Linear and nonlinear parabolic forward-backward problems

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Abstract

The purpose of this paper is to investigate the well-posedness of several linear and nonlinear equations with a parabolic forward-backward structure, and to highlight the similarities and differences between them. The epitomal linear example will be the stationary Kolmogorov equation $y\partial_x u - \partial_{yy} u = f$, which we investigate in a rectangle $(x_0, x_1) \times (-1, 1)$, supplemented with boundary conditions on the "parabolic boundary" of the domain: the top and lower boundaries $\{y = \pm 1\}$, and the lateral boundaries $\{x_0\} \times (0, 1)$ and $\{x_1\} \times (-1, 0)$. We first prove that this equation admits a finite number of singular solutions associated with regular data. These singular solutions, of which we provide an explicit construction, are localized in the vicinity of the points $(x_0, 0)$ and $(x_1, 0)$. Hence, the solutions to the Kolmogorov equation associated with a smooth source term f are regular if and only if f satisfies a finite number of orthogonality conditions. This is similar to well-known phenomena in elliptic problems in polygonal domains.

We then extend this theory to the Vlasov–Poisson–Fokker–Planck system $y\partial_x u + E[u]\partial_y u - \partial_{yy} u = f$, and to two quasilinear equations: the Burgers type equation $u\partial_x u - \partial_{yy} u = f$ in the vicinity of the linear shear flow, and the Prandtl system in the vicinity of a recirculating solution, close to the curve where the horizontal velocity changes sign. We therefore revisit part of a recent work by Iyer and Masmoudi [34, 35]. For the two latter quasilinear equations, we introduce a geometric change of variables which simplifies the analysis. In these new variables, the linear differential operator is very close to the Kolmogorov operator $y\partial_x - \partial_{yy}$. Stepping on the linear theory, we prove existence and uniqueness of regular solutions for data within a manifold of finite codimension, corresponding to some nonlinear orthogonality conditions.

Treating these three nonlinear problems in a unified way also allows us to compare their structures. In particular, we show that the vorticity formulation of the Prandtl system, in an adequate set of variables, is very similar to the Burgers one. As a consequence, solutions of the Prandtl system are actually smoother than the ones of Burgers, which allows us to have a theory of weak solutions of the Prandtl system close to the recirculation zone.

[*]Sorbonne Université, Université Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, Paris
[†]École Normale Supérieure, Université PSL, Département de Mathématiques et applications, Paris
[‡]Univ Rennes, CNRS, IRMAR - UMR 6625, Rennes

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1 Introduction

This manuscript is devoted to the well-posedness of linear and nonlinear equations having a parabolic forward-backward structure. In the linear case, our main example will be the Kolmogorov equation $y\partial_x u - \partial_{yy} u = f$ in the rectangular domain $\Omega := (x_0, x_1) \times (-1, 1)$ where $x_0 < x_1$ and f is an external source term. We will also consider nonlinear perturbations of this linear setting. The easiest nonlinear perturbation we consider is a Vlasov–Poisson–Fokker–Planck type system of the form $y\partial_x u + E[u]\partial_y u - \partial_{yy} u = f$, where $E[u] = \partial_x^{-1} \int u \, dy$. In this case the nonlinearity does not perturb the geometry of the problem, which remains forward parabolic in the region $y > 0$, and backward parabolic in the region $y < 0$.

We will also investigate the existence and uniqueness of sign-changing solutions to the Burgers type equation

$$u\partial_x u - \partial_{yy} u = f \tag{1.1}$$

and to the Prandtl system

$$\begin{aligned} u\partial_x u + v\partial_y u - \partial_{yy} u &= -\partial_x p, \\ u_x + v_y &= 0. \end{aligned} \tag{1.2}$$

A natural solution to (1.1) with a null source term $\mathbb{f} = 0$ is the linear shear flow $\mathbb{u}(x, y) := y$, which changes sign across the horizontal line $\{y = 0\}$. In a similar way, semi-explicit solutions $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$ of the Prandtl system (1.2) such that $\mathbb{u}_{\mathrm{P}}$ changes sign have been exhibited, see the discussion in Section 1.3 below. We are interested in strong solutions to (1.1) (resp. (1.2)) which are close in an appropriate norm to this linear shear flow $\mathbb{u}$ (resp. to the reference solution $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$). Our purpose is to construct such solutions by perturbing the lateral boundary data or the source term.

Since such solutions will change sign across a curve $\{u = 0\}$ lying within Ω, a key feature of this work is that (1.1) and (1.2) must be seen as quasilinear forward-backward parabolic problems in the horizontal direction. Thus, to ensure the existence of a solution, one must be particularly careful as to how one enforces the lateral perturbations. More precisely, the problem is forward parabolic in the domain above the curve $\{u = 0\}$, in which $u > 0$, and therefore we shall prescribe a boundary condition on $\Sigma_0 := \{x = x_0\} \cap \{u > 0\}$; and backward parabolic in the domain below the curve $\{u = 0\}$, and we shall prescribe a boundary condition on $\Sigma_1 := \{x = x_1\} \cap \{u < 0\}$.

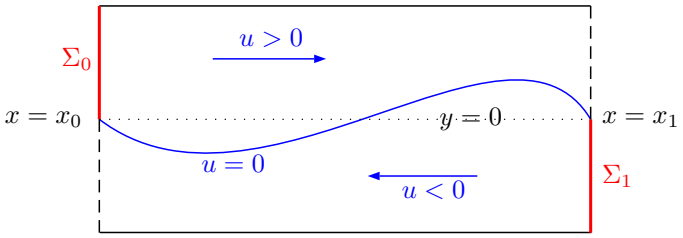

Figure 1: Fluid domain Ω and inflow boundaries $\Sigma_0 \cup \Sigma_1$

We will construct solutions to these problems thanks to an abstract implicit function theorem taking into account the geometry of the problem. More precisely, we will first straighten the free boundary $\{u = 0\}$ by introducing as a new vertical variable $z = u(x, y)$. A suitable change of unknown function will then transform (1.1) and (1.2) into quasilinear equations with an easier-to-handle nonlinearity (see Remark 4.2).

Because of the nonlinearity, we need to work in a high enough regularity space in order to have a suitable control of the derivatives. However, one key difficulty of our work lies in the fact that, even when the source term f is smooth, say in $C_0^\infty(\Omega)$, *solutions to* (1.1) *and* (1.2) *have singularities in general*. Actually, this feature is already present at the linear level, i.e. for the equation $y\partial_x u - \partial_{xx} u = f$. We prove that if f is smooth, the associated weak solution to the linear system inherits the regularity of f if and only if f satisfies *orthogonality conditions* (i.e. the scalar products of f with some identified profiles must vanish). We also describe the singularities that appear when these orthogonality conditions are not satisfied. At the nonlinear level, these orthogonality conditions become a finiteness assumption on the codimension of the data manifold.

All the features described above (orthogonality conditions for linear forward-backward equations, description of the potential singularities, handling of orthogonality conditions for quasilinear systems) appear to be new. We believe that the strategy we use could be extended to other nonlinear settings in which orthogonality conditions appear (elliptic equations in domains with corners, problems in which the linearized operator is Fredholm with negative index, ...)

1.1 Statement of the main results

1.1.1 Linear theory

Due to the forward-backward nature of the problem, we must choose the lateral perturbations and the source term in a particular product space. We therefore introduce the vector space

$$\begin{aligned}\mathcal{X}_B := \Big\{ (f,\delta_0,\delta_1) \in H^1_x H^2_y \times H^5(0,1) \times H^5(-1,0);\ \ f|_{\Sigma_0\cup\Sigma_1} = 0 \\ \text{and } \delta_i(0) = \delta_i((-1)^i) = \delta_i''(0) = \delta_i''((-1)^i) = 0 \Big\},\end{aligned} \tag{1.3}$$

where $\Sigma_0 = \{x_0\}\times(0,1)$ and $\Sigma_1 = \{x_1\}\times(-1,0)$ are the lateral boundaries on which we prescribe boundary conditions. We endow $\mathcal{X}_B$ with its canonical norm

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{X}_B} := \|f\|_{H^1_x H^2_y} + \|\delta_0\|_{H^5} + \|\delta_1\|_{H^5}. \tag{1.4}$$

We establish existence and uniqueness of solutions in the following anisotropic Sobolev space

$$Q^1 := H^{5/3}((x_0,x_1);L^2(-1,1)) \cap H^1((x_0,x_1);H^2(-1,1)). \tag{1.5}$$

In particular, for solutions with such regularity, Equation (1.1) or its linear version $y\partial_x u - \partial_{yy} u = f$ hold in a strong sense, almost everywhere and the various boundary conditions hold in the usual sense of traces. We first state a result concerning the well-posedness in Q^1 of the stationary Kolmogorov equation (see (1.6) below), *up to two orthogonality conditions* (see comments below). Although equation (1.6) has been thoroughly investigated, as we recall in Section 1.2, we could not find this statement in the existing literature.

Theorem 1 (Orthogonality conditions for linear forward-backward parabolic equations)**.** *There exists a vector subspace $\mathcal{X}_{B,\mathrm{sg}}^{\perp} \subset \mathcal{X}_B$ of codimension 2 such that, for each $(f,\delta_0,\delta_1) \in \mathcal{X}_B$, there exists a solution $u \in Q^1$ to the problem*

$$\begin{cases} y\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{1.6}$$

if and only if $(f, \delta_0, \delta_1) \in \mathcal{X}_{B,\mathrm{sg}}^{\perp}$. *Such a solution is unique and satisfies*

$$\|u\|_{Q^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}. \tag{1.7}$$

We emphasize that this result implies that there exist triplets (f, δ_0, δ_1) that can be chosen arbitrarily smooth and compactly supported, and for which there are no Q^1 solutions to (1.6). Furthermore, the vector space $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ can be fully characterized: classically, $\mathcal{X}_{B,\mathrm{sg}}^{\perp} = \ker \overline{\ell^0} \cap \ker \overline{\ell^1}$, where $\overline{\ell^0}$ and $\overline{\ell^1}$ are two linear forms on $\mathcal{X}_B$ which we shall write explicitly. If the data do not belong to $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$, the solution has singularities, which we can describe completely.

Theorem 2 (Decomposition of solutions as a sum of singular profiles and a smooth remainder)**.** *Let* $(f, \delta_0, \delta_1) \in \mathcal{X}_B$. *There exists a unique solution* $u \in H_x^{2/3} L_y^2 \cap L_x^2 H_y^2$ *to equation* (1.6). *Furthermore, this solution admits the following decomposition: there exists* $c_0, c_1 \in \mathbb{R}$, *and* $u_{\mathrm{reg}} \in Q^1$, *such that*

$$u = c_0 \bar{u}^0_{\mathrm{sing}} + c_1 \bar{u}^1_{\mathrm{sing}} + u_{\mathrm{reg}}. \tag{1.8}$$

Each profile $\bar{u}^i_{\mathrm{sing}}$ *is supported in the vicinity of* $(x_i, 0)$ *and is smooth on* $\overline{\Omega} \setminus \{(x_i, 0)\}$. *Furthermore, for* $|x - x_i| \ll 1$ *and* $|y| \ll 1$,

$$\bar{u}^i_{\mathrm{sing}}(x, y) = \left(|y|^2 + |x - x_i|^{\frac{2}{3}} \right)^{\frac{1}{4}} \Lambda_0 \left((-1)^i \frac{y}{|x - x_i|^{\frac{1}{3}}} \right), \tag{1.9}$$

where $\Lambda_0 \in C^\infty(\mathbb{R})$ *is such that* $\Lambda_0(-\infty) = 1$ *and* $\Lambda_0(+\infty) = 0$ *(see Fig. 2 page 30).*

The existence of a weak solution was already known, see in particular [51, 52, 22]. The novelty of the above theorem lies in the identification of the singular profiles $\bar{u}^i_{\mathrm{sing}}$, and in the decomposition of any weak solution. The function Λ_0 is in fact the solution to an ODE, and can be characterized in terms of special functions (namely confluent hypergeometric functions of the second kind, or Tricomi's functions). The assumptions on the data (f, δ_0, δ_1) are not optimal and can be weakened, see Proposition 2.9 and Corollary 2.27 in Section 2 and the definition of the space $\mathcal{H}_K$ in (2.19).

1.1.2 A nonlinear toy model from kinetic theory

As a corollary to Theorem 1, we obtain a similar statement for a (nonlinear) Vlasov–Poisson–Fokker–Planck system in an interval. In order not to burden the introduction, we refer to Section 3 for the presentation of the system and to Theorem 5 for the full statement. The proof of Theorem 5 is rather straightforward since the geometry of the considered problem remains the same as for (1.6), and the nonlinearity is very weak. We nevertheless use this example to set up our nonlinear scheme in Section 3.3 and a general abstract nonlinear existence result in Section 3.5.

1.1.3 The viscous Burgers system

We then turn towards the nonlinear problem (1.1). One of the main results of this paper is the following nonlinear generalization of Theorem 1 for small enough perturbations[1].

Theorem 3 (Existence and uniqueness of strong solutions to (1.1) under orthogonality conditions)**.** *There exists a Lipschitz submanifold* $\mathcal{M}_B$ *of* $\mathcal{X}_B$ *of codimension 2, containing* 0 *and included in*

[1] More precisely, the norm of the perturbation must be smaller than some constant depending only on the size of the domain and on the underlying flow, see Section 1.5.1.

a ball of radius $\eta \ll 1$ in $\mathcal{X}_B$, such that, for every $(f, \delta_0, \delta_1) \in \mathcal{M}_B$, there exists a strong solution $u \in Q^1$ to

$$\begin{cases} u\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = y + \delta_i(y), \\ u_{|y=\pm 1} = \pm 1. \end{cases} \tag{1.10}$$

More precisely, $\mathcal{M}_B$ is modeled on $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ and tangent to it at 0. Such solutions are unique in a small neighborhood of $\mathfrak{u}(x,y) = y$ in Q^1 and satisfy the estimate

$$\|u - \mathfrak{u}\|_{Q^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}. \tag{1.11}$$

In the statement above, the condition that the data (f, δ_0, δ_1) belong to the manifold $\mathcal{M}_B$ is the nonlinear equivalent of the orthogonality conditions from Theorem 1. We emphasize that this is by no means a technical restriction which could be lifted, but actually a necessary condition to solve the equation with smooth solutions, as we state in Proposition 1.1 below. A key difficulty lies in the fact that these orthogonality conditions *depend on the solution itself.*

Proposition 1.1 (Necessity of the orthogonality conditions)**.** *There exists $\eta > 0$ such that the following result holds. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \leq \eta$. Let $u \in Q^1$ be a solution to* (1.10) *such that $\|u - \mathfrak{u}\|_{Q^1} \leq \eta$. Then $(f, \delta_0, \delta_1) \in \mathcal{M}_B$.*

Remark 1.2. *By commodity, the above results are stated using the full triplet (f, δ_0, δ_1), and so is the remainder of this paper. Nevertheless, it is possible to obtain similar results either by fixing $\delta_0 = \delta_1 = 0$ and constructing a submanifold of source terms f yielding regular solutions, or by fixing $f = 0$ and constructing a submanifold of boundary data (δ_0, δ_1). This stems from the independence of the orthogonality conditions, which can be obtained either by Proposition 2.13 or by Proposition 2.31.*

1.1.4 The Prandtl system

We also prove analogous results for the Prandtl system, revisiting the work of Iyer and Masmoudi in [35, 34] (we will comment more thoroughly on the differences between our results in the next sections). Let us now present our mathematical setting; we will provide physical motivation and background for this system in Section 1.3. We consider a reference flow $(\mathfrak{u}_{\mathrm{P}}, \mathfrak{v}_{\mathrm{P}}) \in C^k([x_0, x_1] \times (0, +\infty))$ for some sufficiently large k (say $k = 4$), satisfying the Prandtl system

$$\mathfrak{u}_{\mathrm{P}} \partial_x \mathfrak{u}_{\mathrm{P}} + \mathfrak{v}_{\mathrm{P}} \partial_y \mathfrak{v}_{\mathrm{P}} - \partial_{yy} \mathfrak{v}_{\mathrm{P}} = -\frac{dp}{dx}, \tag{1.12}$$

$$\partial_x \mathfrak{u}_{\mathrm{P}} + \partial_y \mathfrak{v}_{\mathrm{P}} = 0 \tag{1.13}$$

in the whole domain $(x_0, x_1) \times (0, +\infty)$, where p is the trace of the pressure of some outer Euler flow on the boundary $\{y = 0\}$. We assume that there exists a curve $\Gamma := \{y = \gamma_{\mathrm{P}}(x)\}$, with γ_{P} smooth and such that $\inf_{[x_0,x_1]} \gamma_{\mathrm{P}} > 0$, such that $\mathfrak{u}_{\mathrm{P}}$ changes sign on the curve Γ: $\mathfrak{u}_{\mathrm{P}}(x, \gamma_{\mathrm{P}}(x)) = 0$, and $\mathfrak{u}_{\mathrm{P}}(x,y) < 0$ (resp $\mathfrak{u}_{\mathrm{P}}(x,y) > 0$) for $y < \gamma_{\mathrm{P}}(x)$ (resp. $y > \gamma_{\mathrm{P}}(x)$). Our purpose is to construct a solution to the Prandtl system close to $(\mathfrak{u}_{\mathrm{P}}, \mathfrak{v}_{\mathrm{P}})$ and in the vicinity of the curve Γ, by perturbing the inflow/outflow on the lateral boundaries[2].

To that end, we consider $z_b < 0 < z_t$ such that there exist smooth functions $\overline{\gamma_b}, \overline{\gamma_t}$, with $0 < \overline{\gamma_b}(x) < \gamma_{\mathrm{P}}(x) < \overline{\gamma_t}(x)$ for all $x \in [x_0, x_1]$, and such that $\mathfrak{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x)) = z_j$ for $j \in \{b, t\}$. We

[2]In fact, we can perturb either the boundary conditions or the source term.

set $\overline{\Gamma_j} = \{y = \overline{\gamma_j}(x)\}$ for $j \in \{b, t\}$. We consider the Prandtl system (1.2) in a domain Ω_P, which is defined by

$$\Omega_P := \{(x, y) \in (x_0, x_1) \times \mathbb{R}_+;\ \gamma_b(x) < y < \gamma_t(x)\}, \tag{1.14}$$

where γ_b, γ_t are smooth functions, which will actually be free boundaries, corresponding to level sets of the function u. We expect these functions (which are unknowns of the problem) to be smooth functions located in the vicinity of $\overline{\gamma_b}, \overline{\gamma_t}$. We endow system (1.2) with the following boundary conditions, which we will discuss and comment in Section 1.2.

1. *Boundary conditions on the top and bottom free boundaries:* On the bottom boundary $\Gamma_b = \{y = \gamma_b(x)\}$, we enforce

$$\begin{aligned} u|_{\Gamma_b} &= \mathfrak{u}_{\mathrm{P}}|_{\overline{\Gamma_b}} = z_b, \\ \partial_y u|_{\Gamma_b} &= \partial_y \mathfrak{u}_{\mathrm{P}}|_{\overline{\Gamma_b}} + \delta_b, \\ v|_{\Gamma_b} &= \mathfrak{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b, \end{aligned} \tag{1.15}$$

   where δ_b, v_b are small, smooth perturbations.

   Similarly, on the top boundary $\Gamma_t = \{y = \gamma_t(x)\}$, we enforce

$$\begin{aligned} u|_{\Gamma_t} &= z_t, \\ \partial_y u|_{\Gamma_t} &= \partial_y \mathfrak{u}_{\mathrm{P}}|_{\overline{\Gamma_t}} + \delta_t, \end{aligned} \tag{1.16}$$

   where δ_t is, again, a small smooth perturbation.

   Remark 1.3. *Note that on the top and bottom boundary, we prescribe the trace of u and of its normal derivative. Of course it would be impossible to prescribe simultaneously these two boundary conditions if the boundaries Γ_b, Γ_t were fixed. This is only made possible by the fact that these two boundaries are free.*

2. *Lateral boundary conditions:* As in the case of (1.6) and (1.10), we enforce lateral boundary conditions on u, namely

$$u_{|\Sigma_i^P} = \mathfrak{u}_{\mathrm{P}} + \delta_i, \qquad i \in \{0, 1\}, \tag{1.17}$$

   where $\Sigma_i^P = \{(x_i, y),\ (-1)^i \mathfrak{u}_{\mathrm{P}}(x_i, y) > 0, y \in (\gamma_{\mathrm{P}}(x_i), y_i)\}$ for some y_0, y_1 such that $y_0 > \gamma_{\mathrm{P}}(x_0)$, $y_1 < \gamma_{\mathrm{P}}(x_1)$. For simplicity, we assume that $\delta_i(\gamma_{\mathrm{P}}(x_i)) = 0$, and we recall that δ_0, δ_1 are assumed to be small in some sufficiently strong Sobolev norm. Therefore the signs of $\mathfrak{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i$ and of $\mathfrak{u}_{\mathrm{P}}(x_i, \cdot)$ are identical in a vicinity of $\gamma_{\mathrm{P}}(x_i)$.

We will in fact state two different results: one in "low regularity", under merely one orthogonality condition, and another one in higher regularity, under three orthogonality conditions. For the sake of readability, we have stated them under the same regularity and compatibility assumptions on the data, although the assumptions are not optimal in the low regularity case, and the compatibility conditions could be generalized in both cases. We will state a more general result in Section 5 (see Proposition 5.2). Therefore, we take our data in the functional space

$$\begin{aligned} \mathcal{X}_P := \Big\{ (\delta_0, \delta_1, \delta_t, \delta_b, v_b) \in & H^6(\Sigma_0^P) \times H^6(\Sigma_1^P) \times H_0^2(x_0, x_1) \times H_0^2(x_0, x_1) \times H^1(x_0, x_1), \\ & \delta_i \in H_0^4(\Sigma_i^P) \Big\}, \end{aligned}$$

which we endow with its natural norm.

