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Linear Quadratic Control Problems for Mean Field Stochastic Differential Equation with Jumps: Application in Exhaustible Resources Production

Anis MATOUSSI ^{*†} Mohamed MNIF [‡] Chefia ZIRI ^{§¶}

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Abstract

In this paper, we are interested by a stochastic model of production of an exhaustible resource, such as oil. It is known that such reserves are depleted resources, but there is a possibility of exploration and discovery of new reserves which ensure the accumulating or the upkeep of this reserves' level. We modelled the new discoveries by a jump process with intensity given by the exploration effort. We employed a weak formulation of the standard martingale optimality principle to solve a linear quadratic stochastic control problem for mean field stochastic differential equation with jumps in both cases: finite and infinite horizon.

Keywords : Linear quadratic optimal control, Mean field SDEs with jumps, Mean field BSDEs with jumps, Riccati equation, Exhaustible resources.

Introduction

Stochastic optimal control has been widely studied in recent decades due to its applications to mathematical finance, insurance, economics, engineering, etc. There are two main techniques

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to solve a stochastic optimal control: The dynamic programming, developed by Richard Bellman in the 1950s and the stochastic maximum principle formulated by Pontryagin and his group in the 1950s also. The Bellman optimality principle needs that the cost function is the expected value of a functional of the state process, which leads to a so-called time consistent control problem. It is valid only for Markovian systems, where the coefficients are deterministic functions of time and the current value of the state process. In contrast, the Pontryagin principle is well suited for state process and cost functional of mean-field type. It could tackle control problem time inconsistent in the sense that the Bellman optimality principle does not hold. It is a powerful tool to study Non-Markovian systems. i.e. where the coefficients are random. The key idea of the Pontryagin principle is to perturbate the control process to derive a set of necessary condition that must be satisfied by any optimal control.

In recent years, stochastic optimal control problems for mean field stochastic differential equations (in short SDEs) have attracted more attention. The history of mean field type SDEs, also known as McKean-Vlasov systems, can be traced back to the works by Kac in 1956 and McKean in 1966 on stochastic systems with a large number of interacting particles. Thence, many authors have made contributions on optimal control problems for mean field SDEs, see for example the works of Anderson and Djehiche [2] and Carmona and Delarue [5] for stochastic maximum principle approach of optimal control under some convexity assumptions. We can also refer to Alasseur et al. [1] where they develop a model for the optimal management of energy storage and distribution in a smart grid system through an extended MFG. For the stochastic dynamic programming approach, we can refer to Lauriere and Pironneau [15] under some density assumption on the probability law of the state process. In this work, we are interested in a subclass of the mean field control problem in which the cost function is quadratic in all state variables, control variables and mean field terms, while the controlled dynamics is linear and also consists of mean field terms. In 2013, Yong [22] studied a linear quadratic mean field control problem for mean field SDE with deterministic coefficients. The optimality of this system is driven by a variational method. He used a decoupled techniques to obtain two Riccati equation which are uniquely solvable under some conditions. Then he obtained a feedback representation for the optimal control. Since them, Huang et al. in [13] extended the work of Yong [22] for a linear quadratic optimal control problem in an infinite horizon. Other recent works for mean field control have been recently published by Pham and Wei [18, 19], where they developed a dynamic programming technique, with corresponding Hamilton-Jacobi equations on an infinite dimensional space of probability measures, for solving mean field type optimal control problems, for the two cases without and with common noise and by Pham [17] where he consider a linear quadratic McKean Vlasov problem with random coefficients and he extended the approach developed in [18] for the common noise.

In this paper, we consider a linear quadratic optimal control for mean field stochastic differential equation with jumps. We introduce a weak formulation of the standard martingale optimality principle to solve the linear quadratic mean field stochastic control problem. This approach is used in the verification theorem for stochastic control problem which is a crucial step in the classical approach to dynamic programming (see e.g [9, 14]). We prove the exist-

tence of an unique optimal control on finite horizon. The optimality of the control is derived by the stability of a decoupled system of Riccati equations and backward stochastic differential equation with jumps. We also consider the problem with common noise adding up some assumptions on the coefficients. On the other hand, we introduce also the linear quadratic problem on the infinite case. This paper can be regarded as an extension of the work of Basei and Pham [4] to the case with jumps.

To concretize our results, we are motivated by a model of the production of an exhaustible resource with accumulating or maintaining a level of reserves, such as oil. This model is driven by the use of existing reserves and exploration or discovery of new reserves to produce energy. In other words, the level of the resource stock increases through new discoveries of further deposits of the resource, but the resource is still exhaustible. The typical model of the production of an exhaustible resource by a continuum of producers was developed by Guéant et al. in [10] who were inspired from Hotelling's work [12], and it was adapted later by Chan and Sircar in [6, 7]. In 1987, Pindyck [20] extended Hotelling's model for a model of production of exhaustible resources with new discovery, where these new discoveries occur according to a jumps process with intensity given by the exploration effort. We refer also to Deshmukh et al. [8], Arrow and Chang [3], and Keller et al. [11]. This model was adapted by Ludkovski and Sircar in [16] to a stochastic game between an exhaustible producer and a Green producer that has access to an inexhaustible but relatively expensive source, such as solar power.

The paper is organized as follows. In Section 1, we present finite-horizon LQMF problem and we introduce the precise assumptions on the coefficients of our problem. In Section 2, we give a detailed description of the techniques used to show the existence of the unique optimal control, we then use the weak formulation of verification theorem introduced in previous section. In Section 3, we extend our results to the case where a common noise is present. We adapt also our results to the infinite-horizon case, in Section 4. Here, we kept the similar steps as the finite horizon case to apply the verification theorem, except that we should look for the stability of decoupled system of Riccati equation on infinite horizon. Finally, in Section 5, we introduce an application of production of exhaustible resource with accumulating or maintaining a level of reserves where the discovering new resources are modeled by a Poisson process.

1 Preliminary and problem formulation

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a stochastic basis such that \mathcal{F}_0 contains all the P -null sets of \mathcal{F} , $\mathcal{F}_{t+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, and we suppose that the filtration is generated by the two following mutually independent processes :

- (i) a standard real Brownian motion $B := (B_t)_{0 \leq t \leq T}$ and
- (ii) a Poisson random measure π on $\mathbb{R}^+ \times \chi$, where $\chi \subset \mathbb{R} \setminus \{0\}$ is equipped with its Borel field $\mathcal{B}(\chi)$. Throughout this paper the measure $\lambda(\cdot)$ is assumed to be finite on

$(\chi, \mathcal{B}(\chi))$ i.e. $\int_{\chi} \lambda(de) < \infty$. Let $\eta(dt, de) = \lambda(de)dt$ be its compensated process, i.e., $\{\tilde{\pi}([0, t] \times A) = (\pi - \eta)([0, t] \times A)\}_{t \leq T}$ is a martingale for every $A \in \mathcal{B}(\chi)$.

Let $\rho \geq 0$ be the discount factor and define \mathcal{A} the set of admissible controls as follows:

$$\mathcal{A} := \{\alpha : \Omega \times [0, T] \rightarrow \mathbb{R}^m \text{ s.t. } \alpha \text{ is } \mathbb{F}\text{-predictable and } \int_0^T e^{-\rho t} \mathbb{E}[|\alpha_t|^2] dt < \infty\}.$$

Let \mathbb{S}^d be the set of symmetric matrices and $(\mathbb{H}, |\cdot|)$ a normed space. We define the following sets:

- $L^\infty([0, T], \mathbb{H}) := \left\{ \phi : [0, T] \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is measurable and } \sup_{0 \leq s \leq T} |\phi_s| < \infty \right\},$
- $\mathcal{K}^\infty([0, T] \times \chi, \mathbb{H}) := \left\{ \phi : [0, T] \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is measurable and } \sup_{0 \leq s \leq T, e \in \chi} |\phi_s(e)| < \infty \right\},$
- $L^2([0, T], \mathbb{H}) := \{ \phi : [0, T] \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is measurable and } \mathbb{E} \left[\int_0^T e^{-\rho s} |\phi_s|^2 ds \right] < \infty \},$
- $L^2_{\mathcal{F}_T}(\mathbb{H}) := \{ \phi : \Omega \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{E}[|\phi_s|^2] < \infty \},$
- $\mathcal{S}_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{H}) := \{ \phi : \Omega \times [0, T] \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is } \mathbb{F}\text{-progressively measurable and } \mathbb{E}[\text{ess sup}_{s \in [0, T]} |\phi_s|^2] < \infty \},$
- $L^2_{\mathbb{F}}(\Omega \times [0, T], \mathbb{H}) := \{ \phi : \Omega \times [0, T] \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is } \mathbb{F}\text{-progressively measurable and } \int_0^T e^{-\rho s} \mathbb{E}[|\phi_s|^2] ds < \infty \},$
- $\mathcal{K}_{\mathbb{F}}^2(\Omega \times [0, T] \times \chi, \mathbb{H}) := \{ K : \Omega \times [0, T] \times \chi \rightarrow \mathbb{H} \text{ s.t. } K \text{ is } \mathcal{P} \otimes \mathcal{B}(\chi)\text{-measurable process and } \mathbb{E} \left[\int_0^T \int_{\chi} |K_s(e)|^2 \lambda(de) ds \right] < \infty \},$

where \mathcal{P} denote the σ -field of \mathbb{F} -predictable sets on $\Omega \times [0, T]$.

We define the controlled linear mean field stochastic differential equation in \mathbb{R}^d , for a given \mathcal{F} -measurable random variable X_0 independent of W and π , and for a control $\alpha \in \mathcal{A}$, by:

$$\begin{cases} dX_t^\alpha &= b_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t])dt + \sigma_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t])dW_t \\ &+ \int_{\mathcal{X}} R_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t], e) \tilde{\pi}(de, dt), \\ X_0^\alpha &= X_0, \end{cases} \quad (1.1)$$

where for each $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$, $a, \bar{a} \in \mathbb{R}^m$ and $e \in \chi$, we set:

$$\begin{aligned} b_t(x, \bar{x}, a, \bar{a}) &:= \beta_t + A_t x + \tilde{A}_t \bar{x} + B_t a + \tilde{B}_t \bar{a}, \\ \sigma_t(x, \bar{x}, a, \bar{a}) &:= \gamma_t + C_t x + \tilde{C}_t \bar{x} + D_t a + \tilde{D}_t \bar{a}, \\ R_t(x, \bar{x}, a, \bar{a}, e) &:= R_t(x, \bar{x}, a, \bar{a})(e) \\ &:= \delta_t(e) + F_t(e)x + \tilde{F}_t(e)\bar{x} + G_t(e)a + \tilde{G}_t(e)\bar{a}. \end{aligned} \quad (1.2)$$

Here β, γ are vector-valued \mathbb{F} -progressively processes, δ is vector-valued \mathbb{F} -predictable process, and $A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D}, F, \tilde{F}, G, \tilde{G}$ are deterministic matrix-valued functions such that $A, \tilde{A}, C, \tilde{C} : [0, T] \rightarrow \mathbb{R}^{d \times d}$, $B, \tilde{B}, D, \tilde{D} : [0, T] \rightarrow \mathbb{R}^{d \times m}$. In the other hand $F, \tilde{F} : [0, T] \times \Omega \times \chi \rightarrow \mathbb{R}^{d \times d}$ and $G, \tilde{G} : [0, T] \times \chi \rightarrow \mathbb{R}^{m \times d}$ are $\mathcal{P} \otimes \mathcal{B}(\chi)$ -measurable process .

We consider a quadratic cost functional to be minimized over $\alpha \in \mathcal{A}$ of the following type:

$$J(\alpha) := \mathbb{E} \left[\int_0^T e^{-\rho t} f_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t]) dt + e^{-\rho T} g(X_T, \mathbb{E}[X_T]) \right], \quad (1.3)$$

where for each $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $a, \bar{a} \in \mathbb{R}^m$

$$\begin{aligned} f_t(x, \bar{x}, a, \bar{a}) := & (x - \bar{x})^\top Q_t (x - \bar{x}) + \bar{x}^\top (Q_t + \tilde{Q}_t) \bar{x} + 2a^\top I_t (x - \bar{x}) \\ & + 2\bar{a}^\top (I_t + \tilde{I}_t) \bar{x} + (a - \bar{a})^\top N_t (a - \bar{a}) + \bar{a}^\top (N_t + \tilde{N}) \bar{a} + 2M_t^\top x + 2H_t^\top a, \end{aligned} \quad (1.4)$$

and

$$g(x, \bar{x}) := (x - \bar{x})^\top P (x - \bar{x}) + \bar{x}^\top (P + \tilde{P}) \bar{x} + 2L^\top x. \quad (1.5)$$

The coefficients M, H, L defined in (1.4) and (1.5) are vector-valued \mathbb{F} -progressively measurable processes such that $M : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $H : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ and $L : \Omega \rightarrow \mathbb{R}^d$ and the other coefficients $Q, \tilde{Q}, P, \tilde{P}, N, \tilde{N}, I, \tilde{I}$, are deterministic matrix-valued functions such that $Q, \tilde{Q} : [0, T] \rightarrow \mathbb{R}^{d \times d}$, $P, \tilde{P} \in \mathbb{R}^{d \times d}$, $N, \tilde{N} : [0, T] \rightarrow \mathbb{R}^{m \times m}$ and $I, \tilde{I} : [0, T] \rightarrow \mathbb{R}^{m \times d}$. The symbol $^\top$ denotes the transpose operator of any vector or matrix.

Now, we assume the following conditions on the coefficients of the problem :

(H1) The coefficients in equations (1.2) satisfy:

- (i) $\beta, \gamma \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$ and $\delta \in \mathcal{K}_{\mathbb{F}}^2(\Omega \times [0, T] \times \chi, \mathbb{R}^d)$,
- (ii) $A, \tilde{A}, C, \tilde{C} \in L^\infty([0, T], \mathbb{R}^{d \times d})$, $B, \tilde{B}, D, \tilde{D} \in L^\infty([0, T], \mathbb{R}^{d \times m})$,
 $G(\cdot), \tilde{G}(\cdot) \in \mathcal{K}^\infty([0, T] \times \chi, \mathbb{R}^{d \times m})$, $F(\cdot), \tilde{F}(\cdot) \in \mathcal{K}^\infty([0, T] \times \chi, \mathbb{R}^{d \times d})$.

(H2) The coefficients in equations (1.4)-(1.5) satisfy:

- (i) $Q, \tilde{Q} \in L^\infty([0, T], \mathbb{S}^d)$, $P, \tilde{P} \in \mathbb{S}^d$, $N, \tilde{N} \in L^\infty([0, T], \mathbb{S}^m)$, $I, \tilde{I} \in L^\infty([0, T], \mathbb{R}^{m \times d})$,
- (ii) $M \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$, $H \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^m)$, $L \in L_{\mathcal{F}}^2(\mathbb{R}^d)$,
- (iii) there exists $\xi > 0$ such that, for each $t \in [0, T]$,

$$N_t \geq \xi \mathbb{I}_m, \quad P \geq 0, \quad Q_t - I_t^\top N_t^{-1} I_t \geq 0,$$

- (iv) there exists $\xi > 0$ such that, for each $t \in [0, T]$,

$$N_t + \tilde{N}_t \geq \xi \mathbb{I}_m, \quad P + \tilde{P} \geq 0, \quad (Q_t + \tilde{Q}_t) - (I_t + \tilde{I}_t)^\top (N_t + \tilde{N}_t)^{-1} (I_t + \tilde{I}_t) \geq 0.$$

In order to simplify, we denote X_t for X_t^α , \bar{X}_t for $\mathbb{E}[X_t^\alpha]$, $\bar{\alpha}$ for $\mathbb{E}[\alpha_t]$ and $b_t(x, \bar{x})$, $\sigma_t(x, \bar{x})$, $R_t(x, \bar{x})(e)$ for $b_t(x, \bar{x}, \alpha, \bar{\alpha})$, $\sigma_t(x, \bar{x}, \alpha, \bar{\alpha})$ and $R_t(x, \bar{x}, \alpha, \bar{\alpha})(e)$. The following lemma gives an estimate for the solution of mean field SDE (1.2).

Lemma 1.1. *Let Assumption (H1) be satisfied. Then for any admissible control α , the equation (1.1) has a unique solution $(X_t)_t \in \mathcal{S}_{\mathbb{F}}^2$. Moreover, we have the following estimate:*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t|^2 \right] \leq C(1 + \mathbb{E} [|X_0|^2]), \quad (1.6)$$

where C is a positive constant which depends on α .

