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A Quixotic Proof of Fermat's Two Squares Theorem for Prime Numbers

Roland Bacher

October 13, 2022

Abstract

Every odd prime number p can be written in exactly $(p+1)/2$ ways as a sum $ab+cd$ of two ordered products ab and cd such that $\min(a, b) > \max(c, d)$. An easy corollary is a proof of Fermat's Theorem expressing primes in $1 + 4\mathbb{N}$ as sums of two squares ¹.

1 Introduction

Theorem 1.1. *For every odd prime number p there exist exactly $(p+1)/2$ sequences (a, b, c, d) of four elements in the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of non-negative integers such that $p = ab + cd$ and $\min(a, b) > \max(c, d)$.*

As a consequence we obtain a new proof of an old result first observed by Albert Girard (1595-1632) (who wrote moreover that the set of integers which are sums of two squares contains 2 and is closed under product) around 1625. A refined statement including multiplicities was written in 1640 by Pierre Fermat (1607-1665) in a letter addressed to Marin Mersenne (1588-1648). The old rascal did not want to spoil his margins and left the proof (contained in a series of letters and publications dated around 1750) to Leonhard Euler (1707-1783) who had no such qualms. More historical details can for example be found in the entry "Fermat's theorem on sums of two squares" of [12].

Corollary 1.2. *Every prime number in $1 + 4\mathbb{N}$ is a sum of two squares.*

Proof of Corollary 1.2. If p is a prime number congruent to 1 (mod 4), the number $(p+1)/2$ of solutions (a, b, c, d) defined by Theorem 1.1 is odd. The involution $(a, b, c, d) \mapsto (b, a, d, c)$ has therefore a fixed point (a, a, c, c) expressing p as a sum of two squares. \square

Nowadays a venerable old hat, Corollary 1.2 has of course already quite a few proofs. Some are described in [12]. The author enjoyed the presentation of a few "elementary" proofs given in [5].

¹Keywords: Primes, sum of two squares, lattice. Math. class: Primary: 11A41. Secondary: 11H06.

Zagier (based on unpublished notes of Heath-Brown, [8]) published a one sentence proof based on fixed points in [13]. A. Spivak gave an elementary geometric interpretation of Zagier's proof, see [11]. A nice variation on Zagier's proof was given by Dolan in [4]. An interesting discussion on Zagier's proof and variations can be found at [9] which describes moreover A. Spivak's proof in an answer.

Grace gave a very elegant constructive proof, see [6] (essentially equivalent to the fourth proof of Theorem 366 in [7]) which we recall in Section 3 for the convenience of the reader.

Christopher, see [2], gave a proof based on the existence of a fixed point of an involution acting on suitable partitions with parts of exactly two different sizes (amounting essentially to solutions of $p = ab + cd$ without requirements of inequalities).

The set \mathcal{S}_p of solutions defined by Theorem 1.1 is invariant under the action of Klein's Vierergruppe \mathbb{V} (isomorphic to the 2-dimensional vector space over the field of two elements or, equivalently, isomorphic to the unique non-cyclic group of four elements) with non-trivial elements acting by

$$(a, b, c, d) \mapsto (b, a, c, d), (a, b, d, c), (b, a, d, c)$$

(i.e. by exchanging either the first two elements, or the last two elements, or the first two and the last two elements). The following tables list all \mathbb{V} -orbits represented by elements (a, b, c, d) with a, b, c, d decreasing together with the orbit-sizes $\sharp(\mathcal{O})$ occurring in the sets $\mathcal{S}_{29}, \mathcal{S}_{31}, \mathcal{S}_{37}$:

a	b	c	d	$\sharp(\mathcal{O})$	a	b	c	d	$\sharp(\mathcal{O})$	a	b	c	d	$\sharp(\mathcal{O})$
29	1	0	0	2	31	1	0	0	2	37	1	0	0	2
14	2	1	1	2	15	2	1	1	2	18	2	1	1	2
7	4	1	1	2	10	3	1	1	2	12	3	1	1	2
9	3	2	1	4	6	5	1	1	2	9	4	1	1	2
5	5	4	1	2	7	4	3	1	4	6	6	1	1	1
5	5	2	2	1	9	3	2	2	2	7	5	2	1	4
5	4	3	3	2	5	5	3	2	2	11	3	2	2	2
				15					16	7	4	3	3	2
										5	5	4	3	2
														19

Establishing complete lists \mathcal{S}_p of solutions for small primes is a rather pleasant pastime and rates among the author's more confessable procrastinations.

A solution $p = ab + cd$ in \mathcal{S}_p can be visualized as a lattice-tiling with a fundamental domain given by the union of two rectangles of size $a \times b$ and $d \times c$, aligned as in Figure 1. The resulting tiling is invariant by translations in $\mathbb{Z}(a, c) + \mathbb{Z}(-d, b)$. Klein's Vierergruppe \mathbb{V} acts on the set of all such tilings by quarter-turns on rectangles. Tilings associated to \mathbb{V} -invariant solutions correspond to the case where both rectangles are squares.

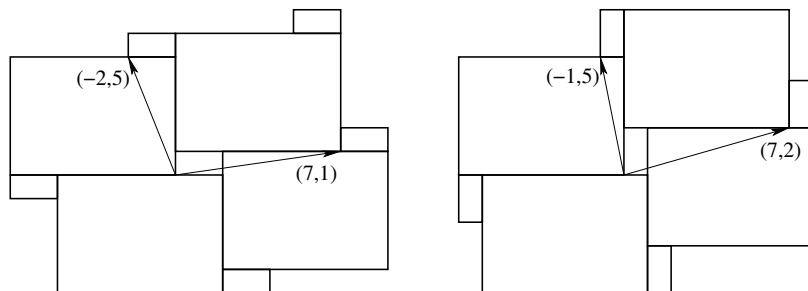


Figure 1: Tilings associated to $7 \cdot 5 + 1 \cdot 2$ and $7 \cdot 5 + 2 \cdot 1$.