Theorem 4. *There exist numbers $\eta > 0$, $z_0 > 0$, depending only on the underlying flow $(\mathtt{u}_\mathrm{P}, \mathtt{v}_\mathrm{P})$, such that if $|z_b|, z_t \leq z_0$, the following results hold.*

1. *There exists a manifold $\mathcal{M}_0$ of codimension 1 in $\mathcal{X}_P$ and included in a ball of radius η in $\mathcal{X}_P$, such that for all $(\delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_0$, equation (1.2) endowed with the boundary conditions (1.15)-(1.16)-(1.17) has a unique solution u such that $u \in L^2_x H^3_y(\Omega_P)$, $(x - x_0)(x - x_1)u \in H^1_x H^3_y(\Omega_P)$, $\partial_y u \in L^\infty(\Omega_P)$, $u\partial_x\partial_y u \in L^2(\Omega_P)$, and*

$$\begin{aligned}&\|u - \mathtt{u}_\mathrm{P}\|_{L^2_x H^3_y} + \|\partial_y(u - \mathtt{u}_\mathrm{P})\|_{L^\infty} + \|(x - x_0)(x - x_1)(u - \mathtt{u}_\mathrm{P})\|_{H^1_x H^3_y}\\ \lesssim &\|(\delta_0, \delta_1, \delta_t, \delta_b, v_b)\|_{\mathcal{X}_P}.\end{aligned}$$

2. *There exists a manifold $\mathcal{M}_1$ of codimension 3 in $\mathcal{X}_P$ and included in a ball of radius η in $\mathcal{X}_P$, such that for all $(\delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_1$, equation (1.2) endowed with the boundary conditions (1.15)-(1.16)-(1.17) has a unique solution u such that $u \in H^{5/3}_x H^1_y \cap H^1_x H^3_y(\Omega^P)$, and*

$$\|u - \mathtt{u}_\mathrm{P}\|_{H^{5/3}_x H^1_y} + \|u - \mathtt{u}_\mathrm{P}\|_{H^1_x H^3_y} \lesssim \|(\delta_0, \delta_1, \delta_t, \delta_b, v_b)\|_{\mathcal{X}_P}.$$

1.2 Comments and previous results

We start with a few comments on our main results and recall related known results.

Problem (1.6), involving the operator $y\partial_x - \partial_{yy}$, can be seen as a particular case of the class of "degenerate second-order elliptic-parabolic linear equations", also referred to as "second-order equations with nonnegative characteristic form" (as opposed to positive definite ones), "forward-backward" or "mixed type" problems. They date back at least to Gevrey [24].

Problem (1.6) itself, as well as these wide classes of equations, has received a lot of attention and has been investigated under different aspects: with variable coefficients or other geometries [22, 52], higher-order operators [42, Ch. 3, 2.6], abstract operators [9, 53], explicit representation formulas [23, 27] or with a focus on numerical analysis [3].

On weak solutions for the linear problem. It is well-known since the work of Fichera [22] that weak solutions to (1.6) with $L^2_x H^1_y$ regularity exist. For general boundary-value problems for elliptic-parabolic second-order equations, one owes to Fichera the systematic separation of the boundary of the domain into three parts: a "noncharacteristic" part, where one sets either Dirichlet or Neumann boundary conditions (here $y = \pm 1$), an "inflow" part, where one sets a Dirichlet boundary condition (here $\Sigma_0 \cup \Sigma_1$) and an "outflow" part, where one cannot set a boundary condition (here, the two sets $\{x_0\} \times (-1, 0)$ and $\{x_1\} \times (0, 1)$).

Baouendi and Grisvard [8] proved the uniqueness of weak solutions to (1.6) with $L^2_x H^1_y$ regularity, by means of a trace theorem and a Green identity (see Appendix A).

On strong solutions for the linear problem. There is an extensive literature on the regularity of solutions to degenerate elliptic-parabolic linear equations, and whether weak solutions are strong. We refer the reader in particular to the book [47] by Oleĭnik and Radkevič. Generally speaking, depending on the exact setting considered, it is quite often possible to prove that the solutions to such equations are regular far from the boundaries of the domain and/or from the regions where the characteristic form is not positive definite. A nice example is Kohn and Nirenberg's work [38], which proves a very general regularity result. A key assumption of their work is that the "outflow"

part of the boundary does not meet the "noncharacteristic" and "inflow" parts (i.e. they are in disjoint connected components of $\partial\Omega$). Hence, it does not apply to (1.6), and hints towards a difficulty near the points $(x_0, 0)$ and $(x_1, 0)$.

In a series of papers [50, 51, 52], Pagani proved the existence of strong solutions to (1.6) (and related equations). More precisely, Pagani proved the existence of solutions such that $y\partial_x u$ and $\partial_{yy} u$ belong to $L^2(\Omega)$. Moreover, he determined the exact regularity of the various traces of such solutions (trace of u at $x = x_i$, at $y = \pm 1$ or $y = 0$, and trace of $\partial_y u$ at $y = 0$). These maximal regularity results play a key role in our analysis and motivate the functional spaces we introduce in Section 1.5.

On orthogonality conditions for higher regularity. As noted by Pyatkov in [54], for such forward-backward problems: "*as a rule, there is no existence theorems for smooth solutions without some additional orthogonality-type conditions on the problem data*". Even for the linear problem (1.6), there have been very few works concerning higher regularity (than the one given by Pagani's framework) in the whole domain. Most of the works focused on higher regularity (such as [54]) involve weighted estimates which entail regularity within the domain but not near the critical points $(x_i, 0)$. An attempt for global regularity is Goldstein and Mazumdar's work [25, Theorem 4.2]; however the proof seems incomplete (see Proposition 2.9 below and its proof for more details).

A misleading aspect is that it is quite easy, assuming the existence of a smooth solution, to prove *a priori* estimates at any order. Such phenomena are usual in the theory of elliptic problems in domains with corners or mixed Dirichlet-Neumann boundary conditions (see for instance [28]). Let us give an illustration of such a phenomenon in a close context. For a source term $f \in C_c^\infty(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ \partial_x u(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ u(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.18}$$

It is classical that such a system has a unique weak solution $u \in H^1(\Omega)$. Moreover, assuming that u is smooth enough, $v := \partial_x u$ satisfies

$$\begin{cases} -\Delta v = \partial_x f & \text{in } \Omega, \\ \partial_x v(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ v(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ v(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.19}$$

For such systems, one has $\|v\|_{H^1} \lesssim \|\partial_x f\|_{L^2}$. Hence $\|\partial_{xx} u\| \lesssim \|\partial_x f\|_{L^2}$, and, using the equation, $\|u\|_{H^2} \lesssim \|f\|_{H^1}$. So one has an *a priori* estimate. However, it is known that there exist source terms for which the unique weak solution $u \in H^1$ does not enjoy H^2 regularity (see [28, Chapter 4] and Section 2.4). The key point is that, when reconstructing u from the solution v to (1.19), say by setting $u(x, y) := \int_{x_0}^x v(x', y)\,\mathrm{d}x'$ for $y > 0$ and $u(x, y) := \int_{x_1}^x v(x', y)\,\mathrm{d}x'$ for $y < 0$, there might be a discontinuity of u or $\partial_y u$ across the line $y = 0$. Such discontinuities prevent u from solving (1.18). Preventing these discontinuities requires that the source term satisfies appropriate orthogonality conditions.

Let us also emphasize that if one wishes to construct solutions of (1.1) with even stronger regularity, say $u \in H^k_x H^1_y$ with $k \geq 1$, then generically, one needs to ensure that $2k$ orthogonality

conditions are satisfied by the source terms (see Lemma 2.15). This situation occurs (at a nonlinear level) in [34].

On orthogonality conditions for nonlinear problems. Of course, such orthogonality conditions make it very difficult to obtain results at a nonlinear level. Generally, one tries to avoid such difficulties when considering nonlinear problems. For instance, for elliptic problems in polygonal domains, the classical textbook [28, Section 8.1] focuses on a nonlinear case where there is no orthogonality condition at the linear level.

Nevertheless, some results are known in the *semilinear* case. For example, for semilinear Fredholm operators with negative index, a theoretical toolbox is known (see e.g. [59, Chapter 11, Section 4.2]) and has been implemented for some reaction-diffusion semilinear systems (see e.g. [60, Chapter 7, Section 2.2], based on [20]).

Outside of the semilinear setting, we are not aware of nonlinear results obtained despite the presence of orthogonality conditions at the linear level prior to our present work (we discuss the recent preprint [34] by Sameer Iyer and Nader Masmoudi in Section 1.3).

Problem (1.1) is only *quasilinear*, and this makes the analysis harder. In an earlier version of this paper, we introduced a nonlinear scheme in which the orthogonality conditions changed at every step. Tracking the evolution of these orthogonality conditions was then a major difficulty.

Following a very helpful remark by several colleagues[3], we have changed our strategy. We first perform a change of unknown which allows us to keep the same linear operator throughout the scheme, and to treat the nonlinearity perturbatively. This greatly simplifies the proof. In turn, this change of variables allows us to revisit the work of Iyer and Masmoudi [35, 34] on the analysis of the Prandtl system in the vicinity of a recirculating flow, since the equation for the vorticity after the change of variables has a very simple structure, see Section 1.4. Let us also recall that at the nonlinear level, the orthogonality conditions are translated in Theorem 3 (resp. Theorem 4) as the fact that the data must lie within the manifold $\mathcal{M}$ (resp. $\mathcal{M}_1$), which can be pictured as a perturbation of the linear subspace $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ of data satisfying the orthogonality conditions for the linear problem.

The proof of both of our main theorems (on Burgers and Prandtl) relies on the same abstract result (see Theorem 6 in Section 3.5) concerning quasilinear equations in a perturbative regime.

On entropy solutions. An entirely different approach to solve (1.1) is to look directly for weak solutions to the nonlinear problem, for example using an entropy formulation. The regularity for such solutions is $u \in L^{\infty}_{x,y} \cap L^2_x H^1_y$ and they are typically obtained as limits of solutions u^{ε} to regularized versions of (1.1), e.g. $u^{\varepsilon}\partial_x u^{\varepsilon} - \partial_{yy} u^{\varepsilon} - \varepsilon \partial_{xx} u^{\varepsilon} = 0$. Such solutions satisfy both the equation and the lateral boundary conditions only in the weak sense of appropriate inequalities linked with "entropy pairs". Given $\delta_0, \delta_1 \in L^{\infty}(-1,1)$, the existence of an entropy solution to

$$\begin{cases} u\partial_x u - \partial_{yy} u = 0, \\ u_{|x=x_i} = \delta_i, \\ u_{|y=\pm 1} = 0 \end{cases} \tag{1.20}$$

was first proved in [11]. More recently, Kuznetsov proved in [39] the uniqueness of the entropy solution to (1.20), determined in which sense the lateral boundary conditions were satisfied and proved a stability estimate of the form

$$\|u - \tilde{u}\|_{L^1(\Omega)} \lesssim \|\delta_0 - \tilde{\delta}_0\|_{L^1(-1,1)} + \|\delta_1 - \tilde{\delta}_1\|_{L^1(-1,1)}. \tag{1.21}$$

[3]Felix Otto, Yann Brenier, and an anonymous referee, whom we warmly thank.

In particular, this stability estimate guarantees that one can construct sign-changing solutions in the vicinity of the linear shear flow.

However, an important drawback of the entropy formulation is that the boundary conditions are only satisfied in a very weak sense. Although functions in $L^\infty_{x,y} \cap L^2_x H^1_y$ do not have classical traces at $x = x_i$, one can give a weak sense to the traces using the equation (see [40] for more details). Unfortunately, it is expected that these weak traces do not coincide with the supplied boundary data on sets of positive measure.

In contrast, since the solutions we construct in this work have (at least) $H^1_x L^2_y$ regularity, they have usual traces $u_{|\Sigma_i} \in L^2(\Sigma_i)$ and the equalities $u_{|\Sigma_i} = \delta_i$ hold in $L^2(\Sigma_i)$, so almost everywhere.

On the choice of the linear shear flow for equation (1.1)**.** We choose to study the well-posedness of (1.1) in the vicinity of the linear shear flow to lighten the computations. Nonetheless, we expect that our results and proofs can be extended to study the well-posedness of (1.1) in the vicinity of any sufficiently regular reference flow $\mathfrak{u}$ changing sign across a single curve $\{\mathfrak{u} = 0\}$, satisfying $\mathfrak{u}_y \geq c_0 > 0$ in Ω (so that (1.6) is the correct toy model) and with either $\|\mathfrak{u}_x\|_\infty$ small enough, or with a restriction on the size of the domain (to ensure *a priori* estimates). In fact, this is precisely what we do when we study the Prandtl system around a recirculating flow: the linearized equation for the vorticity then becomes a forward-backward equation with variable coefficients, see (5.22) and Proposition 5.5.

Moreover, taking a step further in the modeling of recirculation problems in fluid mechanics (see Section 1.3), we also expect that our approach could be extended to an unbounded domain of the form $(x_0, x_1) \times (0, +\infty)$, with a reference flow such that $\mathfrak{u}_{|y=0} = 0$, $\mathfrak{u} < 0$ below some critical curve and then $\mathfrak{u} > 0$ above, with $\mathfrak{u}$ having some appropriate asymptotic behavior as $y \to +\infty$. In such a setting, the Poincaré inequalities in the vertical direction that we use here should probably be replaced with well-suited Hardy inequalities. As mentioned above, one of the issues is then to obtain *a priori* estimates on the linearized system. We comment further on this point at the end of Section 5.6.

On the conditions $\delta_0(0) = \delta_1(0) = 0$ for fixed end-points. It is an important feature of our work that we are able to enforce precisely the exact endpoints of the curve $\{u = 0\}$ at $x = x_0$ and $x = x_1$. Theorem 3 and Theorem 4 are stated for perturbations which satisfy $\delta_i(0) = 0$ (see (1.3)), so that the full boundary data $\mathfrak{u}(x_i, y) + \delta_i(y)$ changes sign exactly at $y = 0$, where $\mathfrak{u} = y$ in Theorem 3 and $\mathfrak{u} = \mathfrak{u}_{\mathrm{P}}$ in Theorem 4. This choice simplifies the definition of the submanifolds $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ of boundary data for which we are able to solve the problem. Nevertheless, given y_0, y_1 sufficiently close to 0 and δ_0, δ_1 such that $y + \delta_i(y)$ changes sign at $y = y_i$, we expect that similar existence results hold, provided that the perturbations are chosen in an appropriate modification of $\mathcal{M}_j$, with suitable modifications to the functional spaces and where, in (1.10), the definitions of Σ_i are generalized by setting $\Sigma_i := \{(x_i, y);\ (-1)^i(\mathfrak{u}(x_i, y) + \delta_i(y)) > 0\}$.

On the boundary conditions (1.15) **and** (1.16) **for the Prandtl system in the recirculation zone.** The boundary conditions we choose for the Prandtl system are mostly meant to simplify the present analysis as much as possible. As stated earlier, they are slightly unconventional since we prescribe both the trace and the normal derivative of u, but we let the boundary remain free. Other choices of boundary conditions are of course possible, and may lead to additional technical difficulties. These boundary conditions are designed in order to have a nice formulation after we have performed a change of variables in order to straighten the curve $\{u = 0\}$.

Note also that we only consider here the Prandtl system in the vicinity of the curve $\{u = 0\}$, and not in the whole infinite strip $(x_0, x_1) \times (0, +\infty)$. The coupling with the outside regions $y < \gamma_b(x)$ and $y > \gamma_t(x)$ leads to additional difficulties, which have been treated by Iyer and Masmoudi in [35], albeit with a different method, since conditions (1.15)-(1.16) are not considered in [35]. We propose in Section 5.6 a possible strategy to solve the Prandtl equation in an infinite strip.

On the compatibility conditions $\delta_i((-1)^i) = 0$ and $\delta_i''(0) = \delta_i''((-1)^i) = 0$ in Theorem 3. These conditions are classical compatibility conditions for solutions to elliptic-parabolic equations. For example, the condition $\delta_0(1) = 0$ in Theorem 3 is intended to match the condition $u_{|y=1} = 0$, and is necessary to have $L^2_x H^2_y$ regularity. The condition $\delta_0''(0) = 0$ comes from the equation. Indeed, if u is a sufficiently regular solution of (1.6) with $f(x_0, 0) = 0$, the equality $\partial_{yy} u = y\partial_x u$ at $(x_0, 0)$ enforces $\partial_{yy} u(x_0, 0) = 0$, so $\delta_0''(0) = 0$. The condition $\delta_0''(1) = 0$ stems similarly from the equation and the fact that $\partial_x u_{|y=1} = 0$. It corresponds to a classical parabolic regularity compatibility condition in order to have $L^2_x H^4_y$ regularity. We have imposed similar conditions for the Prandtl system by requiring $\delta_i \in H^4_0(\Sigma_i^P)$. Note that in the Prandtl case, we actually require extra cancellation assumptions. It is possible that the latter are technical, and could be removed.

On the number of orthogonality conditions for the Prandtl system. Note that the number of orthogonality conditions in Theorem 3 and in Theorem 4 is different. The reason for this is twofold.

Firstly, as previously noted by Iyer and Masmoudi in [35], the "good unknown" for the equation is the vorticity $\partial_y u$, which satisfies an equation which is very similar to (1.6) in a suitable set of variables (see (1.26) and (1.28) below). Therefore, in a sense, (1.2) is smoother than (1.1): indeed, without assuming any orthogonality condition, one can expect the vorticity $\partial_y u$ to belong to the functional space $H^{2/3}_x L^2_y \cap L^2_x H^2_y$ (see Theorem 2), and therefore the solution of the Prandtl system belongs to $H^{2/3}_x H^1_y \cap L^2_x H^3_y$, in which we have gained one vertical derivative. On the contrary, without any orthogonality condition, one cannot expect the solution u of (1.1) to have better regularity than[4] $H^{2/3}_x L^2_y \cap L^2_x H^2_y$. This gain of vertical regularity allows us to have a theory for weaker solutions of the Prandtl system, and therefore to get rid of two of the orthogonality conditions.

Secondly, reconstructing the velocity from the vorticity in the Prandtl system gives rise to one additional orthogonality condition, as we will explain in Section 5. Hence the number of orthogonality conditions in Theorem 4 is odd, while it is even in Theorem 3.

1.3 Motivation from recirculation problems in fluid mechanics

Our original motivation stems from fluid mechanics. Indeed, the stationary Prandtl equation (1.2) describes the behavior of a fluid with small viscosity in the vicinity of a wall. The pressure $p(x)$ is the trace of the pressure of an outer Euler flow. This equation is usually set in a 2d domain of the form $I \times (0, +\infty)$, where $I \subset \mathbb{R}$ is an interval, and $y = 0$ is the solid wall. The equation is endowed with the boundary conditions $u = v = 0$ on $y = 0$, and $\lim_{y\to\infty} u(x, y) = u_E(x)$, where $u_E(x)$ is the trace of the outer Euler flow on the wall, and satisfies $u_E \partial_x u_E = -\partial_x p$.

As long as u remains positive, (1.2) can be seen as a nonlocal, nonlinear diffusion type equation, the variable x being the evolution variable. Using this point of view, Oleinik (see e.g. [48, Theorem

[4]Note that actually, functions in $H^{2/3}_x L^2_y \cap L^2_x H^2_y$ do not have sufficient regularity for a fixed point argument for equation (1.1).

2.1.1]) proved the local well-posedness of a solution to (1.2) when the equation (1.2) is supplemented with a boundary data $u_{|x=0} = u_0$, where $u_0(y) > 0$ for $y > 0$ and such that $u_0'(0) > 0$. Let us mention that such positive solutions exist globally when $\partial_x p \leq 0$, but are only local when $\partial_x p > 0$. More precisely, when $\partial_x p = 1$ for instance, for a large class of boundary data u_0, there exists $x^* > 0$ such that $\lim_{x \to x^*} u_y(x, 0) = 0$. Furthermore, the solution may develop a singularity at $x = x^*$, known as Goldstein singularity. The point x^* is called the separation point: intuitively, if the solution to Prandtl exists beyond x^*, then it must have a negative sign close to the boundary (and therefore change sign). We refer to the seminal works of Goldstein [26] and Stewartson [58] for formal computations on this problem. A first mathematical statement describing separation was given by Weinan E in [21] in a joint work with Luis Cafarelli, but the complete proof was never published. The first author and Nader Masmoudi then gave a complete description of the formation of the Goldstein singularity [17]. The work [57] indicates that this singularity holds for a large class of initial data.

Because of this singularity, it is actually unclear that the Prandtl system is a relevant physical model in the vicinity of the separation point x^*, because the normal velocity v becomes unbounded at $x = x^*$. Consequently, more refined models, such as the triple deck system (see [41] for a presentation of this model, and [36, 18] for a recent mathematical analysis of its time-dependent version), were designed specifically to replace the Prandtl system with a more intricate boundary layer model in the vicinity of the separation point. However, beyond the separation point, i.e. for $x > x^*$, it is expected that the Prandtl system becomes valid again, but with a changing sign solution.

