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Dynamical localization in kicked rotator as a paradigm of other systems: spectral statistics and the localization measure

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Abstract. We study the intermediate statistics of the spectrum of quasi-energies and of the eigenfunctions in the kicked rotator, in the case when the corresponding system is fully chaotic while quantumly localized. As for the eigenphases, we find clear evidence that the spectral statistics is well described by the Brody distribution, notably better than by the Izrailev's one, which has been proposed and used broadly to describe such cases. We also studied the eigenfunctions of the Floquet operator and their localization. We show the existence of a scaling law between the repulsion parameter with relative localization length, but only as a first order approximation, since another parameter plays a role. We believe and have evidence that a similar analysis applies in time-independent Hamilton systems.

1 Introduction

One of the most important manifestations of quantum chaos of low-dimensional classically fully chaotic (ergodic) Hamiltonian systems is the fact that in the (sufficiently deep) semiclassical limit the statistical properties of the discrete energy spectra obey the statistics of Gaussian Random Matrix Theory (RMT). The opposite extreme are classically integrable systems, which quantumly exhibit Poissonian spectral statistics (see [1]).

Quantum kicked rotator (QKR) is a typical example in the field of quantum chaos [2]. A typical property of the QKR is the chaos suppression for sufficiently large time scales. The study of the statistical properties of the classical and quantum (semiclassical) parameters in such systems is of great importance. Here we study in detail the semiclassical region where $k > K > 1$, i.e. the regime of full correspondence between quantum and classical diffusion (on the finite time scale $t \leq t_D$) and the manifested quantum dynamical localization for $t > t_D$. Furthermore, we are focused in the probability level spacing distributions in the regime where the system is classically strongly chaotic ($K \geq 7$) but quantumly localized, i.e. in the *intermediate* or *soft* quantum chaos, as it is described in the literature [5].

2 The quantum kicked rotator model

The QKR model [3] is described by the following function

$$\hat{H} = -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \theta^2} + \varepsilon_0 \cos \theta \sum_{m=-\infty}^{\infty} \delta(t - mT), \quad (1)$$

where \hbar is Planck's constant, I is the moment of inertia of the pendulum and ε_0 is the perturbation strength. The motion after one period T of the ψ wave function then can be described by the following mapping

$$\psi(\theta, t + T) = \hat{U} \psi(\theta, t), \quad (2)$$

$$\hat{U} = \exp\left(i \frac{T\hbar}{4I} \frac{\partial^2}{\partial \theta^2}\right) \exp\left(-i \frac{\varepsilon_0}{\hbar} \cos \theta\right) \exp\left(i \frac{T\hbar}{4I} \frac{\partial^2}{\partial \theta^2}\right), \quad (3)$$

where the ψ function is determined in the middle of the rotation, between two successive kicks. The evolution operator \hat{U} of the system corresponds to one period. Due to the instant action of the perturbation, this evolution can be written as the product of three non-commuting unitary operators, the first and third of which corresponds to the free rotation during half a period $\hat{G}(\tau/2) = \exp\left(i \frac{T\hbar}{4I} \frac{\partial^2}{\partial \theta^2}\right)$, $\tau \equiv \hbar T/I$, while the second $\hat{B}(k) = \exp(-ik \cos \theta)$, $k \equiv \varepsilon_0/\hbar$ describes the kick. The system's behavior depends only on two parameters, i.e. τ and k and its correspondence with the classical systems is described by the relation $K = k\tau = \varepsilon_0 T/I$. In the case $K \equiv k\tau \gg 1$ the motion is well-known to be strongly chaotic. The transition to classical mechanics is described by the limit $k \rightarrow \infty$, $\tau \rightarrow 0$ while $K = \text{const}$. In what follows $\hbar = \tau$ and $T = I = 1$. We shall consider mostly the semiclassical regime $k > K$, where $\tau < 1$.

In order to study how the localization affects the statistical properties of the quasienergy spectra we use the model's representation with a finite number N of levels [4,5]

$$\psi_n(t + T) = \sum_{m=1}^N U_{nm} \psi_m(t), \quad n, m = 1, 2, \dots, N. \quad (4)$$

The finite unitary matrix U_{nm} determines the evolution of a N -dimensional vector (Fourier transform of ψ) of the model

$$U_{nm} = \sum_{n'm'} G_{nm'} B_{n'm'} G_{n'm}, \quad (5)$$

where $G_{ll'} = \exp(i\tau l^2/4) \delta_{ll'}$ is a diagonal matrix corresponding to free rotation during a half period $T/2$ and the matrix $B_{n'm'}$ describing the one kick has the following form

$$B_{n'm'} = \frac{1}{2N+1} \sum_{l=1}^{2N+1} \left\{ \cos \left[(n' - m') \frac{2\pi l}{2N+1} \right] - \cos \left[(n' + m') \frac{2\pi l}{2N+1} \right] \right\} \\ \times \exp \left[-ik \cos \left(\frac{2\pi l}{2N+1} \right) \right].$$

The model (4) with a finite number of states is considered as the quantum analogue of the classical standard mapping on the torus with closed momentum p and phase θ where U_{mn} describes only the odd states of the systems, i.e. $\psi(\theta) = -\psi(-\theta)$.

