$$

Therefore, Q is split by $F(\sqrt{a_1a_2a_3})$.

Suppose now that there exists some $\mu \in F^\times$ such that

$$(5) \quad Q = (a_1, \mu)_F = (a_2, \mu)_F = (a_3, \mu)_F.$$

To obtain a contradiction, we use valuation theory as in [12, §10.2.2]: since $\text{char } k = 0$, we may find on k a dyadic valuation v_0 , with value group some ordered group Γ and residue field \bar{k} of characteristic 2. Consider the Gaussian extension v_1 of v_0 to F , with value group Γ and residue field $\bar{k}(a_1, a_2)$, and let v be the valuation on F obtained by composing v_1 with the $(1 - a_1)$ -adic valuation on $\bar{k}(a_1, a_2)$. The value group of v is $\mathbb{Z} \times \Gamma$ with the right-to-left lexicographic ordering, and the residue field is $\bar{k}(a_2)$. It is clear that v extends uniquely to $F(\sqrt{a_2})$, and this extension is unramified with a purely inseparable residue field extension. In [12, p. 509], it is shown that v also extends uniquely to $F(\sqrt{a_1})$ and $F(\sqrt{a_1 a_3})$, and that these extensions are totally ramified.

Now, since $Q \simeq (a_2, -a_1 a_2)_F$ and (5) holds, we see that $-a_1 a_2 \mu$ is a norm from $F(\sqrt{a_2})$. Because $F(\sqrt{a_2})$ is an unramified extension of F , it follows that $v(-a_1 a_2 \mu) \in 2v(F^\times)$. Scaling μ by the square of an element in F^\times , we may assume $v(-a_1 a_2 \mu) = 0$ and take the residue $\overline{-a_1 a_2 \mu} = a_2 \bar{\mu} \in \bar{k}(a_2)$. (We can omit the sign, since \bar{k} has characteristic 2.) Since $Q \simeq (a_1, -a_1 a_2)_F$, we also derive from (5) that $-a_1 a_2 \mu$ is a norm from $F(\sqrt{a_1})$. As $F(\sqrt{a_1})$ is totally ramified over F , it follows that $\overline{-a_1 a_2 \mu} \in \bar{F}^{\times 2}$, hence $a_2 \bar{\mu} \in \bar{k}^2(a_2^2)$. But (5) also shows that $(a_1 a_3, \mu)_F$ is split, hence μ is a norm from the totally ramified extension $F(\sqrt{a_1 a_3})$, and therefore $\bar{\mu} \in \bar{k}^2(a_2^2)$. We thus reach the conclusion that $a_2 \in \bar{k}^2(a_2^2)$, a contradiction. \square

Corollary 3.13. *Let Q be the quaternion algebra of Example 3.12 and $q_1, q_2, q_3 \in Q$ be pure quaternions satisfying $q_i^2 = a_i$ for $i = 1, 2, 3$. Fix an odd integer $n \geq 3$ and consider as in Proposition 3.6*

$$\widehat{F} = F((t_1)) \dots ((t_n)), \quad \widehat{Q} = Q \otimes_F \widehat{F}, \quad A = M_n(\widehat{Q}),$$

and σ the involution on A adjoint to the skew-hermitian form $\widehat{h} = \langle t_1 q_1, \dots, t_n q_n \rangle$ with $q_4 = \dots = q_n = q_3$. The group $\mathbf{PGO}^+(A, \sigma)$ satisfies (Out 1) but not (Out 3), and it satisfies (Out 2) if and only if $-1 \in k^{\times 2}$.

Therefore, depending on the base field k we started with, we get the required examples.

Proof. Proposition 3.11, together with Proposition 2.5, already shows that the group $\mathbf{PGO}^+(A, \sigma)$ satisfies (Out 1) and not (Out 3), and that it does not satisfy (Out 2) if $-1 \notin \text{Nrd}_Q(Q^\times)$. Therefore, it only remains to show that -1 is not a reduced norm of Q if $-1 \notin k^{\times 2}$, and that σ admits improper similitudes if $-1 \in k^{\times 2}$.

The first part is clear: the reduced norm of Q is the quadratic form

$$n_Q \simeq \langle 1, -a_1, -a_2, a_1 a_2 \rangle$$

over $F = k(a_1, a_2)$. Since a_1 and a_2 are indeterminates, this quadratic form represents -1 if and only if $-1 \in k^{\times 2}$.

Now, assume that $-1 \in k^{\times 2}$. Since $Q = (a_1, a_2)_F$, we have $a_1 \in G_-(a_2)$. Moreover, because $-1 \in k^{\times 2}$ the quaternion algebras $(a_1, a_1)_F$ and $(a_1, a_3)_F$ are split (see (4)), so $a_1 \in G_+(a_1) \cap G_+(a_3)$. Therefore,

$$a_1 \in G_+(a_1) \cap G_-(a_2) \cap G_+(a_3).$$

Proposition 3.6 then shows that σ admits improper similitudes. \square

Remark 3.14. As shown in [6, §15.D], the Clifford algebra construction defines an equivalence of categories from the groupoid $\mathbf{D}_3(F)$ to the groupoid $\mathbf{A}_3(F)$. For any central simple algebra A of degree 6 with orthogonal involution σ over a field of characteristic different from 2, the Clifford algebra $C(A, \sigma)$ has degree 4 and carries a canonical unitary involution $\underline{\sigma}$, and we have canonical isomorphisms (see [6, (15.26), (15.27)])

$$\mathbf{Spin}(A, \sigma) \simeq \mathbf{SU}(C(A, \sigma), \underline{\sigma}), \quad \mathbf{PGO}^+(A, \sigma) \simeq \mathbf{PGU}(C(A, \sigma), \underline{\sigma}).$$

Therefore, Corollary 3.13 with $n = 3$ readily yields examples of groups of type ${}^2\mathbf{A}_3$ that satisfy (Out 1) but not (Out 2), or (Out 2) but not (Out 3). In particular, by Proposition 2.2, it also provides examples of unitary involutions that do not have a descent. In view of Theorem 1.3, we know that the algebra $C(A, \sigma)$ in these examples is a division algebra of degree 4.