Proof. We will use the fixed point argument of an appropriate map l defined from $L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$ into itself such that $(X_t)_t := l((x_t)_t)$ where $(X_t)_t$ is solution of the SDE:

$$dX_t = b_t(x_t, \mathbb{E}[x_t])dt + \sigma_t(x_t, \mathbb{E}[x_t])dW_t + \int_{\chi} R_t(x_t, \mathbb{E}[x_t])(e) \tilde{\pi}(de, dt), \quad (1.7)$$

Given $x^1, x^2 \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$, we define X^1 and X^2 by using equation (1.7). We assume that $X_0^1 = X_0^2$. We define $L = \|A\|_{\infty} \vee \|\tilde{A}\|_{\infty} \vee \|C\|_{\infty} \vee \|\tilde{C}\|_{\infty} \vee \|F\|_{\infty} \vee \|\tilde{F}\|_{\infty}$. Then

$$\begin{aligned} \mathbb{E}[|X_t^1 - X_t^2|^2] &\leq 2\mathbb{E}[|\int_0^t b_s(x_s^1, \mathbb{E}[x_s^1]) - b_s(x_s^2, \mathbb{E}[x_s^2])ds|^2] \\ &\quad + 2\mathbb{E}[|\int_0^t \sigma_s(x_s^1, \mathbb{E}[x_s^1]) - \sigma_s(x_s^2, \mathbb{E}[x_s^2])dB_s|^2] \\ &\quad + 2\mathbb{E}[|\int_0^t R_s(x_{s-}^1, \mathbb{E}[x_{s-}^1])(e) - R_s(x_{s-}^2, \mathbb{E}[x_{s-}^2])(e) \tilde{\pi}(de, ds)|^2]. \end{aligned}$$

From the definition of b_s and using Cauchy Scharwz's inequality, we have:

$$\begin{aligned} |b_s(x_s^1, \mathbb{E}[x_s^1]) - b_s(x_s^2, \mathbb{E}[x_s^2])|^2 &= |A_s(x_s^1 - x_s^2) + \tilde{A}_s(\mathbb{E}[x_s^1] - \mathbb{E}[x_s^2])|^2 \\ &\leq 2L^2(|x_s^1 - x_s^2|^2 + \mathbb{E}[|x_s^1 - x_s^2|^2]). \end{aligned}$$

Using again Cauchy Scharwz's inequality, we have:

$$\mathbb{E}[|\int_0^t b_s(x_s^1, \mathbb{E}[x_s^1]) - b_s(x_s^2, \mathbb{E}[x_s^2])ds|^2] \leq 4L^2T \int_0^t \mathbb{E}[|x_s^1 - x_s^2|^2]ds. \quad (1.8)$$

For the stochastic integral terms, we have:

$$\mathbb{E}[|\int_0^t \sigma_s(x_s^1, \mathbb{E}[x_s^1]) - \sigma_s(x_s^2, \mathbb{E}[x_s^2])dB_s|^2] \leq 4L^2 \int_0^t \mathbb{E}[|x_s^1 - x_s^2|^2]ds \quad (1.9)$$

$$\begin{aligned} &\mathbb{E}[|\int_0^t R_s(x_{s-}^1, \mathbb{E}[x_{s-}^1])(e) - R_s(x_{s-}^2, \mathbb{E}[x_{s-}^2])(e) \tilde{\pi}(de, ds)|^2] \\ &\leq 4L^2\lambda(\chi) \int_0^t \mathbb{E}[|x_s^1 - x_s^2|^2]ds. \end{aligned} \quad (1.10)$$

Plugging inequalities (1.8)-(1.10) into (1.7), we obtain:

$$\mathbb{E}[|X_t^1 - X_t^2|^2] \leq 4L^2(1 + T + \lambda(\chi)) \int_0^t \mathbb{E}[|x_s^1 - x_s^2|^2]ds.$$

For a positive constant c , we define the norm $\|x\|_c^2 := \mathbb{E}[\int_0^T e^{-cs} |x_s|^2 ds]$. Then, we have:

$$\begin{aligned}
\|X^1 - X^2\|_c^2 &:= \int_0^T \mathbb{E}[|X_t^1 - X_t^2|^2] e^{-ct} dt \\
&\leq 4L^2(1+T+\lambda(\chi)) \int_0^T \int_0^t \mathbb{E}[|x_s^1 - x_s^2|^2] ds e^{-ct} dt \\
&\leq 4L^2(1+T+\lambda(\chi)) \int_0^T \mathbb{E}[|x_s^1 - x_s^2|^2] \int_s^T e^{-ct} dt ds \\
&\leq \frac{4L^2(1+T+\lambda(\chi))}{c} \int_0^T \mathbb{E}[|x_s^1 - x_s^2|^2] e^{-cs} ds \\
&= \frac{4L^2(1+T+\lambda(\chi))}{c} \|x^1 - x^2\|_c^2.
\end{aligned}$$

For c large enough, we have $\frac{4L^2(1+T+\lambda(\chi))}{c} < 1$. Therefore l is a contraction mapping on $L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$ into itself, and so l has a unique fixed point $X \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$ which is the unique solution to the mean field SDE (1.1).

We turn to prove estimate (1.6). For $n \in \mathbb{N}$, we define $\tau_n := \inf\{t \geq 0, |X_t - X_0| \geq n\}$ and $f_n(t) := \mathbb{E}[\sup_{s \leq t} |X_{s \wedge \tau_n} - X_0|^2]$. For all $t \leq T$, we have:

$$\begin{aligned}
\sup_{t \leq T} |X_{t \wedge \tau_n} - X_0|^2 &\leq 2(\sup_{t \leq T} |\int_0^{t \wedge \tau_n} b_s(X_s, \mathbb{E}[X_s], \alpha_s, \mathbb{E}[\alpha_s]) ds|^2 \\
&+ \sup_{t \leq T} |\int_0^{t \wedge \tau_n} \sigma_s(X_s, \mathbb{E}[X_s], \alpha_s, \mathbb{E}[\alpha_s]) ds|^2 \\
&+ \sup_{t \leq T} |\int_0^{t \wedge \tau_n} \int_{\chi} R_s(X_{s-}, \mathbb{E}[X_{s-}], \alpha_s, \mathbb{E}[\alpha_s], e) \tilde{\pi}(de, ds)|^2).
\end{aligned}$$

Using Cauchy Scharwz's inequality and Burkholder-Davis-Gundy Inequality, we have:

$$\begin{aligned}
\mathbb{E}[\sup_{t \leq T} |X_{t \wedge \tau_n} - X_0|^2] &\leq C(T \mathbb{E}[\int_0^T |b_{s \wedge \tau_n}(X_{s \wedge \tau_n}, \mathbb{E}[X_{s \wedge \tau_n}], \alpha_{s \wedge \tau_n}, \mathbb{E}[\alpha_{s \wedge \tau_n}])|^2 ds \\
&+ \mathbb{E}[\int_0^T |\sigma_{s \wedge \tau_n}(X_{s \wedge \tau_n}, \mathbb{E}[X_{s \wedge \tau_n}], \alpha_{s \wedge \tau_n}, \mathbb{E}[\alpha_{s \wedge \tau_n}])|^2 ds] \\
&+ \mathbb{E}[\int_0^T \int_{\chi} |R_{s \wedge \tau_n}(X_{s \wedge \tau_n}, \mathbb{E}[X_{s \wedge \tau_n}], \alpha_{s \wedge \tau_n}, \mathbb{E}[\alpha_{s \wedge \tau_n}], e)|^2 \lambda(de) ds]), \tag{1.11}
\end{aligned}$$

where C is a positive constant which could change from line to line. For $\alpha \in \mathcal{A}$ and using the definition of the drift of the state process X (See Equation(1.1)), under (H1), we have:

$$\begin{aligned}
\mathbb{E}[\int_0^T |b_{s \wedge \tau_n}(X_{s \wedge \tau_n}, \mathbb{E}[X_{s \wedge \tau_n}], \alpha_{s \wedge \tau_n}, \mathbb{E}[\alpha_{s \wedge \tau_n}])|^2 ds] &\leq \int_0^T C \mathbb{E}[1 + |X_{s \wedge \tau_n}|^2 + \mathbb{E}[|X_{s \wedge \tau_n}|^2]] ds \\
&\leq \int_0^T C(1 + \mathbb{E}[|X_0|^2] + f_n(s)) ds \tag{1.12}
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T |\sigma_{s\wedge\tau_n}(X_{s\wedge\tau_n}, \mathbb{E}[X_{s\wedge\tau_n}], \alpha_{s\wedge\tau_n}, \mathbb{E}[\alpha_{s\wedge\tau_n}])|^2 ds\right] \\
& + \mathbb{E}\left[\int_0^T \int_{\mathcal{X}} |R_{s\wedge\tau_n}(X_{s\wedge\tau_n}, \mathbb{E}[X_{s\wedge\tau_n}], \alpha_{s\wedge\tau_n}, \mathbb{E}[\alpha_{s\wedge\tau_n}], e)|^2 \lambda(de) ds\right] \\
& \leq \int_0^t C(1 + \mathbb{E}[|X_0|^2] + f_n(s)) ds
\end{aligned} \tag{1.13}$$

Plugging inequalities (1.12) and (1.13) into (1.11) and using Gronwall's lemma, we obtain

$$\mathbb{E}[\sup_{s \leq T} |X_{s\wedge\tau_n} - X_0|^2] \leq C(1 + \mathbb{E}[|X_0|^2]).$$

Sending n to infinity, by monotone convergence theorem, we obtain $\mathbb{E}[\sup_{s \leq T} |X_s - X_0|^2] \leq C(1 + \mathbb{E}[|X_0|^2])$ and so the estimate (1.6) is obtained. \square

2 Linear Quadratic Mean Field Control Problem on finite horizon

To solve a Linear Quadratic Mean Field control problem (LQMF control problem in short), we have to find a strategy $\alpha^* \in \mathcal{A}$, such that

$$V := J(\alpha^*) = \inf_{\alpha \in \mathcal{A}} J(\alpha), \tag{2.14}$$

where the criterion J is defined by (1.3). By Assumption (H2) and the estimate (1.6), the LQMF control problem (2.14) is well defined. The aim of this section to solve this control problem on finite horizon by proving a suitable verification theorem.

We are going to use a weak formulation of the standard martingale optimality principle, see e.g [9, 14].

Lemma 2.1. (*Finite horizon Verification Theorem, Lemma 3.1 in [4]*)

Let $\{\mathcal{W}_t^\alpha, t \in [0, T], \alpha_t \in \mathcal{A}\}$ be a family of \mathbb{F} -adapted process in this form $\mathcal{W}_t^\alpha = w_t(X_t^\alpha, \mathbb{E}[X_t^\alpha])$ for some \mathbb{F} -adapted random field $\{w_t(x, \bar{x}), t \in [0, T], x, \bar{x} \in \mathbb{R}^d\}$ satisfying

$$w_t(x, \bar{x}) \leq C(\mathcal{X}_t + |x|^2 + |\bar{x}|^2), \quad t \in [0, T], \quad x, \bar{x} \in \mathbb{R}^d, \tag{2.15}$$

where C is a positive constant, \mathcal{X} is a non-negative process satisfying $\sup_{t \in [0, T]} \mathbb{E}[|\mathcal{X}_t|] < \infty$, and

$$(i) \quad w_T(x, \bar{x}) = g(x, \bar{x}), \quad x, \bar{x} \in \mathbb{R}^d,$$

(ii) the map $t \in [0, T] \mapsto \mathbb{E}[\mathbf{S}_t^\alpha]$, with $\mathbf{S}_t^\alpha = e^{-\rho t} \mathcal{W}_t^\alpha + \int_0^t e^{-\rho s} f_s(X_s^\alpha, \mathbb{E}[X_s^\alpha], \alpha_t, \mathbb{E}[\alpha_t]) ds$, is non-decreasing for all $\alpha \in \mathcal{A}$,

(iii) the map $t \mapsto \mathbb{E}[\mathbf{S}_t^{\alpha^*}]$ is constant for some $\alpha^* \in \mathcal{A}$.

Then, α^* is an optimal control and $\mathbb{E}[w_0(X_0, \mathbb{E}[X_0])]$ is the value of the LQMF control problem (2.14) i.e.

$$V_0 = \mathbb{E}[w_0(X_0, \mathbb{E}[X_0])] = J(\alpha^*).$$

Moreover, any other optimal control satisfies the condition (iii).

Proof. From the growth condition (2.15) and the estimate (1.6), for all $t \in [0, T]$ and $\alpha \in \mathcal{A}$, $\mathbb{E}[\mathbf{S}_t^\alpha]$ is well defined. From condition (i), we have $\mathbb{E}[e^{-\rho T} \mathcal{W}_T^\alpha] = \mathbb{E}[e^{-\rho T} g(X_T, \bar{X}_T)]$, which implies $\mathbb{E}[\mathbf{S}_T^\alpha] = J(\alpha)$.

From condition (ii), we have

$$\mathbb{E}[\mathcal{W}_0^\alpha(X_0, \bar{X}_0)] = \mathbb{E}[\mathbf{S}_0^\alpha] \leq \mathbb{E}[\mathbf{S}_T^\alpha] = J(\alpha).$$

In the other side, for $\alpha = \alpha^*$ and using condition (iii), we have

$$\mathbb{E}[\mathcal{W}_0^{\alpha^*}(X_0, \bar{X}_0)] = \mathbb{E}[\mathbf{S}_0^{\alpha^*}] = \mathbb{E}[\mathbf{S}_T^{\alpha^*}] = J(\alpha^*),$$

which ensures the optimality of control α^* .

Moreover, we consider an another optimal control $\tilde{\alpha} \in \mathcal{A}$, then we have

$$\mathbb{E}[\mathcal{W}_0^{\tilde{\alpha}}(X_0, \bar{X}_0)] = \mathbb{E}[\mathbf{S}_0^{\tilde{\alpha}}] = \mathbb{E}[\mathbf{S}_0^{\alpha^*}] = \mathbb{E}[\mathbf{S}_T^{\alpha^*}] = J(\alpha^*) = J(\tilde{\alpha}).$$

Since the map $t \rightarrow \mathbb{E}[\mathbf{S}_t^{\tilde{\alpha}}]$ is non-decreasing, so it is constant, and we conclude the proof. \square

We start by introducing the following notations:

$$\begin{aligned} \hat{A} &:= A + \tilde{A}, \quad \hat{B} := B + \tilde{B}, \quad \hat{C} := C + \tilde{C}, \quad \hat{D} := D + \tilde{D}, \quad \hat{F} := F + \tilde{F}, \\ \hat{G} &:= G + \tilde{G}, \quad \hat{I} := I + \tilde{I}, \quad \hat{N} := N + \tilde{N}, \quad \hat{Q} := Q + \tilde{Q}, \quad \hat{P} := P + \tilde{P}. \end{aligned}$$

Our aim is to find the expression of the random field $w_t(x, \bar{x})$ which satisfies the assumptions of Lemma (2.1).