A rough sketch for proving Theorem 1.1 goes along the following lines: Every solution $p = ab + cd$ in \mathcal{S}_p can be encoded by two vectors $u = (a, c), v = (-d, b)$ generating a sublattice $\mathbb{Z}u + \mathbb{Z}v$ of index p in \mathbb{Z}^2 , see above and Figure 1. It is therefore enough to understand the number of solutions encoded by every sublattice of index p in \mathbb{Z}^2 . Sublattices of index p in \mathbb{Z}^2 are in one-to-one correspondence with all $p + 1$ elements of the projective line $\mathbb{F}_p \cup \{\infty\}$ over the finite field \mathbb{F}_p . An element μ encoding the slope $\mu = \frac{b}{a}$ of $[a : b]$ (using the obvious convention for $\mu = \infty$) defines the sublattice $\Lambda_\mu(p) = \{(x, y) \in \mathbb{Z}^2 \mid ax + by \equiv 0 \pmod{p}\}$ of index p in \mathbb{Z}^2 . The two lattices $\Lambda_0(p) = \mathbb{Z}(p, 0) + \mathbb{Z}(0, 1)$ and $\Lambda_\infty(p) = \mathbb{Z}(1, 0) + \mathbb{Z}(0, p)$ with singular slopes $0, \infty \notin \mathbb{F}_p^*$ give rise to the two degenerate solutions $p \cdot 1 + 0 \cdot 0$ and $1 \cdot p + 0 \cdot 0$. All other solutions $p = ab + cd$ correspond to sublattices of index p in \mathbb{Z}^2 generated by $u = (a, c)$ in the open cone delimited by $y = 0$ and $x = y$ (opening up in E-NE directions) and by $v = (-d, c)$ in the open cone delimited by $x = 0$ and $y = -x$ (opening up in N-NW directions). The two lattices $\Lambda_1(p)$ and $\Lambda_{-1}(p)$ with self-inverse slopes 1 and -1 have no such bases and thus do not correspond to a solution. Exactly one lattice in every pair of distinct lattices $\Lambda_\mu(p), \Lambda_{\mu^{-1}}(p)$ with mutually inverse slopes $\mu, \mu^{-1} \in \mathbb{F}_p^* \setminus \{1, -1\}$ has bases with generators in the two open E-NE and N-NW cones. We show then that exactly one of these bases corresponds to a solution in \mathcal{S}_p . Theorem 1.1 follows now easily.

Colouring the set $\{(x, y) \mid xy(x - y)(x + y) > 0\}$ consisting of the four open E-NE, N-NW, W-SW and S-SE cones and their opposites in black we get a picture of the four sails of an old windmill, see Figure 2. We prove therefore Theorem 1.1 by following in Don Quixote's heroic footsteps (see the beginning of Chapter 8 in [1]). Cervantes forgot of course the explicit statement of Theorem 1.1 and botched the proof by sweeping all those bloody details under the rug.

The author's serendipitous encounter with Don Quixote happened as follows: Euclid's algorithm applied to square-matrices of size 2 (replacing iteratively a row/column by itself minus the other row/column) with entries

in the set $\{0, 1, \dots\}$ of non-negative integers yields a set

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = n, \min(a, d) > \max(b, c), a, b, c, d \in \{0, 1, 2, \dots\} \right\}$$

of

$$\sum_{d \mid n, d^2 \geq n} \left(d + 1 - \frac{n}{d} \right) \tag{1}$$

“irreducible” matrices of given determinant $n \geq 1$, see [10].

A sign-change made out of curiosity in a naive program checking Formula (1) for small values of n suggested Theorem 1.1.

A final additional Section links solutions occurring in Theorem 1.1 with geometric properties of the corresponding lattices.

2 A few reminders on lattices in \mathbb{R}^2

This short Section contains a few elementary and well-known results on lattices in \mathbb{R}^2 , recalled for the convenience of the reader.

A lattice denotes henceforth a discrete additive group $\mathbb{Z}e + \mathbb{Z}f$ generated by an arbitrary basis e, f of the Euclidean vector-space \mathbb{R}^2 endowed with the standard scalar product $\langle (u_x, u_y), (v_x, v_y) \rangle = u_x v_x + u_y v_y$ of two vectors in \mathbb{R}^2 . We will mainly work with sublattices of the integral lattice \mathbb{Z}^2 in \mathbb{R}^2 .

A *minimal element* of a lattice Λ is a shortest element in $\Lambda \setminus \{(0, 0)\}$.

An element v of Λ is *primitive* if it is not contained in $k\Lambda$ for some natural integer $k > 1$.

A *basis* of a lattice Λ of rank (or dimension) 2 is a set e, f of two elements in Λ such that $\Lambda = \mathbb{Z}e + \mathbb{Z}f$.

The following result is a special case of Pick’s Theorem²:

Lemma 2.1. *Two linearly independent elements e, f of a 2-dimensional lattice Λ form a basis of the lattice Λ if and only if the triangle with vertices $(0, 0), e, f$ contains no other elements of Λ .*

Proof. The parallelogram \mathcal{P} with vertices $(0, 0), e, f, e + f$ is a fundamental domain for the sublattice $\mathbb{Z}e + \mathbb{Z}f$ of Λ generated by e and f . The two elements e, f generate therefore Λ if and only if Λ intersects \mathcal{P} only in its four vertices.

Since \mathcal{P} and Λ are invariant under the affine involution $x \mapsto -x + e + f$ exchanging the two triangles with vertices $(0, 0), e, f$ and $e, f, e + f$, the parallelogram \mathcal{P} intersects Λ exactly in its vertices if and only if the triangle defined by $(0, 0), e, f$ intersects Λ exactly in its vertices. \square

²Pick’s theorem gives the area $\frac{1}{2}b + i - 1$ of a closed lattice polygon P (with vertices in \mathbb{Z}^2) containing b lattice points $\partial P \cap \mathbb{Z}^2$ in its boundary and i lattice points in its interior.

