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# ORTHOGONAL INVOLUTIONS ON CENTRAL SIMPLE ALGEBRAS AND FUNCTION FIELDS OF SEVERI–BRAUER VARIETIES

ANNE QUÉGUINER-MATHIEU AND JEAN-PIERRE TIGNOL

ABSTRACT. An orthogonal involution  $\sigma$  on a central simple algebra  $A$ , after scalar extension to the function field  $\mathcal{F}(A)$  of the Severi–Brauer variety of  $A$ , is adjoint to a quadratic form  $q_\sigma$  over  $\mathcal{F}(A)$ , which is uniquely defined up to a scalar factor. Some properties of the involution, such as hyperbolicity, and isotropy up to an odd-degree extension of the base field, are encoded in this quadratic form, meaning that they hold for the involution  $\sigma$  if and only if they hold for  $q_\sigma$ . As opposed to this, we prove that there exists non-totally decomposable orthogonal involutions that become totally decomposable over  $\mathcal{F}(A)$ , so that the associated form  $q_\sigma$  is a Pfister form. We also provide examples of nonisomorphic involutions on an index 2 algebra that yield similar quadratic forms, thus proving that the form  $q_\sigma$  does not determine the isomorphism class of  $\sigma$ , even when the underlying algebra has index 2. As a consequence, we show that the  $e_3$  invariant for orthogonal involutions is not classifying in degree 12, and does not detect totally decomposable involutions in degree 16, as opposed to what happens for quadratic forms.

## 1. INTRODUCTION

In characteristic different from 2, every orthogonal involution on a split central simple algebra is the adjoint of a nondegenerate quadratic form. Therefore, the study of orthogonal involutions can be thought of as an extension of quadratic form theory. Reversing the viewpoint, one may try and reduce any question on involutions to a question on quadratic forms by extending scalars to a splitting field of the underlying algebra  $A$ . This is even more relevant as one may generically split  $A$  by tensoring with the function field  $\mathcal{F}(A)$  of its Severi–Brauer variety. Thus, to any orthogonal involution  $\sigma$  on  $A$ , we may associate a quadratic form  $q_\sigma$  on  $\mathcal{F}(A)$ , which is unique up to a scalar factor, and encodes properties of  $\sigma$  over splitting fields of the algebra  $A$ .

Many properties of orthogonal involutions are preserved under field extensions, hence transfer from  $\sigma$  to  $\sigma_{\mathcal{F}(A)}$ , and can be translated into properties of  $q_\sigma$ . For instance, if the involution  $\sigma$  is isotropic or hyperbolic, so is the quadratic form  $q_\sigma$ . Moreover, two conjugate involutions  $\sigma$  and  $\sigma'$  yield similar quadratic forms  $q_\sigma$  and

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$q_{\sigma'}$ . Conversely, some properties of involutions can be tracked down by looking at the involution after scalar extension to  $\mathcal{F}(A)$ , or equivalently at the associated quadratic forms over this function field. For example, Karpenko proved that an involution which is hyperbolic over  $\mathcal{F}(A)$  already is hyperbolic over the base field (see [10, Theorem 1.1]). It is expected that the same holds for isotropy. Only a weaker result is known in general for now, namely, an involution which is isotropic over  $\mathcal{F}(A)$  also is isotropic after an odd degree extension of the base field (see [11, Theorem 1]). In this work, we consider the following property: an involution  $\sigma$  (or the algebra with involution  $(A, \sigma)$ ) is said to be *totally decomposable* if

$$(1) \quad (A, \sigma) \simeq (Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n)$$

for some quaternion algebras with involution  $(Q_i, \sigma_i)$ . Totally decomposable involutions can be considered as an analogue of Pfister forms in quadratic form theory. Indeed, by a theorem of Becher [2], if the algebra  $A$  is split, an orthogonal involution on  $A$  is totally decomposable if and only if it is adjoint to a Pfister form, which means that it admits a decomposition (1) in which each quaternion factor is split. Therefore, every totally decomposable orthogonal involution is adjoint to a Pfister form after generic splitting of the underlying algebra. Whether the converse holds is a classical question, raised in [1, §2.4]:

**Question 1.** *Let  $\sigma$  be an orthogonal involution on a central simple algebra  $A$  of 2-power degree. Suppose  $q_\sigma$  is similar to a Pfister form, i.e.,  $\sigma_{\mathcal{F}(A)}$  is totally decomposable. Does it follow that  $\sigma$  is totally decomposable ?*

If  $\deg A = 4$  or  $8$ , cohomological invariants can be used to give a positive answer. In Section 2 we give examples showing that the answer is negative in degree 16 and index 4 and 8. We even prove a slightly stronger result, namely the involutions in our examples remain non-totally decomposable over any odd degree extension of the base field. We thus disprove a conjecture of Garibaldi [6, Conjecture (16.1)]. See §1.3 for a summary of results on Question 1.

The more general problem underlying questions of the type above is to determine how much information on the involution is lost when the algebra is generically split. In [4], it is proved that if orthogonal involutions  $\sigma$  and  $\sigma'$  on a central simple algebra  $A$  are motivic equivalent over  $\mathcal{F}(A)$ , in an appropriate sense which amounts to motivic equivalence of  $q_\sigma$  and  $q_{\sigma'}$ , then they already are motivic equivalent over the base field. We address the analogous property for isomorphism:

**Question 2.** *Let  $\sigma, \sigma'$  be orthogonal involutions on a central simple algebra  $A$ . Suppose  $q_\sigma$  and  $q_{\sigma'}$  are similar, i.e.,  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$ . Does it follow that  $\sigma \simeq \sigma'$  ?*

As for Question 1, cohomological invariants yield a positive answer for algebras of low degree. Examples showing that the answer to Question 2 is negative in degree 8 and index 4 or 8 were provided in [19, §4]. The case of index 2 is very specific however, as demonstrated by Becher's theorem on total decomposability (see Proposition 1.2), and because function fields of conics are excellent (see [15]). Many questions that are still open in general are solved in index 2, using this peculiarity. For instance, it is known that an orthogonal involution on a central simple algebra  $A$  of index 2 that becomes isotropic over  $\mathcal{F}(A)$  is isotropic over the base field, see [15]. Therefore, we restrict in Section 3 to the case where the index is 2. We identify a few cases where the answer is positive, but we construct examples

in degree 8 and 12 showing that the answer is negative in general: we exhibit skew-hermitian forms of rank 4 or 6 over a quaternion algebra  $Q$  that are similar after scalar extension to the function field  $\mathcal{F}(Q)$  of the corresponding conic, even though they are not similar over  $Q$ . Since the group of isometries  $O(h)$  of a skew-hermitian form  $h$  (which is a classical group of orthogonal type) determines the form up to similarity, we thus have skew-hermitian forms  $h, h'$  such that  $O(h)_{\mathcal{F}(Q)} \simeq O(h')_{\mathcal{F}(Q)}$  even though  $O(h) \not\simeq O(h')$ . By contrast, if  $h_{\mathcal{F}(Q)} \simeq h'_{\mathcal{F}(Q)}$  then  $h \simeq h'$  because scalar extension to  $\mathcal{F}(Q)$  induces an injective map on Witt groups of skew-hermitian forms, as shown by Dejaiffe [5] and Parimala–Sridharan–Suresh [15, Prop. 3.3]. We refer to §1.3 for a summary of results on Question 2.

In addition, our examples have interesting consequences on cohomological invariants of orthogonal involutions. It is well known that the degree 1 and degree 2 invariants, respectively related to discriminants and Clifford algebras, are defined in a similar way for quadratic forms and for involutions and share analogous properties regarding classification and decomposability criteria in both settings (see [23] for precise statements). Using the so-called Rost invariant, which assigns a degree 3 cohomology class to any torsor under some absolutely almost simple simply connected algebraic group, one may also extend the so-called Arason invariant of quadratic form theory to orthogonal involutions, under some additional conditions on the involution and on the underlying algebra. Nevertheless, our examples show that the Arason invariant does not have the same properties in both settings. In particular, it is not classifying for orthogonal involutions on a degree 12 and index 2 algebra, proving that Proposition 4.3 in [20] is optimum (cf. Remark 1.7 (ii)). Moreover, it vanishes for a 16-dimensional quadratic form if and only if it is similar to a Pfister form, but it may vanish for a non-totally decomposable involution on a degree 16 central simple algebra (see Remark 1.4).

All the examples in Sections 2 and 3 have in common the use of the so-called ‘generic sum’ construction, which was introduced in [17], and used there to construct examples of algebraic groups without outer automorphisms. Questions 1 and 2 turn out to be related in a somewhat unexpected way: the examples disproving Garibaldi’s conjecture are generic sums of involutions that provide a negative answer to Question 2. To explain this relation, we develop a notion of ‘unramified algebra with involution’ in Section 2.

The second-named author is grateful to Sofie Beke and Jan Van Geel for motivating discussions on the totally decomposable case in Theorem 1.5(c).