The well-posedness of the Prandtl system (1.2) when the solution u is allowed to change sign has only recently been investigated. Such solutions are called "recirculating solutions", and the zone where $u < 0$ is called a recirculation bubble, the usual convention being that $u_E(x) > 0$, so that the flow is going forward far from the boundary. In the recent preprint [35] by Sameer Iyer and Nader Masmoudi, the authors prove *a priori* estimates in high regularity norms for smooth solutions to the Prandtl equation (1.2) in a domain of the form $I \times (0, +\infty)$, with restrictions on the length of the interval I, in the vicinity of explicit self-similar recirculating flows, called Falkner–Skan profiles. The latter are given by

$$u(x, y) = x^m f'(\zeta), \tag{1.22}$$

$$v(x, y) = -y^{-1}\zeta f(\zeta) - \frac{m-1}{m+1} y^{-1}\zeta^2 f'(\zeta), \tag{1.23}$$

where $\zeta := (\frac{m+1}{2})^{\frac{1}{2}} y x^{\frac{m-1}{2}}$ is the self-similarity variable, m is a real parameter and f is the solution to the Falkner-Skan equation

$$f''' + ff'' + \beta(1 - (f')^2) = 0, \tag{1.24}$$

where $\beta = \frac{2m}{m+1}$, subject to the boundary conditions $f(0) = f'(0) = 0$ and $f'(+\infty) = 1$. Such flows correspond to an outer Euler velocity field $u_E(x) = x^m$. For some particular values of m (or, equivalently, β), these formulas provide physical solutions to (1.2) which exhibit recirculation (see [12]). Obtaining high regulariry *a priori* estimates for recirculating solutions to the Prandtl system (1.2) on the whole infinite strip is a difficult task. This important step was achieved by Sameer Iyer and Nader Masmoudi in [35].

In the present paper, we have chosen to focus on a different type of difficulty, and to consider first the toy-model (1.1), which differs from (1.2) through the lack of the nonlinear transport term $v\partial_y u$ and its associated difficulties (nonlocality, loss of derivative) and the exclusion of the zones close to the wall and far from the wall. For the model (1.1), *a priori* estimates are easy to derive,

see [55, Chapter 4]. The difficulty lies elsewhere, as explained previously. Indeed, in order to construct a sequence of approximate solutions satisfying the *a priori* estimates, we need to ensure that the orthogonality conditions are satisfied all along the sequence. For the Prandtl system (1.2), this difficulty has recently been tackled by Sameer Iyer and Nader Masmoudi in [34], building upon their *a priori* estimates of [35] and the ideas developed in the first version of our present work. We revisit in Section 5 part of their work. Performing the different steps of the analysis (straightening the boundary, linearizing, differentiating with respect to the horizontal or vertical variable), we found a way to substantially simplify the analysis of the system in the vicinity of the recirculating line, and the results that we obtain are slightly different from the ones of [35, 34]. First, we prove a result in a rather *low regularity setting*, in which u_x is not even an L^2 function in the whole domain Ω_P. This result holds under merely one orthogonality condition, whose role is rather different from the ones of Theorem 3, for instance. Indeed, the role of this additional orthonogality condition is not to ensure a certain regularity, but rather to allow for a reconstruction of the velocity from the vorticity. Moreover, we use solely one change of variables, which we present in the next subsection and which is identical to the one for the Burgers equation. Once the adequate change of variables is identified, we retrieve the fact that the vorticity (in these new variables) is a good unknown. In fact, in the appropriate set of variables, the equation for the vorticity becomes remarkably simple (it is a closed, quasilinear equation). We however prescribe lateral boundary conditions on the velocity (rather than the vorticity). Eventually, we do not require any condition on the *horizontal size of the domain* (i.e. on the length $x_1 - x_0$). Our understanding is that such conditions may arise when the Prandtl system in the whole infinite strip is considered. They are linked to the well-posedness of a linearized system in the whole strip. We refer to Section 5.6 for more comments regarding this point.

1.4 Scheme of proof of nonlinear theorems and plan of the paper

The uniqueness of solutions is fairly easy to prove. For the linear problem (1.6), uniqueness already holds at the level of weak solutions (see Proposition 2.2 and Appendix A). For the nonlinear problems, uniqueness is straightforward since we are considering strong solutions. Therefore, the main subject of this paper is the proof of the existence of solutions for the nonlinear problems (1.10) and (1.2) endowed with the boundary conditions (1.15)-(1.16)-(1.17).

A first natural idea would be to prove existence thanks to a nonlinear scheme relying on the linear problem (1.6). For example, concerning equation (1.1), one could wish to construct a sequence of approximate solutions $(u_n)_{n\in\mathbb{N}}$ by setting $u_0 := 0$ (or any other initial guess) and solving

$$\begin{cases} y\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f - (u_n - y)\partial_x u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{1.25}$$

However, this strategy fails. The key point is that the right-hand side contains a full tangential derivative of u_n, whereas the operator $y\partial_x - \partial_{yy}$ only yields a gain of $2/3$ of a derivative in this direction (more precisely, see Proposition 1.7, Remark 1.8 and Proposition 2.4). Hence, this nonlinear scheme would exhibit a "loss of derivative", preventing us from proving a uniform bound on the sequence $(u_n)_{n\in\mathbb{N}}$.

Another drawback of this scheme is that it would not translate well to a setting where one does not assume $\delta_i(0) = 0$. Indeed, in such a case, the inflow boundaries of the problem with the perturbed data $y + \delta_i(y)$ would not match the inflow boundaries of the linear problem (1.6).

Hence, we will rather construct solutions to (1.1) through another iterative scheme. As suggested to us by an anonymous referee and by other colleagues, we first straighten the curve $\{u = 0\}$ by setting as a new vertical variable $z = u(x, y)$. Our new unknown, both for the Burgers equation (1.1) and for the Prandtl system (1.2), is the inverse function of u, i.e. the function Y such that $u(x, Y(x, z)) = z$. In this new set of variables, the equation for Y becomes, in the case of the Burgers equation (1.1),

$$z\partial_x Y - (\partial_z Y)^{-2}\partial_z^2 Y = -\partial_z Y f(x, Y), \tag{1.26}$$

and in the case of the Prandtl system (1.2)

$$z\partial_x Y - \int_{z_b}^{z} \partial_x Y - (\partial_z Y)^{-2}\partial_z^2 Y = -\partial_z Y \frac{dp}{dx} + \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b, \tag{1.27}$$

in which the nonlocal integral term in the left-hand side stems from the transport term $v\partial_y u$ in the original equation. Differentiating the above equation, we find that in the case of the Prandtl system, the vorticity $W = \partial_z Y$ satisfies

$$z\partial_x W + \frac{dp}{dx}\partial_z W + \partial_z^2\left(\frac{1}{W}\right) = 0. \tag{1.28}$$

We immediately see that the linearized operator associated with (1.26) around $Y(x, z) = z$ is equation (1.6). In a similar fashion, the linearized operator of equation (1.28) around the flow $\mathbb{Y}_{\mathrm{P}}$ associated with $\mathbb{u}_{\mathrm{P}}$ is a forward-backward operator with variable coefficients, of the form $z\partial_x + \beta\partial_z - \partial_z^2(\alpha\cdot)$, where $\alpha = (\partial_z \mathbb{Y}_{\mathrm{P}})^{-2}$ and $\beta = dp/dx$. Such forward-backward operators bear strong similarities with the canonical one $z\partial_x - \partial_{zz}$, and therefore we will rely on our linear analysis to study (1.28) (see Lemma 5.3 and Proposition 5.5).

We then construct solutions of (1.26) and (1.27) thanks to an iterative scheme[5], which we now explicit in the Burgers case, the Prandlt one being similar. We define a sequence $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ such that

$$z\partial_x \widetilde{Y}_{n+1} - \partial_{zz}\widetilde{Y}_{n+1} = \frac{\partial_z \widetilde{Y}_n(2 - \partial_z \widetilde{Y}_n)}{(1 - \partial_z \widetilde{Y}_n)^2}\partial_{zz}\widetilde{Y}_n + (1 - \partial_z \widetilde{Y}_n) f(x, z - \widetilde{Y}_n) + g^{n+1}, \tag{1.29}$$

where the additional term g^{n+1} ensures that the orthogonality conditions are satisfied at every step. We then prove that $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space Q^1. Passing to the limit, we obtain a solution $Y = z - \lim_{n\to\infty} \widetilde{Y}_n$ to (1.26) with an additional source term g. The manifold $\mathcal{M}$ is then defined by requiring that the limit term g is zero.

Remark 1.4. *In a first version of this paper [16], we had chosen a strategy which seemed only slightly different, but which led to substantial technical difficulties. However, we believe that this strategy is rather natural, and could be of use in other problems. Therefore we describe it here.*

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence solving the following iterative scheme

$$\begin{cases} u_n\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1}, \\ (u_{n+1})_{|\Sigma_i} = y + \delta_i^{n+1}, \\ (u_{n+1})_{|y=\pm 1} = \pm 1. \end{cases} \tag{1.30}$$

[5]In fact, we will state and use an abstract theorem, whose proof follows precisely the type of scheme.

For this scheme, it is possible to prove a uniform bound for u_n in the space $H^{5/3}_x L^2_y \cap L^2_x H^5_y$ and the convergence of the sequence in an interpolation space $L^2_x H^{7/2}_y \cap H^{7/6}_x L^2_y$. This scheme is similar to the one used to construct solutions to quasilinear symmetric hyperbolic systems, see for instance [7, Section 4.3].

In (1.30)*, the triplet $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ is an appropriate perturbation of the data (f, δ_0, δ_1) tailored to satisfy the orthogonality conditions associated with the linear operator $u_n\partial_x - \partial_{yy}$. In order to define these orthogonality conditions, it is necessary to straighten the curve $\{u_n(x,y) = 0\}$: hence this straightening step is still necessary, but performed* after *the "linearization" of the equation, rather than* before*.*

The issue lies in the fact that the orthogonality conditions change at every step, which is a key difficulty. In particular, in order to allow the sequence u_n to converge, one must prove that these perturbations also converge. This amounts to proving that the linear forms associated with the operator $u_n\partial_x - \partial_{yy}$ depend continuously (and even in a Lipschitz manner) on u_n, for the same topology as the one within which one proves the convergence of the sequence u_n*. This continuity estimate requires identifying quite precisely what the linear forms are.*

We believe that this methodology is rather robust and could be applied to other nonlinear problems in which orthogonality conditions are present at the linearized level, in particular in contexts where there is no nonlinear change of variables such as the one presented above allowing to treat the nonlinearity as a perturbation. For example, the PDE $u(1+\partial_y u)\partial_x u - \partial_{yy} u = f$ could be an example where our previous methodology applies, but not the one exposed in the present paper.

Remark 1.5. *Let us highlight some differences between the strategy of the present paper and the one by Iyer and Masmoudi in [35]. As explained above, there are several mathematical operations which are required to complete the proof of Theorem 4:*

- *Changing variables in order to straighten the free boundary;*
- *Linearizing the equation around some background profile;*
- *Differentiating the equation with respect to the vertical variable in order to obtain an equation for the vorticity;*
- *Differentiating the equation with respect to the tangential variable in order to derive higher order estimates (under compatibility conditions).*

These operations more or less commute at main order, and lead to the study of the equation $z\partial_x u - \partial_{zz} u = f$. However their (lower order) commutators may be a source of substantial technical difficulties. Our understanding is that the authors of [35] perform the operations in the following order (see Section 3 of their paper): 1. Linearize; 2. Differentiate with respect to the horizontal variable; 3. Straighten the free boundary; 4. Differentiate with respect to the vertical variable.

We believe that the computations are much simpler, and the structure is better understood, when the straightening change of variables is performed first. This also allows us to have a more accurate comparison between the Burgers type equation and the Prandtl one.

The plan of this work is as follows. As a preliminary, we introduce in Section 1.5 the functional spaces we will use. First, we study the linear problem (1.6) in Section 2, leading to Theorem 1, and prove that the two orthogonality conditions we expose are indeed nonvoid. We also construct the singular profiles $\bar{u}^i_{\text{sing}}$ and prove Theorem 2. In order to introduce our nonlinear scheme, we extend

these linear results to the Vlasov–Poisson–Fokker–Planck system as an example in Section 3, where we also set up our general nonlinear methodology. We then turn towards the proof of Theorem 3 in Section 4, and the one of Theorem 4 in Section 5. In order to prove the existence of weak solutions of the Prandtl system (i.e. the first point of Theorem 4), we will need an interpolation result: this rather technical step is performed in Section 6. Eventually, in Appendix A, we prove the uniqueness of weak solutions to various linear problems, by adapting an argument due to Baouendi and Grisvard [8]. In Appendix B, we prove various technical results of functional analysis that we use throughout the paper. Appendix C contain the postponed proof of a lemma of Section 5.

As the paper is quite long, a list of notations is provided starting page 105.

1.5 Functional spaces and interpolation results

1.5.1 Notations

Throughout this work, an assumption of the form "$A \ll 1$" will mean that there exists a constant $c > 0$, depending only on Ω such that, if $A \leq c$, the result holds. Similarly, a conclusion of the form "$A \lesssim B$" will mean that there exists a constant $C > 0$, depending only on Ω and on the underlying flow (namely $\mathfrak{u}(x, y) = y$ or $\mathfrak{u}_{\mathrm{P}}$), such that the estimate $A \leq CB$ holds. For ease of reading, we will not keep track of the value of these constants, mostly linked with embeddings of functional spaces. Note in particular that the sizes of these constants will depend on the length $x_1 - x_0$. In particular, it can be checked that the constant C_P appearing implicitely in Pagani's estimate (2.8) in the next Section satisfies $C_P \leq \bar{C}(1 + (x_1 - x_0)^{-1})$ for some universal constant $\bar{C}$.

We will often use the notations $\Omega_{\pm} := \Omega \cap \{\pm z > 0\}$.

1.5.2 Trace spaces for the lateral boundaries

For the traces of the solutions to (1.6) or (1.10) at $x = x_0$ and $x = x_1$, we will need the following spaces, due to [51, 52]. We define $\mathscr{L}^2_z(-1, 1)$ as the completion of $L^2(-1, 1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{L}^2_z} := \left(\int_{-1}^{1} |z| \psi^2(z) \, \mathrm{d}z \right)^{\frac{1}{2}} \tag{1.31}$$

and $\mathscr{H}^1_z(-1, 1)$ as the completion of $H^1_0(-1, 1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{H}^1_z} := \|\psi\|_{\mathscr{L}^2_z} + \|\partial_z \psi\|_{\mathscr{L}^2_z}. \tag{1.32}$$

1.5.3 Pagani's weighted Sobolev spaces

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, and let $\Omega := (x_0, x_1) \times (-1, 1)$. In the works [51, 52] (albeit with swapped variables with respect to our setting), Pagani introduced the space $Z(\mathcal{O})$ of scalar functions ϕ on $\mathcal{O}$ such that ϕ, $\partial_z \phi$, $\partial_{zz} \phi$ and $z \partial_x \phi$ belong to $L^2(\mathcal{O})$ (in the sense of distributions). In this work, we will refer to this space with the notation $Z^0(\mathcal{O})$. It is a Banach space for the following norm

$$\|\phi\|_{Z^0} := \|z \partial_x \phi\|_{L^2} + \|\partial_{zz} \phi\|_{L^2} + \|\partial_z \phi\|_{L^2} + \|\phi\|_{L^2}. \tag{1.33}$$

We will also need the space $Z^1(\mathcal{O})$, which we define as the space of scalar functions ϕ on $\mathcal{O}$ such that ϕ and $\partial_x \phi$ belong to $Z^0(\mathcal{O})$, associated with the following norm

$$\|\phi\|_{Z^1} := \|\phi\|_{Z^0} + \|\partial_x \phi\|_{Z^0}. \tag{1.34}$$

The omitted proofs of the results of this section are postponed to Appendix B. We start with a straightforward extension result, which allows transferring results on $Z^0(\mathbb{R}^2)$ to $Z^0(\Omega)$.

Lemma 1.6. *There exists a continuous extension operator from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.*

The following embedding is the most important result concerning the spaces Z^0. Since solutions to $(z\partial_x - \partial_{zz})u = f$ for $f \in L^2(\Omega)$ belong to $Z^0(\Omega)$ (see Proposition 2.4), the following embedding entails that such solutions belong to $H^{2/3}(\Omega)$.

Proposition 1.7. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $H^{2/3}_x L^2_z$.*

Remark 1.8. *Proposition 1.7 can be seen as an hypoellipticity result for the operator $L = \partial_{zz} - z\partial_x$ in the full space $\mathbb{R}^2$, which is of the form $X_1^2 + X_0$, where $X_1 = \partial_z$, $X_0 = -z\partial_x$ and $[X_0, X_1] = \partial_x$, so the Lie brackets generate the full space and L satisfies Hörmander's sufficient condition of [29] for hypoellipticity. In fact, in the full space $\mathbb{R}^2$, the $H^{2/3}_x L^2_z \cap L^2_x H^2_z$ regularity of solutions to $Lu = f$ for $f \in L^2$ can be derived from the general theory of quadratic operators, which makes a link between the anisotropic gain of regularity and the number of brackets one has to take in order to generate a direction. For instance, this regularity follows from [1, Theorem 2.10] and more precisely Example 2.11 therein applied with*

$$R = 0 \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Lemma 1.9. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $C^0_z(H^{1/2}_x)$.*

Proof. By definition, $Z^0(\mathbb{R}^2) \hookrightarrow H^2_z(L^2_x)$. By Proposition 1.7, $Z^0(\mathbb{R}^2) \hookrightarrow L^2_z(H^{2/3}_x)$. By the "fractional trace theorem" [45, Equation (4.7), Chapter 1], $Z^0(\mathbb{R}^2) \hookrightarrow C^0_z(H^{1/2}_x)$. □

Lemma 1.10. *$Z^0(\Omega)$ is continuously embedded in $C^0([x_0, x_1]; \mathscr{H}^1_z(-1, 1))$.*

Proof. This is contained in the trace result [52, Theorem 2.1]. □

Lemma 1.11. *For $\phi \in Z^0(\Omega)$, $\partial_z \phi|_{z=\pm 1} \in H^{1/4}(x_0, x_1)$.*

Proof. Let $\chi_+ \in C^\infty([-1, 1])$ such that $\chi \equiv 1$ in a neighborhood of $\{z = +1\}$ and $\chi(z) = 0$ for $z < 1/2$. For $\phi \in Z^0(\Omega)$, $\chi_+(z)\phi(x, z) \in H^1_x L^2_z \cap L^2_x H^2_z$. By the "fractional trace theorem" [45, Equation (4.7), Chapter 1], $(\partial_z(\chi_+\phi))|_{z=+1} \in H^{1/4}(x_0, x_1)$. The result follows since $\chi_+ \equiv 1$ near $\{z = +1\}$. The same argument applies for the trace at $z = -1$. □

Remark 1.12. *Although it is "almost" the case, there does not hold $Z^0(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$.*

- *Pagani [51, Theorem 2.1] proves that the operator $\phi \mapsto \phi(\cdot, 0)$ is onto from $Z^0(\mathbb{R}^2)$ to $H^{\frac{1}{2}}(\mathbb{R})$. But $H^{\frac{1}{2}}(\mathbb{R})$ contains unbounded functions of x.*
- *Pagani [51, Theorem 2.3] proves that the operator $\phi \mapsto \phi(0, \cdot)$ is onto from $Z^0(\mathbb{R}^2)$ to the space $\mathscr{H}^1_z(\mathbb{R})$. But this space contains unbounded functions, for example $\psi(z) := (-\ln|z|/2)^s \chi(z)$ for $s < \frac{1}{2}$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $z = 0$.*

1.5.4 Baouendi and Grisvard's weak space

In [8], Baouendi and Grisvard introduce the space

$$\mathcal{B} := \{\phi \in L^2((x_0, x_1), H_0^1(-1,1));\ \ z\partial_x\phi \in L^2_x(H_z^{-1})\}. \tag{1.35}$$

Baouendi and Grisvard proved the uniqueness of solutions to (1.6) in $\mathcal{B}$. They also proved that functions in $\mathcal{B}$ have traces on $\{x = x_i\}$ in $\mathscr{L}^2_z(-1,1)$. These results are recalled in Appendix A, and will be used abundantly throughout the paper.

The following embedding is proved in Appendix B and used in Section 5.5.

Lemma 1.13. *$\mathcal{B}$ is continuously embedded in $H_x^{1/3}L_z^2$.*

Lemma 1.14. *$\mathcal{B}$ is continuously embedded in $C_z^0([-1,1]; H_x^{1/6})$.*

Proof. By definition, $\mathcal{B} \hookrightarrow H_z^1(L_x^2)$. By Lemma 1.13, $\mathcal{B} \hookrightarrow L_z^2(H_x^{1/3})$. Hence, the result follows from the "fractional trace theorem" [45, Equation (4.7), Chapter 1]. □

1.5.5 Anisotropic Sobolev spaces

We will construct solutions to (1.6), (1.10) and (1.2) in various anisotropic Sobolev spaces such as Q^1 of (1.5). Within these spaces, one has heuristically the correspondence $\partial_x \approx \partial_z^3$, which corresponds to the appropriate scaling due to the degeneracy of $z\partial_x$ at $z = 0$.

Indeed, if u is a solution to $z\partial_x u - \partial_{zz} u = 0$ say on the whole plane $\mathbb{R}^2$, then the rescaled functions $u_\lambda(x,z) := u(\lambda^3 x, \lambda z)$ are also solutions. This is also consistent with the shape of the singular profiles $\bar{u}^i_{\text{sing}}$ from Theorem 2, and leads to the rule of thumb "one derivative in x equals three derivatives in z" (which is different from the usual parabolic scaling, because of the cancellation on the line $z = 0$).

In particular, we will use abundantly the following embeddings from the interpolated Pagani spaces to anisotropic Sobolev ones.

Lemma 1.15. *Let $\sigma \in [0,1]$ and $Z^\sigma(\Omega) := [Z^0(\Omega), Z^1(\Omega)]_\sigma$. Then $Z^\sigma \hookrightarrow H_x^{2/3+\sigma}L_z^2 \cap H_x^\sigma L_z^2$. In particular $Z^1 \hookrightarrow Q^1$ defined in* (1.5).

Proof. By Proposition 1.7, $Z^0(\Omega) \hookrightarrow H_x^{2/3}L_z^2$ and $Z^1(\Omega) \hookrightarrow H_x^{5/3}L_z^2$. Hence

$$Z^\sigma(\Omega) \hookrightarrow [H_x^{2/3}L_z^2, H_x^{5/3}L_z^2]_\sigma = H_x^{2/3+\sigma}L_z^2 \tag{1.36}$$

using e.g. [45, Equation (13.4), Chapter 1].