Proposition 2.2. *A lattice M in \mathbb{R}^2 has exactly $p+1$ different sublattices of index a prime number p . These sublattices are in one-to-one correspondence with the set of lines in M/pM representing all elements of the projective line over the finite field \mathbb{F}_p .*

Proof. A sublattice Λ of prime index p in a 2-dimensional lattice M gives rise to a quotient group M/Λ isomorphic to the additive group $\mathbb{Z}/p\mathbb{Z}$. Since pM is contained in the kernel of the quotient map $M \mapsto M/\Lambda$, subgroups of index p in M are in bijection with kernels of linear surjections from the 2-dimensional vector-space M/pM over \mathbb{F}_p onto \mathbb{F}_p considered as a 1-dimensional vector-space. The set of all subgroups of index p in M is thus in bijection with the set of 1-dimensional subspaces in \mathbb{F}_p^2 representing all possible kernels. Such subspaces represent all $p+1$ points of the projective line over \mathbb{F}_p . \square

The Euclidean algorithm computes the positive generator of the cyclic subgroup $\mathbb{Z}a + \mathbb{Z}b$ of \mathbb{Z} generated by two integers a and b . *Gaussian lattice reduction* does essentially the same for lattices in the Euclidean space \mathbb{R}^2 : Given two linearly independent³ vectors e, f in \mathbb{R}^2 , the Gaussian algorithm produces a *reduced basis* r, s of the lattice $\mathbb{Z}e + \mathbb{Z}f = \mathbb{Z}r + \mathbb{Z}s$ defining two (not necessarily unique) distinct shortest pairs $\pm r, \pm s$ of primitive vectors. The Gaussian algorithm starts with a basis e, f of $\Lambda = \mathbb{Z}e + \mathbb{Z}f$ and iterates the following two steps until stabilization:

- Exchange e and f if f is strictly longer than e .
- Replace e by $e + kf$ if $e + kf$ (for k an integer) is strictly shorter than e (the optimal choice for k is given by $k \in \mathbb{Z}$ such that $\left| k + \frac{\langle f, e \rangle}{\langle f, f \rangle} \right|$ is at most equal to $\frac{1}{2}$).

Finally, the following (obvious) result will also be needed a few times:

Proposition 2.3. *The sublattice $\mathbb{Z}(\alpha e + \beta f) + \mathbb{Z}(\gamma e + \delta f)$ of a lattice $\Lambda = \mathbb{Z}e + \mathbb{Z}f$ generated by two linearly independent vectors $\alpha e + \beta f$ and $\gamma e + \delta f$ (with $\alpha, \beta, \gamma, \delta$ in \mathbb{Z}) has index $|\alpha\delta - \beta\gamma|$ in Λ .*

Proof. The result is obvious if $\alpha\beta\gamma\delta = 0$. The general case can be reduced by elementary operations on the generators $u = \alpha e + \beta f$ and $v = \gamma e + \delta f$ to the obvious case. \square

³Gaussian lattice reduction applied to two linearly dependent vectors boils down to the Euclidean algorithm.

3 Grace's proof

For the convenience of the reader, we recall Grace's proof (of Corollary 1.2), as given in [7] (see the fourth proof of Theorem 366 in [7] or [6])⁴.

Proof. Given an odd prime number p we have $\left(\left(\frac{p-1}{2}\right)!\right)^2 (-1)^{(p-1)/2} \equiv (p-1)! \pmod{p}$ which equals $-1 \pmod{p}$ by Wilson's Theorem. For p a prime number congruent to $1 \pmod{4}$, the integer $\iota = \left(\frac{p-1}{2}\right)!$ (and its opposite) is a square root of -1 in the finite field \mathbb{F}_p . The kernel of the homomorphism $\mathbb{Z}^2 \ni (x, y) \mapsto x + \iota y \pmod{p}$ is a sublattice Λ of index p in \mathbb{Z}^2 . Since $\iota(x + \iota y) \equiv -y + \iota x \pmod{p}$, we have $(x, y) \in \Lambda$ if and only if $(-y, x) \in \Lambda$. The lattice Λ is therefore invariant under quarter-turns (rotations of order 4 by ± 90 degrees). Let (a, b) be a non-zero element of minimal length in Λ . Invariance under quarter-turns of Λ implies that $(-b, a)$ is also an element of Λ . Length-minimality of (a, b) and $(-b, a)$ implies that the triangle with vertices $(0, 0), (a, b), (-b, a)$ contains no other element of Λ . Lemma 2.1 shows that the two vectors (a, b) and $(-b, a)$ generate Λ . Proposition 2.3 implies now that $a^2 + b^2$ is equal to the index p of the sublattice Λ in \mathbb{Z}^2 . \square

Grace's proof is effective: Given a prime $p \equiv 1 \pmod{4}$, we can use quadratic reciprocity for finding a non-square n modulo p . We obtain a square root ι of -1 in \mathbb{F}_p by computing $\iota \equiv n^{(p-1)/4} \pmod{p}$ using fast exponentiation. We get now a solution by considering an element (a, b) of a reduced basis (obtained by Gaußian lattice reduction) of the lattice generated by $(p, 0)$ and $(-\iota, 1)$.

4 Interlacedness

Definition 4.1. *Two unordered bases f_1, f_2 and g_1, g_2 of \mathbb{R}^2 are interlaced if f_1, f_2, g_1, g_2 represent four distinct points of the real projective line such that the two projective points represented by f_1, f_2 separate the two projective points represented by g_1, g_2 .*

Interlacedness can be defined equivalently as follows: Colour the two lines $\mathbb{R}f_i$ associated to the first basis f_1, f_2 fuchsia and colour the two lines $\mathbb{R}g_i$ green. Then the set $\mathbb{R}f_1 \cup \mathbb{R}f_2 \cup \mathbb{R}g_1 \cup \mathbb{R}g_2$ should contain four different lines and colours should alternate.

Example 4.2. *The standard basis $(1, 0), (0, 1)$ of \mathbb{R}^2 is interlaced with the basis $(-1, 2), (6, 1)$. The standard basis $(1, 0), (0, 1)$ is however not interlaced with the basis $(2, 3), (-1, -1)$.*

⁴The only difference is the fact that Grace admits the existence of square roots of -1 modulo primes congruent to $1 \pmod{4}$. The author succumbed to the siren song of a well-known explicit construction (based on Wilson's Theorem) for such a square-root.