**1.1. Notations.** All fields in this paper have characteristic different from 2; in particular, the valued fields we consider are non-dyadic. We generally use the notation and terminology in [12], to which we refer for background information on central simple algebras with involution. In particular, for  $n$ -fold Pfister forms we write

$$\langle\langle a_1, \dots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \cdots \langle 1, -a_n \rangle.$$

By an algebra with involution, we always mean a central simple algebra with an involution of the first kind. If  $(D, \gamma)$  is a division algebra with involution, and  $h: M \times M \rightarrow D$  is a nonsingular hermitian or skew-hermitian form with respect to  $\gamma$ , we let  $\text{Ad}(h)$  denote the algebra with involution  $(\text{End}_D M, \text{ad}(h))$ , where  $\text{ad}(h)$  is the adjoint involution of the form  $h$ . In particular, for every nonsingular  $r$ -dimensional quadratic form  $q$  over a field  $k$ ,  $\text{Ad}(q)$  stands for  $(\text{End}_k(k^r), \text{ad}(q))$ . Recall from [12, §4.A] that every algebra with involution can be represented as

$\text{Ad}(h)$  for some nonsingular hermitian or skew-hermitian form. The algebra with involution  $\text{Ad}(h)$  (or the involution  $\text{ad}(h)$ ) is said to be *isotropic* (resp. *hyperbolic*) if the form  $h$  is isotropic (resp. hyperbolic).

The group of *similarity factors* of an algebra with involution  $(A, \sigma)$  over a field  $k$  is defined as

$$G(A, \sigma) = \{\mu \in k^\times \mid \mu = \sigma(g)g \text{ for some } g \in A\}.$$

For a nonsingular hermitian or skew-hermitian form  $h$ , we write  $G(h)$  for  $G(\text{Ad}(h))$ . This group has the following alternative description:

$$G(h) = \{\mu \in k^\times \mid \langle \mu \rangle \cdot h \simeq h\}.$$

**1.2. Cohomological invariants.** Every  $k$ -algebra with orthogonal involution  $(A, \sigma)$  of even degree has a discriminant (see [12, §7.A])

$$d_\sigma = e_1(\sigma) \in k^\times / k^{\times 2},$$

which defines a first cohomological invariant of  $\sigma$ . The discriminant also defines a quadratic étale algebra  $Z = k[X]/(X^2 - d_\sigma)$ , which we also call the discriminant of  $\sigma$  for short. If  $\sigma$  has trivial discriminant, its Clifford algebra  $\mathcal{C}(\sigma)$  is a direct product of two central simple algebras  $\mathcal{C}_+(\sigma) \times \mathcal{C}_-(\sigma)$  whose Brauer classes satisfy  $[\mathcal{C}_+(\sigma)] - [\mathcal{C}_-(\sigma)] = [A]$  (see [12, §9.C] or [23, §3.4]); the  $e_2$  invariant of  $\sigma$  has values in the quotient of the Brauer group  $\text{Br}(k)$  by the subgroup generated by the Brauer class of  $A$ ; it is defined as the image of  $[\mathcal{C}_+(\sigma)]$  or  $[\mathcal{C}_-(\sigma)]$ :

$$e_2(\sigma) = [\mathcal{C}_+(\sigma)] + \langle [A] \rangle = [\mathcal{C}_-(\sigma)] + \langle [A] \rangle \in \text{Br}(k) / \langle [A] \rangle.$$

If  $\sigma$  has trivial  $e_1$  and  $e_2$  invariants and the coindex of  $A$  is even (i.e.,  $\text{ind } A$  divides  $\frac{1}{2} \deg A$ ), one may define an Arason invariant

$$e_3(\sigma) \in H^3(k, \mu_4^{\otimes 2}) / k^\times \cdot [A],$$

by using the Rost invariant of Spin groups (see [23, §3.5]).

Since  $k$  is quadratically closed in  $\mathcal{F}(A)$ , and the kernel of the scalar extension map  $\text{Br}(k) \rightarrow \text{Br}(\mathcal{F}(A))$  is  $\langle [A] \rangle$  by a theorem of Amitsur, the  $e_1$  and the  $e_2$  invariant of an involution are trivial if and only if they are trivial after generic splitting of the algebra  $A$ , and for orthogonal involutions  $\sigma, \sigma'$  on  $A$  we have  $e_1(\sigma) = e_1(\sigma')$  (resp.  $e_2(\sigma) = e_2(\sigma')$ ) if and only if  $e_1(\sigma_{\mathcal{F}(A)}) = e_1(\sigma'_{\mathcal{F}(A)})$  (resp.  $e_2(\sigma_{\mathcal{F}(A)}) = e_2(\sigma'_{\mathcal{F}(A)})$ ). The same applies to the Arason invariant if either  $A$  has index  $\leq 4$ , or  $A$  is Brauer-equivalent to a tensor product of three quaternion algebras, but fails in general (see [9], [8] and [16]).

When the discriminant of the orthogonal involution  $\sigma$  is not trivial (i.e., when  $Z$  is a field), the Clifford algebra  $\mathcal{C}(\sigma)$  is a central simple  $Z$ -algebra. The  $k$ -isomorphism class of  $\mathcal{C}(\sigma)$  has the same property as the  $e_1$  and  $e_2$  invariants:

**Lemma 1.1.** *Let  $\sigma, \sigma'$  be orthogonal involutions on a central simple  $k$ -algebra  $A$  of even degree. If  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$ , then  $\mathcal{C}(\sigma)$  and  $\mathcal{C}(\sigma')$  are isomorphic as  $k$ -algebras.*

*Proof.* As observed above, the isomorphism  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$  implies that  $e_1(\sigma) = e_1(\sigma')$ , hence the centers  $Z$  and  $Z'$  of  $\mathcal{C}(\sigma)$  and  $\mathcal{C}(\sigma')$  are  $k$ -isomorphic. If  $e_1(\sigma) = e_1(\sigma') = 0$ , then  $Z \simeq Z' \simeq k \times k$ , and

$$\mathcal{C}(\sigma) \simeq \mathcal{C}_+(\sigma) \times \mathcal{C}_-(\sigma), \quad \mathcal{C}(\sigma') \simeq \mathcal{C}_+(\sigma') \times \mathcal{C}_-(\sigma').$$

From  $e_2(\sigma) = e_2(\sigma')$ , it follows that  $\mathcal{C}_+(\sigma)$  is isomorphic to  $\mathcal{C}_+(\sigma')$  or  $\mathcal{C}_-(\sigma')$ , hence  $\mathcal{C}(\sigma) \simeq_k \mathcal{C}(\sigma')$ .

For the rest of the proof, assume  $Z$  is a field, and choose an arbitrary isomorphism  $Z \simeq Z'$  to identify  $Z'$  with  $Z$ . After scalar extension to  $Z$  we have  $e_1(\sigma_Z) = e_1(\sigma'_Z) = 0$ , hence the first part of the proof yields  $\mathcal{C}(\sigma_Z) \simeq \mathcal{C}(\sigma'_Z)$  as  $Z$ -algebras. But letting  ${}^t\mathcal{C}(\sigma)$  denote the conjugate  $Z$ -algebra of  $\mathcal{C}(\sigma)$  under the nontrivial  $k$ -automorphism  $\iota$  of  $Z/k$ , we have

$$\mathcal{C}(\sigma_Z) = \mathcal{C}(\sigma) \otimes_k Z \simeq \mathcal{C}(\sigma) \times {}^t\mathcal{C}(\sigma) \quad \text{and likewise} \quad \mathcal{C}(\sigma'_Z) \simeq \mathcal{C}(\sigma') \times {}^t\mathcal{C}(\sigma').$$

Therefore, from  $\mathcal{C}(\sigma_Z) \simeq \mathcal{C}(\sigma'_Z)$  it follows that  $\mathcal{C}(\sigma) \simeq \mathcal{C}(\sigma')$  or  ${}^t\mathcal{C}(\sigma')$  as  $Z$ -algebras, hence  $\mathcal{C}(\sigma) \simeq \mathcal{C}(\sigma')$  as  $k$ -algebras.  $\square$

**1.3. Synopsis of results on Questions 1 and 2.** Question 1 has a positive answer if either the degree or the index of the algebra is small enough. More precisely, we have the following:

**Proposition 1.2.** *Let  $(A, \sigma)$  be a central simple algebra with orthogonal involution over a field  $k$ . Suppose  $\sigma_{\mathcal{F}(A)}$  is totally decomposable. If in addition we have either  $\deg A = 4$  or  $8$ , or  $\text{ind } A = 2$ , then  $\sigma$  is totally decomposable.*

*Proof.* The first assertion follows easily from the cohomological criteria of decomposability that can be established in these degrees, see [1, Proposition 2.10]. The index 2 case is a theorem of Becher [2, Th. 2].  $\square$

As opposed to this, the answer is negative in general in degree 16 and index  $\geq 4$ . More precisely, in Theorem 2.4 below, we prove the following:

**Theorem 1.3.** *There exist central simple algebras with orthogonal involution  $(A, \sigma)$  satisfying all the following conditions:*

- (i)  $\deg A = 16$  and  $\text{ind } A = 4$  or  $8$ , and
- (ii)  $\sigma_{\mathcal{F}(A)}$  is totally decomposable, and
- (iii)  $\sigma$  is not totally decomposable, and remains so over any odd degree extension of the base field.

The algebra with involution  $(A, \sigma)$  of Theorem 2.4 satisfies the conditions above. In particular,  $\sigma_{\mathcal{F}(D)}$  is totally decomposable for  $D$  the division algebra Brauer-equivalent to  $A$ , hence  $\sigma_{\mathcal{F}(A)}$  is totally decomposable, since  $\mathcal{F}(A)$  is an extension of  $\mathcal{F}(D)$ .