3 Intermediate statistics and comparison of probability distributions

Let us first compare the Brody and Izrailev probability distribution functions (PDFs) for the study of the intermediate level statistics. The Brody distribution is defined by the relation

$$P_{\text{BR}}(s) = C_1 s^{\beta_{\text{BR}}} \exp(-C_2 s^{\beta_{\text{BR}}+1}), \quad (6)$$

where the two parameters C_1 and C_2 are determined by the normalization conditions $\int_0^\infty P_{\text{BR}}(s) ds = 1$ and $\int_0^\infty s P_B(s) ds = 1$. Izrailev suggested the following distribution (see [5] and references there for the details and the argumentation)

$$P_{\text{IZ}}(s) = A \left(\frac{1}{2}\pi s\right)^{\beta_{\text{IZ}}} \exp\left[-\frac{1}{16}\beta_{\text{IZ}}\pi^2 s^2 - \left(B - \frac{1}{4}\pi\beta_{\text{IZ}}\right)s\right], \quad (7)$$

in order to describe the intermediate statistics, where the parameters A and B are determined again by the two above normalization conditions. Both β parameters, in the strongly localized regime tend asymptotically to 0 with Poissonian statistics while in the chaotic one tend to 1, which excellently approximates the Gaussian Orthogonal Ensemble (GEO) of the RMT. On the other hand, the non-integer β in the PDFs could be associated with the statistics of the quasienergy states with chaotic localized eigenfunctions. Here, we use $N = 4000$ (which is considerably much larger size compared to the one used in the past studies) and $K = 7$ with $k = 30$. In Fig. 1(a) we show the numerical data (histogram) and the two PDFs. Their repulsion parameters have been calculated with best fit procedure independently. The corresponding values are found to be $\beta_{\text{BR}} \approx 0.424$ and $\beta_{\text{IZ}} \approx 0.419$ for the two PDFs respectively. In the inset figure of Fig. 1(a), we may see how the $P_{\text{BR}}(s)$ manages to capture and describe better the peak of the distribution where the most significant part of quasienergies ω is concentrated. The dot-dashed gray line indicates the Wigner surmise while the dot-dot-dashed one the Poisson distribution. Similar findings have also been found even for smaller sizes of the matrix U_{nm} , where the statistics are improved by sampling more matrices with slightly different values of k as e.g. in [5].

The above statement, regarding the $P_{\text{BR}}(s)$ better agreement with the numerical data, becomes more clear when checking the so-called U -functions $U(s) = (2/\pi) \arccos \sqrt{1 - W(s)}$ of the two above distributions [6]. The $W(s) = \int_0^s P(x) dx$ is the cumulative (or integrated) level spacing distribution function (CDF). The U -function has the advantage that its expected statistical error δU is independent of s , being constant for each s and equal to $\delta U = 1/(\pi\sqrt{N_s})$, where N_s is the total number of objects in the $W(s)$ distribution. The numerical pre-factor