Lemma 4.3. *Two bases f_1, f_2 and g_1, g_2 of a 2-dimensional lattice $\Lambda = \mathbb{Z}f_1 + \mathbb{Z}f_2 = \mathbb{Z}g_1 + \mathbb{Z}g_2$ are never interlaced.*

Proof. Suppose that a lattice $\Lambda = \mathbb{Z}f_1 + \mathbb{Z}f_2 = \mathbb{Z}g_1 + \mathbb{Z}g_2$ has two bases f_1, f_2 and g_1, g_2 which are interlaced. After replacing f_1 and f_2 by their negatives if necessary and perhaps after exchanging f_1 with f_2 we can suppose that f_1 belongs to the open cone spanned by g_1 and $-g_2$ and f_2 belongs to the open cone spanned by g_1 and g_2 . Since we are working with bases of a lattice Λ , there exist strictly positive integers $\alpha, \beta, \gamma, \delta$ such that $f_1 = \alpha g_1 - \beta g_2$ and $f_2 = \gamma g_1 + \delta g_2$ which can be rewritten as $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$.

Proposition 2.3 applied to the inequality $\det \begin{pmatrix} \alpha & -\beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta + \beta\gamma \geq 2$ implies now that $\mathbb{Z}f_1 + \mathbb{Z}f_2$ is a strict sublattice of index at least 2 in $\Lambda = \mathbb{Z}g_1 + \mathbb{Z}g_2$. \square

5 Proof of Theorem 1.1

We consider the eight open cones of \mathbb{R}^2 delimited by the four lines $x = 0$, $y = 0$, $x = y$ and $x = -y$. We call these eight open cones *windmill cones* and we colour them alternately black and white, starting with a black E-NE windmill cone $\{(x, y) \mid 0 < y < x\}$ (using the conventions of wind-roses), as illustrated in Figure 2.

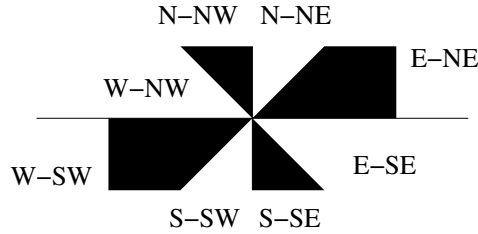


Figure 2: The four black windmill cones E-NE, N-NW, W-SW, S-SE and the four white windmill cones N-NE, W-NW, S-SW, E-SE.

A basis e, f of \mathbb{R}^2 is a *black windmill basis* if e and f are contained in the open upper half-plane and if one element in $\{e, f\}$ belongs to the open black E-NE windmill cone and the other element in $\{e, f\}$ belongs to the open black N-NW windmill cone. Similarly, a *white windmill basis* contains an element in the open white N-NE windmill cone and an element in the open white W-NW windmill cone.

A 2-dimensional lattice Λ in \mathbb{R}^2 has a black (respectively white) windmill basis if $\Lambda = \mathbb{Z}e + \mathbb{Z}f$ is generated by a black (respectively white) windmill basis e, f .

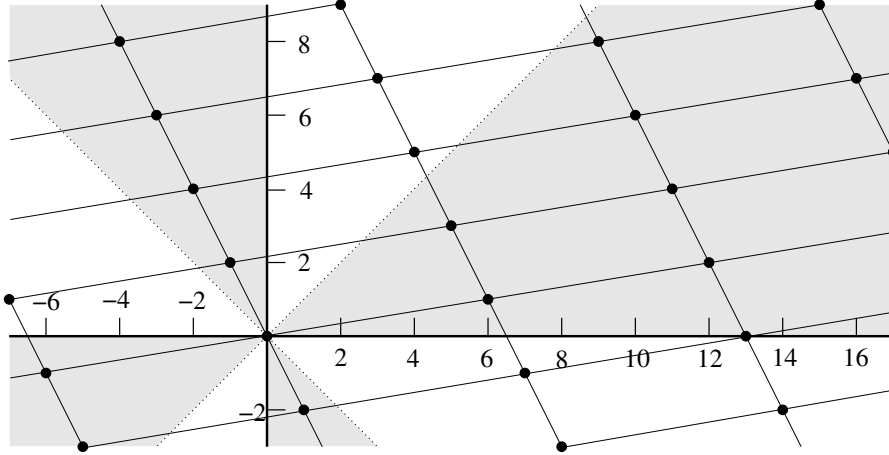


Figure 3: The lattice $\Lambda = \mathbb{Z}(-1, 2) + \mathbb{Z}(6, 1)$.

We illustrate the notion of windmill bases with the following example, henceforth used as a running example: Consider the lattice

$$\Lambda = \{(x, y) \in \mathbb{Z}^2 \mid x + 7y \equiv 0 \pmod{13}\} \quad (2)$$

depicted in Figure 3 with shaded black windmill cones. The lattice Λ has two black windmill bases given by $(-1, 2), (5, 3)$ and by $(-1, 2), (6, 1)$.

The basis $(-1, 2), (-6, -1)$ is not a black windmill basis: $(-6, -1)$ does not belong to the upper half-plane.

The basis $(5, 3), (6, 1)$ is not a windmill basis: Both basis vectors are contained in the open black E-NE windmill cone.

The basis $(-1, 2), (4, 5)$ is not a windmill basis: $(4, 5)$ belongs to the open white N-NE windmill cone and $(-1, 2)$ belongs to the open black N-NW windmill cone.

Lemma 5.1. *All windmill bases of a lattice have the same colour.*

Proof. Otherwise we get a contradiction with Lemma 4.3 since windmill bases of different colours are obviously interlaced. \square

An odd prime number p and an element μ of \mathbb{F}_p (henceforth often identified with $\{0, \dots, p-1\}$) define a sub-lattice

$$\Lambda_\mu(p) = \{(x, y) \in \mathbb{Z}^2 \mid x + \mu y \equiv 0 \pmod{p}\} \quad (3)$$

of index p in \mathbb{Z}^2 . We set $\Lambda_\infty(p) = \{(x, y) \in \mathbb{Z}^2 \mid y \equiv 0 \pmod{p}\}$. We have thus $\Lambda = \Lambda_7(13)$ for our running example given by (2). All $p+1$ lattices $\Lambda_0(p), \dots, \Lambda_{p-1}(p), \Lambda_\infty(p)$ are distinct and \mathbb{Z}^2 contains no other sublattices of prime index p , see Proposition 2.2.