For use in §4.3, we still make a few observations on the square-central similitudes of the skew-hermitian form of Corollary 3.13 in the particular case where $n = 3$, i.e.,

$$\widehat{h} = \langle t_1 q_1, t_2 q_2, t_3 q_3 \rangle$$

with q_1, q_2, q_3 as in Example 3.12.

Lemma 3.15. *Assume $-1 \in k^{\times 2}$. Every square-central similitude g of \widehat{h} is proper and satisfies*

$$g^2 = \mu(g) \in \widehat{F}^{\times 2}.$$

Proof. Let $g^2 = \lambda \in \widehat{F}^\times$. We have $\lambda^2 = \mu(g^2) = \mu(g)^2$, hence $\lambda = \pm\mu(g)$. Scaling g , we may assume by Theorem 3.4 that $\mu(g) \in F^\times$, hence also $\lambda \in F^\times$. By Proposition 3.5 we then have

$$\widetilde{g} = g_1 \oplus g_2 \oplus g_3$$

for some $g_i \in \text{Sim}(h_i)$ with $\mu(g) = \mu(g_1) = \mu(g_2) = \mu(g_3)$ and $\lambda = \widetilde{g}^2 = g_1^2 = g_2^2 = g_3^2$. By Example 3.12 and Proposition 3.11 the similitude g must be proper since it is square-central. Therefore, the number of improper similitudes among g_1, g_2, g_3 is even, so at least one of g_1, g_2, g_3 is a proper similitude. If g_i is proper, then $g_i \in F(q_i)^\times$. Since g_i is square-central, it follows that $g_i \in F^\times \cup q_i F^\times$, hence $g_i^2 \in F^{\times 2} \cup a_i F^{\times 2}$ and $\mu(g_i) = \text{Nrd}_Q(g_i) \in F^{\times 2} \cup (-a_i)F^{\times 2}$. If g_i is improper, then $(a_i, \mu(g_i))_F \simeq Q$: see Lemma 3.2. We now consider the various possibilities:

(1) If g_1 is proper and g_2, g_3 are improper: then $\mu(g) \in F^{\times 2} \cup (-a_1)F^{\times 2}$ and $(a_2, \mu(g))_F \simeq (a_3, \mu(g))_F \simeq Q$. Since $-1 \in k^{\times 2}$, the quaternion algebra $(a_3, -a_1)_F$ is split (see (4)) whereas Q is not split, so this case is impossible.

(2) If g_2 is proper and g_1, g_3 are improper: then $\mu(g) \in F^{\times 2} \cup (-a_2)F^{\times 2}$ and $(a_1, \mu(g))_F \simeq (a_3, \mu(g))_F \simeq Q$. Since Q is not split, we must have $\mu(g) \in (-a_2)F^{\times 2} = a_2 F^{\times 2}$, and we get $(a_1, a_2)_F = (a_3, a_2)_F$, hence $(a_1 a_3, a_2)_F$ is split. By definition of a_3 (see (3)), this means that the quaternion algebra

$$((1 - a_1)((1 - a_1)(1 + a_2)^2 - 4a_2), a_2)_F$$

is split. This is a contradiction, since this quaternion algebra is ramified for the $(1 - a_1)$ -adic valuation.

(3) If g_3 is proper and g_1, g_2 are improper: this case is excluded just like the previous two, because the quaternion algebra $(a_1, a_3)_F$ is split.

The only remaining case is when g_1, g_2 , and g_3 are proper, hence $\mu(g_i) \in F^{\times 2} \cup (-a_i)F^{\times 2}$ for each i . Since a_1, a_2 , and a_3 are in different square classes and $\mu(g_1) = \mu(g_2) = \mu(g_3)$, it follows that $\mu(g_i) \in F^{\times 2}$, hence $g_i \in F^\times$ for all i . Then $\lambda = g_i^2 = \mu(g_i)$ for all i , hence $g^2 = \mu(g) \in F^{\times 2}$. \square

4. OUTER AUTOMORPHISMS AND SIMILITUDES: THE UNITARY CASE

We now turn to the results concerning unitary groups. We already gave in Remark 3.14 examples of groups of type 2A_3 satisfying (Out 1) but not (Out 2), or satisfying (Out 2) but not (Out 3). The other examples we will provide are of the form $\mathbf{PGU}(B, \tau)$ with B of index 2. Unitary involutions on algebras of index 2 are examined in detail in §4.1, and the examples are given in §4.3. They are based on a generic construction of hermitian forms of unitary type which is discussed for division algebras of arbitrary index in §4.2.

The characteristic is arbitrary in §4.1; it is assumed to be different from 2 in §4.2 and §4.3.