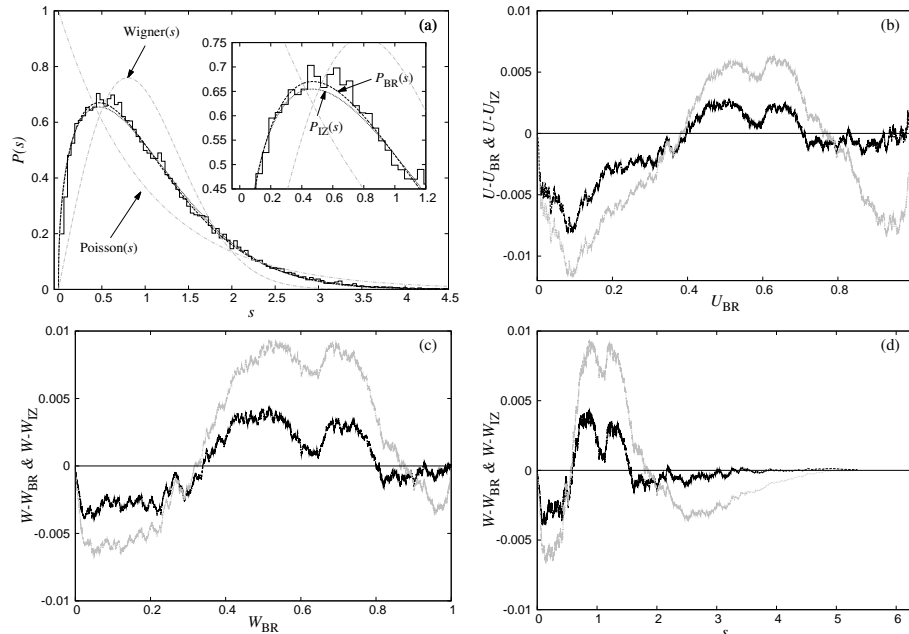


Fig. 1. Intermediate statistics (panel (a)) for distribution $P(s)$ (histogram - black solid line) of the model (4)-(5) fitted with distribution $P_{BR}(s)$ (black dashed line) and $P_{IZ}(s)$ (black dotted line) for $N = 4000$, $K = 7$ and $k = 30$ (see text for discussion). The gray lines indicate the two extreme distributions, i.e. the Poisson and Wigner. In panels (b),(c),(d) we show the comparison of the the Brody (black line) and Izrailev (gray line) PDFs with the numerical using the U -function and W -distribution (see text for discussion).

$2/\pi$ is determined in such a way that $U(s) \in [0, 1]$ when $W(s) \in [0, 1]$. We may note here that the β values for the CDFs may be in principle slightly different compared to those found by the PDFs. In Fig. 1(b), we show the $U_{BR} - U$ and $U_{IZ} - U$ vs. U_{BR} where one may see that the Brody one is in general *closer* to zero (black line) than the Izrailev one (gray color). This fact indicates that the Brody one fits better the numerical data. This is also evident in Fig. 1(c), where the $W_{BR} - W$ and $W_{IZ} - W$ vs. W_{BR} are presented while in Fig. 1(d) the $W_{BR} - W$ and $W_{IZ} - W$ vs. s . The horizontal zero line in these panels indicates the complete agreement between the numerical data and theoretical predictions. The repulsion parameters for the CDFs used in panels (b),(c),(d) are $\beta_{BR} \approx 0.396$ and $\beta_{IZ} \approx 0.366$ respectively.

4 Scaling laws and localized chaotic regimes

A number of different ways to measure and estimate the *localization length* of the eigenfunctions have been proposed in the literature. Here, we adopt the well-

accepted measure described and justified in e.g. [5]: For each N -dimensional eigenvector of the matrix U_{nm} the *information entropy* is $\mathcal{H}_N(u_1, \dots, u_N) = -\sum_{n=1}^N u_n^2 \ln u_n^2$, where $u_n = \text{Re } \varphi_n$ and $\sum_n u_n^2 = 1$. The distribution of u_n^2 for the GOE in the large N -limit tends to the Gaussian distribution and we get $\mathcal{H}_N^{GOE} = \psi(0.5N + 1) - \psi(1.5) \simeq \ln(0.5Na) + O(1/N)$, where $a = 4/\exp(2 - \gamma) \approx 0.96$, while ψ is the digamma function and γ the Euler constant ($\simeq 0.57$). Then the *entropy localization length* l_H is defined as $l_H = N \exp(\mathcal{H}_N - \mathcal{H}_N^{GOE})$. The fluctuations can be minimized when using the *mean localization length* $\langle l_H \rangle \equiv d$, which is computed by averaging over all eigenvectors of the same matrix (or over an ensemble of similar matrices) $d = N \exp(\langle \mathcal{H}_N \rangle - \mathcal{H}_N^{GOE})$.

The parameter that determines the transition from weak to strong quantum chaos is not the strength parameter k but the *ratio of the localization length* l_∞ to the size N of the system, $\Lambda = l_\infty/N$, where $l_\infty = D_{cl}/2\hbar^2$ and D_{cl} is the classical diffusion constant

$$D_{cl} = \begin{cases} \frac{1}{2}K^2 [1 - 2J_2(K)(1 - J_2(K))], & \text{if } K \geq 4.5 \\ 0.30(K - K_{cr})^3, & \text{if } K_{cr} < K \leq 4.5 \end{cases}, \quad (8)$$

where $K_{cr} \simeq 0.9716$ and $J_2(K)$ the Bessel function. The *localization parameter* is then defined as $\beta_{loc} = d/N$. The scaling law we used is $\beta_{loc}(x) = \gamma x / (1 + \gamma x)$, where $x \equiv \Lambda$ and $\gamma \approx 4.2$ which is slightly different (but in agreement) to the one proposed in [7], where $x = k^2/N$ and $K = 5$. In Fig. 2(a), we com-