Proposition 5.2. *The four lattices $\Lambda_0(p), \Lambda_\infty(p), \Lambda_1(p), \Lambda_{p-1}(p)$ have no windmill basis.*

Proof. Each of these four lattices is invariant under an orthogonal reflection with respect to a line separating black and white windmill cones. Such orthogonal reflections, followed by obvious sign changes, exchange white and black windmill bases. Lemma 5.1 shows therefore non-existence of (black or white) windmill bases for these lattices. \square

Proposition 5.2 is optimal by the following result:

Proposition 5.3. *Every lattice $\Lambda_\mu(p)$ with $2 \leq \mu \leq p-2$ has a windmill basis.*

Proof. $\Lambda_\mu(p)$ contains obviously no elements of the form $(\pm m, 0)$ or $(\pm m, \pm p)$ with m in $\{1, 2, \dots, p-1\}$. Since p is prime, $\Lambda_\mu(p)$ contains no elements of the form $(0, \pm m), (\pm p, \pm m)$ with m in $\{1, \dots, p-1\}$. Moreover, for μ in $\{2, \dots, p-2\}$ considered as a subset of the finite field \mathbb{F}_p , the elements $1 + \mu$ and $1 - \mu$ are invertible in \mathbb{F}_p . This implies that $\Lambda_\mu(p)$ has no elements of the form $\pm(m, m), \pm(m, -m)$ for m in $\{1, \dots, p-1\}$. The intersection of a (black or white) windmill cone with $[-p, p]^2$ defines therefore a triangle of area $p^2/2 > p/2$ whose boundary contains no lattice-points of $\Lambda_\mu(p)$ except for its three vertices. Lemma 2.1 implies now that every open (black or white) windmill cone contains a non-zero element (x, y) of $\Lambda_\mu(p)$ with coordinates x, y in $\{\pm 1, \pm 2, \dots, \pm(p-1)\}$.

Thus there exists a parallelogram \mathcal{P} of minimal area with vertices $\pm e, \pm f$ in $\Lambda_\mu(p) \cap \{-p+1, \dots, p-1\}^2$ such that $\{\pm e, \pm f\}$ intersects either all four open black windmill cones or all four open white windmill cones.

Suppose for simplicity that all elements of $\{\pm e, \pm f\}$ are black (i.e. belong to open black windmill cones). (The case where both pairs $\pm e$ and $\pm f$ are white is analogous.)

Since $\Lambda_\mu(p)$ intersects the diagonal $\mathbb{R}(1, 1)$ and the antidiagonal $\mathbb{R}(1, -1)$ in $\mathbb{Z}(p, p)$ and in $\mathbb{Z}(p, -p)$, and since $\Lambda_\mu(p)$ contains obviously no elements of the form $(\pm a, 0), (0, \pm a)$ with a in $\{1, \dots, p-1\}$, all non-zero elements of $\mathcal{P} \cap \Lambda_\mu(p)$ belong to open windmill cones. Suppose that $\mathcal{P} \setminus \{\pm e, \pm f\}$ contains a non-zero element g of $\Lambda_\mu(p)$. Area-minimality of \mathcal{P} and the absence of non-zero elements in $\Lambda_\mu(p) \cap (\mathbb{Z}(1, 1) \cup \mathbb{Z}(1, -1)) \cap \{-p+1, \dots, p-1\}^2$ shows that g is contained in a white windmill cone (under the assumption that e and f are black). After replacing g by $-g$ if necessary, the element g belongs either to the triangle with vertices $(0, 0), e, f$ or to the triangle with vertices $(0, 0), e, -f$. Lemma 2.1 applied to the two sets e, f and $e, -f$ generating the same sublattice $\mathbb{Z}e + \mathbb{Z}f$ of $\Lambda_\mu(p)$ implies thus the existence of a non-zero element h in $\mathcal{P} \cap \Lambda_\mu(p)$ such that $\{\pm g, \pm h\}$ intersects all four open white windmill cones. The parallelogram with vertices $\pm g, \pm h$ in all four open white windmill cones is therefore strictly included in \mathcal{P} in contradiction with area-minimality of \mathcal{P} .

After replacing each of g and f by its negative if necessary, we get that e, f is a windmill basis of $\Lambda_\mu(p)$ by Lemma 2.1. \square

Lemma 5.4. *Let Λ be a lattice with two distinct windmill bases e, f and e, g sharing a common element e . Then Λ has a unique pair of minimal elements given by $\pm e$.*

Proof. Since both ordered bases e, f and e, g start with e and are windmill bases, they induce the same orientation and we have $g = f + ke$ for some non-zero integer k in \mathbb{Z} . The affine line $\mathcal{L} = f + \mathbb{R}e$ intersects therefore the open windmill cone \mathcal{C}_f containing f and g in an open segment of length $l > \sqrt{\langle e, e \rangle}$. Denoting by d the distance of \mathcal{L} to the origin $(0, 0)$ and by α the angle in $(0, \pi/4)$ between the normal line $(\mathbb{R}e)^\perp$ (with $(\mathbb{R}e)^\perp \setminus \{(0, 0)\}$ contained in $\mathcal{C}_f \cup (-\mathcal{C}_f)$) of \mathcal{L} and a boundary line of \mathcal{C}_f (separating \mathcal{C}_f from a windmill cone of the opposite colour) we have the inequalities

$$\begin{aligned} \sqrt{\langle e, e \rangle} &< l \\ &= d(\tan \alpha + \tan(\pi/4 - \alpha)) \\ &= (1 - \tan \alpha \tan(\pi/4 - \alpha))d \tan(\pi/4) \\ &< d \end{aligned}$$

where we have used the addition formula $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ of the tangent function.