*Remark 1.4.* Any algebra with orthogonal involution satisfying conditions (i) and (ii) above satisfies  $e_1(\sigma_{\mathcal{F}(A)}) = e_2(\sigma_{\mathcal{F}(A)}) = e_3(\sigma_{\mathcal{F}(A)}) = 0$  because  $\sigma_{\mathcal{F}(A)}$  is the adjoint involution of a 4-fold Pfister form. Since the algebra  $A$  in Theorem 2.4 is Brauer-equivalent to a tensor product of three quaternion algebras, we have in addition for those examples

$$(2) \quad e_1(\sigma) = e_2(\sigma) = e_3(\sigma) = 0.$$

Therefore, as opposed to what happens for quadratic forms and in smaller degree, the vanishing of the first cohomological invariants does not characterize totally decomposable involutions in degree 16. That is, the examples in Theorem 2.4 disprove Conjecture (16.1) in [6]. Note however that if  $\deg A = 16$  and  $\text{ind } A = 2$ , then by Proposition 1.2 the involution  $\sigma$  is totally decomposable if and only if (2) holds, because (2) is equivalent to the condition that  $\sigma_{\mathcal{F}(A)}$  be totally decomposable.

Question 2 also has a positive answer if the algebra has small enough degree, or under some additional condition on the algebra and on the involution. More precisely, we have the following:

**Theorem 1.5.** *Let  $\sigma$  and  $\sigma'$  be orthogonal involutions on a central simple algebra  $A$  over a field  $k$ , and assume  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$ . If in addition any of the following conditions holds:*

- (a)  $\deg A = 2$  or  $4$ , or
- (b)  $\deg A = 6$  and  $e_1(\sigma) = 0$ , or
- (c)  $\text{ind } A = 2$  and  $\sigma$  is totally decomposable, or
- (d)  $\deg A \equiv 2 \pmod{4}$  and there exists a quadratic field extension of  $k$  over which  $(A, \sigma)$  is split and hyperbolic,

then  $\sigma \simeq \sigma'$ .

*Proof.* In low degree, one may use cohomological invariants to compare involutions. More precisely, if  $\deg A = 2$ , orthogonal involutions on  $A$  are classified by their  $e_1$  invariant, see [12, (7.4)] or [23, Th. 3.6]. If  $\deg A = 4$ , they are classified by their Clifford algebra, see [12, (15.7)]. If  $\deg A = 6$ , orthogonal involutions on  $A$  with trivial  $e_1$  invariant are classified by their  $e_2$  invariant (or their Clifford algebra), see [12, (15.32)] or [23, Th. 3.10]. With this in hand, cases (a) and (b) follow from Lemma 1.1.

Cases (c) and (d) are proved in Proposition 3.5 below. □

Nevertheless, Question 2 has a negative answer in general. More precisely, we may add the following to the theorem above:

**Theorem 1.6.** *There exist central simple algebras  $A$  with orthogonal involutions  $\sigma, \sigma'$  satisfying any of the following conditions:*

- (e)  $\deg A = 8$ ,  $\text{ind } A = 4$  or  $8$ , and  $\sigma$  and  $\sigma'$  are totally decomposable, or
- (f)  $\deg A = 8$ ,  $\text{ind } A = 2$ , and  $e_1(\sigma) = e_1(\sigma') = 0$ , or
- (g)  $\deg A = 12$ ,  $\text{ind } A = 2$ ,  $e_1(\sigma) = e_1(\sigma') = 0$  and  $e_2(\sigma) = e_2(\sigma') = 0$ ,

and such that

$$\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}, \quad \text{and yet} \quad \sigma \not\simeq \sigma'.$$

*Proof.* Case (e) was shown in [19, Ex. 4.2 & 4.3]. The other two cases are new and are explained in Remark 3.2 and Examples 3.6 and 3.8. □

*Remark 1.7.* (i) By (e) and (g), Theorem 1.5(c) does not hold anymore if we drop one of the two assumptions. Moreover, the example of case (f) does not satisfy  $e_2(\sigma) = e_2(\sigma') = 0$ ; otherwise,  $\sigma$  and  $\sigma'$  would be totally decomposable, and this is impossible again by Theorem 1.5 (c).

(ii) In case (g), the  $e_3$  invariant of  $\sigma$  and  $\sigma'$  is defined, and the condition  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$  implies  $e_3(\sigma) = e_3(\sigma')$  since  $\text{ind } A = 2$ . Thus, Example 3.8 shows that the  $e_3$  invariant does not classify orthogonal involutions with trivial  $e_1$  and  $e_2$  invariants on central simple algebras of degree 12 and index 2 (although it is classifying if the algebra  $A$  is split, and for isotropic involutions if the algebra has index 2 by [20, Prop. 4.3]).

(iii) By Lewis [14, Prop. 10], in all three cases, the involutions  $\sigma$  and  $\sigma'$  remain non-isomorphic over any odd degree extension of the base field.

## 2. GENERICALLY PFISTER INVOLUTIONS

The aim of this section is to construct examples proving Theorem 1.3. We start with a few observations on central simple algebras with involution over a complete discretely valued field, introducing a notion of *unramified* algebra with involution.

Throughout this section we let  $K$  denote a field with a discrete valuation  $v: K \rightarrow \Gamma_K \cup \{\infty\}$ . We assume  $K$  is complete for this valuation,  $v(K^\times) = \Gamma_K \simeq \mathbb{Z}$ , and the residue field  $\overline{K}$  has characteristic different from 2. Let  $(D, \gamma)$  be a central division algebra over  $K$  with an involution of the first kind. It is known (see for instance [25, Th. 1.4, Cor. 1.7]) that  $v$  extends to a valuation

$$v_D: D \rightarrow \Gamma_D \cup \{\infty\} \quad \text{where } \Gamma_D = v_D(D^\times) = \frac{1}{\deg D} v(\text{Nrd } D^\times) \subset \frac{1}{\deg D} \Gamma_K.$$

The involution  $\gamma$  induces an involution  $\overline{\gamma}$  on the residue division algebra  $\overline{D}$ . Let  $Z(\overline{D})$  denote the center of  $\overline{D}$ , which is a field extension of  $\overline{K}$ . Since  $\text{char } \overline{K} \neq 2$  and  $\deg D$  is a 2-power, we have

$$(3) \quad [D : K] = [\overline{D} : \overline{K}] \cdot (\Gamma_D : \Gamma_K) \quad \text{and} \quad (\Gamma_D : \Gamma_K) = [Z(\overline{D}) : \overline{K}],$$

see [25, Prop. 8.64]. In particular,  $\Gamma_D = \Gamma_K$  if and only if  $\overline{D}$  is a central division algebra over  $\overline{K}$ ; when this condition holds, we have in addition  $\deg \overline{D} = \deg D$ .

As observed by Scharlau [21, p. 208], there always exists a uniformizing element  $\pi_D$  for  $v_D$  such that  $\gamma(\pi_D) = \pi_D$ , except in the following case:

$$(*) \quad \begin{array}{l} D \text{ is a quaternion algebra, } \gamma \text{ is the canonical (symplectic) involution} \\ \text{on } D, \text{ and } \Gamma_D = \frac{1}{2} \Gamma_K. \end{array}$$

In all cases, Scharlau [21, p. 204] shows that every anisotropic hermitian form  $h$  over  $(D, \gamma)$  defines an anisotropic hermitian form  $\partial_1(h)$  over  $(\overline{D}, \overline{\gamma})$ , called the *first residue form* of  $h$ . If  $(*)$  does not hold, a *second residue form*  $\partial_2(h)$  is defined as the first residue form of  $\pi_D h$  (which is a hermitian form with respect to the involution  $\gamma': d \mapsto \pi_D \gamma(d) \pi_D^{-1}$  on  $D$ ). As an analogue of Springer's theorem, Scharlau proves [21, Satz 3.6] that mapping  $h$  to  $(\partial_1(h), \partial_2(h))$  (resp. to  $\partial_1(h)$ ) yields an isomorphism of Witt groups

$$W(D, \gamma) \xrightarrow{\sim} \begin{cases} W(\overline{D}, \overline{\gamma}) \oplus W(\overline{D}, \overline{\gamma}') & \text{away from case } (*), \\ W(\overline{D}, \overline{\gamma}) & \text{in case } (*). \end{cases}$$

Thus, every hermitian form  $h$  over  $(D, \gamma)$  has a first (and, if  $(*)$  does not hold, a second) residue form, which are either 0 or anisotropic forms with values in  $\overline{D}$ . We call a hermitian form over  $(D, \gamma)$  *unramified* if the following two conditions hold:

$\Gamma_D = \Gamma_K$  (i.e.,  $D$  is unramified over  $K$ ), and the second residue form of  $h$  is 0.

Otherwise the form is said to be *ramified*. Thus, in case  $(*)$  every hermitian form is ramified. We extend this terminology to algebras with involution as follows:  $(A, \sigma)$  is said to be *unramified* if there exists an unramified hermitian form  $h$  such that  $(A, \sigma) \simeq \text{Ad}(h)$ , and otherwise  $(A, \sigma)$  is called *ramified*.

Alternately, one may use the theory of gauges developed in [25] to give a characterization of unramified algebras with involution which does not use representations  $(A, \sigma) \simeq \text{Ad}(h)$ . Recall from [24, Th. 2.2] that if  $(A, \sigma)$  is anisotropic there is a unique map

$$g: A \rightarrow (\Gamma_K \otimes_{\mathbb{Z}} \mathbb{Q}) \cup \{\infty\}$$

with the following properties:



- (i)  $g(a) = \infty$  if and only if  $a = 0$ ;
- (ii)  $g(a + b) \geq \min(g(a), g(b))$  for  $a, b \in A$ ;
- (iii)  $g(a\lambda) = g(a) + v(\lambda)$  for  $a \in A$  and  $\lambda \in F$ ;
- (iv)  $g(1) = 0$  and  $g(ab) \geq g(a) + g(b)$  for  $a, b \in A$ ;
- (v)  $g(\sigma(a)a) = 2g(a)$  for  $a \in A$ .