4.1. Similitudes for unitary hermitian forms over a quaternion algebra.

Let Q be a quaternion division algebra over a field K of arbitrary characteristic, which is a quadratic separable extension of some subfield F . We write ι for the nontrivial automorphism of K over F . Let (B, τ) be an algebra with unitary involution Brauer-equivalent to Q . We have seen in §2.1 that outer automorphisms of $\mathbf{PGU}(B, \tau)$ are given by ι -semilinear automorphisms of (B, τ) . In this section, we describe them explicitly in terms of the underlying hermitian space.

Let U be a finite-dimensional right Q -vector space such that $B = \text{End}_Q U$. By a theorem of Albert [6, (2.22)], unitary involutions on B exist only if Q has a descent to F . We fix a quaternion F -subalgebra $Q_0 \subset Q$ and identify $Q = Q_0 \otimes_F K$. Let also $U_0 \subset U$ be a Q_0 -subspace of U such that $U = U_0 \otimes_F K$. Thus, Q_0 and U_0 are the fixed F -algebra and Q_0 -subspace of the following ι -semilinear automorphisms of Q and U :

$$\iota_Q = \text{Id}_{Q_0} \otimes \iota, \quad \iota_U = \text{Id}_{U_0} \otimes \iota.$$

Similarly, $\text{End}_{Q_0} U_0$ is the F -algebra fixed under the ι -semilinear automorphism of $\text{End}_Q U$ that maps $f \in \text{End}_Q U$ to the endomorphism f^t defined by

$$f^t(x) = \iota_U(f(\iota_U(x))) \quad \text{for all } x \in U.$$

The canonical involution $\bar{}$ on Q commutes with ι_Q because for $x \in Q$

$$\iota_Q(\bar{x}) = \iota_Q(\text{Trd}_Q(x) - x) = \text{Trd}_Q(\iota_Q(x)) - \iota_Q(x) = \overline{\iota_Q(x)}.$$

Let $\theta = \bar{} \circ \iota_Q$, a unitary involution on Q which restricts to the canonical involution on Q_0 . The unitary involution τ on $B = \text{End}_Q U$ is the adjoint involution $\tau = \text{ad}_h$ for some nondegenerate hermitian form $h: U \times U \rightarrow Q$ with respect to θ .

A conjugate hermitian form h^t is defined on U by

$$h^t(x, y) = \iota_Q(h(\iota_U(x), \iota_U(y))) \quad \text{for } x, y \in U.$$

It is readily verified that the adjoint involutions of h and h^t are related as follows:

$$(6) \quad \text{ad}_{h^t}(f)^t = \text{ad}_h(f^t) \quad \text{for all } f \in \text{End}_Q U.$$

We define a map $g \in \text{End}_Q U$ to be a *similitude* $(U, h) \rightarrow (U, h')$ if there exists $\mu \in F^\times$ such that

$$h'(g(x), g(y)) = \mu h(x, y) \quad \text{for all } x, y \in U.$$

The factor μ is said to be the *multiplier* of g . We write $\mu(g)$ for the multiplier of g , and $\text{Sim}(U, h, h')$ or $\text{Sim}(h, h')$ for the set of similitudes $(U, h) \rightarrow (U, h')$.

Proposition 4.1. *Every ι -semilinear automorphism φ of the algebra with unitary involution (B, τ) has the form $\varphi: f \mapsto g f^\iota g^{-1}$ for some $g \in \text{Sim}(U, h, h')$. This automorphism φ has order 2 if and only if $g g^\iota \in F^\times$.*

Proof. It follows from the Skolem–Noether theorem that every ι -semilinear automorphism φ of $\text{End}_Q U$ has the form $\varphi: f \mapsto g f^\iota g^{-1}$ for some $g \in \text{End}_Q U$. Equation (6) shows that φ commutes with ad_h if and only if $\text{Int}(g) \circ \text{ad}_{h'} = \text{ad}_h \circ \text{Int}(g)$. But $\text{Int}(g^{-1}) \circ \text{ad}_h \circ \text{Int}(g)$ is the adjoint involution of the form $(x, y) \mapsto h(g(x), g(y))$, so φ commutes with τ if and only if g is a similitude $(U, h) \rightarrow (U, h')$. The last assertion follows by a straightforward computation. \square

Remark 4.2. For $g \in \text{Sim}(U, h, h')$ we have $g^\iota \in \text{Sim}(U, h', h)$ with $\mu(g^\iota) = \mu(g)$, hence for all $x, y \in U$

$$h'(g g^\iota(x), g g^\iota(y)) = \mu(g) h(g^\iota(x), g^\iota(y)) = \mu(g)^2 h'(x, y).$$

Therefore, if $g g^\iota = \lambda \in F^\times$, then $\lambda^2 = \mu(g)^2$, hence $\lambda = \pm \mu(g)$.