The open strip delimited by the two parallel affine lines \mathcal{L} and $-\mathcal{L}$ and consisting of all points at distance $< d$ from $\mathbb{R}e$ intersects Λ in $\mathbb{Z}e$. All elements of $\Lambda \setminus \mathbb{Z}e$ are therefore at least at distance $d > \sqrt{\langle e, e \rangle}$ from the origin. This shows that $\pm e$ is the unique pair of minimal vectors in Λ . \square

Lemma 5.5. *Two windmill bases of a lattice are never disjoint.*

Proof. Up to replacing a lattice Λ having two disjoint windmill bases by its orthogonal reflection $\sigma(\Lambda)$ with respect to the vertical line $x = 0$, we can assume otherwise that Λ has two disjoint black windmill bases (see Lemma 5.1) given by e, f and g, h with e, g in the open black E-NE windmill cone and f, h in the open black N-NW windmill cone. Since the two bases are not interlaced by Lemma 4.3, we can moreover assume that g and h belong both to the open cone spanned by e, f . If the sum $e + f$ belongs to the open cone spanned by g and h we get two interlaced bases $e, e + f$ and g, h in contradiction with Lemma 4.3.

The non-zero lattice element $e + f$ belongs therefore either to the closed cone spanned by e, g or to the closed cone spanned by f, h . In the first case ($e + f$ in the closed cone spanned by e, g) the element $e + f$ belongs to the open black E-NE windmill cone containing f and g , see Figure 4 for a schematic illustration. (We discuss here only the first case. The second case where $e + f$ belongs to the closed cone spanned by f and h reduces

to the first cases after a quarter-turn of the lattice Λ followed by obvious relabellings and sign-changes. It can also be treated by an easy adaption of the arguments used in the first case.) Since the affine line $\mathcal{L} = e + \mathbb{R}f$

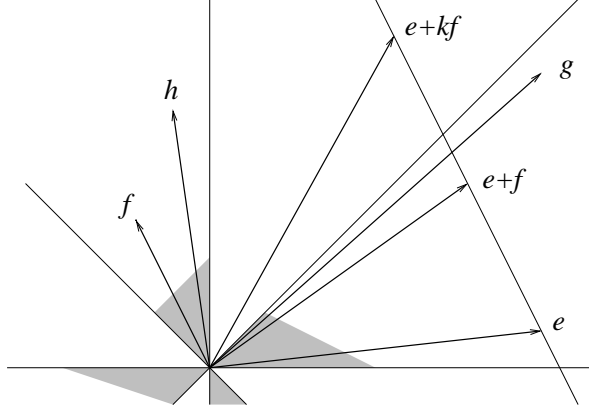


Figure 4: A schematic figure with $e + f$ in the E-NE windmill cone.

has a downward slope strictly steeper than -1 , the intersection of \mathcal{L} with the open white N-NE windmill cone is strictly longer than the intersection of \mathcal{L} with the open black E-NE windmill cone. Since the intersection of \mathcal{L} with the open black E-NE windmill cone contains at least the two elements e and $e + f$ of Λ there exists a natural integer $k > 1$ such that the element $e + kf$ of Λ belongs to the open white N-NE windmill cone. This leads to two interlaced bases $f, e + kf$ and g, h in contradiction with Lemma 4.3. \square

Proposition 5.6. *The following assertions hold if a lattice Λ has at least two windmill bases:*

- Λ has a unique pair $\pm m$ of minimal vectors with m contained in all windmill bases of Λ .
- Λ has only finitely many windmill bases. More precisely, there exists a windmill basis m, f of Λ such that all windmill bases of Λ are given by $m, f + sm$ for s in $\{0, 1, \dots, k - 1\}$ where k is the number of windmill bases of Λ .

Minimal vectors do not necessarily intersect the set $\{e, f\}$ in the case of a lattice Λ with a unique windmill basis e, f .

Both black windmill bases $(-1, 2), (5, 3)$ and $(-1, 2), (6, 1)$ of our running example Λ defined by (2) contain the minimal element $m = (-1, 2)$ of Λ . The running example gives $f = (6, 1)$ and $k = 2$ in the last assertion.

Proof of Proposition 5.6. Lemma 5.5 shows that two windmill bases of Λ intersect in a common element m defining the unique pair $\pm m$ of minimal

vectors of Λ by Lemma 5.4. Thus all windmill bases of Λ are all pairs m, g with g in the set $(f + \mathbb{R}m) \cap \mathcal{C}_m^\perp \cap \Lambda$ where m, f is an arbitrary windmill basis and where \mathcal{C}_m^\perp is the open windmill cone containing f perpendicular to the open windmill cone \mathcal{C}_m defined by m .

Since the vector m does not belong to \mathcal{C}_m^\perp the intersection of the affine line $\mathcal{L} = f + \mathbb{R}m$ with \mathcal{C}_m^\perp is an open interval of bounded length and the set $\mathcal{G} = \mathcal{L} \cap \mathcal{C}_m^\perp \cap \Lambda$ is finite. We replace now f by the element g of \mathcal{G} minimizing the scalar product with m in order to get the result. \square

We call a black windmill basis u, v of a lattice $\Lambda_\mu(p)$ (with μ in $\{2, \dots, p-2\}$) *standard* if $u = (a, c), v = (-d, b)$ with $a, b, c, d \in \mathbb{N}$ such that $\min(a, b) > \max(c, d)$.

The basis $u = (6, 1), v = (-1, 2)$ of our running example (2) is a standard black windmill basis. The inequality $3 \geq 2$ implies that its second black windmill basis $(5, 3), (-1, 2)$ is not standard.

Proposition 5.7. *Given an odd prime number p , a lattice $\Lambda_\mu(p)$ with μ in $\{2, \dots, p-2\}$ has either only white windmill bases or it has a unique standard black windmill basis.*

Proof. Proposition 5.3 shows that such a lattice $\Lambda = \Lambda_\mu(p)$ has windmill bases. They are all of the same colour by Lemma 5.1. We assume now that all windmill bases of Λ are black. A vector w in Λ is of *windmill type* if there exists a windmill basis containing w .

We denote by $u = (a, c) \in \Lambda$ the lowest vector of windmill type in the open black E-NE windmill cone and we denote by $v = (-d, b) \in \Lambda$ the rightmost vector of windmill type in the open black N-NE windmill cone. The vectors u, v form a black windmill basis of Λ : This is obvious if Λ has a unique windmill basis and it follows from the description of all windmill bases given by Proposition 5.6 otherwise.