Moreover, by definition, $Z^0(\Omega) \hookrightarrow L_x^2H_z^2$ and $Z^1(\Omega) \hookrightarrow H_x^1H_z^2$, so $Z^\sigma(\Omega) \hookrightarrow H_x^\sigma H_z^2$. □

Remark 1.16. *The definition* (1.5) *of Q^1 does not contain the "full vertical" regularity $L_x^2H_z^5$, since we do not need it to close our nonlinear estimates. However, assuming sufficient regularity on the source terms, one can build solutions to* (1.6) *and* (1.10) *in $Q^1 \cap L_x^2H_x^5$, and this was in fact what we did in the earlier version [16] of this work.*

2 The case of the linear shear flow

This section concerns the well-posedness of the linear system (1.6) which we restate here for convenience and by using z as a vertical variable rather than y to prepare for the next sections. We thus consider, in $\Omega = (x_0, x_1) \times (-1, 1)$, the system

$$\begin{cases} z\partial_x u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{2.1}$$

where $\Sigma_0 = \{x_0\} \times (0, 1)$ and $\Sigma_1 = \{x_1\} \times (-1, 0)$.

First, in Section 2.1, we recall the theory of weak solutions, due to Fichera for the existence, and to Baouendi and Grisvard for the uniqueness. Then, in Section 2.2, we recall the theory of strong solutions with maximal regularity, due to Pagani. Our contributions regarding this problem are contained in the following subsections. In Section 2.3, we derive orthogonality conditions which are necessary to obtain higher tangential regularity and prove the existence result of Theorem 1. In Section 2.4, we construct explicit singular solutions and prove the decomposition result of Theorem 2. Eventually, in Section 2.5, we state a result concerning the well-posedness of (2.1) with fractional tangential regularity, which will be used in Section 5 and proved in Section 6.

2.1 Existence and uniqueness of weak solutions

Definition 2.1 (Weak solution)**.** *Let $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$ and $\delta_0, \delta_1 \in \mathscr{L}^2_z(-1, 1)$. We say that $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ is a* weak solution *to (2.1) when, for all $v \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds*

$$-\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v = \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.2}$$

Weak solutions in the above sense are known to exist since the work Fichera [22, Theorem XX] (which concerns generalized versions of (2.1), albeit with vanishing boundary data). Uniqueness dates back to [8, Proposition 2] by Baouendi and Grisvard.

Proposition 2.2. *Let $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$ and $\delta_0, \delta_1 \in \mathscr{L}^2_z(-1, 1)$. There exists a unique weak solution $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ to (2.1). Moreover,*

$$\|u\|_{L^2_x H^1_z} \lesssim \|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathscr{L}^2_z} + \|\delta_1\|_{\mathscr{L}^2_z}. \tag{2.3}$$

Proof. The proof of uniqueness is postponed to Appendix A where we adapt Baouendi and Grisvard's arguments to prove the uniqueness of weak solutions to all the linear problems we encounter in this paper in Lemma A.1. It relies on the proof of a trace theorem and a Green identity for the space $\mathcal{B}$ defined in (1.35).

Let us prove the existence. We introduce two Hilbert spaces $\mathscr{V} \hookrightarrow \mathscr{U} \hookrightarrow L^2((x_0, x_1); H^1_0(0, 1))$ as follows. Let $\mathscr{V} := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Omega \setminus (\Sigma_0 \cup \Sigma_1)\}$. Let $\mathscr{U}$ be the completion of $H^1(\Omega) \cap L^2((x_0, x_1); H^1_0(-1, 1))$ with respect to the scalar product

$$\langle u, v\rangle_{\mathscr{U}} := \int_\Omega \partial_z u \partial_z v + \int_{\Sigma_0} zuv - \int_{\Sigma_1} zuv. \tag{2.4}$$

For $u, v \in \mathscr{U} \times \mathscr{V}$, let

$$a(u,v) := -\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v, \tag{2.5}$$

$$b(v) := \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.6}$$

In particular, for every $v \in \mathscr{V}$, integration by parts leads to $a(v,v) = \|v\|^2_{\mathscr{U}}$ and

$$|b(v)| \leq \left(\|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathscr{L}^2_z} + \|\delta_1\|_{\mathscr{L}^2_z} \right) \|v\|_{\mathscr{U}}. \tag{2.7}$$

Hence, $b \in \mathcal{L}(\mathscr{V})$ can be extended as a linear form over $\mathscr{U}$[6] and existence follows from the Lax-Milgram type existence principle Lemma B.2 in Appendix B, which also yields the energy estimate (2.3) thanks to (2.7) and Poincaré's inequality. □

Remark 2.3. *Instead of using the weak Lax-Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation* (2.1) *by vanishing viscosity, and to obtain uniform $L^2_x H^1_z$ estimates on the approximation. This approach will be used in Lemma 5.4 proved in the Appendix, in which we prove $H^1_x L^2_z$ regularity of the weak solutions far from the lateral boundaries.*

2.2 Strong solutions with maximal regularity

We now turn to strong solutions, i.e. solutions for which (2.1) holds almost everywhere. The main result on this topic is due to Pagani.

Proposition 2.4. *Let $f \in L^2(\Omega)$ and $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$. The unique weak solution u to* (2.1) *belongs to $Z^0(\Omega)$ and satisfies*

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z}. \tag{2.8}$$

The boundary conditions $u_{|\Sigma_i} = \delta_i$ hold in the sense of traces in $\mathscr{H}^1_z(\Sigma_i)$ (see Lemma 1.10).

Proof. This is a particular case of [52, Theorem 5.2]. Pagani's proof proceeds by localization. Far from the critical points $(x_0, 0)$ and $(x_1, 0)$, the regularity is rather straightforward. Near these critical points, the regularity stems from the regularity obtained for a similar problem set in a half-space $(0, +\infty) \times \mathbb{R}$ or $\mathbb{R} \times (0, +\infty)$. Pagani studies such half-space problems in [51] where he derives explicit representation formulas for the solutions, using the Mellin transform and the Wiener-Hopf method. We do not reproduce these arguments here for brevity. □

2.3 Orthogonality conditions for higher tangential regularity

We now investigate whether solutions to (2.1) enjoy higher regularity in the horizontal direction. As mentioned in Section 1.2, it is quite easy to obtain *a priori* estimates in the space $Z^1(\Omega)$ (see Proposition 2.5). However, we prove in Proposition 2.9 that the weak solution enjoys such a regularity *if only if the data satisfies appropriate orthogonality conditions*. Eventually, we give statements highlighting the fact that these conditions are non-empty.

[6] Functions in $\mathscr{U}$ *a priori* do not have traces on Σ_i so one could wonder how definition (2.6) makes sense when $v \in \mathscr{U}$. The integrals $\int_{\Sigma_i} z\delta_i v$ make sense precisely because $\mathscr{U}$ is defined as a completion with respect to (2.4). In fact, weak solutions do have traces in a strong sense, as proved in Lemma A.2, thanks to the extra regularity in x provided by the equation.

Proposition 2.5. *Let $f \in H^1((x_0,x_1);H^{-1}(-1,1))$ and $\delta_0,\delta_1 \in \mathscr{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$ and such that $\Delta_0,\Delta_1 \in \mathscr{L}^2_z(-1,1)$, where*

$$\Delta_i(z) := \frac{f(x_i,z) + \partial_z^2\delta_i(z)}{z}. \tag{2.9}$$

If the unique weak solution u to (2.1) belongs to $H^1((x_0,x_1);H^1_0(-1,1))$, then one has the following weak solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{L^2_x H^1_z} \lesssim \|\partial_x f\|_{L^2_x(H^{-1}_z)} + \|\Delta_0\|_{\mathscr{L}^2_z(\Sigma_0)} + \|\Delta_1\|_{\mathscr{L}^2_z(\Sigma_1)}. \tag{2.10}$$

If, moreover, $f \in H^1((x_0,x_1);L^2(-1,1))$, $\Delta_0,\Delta_1 \in \mathscr{H}^1_z(-1,1)$ and $\Delta_0(1) = \Delta_1(-1) = 0$, then $u \in Z^1(\Omega)$ and one has the following strong solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{Z^0} \lesssim \|\partial_x f\|_{L^2} + \|\Delta_0\|_{\mathscr{H}^1_z} + \|\Delta_1\|_{\mathscr{H}^1_z}. \tag{2.11}$$

Proof. The key point is the following argument: if $\partial_x u$ enjoys $L^2_x H^1_z$ regularity, then $\partial_x u$ is the unique weak solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = \partial_x f, \\ w_{|\Sigma_i} = \Delta_i, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{2.12}$$

Then estimate (2.10) follows from (2.3) and estimate (2.11) follows from (2.8).

Hence, let us prove that, if $\partial_x u \in L^2_x H^1_z$, then $\partial_x u$ is a weak solution to (2.12). Let

$$\begin{aligned} \mathscr{V} := \big\{ v \in C^\infty(\overline{\Omega}); \quad & v = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1), \\ & \partial_x v = 0 \text{ on } \{x_0\}\times(-1,0) \text{ and } \{x_1\}\times(0,1)\}. \end{aligned} \tag{2.13}$$

Let $v \in \mathscr{V}$. Then $\partial_x v$ is an admissible test function for Definition 2.1. Hence, since u is the weak solution to (2.1), one has

$$-\int_\Omega zu\partial_x(\partial_x v) + \int_\Omega \partial_z u \partial_z(\partial_x v) = \int_\Omega f(\partial_x v) + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v). \tag{2.14}$$

The $H^1_x H^1_z$ regularity of u legitimates integrations by parts in x in the left-hand side. Thus

$$\begin{aligned} \Big[-\int_{-1}^1 zu\partial_x v\Big]_{x_0}^{x_1} + \int_\Omega z(\partial_x u)\partial_x v + \Big[\int_{-1}^1 \partial_z u \partial_z v\Big]_{x_0}^{x_1} - \int_\Omega \partial_z(\partial_x u)\partial_z v \\ = \Big[\int_{-1}^1 fv\Big]_{x_0}^{x_1} - \int_\Omega f_x v + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v), \end{aligned} \tag{2.15}$$

which, after taking the boundary conditions into account, integrating by parts in z in the boundary terms $\int_{-1}^1 \partial_z u \partial_z v$ and recalling (2.9) yields

$$-\int_\Omega z(\partial_x u)\partial_x v + \int_\Omega \partial_z(\partial_x u)\partial_z v = \int_\Omega f_x v + \int_{\Sigma_0} z\Delta_0 v - \int_{\Sigma_1} z\Delta_1 v. \tag{2.16}$$

Since $\mathscr{V}$ is dense in the set of test functions for Definition 2.1, this proves that $\partial_x u$ is the weak solution to (2.12). □

We start by defining "dual profiles" which are necessary to state our orthogonality conditions.

Lemma 2.6 (Dual profiles)**.** *We define* $\overline{\Phi^0}$, $\overline{\Phi^1} \in Z^0(\Omega_\pm)$ *as the unique solutions to*

$$\begin{cases} -z\partial_x \overline{\Phi^j} - \partial_{zz}\overline{\Phi^j} = 0 \quad \text{in } \Omega_\pm, \\ \left[\overline{\Phi^j}\right]_{|z=0} = \mathbf{1}_{j=1}, \\ \left[\partial_z \overline{\Phi^j}\right]_{|z=0} = -\mathbf{1}_{j=0}, \\ \overline{\Phi^j}_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases} \tag{2.17}$$

Proof. Uniqueness is straightforward. Given $j \in \{0,1\}$ and two solutions to (2.17), let ϕ denote their difference. Then $\phi \in Z^0(\Omega_\pm)$ and both ϕ and $\partial_z\phi$ are continuous across the line $\{z=0\}$. Hence, $\phi \in Z^0(\Omega)$ and ϕ is the solution to a problem of the form (2.1) (with reversed tangential direction). So $\phi = 0$ since weak solutions to such problems are unique in Z^0.

We prove the existence of $\overline{\Phi^0}$. We define $\overline{\Phi^0}(x,z) := -z\mathbf{1}_{z>0}\zeta(z) + \Psi^0(x,z)$, where we choose $\zeta \in C_c^\infty(\mathbb{R})$ such that $\zeta \equiv 1$ in a neighborhood of $z=0$ and $\operatorname{supp}\zeta \subset (-1/2, 1/2)$, and where $\Psi^0 \in L^2((x_0,x_1); H_0^1(-1,1))$ is the unique weak solution to

$$\begin{cases} -z\partial_x\Psi^0 - \partial_{zz}\Psi^0 = -2\mathbf{1}_{z>0}\zeta'(z) - z\mathbf{1}_{z>0}\zeta''(z) & \text{in } \Omega, \\ \Psi^0(x_0,z) = 0 & \text{for } z \in (-1,0), \\ \Psi^0(x_1,z) = z\zeta(z) & \text{for } z \in (0,1), \\ \Psi^0_{|z=\pm 1} = 0. \end{cases} \tag{2.18}$$

By Proposition 2.4, $\Psi^0 \in Z^0(\Omega)$. Hence $\partial_{zz}\overline{\Phi^0} \in L^2(\Omega_\pm)$ and $z\partial_x\overline{\Phi^0} \in L^2(\Omega_\pm)$.

The construction of the profile $\overline{\Phi^1}$ is similar and is left to the reader. For example, one can decompose $\overline{\Phi^1}$ as $\overline{\Phi^1}(x,z) = \mathbf{1}_{z>0}\zeta(z) + \Psi^1(x,z)$, where, similarly, $\Psi^1 \in Z^0(\Omega)$. □

Remark 2.7. *The jump conditions in* (2.17) *prevent the dual profiles from enjoying vertical regularity across the line* $\{z=0\}$*. More subtly, even inside each half-domain, neither the* $\overline{\Phi^j}$ *nor their lifted version the* Ψ^j *enjoy tangential regularity. Indeed, formally,* $\partial_x\overline{\Phi^j}$ *and* $\partial_x\Psi^j$ *satisfy systems of the form* (2.1) *(with reversed tangential direction) with zero source term and zero boundary data. Hence, if they were sufficiently regular, they would be zero by the uniqueness results of Appendix A, and so would* $\overline{\Phi^j}$ *and* Ψ^j *by integration, contradicting* (2.17)*. We will see in Corollary 2.29 that these dual profiles indeed do contain an explicit singular part localized near the endpoints* $(x_i, 0)$*.*

We now turn to the main result of this section, which gives a necessary and sufficient condition for the solutions to enjoy the mentioned tangential regularity. Strangely, we could not find a proof of Proposition 2.9 in the literature, although some works mention orthogonality conditions (see [22, Equation (4.2)] or [54]). Hence, we provide here a full proof. This strategy will be extended in Section 5.2 to equations with smooth variable coefficients (see Proposition 5.5). We prove further that these orthogonality conditions are not empty.

We will work with the following space of data triplets:

$$\begin{aligned} \mathcal{H}_K := \Big\{ (f,\delta_0,\delta_1) \in H_x^1 L_z^2 \times \mathscr{H}_z^1(\Sigma_0) \times \mathscr{H}_z^1(\Sigma_1); \quad (\Delta_0,\Delta_1) \in \mathscr{H}_z^1(\Sigma_0)\times\mathscr{H}_z^1(\Sigma_1) \\ \text{and } \delta_0(1) = \delta_1(-1) = \Delta_0(1) = \Delta_1(-1) = 0 \Big\}, \end{aligned} \tag{2.19}$$

where Δ_i is defined in (2.9), with the associated norm

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K} := \|f\|_{H^1_x L^2_z} + \sum_{i\in\{0,1\}} \|\delta_i\|_{\mathscr{H}^1_z} + \|\Delta_i\|_{\mathscr{H}^1_z}. \tag{2.20}$$

Lemma 2.8. *For $(f,\delta_0,\delta_1)\in\mathcal{H}_K$, one has $\delta_i\in H^2(\Sigma_i)$ with $\|\delta_i\|_{H^2}\lesssim\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K}$.*

Proof. For $i\in\{0,1\}$, recalling (2.9), one has

$$\|\delta_i''\|_{L^2(\Sigma_i)} \le \|\delta_i'' + f(x_i,\cdot)\|_{L^2(\Sigma_i)} + \|f(x_i,\cdot)\|_{L^2(\Sigma_i)} \lesssim \|\Delta_i\|_{\mathscr{L}^2_z} + \|f\|_{H^1_x L^2_z}. \tag{2.21}$$

Moreover, one checks that $\|\delta_i\|_{L^2}\lesssim\|\delta_i\|_{\mathscr{L}^2_z}+\|\delta_i''\|_{L^2}$ (proceeding e.g. as in Lemma B.7). □

Proposition 2.9. *Let $(f,\delta_0,\delta_1)\in\mathcal{H}_K$. The unique weak solution u to (2.1) belongs to $H^1_xH^1_z$ if and only if, for $j=0$ and $j=1$,*

$$\int_\Omega \partial_x f\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j} = \partial_z^j\delta_1(0) - \partial_z^j\delta_0(0), \tag{2.22}$$

where $\overline{\Phi^0}$ and $\overline{\Phi^1}$ are defined in Lemma 2.6.

Furthermore, in this case, u actually belongs to $Z^1(\Omega)$ and the following estimate holds:

$$\|u\|_{Z^1}\lesssim\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K}. \tag{2.23}$$

Proof. ***Step 1.*** *We exhibit possible discontinuities.* Let us consider the unique solution $u\in Z^0(\Omega)$ to (2.1). Following the strategy sketched by Goldstein and Mazumdar[7] [25, Theorem 4.2], we introduce the unique strong solution $w\in Z^0(\Omega)$ to (2.12), so that w is a good candidate for $\partial_x u$. The idea is then to introduce the function u_1 defined by

$$u_1(x,z) := \begin{cases} \delta_0(z) + \int_{x_0}^x w(x',z)\,\mathrm{d}x' & \text{in } \Omega_+,\\ \delta_1(z) - \int_x^{x_1} w(x',z)\,\mathrm{d}x' & \text{in } \Omega_- \end{cases} \tag{2.24}$$

so that $\partial_x u_1 = w$ almost everywhere. Furthermore, it can be easily proved that, in $\mathcal{D}'(\Omega_\pm)$,

$$z\partial_x u_1 - \partial_{zz}u_1 = f. \tag{2.25}$$

However, this does not entail that u_1 is a solution to this equation in the whole domain. Indeed, u_1 and $\partial_z u_1$ may have discontinuities across the line $\{z=0\}$. One checks that u_1 and $\partial_z u_1$ are continuous across $z=0$ if and only if

$$\begin{aligned} \int_{x_0}^{x_1} w(x,0)\,\mathrm{d}x &= \delta_1(0)-\delta_0(0),\\ \int_{x_0}^{x_1} w_z(x,0)\,\mathrm{d}x &= \partial_z\delta_1(0) - \partial_z\delta_0(0). \end{aligned} \tag{2.26}$$

The two integrals are well-defined since w_z and w_{zz} belong to $L^2(\Omega)$.

[7] Oddly, Goldstein and Mazumdar do not mention the orthogonality conditions (2.22). They merely state that, "since $\partial_{zz}u_1 = z\partial_x u_1 - f$ in $\mathcal{D}'(\Omega_\pm)$ and since $zu_1, f\in C^0([x_0,x_1];L^2(-1,1))$, consequently $z\partial_x u_1 - \partial_{zz}u_1 = f$ in $L^2(\Omega)$". However, these orthogonality conditions are non-empty, as we show below (see Proposition 2.13).

Step 2. *We compute the horizontal mean value of w and w_z using the dual profiles.* Let $\phi \in Z^0(\Omega_\pm)$ such that $\phi_{|\partial\Omega\setminus(\Sigma_0\cap\Sigma_1)} = 0$. Since $w \in Z^0(\Omega)$, it satisfies (2.12) almost everywhere, so that we can multiply the equation by ϕ and integrate over Ω_+. Hence,

$$\int_{\Omega_+} f_x\phi = \int_{\Omega_+} (z\partial_x w - \partial_{zz} w)\phi, \tag{2.27}$$

where, on the one hand,

$$\int_{\Omega_+} z(\partial_x w)\phi = \int_{\Sigma_1} z\Delta_1\phi - \int_{\Omega_+} zw\partial_x\phi \tag{2.28}$$

and on the other hand,

$$-\int_{\Omega_+} (\partial_{zz} w)\phi = \int_{x_0}^{x_1} (\partial_z w\phi - w\partial_z\phi)(x,0^+)\,\mathrm{d}x - \int_{\Omega_+} w\partial_{zz}\phi. \tag{2.29}$$

Thus, performing the same computation on Ω_- and summing both contributions yields

$$\begin{aligned}\int_{x_0}^{x_1} (\partial_z w[\phi]_{|z=0} - w[\partial_z\phi]_{|z=0})(x,0)\,\mathrm{d}x = \int_\Omega f_x\phi + \int_{\Sigma_0} z\Delta_0\phi - \int_{\Sigma_1} z\Delta_1\phi \\ + \sum_\pm \int_{\Omega_\pm} w(z\partial_x\phi + \partial_{zz}\phi).\end{aligned} \tag{2.30}$$

Hence, for $j \in \{0,1\}$,

$$\int_{x_0}^{x_1} \partial_z^j w(x,0)\,\mathrm{d}x = \int_\Omega f_x\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j}, \tag{2.31}$$

where the dual profiles $\overline{\Phi^0}$ and $\overline{\Phi^1}$ are defined in Lemma 2.6.

Step 3. *Conclusion.* Assume that the orthogonality conditions (2.22) are satisfied for $j = 0$ and $j = 1$. Then (2.26) holds, and a consequence, $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$. Thus $u_1 \in L^2((x_0,x_1);H_0^1(-1,1))$ is a weak solution to (2.1). We infer from the uniqueness of weak solutions that $u = u_1$, and therefore $\partial_x u = w \in Z^0$. Hence $u \in H^1((x_0,x_1);H_0^1(-1,1))$. Estimate (2.23) follows from (2.8) and (2.11).