Of course, in the discussion above the choice of Q_0 is arbitrary, and h is defined up to a scalar factor. Multiplying h by some nonzero central element α such that $\iota(\alpha) = -\alpha$, we may assume h is skew-hermitian instead of hermitian. More generally, for any $q \in Q^\times$ such that $\theta(q) = -q$, we may consider $\theta' = \text{Int}(q) \circ \theta$ and set

$$h'(x, y) = q h(x, y) \quad \text{for } x, y \in U.$$

Then h' is a nondegenerate skew-hermitian form with respect to θ' , and clearly $\text{ad}_{h'} = \text{ad}_h$. Let also $\iota'_Q = \text{Int}(q) \circ \iota_Q$. The condition $\theta_Q(q) = -q$ yields $\iota_Q(q) = -\bar{q}$, hence $q \iota_Q(q) \in F^\times$ and therefore ι'_Q is a ι -semilinear automorphism of Q of order 2. Letting Q'_0 denote the F -subalgebra of Q fixed under ι'_Q , we have

$$Q = Q'_0 \otimes_F K \quad \text{and} \quad \theta' = {}^- \circ \iota'_Q = \iota'_Q \circ {}^-.$$

Here is one case where an appropriate choice of q may lead to a substantial simplification:

Proposition 4.3. *Let e_1, \dots, e_n be an orthogonal Q -base of (U, h) , and let*

$$h = \langle q_1, \dots, q_n \rangle$$

be the corresponding diagonalization of h . If the K -span of the quaternions q_1, \dots, q_n has dimension at most 3, then there is a quaternion $q \in Q^\times$ such that the skew-hermitian form $h' = qh$ over (Q, θ') has a diagonalization

$$h' = \langle qq_1, \dots, qq_n \rangle$$

with $qq_i \in Q'_0$ for $i = 1, \dots, n$. The skew-hermitian form h' then restricts to a nondegenerate skew-hermitian form h'_0 (over $(Q'_0, {}^-)$) on the Q'_0 -span U'_0 of e_1, \dots, e_n , and we have

$$(B, \tau) = (\text{End}_Q U, \text{ad}_h) = (\text{End}_{Q'_0} U'_0, \text{ad}_{h'_0}) \otimes_F (K, \iota).$$

Proof. Let $S \subset Q$ be the K -span of q_1, \dots, q_n , and let $S^\perp \subset Q$ be the orthogonal of S for the norm form on Q . Since $\text{Nrd}_Q(q) = q\bar{q} = \bar{q}q$ for every $q \in Q$, we have

$$S^\perp = \{s \in Q \mid s\bar{q}_i + q_i\bar{s} = 0 \text{ for all } i\} = \{s \in Q \mid \bar{s}q_i + \bar{q}_i s = 0 \text{ for all } i\}.$$

The K -space S^\perp is stable under θ because $\theta(q_i) = q_i$ for all i and θ commutes with $\bar{}$. If $\dim S \leq 3$, then $\dim S^\perp \geq 1$, hence we may find $q \in Q^\times$ such that $q^{-1} \in S^\perp$ and $\theta(q) = -q$. (Take $q = (s - \theta(s))^{-1}$ for any $s \in S^\perp$ such that $\theta(s) \neq s$; if no such s exists we must have $S^\perp = \{0\}$ because θ is ι -semilinear.) Since $\theta(q_i) = q_i$ and $\theta(q) = -q$ we have $\bar{q}_i = \iota_Q(q_i)$ and $\bar{q} = -\iota_Q(q)$, hence

$$\iota_Q(q_i)q^{-1} = \bar{q}_i q^{-1} = -\bar{q}^{-1} q_i = \iota_Q(q)^{-1} q_i \quad \text{for } i = 1, \dots, n.$$

Therefore,

$$\iota'_Q(qq_i) = q\iota_Q(q)\iota(q_i)q^{-1} = qq_i \quad \text{for } i = 1, \dots, n.$$

We have thus shown $qq_i \in Q'_0$ for $i = 1, \dots, n$; the other assertions readily follow. \square

The condition on the dimension of the K -span of q_1, \dots, q_n is automatically satisfied if $n \leq 3$. Therefore, Theorem 1.3 for B of index 2 follows from Proposition 4.3. The case where B is split was already considered in Corollary 2.4.

Note that the proof does not require any hypothesis on the characteristic. (Of course, skew-hermitian forms are hermitian in characteristic 2.)

4.2. Generic construction of hermitian forms of unitary type. In this section, we fix a central division algebra with involution of the first kind (D, ρ) over an arbitrary field F of characteristic different from 2. Adjoining to F an indeterminate t , we consider the fields of Laurent series

$$\widehat{K} = F((t)) \quad \text{and} \quad \widehat{F} = F((t^2)) \subset \widehat{K}.$$

We let ι denote the nontrivial \widehat{F} -automorphism of \widehat{K} and

$$(\widehat{D}, \widehat{\rho}) = (D, \rho) \otimes_F (\widehat{K}, \iota).$$

Thus, $(\widehat{D}, \widehat{\rho})$ is a central division algebra over \widehat{K} with unitary involution. Over this division algebra, we construct hermitian forms of a particular type, as follows: let (V_1, h_1) be a hermitian space over (D, ρ) and let (V_2, h_2) be a skew-hermitian space over (D, ρ) . Extending scalars, we obtain a hermitian form \widehat{h}_1 on $\widehat{V}_1 = V_1 \otimes_F \widehat{K}$ and a skew-hermitian form \widehat{h}_2 on $\widehat{V}_2 = V_2 \otimes_F \widehat{K}$ (over $(\widehat{D}, \widehat{\rho})$). We then set

$$(\widehat{U}, \widehat{h}) = (\widehat{V}_1 \oplus \widehat{V}_2, \widehat{h}_1 \perp \langle t \rangle \widehat{h}_2).$$