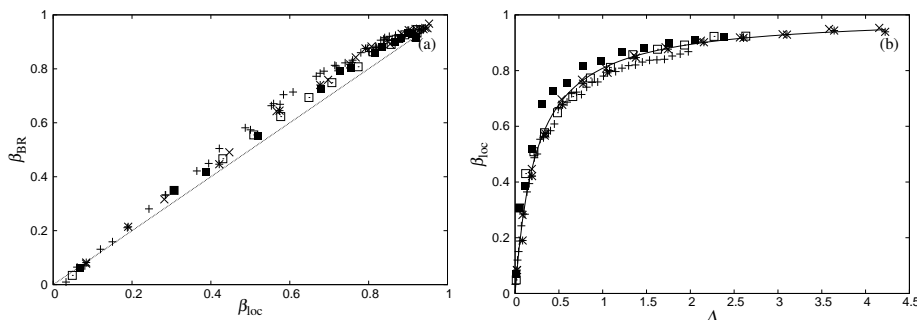


Fig. 2. (a) The fit parameter β_{BR} as a function of β_{loc} for 161×398 elements, $K = 7(+)$, $14(\times)$, $20(*)$, $30(\square)$, $35(\blacksquare)$ for a wide range of k values. (b) The parameter β_{loc} vs. Λ where the scaling law (see text) is shown with the black line.

pare β_{BR} repulsion parameter of the $P_{BR}(s)$ with the *localization parameter* β_{loc} through the localization length $d = \langle l_H \rangle$. For the numerical calculations and results regarding the spacing distributions $P(s)$ for the quasienergies, we have considered a wide range of the quantum perturbation parameter k keeping the classical parameter fixed (classically always fully chaotic). In order to ameliorate the statistics, we considered a sample of 161 matrices U_{nm} of size $N = 398$

($\approx 64,000$ elements), in a similar manner as e.g. in [5]) with slightly different values of k ($\Delta k = \pm 0.00125 \ll k$).

5 Summary

We studied aspects of dynamical localization in the kicked rotator, following [4,5,7], and largely confirm these results. We here considered the case with $K \geq 7$, where the dynamics is already fully chaotic (ergodic). The fractional power law level repulsion is clearly manifested, and globally the level spacing distribution is very well described by the Brody or by the Izrailev distribution, with a clear and systematic (although not very large) trend towards Brody rather than Izrailev. We show that the scaling law (β_{loc} vs. Λ) exists, but only as a first order approximation, as we see some scattering of data around the scaling curve. It seems that with increasing dimension of the matrices the scaling curve asymptotes to the limiting curve with only statistical scattering of the data points left. Further research confirms that a similar picture describing the dynamical localization applies also in time-independent systems, like e.g. billiards [8].

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References

1. Haake F.: Quantum Signatures of Chaos. (Berlin: Springer) (2001)
2. Stöckmann, H. J.: Quantum Chaos - An Introduction (Cambridge: Cambridge University Press) (1999).
3. Casati, G., Chirikov, B. V., Izrailev, F. M., Ford, J.: Stochastic behavior of a quantum pendulum under a periodic perturbation. In Stochastic Behaviour in Classical and Quantum Hamiltonian Systems. Proc. Como Conf. (1977) eds G. Casati and J. Ford. Lecture Notes in Physics (Berlin: Springer). **93** (1979) p. 334-352
4. Izrailev, F. M.: Quantum localization and statistics of quasienergy spectrum in a classically chaotic system. Phys. Lett. A. **134** (1988) 13–18
5. Izrailev, F. M.: Simple models of quantum chaos: Spectrum and eigenfunctions. Phys. Rep. **196** (1990) 299–392
6. Prosen, T., Robnik, M.: Semiclassical energy level statistics in the transition region between integrability and chaos: transition from Brody-like to Berry-Robnik behaviour. J. Phys. A: Math. Gen. **27** (1994) 8059–8077
7. Izrailev, F. M.: Quantum chaos, localization and band random matrices. In Quantum Chaos: Between order and disorder (Cambridge University Press), eds G. Casati and B. Chirikov, (1995) p. 557-576
8. Batistić, B., Robnik, M.: Semiempirical theory of level spacing distribution beyond the Berry-Robnik regime: modeling the localization and the tunneling effects. J. Phys. A: Math. Theor. **43** (2010) 215101