Conversely, if u is a solution to (2.1) with $H^1((x_0,x_1);H_0^1(-1,1))$ regularity, then $\partial_x u$ is a weak solution to (2.12) (see the proof of Proposition 2.5) and u is given in terms of $\partial_x u$ by (2.24) almost everywhere. Thus $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$. Hence $\int_{x_0}^{x_1} u_x(x,0)\,\mathrm{d}x = \delta_1(0) - \delta_0(0)$ and $\int_{x_0}^{x_1} u_{xz}(x,0)\,\mathrm{d}x = \partial_z\delta_1(0) - \partial_z\delta_0(0)$, and thus the orthogonality conditions (2.22) are satisfied. □

Theorem 1 follows easily from Proposition 2.9. Indeed, one easily checks from (1.3) and (2.19) that $\mathcal{X}_B \hookrightarrow \mathcal{H}_K$. Moreover, by Lemma 1.15, $Z^1(\Omega) \hookrightarrow Q^1$ (defined in (1.5)). The fact that $\mathcal{X}_{B,\mathrm{sg}}^\perp$ is of codimension 2 follows from Proposition 2.13 or Proposition 2.31 below.

Definition 2.10. *In the sequel, we denote by $\overline{\ell^j}$ the linear forms associated with the orthogonality conditions* (2.22) *for the linear shear flow problem, i.e., for $(f,\delta_0,\delta_1) \in \mathcal{H}_K$, we set*

$$\overline{\ell^j}(f,\delta_0,\delta_1) := \partial_z^j\delta_0(0) - \partial_z^j\delta_1(0) + \int_\Omega \partial_x f\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j}. \tag{2.32}$$

Lemma 2.11. *The linear forms $\overline{\ell^j}$ for $j \in \{0,1\}$ are continuous over $\mathcal{H}_K$.*

Proof. First, by Lemma 2.8, for $(f,\delta_0,\delta_1)\in\mathcal{H}_K$, $\delta_i\in H^2(\Sigma_i)$ so that $\delta_i(0)$ and $\delta_i'(0)$ depend continuously on $(f,\delta_0,\delta_1)\in\mathcal{H}_K$. Second, by Lemma 2.6, $\overline{\Phi^j}\in Z^0(\Omega_\pm)$ so, in particular $\overline{\Phi^j}\in L^2(\Omega)$. Hence $f\mapsto\int_\Omega\partial_x f\overline{\Phi^j}$ is continuous on $H^1_xL^2_z$. Eventually, by Lemma 1.10, $\overline{\Phi^j}(x_i,\cdot)\in\mathscr{H}^1_z(\Sigma_i)\hookrightarrow\mathscr{L}^2_z(\Sigma_i)$, so $(f,\delta_0,\delta_1)\mapsto\int_{\Sigma_i}z\Delta_i(z)\overline{\Phi^j}(x_i,z)\,\mathrm{d}z$ is continuous on $\mathcal{H}_K$. □

Remark 2.12. *Although this continuity result will be sufficient for most of our purpose, the linear forms $\overline{\ell^j}$ are in fact continuous for weaker topologies than the one of $\mathcal{H}_K$. In particular, one does not need $f\in H^1_xL^2_z$ (see Remark 2.30).*

We now prove that the orthogonality conditions (2.22) are non-empty and independent.

Proposition 2.13 (Independence of the orthogonality conditions)**.** *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent over $C^\infty_c(\Omega)\times\{0\}\times\{0\}\subset\mathcal{H}_K$.*

Proof. Proceeding by contradiction, let $(c_0,c_1)\in\mathbb{R}^2$ such that, for every $f\in C^\infty_c(\Omega)$, there holds $c_0\overline{\ell^0}(f,0,0)+c_1\overline{\ell^1}(f,0,0)=0$. Then $\Phi^c:=c_0\overline{\Phi^0}+c_1\overline{\Phi^1}$ satisfies $\int_\Omega\partial_x f\Phi^c=0$ for every $f\in C^\infty_c(\Omega)$, so $\partial_x\Phi^c=0$ in $\mathcal{D}'(\Omega_+)$. Since $\Phi^c(x_1,z)=0$ for $z\in(0,1)$ and $\Phi^c\in Z^0(\Omega_+)$, this implies that $\Phi^c=0$ in Ω_+ (since Z^0 functions have traces in the usual sense, see Lemma 1.10). The same holds in Ω_-. Hence $[\Phi^c]_{|z=0}=[\partial_z\Phi^c]_{|z=0}=0$, which implies $c_0=c_1=0$. □

Remark 2.14. *Proposition 2.13 of course implies that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\mathcal{H}_K$. Although Proposition 2.13 gives a prominent role to the source term f, we will actually also prove that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\{0\}\times C^\infty_c(\Sigma_0)\times C^\infty_c(\Sigma_1)\subset\mathcal{H}$. This property relies on the structure of the dual profiles $\overline{\Phi^j}$ near the points $(x_i,0)$, and will be proved at the end of this section (see Proposition 2.31).*

Similarly, it can be easily checked that the control of k derivatives in x requires the cancellation of $2k$ independent conditions. Although controlling a single x-derivative will be sufficient in the sequel to obtain our nonlinear result, we establish here this short higher-regularity statement as an illustration. More precisely, we have the following result.

Lemma 2.15. *Let $k\geq1$. Let $f\in C^\infty(\overline{\Omega})$, $\delta_i\in C^\infty(\overline{\Sigma_i})$. Define recursively Δ^n_i for $0\leq n\leq k$ and $z\in\Sigma_i$ by*

$$\Delta^0_i(z):=\delta_i(z), \tag{2.33}$$

$$\Delta^n_i(z):=\frac{1}{z}\left(\partial_x^{n-1}f(x_i,z)+\partial_{zz}\Delta^{n-1}_i(z)\right). \tag{2.34}$$

Assume that the following compatibility conditions are satisfied:

$$\forall n\in\{0,\cdots,k\},\quad \Delta^n_0(1)=\Delta^n_1(-1)=0. \tag{2.35}$$

Assume furthermore that for all $n\in\{0,\cdots,k\}$, $\Delta^n_i\in\mathscr{H}^1_z(\Sigma_i)$.

Let u be the unique solution to (1.6). Then $u\in H^k_xH^2_z$ if and only if the following orthogonality conditions are satisfied

$$\overline{\ell^j}(\partial^n_x f,\Delta^n_0,\Delta^n_1)=0,\quad\forall n\in\{0,\cdots,k-1\},\quad j\in\{0,1\}. \tag{2.36}$$

Furthermore, these $2k$ orthogonality conditions are linearly independent.

Proof. First, notice that $\partial_x^n u$ satisfies formally

$$\begin{cases} (z\partial_x - \partial_{zz})\partial_x^n u = \partial_x^n f \quad \text{in } \Omega, \\ \partial_x^n u_{|z=\pm 1} = 0, \\ \partial_x^n u_{|\Sigma_i} = \Delta_i^n. \end{cases}$$

The first part of the statement follows easily from Proposition 2.9 and Proposition 2.5 and from an induction argument.

Let us now check the independence of the orthogonality conditions. We extend the methodology used in the proof of Proposition 2.13. Assume that there exist $c_n^j \in \mathbb{R}$, $0 \leq n \leq k-1$, $j = 0, 1$ such that for all (f, δ_0, δ_1) satisfying the assumptions of the lemma,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, \Delta_0^n, \Delta_1^n) = 0.$$

In particular, for any $f \in C_c^\infty(\Omega)$,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, 0, 0) = 0,$$

i.e.

$$\sum_{n=0}^{k-1} \int_\Omega \partial_x^n f \left(\sum_{j=0,1} c_n^j \overline{\Phi^j} \right) = 0.$$

This means that

$$\sum_{j=0,1} \sum_{n=0}^{k-1} (-1)^n c_n^j \partial_x^n \overline{\Phi^j} = 0$$

in the sense of distributions. Since $[\partial_x^n \overline{\Phi^j}]_{|z=0} = [\partial_x^n \partial_z \overline{\Phi^j}]_{|z=0} = 0$ for $n \geq 1$, we infer that

$$\left[c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1} \right]_{|z=0} = \left[\partial_z (c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1}) \right]_{|z=0} = 0.$$

Once again, using the jump conditions on $\overline{\Phi^j}$, we deduce that $c_0^j = 0$, and thus

$$\partial_x \left(\sum_{j=0,1} \sum_{n=1}^{k-1} (-1)^n c_n^j \partial_x^{n-1} \overline{\Phi^j} \right) = 0.$$

It follows that

$$\sum_{j=0,1} \sum_{n=1}^{k-1} (-1)^n c_n^j \partial_x^{n-1} \overline{\Phi^j} = p(z)$$

for some function p. Note that by parabolic regularity, the profiles $\overline{\Phi^j}$ (and therefore the function p) are smooth away from the line $\{z = 0\}$. Taking the trace of the above identity on $\{x_0\} \times (-1, 0) \cup \{x_1\} \times (0, 1)$, we find that $p = 0$. Arguing by induction, we infer eventually that $c_n^j = 0$ for all $0 \leq n \leq k-1$, $j = 0, 1$. □

Corollary 2.16 (Biorthogonal basis)**.** *There exist* $\Xi^k = (f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}_K$ *for* $k \in \{0,1\}$ *such that, for every* $j,k \in \{0,1\}$*,*

$$\overline{\ell^j}(\Xi^k) = \overline{\ell^j}(f^k, \delta_0^k, \delta_1^k) = \mathbf{1}_{j=k} \tag{2.37}$$

and such that, within $\mathcal{H}_K$*,*

$$\big(\mathbb{R}\Xi^0 + \mathbb{R}\Xi^1\big)^\perp = \ker \overline{\ell^0} \cap \ker \overline{\ell^1} \tag{2.38}$$

is a vector subspace of codimension 2.

Proof. Since $\overline{\ell^0}$ and $\overline{\ell^1}$ are continuous linear forms on $\mathcal{H}_K$, by the Riesz representation theorem, they can be written as scalar products with two given triplets, say $\overline{\Xi^0}, \overline{\Xi^1} \in \mathcal{H}_K$ which are linearly independent thanks to Proposition 2.13. Then one looks for $\Xi^k = (f^k, \delta_0^k, \delta_1^k)$ as $a_k\overline{\Xi^0} + b_k\overline{\Xi^1}$ where $a_k, b_k \in \mathbb{R}^2$ are such that $a_k\langle\overline{\Xi^j};\overline{\Xi^0}\rangle + b_k\langle\overline{\Xi^j};\overline{\Xi^1}\rangle = \mathbf{1}_{j=k}$. These systems can be solved since $\overline{\Xi^0}$ and $\overline{\Xi^1}$ are free. This proves the equality (2.38). The independence of the linear forms guarantees that (2.38) is of codimension 2 in $\mathcal{H}_K$. □

2.4 Singular radial solutions in the half-plane and profile decomposition

In this subsection, we give a full description of the singularities that appear when the orthogonality conditions are not satisfied. We start by constructing singular solutions to the homogeneous equation set in the half-plane, using separation of variables in polar-like coordinates. We then localize these solutions near the critical points $(x_i, 0)$ to obtain the decomposition result of Theorem 2.

Our approach is similar to the one developed by Grisvard in [28, Section 4.4] for elliptic problems in polygonal domains (see in particular the singular profiles of equation (4.4.3.7) and the decomposition result of Theorem 4.4.3.7 therein). The main difference is that we cannot use usual polar coordinates and that the construction of the elementary singular profiles is much more technical than, for instance, the classical solution of the form $r^{\frac{1}{2}}\sin(\theta/2)$ which is involved in the resolution of Dirichlet-Neumann junctions as in the elliptic problem (1.18) mentioned in the introduction.

2.4.1 Construction of singular solutions in the half-plane

In this paragraph, we look for elementary singular radial solutions to the following problem without source-term in the half-plane:

$$\begin{cases} z\partial_x u - \partial_{zz} u = 0 & x \geq 0, z \in \mathbb{R}, \\ u(0,z) = 0 & z > 0. \end{cases} \tag{2.39}$$

Remark 2.17. *In [23], Fleming considered the related problem of finding a "fairly explicit formula" for solutions to* $z\partial_x u - \partial_{zz} u = 0$ *in a strip* $(0,1)\times\mathbb{R}$*, with prescribed boundary data at* $x = 0$*,* $z > 0$ *and* $x = 1$*,* $z < 0$*. His proof involves Whittaker functions, which are related to the confluent hypergeometric functions we use below.*

In [27], Gor'kov computes a representation formula for solutions to (2.39) *with a non-zero source term and boundary data, and proves uniqueness of such solutions, under a growth assumption of the form* $|u(0,z)| \lesssim |z|^\sigma$ *for* $0 \leq \sigma < \frac{1}{2}$ *on the line* $\{x = 0\}$*, for which he claims that uniqueness holds. The threshold* $\sigma = \frac{1}{2}$ *is precisely the scaling (at which uniqueness indeed breaks) of the first fundamental singular solution* v_0 *which we construct below.*

Our setting is a little different from the works mentioned above, as we look for (non-zero) solutions to the homogeneous equation. Similar computations were also performed in [30, 31], albeit with different boundary conditions, and therefore with a different exponent for r*, and a*

different asymptotic behavior for the profile Λ in Proposition 2.18. However we were not able to find the specific expression of the profiles from Proposition 2.18 in previous works.

Near the point $(0,0)$ which is expected to be singular, balancing the terms $z\partial_x$ and ∂_{zz} leads to the natural scaling $z \sim x^{\frac{1}{3}}$. Thus, we introduce the following polar-like coordinates $(r,t) \in [0,+\infty) \times \mathbb{R}$:

$$r := (z^2 + x^{\frac{2}{3}})^{\frac{1}{2}} \quad \text{and} \quad t := zx^{-\frac{1}{3}} \tag{2.40}$$

The reverse change of coordinates is given by

$$x = \frac{r^3}{(1+t^2)^{\frac{3}{2}}} \quad \text{and} \quad z = \frac{rt}{(1+t^2)^{\frac{1}{2}}}. \tag{2.41}$$

Since it will be convenient to switch from cartesian coordinates (x,z) to the polar-like coordinates (r,t), we compute the Jacobian

$$J(r,t) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{3x^{\frac{1}{3}}r} & \frac{z}{r} \\ -\frac{t}{3x} & \frac{1}{x^{\frac{1}{3}}} \end{pmatrix} = \begin{pmatrix} \frac{(1+t^2)^{\frac{1}{2}}}{3r^2} & \frac{t}{(1+t^2)^{\frac{1}{2}}} \\ -\frac{t(1+t^2)^{\frac{3}{2}}}{3r^3} & \frac{(1+t^2)^{\frac{1}{2}}}{r} \end{pmatrix} \tag{2.42}$$

where we have used the equalities (2.41). In particular,

$$\det J(r,t) = \frac{(1+t^2)^2}{3r^3}, \tag{2.43}$$

which we will use to compute integrals using the (r,t) variables.

By (2.42), for any C^1 function φ,

$$\partial_x \varphi = \frac{(1+t^2)^{\frac{1}{2}}}{3r^2}\partial_r \varphi - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\partial_t \varphi, \tag{2.44}$$

$$\partial_z \varphi = \frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r \varphi + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t \varphi. \tag{2.45}$$

In particular, if $u(r,t) = r^\lambda \Lambda(t)$,

$$z\partial_x u = \frac{r^{\lambda-2}}{3}\left[\lambda t\Lambda(t) - t^2(1+t^2)\partial_t \Lambda(t)\right], \tag{2.46}$$

$$\begin{aligned} \partial_{zz} u &= \left[\frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t\right]\left(r^{\lambda-1}\left(\frac{\lambda t}{(1+t^2)^{\frac{1}{2}}}\Lambda(t) + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\right) \\ &= r^{\lambda-2}\left[(\lambda-1)\left(\frac{\lambda t^2}{1+t^2}\Lambda(t) + t\partial_t\Lambda(t)\right) + (1+t^2)^{\frac{1}{2}}\partial_t\left(\frac{\lambda t}{(1+t^2)^{\frac{1}{2}}}\Lambda(t) + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\right]. \end{aligned} \tag{2.47}$$

We are now ready to construct solutions to (2.39) using these coordinates.

Proposition 2.18. *For every $k \in \mathbb{Z}$, equation (2.39) has a solution of the form*

$$v_k := r^{\frac{1}{2}+3k}\Lambda_k(t) \tag{2.48}$$

with the variables (r,t) of (2.40) and $\Lambda_k \in C^\infty(\mathbb{R};\mathbb{R})$ is a smooth bounded function satisfying $\Lambda_k(-\infty) = 1$ and $\Lambda_k(+\infty) = 0$. The profile Λ_0 is presented in Fig. 2.

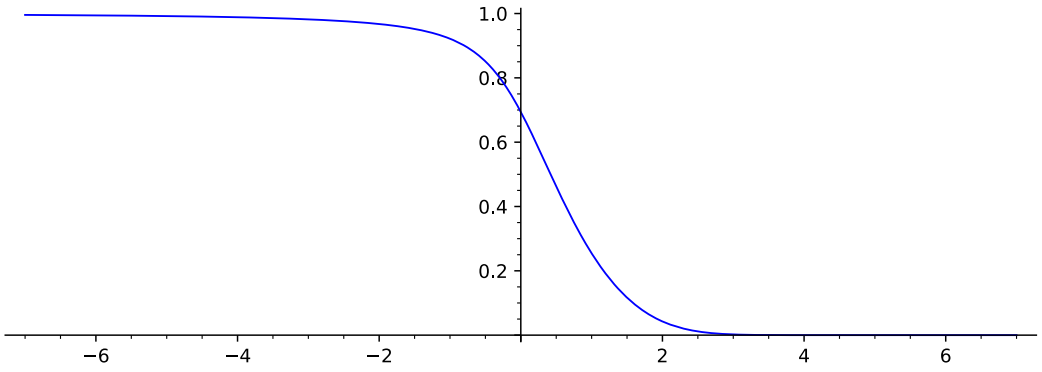

Figure 2: Plot of $t \mapsto \Lambda_0(t)$ for $t \in (-7, 7)$, highlighting the main properties: Λ_0 is a smooth, monotone decreasing function on $\mathbb{R}$, such that $\Lambda_0(-\infty) = 1$ and $\Lambda_0(+\infty) = 0$

Proof. By separation of variables, we look for a solution to (2.39) under the form $u := r^\lambda \Lambda(t)$ where $\lambda \in \mathbb{R}$ and $\Lambda : \mathbb{R} \to \mathbb{R}$ is a smooth function. The boundary condition $u(0, z) = 0$ for $z > 0$ translates to $\Lambda(+\infty) = 0$. From (2.46) and (2.47) above, one checks that such a u satisfies $z\partial_x u - \partial_{zz} u = 0$ if and only if

$$\partial_t^2 \Lambda(t) + \left(\frac{t^2}{3} + \frac{2\lambda t}{1+t^2} \right) \partial_t \Lambda(t) + \lambda \left(-\frac{1}{3} \frac{t}{1+t^2} + \frac{1 + (\lambda - 1)t^2}{(1+t^2)^2} \right) \Lambda(t) = 0. \tag{2.49}$$

To absorb the $(1 + t^2)$ factors, we perform the change of unknown $\Lambda(t) =: (1 + t^2)^{-\frac{\lambda}{2}} H(t)$. Then, Λ satisfies (2.49) if and only if H is a solution to

$$\partial_t^2 H(t) + \frac{t^2}{3} \partial_t H(t) - \frac{\lambda t}{3} H(t) = 0. \tag{2.50}$$

Moreover, for $t \neq 0$, using the change of variable $\zeta := -t^3/9$, and looking for $H(t) =: W(-t^3/9)$, we obtain that H solves (2.50) on $\mathbb{R} \setminus \{0\}$ if and only if W is a solution to

$$\zeta \partial_\zeta^2 W(\zeta) + \left(\frac{2}{3} - \zeta \right) \partial_\zeta W(\zeta) - \left(-\frac{\lambda}{3} \right) W(\zeta) = 0 \tag{2.51}$$

which corresponds to Kummer's equation, with $a = -\frac{\lambda}{3}$ and $b = \frac{2}{3}$. It is known (see [49, Section 13.2]) that (2.51) has a unique solution behaving like ζ^{-a} as $\zeta \to \infty$. This (complex valued) solution is usually denoted by $U(a, b, \zeta)$ and called *confluent hypergeometric function of the second kind*, or *Tricomi's function*. In general, U has a branch point at $\zeta = 0$. More precisely, the asymptotic ζ^{-a} holds in the region $|\arg \zeta| < \frac{3\pi}{2}$ and the principal branch of $U(a, b, \zeta)$ corresponds to the principal value of ζ^{-a}. Moreover, when b is not an integer, which is our case, one has (see [49, Equation 13.2.42]),

$$U(a, b, \zeta) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, \zeta) + \frac{\Gamma(b-1)}{\Gamma(a)} \zeta^{1-b} M(a-b+1, 2-b, \zeta), \tag{2.52}$$

where M is the *confluent hypergeometric function of the first kind* or *Kummer's function*,

$$M(a,b,\zeta) := \sum_{n\in\mathbb{N}} \frac{(a)_n}{(b)_n}\frac{\zeta^n}{n!}, \tag{2.53}$$

where $(a)_n$ and $(b)_n$ denote the rising factorial. In particular, M is an entire function of ζ. From (2.52), we see that the singularity in Tricomi's function U stems from the fractional power $\zeta^{1-b} = \zeta^{\frac{1}{3}}$. When $\zeta = -\rho$ (for $\rho > 0$), $\zeta^{\frac{1}{3}} = e^{\frac{i\pi}{3}}\rho^{\frac{1}{3}}$.