Since $\iota(t) = -t$ and \widehat{h}_2 is skew-hermitian, the form $\langle t \rangle \widehat{h}_2$ is hermitian, hence \widehat{h} is a hermitian form on \widehat{U} over $(\widehat{D}, \widehat{\rho})$. Set $\widehat{D}_0 = D \otimes_F \widehat{F}$; we have $\widehat{D} = \widehat{D}_0 \otimes_{\widehat{F}} \widehat{K}$; hence, the algebra \widehat{D} has a descent. Define $\iota_{\widehat{D}} = \text{Id}_D \otimes \iota = \text{Id}_{\widehat{D}_0} \otimes \iota$, and $\widehat{U}_0 = (V_1 \oplus V_2) \otimes_F \widehat{F}$, $\iota_{\widehat{U}} = \text{Id}_{\widehat{U}_0} \otimes \iota$. Every vector $x \in \widehat{U}$ has a unique expression as a series $x = \sum_i x_i \otimes t^i$ with $x_i \in V_1 \oplus V_2$ for all i , and $\iota_{\widehat{U}}(x) = \sum_i x_i \otimes (-t)^i$. The conjugate hermitian form \widehat{h}^t is

$$(7) \quad \widehat{h}^t = \widehat{h}_1 \perp \langle -t \rangle \widehat{h}_2.$$

For the rest of this section, we assume h_1 and h_2 are anisotropic, hence \widehat{h} is anisotropic. As in §3.2, we use the t -adic valuation to obtain information on the set of similitudes $\text{Sim}(\widehat{U}, \widehat{h}, \widehat{h}^t)$. More precisely, we prove:

Proposition 4.4. *Let $(\widehat{U}, \widehat{h})$ be defined as above by $\widehat{h} = \widehat{h}_1 \perp \langle t \rangle \widehat{h}_2$, where h_1 (respectively h_2) is an anisotropic hermitian (respectively skew-hermitian) form over (D, ρ) . Every similitude $g \in \text{Sim}(\widehat{U}, \widehat{h}, \widehat{h}^t)$ has the form $g = \lambda g'$ for some $\lambda \in \widehat{K}^\times$ and some similitude $g' \in \text{Sim}(\widehat{U}, \widehat{h}, \widehat{h}^t)$ with $\mu(g') \in F^\times$. Moreover, on the graded module $\text{gr}(\widehat{U})$ associated to a suitable norm on \widehat{U} , the map g' induces a map \widetilde{g}' of the form $\widetilde{g}' = g_1 \oplus g_2$ for some similitudes $g_1 \in \text{Sim}(V_1, h_1)$, $g_2 \in \text{Sim}(V_2, h_2)$ with*

$$\mu(g') = \mu(g_1) = -\mu(g_2).$$

Proof. Let v be the t -adic valuation on \widehat{K} . We write again v for its extension to \widehat{D} and define a v -norm on \widehat{U} by

$$\nu(x) = \frac{1}{2}v(\widehat{h}(x, x)) \quad \text{for } x \in \widehat{U}.$$

Thus, we have $\nu(x_1) \in \mathbb{Z}$ for $x_1 \in \widehat{V}_1$, $\nu(x_2) \in \frac{1}{2} + \mathbb{Z}$ for $x_2 \in \widehat{V}_2$, and

$$\nu(x_1 + x_2) = \min(\nu(x_1), \nu(x_2)) \in \frac{1}{2}\mathbb{Z} \quad \text{for } x_1 \in \widehat{V}_1 \text{ and } x_2 \in \widehat{V}_2.$$

In view of (7) it follows that $v(\widehat{h}(x, x)) = v(\widehat{h}^t(x, x))$, hence

$$(8) \quad \nu(x) = \frac{1}{2}v(\widehat{h}^t(x, x)) \quad \text{for } x \in \widehat{U}.$$

The graded module $\text{gr}(\widehat{U})$ is defined as in §3.2. It carries a hermitian form \widetilde{h} and we have

$$(\text{gr}(\widehat{U}), \widetilde{h}) = (\text{gr}(\widehat{V}_1), \widetilde{h}_1) \perp (\text{gr}(\widehat{V}_2), \widetilde{h}_2), \quad (\text{gr}(\widehat{U}), \widetilde{h}^t) = (\text{gr}(\widehat{V}_1), \widetilde{h}_1) \perp (\text{gr}(\widehat{V}_2), -\widetilde{h}_2)$$

where the hermitian forms $\widetilde{h}_1, \widetilde{h}_2$ are given by

$$\widetilde{h}_1(x_1, y_1) = h_1(x_1, y_1) \quad \text{and} \quad \widetilde{h}_2(x_2, y_2) = \widetilde{t}h_2(x_2, y_2)$$

for $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$.

Now, suppose $g: (\widehat{U}, \widehat{h}) \rightarrow (\widehat{U}, \widehat{h}^t)$ is a similitude. From $\widehat{h}^t(g(x), g(x)) = \mu(g)\widehat{h}(x, x)$ it follows by (8) that

$$\nu(g(x)) = \nu(x) + \frac{1}{2}v(\mu(g)) \quad \text{for } x \in \widehat{U}.$$

Therefore, g induces a similitude $\widetilde{g}: (\text{gr}(\widehat{U}), \widetilde{h}) \rightarrow (\text{gr}(\widehat{U}), \widetilde{h}^t)$, which shifts the grading by $\frac{1}{2}v(\mu(g))$. Note that $\frac{1}{2}v(\mu(g)) \in \mathbb{Z}$ because $\mu(g) \in \widehat{F} \subset \widehat{K}$. Therefore, $\text{gr}(\widehat{V}_1)$ and $\text{gr}(\widehat{V}_2)$ are invariant under \widetilde{g} . If $\mu(g) \in F^\times$, the restriction of \widetilde{g} to $V_1 \subset \text{gr}(\widehat{V}_1)$ (resp. to $V_2 \subset \text{gr}(\widehat{V}_2)$) is a similitude $g_1 \in \text{Sim}(V_1, h_1)$ (resp. $g_2 \in \text{Sim}(V_2, h_2)$), and we write (with a slight abuse of notation) $\widetilde{g} = g_1 \oplus g_2$.