Using  $g$ , one defines a  $\overline{K}$ -algebra  $A_0$  as follows:

$$A_0 = \frac{\{a \in A \mid g(a) \geq 0\}}{\{a \in A \mid g(a) > 0\}}.$$

The algebra  $A_0$  is semisimple because  $g$  is a  $v$ -gauge in the sense of [25, §3.2.2], see [24, Th. 2.2]. We call  $g$  the  $\sigma$ -special gauge on  $A$ . It satisfies  $g(\sigma(a)) = g(a)$  for all  $a \in A$ , and is actually characterized by this property among all the  $v$ -gauges on  $A$ .

**Proposition 2.1.** *The anisotropic algebra with involution  $(A, \sigma)$  is unramified if and only if  $A_0$  is a central simple  $\overline{K}$ -algebra. When this condition holds, we have  $\deg A_0 = \deg A$ .*

*Proof.* Suppose  $(A, \sigma) = (\text{End}_D M, \text{ad}(h))$  for some hermitian space  $(M, h)$  over a division algebra with involution  $(D, \gamma)$ . Inspection of the proof of [24, Th. 2.2] shows that the map  $g$  is defined as follows: for  $a \in \text{End}_D M$ ,

$$g(a) = \min\{\alpha(a(m)) - \alpha(m) \mid m \in M, m \neq 0\},$$

where, for all  $m \in M$ ,

$$\alpha(m) = \frac{1}{2}v_D(h(m, m)) \in \frac{1}{2}\Gamma_D \cup \{\infty\}.$$

Thus,  $g = \text{End}(\alpha)$  in the notation of [25, p. 104], and the set

$$\Gamma_M = \{\alpha(m) \mid m \in M \setminus \{0\}\}$$

is a union of cosets of  $\Gamma_D$ . Let  $r$  be the number of cosets of  $\Gamma_D$  in  $\Gamma_M$ . Since  $\Gamma_D \simeq \mathbb{Z}$  and  $\Gamma_M \subset \frac{1}{2}\Gamma_D$ , we have  $r = 1$  or  $2$ . By [25, Prop. 3.34 and Prop. 2.41] the algebra  $A_0$  is semisimple, and its center is a product of  $r$  copies of  $Z(\overline{D})$ . Therefore,  $A_0$  is central simple over  $\overline{K}$  if and only if  $r = 1$  and  $Z(\overline{D}) = \overline{K}$ . This last condition implies that  $D$  is unramified, in particular  $(*)$  does not hold.

For the rest of the proof, assume  $D$  is unramified. This implies that  $Z(\overline{D}) = \overline{K}$ , and we need to prove that  $(A, \sigma)$  is unramified if and only if  $r = 1$ . Let  $\pi_D$  be a uniformizing element for  $v_D$  such that  $\gamma(\pi_D) = \pi_D$ . Note that the map  $\alpha$  is the norm on  $M$  used by Scharlau in his definition of the first residue  $\partial_1(h)$ ; see [21, Prop. 3.1]. If  $(A, \sigma)$  is unramified we may assume  $h$  is unramified. Since  $\partial_2(h) = 0$  it follows that  $v(h(m, m)) \in 2\Gamma_D$  for all nonzero  $m \in M$ , hence  $r = 1$  and  $A_0$  is central simple over  $\overline{K}$ . Conversely, if  $r = 1$  then either  $\Gamma_M = \Gamma_D$  or  $\Gamma_M = \frac{1}{2}v_D(\pi_D) + \Gamma_D$ . In the first case  $h$  is unramified, hence  $(A, \sigma)$  is unramified. In the second case  $\pi_D h$  is unramified. Since  $\text{ad}(h) = \text{ad}(\pi_D h)$  we may substitute  $\pi_D h$  for  $h$  and thus again conclude that  $(A, \sigma)$  is unramified.  $\square$

We now establish a sufficient condition for a totally decomposable algebra to be unramified. This provides a tool for proving that some ramified algebras with involution are not totally decomposable.

**Proposition 2.2.** *A totally decomposable algebra with involution  $(A, \sigma)$  is unramified if  $v(G(A, \sigma)) \subset 2\Gamma_K$ .*

*Proof.* We first consider the case where  $A$  is a quaternion algebra. If  $A$  is a split quaternion algebra, then  $A \simeq \text{Ad}(q)$  for some binary quadratic form  $q$  over  $K$ . The determinant of  $q$  is a similarity factor of  $q$ , hence the condition  $v(G(A, \sigma)) \subset 2\Gamma_K$  implies that  $\text{Ad}(q)$  is unramified. Now, assume  $A$  is a quaternion division algebra. The  $\sigma$ -special gauge  $g$  on  $A$  does not depend on  $\sigma$  and coincides with the valuation  $v_A$  extending  $v$ , hence  $A_0$  is the residue division algebra  $\bar{A}$ . If  $\sigma$  is the canonical involution, then  $G(A, \sigma) = \text{Nrd}(A^\times)$ , hence the condition  $v(G(A, \sigma)) \subset 2\Gamma_K$  implies  $\Gamma_A = \Gamma_K$ . It then follows by (3) that  $A_0$  is a central simple  $\bar{K}$ -algebra, hence  $(A, \sigma)$  is unramified. If  $\sigma$  is an orthogonal involution, then as observed above  $A$  contains a uniformizing element  $\pi_A$  such that  $\sigma(\pi_A) = \pi_A$  by [21, p. 208]. Let  $\bar{\pi}_A = \text{Trd}(\pi_A) - \pi_A \in A$  be the conjugate quaternion. Suppose  $\Gamma_A \neq \Gamma_K$ . Then  $v(\pi_A + \bar{\pi}_A) \in \Gamma_K$  whereas  $v(\pi_A) = v(\bar{\pi}_A) \notin \Gamma_K$ , hence  $v(\pi_A + \bar{\pi}_A) > v(\pi_A)$ , and therefore

$$v(\pi_A - \bar{\pi}_A) = v(2\pi_A - (\pi_A + \bar{\pi}_A)) = v(\pi_A) \notin \Gamma_K.$$

Now,

$$\sigma(\pi_A - \bar{\pi}_A) \cdot (\pi_A - \bar{\pi}_A) = (\pi_A - \bar{\pi}_A)^2 \in K^\times,$$

hence  $(\pi_A - \bar{\pi}_A)^2 \in G(A, \sigma)$ . But  $v((\pi_A - \bar{\pi}_A)^2) = 2v(\pi_A) \notin 2\Gamma_K$ , in contradiction with the hypothesis that  $v(G(A, \sigma)) \subset 2\Gamma_K$ . Therefore  $\Gamma_A = \Gamma_K$  and it follows as in the previous case that  $(A, \sigma)$  is unramified. We have thus proved the proposition in the case where  $A$  is a single quaternion algebra.

Now, let  $(A, \sigma) = (Q_1, \sigma_1) \otimes_K \cdots \otimes_K (Q_n, \sigma_n)$ , where each  $Q_i$  is a quaternion  $K$ -algebra, and assume  $v(G(A, \sigma)) \subset 2\Gamma_K$ . Then  $(A, \sigma)$  is not hyperbolic because otherwise  $G(A, \sigma) = K^\times$ . It follows by [3, Cor. 3.2] that  $(A, \sigma)$  is anisotropic, hence each  $(Q_i, \sigma_i)$  is anisotropic. Each  $Q_i$  then carries a  $\sigma_i$ -special gauge  $g_i$ , and since  $G(Q_i, \sigma_i) \subset G(A, \sigma)$  the first part of the proof shows that each  $(Q_i, \sigma_i)$  is unramified. The tensor product  $g = g_1 \otimes \cdots \otimes g_n$  is a  $v$ -gauge on  $A$  by [25, Prop. 3.41], and it satisfies  $g \circ \sigma = g$  by [24, Prop. 1.3] because  $g_i \circ \sigma_i = g_i$  for all  $i$  by the uniqueness property of the  $\sigma_i$ -special gauge. The map  $g$  is therefore the  $\sigma$ -special gauge on  $A$  by [24, Th. 2.2]. By [25, Prop. 3.41] or [24, Prop. 1.3] we have

$$A_0 = (Q_1)_0 \otimes_{\bar{K}} \cdots \otimes_{\bar{K}} (Q_n)_0.$$

Since each  $(Q_i, \sigma_i)$  is unramified, it follows that each  $(Q_i)_0$  is a central quaternion  $\bar{K}$ -algebra, hence  $A_0$  is a central simple  $\bar{K}$ -algebra. Therefore, Proposition 2.1 shows that  $(A, \sigma)$  is unramified.  $\square$

Let us now use Proposition 2.2 to construct examples of algebras with involution that are not totally decomposable, and that remain non-totally decomposable after odd-degree scalar extensions. Let  $(D, \gamma)$  be a central division algebra with an involution of the first kind over an arbitrary field  $k$  (of characteristic different from 2), and let  $h_1, h_2$  be nonsingular hermitian forms over  $(D, \gamma)$ . Consider the field  $F = k((t))$  of Laurent series in one indeterminate over  $k$ . Extending scalars to  $F$ , we obtain the central division algebra with involution  $(\widehat{D}, \widehat{\gamma}) = (D, \gamma)_F$  over  $F$  and the hermitian forms  $\widehat{h}_1 = (h_1)_F$  and  $\widehat{h}_2 = (h_2)_F$  over  $(\widehat{D}, \widehat{\gamma})$ . Consider the following hermitian form over  $(\widehat{D}, \widehat{\gamma})$  (cf. [17, §3.2]):

$$h = \widehat{h}_1 \perp \langle t \rangle \widehat{h}_2.$$

**Proposition 2.3.** *Let  $K$  be an odd-degree field extension of  $F$ , and let  $v: K \rightarrow \Gamma_K \cup \{\infty\}$  denote the discrete valuation on  $K$  extending the  $t$ -adic valuation on*

*F.* If  $h_1$  and  $h_2$  are not hyperbolic, then the algebra with involution  $\text{Ad}(h)_K$  is ramified. If  $h_1$  and  $h_2$  are not similar, then  $v(G(h_K)) \subset 2\Gamma_K$ . If  $h_1$  and  $h_2$  are not hyperbolic and are not similar, then  $\text{Ad}(h)_K$  is not totally decomposable.