From the quadratic form of the cost functional f_t in (1.4) and the terminal cost g_t in (1.5), we guess the quadratic form of random field $\{w_t(x, \bar{x}), t \in [0, T], x, \bar{x} \in \mathbb{R}^d\}$ i.e.

$$w_t(x, \bar{x}) = (x - \bar{x})^\top K_t (x - \bar{x}) + \bar{x}^\top \Lambda_t \bar{x} + 2Y_t^\top x + \varphi_t, \quad (2.16)$$

where (K, Λ, Y, φ) are valued in $\mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}$ and solution of the following system:

$$\begin{cases} dK_t = \dot{K}_t dt, \quad 0 \leq t \leq T, \quad K_T = P, \\ d\Lambda_t = \dot{\Lambda}_t dt, \quad 0 \leq t \leq T, \quad \Lambda = \hat{P}, \\ dY_t = \dot{Y}_t dt + Z_t^Y dW_t + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), \quad 0 \leq t \leq T, \quad Y_T = L, \\ d\varphi_t = \dot{\varphi}_t dt, \quad 0 \leq t \leq T, \quad \varphi_T = 0. \end{cases} \quad (2.17)$$

The deterministic functions $(\dot{K}, \dot{\Lambda}, \dot{\varphi})$ are valued in $\mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}$, the processes (\dot{Y}, Z^Y) are \mathbb{F} -adapted valued in $\mathbb{R}^d \times \mathbb{R}^d$ and R^Y is \mathbb{F} -predictable process valued in \mathbb{R}^d . For $\alpha \in \mathcal{A}$, we consider \mathbf{S}^α defined as:

$$\mathbf{S}_t^\alpha = e^{-\rho t} w_t(X_t, \bar{X}_t) + \int_0^t e^{-\rho s} f_s(X_s, \bar{X}_s, \alpha_t, \bar{\alpha}_t) ds, \quad t \in [0, T]. \quad (2.18)$$

Then $\mathbb{E}[\mathbf{S}_t^\alpha]$ satisfies the following ODE:

$$d\mathbb{E}[\mathbf{S}_t^\alpha] = e^{-\rho t} \mathbb{E}[\mathcal{D}_t^\alpha] dt,$$

where the drift $\mathbb{E}[\mathcal{D}_t^\alpha]$ is defined as follows:

$$\mathbb{E}[\mathcal{D}_t^\alpha] = \mathbb{E} \left[-\rho w_t(X_t, \bar{X}_t) + \frac{d}{dt} \mathbb{E}[w_t(X_t, \bar{X}_t)] + f_t(X_t, \bar{X}_t, \alpha_t, \bar{\alpha}_t) \right].$$

From the dynamics of X_t (See equation (1.1)), we have:

$$d\bar{X}_t = [\bar{\beta}_t + \hat{A}_t \bar{X}_t + \hat{B}_t \bar{\alpha}_t] dt, \quad (2.19)$$

and

$$\begin{aligned} d(X_t - \bar{X}_t) &= [(\beta_t - \bar{\beta}_t) + A_t(X_t - \bar{X}_t) + B_t(\alpha_t - \bar{\alpha}_t)] dt \\ &\quad + [\gamma_t + C_t(X_t - \bar{X}_t) + \hat{C}_t \bar{X}_t + D_t(\alpha_t - \bar{\alpha}_t) + \hat{D}_t \bar{\alpha}_t] dW_t \\ &\quad + \int_{\mathcal{X}} [\delta_t + F_t(e)(X_{t-} - \bar{X}_{t-}) + \hat{F}_t(e) \bar{X}_{t-} + G_t(e)(\alpha_t - \bar{\alpha}_t) + \hat{G}_t(e) \bar{\alpha}_t] \tilde{\pi}(de, dt). \end{aligned} \quad (2.20)$$

We apply the Itô's formula to $w_t(X_t, \bar{X}_t)$, we use the quadratic form of the running cost f_t and the dynamics of equations (2.19), (2.20) and (1.1), we obtain :

$$\begin{aligned} \mathbb{E}[\mathcal{D}_t^\alpha] &= \mathbb{E} \left[(X_t - \bar{X}_t)^\top (\dot{K}_t + \phi_t)(X_t - \bar{X}_t) + \bar{X}_t^\top (\dot{\Lambda}_t + \psi_t) \bar{X}_t \right. \\ &\quad \left. + 2(\dot{Y}_t + \Delta_t)^\top X_t + \dot{\varphi}_t - \rho \varphi_t + \bar{\Gamma}_t + \kappa_t(\alpha) \right], \end{aligned} \quad (2.21)$$

where, for $t \in [0, T]$,

$$\left\{ \begin{array}{l} \phi_t := -\rho K_t + K_t A_t + A_t^\top K_t + C_t^\top K_t C_t + Q_t + \int F_t^\top(e) K_t F_t(e) \lambda(de) = \phi_t(K_t), \\ \psi_t := -\rho \Lambda_t + \Lambda_t \hat{A}_t + \hat{A}_t^\top \Lambda_t + \hat{C}_t^\top K_t \hat{C}_t + \hat{Q}_t + \int \hat{F}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) = \psi_t(K_t, \Lambda_t), \\ \Delta_t := -\rho Y_t + A_t^\top (Y_t - \bar{Y}_t) + \hat{A}_t^\top \bar{Y}_t + C_t^\top (Z_t^Y - \bar{Z}_t^Y) + \hat{C}_t^\top \bar{Z}_t^Y \\ \quad + K_t(\beta_t - \bar{\beta}_t) + \Lambda_t \bar{\beta}_t + C_t^\top K_t(\gamma_t - \bar{\gamma}_t) + \hat{C}_t^\top K_t \bar{\gamma}_t + M_t \\ \quad + \int_{\mathcal{X}} F_t^\top(e) (R_t^Y(e) - \bar{R}_t^Y(e)) \lambda(de) + \int_{\mathcal{X}} \hat{F}_t^\top(e) \bar{R}_t^Y(e) \lambda(de) \\ \quad + \int_{\mathcal{X}} F_t^\top(e) K_t (\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) + \int_{\mathcal{X}} \bar{F}_t^\top(e) K_t \bar{\delta}_t(e) \lambda(de), \\ \quad = \Delta_t(K_t, \Lambda_t, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y), \\ \Gamma_t := \gamma_t^\top K_t \gamma_t + 2\beta_t^\top Y_t + 2\gamma_t^\top Z_t^Y + \int_{\mathcal{X}} \delta_t(e)^\top K_t \delta_t(e) \lambda(de) + 2 \int_{\mathcal{X}} \delta_t(e)^\top R_t^Y(e) \lambda(de) \\ \quad = \Gamma_t(K_t, Y_t, Z_t^Y, R_t^Y), \end{array} \right.$$

and

$$\kappa_t(\alpha) := (\alpha_t - \bar{\alpha}_t)^\top S_t (\alpha_t - \bar{\alpha}_t) + \bar{\alpha}_t^\top \hat{S}_t \bar{\alpha}_t + 2(U_t(X_t - \bar{X}_t) + V_t \bar{X}_t + \Theta_t + \epsilon_t - \bar{\epsilon}_t) \alpha_t. \quad (2.22)$$

Here $S_t, \hat{S}_t, U_t, V_t, \Theta_t$ which appear in (2.22), are defined as follows

$$\left\{ \begin{array}{l} S_t := N_t + D_t^\top K_t D_t + \int G_t^\top(e) K_t G_t(e) \lambda(de), \\ \hat{S}_t := \hat{N}_t + \hat{D}_t^\top K_t \hat{D}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{G}_t(e) \lambda(de), \\ U_t := I_t + D_t^\top K_t C_t + B_t^\top K_t + \int_{\mathcal{X}} G_t^\top(e) K_t F_t(e) \lambda(de), \\ V_t := \hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \hat{B}_t^\top \Lambda_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de), \\ \Theta_t := \bar{H}_t + \hat{D}_t^\top K_t \bar{\gamma}_t + \hat{B}_t^\top \bar{Y}_t + \hat{D}_t^\top \bar{Z}_t^Y + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \bar{\delta}_t(e) \lambda(de) + \int_{\mathcal{X}} \hat{G}_t^\top(e) \bar{R}_t^Y(e) \lambda(de), \end{array} \right. \quad (2.23)$$

and

$$\left\{ \begin{array}{l} \epsilon_t := H_t + D_t^\top K_t \gamma_t + B_t^\top Y_t + D_t^\top Z_t^Y + \int_{\mathcal{X}} G_t^\top(e) K_t \delta_t(e) \lambda(de) + \int_{\mathcal{X}} G_t^\top(e) R_t^Y(e) \lambda(de), \\ \bar{\epsilon}_t := \bar{H}_t + D_t^\top K_t \bar{\gamma}_t + B_t^\top \bar{Y}_t + D_t^\top \bar{Z}_t^Y + \int_{\mathcal{X}} G_t^\top(e) K_t \bar{\delta}_t(e) \lambda(de) + \int_{\mathcal{X}} G_t^\top(e) \bar{R}_t^Y(e) \lambda(de). \end{array} \right. \quad (2.24)$$

We notice that the matrices S_t and \hat{S}_t are definite positive in \mathbb{S}^m . This follows from the non-negativity of the matrix K , conditions (iii)-(iv) in (H2) and the non-negativity of the integral

$\int_{\mathcal{X}} G_t^\top(e) K_t G_t(e) \pi(dt, de)$. In this case, one could find a deterministic $\mathbb{R}^{m \times m}$ -valued Υ such that for all $t \in [0, T]$

$$\Upsilon_t S_t \Upsilon_t^\top = \hat{S}_t,$$

for all $t \in [0, T]$. This implies, that we can rewrite the expectation of $\kappa_t(\alpha)$ as

$$\mathbb{E}[\kappa_t(\alpha)] = \mathbb{E} \left[(\alpha_t - \bar{\alpha}_t + \Upsilon_t^\top \bar{\alpha}_t - \eta_t)^\top S_t (\alpha_t - \bar{\alpha}_t + \Upsilon_t^\top \bar{\alpha}_t - \eta_t) - \zeta_t \right],$$

where

$$\eta_t = a_t^0(X_t, \bar{X}_t) + \Upsilon_t^\top a_t^1(\bar{X}_t) dt \otimes d\mathbb{P} \text{ a.e.}, \quad (2.25)$$

with $a_t^0(X_t, \bar{X}_t)$ a centred random variable and a_t^1 a deterministic function

$$a_t^0(x, \bar{x}) := -S_t^{-1} U_t (x - \bar{x}) - S_t^{-1} (\epsilon_t - \bar{\epsilon}_t), \quad a_t^1(\bar{x}) := -\hat{S}_t^{-1} (V_t \bar{x} + \Theta_t),$$

and

$$\begin{aligned} \zeta_t := & (X_t - \bar{X}_t)^\top (U_t^\top S_t^{-1} U_t) (X_t - \bar{X}_t) + \bar{X}_t^\top (V_t \hat{S}_t^{-1} V_t) \bar{X}_t + 2(U_t^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) \\ & + V_t^\top \hat{S}_t^{-1} \Theta_t)^\top X_t + (\epsilon_t - \bar{\epsilon}_t)^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) + \Theta_t^\top \hat{S}_t^{-1} \Theta_t. \end{aligned}$$

It yields that one could write (2.21) in the following form :

$$\begin{aligned} \mathbb{E}[\mathcal{D}_t^\alpha] = & \mathbb{E} \left[(X_t - \bar{X}_t)^\top (\dot{K}_t + \phi_t^0) (X_t - \bar{X}_t) + \bar{X}_t^\top (\dot{\Lambda}_t + \psi_t^0) \bar{X}_t \right. \\ & + 2(\dot{Y}_t + \Delta_t^0)^\top X_t + \dot{\varphi}_t - \rho \varphi_t + \bar{\Gamma}_t^0 \\ & + (\alpha_t - a_t^0(X_t, \bar{X}_t) - \bar{\alpha}_t + \Upsilon_t^\top (\bar{\alpha}_t - a_t^1(\bar{X}_t)))^\top S_t \\ & \left. (\alpha_t - a_t^0(X_t, \bar{X}_t) - \bar{\alpha}_t + \Upsilon_t^\top (\bar{\alpha}_t - a_t^1(\bar{X}_t))) \right], \end{aligned}$$

where

$$\begin{cases} \phi_t^0 := & \phi_t - U_t^\top S_t^{-1} U_t = \phi_t^0(K_t), \\ \psi_t^0 := & \psi_t - V_t^\top \hat{S}_t^{-1} V_t = \psi_t^0(K_t, \Lambda_t), \\ \Delta_t^0 := & \Delta_t - U_t^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) - V_t^\top \hat{S}_t^{-1} \Theta_t = \Delta_t^0(K_t, \Lambda_t, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y), \\ \Gamma_t^0 := & \Gamma_t - (\epsilon_t - \bar{\epsilon}_t)^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) - \Theta_t^\top \hat{S}_t^{-1} \Theta_t = \Gamma_t^0(K_t, Y_t, Z_t^Y, R_t^Y). \end{cases} \quad (2.26)$$

By choosing (K, Λ, Y, φ) s.t. for all $t \in [0, T]$, we have

$$\dot{K}_t + \phi_t^0 = 0, \quad \dot{\Lambda}_t + \psi_t^0 = 0, \quad \dot{Y}_t + \Delta_t^0 = 0, \quad \dot{\varphi}_t - \rho \varphi_t + \bar{\Gamma}_t^0 = 0, \quad (2.27)$$

we obtain

$$\begin{aligned} \mathbb{E}[\mathcal{D}_t^\alpha] = & \mathbb{E} \left[(\alpha_t - a_t^0(X_t, \bar{X}_t) - \bar{\alpha}_t + \Upsilon_t^\top (\bar{\alpha}_t - a_t^1(\bar{X}_t)))^\top S_t (\alpha_t - a_t^0(X_t, \bar{X}_t) \right. \\ & \left. - \bar{\alpha}_t + \Upsilon_t^\top (\bar{\alpha}_t - a_t^1(\bar{X}_t))) \right], \end{aligned} \quad (2.28)$$

which is non-negative for all $t \in [0, T]$, $\alpha \in \mathcal{A}$. This shows that \mathbf{S}^α satisfies the condition (ii) of the verification theorem (See Lemma 2.1).

In the following lemma, we study the existence of a unique solution to the system of ODEs-BSDE (2.17) satisfying (2.27).

Lemma 2.2. *The decoupled system*

$$\begin{cases} dK_t = -\phi_t^0(K_t)dt, & 0 \leq t \leq T, & K_T = P, \\ d\Lambda_t = -\psi_t^0(K_t, \Lambda_t)dt, & 0 \leq t \leq T, & \Lambda_T = \hat{P}, \\ dY_t = -\Delta_t^0(K_t, \Lambda_t, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y)dt + Z_t^Y dW_t, \\ \quad + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), & 0 \leq t \leq T, & Y_T = L, \\ d\varphi_t = (\rho\varphi_t - \mathbb{E}[\Gamma_t^0(K_t, Y_t, Z_t^Y, R_t^Y)])dt, & 0 \leq t \leq T, & \varphi_T = 0, \end{cases} \quad (2.29)$$

where the processes ϕ^0 , ψ^0 , Δ^0 and Γ^0 are defined in (2.26), admits a unique solution $(K, \Lambda, Y, Z^Y, R^Y, \varphi)$ in $L^\infty([0, T], \mathbb{S}^d) \times L^\infty([0, T], \mathbb{S}^d) \times \mathcal{S}_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d) \times L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d) \times \mathcal{K}_{\mathbb{F}}^2(\Omega \times [0, T] \times \mathcal{X}, \mathbb{R}^d) \times L^\infty([0, T], \mathbb{S}^d)$.

Proof. • We start by introducing the following multi-dimensional Riccati-type equations

$$\begin{cases} \dot{K}_t + Q_t - \rho K_t + K_t A_t + A_t^\top K_t + C_t^\top K_t C_t + \int_{\mathcal{X}} F_t^\top(e) K_t F_t(e) \lambda(de) \\ \quad - \left(I_t + D_t^\top K_t C_t + B_t^\top K_t + \int_{\mathcal{X}} G_t^\top(e) K_t F_t(e) \lambda(de) \right)^\top \\ \quad \left(N_t + D_t^\top K_t D_t + \int_{\mathcal{X}} G_t^\top(e) K_t G_t(e) \lambda(de) \right)^{-1} \\ \quad \left(I_t + D_t^\top K_t C_t + B_t^\top K_t + \int_{\mathcal{X}} G_t^\top(e) K_t F_t(e) \lambda(de) \right) = 0, \\ K_T = P, \end{cases} \quad (2.30)$$

Where the unknown is the matrix K . It is known that the equations (2.30) are related to the linear quadratic stochastic control problem:

$$\begin{aligned} v_t(x) &:= \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^T e^{-\rho s} \left((\tilde{X}_s^{\alpha, x})^\top Q_s \tilde{X}_s^{\alpha, x} + 2\alpha_s^\top I \tilde{X}_s^{\alpha, x} + \alpha_s^\top N \alpha_s \right) ds \right. \\ &\quad \left. + e^{-\rho T} (\tilde{X}_T^{\alpha, x})^\top P \tilde{X}_T^{\alpha, x} \right], \end{aligned}$$

where for $\alpha \in \mathcal{A}$, the process \tilde{X} is solution of the following SDE:

$$\begin{cases} d\tilde{X}_s = (A\tilde{X}_s + B\alpha_s)ds + (C\tilde{X}_s + D\alpha_s)dW_s + \int_{\mathcal{X}} (F_s(e)\tilde{X}_s^{\alpha, x} + G_s(e)\alpha_s) \tilde{\pi}(de, ds), \\ \tilde{X}_0 = x. \end{cases} \quad (2.31)$$

A straightforward extension of Yong and Zhou ([23], Theorem 7.2 p.320) to the jump diffusion case, and under the assumptions (H1) and (H2) (i)-(iii), there exists a unique solution $K \in L^\infty([0, T], \mathbb{S}^d)$ to the equation (2.30) and the solution satisfies $K_t \geq 0$.