Since $\mu(g) \in \widehat{F}^\times$ we have $v(\mu(g)) \in 2\mathbb{Z}$ hence there exists $\lambda_0 \in \widehat{K}^\times$ such that $v(\mu(g)) = 2v(\lambda_0)$. Then $v(\mu(\lambda_0^{-1}g)) = 0$ and we may find $a \in F^\times$, $m \in \widehat{F}^\times$ with $\mu(\lambda_0^{-1}g) = a(1+m)$ and $v(m) > 0$. Arguing as in the proof of the first assertion of Theorem 3.4, we find $\lambda_1 \in \widehat{F}^\times$ such that $\lambda_1^2 = 1+m$, and set $\lambda = \lambda_0\lambda_1$. Then $g' = \lambda^{-1}g \in \text{Sim}(\widehat{U}, \widehat{h}, \widehat{h}^t)$ and $\mu(g') = a \in F^\times$. The equation

$$\widetilde{h}^t(g'(x), g'(y)) = a\widetilde{h}(x, y) \quad \text{for } x, y \in \text{gr}(\widehat{U})$$

yields in particular

$$h_1(g'(x_1), g'(y_1)) = ah_1(x_1, y_1) \quad \text{for } x_1, y_1 \in V_1$$

and

$$-\widetilde{t}h_2(g'(x_2), g'(y_2)) = a\widetilde{t}h_2(x_2, y_2) \quad \text{for } x_2, y_2 \in V_2.$$

Therefore, the restriction g_1 of \tilde{g} to V_1 is a similitude with $\mu(g_1) = a$, and the restriction g_2 of \tilde{g} to V_2 is a similitude with $\mu(g_2) = -a$. \square

Remark 4.5. It is readily verified that $\widehat{\iota_{\tilde{g}}(x)} = \tilde{x}$ for all $x \in V_1 \oplus V_2$. Therefore, $\tilde{g}^t = \tilde{g}$ if $\mu(g) \in F^\times$.

4.3. Examples of groups of type 2A_n . In this section, we use Example 3.12 together with the generic construction of §4.2 to build examples of unitary groups for which (Out 1) holds and (Out 2) fails, or (Out 2) holds and (Out 3) fails.

Let $n \geq 7$ be an odd integer. Write $n = 5 + 2m$, where $m \geq 1$. We construct groups of type 2A_n as unitary groups of hermitian forms of dimension $3 + m$ over a quaternion division algebra with unitary involution. Since the index of the endomorphism algebra is 2, these groups satisfy (Out 1).

Adjoining independent indeterminates to an arbitrary field k of characteristic 0, we form the field

$$F = k(a_1, a_2, x_1, \dots, x_m)((t_1))((t_2))((t_3))$$

and the quaternion algebra

$$Q = (a_1, a_2)_F$$

with its conjugation involution $\bar{}$. Let $a_3 \in F$ be defined by Equation (3). Recall from Example 3.12 that Q contains pure quaternions q_1, q_2, q_3 with $q_i^2 = a_i$ for $i = 1, 2, 3$. Adjoining to F another indeterminate t , form

$$\widehat{K} = F((t)), \quad \widehat{F} = F((t^2)) \subset \widehat{K}, \quad \widehat{Q} = Q \otimes_F \widehat{K}.$$

Let ι be the nontrivial \widehat{F} -automorphism of \widehat{K} . Consider the unitary involution $\widehat{\rho} = \bar{} \otimes \iota$ on \widehat{Q} and the following hermitian form over $(\widehat{Q}, \widehat{\rho})$:

$$\widehat{h} = \langle x_1, \dots, x_m \rangle \perp \langle t \rangle \langle t_1 q_1, t_2 q_2, t_3 q_3 \rangle.$$

Let $\tau = \text{ad}_{\widehat{h}}$ be its adjoint involution on $B = M_{m+3}(\widehat{Q})$.

Proposition 4.6. *The algebra with involution (B, τ) does not admit any ι -semilinear automorphism of order 2. It admits ι -semilinear automorphisms if and only if $-1 \in k^{\times 2}$.*

In view of Proposition 2.1, this provides a group $\mathbf{PGU}(B, \tau)$ which does not satisfy (Out 3), and satisfies (Out 2) if and only if $-1 \in k^{\times 2}$.

Proof. Proposition 4.1 translates the conditions on semilinear automorphisms of (B, τ) into conditions on similitudes of \widehat{h} . Thus, we have to show that there are no similitudes $g \in \text{Sim}(\widehat{h}, \widehat{h}^t)$ such that $gg^t \in \widehat{F}$, and that $\text{Sim}(\widehat{h}, \widehat{h}^t)$ is nonempty if and only if $-1 \in k^{\times 2}$.