*Proof.* Let  $e = (v(K^\times) : v(F^\times))$  be the ramification index of  $K/F$ , and let  $\pi \in K$  be a uniformizing element of  $K$ , hence also of  $\widehat{D}_K = D \otimes_k K$ . We have  $\pi^e = ut$  for some  $u \in K^\times$  with  $v(u) = 0$ , and  $t \equiv u\pi \pmod{K^{\times 2}}$  since  $e$  is odd. Therefore,

$$h_K \simeq (\widehat{h}_1)_K \perp \langle u\pi \rangle (\widehat{h}_2)_K,$$

hence

$$\partial_1(h_K) = (h_1)_{\overline{K}} \quad \text{and} \quad \partial_2(h_K) = \langle \overline{u} \rangle (h_2)_{\overline{K}} \quad \text{in the Witt group } W(D_{\overline{K}}, \gamma_{\overline{K}}).$$

If  $h_1$  and  $h_2$  are not hyperbolic, then  $(h_1)_{\overline{K}}$  and  $(h_2)_{\overline{K}}$  are not hyperbolic because  $[\overline{K} : k]$  is odd, hence no scalar multiple of  $h_K$  is unramified. Therefore,  $\text{Ad}(h)_K$  is ramified.

Now, let  $\mu \in G(h_K)$ . If  $v(\mu) \notin 2\Gamma_K$ , then there exists  $\mu_0 \in K^\times$  such that  $v(\mu_0) = 0$  and  $\mu \equiv \mu_0\pi \pmod{K^{\times 2}}$ . Then from  $\langle \mu \rangle h_K \simeq h_K$  it follows that

$$\langle \mu_0 u \rangle (\widehat{h}_2)_K \perp \langle \mu_0 \pi \rangle (\widehat{h}_1)_K \simeq (\widehat{h}_1)_K \perp \langle u\pi \rangle (\widehat{h}_2)_K.$$

Comparing the residues of each side, we obtain

$$\langle \mu_0 u \rangle (h_2)_{\overline{K}} = (h_1)_{\overline{K}} \quad \text{in } W(D_{\overline{K}}, \gamma_{\overline{K}}).$$

Since  $[\overline{K} : k]$  is odd, a transfer argument shows that  $h_1$  and  $h_2$  are similar; see [14, Prop. 10].

Hence if the hermitian forms  $h_1$  and  $h_2$  are not hyperbolic and not similar,  $\text{Ad}(h)_K$  is ramified and  $v(G(\text{Ad}(h)_K)) \subset 2\Gamma_K$ . It follows by Proposition 2.2 that  $\text{Ad}(h)_K$  is not totally decomposable.  $\square$

To obtain particularly significant instances of this construction, recall from [19, Ex. 4.2 & 4.3] that there exist central simple  $k$ -algebras  $A$  of degree 8 with orthogonal involutions  $\sigma_1, \sigma_2$  such that

- (i)  $\sigma_1$  and  $\sigma_2$  are not hyperbolic and are not isomorphic, and
- (ii) over the function field  $\mathcal{F}(A)$  of the Severi–Brauer variety of  $A$ , there is a 3-fold Pfister form  $\varphi$  such that

$$(\sigma_1)_{\mathcal{F}(A)} \simeq (\sigma_2)_{\mathcal{F}(A)} \simeq \text{ad}(\varphi).$$

In these examples, the field  $k$  has characteristic zero, the index of  $A$  is 4 or 8, and  $A$  is a tensor product of three quaternion algebras. Now, choose a division algebra  $D$  Brauer-equivalent to  $A$  and an orthogonal involution  $\gamma$  on  $D$ . We may then find hermitian forms  $h_1, h_2$  over  $(D, \gamma)$  such that  $\sigma_1 \simeq \text{ad}(h_1)$  and  $\sigma_2 \simeq \text{ad}(h_2)$ . With this choice of  $h_1$  and  $h_2$ , the construction preceding Proposition 2.3 yields a hermitian form  $h = \widehat{h}_1 \perp \langle t \rangle \widehat{h}_2$  over the division algebra with involution  $(\widehat{D}, \widehat{\gamma}) = (D, \gamma)_F$ , where  $F = k((t))$ .

**Theorem 2.4.** *With the notation above,  $\text{Ad}(h)$  is a central simple  $F$ -algebra of degree 16 with orthogonal involution that is not totally decomposable over  $F$  nor over any odd-degree extension of  $F$ . Yet, over the function field  $\mathcal{F}(\widehat{D})$  of the Severi–Brauer variety of  $\widehat{D}$ , the algebra with involution  $\text{Ad}(h)_{\mathcal{F}(\widehat{D})}$  is totally decomposable.*

*Proof.* Since  $(A, \sigma_1)_F \simeq \text{Ad}(\widehat{h}_1)$  and  $(A, \sigma_2)_F \simeq \text{Ad}(\widehat{h}_2)$  are central simple algebras of degree 8 with orthogonal involutions, it is clear that  $\text{Ad}(h)$  is a central simple algebra of degree 16 with orthogonal involution. Condition (i) on  $\sigma_1$  and  $\sigma_2$  implies that  $h_1$  and  $h_2$  are not hyperbolic and are not similar. Therefore, it follows from Proposition 2.3 that  $\text{Ad}(h)$  is not totally decomposable over  $F$  nor over any odd-degree extension of  $F$ . To prove the last statement, note that  $\mathcal{F}(A)$  is a purely transcendental extension of  $\mathcal{F}(D)$ , hence for the Pfister form  $\varphi$  in condition (ii) we may choose a form defined over  $\mathcal{F}(D)$ . Choosing a minimal left ideal in the split algebra  $D_{\mathcal{F}(D)}$ , we set up a Morita equivalence between hermitian forms over  $(D_{\mathcal{F}(D)}, \gamma_{\mathcal{F}(D)})$  and quadratic forms over  $\mathcal{F}(D)$ , hence also between hermitian forms over  $(\widehat{D}_{\mathcal{F}(\widehat{D})}, \widehat{\gamma}_{\mathcal{F}(\widehat{D})})$  and quadratic forms over  $\mathcal{F}(\widehat{D})$ , which yield identifications  $W(D_{\mathcal{F}(D)}, \gamma_{\mathcal{F}(D)}) = W(\mathcal{F}(D))$  and  $W(\widehat{D}_{\mathcal{F}(\widehat{D})}, \widehat{\gamma}_{\mathcal{F}(\widehat{D})}) = W(\mathcal{F}(\widehat{D}))$ . Condition (ii) shows that there are  $\alpha_1, \alpha_2 \in \mathcal{F}(D)^\times$  such that

$$(h_1)_{\mathcal{F}(D)} = \langle \alpha_1 \rangle \varphi \quad \text{and} \quad (h_2)_{\mathcal{F}(D)} = \langle \alpha_2 \rangle \varphi \quad \text{in } W(\mathcal{F}(D)),$$

hence

$$h_{\mathcal{F}(\widehat{D})} = \langle \alpha_1, \alpha_2 t \rangle \varphi_{\mathcal{F}(\widehat{D})} \quad \text{in } W(\mathcal{F}(\widehat{D})).$$

Since the right side is a multiple of a Pfister form, it follows that  $\text{Ad}(h)_{\mathcal{F}(\widehat{D})}$  is totally decomposable.  $\square$

*Remarks 2.5.* (1) The algebra  $\text{Ad}(h)$  in Theorem 2.4 has index 4 or 8. By contrast, Becher's theorem [2, Th. 2] shows that if a central simple algebra with orthogonal involution of index at most 2 is totally decomposable after generic splitting, then it is totally decomposable (see Proposition 1.2). Therefore, the construction used in Theorem 2.4 is impossible if the index of  $A$  is 2, which means that there are no orthogonal involutions  $\sigma_1, \sigma_2$  satisfying conditions (i) and (ii) if the index of  $A$  is 2. In §3, we provide a direct proof of this fact (see Proposition 3.5).

(2) Let  $L$  be an arbitrary field extension of the center  $F$  of the algebra  $\text{Ad}(h)$  of Theorem 2.4. If  $\text{Ad}(h)_L$  is isotropic, then it is also isotropic after generic splitting of  $\widehat{D}_L$ . But since  $\text{Ad}(h)$  is totally decomposable over  $\mathcal{F}(\widehat{D})$  it is also totally decomposable over  $\mathcal{F}(\widehat{D}_L)$ , hence  $\text{Ad}(h)_{\mathcal{F}(\widehat{D}_L)}$  is hyperbolic, and it follows from [10, Theorem 1.1] that  $\text{Ad}(h)_L$  is hyperbolic. Thus,  $\text{Ad}(h)$  satisfies the 'Isotropy  $\Rightarrow$  Hyperbolicity' condition introduced in [1], and Theorem 2.4 shows that this condition is necessary but not sufficient for an algebra with involution to be totally decomposable.