- Given K , we consider the following equation for Λ :

$$\left\{ \begin{array}{l} \dot{\Lambda}_t + \hat{Q}_t - \rho\Lambda_t + \Lambda_t \hat{A}_t + \hat{A}_t^\top \Lambda_t + \hat{C}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{F}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) \\ \quad - \left(\hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \hat{B}_t \Lambda_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) \right)^\top \\ \quad \left(\hat{N}_t + \hat{D}_t^\top K_t \hat{D}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{G}_t(e) \lambda(de) \right)^{-1} \\ \quad \left(\hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \hat{B}_t \Lambda_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) \right) = 0, \\ \Lambda_T = \hat{P}. \end{array} \right. \quad (2.32)$$

We rewrite this multi-dimensional Riccati equations as follows:

$$\left\{ \begin{array}{l} \dot{\Lambda}_t + \hat{Q}_t^K - \rho\Lambda_t + \Lambda_t \hat{A}_t + \hat{A}_t^\top \Lambda_t - (\hat{I}_t^K + \hat{B}_t^\top \Lambda_t)^\top (\hat{N}_t^K)^{-1} (\hat{I}_t^K + \hat{B}_t^\top \Lambda_t) = 0 \\ \Lambda_T = \hat{P}, \end{array} \right. \quad (2.33)$$

where $\hat{Q}_t^K, \hat{I}_t^K, \hat{N}_t^K$ are defined by

$$\left\{ \begin{array}{l} \hat{Q}_t^K := \hat{Q}_t + \hat{C}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{F}_t^\top(e) K_t \hat{F}_t(e) \lambda(de), \\ \hat{I}_t^K := \hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de), \\ \hat{N}_t^K := \hat{N}_t + \hat{D}_t^\top K_t \hat{D}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{G}_t(e) \lambda(de). \end{array} \right.$$

From Assumption (H2)(iv), we have $\hat{P} \geq 0$, and $\hat{N}_t \geq \xi \mathbb{I}_m$. As $K \in L^\infty([0, T], \mathbb{S}^d)$ and $K_t \geq 0$ for all $t \in [0, T]$, then $\hat{N}_t^K \geq \xi \mathbb{I}_m$. It remains to check that for all $t \in [0, T]$, we

have $\hat{Q}_t^K - \hat{I}_t^K (\hat{N}_t^K)^{-1} \hat{I}_t^K \geq 0$. In fact, we have

$$\begin{aligned}
& \hat{Q}_t^K - (\hat{I}_t^K)^\top (\hat{N}_t^K)^{-1} \hat{I}_t^K \tag{2.34} \\
= & \hat{Q}_t + \hat{C}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{F}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) - (\hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de))^\top \\
& (\hat{N}_t + \hat{D}_t^\top K_t \hat{D}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{G}_t(e) \lambda(de))^{-1} \\
& (\hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{F}_t(e) \lambda(de)) \\
= & \hat{Q}_t - (\hat{I}_t)^\top (\hat{N}_t)^{-1} \hat{I}_t + (\hat{C}_t - \hat{D}_t \hat{N}_t^{-1} \hat{I}_t)^\top K (\hat{C}_t - \hat{D}_t \hat{N}_t^{-1} \hat{I}_t) \\
& - (\hat{D}_t^\top K (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t))^\top (\hat{N}_t + \hat{D}_t^\top K \hat{D}_t)^{-1} (\hat{D}_t^\top K_t (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t)) \\
& + \int_{\mathcal{X}} (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t)^\top K (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t) \lambda(de) \\
& - \int_{\mathcal{X}} (\hat{G}_t(e)^\top K_t (\hat{F}_t(e) - \hat{G}_t^\top(e) (\hat{N}_t)^{-1} \hat{I}_t))^\top \lambda(de) (\hat{N}_t^K)^{-1} \\
& \int_{\mathcal{X}} (\hat{G}_t(e)^\top K (\hat{F}_t(e) - \hat{G}_t^\top(e) (\hat{N}_t)^{-1} \hat{I}_t)) \lambda(de)
\end{aligned}$$

As $K_t \geq 0$, we have $\hat{N}_t^K \geq \hat{N}_t \geq \xi \mathbb{I}_m > 0$, then

$$\begin{aligned}
& (\hat{D}_t^\top K (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t))^\top (\hat{N}_t^K)^{-1} (\hat{D}_t^\top K_t (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t)) \tag{2.35} \\
\leq & (\hat{D}_t^\top K (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t))^\top (\hat{D}_t^\top K \hat{D}_t)^{-1} (\hat{D}_t^\top K_t (\hat{C}_t - \hat{D}_t^\top (\hat{N}_t)^{-1} \hat{I}_t))
\end{aligned}$$

As $K \in L^\infty([0, T], \mathbb{S}^d)$ and $K_t \geq 0$, then by Cauchy Schwarz inequality, we have

$$\begin{aligned}
& \int_{\mathcal{X}} (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t)^\top K_t (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t) \lambda(de) \tag{2.36} \\
\geq & \int_{\mathcal{X}} (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t)^\top \lambda(de) K_t \int_{\mathcal{X}} (\hat{F}_t(e) - \hat{G}_t(e) \hat{N}_t^{-1} \hat{I}_t) \lambda(de)
\end{aligned}$$

From inequalities (2.34)-(2.36), by algebraic manipulations, we deduce

$$\hat{Q}_t^K - (\hat{I}_t^K)^\top (\hat{N}_t^K)^{-1} \hat{I}_t^K \geq \hat{Q}_t + \hat{I}_t^\top (\hat{N}_t)^{-1} \hat{I}_t \geq 0$$

As in the first step and according to [21] and to Yong and Zhou [23], we deduce that equation (2.33) admits a unique solution $\Lambda \in L^\infty([0, T], \mathbb{S}^d)$ with $\Lambda_t \geq 0$.

- Given (K, Λ) , we consider the following mean field BSDE

$$\begin{cases} dY_t = -\Delta_t^0(K_t, \Lambda_t, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y) dt + Z_t^Y dW_t + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt) \\ Y_T = L. \end{cases} \tag{2.37}$$

It could be written in the following form

$$\left\{ \begin{array}{l} dY_t = (c_t + \theta_t^\top (Y_t - \mathbb{E}[Y_t]) + \hat{\theta}_t^\top \mathbb{E}[Y_t] + \vartheta_t^\top (Z_t^Y - \mathbb{E}[Z_t^Y]) + \hat{\vartheta}_t^\top \mathbb{E}[Z_t^Y] \\ \quad + \int_{\mathcal{X}} \varrho_t^\top(e) (R_t^Y(e) - \mathbb{E}[R_t^Y(e)]) \lambda(de) + \int_{\mathcal{X}} \hat{\varrho}_t^\top(e) \mathbb{E}[R_t^Y(e)] \lambda(de) \Big) dt \\ \quad + Z_t^Y dW_t + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), \\ Y_T = L, \end{array} \right. \quad (2.38)$$

where the stochastic process $(c_t)_t \in L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d)$ is defined by, $\forall t \in [0, T]$ \mathbb{P} p.s

$$\begin{aligned} c_t := & -M_t - K_t(\beta_t - \bar{\beta}_t) - \Lambda_t \bar{\beta}_t - C_t^\top K_t(\gamma_t - \bar{\gamma}_t) - \hat{C}_t^\top K_t \bar{\gamma}_t \\ & - \int_{\mathcal{X}} F_t^\top(e) K_t(\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) - \int_{\mathcal{X}} \hat{F}_t^\top(e) K_t \bar{\delta}_t(e) \lambda(de) \\ & + U_t^\top S_t^{-1} \left(H_t - \bar{H}_t + D_t^\top K_t(\gamma_t - \bar{\gamma}_t) + \int_{\mathcal{X}} G_t(e)^\top K_t(\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) \right) \\ & + V_t^\top S_t^{-1} (\bar{H}_t + \hat{D}_t^\top K_t \bar{\gamma}_t + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \bar{\delta}_t(e) \lambda(de)), \end{aligned}$$

and the deterministic coefficients $\theta, \hat{\theta}, \vartheta, \hat{\vartheta} \in L^\infty([0, T], \mathbb{R}^{d \times d})$, and $\varrho, \hat{\varrho} \in \mathcal{K}^\infty([0, T] \times \mathcal{X}, \mathbb{R}^{d \times d})$ are defined by

$$\begin{aligned} \theta_t &:= \rho \mathbb{I}_d - A_t + B_t S_t^{-1} U_t, & \hat{\theta}_t &:= \rho \mathbb{I}_d - \hat{A}_t + \hat{B}_t \hat{S}_t^{-1} V_t, \\ \vartheta_t &:= -C_t + D_t S_t^{-1} U_t, & \hat{\vartheta}_t &:= -\hat{C}_t + \hat{D}_t \hat{S}_t^{-1} V_t, \\ \varrho_t(e) &:= -F_t(e) + G_t(e) S_t^{-1} U_t, & \hat{\varrho}_t(e) &:= -\hat{F}_t(e) + \hat{G}_t(e) \hat{S}_t^{-1} V_t. \end{aligned}$$

By Tang and Meng [21] (see Lemma 2.2.), under (H1) the BSDE (2.37) admits a unique solution $(Y, Z^Y, R^Y) \in \mathcal{S}_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d) \times L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^d) \times \mathcal{K}_{\mathbb{F}}^2(\Omega \times [0, T] \times \mathcal{X}, \mathbb{R}^d)$.

- Given $(K, \Lambda, Y, Z^Y, R^Y)$, the linear ordinary differential equation for φ :

$$d\varphi_t = (\rho\varphi_t - \mathbb{E}[\Gamma_t^0])dt, \quad \varphi_T = 0, \quad (2.39)$$

where Γ_t^0 is defined by (2.26), admits a unique explicit solution given by:

$$\varphi_t = \int_t^\top e^{-\rho(s-t)} l_s ds.$$

The deterministic function l is defined, for all $t \in [0, T]$ by:

$$\begin{aligned} l_t := & -\mathbb{E} \left[\gamma_t^\top K_t \gamma_t + 2\beta_t^\top Y_t + 2\delta_t^\top Z_t^Y + 2 \int_{\mathcal{X}} \delta_t^\top R_t^Y(e) \lambda(de) \right. \\ & \left. + \int_{\mathcal{X}} \delta_t^\top K_t \delta_t \lambda(de) - \epsilon_t^\top S_t^{-1} \epsilon_t \right] - \bar{\epsilon}_t^\top S_t^{-1} \bar{\epsilon}_t + \Theta_t^\top \hat{S}_t^{-1} \Theta_t. \end{aligned}$$

□

The next theorem gives a connexion between the solution to the decoupled system (2.29) and the solution to the LQMF problem (2.14).

Theorem 2.3. *Under Assumptions (H1)-(H2), there exists an optimal control α^* for LQMF problem (2.14) giving by the following explicit form*

$$\alpha_t^* = -S_t^{-1}U_t(X_{t-}^* - \bar{X}_{t-}^*) - S_t^{-1}(\epsilon_{t-} - \bar{\epsilon}_{t-}) - S_t^{-1}(V_t\bar{X}_{t-}^* + \Theta_t), dt \otimes \mathbb{P} - a.e.$$

where $X^* = X^{\alpha^*}$ is the state process, and α^* is the optimal control. The corresponding value of the problem is

$$V_0 = J(\alpha^*) = \mathbb{E} \left[(X_0 - \bar{X}_0)^T K_0 (X_0 - \hat{X}_0) + \bar{X}_0^T \Lambda_0 \bar{X}_0 + 2Y_0^T X_0 + \varphi_0 \right].$$

Proof. We proved in Lemma 2.2 the existence of $(K, \Lambda, Y, Z^Y, R^Y, \varphi)$ solution to the system (2.29). We consider the candidate $\{w_t(x, \bar{x})\}$ given by (2.16). As K, Λ, R are bounded and Y satisfies a square-integrability condition, then the growth condition of the random field $\{w_t(x, \bar{x})\}$ i.e condition (2.15)) in the verification theorem (see Lemma 2.2) is satisfied. Thanks to the choice of $(K, \Lambda, Y, Z^Y, R^Y, \varphi)$, the terminal conditions satisfied $\{w_T(x, \bar{x})\} = g(x, \bar{x})$, and so Assumption (i) of Lemma 2.2 is satisfied. From equation (2.28), we deduce that $\mathbb{E}[D_t^\alpha]$ is non-negative for all $\alpha \in \mathcal{A}$. It yields that Assumption (ii) of the verification in Lemma 2.2 is satisfied. Moreover, one could prove that $\mathbb{E}[D_t^{\alpha^*}] = 0$ for some $\alpha = \alpha^*$ if and only if

$$\alpha_t^* - a_t^0(X_t^*, \bar{X}_t^*) - \bar{\alpha}_t^* + \Upsilon_t^\top (\bar{\alpha}_t^* - a_t^1(\bar{X}_t^*)) = 0 dt \otimes d\mathbb{P} \text{ a.e.} \quad (2.40)$$

Taking the expectation and using the fact that $\mathbb{E}[a_t^0(X_t^*, \bar{X}_t^*)] = 0$ and Υ_t is an invertible matrix, we get $\bar{\alpha}_t^* - a_t^1(\bar{X}_t^*) = 0$. Then we obtain $\alpha_t^* = a_t^0(X_t^*, \bar{X}_t^*) + a_t^1(\bar{X}_t^*)$. As the strategy α^* must be \mathbb{F} -predictable and the number of jumps of the state process X is finite a.s. over the time interval $[0, T]$, then the optimal strategy α^* satisfies

$$\alpha_t^* = -S_t^{-1}U_t(X_{t-}^* - \bar{X}_{t-}^*) - S_t^{-1}(\epsilon_{t-} - \bar{\epsilon}_{t-}) - S_t^{-1}(V_t\bar{X}_{t-}^* + \Theta_t).$$

Under Assumptions (H1)-(H2), $S^{-1}, \hat{S}^{-1}, U, V$ are bounded and Θ, ϵ are square-integrable respectively in $L^2([0, T], \mathbb{R}^m)$ and $L_{\mathbb{F}}^2(\Omega \times [0, T], \mathbb{R}^m)$. As X^* satisfies the square integrability condition (See inequality (1.6)), then $\alpha^* \in \mathcal{A}$. We proved that there exists a random field $w_t(x, \bar{x})$ that satisfies the assumptions of Lemma 2.2. Then, by the verification theorem, we conclude that α^* is the optimal control for the LQMF problem (2.14). \square

3 Extensions of LQMF Problem on the case of common noise

In this section, we extend the results of Theorem 2.3, to the case with a common noise. Let W and W^0 be two independent real Brownian motions and $\tilde{\pi}$ be an Poisson random measure defined on the same filtered probability space $(\Omega, \mathcal{F}_t, (\mathbb{P}, \mathbb{P}^0))$ where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the

filtration generated by the $(W, W^0, \tilde{\pi})$ and we denote by $\mathbb{F}^0 = \{\mathcal{F}_t^0\}_{t \in [0, T]}$ the filtration generated by W^0 . As in section (2), for any r.v. X_0 and $\alpha \in \mathcal{A}$, the controlled process X_t^α is defined by:

$$\begin{cases} dX_t^\alpha = b_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | W_t^0])dt + \sigma_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | \mathcal{F}_t^0])dW_t + \\ \sigma_t^0(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | \mathcal{F}_t^0])dW_t^0 + \int_{\mathcal{X}} R_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | \mathcal{F}_t^0])(e)\tilde{\pi}(de, dt), \\ X_0^\alpha = X_0, \end{cases} \quad (3.41)$$

where for each $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $a, \bar{a} \in \mathbb{R}^m$ we set:

$$\begin{aligned} b_t(x, \bar{x}, a, \bar{a}) &:= \beta_t + A_t x + \tilde{A}_t \bar{x} + B_t a + \tilde{B}_t \bar{a}, \\ \sigma_t(x, \bar{x}, a, \bar{a}) &:= \gamma_t + C_t x + \tilde{C}_t \bar{x} + D_t a + \tilde{D}_t \bar{a}, \\ \sigma_t^0(x, \bar{x}, a, \bar{a}) &:= \gamma_t^0 + C_t^0 x + \tilde{C}_t^0 \bar{x} + D_t^0 a + \tilde{D}_t^0 \bar{a}, \\ R_t(x, \bar{x}, a, \bar{a})(e) &:= \delta_t(e) + F_t(e)x + \tilde{F}_t(e)\bar{x} + G_t(e)a + \tilde{G}_t(e)\bar{a}. \end{aligned} \quad (3.42)$$

For this case, we assume that

- (H3) (i) $A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D}, C^0, \tilde{C}^0, D^0, \tilde{D}^0, F, \tilde{F}, G, \tilde{G}$ are essentially bounded \mathbb{F}^0 -adapted processes,
(ii) β, γ, γ^0 are square-integrable \mathbb{F} -adapted processes and δ is square-integrable \mathbb{F} -predictable process.

The LQMF control problem is to find $\alpha^* \in \mathcal{A}$ s.t.

$$V_0 := J(\alpha^*) = \inf_{\alpha \in \mathcal{A}} J(\alpha), \quad (3.43)$$

where

$$J(\alpha) := \mathbb{E} \left[\int_0^T e^{-\rho t} f_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | \mathcal{F}_t^0])dt + e^{-\rho T} g(X_T^\alpha, \mathbb{E}[X_T^\alpha | \mathcal{F}_T^0]) \right], \quad (3.44)$$

and the coefficients f, g defined in (1.4)-(1.5), satisfy the following assumptions:

- (H4) (i) $Q, \tilde{Q}, I, \tilde{I}, N, \tilde{N}$ are essentially bounded \mathbb{F}^0 -adapted processes,
(ii) P, \tilde{P} are essentially bounded \mathcal{F}_T^0 -measurable random variables,
(iii) M, H are square-integrable \mathbb{F} -adapted processes,
and L is a square-integrable \mathcal{F}_T -measurable random variables.