We therefore set

$$W(\zeta) := \Re\left\{e^{\frac{i\pi}{3}} U\left(-\frac{\lambda}{3}, \frac{2}{3}, \zeta\right)\right\}. \tag{2.54}$$

By linearity, W is still a solution to (2.51). Moreover, by [49, Equation 13.7.3], as $\zeta \to \infty$,

$$W(\zeta) = \Re\left\{e^{\frac{i\pi}{3}}\zeta^{-a}\left(1 + O\left(\frac{1}{|\zeta|}\right)\right)\right\}. \tag{2.55}$$

In particular, when $\lambda = \frac{1}{2} + 3k$ for $k \in \mathbb{Z}$ (and only in this situation), as $\rho \to +\infty$,

$$W(-\rho) = O(\rho^{-a-1}), \tag{2.56}$$

because $\Re\{e^{i\pi/3}e^{-ia\pi}\rho^{-a}\} = \Re\{(-1)^k e^{i\pi/3}e^{i\pi/6}\rho^{-a}\} = (-1)^k\rho^{-a}\Re\{i\} = 0$. Defining $H(t) := W(-t^3/9)$ for W as in (2.54) and recalling that $\Lambda(t) = (1+t^2)^{-\lambda/2}H(t)$ implies that $\Lambda(+\infty) = 0$. Indeed, as $t \to +\infty$,

$$\Lambda(t) = (1+t^2)^{-\frac{\lambda}{2}} O\left(t^{3(\frac{\lambda}{3}-1)}\right) = O(t^{-3}). \tag{2.57}$$

Moreover, from (2.55), we obtain that Λ is bounded as $t \to -\infty$. Indeed, as $t \to -\infty$,

$$\begin{aligned}\Lambda(t) &= (1+t^2)^{-\frac{\lambda}{2}}\Re\left\{e^{\frac{i\pi}{3}}\left(-\frac{t^3}{9}\right)^{-a}\left(1 + O\left(\frac{1}{|t|^3}\right)\right)\right\} \\ &= \frac{1}{2}9^{-\frac{1}{6}-k}(1+t^2)^{-\frac{\lambda}{2}}|t|^{-3a}\left(1+O(|t|^{-3})\right) = \frac{1}{2}9^{-\frac{1}{6}-k} + O(|t|^{-2}).\end{aligned} \tag{2.58}$$

Eventually, let us check that H is an entire function of t, which will entail that Λ is smooth. First, note that $M(-\lambda/3, 2/3, -t^3/9)$ and $M(-(\lambda-1)/3, 4/3, -t^3/9)$ are real valued and entire functions of t. Additionally,

$$\begin{aligned}\Re\left\{e^{i\pi/3}\left(-\frac{t^3}{9}\right)^{1/3}\right\} &= 9^{-1/3} \times \begin{cases} |t|/2 & \text{if } t < 0, \\ t\Re(e^{2i\pi/3}) & \text{if } t < 0 \end{cases} \\ &= -\frac{1}{2}\frac{t}{9^{1/3}}.\end{aligned}$$

Using (2.54) and (2.52), we obtain

$$H(t) = \frac{1}{2}\frac{\Gamma(1-b)}{\Gamma(a-b+1)}M\left(a, b, -\frac{t^3}{9}\right) - \frac{1}{2}\frac{t}{9^{1/3}}\frac{\Gamma(b-1)}{\Gamma(a)}M\left(a-b+1, 2-b, -\frac{t^3}{9}\right), \tag{2.59}$$

so that H is entire because M is. This also entails that H solves (2.50) even across $t = 0$. Moreover, (2.57) and (2.58) imply that Λ is bounded on $\mathbb{R}$. Eventually, using (2.58), we can define Λ_k as $2 \cdot 9^{\frac{1}{6}+k}\Lambda$, which ensures that $\Lambda_k(-\infty) = 1$. For this normalization, one deduces from (2.59) that

$$\Lambda_k(0) = 9^{\frac{1}{6}+k}\frac{\Gamma(1/3)}{\Gamma(1/6-k)}, \tag{2.60}$$

which will be used below. □

If u is a solution to (2.39), then, formally, $\partial_x u$ too (the operator $z\partial_x - \partial_{zz}$ commutes with ∂_x, and the boundary condition at $x = 0$ and $z > 0$ is satisfied thanks to the equation). This property entails that the solutions $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$ are related by a recurrence relation on the profiles Λ_k.

Lemma 2.19 (Recurrence relations)**.** *Let $k \in \mathbb{Z}$ and $c_k := \frac{1}{4} - 9k^2$. One has*

$$\partial_x v_k = c_k v_{k-1}. \tag{2.61}$$

Moreover, for every $t \in \mathbb{R}$,

$$c_k\Lambda_{k-1}(t) = \frac{(1+t^2)^{\frac{1}{2}}}{3}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - t(1+t^2)\Lambda_k'(t)\right), \tag{2.62}$$

or, equivalently,

$$\Lambda_k'(t) = \frac{1}{t(1+t^2)}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - \frac{3c_k\Lambda_{k-1}(t)}{(1+t^2)^{\frac{1}{2}}}\right). \tag{2.63}$$

Proof. By (2.44), one has $\partial_x v_k = r^{\frac{1}{2}+3(k-1)}H_k(t)$, where $H_k(t)$ is the right-hand side of (2.62). Thus $\partial_x v_k$ is a solution to (2.39) of the form studied in Proposition 2.18. Since the proof of Proposition 2.18 proceeds by equivalence, v_{k-1} is the only solution of the form $r^{\frac{1}{2}+3(k-1)}$. This entails that $H_k(t)$ is proportional to $\Lambda_{k-1}(t)$ and the constant can be identified by comparing the values at 0 using (2.60), yielding (2.62), (2.61) and (2.63) (which are all equivalent) with $c_k = \frac{1}{4} - 9k^2$.

Actually, these identities are linked with recurrence relations on Tricomi's function U. Let us give another proof of (2.63) using this approach. By the proof of Proposition 2.18,

$$\Lambda_k(t) = 2\cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k}\cdot\Re\left\{e^{\frac{i\pi}{3}}U\left(-\frac{1}{6}-k,\frac{2}{3},-\frac{t^3}{9}\right)\right\}. \tag{2.64}$$

First, using the relation $\partial_\zeta U(a-1,b,\zeta) = (1-a)U(a,b+1,\zeta)$ (see [49, Equation (13.3.22)]),

$$\begin{aligned}\Lambda_k'(t) = &-\left(\frac{1}{2}+3k\right)\frac{t}{1+t^2}\Lambda_k(t)\\ &+2\cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k}\cdot\frac{3}{t}\left(k+\frac{1}{6}\right)\Re\left\{e^{\frac{i\pi}{3}}\left(-\frac{t^3}{9}\right)U\left(-\frac{1}{6}-k+1,\frac{5}{3},-\frac{t^3}{9}\right)\right\}.\end{aligned} \tag{2.65}$$

Eventually, (2.63) follows from the relation $(b-a)U(a,b,\zeta) + U(a-1,b,\zeta) - \zeta U(a,b+1,\zeta) = 0$ (see [49, Equation (13.3.10)]). □

Remark 2.20. *We will see below that v_0 is linked with two solutions to (2.1) which have Z^0 regularity, but do not belong to $H^1_x H^1_z$. Similarly, for each $k \geq 0$, v_k is linked with solutions u such that $\partial_x^k u \in Z^0(\Omega)$ but $u \notin H^{k+1}_x H^1_z$. Conversely, for $k = -1$, one could expect to be able to construct a very weak solution u based on v_{-1} which would entail that uniqueness fails for solutions with less than $L^2_x H^1_z$ regularity.*

Lemma 2.19 entails the following decay estimates, which will be useful in the sequel.

Lemma 2.21. *For every $k \in \mathbb{Z}$, there exists $C_k > 0$ such that, for every $t \in \mathbb{R}$,*

$$|\Lambda_k(t)| + |t^3\Lambda_k'(t)| + |t^4\Lambda_k''(t)| + |t^5\Lambda_k'''(t)| \leq C_k. \tag{2.66}$$

Proof. For all $k \in \mathbb{Z}$, the bound $|\Lambda_k(t)| \leq C_k$ is already contained in Proposition 2.18 which claims that Λ_k is bounded. Since Λ_{k-1} and Λ_k are uniformly bounded over $\mathbb{R}$, we deduce from (2.63) that $t^3\Lambda_k'(t)$ is also bounded on $\mathbb{R}$. Eventually, differentiating (2.63) with respect to t leads to a uniform bound for $|t^4\Lambda_k''(t)|$ and $|t^5\Lambda_k'''(t)|$ over $\mathbb{R}$. □

Moreover, the recurrence relations of Lemma 2.19 also imply that the solutions v_k to (2.39) are smooth, up to the boundary $\{x = 0\}$, except at the origin $(0, 0)$.

Lemma 2.22. *For every $k \in \mathbb{Z}$, $v_k \in C^\infty(P_*)$, where $P_* := ([0, +\infty) \times \mathbb{R}) \setminus \{(0, 0)\}$.*

Proof. The smoothness inside the half-plane $\{x > 0\}$ follows directly from Proposition 2.18 since $\Lambda_k \in C^\infty(\mathbb{R})$ and the function $r \mapsto r^{\frac{1}{2}+3k}$ as well as the change of coordinates of (2.40) are smooth inside this domain.

By Proposition 2.18, since Λ_k is continuous on $\mathbb{R}$ and has limits at $t = \pm\infty$, we obtain that $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$ is continuous up to the boundary $\{x = 0\}$, except at the origin: $v_k \in C^0(P_*)$.

We now turn to the continuity of derivatives. Using (2.45),

$$\partial_z v_k = r^{-\frac{1}{2}+3k}\left[\left(\frac{1}{2}+3k\right)\frac{t}{(1+t^2)^{\frac{1}{2}}}\Lambda_k(t) + (1+t^2)^{\frac{1}{2}}\Lambda_k'(t)\right]. \tag{2.67}$$

Since Λ_k has limits at $t = \pm\infty$ and since, by Lemma 2.21, $t^3\Lambda_k'(t) = O(1)$, we obtain that $\partial_z v_k$ has limits at $t = \pm\infty$. Hence $\partial_z v_k \in C^0(P_*)$.

Eventually, the $C^\infty(P_*)$ regularity follows from an induction argument. Indeed, by (2.61), $\partial_x v_k = c_k v_{k-1}$, so $\partial_x v_k \in C^0(P_*)$ because $v_{k-1} \in C^0(P_*)$. And, similarly, in the vertical direction, using (2.39), $\partial_{zz} v_k = z\partial_x v_k = z c_k v_{k-1}$ so $\partial_{zz} v_k \in C^0(P_*)$. Iterating the argument concludes the proof. □

2.4.2 Localization and decomposition

We now introduce singular profiles $\bar{u}^i_{\text{sing}}$, for $i = 0, 1$, localized in the vicinity of $(x_i, 0)$ and based on the singular profiles of the previous paragraph. Let $\chi_i \in C^\infty(\overline{\Omega})$ be a cut-off function such that $\chi_i \equiv 1$ in a neighborhood of $(x_i, 0)$, and $\operatorname{supp}\chi \subset B((x_i, 0), \bar{R})$ for some $\bar{R} < \min(1, x_1 - x_0)/2$. These localized profiles are the ones involved in the main decomposition result of Theorem 2.

Definition 2.23. *For $i \in \{0, 1\}$, let*

$$\bar{u}^i_{\text{sing}}(x, z) := r_i^{\frac{1}{2}}\Lambda_0(t_i)\chi_i(x, z), \tag{2.68}$$

where Λ_0 is constructed in Proposition 2.18 and

$$r_i := \left(z^2 + |x - x_i|^{\frac{2}{3}}\right)^{\frac{1}{2}} \quad \text{and} \quad t_i := (-1)^i z|x - x_i|^{-\frac{1}{3}}. \tag{2.69}$$

Lemma 2.24. *For $i \in \{0, 1\}$, there exists $\overline{f_i} \in C^\infty(\overline{\Omega})$, with $\overline{f_i} \equiv 0$ in neighborhoods of $(x_i, 0)$ and $\{z = \pm 1\}$, such that $\bar{u}^i_{\text{sing}}$ is the unique solution with $Z^0(\Omega)$ regularity to*

$$\begin{cases} z\partial_x \bar{u}^i_{\text{sing}} - \partial_{zz}\bar{u}^i_{\text{sing}} = \overline{f_i}, \\ \bar{u}^i_{\text{sing}|\Sigma_0\cup\Sigma_1} = 0, \\ \bar{u}^i_{\text{sing}|z=\pm 1} = 0. \end{cases} \tag{2.70}$$

Moreover, $\bar{u}^i_{\text{sing}} \in C^\infty(\overline{\Omega} \setminus \{(x_i, 0)\})$ but $\bar{u}^i_{\text{sing}} \notin H^1_x H^1_z$.

Proof. By symmetry, we only prove the statement for $\overline{f_0}$ and $\bar{u}^0_{\rm sing}$. In order to alleviate the notation, we drop the index 0 in r_0, t_0 and χ_0. We introduce positive numbers $0 < r_- < r_+$ such that $\chi \equiv 1$ for $r \leq r_-$ and $\chi \equiv 0$ for $r \geq r_+$. In particular, all derivatives of χ are smooth, bounded, and supported in $\mathbf{1}_{r_-<r<r_+}$.

Straightforward computations lead to (2.70), provided that one defines

$$\overline{f_0} := r^{\frac{1}{2}}\Lambda_0(t)\left(z\partial_x\chi - \partial_{zz}\chi\right) - 2\partial_z(r^{\frac{1}{2}}\Lambda_0(t))\partial_z\chi = v_0\left(z\partial_x\chi - \partial_{zz}\chi\right) - 2\partial_z v_0\partial_z\chi. \tag{2.71}$$

Since the derivatives of χ are supported away from the point $(x_0, 0)$, the $C^\infty(\overline{\Omega})$ regularity of $\overline{f_0}$ follows directly from the smoothness of v_0 away from the origin proved in Lemma 2.22. Since $\bar{u}^0_{\rm sing}(x,z) = v_0(x,z)\chi(x,z)$, the $C^\infty(\overline{\Omega}\setminus\{(x_i,0)\})$ regularity of $\bar{u}^0_{\rm sing}$ follows from Lemma 2.22.

Therefore, to prove the lemma, there remains to prove that $\bar{u}^0_{\rm sing}$, $\partial_{zz}\bar{u}^0_{\rm sing}$ and $z\partial_x\bar{u}^0_{\rm sing}$ are in $L^2(\Omega)$ but $\partial_x\partial_z\bar{u}^0_{\rm sing} \notin L^2(\Omega)$. We will use the change of coordinates from cartesian to polar-like ones of Jacobian given by (2.43), so that, for $\varphi : \Omega \to \mathbb{R}$,

$$\|\varphi\|^2_{L^2(\Omega)} = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\varphi(r,t)^2 \,\mathrm{d}t\,\mathrm{d}r. \tag{2.72}$$

In particular, we have the following integrability criterion. Assume that φ is of the form $r^\mu H(t)\psi$ where $H(t) = O_{t\to\pm\infty}(|t|)$ and $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. If $\mu > -2$ or $\operatorname{supp}\psi \subset \mathbf{1}_{r_-<r}$, then $\varphi \in L^2(\Omega)$.

Step 1. *Preliminary estimates.* Let ψ such that $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. By the previous integrability criterion, since $\Lambda_0(t) = O(1)$, $r^{\frac{1}{2}}\Lambda_0(t)\psi \in L^2(\Omega)$. Using (2.45),

$$\partial_z\left(r^{\frac{1}{2}}\Lambda_0(t)\right) = r^{-\frac{1}{2}}\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right]. \tag{2.73}$$

By Lemma 2.21, $|t|\Lambda_0'(t) = O(|t|^{-2})$. Thus, $\partial_z(r^{\frac{1}{2}}\Lambda_0(t))\psi \in L^2(\Omega)$. Using (2.45) again,

$$\begin{aligned}\partial_{zz}\left(r^{\frac{1}{2}}\Lambda_0(t)\right) = -\frac{1}{2}r^{-\frac{3}{2}}\frac{t}{(1+t^2)^{\frac{1}{2}}}&\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right]\\ &+ r^{-\frac{3}{2}}(1+t^2)^{\frac{1}{2}}\partial_t\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right].\end{aligned} \tag{2.74}$$

Using (2.44),

$$\partial_x\left(r^{\frac{1}{2}}\Lambda_0(t)\right) = r^{-\frac{5}{2}}\left[\frac{(1+t^2)^{\frac{1}{2}}}{6}\Lambda_0(t) - \frac{t(1+t^2)^{\frac{3}{2}}}{3}\Lambda_0'(t)\right]. \tag{2.75}$$

By Lemma 2.21, $\Lambda_0'(t) = O(|t|^{-3})$. Hence, $|t|\Lambda_0(t) = O(|t|)$ and $|t|^4\Lambda_0'(t) = O(|t|)$ so, assuming additionally that $\operatorname{supp}\psi \subset \mathbf{1}_{r_-<r<r_+}$, one concludes that $\partial_x(r^{\frac{1}{2}}\Lambda_0(t))\psi \in L^2(\Omega)$.

Eventually, using (2.45),

$$\partial_{xz}(r^{\frac{1}{2}}\Lambda_0(t)) = r^{-\frac{7}{2}}\left(-\frac{t}{4}\Lambda_0(t) - \frac{1}{6}\Lambda_0'(t)(1+t^2)(1+3t^2) - \frac{t}{3}(1+t^2)^2\Lambda_0''(t)\right). \tag{2.76}$$

Step 2. *Z^0 estimates on $\bar{u}^0_{\rm sing}$.* By Step 1, $\bar{u}^0_{\rm sing}$ and $\partial_{zz}\bar{u}^0_{\rm sing}$ belong to $L^2(\Omega)$. Since $z\partial_x\bar{u}^0_{\rm sing} = \overline{f_0} + \partial_{zz}\bar{u}^0_{\rm sing}$ and $\overline{f_0} \in L^2(\Omega)$, we infer that $z\partial_x\bar{u}^0_{\rm sing} \in L^2(\Omega)$. Hence $\bar{u}^0_{\rm sing} \in Z^0(\Omega)$.

Step 3. *Lack of $H^1_x H^1_z$ estimate for $\bar{u}^0_{\rm sing}$.* Recalling (2.76),

$$\partial_x \partial_z \bar{u}^0_{\rm sing} = r^{-\frac{7}{2}} h(t)\chi + \partial_x(r^{\frac{1}{2}}\Lambda_0(t))\partial_z\chi + \partial_z(r^{\frac{1}{2}}\Lambda_0(t)\partial_x\chi), \tag{2.77}$$

where, by (2.76), the function h is given by

$$h(t) = -\frac{t}{4}\Lambda_0(t) - \frac{1}{6}\Lambda_0'(t)(1+t^2)(1+3t^2) - \frac{t}{3}(1+t^2)^2\Lambda_0''(t).$$

Using (2.63) together with the relation $\Lambda_k(-\infty) = 9^{-\frac{1}{6}-k}/2$, we find that as $t \to -\infty$,

$$\begin{aligned}
\Lambda_0(t) =& a + \frac{b}{t^2} + \frac{c}{t^3} + O(t^{-4}),\\
\Lambda_0'(t) =& -\frac{2b}{t^3} - \frac{3c}{t^4} + O(t^{-5}),\\
\Lambda_0''(t) =& \frac{6b}{t^4} + \frac{12c}{t^5} + O(t^{-6}),
\end{aligned}$$

and the coefficients a, b, c are defined by $a = \Lambda_0(-\infty)$, $-2b = a/2$, and $-3c = 3c_0\Lambda_{-1}(-\infty)$. We infer that as $t \to -\infty$,

$$h(t) = \frac{3c}{2} - \frac{12c}{3} + O(t^{-1}) \sim \frac{5}{2}c_0\Lambda_{-1}(-\infty) \neq 0.$$

Hence $h \neq 0$.

The last two terms in the right-hand side of (2.77) belong to $L^2(\Omega)$ according to the previous computations. Since $h \neq 0$, the L^2 norm of the first term is bounded from below by

$$c\int_0^{r_-} r^{-7}r^3 \,\mathrm{d}r = +\infty. \tag{2.78}$$

and thus $\partial_x\partial_z\bar{u}^0_{\rm sing} \notin L^2(\Omega)$. □

Actually, we have the following regularity on the profiles $\bar{u}^i_{\rm sing}$, which is slightly better than Z^0.

Lemma 2.25. *For all $\sigma < \frac{1}{2}$, $\bar{u}^i_{\rm sing} \in H_x^{\frac{2+\sigma}{3}} L^2_z \cap L^2_x H^{2+\sigma}_z \hookrightarrow H_x^{\frac{2+\sigma}{3}} L^2_z \cap H_x^{\frac{\sigma}{3}} H^2_z$ and this is optimal. More precisely, $\bar{u}^i_{\rm sing} \notin H_x^{\frac{5}{6}} L^2_z \cap H_x^{\frac{1}{6}} H^2_z$.*

Proof. The proof follows from an easy scaling argument. We start with the z derivative and focus on $\bar{u}^0_{\rm sing}$. Dropping the index 0 in r_0 and t_0 as in the previous proof, we have, using (2.40) and Definition 2.23, and setting $\chi(x,z) := \chi_0(x_0+x,z)$,

$$\bar{u}^0_{\rm sing}(x_0+x,z) = x^{\frac{1}{6}}\varphi\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi(x,z),$$

where $\varphi(t) = (1+t^2)^{\frac{1}{4}}\Lambda_0(t)$. Therefore,

$$\partial_z^2\bar{u}^0_{\rm sing}(x_0+x,z) = x^{-\frac{1}{2}}\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi + 2x^{-\frac{1}{6}}\varphi'\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi_z + x^{\frac{1}{6}}\varphi\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi_{zz}. \tag{2.79}$$

We focus on the regularity of the first term, which is the most singular. We have, for any $\sigma > 0$,

$$\left\| x^{-\frac{1}{2}} \varphi'' \left(\frac{z}{x^{\frac{1}{3}}} \right) \chi(x,z) \right\|^2_{L^2_x H^\sigma_z} \leq \int \frac{1}{x} \frac{1}{|z-z'|^{1+2\sigma}} \left(\varphi'' \left(\frac{z}{x^{\frac{1}{3}}} \right) - \varphi'' \left(\frac{z'}{x^{\frac{1}{3}}} \right) \right)^2 \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}z'.$$

Changing variables in the above integral, we get

$$\left\| x^{-\frac{1}{2}} \varphi'' \left(\frac{z}{x^{\frac{1}{3}}} \right) \chi(x,z) \right\|^2_{L^2_x H^\sigma_z} \lesssim \|\varphi''\|^2_{H^\sigma(\mathbb{R})} \int_0^{x_1 - x_0} |x|^{-\frac{2}{3} - \frac{2\sigma}{3}} \, \mathrm{d}x.$$

The integral in the right-hand side is finite if and only if $\sigma < \frac{1}{2}$. Moreover, $\|\varphi''\|^2_{H^\sigma(\mathbb{R})} \leq \|\varphi''\|^2_{H^1(\mathbb{R})}$. From the definition of φ and the decay bounds of Lemma 2.21, we infer that $\varphi'' \in H^1(\mathbb{R})$. This shows that $\bar{u}^i_{\mathrm{sing}} \in L^2_x H^{2+\sigma}_z$ for $\sigma < \frac{1}{2}$.