### 3. GENERIC SPLITTING IN INDEX 2

Throughout this section,  $Q$  denotes a quaternion algebra over a field  $k$  (of characteristic different from 2). We address Question 2, considering only algebras with orthogonal involution that are split and hyperbolic over a quadratic extension of  $k$  (and are therefore of index at most 2). They can be described explicitly as follows:

**Lemma 3.1.** *Let  $\sigma$  be an orthogonal involution on  $A = M_r(Q)$ . If there exists a quadratic étale extension  $Z/k$  over which  $(A, \sigma)$  is split and hyperbolic, then  $Q$  admits an orthogonal involution  $\rho$  of discriminant  $Z$ , and  $(A, \sigma)$  decomposes as*

$$(A, \sigma) \simeq (Q, \rho) \otimes \text{Ad}(\varphi)$$

for some  $r$ -dimensional quadratic form  $\varphi$  over  $k$ .

*Proof.* If  $Z = k \times k$ , then  $Q$  is split and  $\sigma$  is hyperbolic. Therefore the lemma holds with  $(Q, \rho) = \text{Ad}(\langle 1, -1 \rangle)$  and  $\varphi$  arbitrary. Assume now that  $Z = k(\sqrt{a})$  is a quadratic field extension of  $k$ . If  $(A, \sigma)$  is split and hyperbolic over  $Z$ , then  $A$  contains a skew-symmetric element  $g$  such that  $g^2 = a$  and the centralizer of  $g$  in  $A$  is split. Therefore, by [18, Appendix], there exists a pure quaternion  $i \in Q$  such that  $i^2 = a$ , and some elements  $\alpha_i \in k^\times$  for  $i = 1, \dots, r$ , such that  $\sigma$  is adjoint to the skew-hermitian form over  $(Q, -)$  defined by  $h = \langle i\alpha_1, \dots, i\alpha_r \rangle$ , where  $-$  is the canonical involution of  $Q$ . The lemma follows, with  $\rho = \text{Int}(i) \circ -$  and  $\varphi = \langle \alpha_1, \dots, \alpha_r \rangle$ .  $\square$

*Remark 3.2.* For  $A = M_r(Q)$ , the field  $\mathcal{F}(A)$  is a purely transcendental extension of  $\mathcal{F}(Q)$ , hence for involutions  $\sigma$  and  $\sigma'$  on  $A$  we have  $\sigma_{\mathcal{F}(A)} \simeq \sigma'_{\mathcal{F}(A)}$  if and only if  $\sigma_{\mathcal{F}(Q)} \simeq \sigma'_{\mathcal{F}(Q)}$  by [19, Lemma 4.1]. Therefore, to establish Theorem 1.6(f) and (g) we may (and will) substitute  $\mathcal{F}(Q)$  for  $\mathcal{F}(A)$ .

**Lemma 3.3.** *Let  $(A, \sigma) = (Q, \rho) \otimes \text{Ad}(\varphi)$  be as in the previous lemma. Any orthogonal involution  $\sigma'$  on  $A = M_r(Q)$  such that  $\sigma'_{\mathcal{F}(Q)} \simeq \sigma_{\mathcal{F}(Q)}$  also decomposes as  $\sigma' \simeq \rho \otimes \text{ad}(\varphi')$  for some  $r$ -dimensional quadratic form  $\varphi'$  over  $k$ .*

*Proof.* Let  $Z$  be the quadratic extension of  $k$  defined by the discriminant of  $\rho$ , and denote by  $Z_Q = Z \cdot \mathcal{F}(Q)$  the corresponding quadratic extension of  $\mathcal{F}(Q)$ . Since  $Q_Z$  is split,  $Z_Q$  is a purely transcendental extension of  $Z$ . Therefore,  $\sigma_{\mathcal{F}(Q)} \simeq \sigma'_{\mathcal{F}(Q)}$  implies  $\sigma_{Z_Q} \simeq \sigma'_{Z_Q}$ , which in turn implies  $\sigma_Z \simeq \sigma'_Z$  by [19, Lemma 4.1]. So  $(A, \sigma')$  also is split and hyperbolic over  $Z$  and the previous lemma finishes the proof.  $\square$

With this in hand, Question 2 for involutions that become hyperbolic over a quadratic splitting field of  $Q$  boils down to a quadratic form question over  $\mathcal{F}(Q)$ , as the next proposition shows:

**Proposition 3.4.** *Let  $\rho$  be an orthogonal involution of  $Q$  of discriminant  $k(\sqrt{a})$ . Given two  $r$ -dimensional quadratic forms  $\varphi$  and  $\varphi'$  over  $k$ , consider the orthogonal involutions  $\sigma = \rho \otimes \text{ad}(\varphi)$  and  $\sigma' = \rho \otimes \text{ad}(\varphi')$  of  $A = M_r(Q)$ .*

(i)  $\sigma'_{\mathcal{F}(Q)} \simeq \sigma_{\mathcal{F}(Q)}$  if and only if there exists  $\lambda \in \mathcal{F}(Q)^\times$  such that

$$\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle \langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}.$$

(ii)  $\sigma' \simeq \sigma$  if and only if there exists  $\nu \in k^\times$  such that

$$\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)} \simeq \langle \nu \rangle \langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}.$$

*Proof.* Since orthogonal involutions on a quaternion algebra are classified by their discriminant [12, (7.4)], and  $k$  is quadratically closed in  $\mathcal{F}(Q)$ , there exists an isomorphism

$$(Q, \rho)_{\mathcal{F}(Q)} \simeq \text{Ad}(\langle\langle a \rangle\rangle_{\mathcal{F}(Q)}).$$

Hence, for every quadratic form  $\psi$  over  $k$ , we have

$$((Q, \rho) \otimes \text{ad}(\psi))_{\mathcal{F}(Q)} \simeq \text{Ad}(\langle\langle a \rangle\rangle \psi_{\mathcal{F}(Q)}).$$

In particular,  $\sigma_{\mathcal{F}(Q)}$  and  $\sigma'_{\mathcal{F}(Q)}$  are respectively adjoint to the quadratic forms  $\langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}$  and  $\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)}$ . Assertion (i) follows immediately.

To prove assertion (ii) we may assume  $k(\sqrt{a})$  is a field, for otherwise  $\rho$  is hyperbolic and (ii) trivially holds. Let  $i \in Q$  be a quaternion such that  $\rho(i) = -i \neq 0$ . Then  $i$  is pure,  $i^2 \equiv a \pmod{k^{\times 2}}$ , and  $\sigma$  (resp.  $\sigma'$ ) is adjoint to the skew-hermitian

form  $\langle i \rangle \varphi$  (resp.  $\langle i \rangle \varphi'$ ) over  $(Q, -)$ . Therefore,  $\sigma$  and  $\sigma'$  are isomorphic if and only if there exists  $\nu \in k^\times$  such that

$$(4) \quad \langle i \rangle \varphi' \simeq \langle \nu i \rangle \varphi.$$

Since the scalar extension map  $W^-(Q, -) \rightarrow W^-(Q_{\mathcal{F}(Q)}, -)$  is injective (see [5] or [15, Prop. 3.3]), and since  $W^-(Q_{\mathcal{F}(Q)}, -) \simeq W(\mathcal{F}(Q))$  by a Morita-equivalence that carries  $\langle i \rangle_{\mathcal{F}(Q)}$  to  $\langle\langle a \rangle\rangle_{\mathcal{F}(Q)}$ , the existence of the isomorphism (4) is equivalent to

$$\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)} \simeq \langle \nu \rangle \langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}. \quad \square$$

We use Proposition 3.4 to prove (c) and (d) in Theorem 1.5.

**Proposition 3.5.** *Let  $A = M_r(Q)$  be endowed with two orthogonal involutions  $\sigma$  and  $\sigma'$ . We assume that either  $(A, \sigma)$  is totally decomposable, or  $r$  is odd and there exists a quadratic extension  $Z/k$  over which  $(A, \sigma)$  is split and hyperbolic. Under any of those two conditions, if  $\sigma_{\mathcal{F}(Q)}$  and  $\sigma'_{\mathcal{F}(Q)}$  are isomorphic, then  $\sigma \simeq \sigma'$ .*

*Proof.* By [2, Th. 2], if  $(A, \sigma)$  is totally decomposable, it admits a decomposition

$$(A, \sigma) \simeq (Q, \rho) \otimes \text{ad}(\pi),$$

for some orthogonal involution  $\rho$  of  $Q$  and some Pfister quadratic form  $\pi$  over  $k$ . Therefore in both cases there exists a quadratic extension  $Z/k$  over which  $(A, \sigma)$  is split and hyperbolic:  $Z$  is the discriminant of  $\rho$  in the totally decomposable case. In view of Lemmas 3.1 and 3.3, we may thus assume that  $\sigma = \rho \otimes \text{ad}(\varphi)$  and  $\sigma' = \rho \otimes \text{ad}(\varphi')$  for some quadratic forms  $\varphi$  and  $\varphi'$  over  $k$ , and apply Proposition 3.4.