As in the previous section, we guess a quadratic form for $w_t(x, \bar{x})$ i.e.

$$w_t(x, \hat{x}) = (x - \hat{x})^T K_t (x - \hat{x}) + \bar{x}^T \Lambda_t \bar{x} + 2Y_t^T x + \varphi_t,$$

where the processes K , Λ , Y and φ are to be determined later. As the quadratic terms in f_t and g are \mathbb{F}^0 -adapted, we guess that K and Λ are \mathbb{F}^0 -adapted. Since the affine coefficients in b_t , σ_t and σ_t^0 and the linear coefficients in f_t and g are \mathbb{F} -adapted, then Y is \mathbb{F} -adapted i.e. depends on W , W^0 and $\tilde{\pi}$. We look for processes (K, Λ, Y, φ) valued in $\mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}$ and satisfy the following system:

$$\begin{cases} dK_t = \dot{K}_t dt + Z_t^K dW_t^0, & 0 \leq t \leq T, \quad K_T = P, \\ d\Lambda_t = \dot{\Lambda}_t dt + Z_t^\Lambda dW_t^0, & 0 \leq t \leq T, \quad \Lambda_T = \hat{P}, \\ dY_t = \dot{Y}_t dt + Z_t^Y dW_t + Z_t^{Y,0} dW_t^0 + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), & 0 \leq t \leq T, \quad Y_T = L, \\ d\varphi_t = \dot{\varphi}_t, & 0 \leq t \leq T, \quad \varphi_T = 0, \end{cases} \quad (3.45)$$

for some \mathbb{F}^0 -adapted processes \dot{K} , $\dot{\Lambda}$, Z^K , Z^Λ valued in \mathbb{S}^d and some \mathbb{F} -adapted processes \dot{Y} , Z^Y , $Z^{Y,0}$ valued in \mathbb{R}^d , an \mathbb{F} -predictable process R^Y valued in \mathbb{R}^d and a continuous function $\dot{\varphi}$ valued in \mathbb{R} .

We keep the notations of section 2 and we add the following notations:

$$\hat{C}_t^0 = C_t^0 + \tilde{C}_t^0, \quad \hat{D}_t^0 = D_t^0 + \tilde{D}_t^0.$$

For $\alpha \in \mathcal{A}$ and $t \in [0, T]$, we set \mathbf{S}^α in the following form:

$$\mathbf{S}_t^\alpha = e^{-\rho t} W_t^\alpha + \int_0^t e^{-\rho s} f_s(X_s^\alpha, \mathbb{E}[X_s^\alpha | \mathcal{F}_s^0], \alpha_s, \mathbb{E}[\alpha_s | \mathcal{F}_s^0]) ds. \quad (3.46)$$

As in Section 2, we compute

$$d \mathbb{E}[\mathbf{S}_t^\alpha] = e^{-\rho t} \mathbb{E}[\mathcal{D}_t^\alpha] dt,$$

where

$$\mathbb{E}[\mathcal{D}_t^\alpha] = \mathbb{E} \left[-\rho w_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0]) + \frac{d}{dt} \mathbb{E}[w_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0])] + f_t(X_t^\alpha, \mathbb{E}[X_t^\alpha | \mathcal{F}_t^0], \alpha_t, \mathbb{E}[\alpha_t | \mathcal{F}_t^0]) \right].$$

As in the previous section, we denote by \bar{X}_t for $\mathbb{E}[X_t^\alpha | \mathcal{F}_t^0]$ and $\bar{\alpha}_t$ for $\mathbb{E}[\alpha_t | \mathcal{F}_t^0]$. By applying Itô's formula to \mathbf{S}_t^α and using the quadratic form of the cost f_t and the dynamics of \bar{X}_t and $X_t - \bar{X}_t$, we obtain:

$$d\bar{X}_t = [\beta_t + \hat{A}_t \bar{X}_t + \hat{B}_t \bar{\alpha}_t] dt, \quad (3.47)$$

and

$$\begin{aligned} d(X_t - \bar{X}_t) = & [(\beta_t - \bar{\beta}_t) + A_t(X_t - \bar{X}_t) + B_t(\alpha_t - \bar{\alpha}_t)] dt \\ & + [\gamma_t + C_t(X_t - \bar{X}_t) + \hat{C}_t \bar{X}_t + D_t(\alpha_t - \bar{\alpha}_t) + \hat{D}_t \bar{\alpha}_t] dW_t \\ & + [\gamma_t^0 + C_t^0(X_t - \bar{X}_t) + \hat{C}_t^0 \bar{X}_t + D_t^0(\alpha_t - \bar{\alpha}_t) + \hat{D}_t^0 \bar{\alpha}_t] dW_t^0 \\ & + \int_{\mathcal{X}} [\delta_t + F_t(e)(X_{t-} - \bar{X}_{t-}) + \hat{F}_t(e) \bar{X}_{t-} + G_t(e)(\alpha_t - \bar{\alpha}_t) + \hat{G}_t(e) \bar{\alpha}_t] \tilde{\pi}(de, dt). \end{aligned} \quad (3.48)$$

Using the same arguments as in the previous section, we obtain:

$$\begin{aligned} \mathbb{E}[\mathcal{D}_t^\alpha] = & \mathbb{E} \left[(X_t - \bar{X}_t)^T (\dot{K}_t + \phi_t) (X_t - \bar{X}_t) + \bar{X}_t^\top (\dot{\Lambda}_t + \psi_t) \bar{X}_t \right. \\ & \left. + 2(\dot{Y}_t + \Delta_t) X_t + \dot{\varphi}_t - \rho \varphi_t + \bar{\Gamma}_t + \kappa_t(\alpha) \right], \end{aligned} \quad (3.49)$$

where

$$\left\{ \begin{aligned} \phi_t & := -\rho K_t + K_t A_t + A_t^\top K_t + Z_t^K C_t^0 + (C_t^0)^\top Z_t^K + C_t^\top K_t C_t + (C_t^0)^\top K_t C_t^0 + Q_t \\ & \quad + \int_{\mathcal{X}} F_t^\top(e) K_t F_t(e) \lambda(de) = \phi_t(K_t, Z_t^K), \\ \psi_t & := -\rho \Lambda_t + \Lambda_t \hat{A}_t + \hat{A}_t^\top \Lambda_t + Z_t^\Lambda C_t^0 + (C_t^0)^\top Z_t^\Lambda + \hat{C}_t^\top K_t \hat{C}_t + (C_t^0)^\top \Gamma_t C_t^0 + \hat{Q}_t \\ & \quad + \int_{\mathcal{X}} \hat{F}_t^\top(e) K_t \hat{F}_t(e) \lambda(de) = \psi_t(K_t, \Lambda_t), \\ \Delta_t & := -\rho Y_t + A_t^\top Y_t + \bar{A}_t^\top \bar{Y}_t + C_t^\top Z_t^Y + (C_t^0)^\top Z_t^{Y,0} + (\tilde{C}_t^0)^\top \bar{Z}_t^{Y,0} + \tilde{C}_t^\top \bar{Z}_t^Y \\ & \quad + K_t(\beta_t - \bar{\beta}_t) + \Lambda_t \bar{\beta}_t + C_t^\top K_t(\gamma_t - \bar{\gamma}_t) + \hat{C}_t^\top K_t \bar{\gamma}_t + M_t + Z_t^K(\gamma_t^0 - \bar{\gamma}_t^0) + Z_t^\Lambda \bar{\gamma}_t^0 \\ & \quad + (C_t^0)^\top K_t(\gamma_t^0 - \bar{\gamma}_t^0) + (\hat{C}_t^0)^\top \Lambda_t \bar{\gamma}_t^0 + \int_{\mathcal{X}} F_t^\top(e) (R_t^Y(e) - \bar{R}_t^Y(e)) \lambda(de) \\ & \quad + \int_{\mathcal{X}} \hat{F}_t(e)^\top \bar{R}_t^Y(e) \lambda(de) + \int_{\mathcal{X}} F_t^\top(e) (\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) \\ & \quad + \int_{\mathcal{X}} \hat{F}_t(e)^\top \bar{\delta}_t(e) \lambda(de) = \Delta_t(K_t, Z_t^K, \Lambda_t, Z_t^\Lambda, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y), \\ \Gamma_t & := \gamma_t^\top K_t \gamma_t + (\gamma_t^0 - \bar{\gamma}_t^0)^\top K_t (\gamma_t^0 - \bar{\gamma}_t^0) + 2(\delta_t)^\top Z_t^{Y,0} + (\bar{\gamma}_t^0)^\top \Lambda_t (\bar{\gamma}_t^0) + 2\beta_t^\top Y_t \\ & \quad + 2\gamma_t^\top Z_t^Y + \int_{\mathcal{X}} \delta_t^\top(e) K_t \delta_t(e) \lambda(de) + 2 \int_{\mathcal{X}} \delta_t(e)^\top R_t^Y(e) \lambda(de) \\ & = \Gamma_t(K_t, Y_t, Z_t^Y, R_t^Y), \end{aligned} \right.$$

and

$$\kappa_t(\alpha) := (\alpha_t - \bar{\alpha}_t)^\top S_t (\alpha_t - \bar{\alpha}_t) + \bar{\alpha}_t^\top \hat{S}_t \bar{\alpha}_t + 2(U_t(X_t - \bar{X}_t) + V_t \bar{X}_t + \Theta_t + \epsilon_t - \bar{\epsilon}_t) \alpha_t. \quad (3.50)$$

Here $S_t, \hat{S}_t, U_t, V_t, \Theta_t$ are defined as follows:

$$\left\{ \begin{aligned} S_t & := N_t + D_t^\top K_t D_t + (D_t^0)^\top K_t D_t^0 + \int_{\mathcal{X}} G_t^\top(e) K_t G_t(e) \lambda(de), \\ \hat{S}_t & := \hat{N}_t + \hat{D}_t^\top K_t \hat{D}_t + (\hat{D}_t^0)^\top K_t \hat{D}_t^0 + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \hat{G}_t(e) \lambda(de), \\ U_t & := I_t + D_t^\top K_t C_t + B_t^\top K_t + (D_t^0)^\top K_t C_t^0 + (D_t^0)^\top Z_t^K + \int_{\mathcal{X}} G_t^\top(e) K_t F_t(e) \lambda(de), \\ V_t & := \hat{I}_t + \hat{D}_t^\top K_t \hat{C}_t + \hat{B}_t^\top \Lambda_t + (D_t^0)^\top Z_t^\Lambda + (D_t^0)^\top K_t (C_t^0) + \int_{\mathcal{X}} \hat{G}_t(e) K_t \hat{F}_t(e) \lambda(de), \\ \Theta_t & := \bar{H}_t + \hat{D}_t^\top K_t \bar{\gamma}_t + (\hat{D}_t^0)^\top K_t \bar{\gamma}_t + \hat{B}_t^\top \bar{Y}_t + (\hat{D}_t^0)^\top \bar{Z}_t^{Y,0} + \hat{D}_t^\top \bar{Z}_t^Y \\ & \quad + \int_{\mathcal{X}} \hat{G}_t^\top(e) K_t \bar{\delta}_t \lambda(de) + \int_{\mathcal{X}} \hat{G}_t^\top(e) \bar{R}_t^Y(e) \lambda(de), \end{aligned} \right. \quad (3.51)$$

$$\begin{cases} \epsilon_t := & H_t + D_t^\top K_t \gamma_t + B_t^\top Y_t + D_t^\top Z_t^Y + (D_t^0)^\top Z_t^{Y,0} + (D_t^0)^\top K_t(\gamma_t) \\ & + \int_{\mathcal{X}} G_t^\top(e) K_t \delta_t \lambda(de) + \int_{\mathcal{X}} G_t^\top(e) R_t^Y(e) \lambda(de), \\ \bar{\epsilon}_t := & \bar{H}_t + D_t^\top K_t \bar{\gamma}_t + B_t^\top \bar{Y}_t + D_t^\top \bar{Z}_t^Y + (D_t^0)^\top \bar{Z}_t^{Y,0} + (D_t^0)^\top K_t(\bar{\gamma}_t) \\ & + \int_{\mathcal{X}} G_t^\top(e) K_t \bar{\delta}_t \lambda(de) + \int_{\mathcal{X}} G_t^\top(e) \bar{R}_t^Y(e) \lambda(de). \end{cases} \quad (3.52)$$

Completing the square in the expression of $\mathbb{E}[D_t^\alpha]$ (See equation (3.49)) and getting rid of the terms which not depend in α , the computations lead to the following decoupled system,

$$\begin{cases} dK_t &= -\phi_t^0(K_t, Z^k)dt + Z^k dW_t^0, \quad 0 \leq t \leq T, \quad K_T = P, \\ d\Lambda_t &= -\psi_t^0(K_t, \Lambda_t, Z^\Lambda)dt + Z^\Lambda dW_t^0, \quad 0 \leq t \leq T, \quad \Lambda_T = \hat{P}, \\ dY_t &= -\Delta_t^0(K_t, Z_t^K, \Lambda_t, Z_t^\Lambda, Y_t, \bar{Y}_t, Z_t^Y, \bar{Z}_t^Y, R_t^Y, \bar{R}_t^Y)dt + Z_t^Y dW_t \\ &+ Z_t^{Y,0} dW_t^0 + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), \quad 0 \leq t \leq T, \quad Y_T = L, \\ d\varphi_t &= (\rho\varphi_t - \mathbb{E}[\Gamma_t^0])dt, \quad 0 \leq t \leq T, \quad \varphi_T = 0, \end{cases} \quad (3.53)$$

with

$$\begin{cases} \phi_t^0 &:= \phi_t - U_t^\top S_t^{-1} U_t, \\ \psi_t^0 &:= \psi_t - V_t^\top \hat{S}_t^{-1} V_t, \\ \Delta_t^0 &:= \Delta_t - V_t^\top \hat{S}_t^{-1} \Theta_t - U_t^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t), \\ \Gamma_t^0 &:= \Gamma_t - (\epsilon_t - \bar{\epsilon}_t)^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) - \Theta_t^\top \hat{S}_t^{-1} \Theta_t. \end{cases} \quad (3.54)$$

One could prove that (K, Λ) is the unique solution to the Stochastic Backward Riccati Equation in (3.53). Given (K, Λ) , we have also the existence of a unique solution $(Y, Z^Y, Z^{Y,0}, R^Y)$ of the mean field backward stochastic differential equation in (3.53). Then from Lemma 2.1, we have the following proposition which gives the structure of the optimal control α^* .

Proposition 3.1. *Under Assumptions (H3)-(H4), the optimal control α^* for optimal problem (3.43)-(3.44) is defined as follows*

$$\alpha_t^* = -S_t^{-1} U_t (X_{t-}^* - \mathbb{E}[X_{t-}^* | \mathcal{F}_t^0]) - S_t^{-1} (\epsilon_{t-} - \mathbb{E}[\epsilon_{t-} | \mathcal{F}_t^0]) - \hat{S}_t^{-1} (V_t - \mathbb{E}[X_{t-}^* | \mathcal{F}_t^0] + \Theta_t).$$

Proof. The proof is similar to that of Theorem 2.3. \square

4 Linear Quadratic Mean Field Control Problem on infinite horizon

Let us consider the infinite horizon case. For $\rho > 0$, we define the set of admissible controls as follows:

$$\mathcal{A} := \{\alpha : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m \text{ s.t } \alpha \text{ is } \mathbb{F}\text{-predictable and } \int_0^\infty e^{-\rho t} \mathbb{E}[|\alpha_t|^2] dt < \infty\}.$$

The controlled process is defined on $[0, \infty)$ by:

$$\begin{cases} dX_t^\alpha &= b_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t])dt + \sigma_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t])dW_t \\ &+ \int_{\mathcal{X}} R_t(X_t^{\alpha-}, \mathbb{E}[X_t^{\alpha-}], \alpha_t, \mathbb{E}[\alpha_t])(e)\tilde{\pi}(de, dt), \\ X_0^\alpha &= X_0, \end{cases} \quad (4.55)$$

where for each $t \geq 0, x, \bar{x} \in \mathbb{R}^d$, we use the same formulation as in Section 2:

$$\begin{aligned} b_t(x, \bar{x}, a, \bar{a}) &:= \beta_t + Ax + \tilde{A}\bar{x} + Ba + \tilde{B}\bar{a}, \\ \sigma_t(x, \bar{x}, a, \bar{a}) &:= \gamma_t + Cx + \tilde{C}\bar{x} + Da + \tilde{D}\bar{a}, \\ R_t(x, \bar{x}, a, \bar{a})(e) &:= \delta_t(e) + F(e)x + \tilde{F}(e)\bar{x} + G(e)a + \tilde{G}(e)\bar{a}. \end{aligned} \quad (4.56)$$

For the infinite horizon case, the coefficients of the linear terms are constant vectors, and the coefficients β, γ and δ are stochastic processes. The control problem on infinite horizon is formulated as:

$$V_0 := \inf_{\alpha \in \mathcal{A}} J(\alpha), \quad (4.57)$$

where

$$J(\alpha) := \mathbb{E} \left[\int_0^\infty e^{-\rho t} f_t(X_t^\alpha, \mathbb{E}[X_t^\alpha], \alpha_t, \mathbb{E}[\alpha_t]) dt \right], \quad (4.58)$$

and, for each $t \geq 0, x, \bar{x} \in \mathbb{R}^d$ and $a, \bar{a} \in \mathbb{R}^m$, we define f as in Section 1 i.e.

$$\begin{aligned} f_t(x, \bar{x}, a, \bar{a}) &:= (x - \bar{x})^\top Q(x - \bar{x}) + \bar{x}^\top (Q + \tilde{Q})\bar{x} + 2a^\top I(x - \bar{x}) \\ &+ 2\bar{a}^\top (I + \tilde{I})\bar{x} + (a - \bar{a})^\top N(a - \bar{a}) + \bar{a}^\top (N + \tilde{N})\bar{a} + 2M_t^\top x + 2H_t^\top a. \end{aligned} \quad (4.59)$$

Here, we note that the coefficients of the quadratic terms are constant matrices, and the coefficients H and M may be stochastic processes.