For our running example (2) we get $u = (6, 1)$ and $v = (-1, 2)$.

We claim that u, v is a standard black windmill basis of Λ : We have $a > c$ since $u = (a, c)$ belongs to the open black E-NE windmill cone and $b > d$ since $v = (-d, b)$ belongs to the open black N-NW windmill cone.

Since $u - v = (a + d, c - b)$ is lower than u , the basis $u - v = (a + d, c - b), v = (-d, b)$ of Λ is not a windmill basis and we have therefore $b \geq c$. If $b = c$, the vectors $u - v = (a + d, 0), v = (-d, b)$ are a basis of Λ . Since Λ intersects $\mathbb{Z}(1, 0)$ in $\mathbb{Z}(p, 0)$ we have $a + d = p$ which implies $u - v = (p, 0)$. Since $u - v = (p, 0)$ and $v = (-d, b)$ is a basis of Λ we get $b = 1$ in contradiction with the inequalities $1 \leq d < b$. We have thus $b > c$.

Similarly, since $v + u = (a - d, b + c)$ is at the right of v , the basis $u = (a, c), v + u = (a - d, b + c)$ of Λ is not a windmill basis and we have $a \geq d$. If $a = d$, the vectors $u = (a, c), v + u = (0, b + c)$ are a basis of Λ . This implies $b + c = p$ and $a = 1$ contradicting the inequalities $1 \leq c < a$. This shows $a > d$.

Unicity follows easily from the description of all windmill bases given by the last assertion of Proposition 5.6. \square

Proof of Theorem 1.1. Given an odd prime number p , we denote by \mathcal{S}_p the set of all solutions (a, b, c, d) as defined by Theorem 1.1.

We associate to a solution (a, b, c, d) in \mathcal{S}_p the two vectors $u = (a, c)$, $v = (-d, b)$ and we consider the sublattice $\Lambda = \mathbb{Z}u + \mathbb{Z}v$ of index $p = ab - c(-d)$ in \mathbb{Z}^2 generated by u and v . Since p is prime, there are exactly two solutions with $cd = 0$, given by $(p, 1, 0, 0)$ and $(1, p, 0, 0)$ corresponding to the lattices $\mathbb{Z}(p, 0) + \mathbb{Z}(0, 1)$ and $\mathbb{Z}(1, 0) + \mathbb{Z}(0, p)$.

We suppose henceforth $cd > 0$. The vectors u and v are then contained respectively in the open black E-NE and in the open black N-NW windmill cone and form therefore a standard black windmill basis of the lattice Λ .

Sub-lattices of prime-index p in \mathbb{Z}^2 are in bijection with the set of all $p+1$ points on the projective line $\mathbb{P}^1\mathbb{F}_p$ over the finite field \mathbb{F}_p , cf. Proposition 2.2. More precisely, a point $[a : b]$ of the projective line defines the lattice

$$\Lambda_{[a:b]} = \{(x, y) \in \mathbb{Z}^2 \mid ax + by \equiv 0 \pmod{p}\}$$

which is equal to the lattice $\Lambda_\mu(p)$ defined by (3) for $\mu \equiv b/a \pmod{p}$ using obvious conventions. We have already considered lattices associated to the two solutions with $cd = 0$. By Proposition 5.2, the two lattices given by $\mu \equiv \pm 1 \pmod{p}$ have no windmill basis and yield thus no solutions. All $(p-3)$ lattices $\Lambda_\mu(p)$ with $\mu \in \{2, \dots, p-2\}$ have windmill bases by Proposition 5.3.

Since $\Lambda_\mu(p)$ and $\Lambda_{p-\mu}(p)$ (respectively $\Lambda_{\mu^{-1} \pmod{p}}(p)$) differ by a horizontal (respectively diagonal) reflection, they have windmill bases of different colours. Proposition 5.7 shows that exactly $(p-3)/2$ values of μ in $\{2, \dots, p-2\}$ correspond to lattices $\Lambda_\mu(p)$ with unique standard black windmill bases. These $(p-3)/2$ lattice are therefore in one-to-one correspondence with solutions in (a, b, c, d) in \mathcal{S}_p such that $cd > 0$. Taking into account the two degenerate solutions $p \cdot 1 + 0 \cdot 0$ and $1 \cdot p + 0 \cdot 0$, we get a total number of $(p-3)/2 + 2 = (p+1)/2$ solutions in \mathcal{S}_p . \square

6 Complement: Voronoi cells and windmill bases

A lattice Λ induces a *Voronoi* diagram tiling the plane \mathbb{R}^2 with *Voronoi cell* bounded by points of \mathbb{R}^2 having more than a unique closest lattice point. Points at locally maximal distance to Λ are vertices of the Voronoi diagram for Λ , see for example the monograph [3] for more on Voronoi cells of lattices and sphere packings.

We denote by \mathcal{C}_V the Voronoi cell consisting of all points of \mathbb{R}^2 with closest lattice-point the trivial element $(0, 0)$ of Λ . The plane \mathbb{R}^2 is tiled by Λ -translates of \mathcal{C}_V which is a rectangle if Λ has a reduced basis of two

orthogonal vectors and which is a centrally symmetric hexagon otherwise. A reduced basis e, f defines normal vectors to two pairs of parallel sides of \mathcal{C}_V . A normal vector g to the remaining pair of sides in the hexagonal case is given by $e - \epsilon f$ where $\epsilon = \frac{\langle e, f \rangle}{|\langle e, f \rangle|}$ is the sign of the scalar product $\langle e, f \rangle$ between e and f . We call the set $\{\pm e, \pm f\}$, respectively $\{\pm e, \pm f, \pm g\}$ of primitive lattice elements normal to sides of \mathcal{C}_V the set of *Voronoi vectors*. The Voronoi cell \mathcal{C}_V is defined by the inequalities $2\langle x, v \rangle \leq \langle v, v \rangle$ for all elements v of the set \mathcal{V} of Voronoi vectors. Two linearly independent Voronoi vectors u, v in \mathcal{V} generate Λ .