Assume first that  $\sigma$  is totally decomposable. Then we may assume  $\varphi = \pi$  is a Pfister form, and modifying it by a scalar if necessary, we may also assume  $\varphi'$  represents 1. If  $\sigma_{\mathcal{F}(Q)}$  and  $\sigma'_{\mathcal{F}(Q)}$  are isomorphic, assertion (i) in Proposition 3.4 says that

$$\langle\langle a \rangle\rangle \pi_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle \langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)}$$

for some  $\lambda \in \mathcal{F}(Q)^\times$ . But a quadratic form that is similar to a Pfister form and represents 1 actually is isomorphic to this Pfister form. Therefore, we may take  $\lambda = 1$ , so that the equivalent conditions of assertion (ii) hold, with  $\nu = 1$ . This concludes the proof in this case.

Assume now that  $r$  is odd. Since the quadratic forms  $\varphi$  and  $\varphi'$  are  $r$ -dimensional, and only defined up to a scalar factor, we may assume they have trivial discriminant. If  $\sigma_{\mathcal{F}(Q)}$  and  $\sigma'_{\mathcal{F}(Q)}$  are isomorphic, assertion (i) says that there exists  $\lambda \in \mathcal{F}(Q)^\times$  such that

$$\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle \langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}.$$

Computing the Clifford invariant of both forms as in [13, V.3], and taking into account the fact that  $r$  is odd and  $\varphi$  and  $\varphi'$  have trivial discriminant, we get  $(\lambda, a) = 0 \in \text{Br}_2(k)$ . Therefore,  $\lambda$  is represented by  $\langle\langle a \rangle\rangle$ , so that

$$\langle \lambda \rangle \langle\langle a \rangle\rangle \simeq \langle\langle a \rangle\rangle.$$

Hence, again we may take  $\lambda = 1$ , and the equivalent conditions of assertion (ii) hold.  $\square$

In the rest of this paper, we use Proposition 3.4 to produce examples of pairs of orthogonal involutions for which the answer to Question 2 is negative. For this, we will exhibit quadratic forms  $\varphi$  and  $\varphi'$  defined over  $k$  such that

$$\langle\langle a \rangle\rangle \varphi'_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle \langle\langle a \rangle\rangle \varphi_{\mathcal{F}(Q)}$$

for some  $\lambda \in \mathcal{F}(Q)^\times$ , but not for any  $\lambda \in k^\times$ . The heuristic idea behind this construction is the following. Suppose  $\varphi_1, \varphi'_1, \varphi_2, \varphi'_2$  are quadratic forms over  $k$  satisfying assertion (i) of Proposition 3.4 with the same factor  $\lambda \in \mathcal{F}(Q)^\times$ :

$$(\langle\langle a \rangle\rangle \varphi'_1)_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle (\langle\langle a \rangle\rangle \varphi_1)_{\mathcal{F}(Q)} \quad \text{and} \quad (\langle\langle a \rangle\rangle \varphi'_2)_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle (\langle\langle a \rangle\rangle \varphi_2)_{\mathcal{F}(Q)}.$$

Then for every  $t \in k^\times$  we also have

$$(\langle\langle a \rangle\rangle \varphi'_1 \perp \langle t \rangle \langle\langle a \rangle\rangle \varphi'_2)_{\mathcal{F}(Q)} \simeq \langle \lambda \rangle (\langle\langle a \rangle\rangle \varphi_1 \perp \langle t \rangle \langle\langle a \rangle\rangle \varphi_2)_{\mathcal{F}(Q)}.$$

If  $\varphi_1, \varphi'_1$  and  $\varphi_2, \varphi'_2$  do not yield a negative answer to Question 2, then we may find  $\nu_1, \nu_2 \in k^\times$  such that

$$(\langle\langle a \rangle\rangle \varphi'_1)_{\mathcal{F}(Q)} \simeq (\langle \nu_1 \rangle \langle\langle a \rangle\rangle \varphi_1)_{\mathcal{F}(Q)} \quad \text{and} \quad (\langle\langle a \rangle\rangle \varphi'_2)_{\mathcal{F}(Q)} \simeq (\langle \nu_2 \rangle \langle\langle a \rangle\rangle \varphi_2)_{\mathcal{F}(Q)}.$$

But if the  $\nu_1$  and  $\nu_2$  satisfying these equations are sufficiently distinct, and if  $t$  is 'generic', there may not be any  $\nu \in k^\times$  such that

$$(\langle\langle a \rangle\rangle \varphi'_1 \perp \langle t \rangle \langle\langle a \rangle\rangle \varphi'_2)_{\mathcal{F}(Q)} \simeq \langle \nu \rangle (\langle\langle a \rangle\rangle \varphi_1 \perp \langle t \rangle \langle\langle a \rangle\rangle \varphi_2)_{\mathcal{F}(Q)}.$$

*Example 3.6.* Let  $k_0$  be a field of characteristic different from 2 and  $k = k_0(\langle\langle a \rangle\rangle)(\langle\langle t \rangle\rangle)$  the iterated Laurent series field in two variables over  $k_0$ . Pick  $b \in k_0^\times$  and let  $Q = (a, b)_k$ . Assume  $b$  is not a square in  $k_0$ ; then  $Q$  is a division algebra, which admits an orthogonal involution  $\rho$  of discriminant  $a$ . The function field  $\mathcal{F}(Q)$  may be represented as  $k(X, Y)$  where  $X$  and  $Y$  satisfy

$$(5) \quad X^2 - aY^2 + ab = 0.$$

Fix an element  $c \in k_0^\times$  and let

$$\varphi = \langle\langle b + 1 \rangle\rangle \perp \langle t \rangle \langle\langle b + c^2 \rangle\rangle, \quad \text{and} \quad \varphi' = \langle\langle b + 1 \rangle\rangle \perp \langle ct \rangle \langle\langle b + c^2 \rangle\rangle.$$

In the following proposition, we use the notation  $D_K(\theta)$  for the set of represented values of a quadratic form  $\theta$  over a field  $K$ .

**Proposition 3.7.** *The involutions  $\sigma = \rho \otimes \text{ad}(\varphi)$  and  $\sigma' = \rho \otimes \text{ad}(\varphi')$  satisfy  $\sigma_{\mathcal{F}(Q)} \simeq \sigma'_{\mathcal{F}(Q)}$ . If  $\sigma \simeq \sigma'$ , then*

$$(6) \quad c \in (k_0^\times \cap D_{k_0(\sqrt{b})}(\langle\langle b + 1 \rangle\rangle)) \cdot (k_0^\times \cap D_{k_0(\sqrt{b})}(\langle\langle b + c^2 \rangle\rangle)).$$

Note that, in contrast to (6), we always have

$$(\sqrt{b} + c)^2 - (b + c^2) = 2c\sqrt{b} \quad \text{and} \quad \left(\frac{\sqrt{b} + 1}{2\sqrt{b}}\right)^2 - (b + 1) \left(\frac{1}{2\sqrt{b}}\right)^2 = \frac{1}{2\sqrt{b}},$$

hence  $(2\sqrt{b})^{-1} \in D_{k_0(\sqrt{b})}(\langle\langle b + 1 \rangle\rangle)$  and  $2c\sqrt{b} \in D_{k_0(\sqrt{b})}(\langle\langle b + c^2 \rangle\rangle)$ , and therefore

$$(7) \quad c = (2\sqrt{b})^{-1}(2c\sqrt{b}) \in k_0^\times \cap [D_{k_0(\sqrt{b})}(\langle\langle b + 1 \rangle\rangle) \cdot D_{k_0(\sqrt{b})}(\langle\langle b + c^2 \rangle\rangle)].$$

*Proof of Proposition 3.7.* To establish the first claim, observe that  $\langle\langle a, b + 1 \rangle\rangle_{\mathcal{F}(Q)}$  represents

$$X^2 - a(Y + 1)^2 + a(b + 1) = -2aY,$$

while  $\langle\langle a, b + c^2 \rangle\rangle_{\mathcal{F}(Q)}$  represents

$$X^2 - a(Y + c)^2 + a(b + c^2) = -2aYc.$$

Therefore  $-2aY$  (respectively  $-2aYc$ ) is a similarity factor of  $\langle\langle a, b + 1 \rangle\rangle_{\mathcal{F}(Q)}$  (respectively  $\langle\langle a, b + c^2 \rangle\rangle_{\mathcal{F}(Q)}$ ), and we get

$$\langle\langle a, b + 1 \rangle\rangle \simeq \langle -2aY \rangle \langle\langle a, b + 1 \rangle\rangle \quad \text{and} \quad \langle c \rangle \langle\langle a, b + c^2 \rangle\rangle \simeq \langle -2aY \rangle \langle\langle a, b + c^2 \rangle\rangle,$$

hence

$$\langle\langle a \rangle\rangle\varphi'_{\mathcal{F}(Q)} \simeq \langle -2aY \rangle\langle\langle a \rangle\rangle\varphi_{\mathcal{F}(Q)}.$$

By Proposition 3.4, this implies  $\sigma_{\mathcal{F}(Q)} \simeq \sigma'_{\mathcal{F}(Q)}$ .