Note that the form \widehat{h} is obtained by the generic construction of §4.2, with $(D, \rho) = (Q, \bar{})$ and

$$h_1 = \langle x_1, \dots, x_m \rangle, \quad h_2 = \langle t_1 q_1, t_2 q_2, t_3 q_3 \rangle.$$

Suppose first $-1 \notin k^{\times 2}$ and $g \in \text{Sim}(\widehat{h}, \widehat{h}^t)$. By Proposition 4.4 we may assume $\mu(g) \in F^\times$, hence $\tilde{g} = g_1 \oplus g_2$ for some similitudes $g_1 \in \text{Sim}(h_1)$, $g_2 \in \text{Sim}(h_2)$ with $\mu(g_1) = -\mu(g_2)$. Since by Corollary 3.13 h_2 does not admit improper similitudes, the similitude g_2 must be proper, hence by [6, (13.38)] $\mu(g_2)$ is a norm from the

discriminant extension, which is $F(\sqrt{a_1 a_2 a_3})$. As this extension splits Q , it follows that $\mu(g_2)$ is a reduced norm of Q , hence

$$(9) \quad \langle \mu(g_2) \rangle \langle 1, -a_1, -a_2, a_1 a_2 \rangle \simeq \langle 1, -a_1, -a_2, a_1 a_2 \rangle.$$

On the other hand, since g_1 is a similitude of h_1 with multiplier $-\mu(g_2)$, we have

$$\langle -\mu(g_2) \rangle h_1 \simeq h_1.$$

It follows that $-\mu(g_2)$ is also the multiplier of a similitude of the ‘‘trace’’ quadratic form $\varphi(x) = h_1(x, x)$, which is

$$\varphi \simeq \langle 1, -a_1, -a_2, a_1 a_2 \rangle \langle x_1, \dots, x_m \rangle.$$

Taking into account (9), we see that

$$\langle -1 \rangle \langle 1, -a_1, -a_2, a_1 a_2 \rangle \langle x_1, \dots, x_m \rangle \simeq \langle 1, -a_1, -a_2, a_1 a_2 \rangle \langle x_1, \dots, x_m \rangle.$$

This is impossible because $-1 \notin k^{\times 2}$ and $a_1, a_2, x_1, \dots, x_m$ are indeterminates. Therefore, $\text{Sim}(\widehat{h}, \widehat{h}^t) = \emptyset$ if $-1 \notin k^{\times 2}$.

Suppose next $-1 \in k^{\times 2}$. Then $\widehat{h} = \widehat{h}_1 \perp \langle t \rangle \widehat{h}_2$ is clearly isometric to $\widehat{h}^t = \widehat{h}_1 \perp \langle -t \rangle \widehat{h}_2$, hence $\text{Sim}(\widehat{h}, \widehat{h}^t)$ is not empty. Assume $g \in \text{Sim}(\widehat{h}, \widehat{h}^t)$ satisfies $gg^t = \lambda \in \widehat{F}^\times$. As above, we may scale g and assume $\mu(g) \in F^\times$, hence also $\lambda \in F^\times$ since $\lambda = \pm \mu(g)$ by Remark 4.2. By Proposition 4.4 we have

$$\widetilde{g} = g_1 \oplus g_2$$

for some $g_1 \in \text{Sim}(h_1)$, $g_2 \in \text{Sim}(h_2)$ with $\mu(g) = \mu(g_1) = -\mu(g_2)$. By Remark 4.5, the equation $gg^t = \lambda$ yields $\widetilde{g}^2 = \lambda$, hence we also have $g_1^2 = g_2^2 = \lambda$. Now, by Lemma 3.15 the similitude g_2 must be proper and satisfy $g_2^2 = \mu(g_2) \in F^{\times 2}$ since it is square-central. Scaling again, we may assume

$$\mu(g) = \mu(g_1) = -\mu(g_2) = -1 \quad \text{and} \quad \widetilde{g}^2 = g_1^2 = g_2^2 = 1.$$

The following lemma shows that h_1 does not have any similitude g_1 such that $g_1^2 = -\mu(g_1) = 1$, hence the existence of g leads to a contradiction and the proof of Proposition 4.6 is complete:

Lemma 4.7. *There is no similitude $g \in \text{Sim}(h_1)$ such that $g^2 = -\mu(g) = 1$.*

Proof. Extending scalars to $k(a_1, a_2)((x_1)) \dots ((x_m))((t_1))((t_2))((t_3))$, we may regard h_1 as a generic orthogonal sum of m times the hermitian form $\langle 1 \rangle$ over the quaternion algebra $H = (a_1, a_2)_{k(a_1, a_2)}$, and use the results of §3.2. If $g \in \text{Sim}(h_1)$ is such that $g^2 = -\mu(g) = 1$, then by Lemma 3.5 we have

$$\widetilde{g} = g_1 \oplus \dots \oplus g_m$$

for some $g_i \in \text{Sim}(\langle 1 \rangle) = \text{Sim}(H, \text{---})$ with $g_i^2 = -\mu(g_i) = 1$. Each g_i is a pure quaternion because $g_i^2 = -\mu(g_i)$, and H does not contain any pure quaternion with square 1 because it is not split. We thus obtain a contradiction. \square

APPENDIX: TRIALITARIAN GROUPS

Let G be an algebraic group scheme of adjoint type D_4 over an arbitrary field F . Via the $*$ -action of the absolute Galois group of F on the Dynkin diagram Δ of G (see [11, §15.5]) we may associate to G a cubic étale F -algebra L such that

$$\mathbf{Aut}(\Delta) = \mathbf{Aut}_F(L).$$

If g is the index of the kernel of the Galois action, the type of G is denoted by ${}^g\mathbf{D}_4$. Thus, if G is of type ${}^6\mathbf{D}_4$, then L is a noncyclic separable cubic field extension of F , so $\text{Aut}_F(L) = \{\text{Id}\}$ and G does not have any outer automorphism defined over F . If G is of type ${}^2\mathbf{D}_4$, then $L \simeq F \times Z$ for some separable quadratic field extension Z of F , and $G \simeq \mathbf{P}\mathbf{G}\mathbf{O}^+(A, \sigma, f)$ for some quadratic pair (σ, f) with discriminant Z over a central simple F -algebra A of degree 8: see [11, §17.3.13]. This case has been discussed in §3. For the rest of this appendix, we focus on types ${}^1\mathbf{D}_4$ and ${}^3\mathbf{D}_4$.