The bound in $H_x^{\frac{2+\sigma}{3}} L^2_z$ is obtained similarly and left to the reader.

Conversely, if one had $\bar{u}^i_{\mathrm{sing}} \in H^{\frac{5}{6}}_x L^2_z \cap H^{\frac{1}{6}}_x H^2_z$, by the fractional trace theorem [42, Equation (4.7), Chapter 1], one would have $\bar{u}^i_{\mathrm{sing}} \in C^0_z(H^{2/3}_x)$. In particular, $\bar{u}^i_{\mathrm{sing}}(\cdot, 0) \in H^{2/3}(x_0, x_1)$. But, in a neighborhood of $x = 0$, $\bar{u}^i_{\mathrm{sing}}(x_0 + x, 0) = \Lambda_0(0) x^{\frac{1}{6}}$ with $\Lambda_0(0) \neq 0$. One checks that $x \mapsto x^{\frac{1}{6}} \in H^s(0,1)$ if and only if $s < 2/3$, which completes the proof. □

Eventually, we introduce the following 2×2 nonsingular matrix which translates the fact that $\bar{u}^0_{\mathrm{sing}}$ and $\bar{u}^1_{\mathrm{sing}}$ are indeed independent elementary solutions related with the non-satisfaction of the orthogonality constraints associated with $\overline{\ell^0}$ and $\overline{\ell^1}$. We will use this reference matrix multiple times in the sequel for perturbations of this shear flow situation.

Lemma 2.26. *Let $\overline{f_0}$, $\overline{f_1}$ as in Lemma 2.24 and $\overline{\Phi^0}$, $\overline{\Phi^1}$ as in Lemma 2.6. The matrix*

$$\overline{M} := \left(\int_\Omega \partial_x \overline{f_j} \, \overline{\Phi^i} \right)_{0 \leq i,j \leq 1} \in \mathcal{M}_2(\mathbb{R}) \tag{2.80}$$

is invertible.

Proof. Let $c \in \mathbb{R}^2$ such that $\overline{M} c = 0$. Then, for $j = 0, 1$,

$$\int_\Omega \partial_x (c_0 \overline{f_0} + c_1 \overline{f_1}) \overline{\Phi^j} = 0.$$

Thus, the source term for the function $c_0 \bar{u}^0_{\mathrm{sing}} + c_1 \bar{u}^1_{\mathrm{sing}}$ satisfies the orthogonality conditions (2.22) (note that in this case, the boundary data are null). It then follows from Proposition 2.9 that $c_0 \bar{u}^0_{\mathrm{sing}} + c_1 \bar{u}^1_{\mathrm{sing}} \in H^1_x H^1_z$. Localizing in the vicinity of $(x_i, 0)$, we infer that $c_i \bar{u}^i_{\mathrm{sing}} \in H^1_x H^1_z$, which, since $\bar{u}^i_{\mathrm{sing}} \notin H^1_x H^1_z$ (by Lemma 2.24), implies that $c_i = 0$. Therefore, $c = 0$ and $\overline{M}$ is invertible. □

Corollary 2.27 (Decomposition into singular profiles)**.** *Let $(f, \delta_0, \delta_1) \in \mathcal{H}_K$ and $u \in Z^0(\Omega)$ be the unique solution to* (2.1)*. Then there exists two real constants c_0, c_1 and a function $u_{\mathrm{reg}} \in Z^1(\Omega)$, as defined in* (1.34)*, such that*

$$u = c_0 \bar{u}^0_{\mathrm{sing}} + c_1 \bar{u}^1_{\mathrm{sing}} + u_{\mathrm{reg}}. \tag{2.81}$$

Proof. We recall the definition of the matrix $\overline{M}$ from Lemma 2.26. Since $\overline{M}$ is invertible, we may define $c = (c_0, c_1)$ such that

$$\overline{M}c = \begin{pmatrix} \overline{\ell^0}(f, \delta_0, \delta_1) \\ \overline{\ell^1}(f, \delta_0, \delta_1) \end{pmatrix}. \tag{2.82}$$

Let $\overline{f_0}$ and $\overline{f_1}$ as in Lemma 2.24. By construction, the triplet $(f - c_0\overline{f_0} - c_1\overline{f_1}, \delta_0, \delta_1)$ satisfies the orthogonality conditions from Proposition 2.9. It follows that the solution u_{reg} to

$$\begin{cases} z\partial_x u_{\mathrm{reg}} - \partial_{zz} u_{\mathrm{reg}} = f - c_0\overline{f_0} - c_1\overline{f_1} \quad \text{in } \Omega, \\ u_{\mathrm{reg}|\Sigma_i} = \delta_i, \\ u_{\mathrm{reg}|z=\pm 1} = 0 \end{cases} \tag{2.83}$$

satisfies $u_{\mathrm{reg}} \in H^1_x(H^1_z)$. Thus, estimate (2.11) of Proposition 2.5 ensures that $\partial_x u_{\mathrm{reg}} \in Z^0(\Omega)$, i.e. $u_{\mathrm{reg}} \in Z^1$. Now, u and $u_{\mathrm{reg}} + c_0\bar{u}^0_{\mathrm{sing}} + c_1\bar{u}^1_{\mathrm{sing}}$ both belong to $Z^0(\Omega)$ and satisfy system (2.1). By the uniqueness result of Proposition 2.2, the result follows. □

Theorem 2 follows easily from Corollary 2.27. Indeed, one easily checks from (1.3) and (2.19) that $\mathcal{X}_B \hookrightarrow \mathcal{H}_K$. Moreover, by Proposition 1.7, $Z^0 \hookrightarrow H^{2/3}_x L^2_z \cap L^2_x H^2_z$ and, by Lemma 1.15, $Z^1 \hookrightarrow Q^1$ (defined in (1.5)). The rest of the conclusions on $\bar{u}^i_{\mathrm{sing}}$ are derived in Lemma 2.24.

Remark 2.28. *The constants c_0, c_1 from Corollary 2.27 depend (linearly) on u, but do not depend on the choice of the truncation functions χ_i. Indeed, if χ_0', χ_1' is another truncation, associated with constants c_0', c_1', then applying Corollary 2.27 twice yields*

$$c_0\bar{u}^0_{\mathrm{sing}} + c_1\bar{u}^1_{\mathrm{sing}} - c_0'(\bar{u}^0_{\mathrm{sing}})' - c_1'(\bar{u}^1_{\mathrm{sing}})' \in Z^1. \tag{2.84}$$

Therefore, in a small neighborhood $V_i = \chi_i^{-1}(\{1\}) \cap (\chi_i')^{-1}(\{1\})$ of $(x_i, 0)$, we obtain

$$(c_i - c_i')r_i^{\frac{1}{2}}\Lambda_0(t_i) \in H^1_x H^1_z(V_i), \tag{2.85}$$

and therefore $c_i = c_i'$.

As already claimed in Remark 2.7, we can also prove a related decomposition result for the dual profiles $\overline{\Phi^j}$ defined in Lemma 2.6. Here, the decomposition *always* involves a singular part.

Corollary 2.29. *Let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$. There exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Phi_{\mathrm{reg}} \in Z^1$, as defined in* (1.34)*, such that*

$$c_0\overline{\Phi^0} + c_1\overline{\Phi^1} = (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z) + d_0\bar{u}^0_{\mathrm{sing}}(x, -z) + d_1\bar{u}^1_{\mathrm{sing}}(x, -z) + \Phi_{\mathrm{reg}}, \tag{2.86}$$

where ζ is a smooth cut-off function, equal to 1 near $z = 0$ and compactly supported in $(-1, 1)$.

Proof. Using the same decomposition as in Lemma 2.6, set

$$\Psi^c := c_0\overline{\Phi^0} + c_1\overline{\Phi^1} - (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z). \tag{2.87}$$

Then $\widetilde{\Psi^c}(x, z) := \Psi^c(x, -z)$ is the solution to

$$\begin{cases} z\partial_x\widetilde{\Psi^c} - \partial_{zz}\widetilde{\Psi^c} = g_c & \text{in } \Omega, \\ \Psi^c(x_0, z) = 0 & \text{for } z \in (0, 1), \\ \Psi^c(x_1, z) = (-c_0 z - c_1)\zeta(-z) & \text{for } z \in (-1, 0), \\ \Psi^c_{|z=\pm 1} = 0, \end{cases} \tag{2.88}$$

where $g_c = (c_1\zeta''(-z) - 2c_0\zeta'(-z) + c_0 z\zeta''(-z))\mathbf{1}_{z<0}$. Thus, (2.86) follows from Corollary 2.27, applied with $f = g_c \in C^\infty(\overline{\Omega})$, $\delta_0 = \Delta_0 = 0$ and $\delta_1(z) = (-c_0 z - c_1)\zeta(-z)$ and $\Delta_1 = 0$.

It remains to prove that $(d_0, d_1) \neq (0,0)$. By Proposition 2.9, $\widetilde{\Psi^c} \in H^1_x H^1_z$ if and only if $\overline{\ell^j}(g_c, 0, \delta_1) = 0$ for $j = 0, 1$. By Definition 2.10, since $\partial_x g_c = 0$ and $\Delta_0 = \Delta_1 = 0$,

$$\overline{\ell^j}(g_c, 0, \delta_1) = 0 \Longleftrightarrow \partial_z^j \delta_0(0) - \partial_z^j \delta_1(0) = 0. \tag{2.89}$$

Since $\delta_0 = 0$ and $\delta_1(0) = -c_1$ and $\delta_1'(0) = -c_0$, $(d_0, d_1) = (0,0)$ if and only if $\widetilde{\Psi^c} \in H^1_x H^1_z$, if and only if $(c_0, c_1) = 0$. □

Remark 2.30. *Using Corollary 2.29 and the regularity result Lemma 2.25 on $\bar{u}^i_{\mathrm{sing}}$, we see that $f \mapsto \overline{\ell^j}(f, 0, 0) = \int_\Omega \partial_x f \overline{\Phi^j}$ is not only continuous on $H^1_x L^2_z$ but also on $H^\sigma_x L^2_z$ for every $\sigma > \frac{1}{6}$. We will encounter a related threshold of tangential regularity in Proposition 2.33.*

Using the decomposition of the dual profiles, we can show that the orthogonality conditions are also independent when considering only variations of the inflow boundary data.

Proposition 2.31. *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are independent on $\{0\} \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1)$.*

Proof. By contradiction, let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$ such that, for every $\delta_0 \in C^\infty_c(\Sigma_0)$ and $\delta_1 \in C^\infty_c(\Sigma_1)$,

$$c_0 \overline{\ell^0}(0, \delta_0, \delta_1) + c_1 \overline{\ell^1}(0, \delta_0, \delta_1) = 0. \tag{2.90}$$

Let $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Phi_{\mathrm{reg}} \in Z^1$ be given by Corollary 2.29. By symmetry, assume that $d_0 \neq 0$. Then, by Definition 2.10, for every $\delta_0 \in C^\infty_c(\Sigma_0)$, defining $\Delta_0(z) := \delta_0''(z)/z$

$$\begin{aligned} 0 &= c_0 \overline{\ell^0}(0, \delta_0, 0) + c_1 \overline{\ell^1}(0, \delta_0, 0) \\ &= \int_{\Sigma_0} z\Delta_0 \left[(-c_0 z + c_1)\zeta(z) + d_0 \bar{u}^0_{\mathrm{sing}}(x_0, -z) + \Phi_{\mathrm{reg}}(x_0, z)\right] \\ &= \int_{\Sigma_0} \delta_0'' \left[(-c_0 z + c_1)\zeta(z) + d_0 \bar{u}^0_{\mathrm{sing}}(x_0, -z) + \Phi_{\mathrm{reg}}(x_0, z)\right]. \end{aligned} \tag{2.91}$$

Let $\bar{z} > 0$ small enough, one can ensure that $\zeta \equiv 1$ on $(0, \bar{z})$ and $\bar{u}^0_{\mathrm{sing}}(x_0, -z) = z^{\frac{1}{2}} \Lambda_0(-\infty) = z^{\frac{1}{2}}$ (see Definition 2.23 and Proposition 2.18). Since $Z^1 \hookrightarrow H^1_x H^2_z$, $\Phi_{\mathrm{reg}|\Sigma_0} \in H^2(\Sigma_0)$. If $\operatorname{supp} \delta_0 \subset (0, \bar{z})$ for $\bar{z} > 0$, integrating by parts yields

$$0 = d_0 \int_0^1 \left[-\frac{1}{4} z^{-\frac{3}{2}} + \varphi(z)\right] \delta_0(z), \tag{2.92}$$

where $\varphi(z) := \partial_{zz}\Phi_{\mathrm{reg}}(x_0, z) \in L^2(\Sigma_0)$. Since $z \mapsto z^{-\frac{3}{2}}$ does not belong to $L^2(0, \bar{z})$ but φ does, one easily deduces that there exists $\delta_0 \in C^\infty_c((0, \bar{z}))$ such that the right-hand side is non-zero, reaching a contradiction. □

Let us conclude this section with an easy consequence of the decomposition result from Corollary 2.27, which will be used in Section 5.2.

Corollary 2.32 (Single orthogonality condition for localized solutions)**.** *There exists a couple $(a_0, a_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that the following result holds.*

Let $(f, \delta_0, \delta_1) \in \mathcal{H}_K$ and let $u \in Z^0(\Omega)$ be the unique solution to (2.1). Assume that there exists $0 < r < \min(x_1 - x_0, 1)$ such that $\operatorname{supp} u \subset B((x_1, 0), r)^c$. Then $u \in Z^1(\Omega)$ if and only if

$$\left(a_0 \overline{\ell^0} + a_1 \overline{\ell^1}\right)(f, \delta_0, \delta_1) = 0. \tag{2.93}$$

Proof. Let us choose the cut-off function χ_1 from Definition 2.23 such that $\operatorname{supp} u \cap \operatorname{supp} \chi_1 = \emptyset$. According to Corollary 2.27, there exists $(c_0, c_1) \in \mathbb{R}^2$ and $u_{\rm reg} \in Z^1(\Omega)$ such that $u = c_0 \bar{u}^0_{\rm sing} + c_1 \bar{u}^1_{\rm sing} + u_{\rm reg}$. Multiplying this identity by χ_1, we infer that $c_1 \bar{u}^1_{\rm sing}\chi_1 = -u_{\rm reg}\chi_1 \in Z^1(\Omega)$. Lemma 2.24 then entails that $c_1 = 0$.

Therefore $u \in Z^1(\Omega)$ if and only if $c_0 = 0$. We then recall (2.82), and we denote by (a_0, a_1) the two coefficients in the first line of $\overline{M}^{-1}$. The result follows. □

2.5 Interpolation and fractional regularity

For further purposes, we will also need some fractional regularity results. Their proof relies on interpolation arguments, and therefore on the explicit expressions of the singular profiles. Due to a subtle technical difficulty, the proof of these results are postponed to Section 6.

Proposition 2.33. *Let $\sigma \in (0,1) \setminus \{1/6, 1/2\}$. Let $f \in H^\sigma_x L^2_z$, $\delta_0 \in H^2(\Sigma_0)$, $\delta_1 \in H^2(\Sigma_1)$ such that $\delta_0(1) = \delta_1(-1) = 0$.*

- *If $\sigma > 1/6$, assume that $\overline{\ell^0}(f, \delta_0, \delta_1) = \overline{\ell^1}(f, \delta_0, \delta_1) = 0$.*
- *If $\sigma > 1/2$, assume also that $\Delta_i \in \mathscr{H}^1_z(\Sigma_i)$ and $\Delta_1(-1) = \Delta_0(1) = 0$ (recall (2.9)).*

The unique strong solution $u \in Z^0(\Omega)$ to (2.1) satisfies $u \in Z^\sigma(\Omega) := [Z^0(\Omega), Z^1(\Omega)]_\sigma$, with

$$\|u\|_{Z^\sigma} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2} + \|\delta_1\|_{H^2} + \mathbf{1}_{\sigma>1/2}\left(\|\Delta_0\|_{\mathscr{H}^1_z} + \|\Delta_1\|_{\mathscr{H}^1_z}\right). \tag{2.94}$$

Remark 2.34. *The case $\sigma = 1/6$ is not covered in the above result. This critical level of regularity corresponds to the maximal continuity of the orthogonality conditions. Such critical levels are excluded from the abstract interpolation results on which we rely (see Lemma 6.1). In this case, one would expect a similar result to hold, but with a supplementary norm on the data, in the spirit of [5, 6]. The case $\sigma = 1/2$ is also excluded, but it would be possible to include it provided one introduces an appropriate additional norm.*

Remark 2.35. *The regularity assumptions on the δ_i's are not optimal and could be weakened.*

We also obtain the following analogue of Corollary 2.32 in fractional regularity.

Corollary 2.36 (Single orthogonality condition for localized solutions in fractional regularity)**.** *Let $(a_0, a_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be the couple from Corollary 2.32. Let $\sigma \in (1/6, 1) \setminus \{1/2\}$. Let $f \in H^\sigma_x L^2_z$, $\delta_0 \in H^2(\Sigma_0)$ such that $\delta_0(1) = 0$. For $\sigma > 1/2$, assume also that $\Delta_0 \in \mathscr{H}^1_z(\Sigma_0)$ and $\Delta_0(1) = 0$. Let $u \in Z^0(\Omega)$ be the unique solution to (2.1) associated with $(f, \delta_0, 0)$.*

Assume that there exists $0 < r < \min(x_1 - x_0, 1)$ such that $\operatorname{supp} u \subset B((x_1, 0), r)^c$. Then $u \in Z^\sigma(\Omega)$ if and only if

$$\left(a_0\overline{\ell^0} + a_1\overline{\ell^1}\right)(f, \delta_0, 0) = 0, \tag{2.95}$$

and in this case

$$\|u\|_{Z^\sigma} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2(\Sigma_0)} + \mathbf{1}_{\sigma>1/2}\|\Delta_0\|_{\mathscr{H}^1_z(\Sigma_0)}. \tag{2.96}$$

Proof. The proof follows the same structure as the Z^1 case.

Using Proposition 2.33, we first prove an analogue of the decomposition result Corollary 2.27 for source terms $f \in H^\sigma_x L^2_z$ with $\sigma \in (1/6, 1)$, where the conclusion is that $u_{\rm reg} \in Z^\sigma$.

The conclusion then stems from the fact that $\bar{u}^0_{\rm sing} \notin Z^\sigma$. Indeed, by Lemma 1.15, for $\sigma \geq 1/6$, $Z^\sigma \hookrightarrow H^{\frac{5}{6}}_x L^2_z \cap H^{\frac{1}{6}}_x H^2_z$. But, from Lemma 2.25, $\bar{u}^0_{\rm sing} \notin H^{\frac{5}{6}}_x L^2_z \cap H^{\frac{1}{6}}_x H^2_z$. □

3 A first nonlinear example in kinetic theory

In this section, we explain how the linear theory of Section 2 can be used in a simple nonlinear context. Before moving on to nonlinear examples from fluid mechanics in Sections 4 and 5 (which involve additional difficulties), we encourage the reader to start by reading this section where we set up the basics of our method to construct perturbative solutions to semilinear or quasilinear problems despite orthogonality conditions. In particular, we formulate a black-box abstract result in Section 3.5 which we will use in the sequel.