To prove the second claim, assume  $\sigma \simeq \sigma'$ . Proposition 3.4(ii) yields  $\nu \in k^\times$  such that the forms  $\langle\langle a \rangle\rangle\varphi'_{\mathcal{F}(Q)}$  and  $\langle\nu\rangle\langle\langle a \rangle\rangle\varphi_{\mathcal{F}(Q)}$  are isomorphic. Since the kernel of the restriction map  $W(k) \rightarrow W(\mathcal{F}(Q))$  is  $\langle\langle a, b \rangle\rangle W(k)$  (see [13, Ch. X, Cor. 4.28]), it follows that

$$(8) \quad \langle\langle \nu, a, b+1 \rangle\rangle + \langle ct, -\nu t \rangle\langle\langle a, b+c^2 \rangle\rangle \in \langle\langle a, b \rangle\rangle W(k).$$

Since  $k$  is the field of iterated Laurent series in  $a$  and  $t$  over  $k_0$ , every square class in  $k$  is represented by an element of the form  $\nu_0, a\nu_0, t\nu_0$  or  $at\nu_0$  for some  $\nu_0 \in k_0^\times$ ; see [13, Ch. VI, Cor. 1.3]. If  $\nu_1 = -a\nu$ , then  $\langle\langle \nu_1, a \rangle\rangle \simeq \langle\langle \nu, a \rangle\rangle$  and  $\langle -\nu_1 t \rangle\langle\langle a \rangle\rangle \simeq \langle -\nu t \rangle\langle\langle a \rangle\rangle$ . Therefore, substituting  $-a\nu$  for  $\nu$  if  $\nu = a\nu_0$  or  $at\nu_0$ , we may assume  $\nu = \nu_0$  or  $t\nu_0$  with  $\nu_0 \in k_0^\times$ . We consider these two cases separately.

**Case 1:** Suppose  $\nu = t\nu_0$  with  $\nu_0 \in k_0^\times$ . Taking the first residue of (8) for the  $t$ -adic valuation, and the first residue of the resulting relation for the  $a$ -adic valuation, we obtain

$$\langle\langle b+1 \rangle\rangle + \langle -\nu_1 \rangle\langle\langle b+c^2 \rangle\rangle \in \langle\langle b \rangle\rangle W(k_0).$$

Comparing discriminants yields  $(b+1)(b+c^2) \in k_0^{\times 2} \cup (bk_0^{\times 2})$ , hence  $b+1 \equiv b+c^2 \pmod{k_0(\sqrt{b})^{\times 2}}$ . Then (7) yields (6).

**Case 2:** Suppose  $\nu \in k_0^\times$ . Taking the residues of (8) for the  $t$ -adic valuation, we obtain

$$\langle\langle \nu, a, b+1 \rangle\rangle \in \langle\langle a, b \rangle\rangle W(k_0((a))) \quad \text{and} \quad \langle c, -\nu \rangle\langle\langle a, b+c^2 \rangle\rangle \in \langle\langle a, b \rangle\rangle W(k_0((a))).$$

We next take the first residue for the  $a$ -adic valuation, and get

$$\langle\langle \nu, b+1 \rangle\rangle \in \langle\langle b \rangle\rangle W(k_0) \quad \text{and} \quad \langle c, -\nu \rangle\langle\langle b+c^2 \rangle\rangle \in \langle\langle b \rangle\rangle W(k_0).$$

It follows that the forms  $\langle\langle \nu, b+1 \rangle\rangle$  and  $\langle\langle \nu c, b+c^2 \rangle\rangle$  become hyperbolic over  $k_0(\sqrt{b})$ . This means that  $\nu$  is represented by the form  $\langle\langle b+1 \rangle\rangle$  over  $k_0(\sqrt{b})$ , and  $\nu c$  by the form  $\langle\langle b+c^2 \rangle\rangle$  over  $k_0(\sqrt{b})$ , so the equation  $c = \nu^{-1}(\nu c)$  yields (6).  $\square$

To conclude, it remains to find fields  $k_0$  and elements  $b, c \in k_0^\times$  such that (6) does not hold. Quadratic extensions for which the ‘Common Value Property’ investigated in [22] (see also [7, §6]) fails yield such examples:

- we may take  $k_0 = \mathbb{Q}(b)$ , where  $b$  is an indeterminate, and  $c = 2$ : see [22, Rem. 5.4];
- we may take  $k_0 = \ell(b, c)$ , where  $b, c$  are independent indeterminates over an arbitrary field  $\ell$  of characteristic 0: see [22, Rem. 5.10].

*Example 3.8.* It is easy to modify Example 3.6 to obtain nonisomorphic orthogonal involutions on  $M_6(Q)$  with trivial discriminant and trivial  $e_2$ -invariant that are isomorphic after generic splitting. From (5), derive

$$X^2 - a(Y+1)^2 = -2aY - a(b+1)$$

and

$$-(b+1)(b+c^2) \left( \left( \frac{X}{b+c^2} \right)^2 - a \left( \frac{Y+c}{b+c^2} \right)^2 \right) = -(b+1) \left( \frac{-2acY}{b+c^2} - a \right).$$



Summing these two equations, we see that  $\langle\langle a, (b+1)(b+c^2) \rangle\rangle$  represents

$$-2aY - a(b+1) + (b+1) \left( \frac{2acY}{b+c^2} + a \right) = -2aYc', \quad \text{with } c' = 1 - \frac{(b+1)c}{b+c^2}.$$

If  $c' = 0$ , then  $b+c^2 = (b+1)c$  and  $\langle\langle a, (b+1)(b+c^2) \rangle\rangle = \langle\langle a, c \rangle\rangle$  is hyperbolic. Since  $a$  is an indeterminate over  $k_0$ , it follows that  $c \in k_0^{\times 2}$ , hence (6) trivially holds. This is a contradiction; therefore  $c' \neq 0$  and

$$\langle -2aYc' \rangle \langle\langle a, (b+1)(b+c^2) \rangle\rangle \simeq \langle\langle a, (b+1)(b+c^2) \rangle\rangle.$$

Let  $k_1 = k((u))$ , where  $u$  is a new indeterminate, and  $Q_1 = Q \otimes_k k_1$ . Consider the following quadratic forms over  $k_1$ :

$$\begin{aligned} \psi &= \langle\langle b+1 \rangle\rangle \perp \langle t \rangle \langle\langle b+c^2 \rangle\rangle \perp \langle u \rangle \langle\langle (b+1)(b+c^2) \rangle\rangle, \\ \psi' &= \langle\langle b+1 \rangle\rangle \perp \langle ct \rangle \langle\langle b+c^2 \rangle\rangle \perp \langle c'u \rangle \langle\langle (b+1)(b+c^2) \rangle\rangle \end{aligned}$$

and the involutions  $\tau = \rho \otimes \text{ad}(\psi)$  and  $\tau' = \rho \otimes \text{ad}(\psi')$  over  $M_6(Q_1)$ . Proposition 3.4(i) shows that they satisfy

$$\tau_{\mathcal{F}(Q_1)} \simeq \tau'_{\mathcal{F}(Q_1)},$$

because

$$\langle\langle a \rangle\rangle \psi'_{\mathcal{F}(Q)} \simeq \langle -2aY \rangle \langle\langle a \rangle\rangle \psi_{\mathcal{F}(Q)}.$$

It remains to prove that  $\tau$  and  $\tau'$  are not isomorphic. Using again Proposition 3.4(ii), we need to prove that there is no  $\nu \in k_1^\times$  such that

$$\begin{aligned} \langle \nu, a, b+1 \rangle + \langle ct, -\nu t \rangle \langle\langle a, b+c^2 \rangle\rangle + \langle c'u, -\nu u \rangle \langle\langle a, (b+1)(b+c^2) \rangle\rangle \\ \in \langle\langle a, b \rangle\rangle W(k((u))). \end{aligned}$$

We may assume  $\nu \in k^\times$  or  $\nu = u\nu_1$  with  $\nu_1 \in k^\times$ . If  $\nu \in k^\times$ , taking the first residue for the  $u$ -adic valuation yields (8), which has no solution. If  $\nu = u\nu_1$  with  $\nu_1 \in k^\times$ , taking the first residue yields

$$(9) \quad \langle\langle a, b+1 \rangle\rangle + \langle ct \rangle \langle\langle a, b+c^2 \rangle\rangle + \langle -\nu_1 \rangle \langle\langle a, (b+1)(b+c^2) \rangle\rangle \in \langle\langle a, b \rangle\rangle W(k).$$

Recall  $k = k_0((a))((t))$ , so every square class in  $k$  is represented by an element of the form  $\nu_0, a\nu_0, t\nu_0$  or  $at\nu_0$  for some  $\nu_0 \in k_0^\times$ . Substituting  $-a\nu_1$  for  $\nu_1$  if  $\nu_1 = a\nu_0$  or  $at\nu_0$ , we may assume  $\nu_1 = \nu_0$  or  $t\nu_0$  with  $\nu_0 \in k_0^\times$ . If  $\nu_1 = t\nu_0$ , then taking the first residue of (9) for the  $t$ -adic valuation and for the  $a$ -adic valuation in the resulting relation, we obtain  $\langle\langle b+1 \rangle\rangle \in \langle\langle b \rangle\rangle W(k_0)$ . This implies  $b+1 \in k_0(\sqrt{b})^{\times 2}$ , hence (6) holds, a contradiction. If  $\nu_1 \in k_0^\times$ , then we take the second residue of (9) for the  $t$ -adic valuation and the first residue for the  $a$ -adic valuation, and find  $\langle c \rangle \langle\langle b+c^2 \rangle\rangle \in \langle\langle b \rangle\rangle W(k_0)$ , hence  $b+c^2 \in k_0(\sqrt{b})^{\times 2}$ , which also yields a contradiction. Therefore  $\tau$  and  $\tau'$  are not isomorphic.

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UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, LAGA - CNRS (UMR 7539), F-93430 VILLETANEUSE, FRANCE

*E-mail address:* `queguin@math.univ-paris13.fr`

ICTEAM INSTITUTE, BOX L4.05.01, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, B-1348 LOUVAIN-LA-NEUVE, BELGIUM

*E-mail address:* `jean-pierre.tignol@uclouvain.be`