Let $(\mathbb{H}, |\cdot|)$ be a normed space, and $\hat{\rho}$ a positive constant. We define the new spaces:

- $L^\infty(\mathbb{R}^+, \mathbb{H}) := \{ \phi : \mathbb{R}^+ \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is measurable and } \sup_{t \geq 0} |\phi_t| < \infty \text{ a.e.} \},$
- $\mathcal{K}^\infty(\chi, \mathbb{H}) := \{ K : \chi \rightarrow \mathbb{H}, K \text{ is } \mathcal{B}(\chi)\text{-measurable and } \sup_{e \in \chi} |K(e)| < \infty \},$
- $L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{H}) := \{ \phi : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{H} \text{ s.t. } \phi \text{ is } \mathbb{F}\text{-progressively measurable and } \int_0^\infty e^{\hat{\rho}t} \mathbb{E}[|\phi_t|^2] dt < \infty \}$
- $\mathcal{K}_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+ \times \chi, \mathbb{H}) := \{ K : \Omega \times \mathbb{R}^+ \times \chi \rightarrow \mathbb{H}, K \text{ is } \mathcal{P} \otimes \mathcal{B}(\chi)\text{-measurable processes and } \mathbb{E} \left[\int_0^\infty \int_{\chi} e^{\hat{\rho}t} |K_t(e)|^2 \lambda(de) dt \right] < \infty \},$

where \mathcal{P} denote the σ -field of \mathbb{F} -predictable sets on $\Omega \times \mathbb{R}^+$.

We consider the following assumptions on the coefficients of the problem in the infinite horizon case.

(H1') The coefficients in equation (4.55) satisfy:

- (i) $\beta, \gamma \in L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{R}^d)$ and $\delta \in \mathcal{K}_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+ \times \chi, \mathbb{R}^d)$,
- (ii) $A, \tilde{A}, C, \tilde{C} \in \mathbb{R}^{d \times d}$ and $B, \tilde{B}, D, \tilde{D}, J, \tilde{J} \in \mathbb{R}^{d \times m}$,
- (iii) $F, \tilde{F} \in \mathcal{K}^\infty(\chi, \mathbb{R}^{d \times d})$ and $G, \tilde{G} \in \mathcal{K}^\infty(\chi, \mathbb{R}^{d \times m})$.

(H2') The coefficients in equation (4.59) satisfy:

- (i) $Q, \tilde{Q} \in \mathbb{S}^d, N, \tilde{N} \in \mathbb{S}^m, I, \tilde{I} \in \mathbb{R}^{m \times d}$,
- (ii) $M \in L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{R}^d), H \in L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{R}^m)$,
- (iii) $N > 0, \quad Q + I^T N^{-1} I \geq 0$,
- (iv) $N + \tilde{N} > 0, \quad (Q + \tilde{Q}) - (I + \tilde{I})^T (N + \tilde{N})^{-1} (I + \tilde{I}) \geq 0$.

(H3') $\rho > 2 \max\{|A| + |C|^2 + \int_{\chi} |F(e)|^2 \lambda(de), |\hat{A}|\}$.

Proposition 4.1. *Under (H1') and (H3'), the following estimate holds for each square-integrable variable X_0 and $\alpha \in \mathcal{A}$,*

$$\int_0^\infty e^{-\rho t} \mathbb{E}[|X_t^\alpha|^2] dt \leq C_\alpha (1 + \mathbb{E}[|X_0|^2]), \quad (4.60)$$

where C_α is a positive constant.

Proof. By Itô's formula and Young's inequality, we have the following estimate

$$\begin{aligned} & \frac{d}{dt} \mathbb{E}[e^{-\rho t} |\bar{X}_t|^2] \\ &= e^{-\rho t} (-\rho |\bar{X}_t|^2 + 2\bar{b}_t^\top \bar{X}_t) \\ &\leq e^{-\rho t} (-\rho |\bar{X}_t|^2 + 2(|\bar{\beta}_t| |\bar{X}_t| + |\hat{B}| |\bar{\alpha}_t| |\bar{X}_t| + \bar{X}_t^\top \hat{A} \bar{X}_t)) \\ &\leq e^{-\rho t} [(-\rho + 2|\hat{A}| + \epsilon) |\bar{X}_t|^2 + c_\epsilon (|\bar{\beta}_t|^2 + |\bar{\alpha}_t|^2)], \end{aligned}$$

where c_ϵ is a positive constant. We define:

$$\zeta_\epsilon := |\bar{X}_0|^2 + c_\epsilon \int_0^\infty e^{-\rho t} \mathbb{E}[|\beta_t^2| + |\alpha_t^2|], \quad \eta_\epsilon := \rho - 2|\hat{A}| - \epsilon.$$

Under (H1') and as $\alpha \in \mathcal{A}$, we have $\zeta_\epsilon < \infty$ and for ϵ small enough, we have $\eta_\epsilon > 0$. By Gronwall's lemma, we get:

$$\int_0^\infty e^{-\rho t} |\bar{X}_t|^2 dt \leq c_{\alpha, \epsilon} (1 + |\bar{X}_0|^2), \quad (4.61)$$

where $c_{\alpha, \epsilon}$ is positive constant. Using again Itô's formula and Young's inequality, we have the

following estimate:

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E}[e^{-\rho t} |X_t - \bar{X}_t|^2] \\
&= e^{-\rho t} \mathbb{E}[-\rho |X_t - \bar{X}_t|^2 + 2(b_t - \bar{b}_t)^\top (X_t - \bar{X}_t) + |\sigma_t|^2 + \int_{\mathcal{X}} |R_t(e)|^2 \lambda(de)] \\
&\leq e^{-\rho t} \mathbb{E}[-\rho |X_t - \bar{X}_t|^2 + 2(|\beta_t - \bar{\beta}_t| |X_t - \bar{X}_t| + |B| |\alpha_t - \bar{\alpha}_t| |X_t - \bar{X}_t| + (X_t - \bar{X}_t)^\top A (X_t - \bar{X}_t)) \\
&+ 2(|\gamma_t|^2 + |C|^2 |X_t - \bar{X}_t|^2 + |\hat{C}|^2 |\bar{X}_t|^2 + |D|^2 |\alpha_t|^2 + |\tilde{D}|^2 |\bar{\alpha}_t|^2) \\
&+ 2(\int_{\mathcal{X}} |\delta_t(e)|^2 \lambda(de) + \int_{\mathcal{X}} |F(e)|^2 \lambda(de) |X_t - \bar{X}_t|^2 + \int_{\mathcal{X}} |\hat{F}(e)|^2 \lambda(de) |\bar{X}_t|^2 + \int_{\mathcal{X}} |G(e)|^2 \lambda(de) |\alpha_t|^2 \\
&+ \int_{\mathcal{X}} |\tilde{G}(e)|^2 \lambda(de) |\bar{\alpha}_t|^2)] \\
&\leq e^{-\rho t} \mathbb{E}[(-\rho + 2|A| + 2|C|^2 + 2 \int_{\mathcal{X}} |F(e)|^2 \lambda(de) + \epsilon) |X_t - \bar{X}_t|^2 \\
&+ c'_\epsilon (|\beta_t|^2 + |\gamma_t|^2 + \int_{\mathcal{X}} |\delta_t(e)|^2 \lambda(de) + |\alpha_t|^2 + |\bar{X}_t|^2)],
\end{aligned}$$

where c_ϵ is a positive constant. We define:

$$\begin{aligned}
\zeta'_\epsilon &:= c'_\epsilon \int_0^\infty e^{-\rho t} \mathbb{E}[|\beta_t|^2 + |\gamma_t|^2 + \int_{\mathcal{X}} |\delta_t(e)|^2 \lambda(de) + |\alpha_t|^2 + |\bar{X}_t|^2] dt, \\
\eta'_\epsilon &:= \rho - 2|A| - 2|C|^2 - 2 \int_{\mathcal{X}} |F(e)|^2 \lambda(de) - \epsilon.
\end{aligned}$$

Under (H1'), inequality (4.61) and as $\alpha \in \mathcal{A}$, we have $\zeta'_\epsilon < \infty$ and for ϵ small enough, we have $\eta'_\epsilon > 0$. By Gronwall's lemma, we get

$$\mathbb{E}[\int_0^\infty e^{-\rho t} |X_t - \bar{X}_t|^2 dt] \leq c'_{\alpha, \epsilon} (1 + |\bar{X}_0|^2), \quad (4.62)$$

for a suitable $c'_{\alpha, \epsilon} > 0$. From inequalities (4.61) and (4.62), we obtain (4.60). \square

From assumption (H2') and the estimate (4.60), the problem (4.57) is well defined. The following lemma is a verification theorem based on the weak martingale approach in the infinite horizon case.

Lemma 4.2. *Let $\{\mathcal{W}_t^\alpha, t \geq 0, \alpha \in \mathcal{A}\}$ be a family of \mathbb{F} -adapted process in the form $\mathcal{W}_t^\alpha = w_t(X_t^\alpha, \mathbb{E}[X_t^\alpha])$ for some \mathbb{F} -adapted random field $\{w_t(x, \bar{x}), t \in [0, T], x, \bar{x} \in \mathbb{R}^d\}$ satisfying*

$$w_t(x, \bar{x}) \leq C(\chi_t + |x|^2 + |\bar{x}|^2), \quad t \in [0, \infty), \quad x, \bar{x} \in \mathbb{R}^d,$$

for some positive constant C , and non-negative process χ s.t. $e^{-\rho t} \mathbb{E}[\chi_t]$ converge to zero as $t \rightarrow \infty$, and such that:

(i) the map $t \in \mathbb{R}^+ \mapsto \mathbb{E}[\mathbf{S}_t^\alpha]$, with $\mathbf{S}_t^\alpha = e^{-ct}W_t^\alpha + \int_0^t e^{-cs} f_s(X_s^\alpha, \mathbb{E}[X_s^\alpha], \alpha_t, \mathbb{E}[\alpha_t])ds$,
is non-decreasing for all $\alpha \in \mathcal{A}$,

(ii) the map $t \in \mathbb{R}^+ \mapsto \mathbb{E}[\mathbf{S}_t^{\alpha^*}]$ is constant for some $\alpha^* \in \mathbf{A}$.

Then, α^* is an optimal control and $\mathbb{E}[w_0(X_0, \mathbb{E}[X_0])]$ is the value of the LQMKV control problem (4.57) i.e.

$$V_0 = \mathbb{E}[w_0(X_0, \mathbb{E}[X_0])] = J(\alpha^*).$$

Moreover, any other optimal control satisfies the condition (ii).

We extend the results in Theorem 2.3 to the infinite horizon case, where we kept the steps similar to the finite horizon case. We prove our result by applying the lemma 4.2. We should look for the stability of decoupled system on infinite horizon.

We adopt the same approach as in the finite time horizon. We consider a candidate for the random field $w_t(x, \bar{x})$ in the form:

$$w_t(x, \hat{x}) = (x - \bar{x})^T K_t (x - \bar{x}) + \bar{x}^T \Lambda_t \bar{x} + 2Y_t^T x + \varphi_t,$$

where (K, Λ, Y, φ) valued in $\mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}$ satisfying the following system:

$$\begin{cases} dK_t = -\phi_t^0 dt, & t \geq 0, \\ d\Lambda_t = -\psi_t^0 dt, & t \geq 0, \\ dY_t = -\Delta_t^0 dt + Z_t^Y dW_t + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), & t \geq 0, \\ d\varphi_t = (\rho\varphi_t - \mathbb{E}[\Gamma_t^0])dt, & t \geq 0. \end{cases} \quad (4.63)$$

The maps $\phi^0, \psi^0, \Delta^0, \Gamma^0$ are defined by (2.26), where in this case the coefficients $A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D}, Q, \tilde{Q}, N, \tilde{N}, I, \tilde{I}, F(\cdot), \tilde{F}(\cdot), G(\cdot), \tilde{G}(\cdot)$ are constant i.e. independent of time.

We note that there are no terminal conditions in the system, as we are in the infinite horizon case. We need to show the existence of a solution to the system (4.63).

Lemma 4.3. *We assume $(H1')$ - $(H2')$. Then, the system (4.63) admits a solution $(K, \Lambda, Y, Z^Y, R^Y, \varphi) \in L^\infty([0, \infty], \mathbb{S}^d) \times L^\infty([0, \infty], \mathbb{S}^d) \times L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d) \times L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d) \times \mathcal{K}^{2, \hat{\rho}}(\Omega \times [0, \infty] \times \mathcal{X}, \mathbb{R}^d) \times L^\infty([0, \infty], \mathbb{S}^d)$.*

Proof. We prove the existence of a solution to the decoupled system (4.63).

- We introduce the following Riccati-type equation:

$$\begin{aligned}
& Q - \rho K + KA + A^\top K + C^\top KC + \int_{\mathcal{X}} F^\top(e)KF(e)\lambda(de) \\
& - \left(I + D^\top KC + B^\top K + \int_{\mathcal{X}} G^\top(e)KF(e)\lambda(de) \right)^\top \\
& \left(N + D^\top KD + \int_{\mathcal{X}} G^\top(e)KG(e)\lambda(de) \right)^{-1} \\
& \left(I + D^\top KC + B^\top K + \int_{\mathcal{X}} G^\top(e)KF(e)\lambda(de) \right) = 0. \tag{4.64}
\end{aligned}$$

We prove the existence of a solution to (4.64) by relating it to a suitable infinite-horizon linear-quadratic control problem. For $T \in \mathbb{R}_+ \cup \{\infty\}$ and $x \in \mathbb{R}^d$, we consider the following control problem:

$$V^T(x) := \inf_{\alpha \in \mathcal{A}_T} \mathbb{E} \left[\int_0^T e^{-\rho t} \left((\tilde{X}_t^{\alpha,x})^\top Q \tilde{X}_t^{\alpha,x} + 2\alpha_t^\top I \tilde{X}_t^{\alpha,x} + \alpha_t^\top N \alpha_t \right) dt \right],$$

where \mathcal{A}_T is defined by:

$$\mathcal{A}_T := \{ \alpha : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m \text{ s.t. } \alpha \text{ is } \mathbb{F} \text{ predictable and } \int_0^T e^{-\rho t} \mathbb{E}[|\alpha_t|^2] dt < \infty \},$$

and for $\alpha \in \mathcal{A}_T$, the process $\tilde{X} := \tilde{X}^{\alpha,x}$ is solution of the following SDE:

$$d\tilde{X}_t = (A\tilde{X}_t + B\alpha_t)dt + (C\tilde{X}_t + D\alpha_t)dW_t + \int_{\mathcal{X}} (F(e)\tilde{X}_t + G(e)\alpha_t)\tilde{\pi}(de, dt), \quad \tilde{X}_0 = 0. \tag{4.65}$$

Thanks to the integrability condition for $\alpha \in \mathcal{A}_T$, we have $\int_0^T e^{-\rho t} \mathbb{E}[|\tilde{X}_t^{\alpha,x}|^2] dt < \infty$, and so the problems V^T are well-defined for any $T \in \mathbb{R}_+ \cup \{\infty\}$. If $T < \infty$, as already recalled in the finite-horizon case, (H1')-(H2') imply that there exists a unique symmetric solution $(K_t^T)_{t \in [0, T]}$ to Riccati equations:

$$\left\{ \begin{array}{l} \frac{d}{dt} K_t^T + Q - \rho K_t^T + K_t^T A + A^\top K_t^T + C^\top K_t^T C + \int_{\mathcal{X}} F^\top(e)K_t^T F(e)\lambda(de) \\ - \left(I + D^\top K_t^T C + B^\top K_t^T + \int_{\mathcal{X}} G^\top(e)K_t^T F(e)\lambda(de) \right)^\top \\ \left(N + D^\top K_t^T D + \int_{\mathcal{X}} G^\top(e)K_t^T G(e)\lambda(de) \right)^{-1} \\ \left(I + D^\top K_t^T C + B^\top K_t^T + \int_{\mathcal{X}} G^\top(e)K_t^T F(e)\lambda(de) \right) = 0, \\ K_T^T = 0, \end{array} \right. \tag{4.66}$$

and that for every $x \in \mathbb{R}^d$ we have $V^T(x) = x^\top K_0^T x$. It is easy to check from the definition of V^T that $V^T(x) \rightarrow V^\infty(x)$ as T goes to infinity, from which we deduce that

$$V^\infty(x) = \lim_{T \rightarrow \infty} x^\top K_0^T x = x \left(\lim_{T \rightarrow \infty} K_0^T \right) x, \text{ for all } x \in \mathbb{R}^d.$$

This implies the existence of the limit $K = \lim_{T \rightarrow \infty} K_0^T$. By passing to the limit in T in ODE (4.66) at $t = 0$, we obtain by standard arguments (see Lemma 2.8 in [?]), that K satisfies (4.64). Moreover, $K \in \mathbb{S}^d$ and $K \geq 0$.