Type ${}^1\mathbf{D}_4$. In this case $L \simeq F \times F \times F$, hence $\text{Aut}(\Delta)$ is the symmetric group \mathfrak{S}_3 , and G may have outer automorphisms of order 2 or 3.

Proposition 4.8. *Let G be an algebraic group scheme of adjoint type ${}^1\mathbf{D}_4$. For every nontrivial subgroup $H \subset \text{Aut}(\Delta)$, the following conditions are equivalent:*

- (1) every element in H fixes the Tits class t_G ;
- (2) H is contained in the image of the canonical map $\text{Aut}(G) \rightarrow \text{Aut}(\Delta)$;
- (3) there is a subgroup $H' \subset \text{Aut}(G)$ isomorphic to H under the canonical map $\text{Aut}(G) \rightarrow \text{Aut}(\Delta)$.

When $|H| = 2$ the conditions above hold if and only if $G = \mathbf{P}\mathbf{G}\mathbf{O}^+(q)$ for some 8-dimensional quadratic form q with trivial discriminant. When $|H| = 3$ or 6, they hold if and only if $G = \mathbf{P}\mathbf{G}\mathbf{O}^+(q)$ for some 3-fold quadratic Pfister form q .

Note that the conditions (1), (2), (3) are analogues of (Out 1), (Out 2), and (Out 3) respectively.

Proof. The implications (3) \Rightarrow (2) \Rightarrow (1) are clear, hence it suffices to prove (1) \Rightarrow (3). Choose a representation $G \simeq \mathbf{P}\mathbf{G}\mathbf{O}^+(A, \sigma, f)$ for some central simple algebra A of degree 8 with a quadratic pair (σ, f) of trivial discriminant. If H contains an element α of order 2, we may choose the representation of G in such a way that the action of α on the Tits algebras interchanges the two components $C_{\pm}(A, \sigma, f)$ of $C(A, \sigma, f)$, see [6, (42.3)]. Then (1) implies $C_+(A, \sigma, f) \simeq C_-(A, \sigma, f)$. Similarly, any element of order 3 in H permutes A , $C_+(A, \sigma, f)$, and $C_-(A, \sigma, f)$. Thus, in each case we have $C_+(A, \sigma, f) \simeq C_-(A, \sigma, f)$ if (1) holds. Using the fundamental relations between A and $C(A, \sigma, f)$ in [6, (9.12)], we get that A is split if (1) holds for any nontrivial H , and we may then represent G as $\mathbf{P}\mathbf{G}\mathbf{O}^+(q)$ for some 8-dimensional quadratic form q of trivial discriminant. Since every quadratic space admits square-central improper isometries, as pointed out in Remark 2.7, condition (3) holds if $|H| = 2$. The proof is thus complete in this case.

If $|H| = 3$ or 6, the preceding arguments show that $C_+(A, \sigma, f)$ and $C_-(A, \sigma, f)$ are isomorphic to A when (1) holds, hence they are also split; this means that by scaling q we may assume q is a 3-fold Pfister form. Now, for any 3-fold Pfister form q we may choose a para-Cayley algebra with norm form q , and use the multiplication in the algebra to define outer automorphisms of $\mathbf{P}\mathbf{G}\mathbf{O}^+(q)$ of order 3, see [6, (35.9)]. Using in addition the conjugation in the para-Cayley algebra, we may also define a subgroup of $\text{Aut}(G)(F)$ isomorphic to S_3 , see [6, (35.15)]. \square

Type ${}^3\mathbf{D}_4$. In this case L is a cyclic cubic field extension of F , hence $\text{Aut}_F(L) \simeq \mathbb{Z}/3\mathbb{Z}$. We may then again consider the conditions (Out 1), (Out 2), and (Out 3), with the following slight modification: in (Out 3), the outer automorphism has order 3 instead of 2. If $\text{char } F \neq 2$, the group G can be represented in the form

$G = \mathbf{PGO}^+(T)$ for some trialitarian algebra¹ T , see [6, (44.8)]. The Allen invariant of G is a central simple L -algebra of degree 8.

Proposition 4.9. *For $G = \mathbf{PGO}^+(T)$ of type ${}^3\mathbf{D}_4$, conditions (Out 1) and (Out 2) are equivalent; they hold if and only if the Allen invariant of T is split. Condition (Out 3) holds if and only if T is the endomorphism algebra of a cyclic composition induced by a symmetric composition over F .*

The first assertion is the main Theorem A in Garibaldi–Petersson [5]. The second assertion is proved in [7, Theorem 4.3].

As a result of this proposition, it is easy to find examples of groups of type ${}^3\mathbf{D}_4$ for which (Out 1) and (Out 2) hold while (Out 3) fails: see [7, Remark 2.1].

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¹Trialitarian algebras are not defined in characteristic 2.