We illustrate these notions on our running example $\Lambda = \mathbb{Z}(-1, 2) + \mathbb{Z}(6, 1)$ defined by (3): A reduced basis for Λ is given by the minimal vector $e = (-1, 2)$ and by $f = (5, 3)$. Voronoi vectors are given by $\{\pm e, \pm f, \pm g\}$ for $g = f - e = (6, 1)$. The lattice Λ has two black windmill bases given by $\{e, f\}$ and $\{e, g\}$. The last basis $\{e, g\}$ is the standard basis of Λ . The Voronoi cell \mathcal{C}_V of Λ is the hexagon with vertices $\pm(53/26, 59/26), \pm(77/26, 19/26), \pm(79/26, 7/26)$.

Voronoi vectors are related to windmill bases by the following result:

Proposition 6.1. *Every lattice Λ in \mathbb{R}^2 with windmill bases has a windmill basis contained in its set of Voronoi vectors.*

Proposition 6.1 (together with Proposition 5.6 and Gaussian lattice reduction) gives a fast algorithm (using $O(\log p)$ operations on integers not exceeding p) for computing the solution of \mathcal{S}_p associated to $\Lambda_{\pm\mu}(p)$ for μ in $\mathbb{F}_p \setminus \{0, \pm 1\}$: Compute a reduced basis of $\Lambda_\mu(p)$ and use it to construct the associated set \mathcal{V} of Voronoi vectors which contains a windmill basis by Proposition 5.3 and Proposition 6.1. If the windmill basis is white, replace $\Lambda_\mu(p)$ by $\Lambda_{-\mu}(p)$ (using for example the vertical reflection $\sigma(x, y) = (-x, y)$ of \mathbb{R}^2) in order to get a black windmill basis of $\Lambda_{-\mu}(p)$. Use now Proposition 5.6 for constructing the unique standard basis $(a, c), (-d, b)$ encoding the solution $p = ab + cd$ of \mathcal{S}_p .

The main tool for proving Proposition 6.1 is the following result which is perhaps of independent interest:

Lemma 6.2. *Assume that the set \mathcal{V} of Voronoi vectors of a lattice Λ does not intersect the set of boundary lines separating black and white windmill cones. Then Λ has a windmill basis contained in \mathcal{V} .*

Proof of Lemma 6.2. We suppose first that the Voronoi domain of Λ is a rectangle. This implies $\mathcal{V} = \{\pm e, \pm f\}$ with e and f two elements of open windmill cones in the upper halfplane forming a reduced orthogonal basis of Λ . Since e and f are orthogonal they belong to two distinct open windmill cones of the same colour. They form therefore a windmill basis.

We consider now $\mathcal{V} = \{\pm e, \pm f, \pm g\}$ with e, f, g in open windmill cones of the upper half-plane. We assume moreover that e is a (perhaps not unique) minimal vector of Λ . Minimality of e and the inequalities $|\langle e, f \rangle| < \langle e, e \rangle$

and $|\langle e, g \rangle| < \langle e, e \rangle$ imply that the line $\mathbb{R}e$ crosses both lines $\mathbb{R}f$ and $\mathbb{R}g$ with angles strictly larger than $\pi/4$. The open windmill cone \mathcal{C}_e containing e has an opening angle of $\pi/4$ and is therefore distinct from the (not necessarily distinct) open windmill cones \mathcal{C}_f and \mathcal{C}_g containing f , respectively g . We get a windmill basis e, h for h in $\{f, g\}$ such that \mathcal{C}_e and \mathcal{C}_h are of the same colour. If such an element h does not exist, then \mathcal{C}_f and \mathcal{C}_g have the same colour opposite to the colour of \mathcal{C}_e . Since f and g are either separated by the line $\mathbb{R}e$ or by its orthogonal $(\mathbb{R}e)^\perp$ (with $(\mathbb{R}e)^\perp \setminus \{(0, 0)\}$ contained in the two open windmill cones orthogonal to \mathcal{C}_e and of the same colour as \mathcal{C}_e), the elements f and g of the upper half-plane belong to different open windmill cones of the same colour and form therefore a windmill basis. \square

Proof of Proposition 6.1. The result holds by Lemma 6.2 if \mathcal{V} contains no elements on boundary lines separating black and white windmill cones.

Otherwise, if $\mathcal{V} = \{\pm e, \pm f\}$ is reduced to two pairs of orthogonal elements, then e, f are both elements in the boundary of black and white windmill cones. The reflection $ae + bf \mapsto ae - bf$ induces therefore a lattice isomorphism of Λ which exchanges colours of windmill cones. Such a lattice has no windmill bases by Lemma 5.1.

We suppose now that Λ has a windmill basis u, v not contained in the set $\mathcal{V} = \{\pm e, \pm f, \pm g\}$ of Voronoi vectors, with e, f, g in the closed upper half-plane. A sufficiently small rotation ρ (suitably chosen if $\{e, f, g\}$ intersects the horizontal line $y = 0$) sends the windmill basis u, v of Λ to a windmill basis (of the same colour) $\rho(u), \rho(v)$ of $\rho(\Lambda)$ and sends \mathcal{V} to a set $\rho(\mathcal{V})$ of Voronoi vectors having no elements on boundary lines separating windmill cones. Lemma 6.2 shows that $\rho(\Lambda)$ has an additional windmill basis contained in $\rho(\mathcal{V})$ distinct from the windmill basis $\rho(u), \rho(v)$ not contained in $\rho(\mathcal{V})$. Proposition 5.6 implies therefore that $\rho(\Lambda)$ has a unique pair $\pm\rho(m)$ of minimal vectors intersecting every windmill basis of $\rho(\Lambda)$. We can therefore assume (perhaps after a permutation among the elements e, f, g) that e is the unique minimal element in the upper half-plane of Λ and that e is contained in every windmill basis of Λ . Since $\pm f$ and $\pm g$ are the elements of $\pm f + \mathbb{Z}e$ which are closest to the line $(\mathbb{R}e)^\perp$ orthogonal to $\mathbb{R}e$, the last assertion of Proposition 5.6 implies that either e, f or e, g is a windmill basis of Λ . \square

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