- Given K , we consider the following equation of Λ :

$$\hat{Q}^K - \rho\Lambda + \Lambda\hat{A} + \hat{A}^\top\Lambda - (\hat{I}^K + \hat{B}^\top\Lambda)^\top(\hat{N}^K)^{-1}(\hat{I}^K + \hat{B}^\top\Lambda) = 0, \quad (4.67)$$

where

$$\begin{cases} \hat{Q}^K & := \hat{Q} + \hat{C}^\top K \hat{C} + \int \hat{F}^\top(e) K \hat{F}(e) \lambda(de), \\ \hat{I}^K & := \hat{I} + \hat{D}^\top K \hat{C} + \int_{\mathcal{X}} \hat{G}^\top(e) K \hat{F}(e) \lambda(de), \\ \hat{N}^K & := \hat{N} + \hat{D}^\top K \hat{D} + \int_{\mathcal{X}} \hat{G}^\top(e) K \hat{G}(e) \lambda(de). \end{cases}$$

Existence of a solution to (4.67) is obtained by the same arguments used for (4.64) under $(H2')$.

- Given (K, Λ) , we consider the following mean field BSDE with jumps on infinite horizon:

$$\begin{aligned} dY_t &= \left(c_t + (\rho\mathbb{I}_d + \theta)^\top (Y_t - \mathbb{E}[Y_t]) + (\rho\mathbb{I}_d + \hat{\theta})^\top \mathbb{E}[Y_t] + \vartheta^\top (Z_t^Y - \mathbb{E}[Z_t^Y]) + \hat{\vartheta}^\top \mathbb{E}[Z_t^Y] \right. \\ &+ \int_{\mathcal{X}} \varrho^\top(e) (R_t^Y(e) - \mathbb{E}[R_t^Y(e)]) \lambda(de) + \int_{\mathcal{X}} \hat{\varrho}^\top(e) \mathbb{E}[R_t^Y(e)] \lambda(de) \Big) dt \\ &+ Z_t^Y dW_t + \int_{\mathcal{X}} R_t^Y(e) \tilde{\pi}(de, dt), \end{aligned} \quad (4.68)$$

where the stochastic process $c \in L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d)$ is defined by:

$$\begin{aligned} c_t &:= -M_t - K(\beta_t - \bar{\beta}_t) - \Lambda \bar{\beta}_t - C^\top K(\gamma_t - \bar{\gamma}_t) - \hat{C}^\top K \bar{\gamma}_t \\ &- \int_{\mathcal{X}} \bar{F}^\top(e) K(\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) - \int_{\mathcal{X}} \bar{F}^\top(e) K \bar{\delta}_t(e) \lambda(de) \\ &+ U^\top S^{-1} \left(H_t - \bar{H}_t + D^\top K(\gamma_t - \bar{\gamma}_t) + \int_{\mathcal{X}} G(e)^\top K(\delta_t(e) - \bar{\delta}_t(e)) \lambda(de) \right) \\ &+ V^\top \hat{S}^{-1} \left(\bar{H}_t + \hat{D}^\top K \bar{\gamma}_t + \int_{\mathcal{X}} \hat{G}^\top(e) K \bar{\delta}_t(e) \lambda(de) \right), \end{aligned}$$

the coefficients $\theta, \hat{\theta}, \vartheta, \hat{\vartheta}$ are constant in \mathbb{R}^d and are defined by:

$$\begin{aligned}\theta &:= -A + B S^{-1}U, \\ \hat{\theta} &:= -\hat{A} + \hat{B} \hat{S}^{-1}V, \\ \vartheta &:= -C + D S^{-1}U, \\ \hat{\vartheta} &:= -\hat{C} + \hat{D} \hat{S}^{-1}V,\end{aligned}$$

and $\varrho, \hat{\varrho} : \chi \rightarrow \mathbb{R}^d$ are defined by:

$$\begin{aligned}\varrho(e) &:= -F(e) + G(e)S^{-1}U, \forall e \in \chi, \\ \hat{\varrho}(e) &:= -\hat{F}(e) + \hat{G}(e)\hat{S}^{-1}V, \forall e \in \chi.\end{aligned}$$

To simplify the notations let us denote:

$$\begin{aligned}-f(t, Y_t, Z_t, R_t, \mathbb{E}[Y_t], \mathbb{E}[Z_t], \mathbb{E}[R_t]) &= c_t + (\rho \mathbb{I}_d + \theta)^\top (Y_t - \mathbb{E}[Y_t]) + (\rho \mathbb{I}_d + \hat{\theta})^\top \mathbb{E}[Y_t] \\ &+ \vartheta^\top (Z_t - \mathbb{E}[Z_t]) + \hat{\vartheta}^\top \mathbb{E}[Z_t] + \int_{\chi} \varrho^\top(e) (R_t(e) - \mathbb{E}[R_t(e)]) \lambda(de) + \int_{\chi} \hat{\varrho}^\top(e) \mathbb{E}[R_t(e)] \lambda(de).\end{aligned}$$

We prove that the following linear BSDE with jumps defined by: for $t \geq 0$,

$$\begin{cases} dY_t = -f(t, Y_t, Z_t, R_t, \mathbb{E}[Y_t], \mathbb{E}[Z_t], \mathbb{E}[R_t]) dt + Z_t dW_t + \int_{\chi} R_t(e) \tilde{\pi}(de, dt) \\ \lim_{t \rightarrow \infty} Y_t = 0. \end{cases} \quad (4.69)$$

has a solution (Y, Z, R) in $L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{R}^d) \times L_{\mathbb{F}}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+, \mathbb{R}^d) \times \mathcal{K}^{2, \hat{\rho}}(\Omega \times \mathbb{R}^+ \times \chi, \mathbb{R}^d)$, where $\hat{\rho}$ is a positive constant which will be fixed later.

Existence: Let (Y^n, Z^n, R^n) be a solution on $[0, n]$ of the following BSDE

$$\begin{aligned}Y_t^n &= \int_t^n f^n(s, Y_s^n, Z_s^n, R_s^n, \mathbb{E}[Y_s^n], \mathbb{E}[Z_s^n], \mathbb{E}[R_s^n]) ds - \int_t^n Z_s^n dB_s \\ &\quad - \int_t^n \int_{\chi} R_s^n(e) \tilde{\pi}(de, ds), \quad t \in [0, n],\end{aligned}$$

where

$$\begin{aligned}f^n(t, Y_t^n, Z_t^n, R_t^n, \mathbb{E}[Y_t^n], \mathbb{E}[Z_t^n], \mathbb{E}[R_t^n]) &= c_t \mathbf{1}_{[0, n]}(t) + (\rho \mathbb{I}_d + \theta)^\top (Y_t^n - \mathbb{E}[Y_t^n]) \\ &+ (\rho \mathbb{I}_d + \hat{\theta})^\top \mathbb{E}[Y_t^n] + \vartheta^\top (Z_t^n - \mathbb{E}[Z_t^n]) + \hat{\vartheta}^\top \mathbb{E}[Z_t^n] + \int_{\chi} \varrho^\top(e) (R_t^n(e) - \mathbb{E}[R_t^n(e)]) \lambda(de) \\ &+ \int_{\chi} \hat{\varrho}^\top(e) \mathbb{E}[R_t^n(e)] \lambda(de),\end{aligned}$$

and we take $(Y_t^n, Z_t^n, R_t^n) = (0, 0, 0)$ on (n, ∞) .

We fix $m > n$. Applying Itô's formula to $e^{\hat{\rho}t}|Y_t^m - Y_t^n|^2$, we get for all $t \geq 0$,

$$\begin{aligned} & e^{\hat{\rho}T}|Y_T^m - Y_T^n|^2 - e^{\hat{\rho}t}|Y_t^m - Y_t^n|^2 \\ &= \int_t^T e^{\hat{\rho}s} \left(\hat{\rho}|Y_s^m - Y_s^n|^2 + |Z_s^m - Z_s^n|^2 + \int_{\mathcal{X}} |(R_s^m - R_s^n)(e)|^2 \lambda(de) \right) ds \\ &- 2 \int_t^T e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top \Delta^{n,m} f_s ds \\ &+ 2 \int_t^T e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top (Z_s^m - Z_s^n) dB_s + \int_{\mathcal{X}} e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top (R_s^m - R_s^n)(e) \tilde{\pi}(de, ds), \end{aligned}$$

where

$$\Delta^{n,m} f_s := f^m(s, Y_s^m, Z_s^m, R_s^m, \mathbb{E}[Y_s^m], \mathbb{E}[Z_s^m], \mathbb{E}[R_s^m]) - f^n(s, Y_s^n, Z_s^n, R_s^n, \mathbb{E}[Y_s^n], \mathbb{E}[Z_s^n], \mathbb{E}[R_s^n]).$$

We focus on the dependence in Y , we obtain:

$$\begin{aligned} & e^{\hat{\rho}T}|Y_T^m - Y_T^n|^2 - e^{\hat{\rho}t}|Y_t^m - Y_t^n|^2 \\ &= \int_t^T e^{\hat{\rho}s} \left(\hat{\rho}|Y_s^m - Y_s^n|^2 + |Z_s^m - Z_s^n|^2 + \int_{\mathcal{X}} |(R_s^m - R_s^n)(e)|^2 \lambda(de) \right) ds \\ &- 2 \int_t^T e^{\hat{\rho}s} ((Y_s^m - Y_s^n) - \mathbb{E}[Y_s^m - Y_s^n])^\top (\rho \mathbb{I}_d + \theta) (Y_s^m - Y_s^n) ds \\ &- 2 \int_t^T e^{\hat{\rho}s} \mathbb{E}[Y_s^m - Y_s^n]^\top (\rho \mathbb{I}_d + \hat{\theta}) (Y_s^m - Y_s^n) ds - 2 \int_t^T e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top \Delta^{n,m} f_s^0 ds \\ &+ 2 \int_t^T e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top (Z_s^m - Z_s^n) dB_s + \int_{\mathcal{X}} e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top (R_s^m - R_s^n)(e) \tilde{\pi}(de, ds). \end{aligned}$$

where

$$\Delta^{n,m} f_s^0 := f^m(s, 0, Z_s^m, R_s^m, 0, \mathbb{E}[Z_s^m], \mathbb{E}[R_s^m]) - f^n(s, 0, Z_s^n, R_s^n, 0, \mathbb{E}[Z_s^n], \mathbb{E}[R_s^n]).$$

Taking the expectation, the contribution of the stochastic integrals vanishes. Using the Young's inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, where $\epsilon > 0$, and Cauchy Schwarz's inequality, we obtain:

$$\begin{aligned} & 2\mathbb{E} \left[\int_t^T e^{\hat{\rho}s} ((Y_s^m - Y_s^n) - \mathbb{E}[Y_s^m - Y_s^n])^\top (\rho \mathbb{I}_d + \theta) (Y_s^m - Y_s^n) ds \right] \\ &+ 2\mathbb{E} \left[\int_t^T e^{\hat{\rho}s} \mathbb{E}[Y_s^m - Y_s^n]^\top (\rho \mathbb{I}_d + \hat{\theta}) (Y_s^m - Y_s^n) ds \right] \\ &\leq 2\rho \mathbb{E} \left[\int_t^T e^{\hat{\rho}s} |Y_s^m - Y_s^n|^2 ds \right] + 2 \int_t^T e^{\hat{\rho}s} |\theta| E [|Y_s^m - Y_s^n|^2] ds \\ &+ 2 \int_t^T e^{\hat{\rho}s} |\theta| E [|Y_s^m - Y_s^n|^2] ds + 2 \int_t^T e^{\hat{\rho}s} |\hat{\theta}| E [|Y_s^m - Y_s^n|^2] ds \\ &\leq 2\rho \mathbb{E} \left[\int_t^T e^{\hat{\rho}s} |Y_s^m - Y_s^n|^2 ds \right] + 2 \int_t^T e^{\hat{\rho}s} (2|\theta| + |\hat{\theta}|) E [|Y_s^m - Y_s^n|^2] ds. \end{aligned}$$

Similar calculus for the term $\int_t^T e^{\hat{\rho}s} (Y_s^m - Y_s^n)^\top \Delta^{n,m} f_s^0 ds$, shows that:

$$\begin{aligned} & \mathbb{E} \left[\int_0^T e^{\hat{\rho}s} \left(\hat{\rho} |Y_s^m - Y_s^n|^2 + |Z_s^m - Z_s^n|^2 + \int_{\mathcal{X}} |(R_s^m - R_s^n)(e)|^2 \lambda(de) \right) ds \right] \\ & \leq \mathbb{E} \left[e^{\hat{\rho}T} |Y_T^m - Y_T^n|^2 \right] + \left(\delta + \frac{1}{\epsilon} + 2\rho + C \right) \mathbb{E} \left[\int_0^T e^{\hat{\rho}s} |Y_s^m - Y_s^n|^2 ds \right] \\ & \quad + C\epsilon \left(\mathbb{E} \left[\int_0^T e^{\hat{\rho}s} |Z_s^m - Z_s^n|^2 ds \right] \right) + \mathbb{E} \left[\int_0^T \int_{\mathcal{X}} e^{\hat{\rho}s} |(R_s^m - R_s^n)(e)|^2 \lambda(de) ds \right] \\ & \quad + \frac{1}{\delta} \mathbb{E} \left[\int_0^T e^{\hat{\rho}s} |c_s|^2 \mathbf{1}_{[n,m]}(s) ds \right]. \end{aligned}$$

where $C := 2|\theta| + |\hat{\theta}| + 2|\vartheta| + |\hat{\vartheta}| + 2\int_{\mathcal{X}} |\varrho(e)|^2 \lambda(de) + \int_{\mathcal{X}} |\hat{\varrho}(e)|^2 \lambda(de)$. Under (H1')(iii), C is finite. By choosing $0 < \epsilon < \frac{1}{2C}$, $\hat{\rho}$ and $\delta > 0$ s.t. $\hat{\rho} > \delta + \frac{1}{\epsilon} + 2\rho + C + \frac{1}{2}$, we deduce:

$$\begin{aligned} & \mathbb{E} \left[\int_0^T e^{\hat{\rho}s} \left(|Y_s^m - Y_s^n|^2 + |Z_s^m - Z_s^n|^2 + \int_{\mathcal{X}} |(R_s^m - R_s^n)(e)|^2 \lambda(de) \right) ds \right] \\ & \leq \frac{2}{\delta} \mathbb{E} \left[\int_0^T e^{\hat{\rho}s} |c_s|^2 \mathbf{1}_{[n,m]}(s) ds \right]. \end{aligned}$$

Sending T to infinity, by the monotone convergence theorem, we obtain:

$$\begin{aligned} & \mathbb{E} \left[\int_0^\infty e^{\hat{\rho}s} \left(|Y_s^m - Y_s^n|^2 + |Z_s^m - Z_s^n|^2 + \int_{\mathcal{X}} |(R_s^m - R_s^n)(e)|^2 \lambda(de) \right) ds \right] \\ & \leq \frac{2}{\delta} \mathbb{E} \left[\int_0^\infty e^{\hat{\rho}s} |c_s|^2 \mathbf{1}_{[n,m]}(s) ds \right]. \end{aligned}$$

As $c \in L_{\mathbb{F}}^{2,\hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d)$, then $|c_s|^2 \mathbf{1}_{[n,m]}(s) \rightarrow 0$, $dt \otimes d\mathbb{P}$ a.e., $s \geq 0$ when n goes to infinity. By using the dominated convergence theorem for the right hand side, we deduce that the sequence (Y^n, Z^n, K^n) is a Cauchy sequence in $L_{\mathbb{F}}^{2,\hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d) \times L_{\mathbb{F}}^{2,\hat{\rho}}(\Omega \times [0, \infty], \mathbb{R}^d) \times \mathcal{K}^{2,\hat{\rho}}(\Omega \times [0, \infty] \times \mathcal{X}, \mathbb{R}^d)$ and that the limit (Y, Z, K) is a solution of a MF BSDE with jumps (4.69).

- Given $(K, \Lambda, Y, Z^Y, R^Y)$, the linear ordinary differential equation for φ

$$d\varphi_t = (\rho\varphi_t - \mathbb{E}[\Gamma_t^0])dt. \quad (4.70)$$

where $\Gamma_t^0 = \Gamma_t - (\epsilon_t - \bar{\epsilon}_t)^\top S_t^{-1} (\epsilon_t - \bar{\epsilon}_t) - \Theta_t^\top \hat{S}_t^{-1} \Theta_t$ is defined in (2.26), admits a unique explicit solution given by: $\varphi_t = \int_t^\infty e^{-\rho(s-t)} \mathbb{E}[\Gamma_s^0] ds$.

Then, we proved the existence of solutions of the decoupled system (4.63). \square

The following theorem gives the structure of the optimal control for LQMF problem (4.57).

Theorem 4.4. *Under Assumptions (H1')-(H2'), the optimal control for LQMF problem (4.57) is given by*

$$\alpha_t^* = -S^{-1}U(X_{t-}^* - \bar{X}_{t-}^*) - S^{-1}(\epsilon_{t-} - \bar{\epsilon}_{t-}) - S^{-1}(V\bar{X}_{t-}^* + \Theta_t),$$

where $X^* = X^{\alpha^*}$ is the state process where the α^* is the optimal control and the deterministic coefficients S, \hat{S}, U and V and the stochastic coefficients ϵ_t and Θ_t are defined in Section 2.

5 Application to production of an exhaustible resource

In this section, we study a model of production of exhaustible resource with accumulating or maintaining a level of reserves, inspired by a serie of works extented from the Hotelling's model [12]. In the classic Hotelling's model, the dynamics market's evolution is driven by the use of existing reserves of an exhaustible reserves to produce energy without possibility to exploration and/or discovery of new reserves. But many studies have made it possible to ensure that there are still resources to be explored over time, that is to say that the reservation rate can be increased. We can refer to the series of works extended from Prindyck's model [20], Deshmukh et al. [8], Arrow and Chang [3], and Keller et al. [11]. The increase in reserve discoveries occurs stochastically via the Poisson process. It should be noted that this increase is smaller, and it is the reason that the resources always remain exhaustible.

We consider an energy market with N producers (players). Each producer uses exhaustible resources, such as oil, to produce energy. The quantity X_t^i represents the reserve's level of player i , at time t , $i = 1, \dots, N$. It takes values in the set \mathbb{R}^+ . The reserve level X_t^i decreases at a controlled production rate $\alpha_t^i \geq 0$ $dt \otimes dP$ a.e., and also has random discrete increment due to exploration. We use N independent Poisson point process π^i , $i = 1 \dots N$ to model the new discoveries and we denote by $\lambda^i(de)dt$ the associated compensator. We assume that the dynamics of the reserve has a noise which is proportional to the current level of the reserve. The reserve's dynamics of each producer i is given by the following stochastic differential equation:

$$\begin{cases} dX_t^i &= -\alpha_t^i dt + \sigma X_t^i dW_t^i + \eta \int_{\mathcal{X}} X_{t-}^i \tilde{\pi}^i(de, dt), \\ X_0^i &= x_0^i, \end{cases} \quad (5.71)$$

where x_0^i is the initial reserve's level of player i , $\sigma > 0$, W^i , $i = 1 \dots N$ are independent standard Brownian motion and independent of π^i , $i = 1 \dots N$, and $\eta > 0$ is the rate of new discoveries.

The cost functional for producer i is given by:

$$J^i(\alpha^1, \dots, \alpha^n) := \mathbb{E} \left(\int_0^\infty e^{-\rho t} [-\alpha_t^i P_t(\alpha_t^i) + C_p(\alpha_t^i) + C_{ex}(\alpha_t^i, X_t^i)] dt \right), \quad (5.72)$$

where

- P_t^i is the selling price for producer i . It follows a linear inverse demand rule, defined as:

$$P_t^i := P_t(\alpha_t^i) = P_t^0 - \delta \alpha_t^i - \varepsilon \int_0^t \frac{1}{N} \sum_{j=1}^N \alpha_s^j ds,$$

Here δ, ε are positive constants and P^0 is a determinist function. It will be the same for all producers. The price P^i of producer i is related to his production and also to the production of all other producers.

- C_p is the cost functions of production, defined as:

$$C_p(\alpha_t^i) := c_1 \text{Var}(\alpha_t^i).$$

- C_{ex} is the cost functions of exploration defined as:

$$C_{\text{ex}}(\alpha_t^i, X_t^i) := c_2 \alpha_t^i \left(\frac{x_0^i - X_t^i}{x_0^i} \right).$$

The constants c_1 and c_2 are positive and represent respectively the cost of production and the cost of extraction. They are the same for the all producers. From the theory of propagation of chaos, the individual level of reserve X^i and the price process P^i , $i = 1, \dots, N$, become independent and identically distributed, when N goes to infinity, with a common distribution given by the law of the solution (X, P) to the stochastic McKean-Vlasov equation

$$dX_t = -\alpha_t dt + \sigma X_t dW_t + \eta \int_{\mathcal{X}} X_t - \bar{\pi}(de, dt), X_0 = x_0, \quad (5.73)$$

$$P_t = P_t^0 - \delta \alpha_t - \varepsilon \int_0^t \bar{\alpha}_s ds, \quad (5.74)$$

where W is a Brownian motion, and $\alpha_t = a(t, X_t)$, $t \geq 0$, for some measurable function a on $\mathbb{R}_+ \times \mathbb{R}$. We reduce the problem to a representative producer with initial reserve $x_0 > 0$. The state process is given X (see (5.73)). The control processes are given by (α, P) , where P satisfies (5.74). The aim of the representative producer is to minimise the cost functional given by:

$$J(\alpha) := \mathbb{E} \left[\int_0^\infty e^{-\rho t} \alpha_t \left(-P_t^0 + \delta \alpha_t + \varepsilon \int_0^t \bar{\alpha}_s ds \right) + c_1 \text{Var}(\alpha_t) + c_2 \alpha_t \left(\frac{x_0 - X_t}{x_0} \right) dt \right], \quad (5.75)$$

under the constraints that $\alpha_t \geq 0$ and $X_t \geq 0$ \mathbb{P} a.s. for all $t \geq 0$. As $\bar{X}_t = x_0 - \int_0^t \bar{\alpha}_s ds$, then

$$J(\alpha) = \mathbb{E} \left[\int_0^\infty e^{-\rho t} \alpha_t \left(-P_t^0 + \delta \alpha_t + \varepsilon(x_0 - \bar{X}_t) \right) + c_1 \text{Var}(\alpha_t) + c_2 \alpha_t \left(\frac{x_0 - X_t}{x_0} \right) dt \right],$$

and we are in the framework of Section 4 with $d = m = 1$ (one-dimensional state variable and control), the coefficients of the state process and the cost functional are given by

$$B = -1, \quad C = \sigma, \quad F(e) = \eta, \quad \text{for all } e \in \chi$$

and

$$N + \tilde{N} = \delta, \quad N = \delta + c_1, \quad I + \tilde{I} = -\frac{c_2 + \varepsilon x_0}{2x_0}, \quad I = -\frac{c_2}{2x_0}, \quad H_t = \frac{c_2 + \varepsilon x_0 - P_t^0}{2},$$

where the other coefficient are equal to zero. We define $\lambda(\chi) := \int_{\chi} \lambda(de)$. Notice that under the assumption $\rho > \sigma^2 + \lambda(\chi)\eta^2$, (H1') and (H3') are satisfied. By following the approach developed in section 4, the optimal control is given explicitly. We have to solve the decoupled system of Riccati equations and BSDEs with jumps (4.63). The Riccati equations (4.64) for K and Λ (4.67) are given by:

$$\frac{\left(K + \frac{c_2}{2x_0}\right)^2}{\delta + c_1} + (\rho - \sigma^2 - \lambda(\chi)\eta^2)K = 0, \quad \frac{\left(\Lambda + \frac{c_2 + \varepsilon x_0}{2x_0}\right)^2}{\delta} + \rho\Lambda - (\sigma^2 + \lambda(\chi)\eta^2)K = 0. \quad (5.76)$$

Let us also remark that the condition (H2') is not satisfied, but we have the existence of a solution (K, Λ) to (5) such that $K_{c_1, c_2} := \frac{K + \frac{c_2}{2x_0}}{\delta + c_1} > 0$ and $\Lambda_{\varepsilon} := \frac{\Lambda + \frac{c_2 + \varepsilon x_0}{2x_0}}{\delta} > 0$, and given by:

$$K_{c_1, c_2} = \frac{-\rho - \sigma^2 - \lambda(\chi)\eta^2 + \sqrt{(\rho - \sigma^2 - \lambda(\chi)\eta^2)^2 + 2c_2 \frac{\rho - \sigma^2 - \lambda(\chi)\eta^2}{x_0(\delta + c_1)}}}{2} > 0,$$

and

$$\Lambda_{\varepsilon} = \frac{-\rho + \sqrt{\rho^2 + 2 \frac{\rho(c_2 + \varepsilon x_0) + 2(\sigma^2 + \lambda(\chi)\eta^2)K}{\delta x_0}}}{2} > 0.$$

Therefore, we can write the linear BSDE (4.69) with jumps as:

$$\begin{aligned} -dY_t = & \left(\frac{\Lambda_{\varepsilon}}{2}(c_2 + \varepsilon x_0 - P_t^0) - (\rho + K_{c_1, c_2})Y_t + (K_{c_1, c_2} - \Lambda_{\varepsilon})\bar{Y}_t \right. \\ & \left. - \sigma Z_t^Y + \eta \int_{\chi} R_t^Y(e) \lambda(de) \right) dt - Z_t^Y dW_t - \int_{\chi} R_t^Y(e) \tilde{\pi}(de, dt). \end{aligned} \quad (5.77)$$

One could check that a solution of the BSDE (5.77) is given by:

$$(Y, Z^Y, R^Y) = \left(\int_t^{\infty} e^{-(\rho + \Lambda_{\varepsilon})(s-t)} \left(\Lambda_{\varepsilon} \frac{c_2 + \varepsilon x_0 - P_s^0}{2} \right) ds, 0, 0 \right)_{0 \leq t \leq T}. \quad (5.78)$$

In the remaining part of the paper, we assume that $P_t^0 = p^0$ for all $t \geq 0$: p^0 is interpreted as a substitute price for the exhaustible resource. We study two cases. The first one, when

$p^0 = c_2 + \varepsilon x_0$ i.e. p^0 coincides with $c_2 + \varepsilon x_0$ which is the cost of extraction for the last unit of resource. In other words, the Hotelling rent $H_r := p^0 - c_2 - \varepsilon x_0$ is equal to zero. The second case when $p^0 < c_2 + \varepsilon x_0$ i.e. the Hotelling rent is negative. The next proposition gives an explicit solution to the problem (5.75) when $H_r = 0$.

Proposition 5.1. *We assume that $p^0 = c_2 + \varepsilon x_0$ for all $t \geq 0$, x_0 is large enough and $1 > \rho^2 + \frac{2\varepsilon}{\delta}$. Then the solution of (5.75) is given by*

$$\alpha_t^* = K_{c_1, c_2} X_{t-}^* + (\Lambda_\varepsilon - K_{c_1, c_2}) \bar{X}_{t-}^*.$$

Proof. Since (H1') and (H3') are satisfied and the Riccati equations have a solution, then, by Theorem (4.4), the optimal control is then given by:

$$\alpha_t^* = K_{c_1, c_2} (X_{t-}^* - \bar{X}_{t-}^*) + \Lambda_\varepsilon \bar{X}_{t-}^* - \frac{1}{2\delta} (c_2 + \varepsilon x_0 - P_t^0 - Y_t).$$

As $P_t^0 = p^0 = c_2 + \varepsilon x_0$ for all $t \geq 0$, then the solution of the BSDE (5.78) satisfies $Y_t = 0$ for all $t \geq 0$ which yields:

$$\alpha_t^* = K_{c_1, c_2} (X_{t-}^* - \bar{X}_{t-}^*) + \Lambda_\varepsilon \bar{X}_{t-}^*.$$

It remains to show that the optimal strategy satisfies the constraint $\alpha_t^* \geq 0$ \mathbb{P} a.s. for all $t \geq 0$. As x_0 is large, by using Taylor's formula, we have

$$\begin{aligned} 2 \quad (\Lambda_\varepsilon - K_{c_1, c_2}) &= -\rho + \sqrt{\rho^2 + \frac{2\varepsilon}{\delta}} \left(1 + \frac{\rho c_2 + 2(\sigma^2 + \lambda(\chi)\eta^2)K}{\delta x_0(\rho^2 + \frac{2\varepsilon}{\delta})}\right) - \frac{c_2 \rho}{x_0(\delta + c_1)} + o\left(\frac{1}{x_0}\right) \\ &= -\rho + \sqrt{\rho^2 + \frac{2\varepsilon}{\delta}} + \frac{c_2 \rho}{x_0} \left(\frac{1}{\delta \sqrt{\rho^2 + \frac{2\varepsilon}{\delta}}} - \frac{1}{\delta + c_1}\right) + \frac{2(\sigma^2 + \lambda(\chi)\eta^2)K}{\delta x_0 \sqrt{\rho^2 + \frac{2\varepsilon}{\delta}}} + o\left(\frac{1}{x_0}\right). \end{aligned}$$

As $1 > \rho^2 + \frac{2\varepsilon}{\delta}$, then

$$\Lambda_\varepsilon - K_{c_1, c_2} \geq 0. \tag{5.79}$$

We define the stopping time τ^* as follows:

$$\tau^* := \inf\{t \geq 0 \text{ s.t. } X_t^* \leq 0\}.$$

Then on the set $\{t < \tau^*\}$, from inequality (5.79), we have $\alpha_t^* \geq 0$ \mathbb{P} a.s. On the set $\{t = \tau^*\}$, the state process $X_{\tau^*}^* = 0$, which implies $\alpha_{\tau^*}^* = 0$. Since the drift, the diffusion and the jump terms of the state process are equal to zero, then the process X^* remains at the level 0 for all $t \geq \tau^*$ and the optimal strategy α^* is the null strategy for all $t \geq \tau^*$. \square

In the second case, we assume that $p^0 < c_2 + \varepsilon x_0$. It is not obvious to check the positivity of the the state process and the optimal strategy. We study the stationary level of the reserve and the optimal production rate in mean. From the definition of X^* , we have

$$\begin{aligned} \bar{X}_t^* &= x_0 - \int_0^t \alpha_s^* ds = x_0 - \int_0^t \Lambda_\varepsilon \bar{X}_s^* ds \\ &+ \int_0^t \frac{1}{2\delta} \left(c_2 + \varepsilon x_0 - p^0 - \int_s^\infty e^{-(\rho + \Lambda_\varepsilon)(u-s)} \left(\Lambda_\varepsilon \frac{c_2 + \varepsilon x_0 - p^0}{2} \right) du \right) ds \end{aligned}$$

which implies

$$\begin{aligned}\bar{X}_t^* &= x_0 + \frac{(2\rho + \Lambda_\varepsilon)(c_2 + \varepsilon x_0)}{2\delta} \frac{1 - e^{-\Lambda_\varepsilon t}}{2\Lambda_\varepsilon(\rho + \Lambda_\varepsilon)} \\ &\quad - \frac{p^0}{2\delta} \int_0^t e^{-\Lambda_\varepsilon(t-s)} \left(1 - \int_s^\infty \frac{\Lambda_\varepsilon}{2} e^{(\rho + \Lambda_\varepsilon)(u-s)} du\right) ds.\end{aligned}$$

It yields that $\lim_{t \rightarrow \infty} \bar{X}_t^* = \frac{(2\rho + \Lambda_\varepsilon)(c_2 + \varepsilon x_0 - p^0)}{4\delta\Lambda_\varepsilon(\rho + \Lambda_\varepsilon)}$. As the Hotelling rent $H_r := p^0 - c_2 - \varepsilon x_0$

is negative, then $\lim_{t \rightarrow \infty} \bar{X}_t^*$ exists and is positive. As $\lim_{t \rightarrow \infty} \bar{X}_t^* = x_0 - \int_0^\infty \bar{\alpha}_s^* ds$, then $\lim_{t \rightarrow \infty} \bar{\alpha}_t^* = 0$. It means that when we switch to substitute good, there is a remaining resource and we stop the production of exhaustible resource.

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