3.1 Description of the model and main result

As an example, we will show how one can build regular solutions to a stationary nonlinear system of Vlasov–Poisson–Fokker–Planck type, set on a bounded interval. For the sake of readability, we will focus on the following system:

$$\begin{cases} z\partial_x u + E[u]\partial_z u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{3.1}$$

where $E[u]$ is an electric force deriving from a potential $V[u]$ satisfying a Poisson equation:

$$E = \partial_x V \quad \text{where} \quad \begin{cases} \partial_{xx} V(x) = \int_{-1}^{1} u(x,z)\,\mathrm{d}z & \text{for } x \in (x_0, x_1), \\ \partial_x V_{|x=x_0} = 0. \end{cases} \tag{3.2}$$

In this toy model, the term $E[u]\partial_z u$ corresponds to a semilinear contribution, which is easily estimated since explicit integration of (3.2) and the Cauchy–Schwarz inequality yield

$$\|E[u]\|_{L^\infty(x_0,x_1)} \lesssim \|E[u]\|_{H^1(x_0,x_1)} \lesssim \|u\|_{L^2(\Omega)}. \tag{3.3}$$

Remark 3.1. *Our toy kinetic model* (3.1)-(3.2) *departs from classical kinetic models such as the one studied in [32] in the following ways:*

- *As mentioned before, the variable z is more commonly denoted by v and represents the velocity of the particles. We keep the notation z by consistency with the remainder of the paper.*
- *Usually, even if the position variable x lives in a bounded domain, the velocity variable z lives in $\mathbb{R}$ so that particles can take arbitrary speeds. Since our motivation is to understand what happens near the critical line $\{z = 0\}$ we focus here on the region $z \in [-1, 1]$. We expect that our techniques can be applied to the unbounded case to obtain similar results, provided that one works in the appropriate functional spaces to encode decay as $|z| \to \infty$.*
- *One could also enforce a non-zero Neumann boundary condition for the potential V at the left endpoint $\partial_x V_{|x=x_0} = g_0 \in \mathbb{R}$ as in [32]. This is a straightforward adaptation of the results presented below.*

The goal of the next paragraphs is to prove the following counterparts of Proposition 2.4 and Proposition 2.9 concerning the linear model (2.1) for our nonlinear toy model. We will work with the following spaces of data triplets:

$$\mathcal{H}_{FP} := \big\{(f, \delta_0, \delta_1) \in \mathcal{H}_K; \quad \delta_i'(z)/z \in \mathscr{H}^1_z(\Sigma_i) \text{ and } \delta_i'((-1)^i) = 0 \text{ for } i \in \{0, 1\}\big\} \tag{3.4}$$

with the norm

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_{FP}} := \|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K} + \|\delta_0'(z)/z\|_{\mathscr{H}^1_z} + \|\delta_1'(z)/z\|_{\mathscr{H}^1_z}, \tag{3.5}$$

where we recall that the space $\mathscr{H}^1_z$ is defined in (1.32). We also define

$$\mathcal{H}^{\perp}_{FP,\mathrm{sg}} := \{(f,\delta_0,\delta_1)\in\mathcal{H}_{FP};\quad \overline{\ell^0}(f,\delta_0,\delta_1) = \overline{\ell^1}(f,\delta_0,\delta_1) = 0\}. \tag{3.6}$$

Theorem 5. *There exists a constant $\eta>0$, and a Lipschitz submanifold $\mathcal{M}_{FP}$ of $\mathcal{H}_{FP}$ of codimension 2, containing 0 and included in the ball of radius η in $\mathcal{H}_{FP}$, modeled on $\mathcal{H}^{\perp}_{FP,\mathrm{sg}}$ and tangent to it at 0 (see Remark 3.12), such that the following statements hold:*

1. *For all $(f,\delta_0,\delta_1)\in L^2(\Omega)\times\mathscr{H}^1_z(\Sigma_0)\times\mathscr{H}^1_z(\Sigma_1)$ with $\delta_0(1)=\delta_1(-1)=0$ such that*

$$\|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z} \le \eta, \tag{3.7}$$

system (3.1)-(3.2) has a solution $u\in Z^0(\Omega)$ satisfying

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z} \tag{3.8}$$

and which is unique in a neighborhood of 0 in $Z^0(\Omega)$.

2. *For all $(f,\delta_0,\delta_1)\in\mathcal{H}_{FP}$ such that*

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_{FP}} \le \eta, \tag{3.9}$$

the locally unique solution $u\in Z^0(\Omega)$ to (3.1)-(3.2) enjoys $Z^1(\Omega)$ regularity if and only if $(f,\delta_0,\delta_1)\in\mathcal{M}_{FP}$, which corresponds to two nonlinear orthogonality conditions.

For such data, one has

$$\|u\|_{Z^1} \lesssim \|(f,\delta_0,\delta_1)\|_{\mathcal{H}_{FP}}. \tag{3.10}$$

Remark 3.2. *The nonlinearity of the Vlasov–Poisson–Fokker–Planck system (3.1) is sufficiently mild to allow for a theory of weak solutions, leading to the first statement of Theorem 5. The Prandtl system in the vicinity of the recirculation zone enjoys the same feature, accounting for the first part of Theorem 4. However, the nonlinearity in the Burgers system (1.1) is stronger, and prevents us from proving the analogue of the first statement of the above theorem.*

3.2 Well-posedness theory with low regularity

We prove in this subsection Item 1 of Theorem 5, which corresponds to the well-posedness theory at regularity Z^0, and is therefore a nonlinear counterpart of Proposition 2.4.

Lemma 3.3 (Existence of Z^0 solutions of (3.1)-(3.2))**.** *There exists $\eta>0$ such that, for any $(f,\delta_0,\delta_1)\in L^2(\Omega)\times\mathscr{H}^1_z(\Sigma_0)\times\mathscr{H}^1_z(\Sigma_1)$ with $\delta_0(1)=\delta_1(-1)=0$ satisfying (3.7), there exists a solution $u\in Z^0(\Omega)$ to (3.1)-(3.2) with (3.8).*

Proof. Let $(f,\delta_0,\delta_1)\in L^2(\Omega)\times\mathscr{H}^1_z(\Sigma_0)\times\mathscr{H}^1_z(\Sigma_1)$ with $\delta_0(1)=\delta_1(-1)=0$ satisfying (3.7) for some $\eta>0$ small enough to be chosen later.

- **Definition of the sequence.** We construct a sequence by setting $u_0 := 0$ and, for all $n \in \mathbb{N}$, we define $u_{n+1} \in Z^0(\Omega)$ by induction as the solution to

$$\begin{cases} z\partial_x u_{n+1} - \partial_{zz} u_{n+1} = f - E_n \partial_z u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|z=\pm 1} = 0, \end{cases} \tag{3.11}$$

where $E_n := E[u_n]$. At each step, by (3.3), $E_n \in L^\infty(x_0, x_1)$. Hence, since $u_n \in Z^0(\Omega)$, $f - E_n \partial_z u_n \in L^2(\Omega)$, so the existence of $u_{n+1} \in Z^0(\Omega)$ follows from Proposition 2.4.

- **Uniform bound in Z^0.** Let us prove by induction that $\|u_n\|_{Z^0} \leq 2C_P\eta$ for all $n \in \mathbb{N}$, where C_P is the constant in Pagani's estimate (2.8), provided that η is small enough. The statement is true for $n = 0$. For $n \geq 0$, by (3.3), $\|E_n\|_{L^\infty} \lesssim \|u_n\|_{L^2} \lesssim \eta$. As a consequence, it follows from Proposition 2.4 that

$$\|u_{n+1}\|_{Z^0} \leq C_P(\|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z} + \|E_n\|_\infty \|\partial_z u_n\|_{L^2}) \leq C_P\eta + C\eta^2, \tag{3.12}$$

for some C depending only on Ω. Therefore, if $C\eta < C_P$, the bound propagates by induction.

- **Convergence.** Now, let $w_n := u_{n+1} - u_n$. Then, for $n \geq 1$, w_n is a solution to

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = -(E_n - E_{n-1})\partial_z u_{n-1} - E_n \partial_z (u_n - u_{n-1}), \\ (w_n)_{|\Sigma_i} = 0, \\ (w_n)_{|z=\pm 1} = 0. \end{cases} \tag{3.13}$$

By (3.3), $\|E_n - E_{n-1}\|_{L^\infty} \lesssim \|u_n - u_{n-1}\|_{L^2}$ and $\|E_n\|_{L^\infty} \lesssim \|u_n\|_{L^2}$. Hence, by Proposition 2.4,

$$\|w_n\|_{Z^0} \lesssim \|(E_n - E_{n-1})\partial_z u_{n-1}\|_{L^2} + \|E_n \partial_z (u_n - u_{n-1})\|_{L^2} \lesssim \eta \|w_{n-1}\|_{Z^0} \tag{3.14}$$

and thus $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Z^0(\Omega)$ provided that η is small enough. Passing to the limit as $n \to \infty$, we obtain a strong solution $u \in Z^0$ with $\|u\|_{Z^0} \leq 2C_P\eta$ to (3.1)-(3.2).

Eventually, the uniform bound propagated on the sequence also passes to the limit and implies (3.8). □

Lemma 3.4 (Uniqueness of Z^0 solutions of (3.1)-(3.2))**.** *There exists $\eta > 0$ such that, for any $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathscr{H}^1_z(\Sigma_0) \times \mathscr{H}^1_z(\Sigma_1)$, (3.1)-(3.2) has at most one solution $u \in Z^0(\Omega)$ such that $\|u\|_{Z^0} \leq \eta$.*

Proof. Let $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathscr{H}^1_z(\Sigma_0) \times \mathscr{H}^1_z(\Sigma_1)$ and $u, u' \in Z^0(\Omega)$ be two solutions to (3.1)-(3.2). Then $w := u - u' \in Z^0(\Omega)$ is a solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = (E[u'] - E[u])\partial_z u' - E[u]\partial_z w, \\ w_{|\Sigma_i} = 0, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{3.15}$$

Multiplying (3.15) by w, integrating by parts and using the boundary conditions and $\partial_z E[u] = 0$, we obtain

$$\int_\Omega (\partial_z w)^2 \leq \int_\Omega |(E[u'] - E[u])\partial_z u' w| \,. \tag{3.16}$$

By (3.3), $\|E[u'] - E[u]\|_{L^\infty} \lesssim \|w\|_{L^2}$. Hence, since $w_{|z=\pm 1} = 0$, Poincaré's inequality entails that

$$\|w\|_{L^2}^2 \lesssim \|\partial_z w\|_{L^2}^2 \lesssim \|E[u'] - E[u]\|_{L^\infty} \|\partial_z u'\|_{L^2} \|w\|_{L^2} \lesssim \|\partial_z u'\|_{L^2} \|w\|_{L^2}^2. \tag{3.17}$$

Hence, there exists $C_1 > 0$ (depending only on Ω) such that

$$\|w\|_{L^2}^2 \leq C_1 \|\partial_z u'\|_{L^2} \|w\|_{L^2}^2. \tag{3.18}$$

If $C_1 \|\partial_z u'\|_{L^2} < 1$, (3.18) implies $w = 0$, so uniqueness holds in the ball of radius $1/C_1$ of $Z^0(\Omega)$. □

3.3 Nonlinear orthogonality conditions for higher regularity

We now prove Item 2 of Theorem 5, which corresponds to the well-posedness theory at regularity Z^1, under orthogonality conditions, and is therefore a nonlinear counterpart of Proposition 2.9.

Lemma 3.5. *There exists $(f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}_{FP}$ such that, for $j, k \in \{0, 1\}$, $\overline{\ell^j}(f^k, \delta_0^k, \delta_1^k) = \mathbf{1}_{j=k}$.*

Proof. As Corollary 2.16, this follows from Proposition 2.13. □

Proposition 3.6. *There exist $\eta > 0$ and maps $U_{FP} : B_\eta \to Z^1(\Omega)$ and $(\nu_{FP}^0, \nu_{FP}^1) : B_\eta \to \mathbb{R}^2$, where B_η is the ball of radius η in $\mathcal{H}_{FP}$ such that, for any $(f, \delta_0, \delta_1) \in B_\eta$, $u := U_{FP}(f, \delta_0, \delta_1) \in Z^1(\Omega)$ and $\nu^j := \nu_{FP}^j(f, \delta_0, \delta_1)$ obey the equation*

$$\begin{cases} z\partial_x u + E[u]\partial_z u - \partial_{zz} u = f + \nu^0 f^0 + \nu^1 f^1, \\ u_{|\Sigma_i} = \delta_i + \nu^0 \delta_i^0 + \nu^1 \delta_i^1, \\ u_{|z=\pm 1} = 0 \end{cases} \tag{3.19}$$

where the triplets $(f^k, \delta_0^k, \delta_1^k)$ for $k \in \{0, 1\}$ are defined in Lemma 3.5. Furthermore, u and ν satisfy the estimate

$$\|u\|_{Z^1} + |\nu^0| + |\nu^1| \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \tag{3.20}$$

and the orthogonality conditions

$$\nu^j = -\overline{\ell^j}(f - E[u]\partial_z u, \delta_0, \delta_1) \quad \text{for } j \in \{0, 1\}. \tag{3.21}$$

Proof. Let $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ small enough to be chosen later on. We modify our iterative scheme to construct Z^1 solutions using Proposition 2.9 and accommodate for the two orthogonality conditions at each step.

- **Definition of the sequence.** More precisely, we take $u_0 := 0$ and, for $n \in \mathbb{N}$, given $u_n \in Z^1(\Omega)$, we define $u_{n+1} \in Z^1(\Omega)$ as the solution to

$$\begin{cases} z\partial_x u_{n+1} - \partial_{zz} u_{n+1} = f - E_n \partial_z u_n + \nu_{n+1}^0 f^0 + \nu_{n+1}^1 f^1, \\ (u_{n+1})_{|\Sigma_i} = \delta_i + \nu_{n+1}^0 \delta_i^0 + \nu_{n+1}^1 \delta_i^1, \\ (u_{n+1})_{|z=\pm 1} = 0, \end{cases} \tag{3.22}$$

  where $E_n := E[u_n]$, the triplets $(f^k, \delta_0^k, \delta_1^k)$ for $k \in \{0, 1\}$ are defined in Lemma 3.5 and

$$\nu_{n+1}^j := -\overline{\ell^j}(f - E_n \partial_z u_n, \delta_0, \delta_1). \tag{3.23}$$

This choice ensures that the two orthogonality conditions

$$\overline{\ell^j}\big(f - E_n\partial_z u_n + \nu^0_{n+1}f^0 + \nu^1_{n+1}f^1, \delta_0 + \nu^0_{n+1}\delta^0_0 + \nu^1_{n+1}\delta^1_0, \delta_1 + \nu^0_{n+1}\delta^0_1 + \nu^1_{n+1}\delta^1_1\big) = 0 \quad (3.24)$$

are satisfied.

We now verify that the data of (3.22) satisfy the assumptions of Proposition 2.9. This mostly follows from the inclusion $\mathcal{H}_{FP} \subset \mathcal{H}_K$. It only remains to check that $(-E_n\partial_z u_n, 0, 0) \in \mathcal{H}_K$, i.e. that $-E_n\partial_z u_n \in H^1_x L^2_z$, $(-E_n\partial_z u_n/z)_{|\Sigma_i} \in \mathscr{H}^1_z(\Sigma_i)$ and $(-E_n\partial_z u_n/z)(x_i,(-1)^i) = 0$. We estimate these norms in the next item. The condition $\partial_z u_n(x_i,(-1)^i) = 0$ is guaranteed by the constraint $\delta_i'((-1)^i) = 0$ contained in definition (3.4) which also entails $(\delta^k_i)'((-1)^i) = 0$. By Proposition 2.9, we conclude that $u_{n+1} \in Z^1(\Omega)$.

- **Uniform bound in Z^1.** Let us prove by induction that there exists a constant $C_1 > 0$ such that, if η is small enough, then, for all $n \in \mathbb{N}$,

$$U_n := \|u_n\|_{Z^1} + \sum_{i\in\{0,1\}} \|\partial_z u_n(x_i,z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} \le 2C_1\eta. \quad (3.25)$$

This holds for $n = 0$. For $n \in \mathbb{N}$, it follows from (2.23) that

$$\|u_{n+1}\|_{Z^1} \lesssim \|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K} + |\nu_{n+1}| + \|E_n\partial_z u_n\|_{H^1_x L^2_z} + \sum_{i\in\{0,1\}} \|(E_n\partial_z u_n)|_{\Sigma_i}/z\|_{\mathscr{H}^1_z(\Sigma_i)}. \quad (3.26)$$

Moreover, from the lateral boundary conditions, we derive that

$$\begin{aligned}\|\partial_z u_{n+1}(x_i,z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} &\le \|\delta_i'(z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} + \sum_{k\in\{0,1\}} |\nu^k_{n+1}| \cdot \|(\delta^k_i)'(z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} \\ &\lesssim \|(f,\delta_0,\delta_1)\|_{\mathcal{H}_{FP}} + |\nu^0_{n+1}| + |\nu^1_{n+1}|.\end{aligned} \quad (3.27)$$

We obtain from (3.23) and Lemma 2.11 that

$$|\nu^j_{n+1}| \lesssim \|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K} + \|E_n\partial_z u_n\|_{H^1_x L^2_z} + \sum_{i\in\{0,1\}} \|(E_n\partial_z u_n)|_{\Sigma_i}/z\|_{\mathscr{H}^1_z(\Sigma_i)}. \quad (3.28)$$

First, using (3.3),

$$\|E_n\partial_z u_n\|_{H^1_x L^2_z} \lesssim \|E_n\|_{H^1_x}\|\partial_z u_n\|_{L^\infty_x L^2_z} + \|E_n\|_{L^\infty_x}\|\partial_z u_n\|_{H^1_x L^2_z} \lesssim \|u_n\|_{L^2}\|u_n\|_{Z^1}. \quad (3.29)$$

Second, for $i \in \{0,1\}$, since E_n does not depend on z,

$$\|E_n(x_i)\partial_z u_n(x_i,z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} \le \|E_n\|_{L^\infty}\|\partial_z u_n(x_i,z)/z\|_{\mathscr{H}^1_z(\Sigma_i)}. \quad (3.30)$$

Hence, there exists $C_1 > 0$ (depending only on Ω) such that

$$U_{n+1} \le C_1\left(\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_{FP}} + U_n^2\right). \quad (3.31)$$

Thus, if $\eta \le 1/(4C_1^2)$, then the bound $U_n \le 2C_1\eta$ propagates by induction.

- **Convergence.** As in the low regularity case, we let $w_n := u_{n+1} - u_n$. Then, for $n \geq 1$, w_n is now a solution to:

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = -(E_n - E_{n-1})\partial_z u_{n-1} - E_n \partial_z (u_n - u_{n-1}) \\ \qquad\qquad + (\nu^0_{n+1} - \nu^0_n) f^0 + (\nu^1_{n+1} - \nu^1_n) f^1, \\ (w_n)_{|\Sigma_i} = (\nu^0_{n+1} - \nu^0_n)\delta^0_i + (\nu^1_{n+1} - \nu^1_n)\delta^1_i, \\ (w_n)_{|z=\pm 1} = 0. \end{cases} \tag{3.32}$$

  Using the same type of proof as above, we derive from (3.23) and Definition 2.10 that

$$|\nu^j_{n+1} - \nu^j_n| \lesssim \eta \|w_{n-1}\|_{Z^1}. \tag{3.33}$$

  Therefore, estimate (2.23) of Proposition 2.9 entails that

$$\|w_n\|_{Z^1} \lesssim \eta \|w_{n-1}\|_{Z^1}. \tag{3.34}$$

  Thus $(u_n)_{n\in\mathbb{N}}$ and $(\nu^j_n)_{n\in\mathbb{N}}$ are Cauchy sequences. Passing to the limit, we deduce that there exist $u \in Z^1(\Omega)$ and $(\nu^0, \nu^1) \in \mathbb{R}^2$ satisfying (3.19), (3.20) and (3.21). □

Definition 3.7. *For $\eta > 0$ small enough, we define $\mathcal{M}_{FP}$ as*

$$\mathcal{M}_{FP} := \{(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}; \quad \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta \text{ and } \nu_{FP}(f, \delta_0, \delta_1) = (0, 0)\}. \tag{3.35}$$

By definition, for any $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$, there exists a solution $u \in Z^1(\Omega)$ to (3.1) (since (3.19) is satisfied with $\nu^0 = \nu^1 = 0$), which satisfies (3.10) thanks to (3.20).

Proposition 3.8. *There exists $\eta > 0$ such that, for any $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ and $u \in Z^1(\Omega)$ solution to (3.1)-(3.2) satisfying $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ and $\|u\|_{Z^1} \leq \eta$, one has $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$.*

Proof. Let $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ and $u \in Z^1(\Omega)$ be a solution to (3.1)-(3.2) satisfying $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ and $\|u\|_{Z^1} \leq \eta$ for some $\eta > 0$ small enough to be chosen later.

Since $u \in Z^1(\Omega)$, one has $-E[u]\partial_z u \in H^1_x L^2_z$. Thus, viewing (3.1) as a linear equation with source term $f - E[u]\partial_z u$, Proposition 2.9 implies that

$$\overline{\ell^j}(f - E[u]\partial_z u, \delta_0, \delta_1) = 0. \tag{3.36}$$

Now, let $(\tilde{u}, \nu^0, \nu^1) \in Z^1(\Omega) \times \mathbb{R}^2$ be the solution to (3.19) constructed from (f, δ_0, δ_1) in Proposition 3.6. By (3.21),

$$\nu^j = -\overline{\ell^j}(f - E[\tilde{u}]\partial_z \tilde{u}, \delta_0, \delta_1). \tag{3.37}$$

Combining both equalities leads to

$$|\nu^0| + |\nu^1| \lesssim (\|u\|_{Z^1} + \|\tilde{u}\|_{Z^1}) \|u - \tilde{u}\|_{Z^1}. \tag{3.38}$$

Therefore, writing the system satisfied by $w := u - \tilde{u}$ and applying estimate (1.7) of Proposition 2.9 leads to

$$\|w\|_{Z^1} \lesssim \eta \|w\|_{Z^1}. \tag{3.39}$$

If $\eta > 0$ is small enough, this implies that $w = 0$, so $\nu^0 = \nu^1 = 0$, and $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$. □

Remark 3.9 (An alternative approach)**.** *Another potential proof of Item 2 of Theorem 5 could be the following. Consider the map*

$$\nu^j_{FP} : (f,\delta_0,\delta_1) \in \mathcal{H}_{FP} \mapsto -\overline{\ell^j}(f,\delta_0,\delta_1) + \overline{\ell^j}(E[u]\partial_z u, 0, 0) \in \mathbb{R}^2, \tag{3.40}$$

where $u \in Z^0$ is the unique solution to system (3.1)-(3.2)*, provided by Lemma 3.3.*

Since $u \in Z^0$, $E[u]\partial_z u \in H^{1/3}_x L^2_z$, and therefore ν^j_{FP} is well-defined thanks to Remark 2.30. We then set $\mathcal{M}_{FP} := \{(f,\delta_0,\delta_1) \in \mathcal{H}_{FP};\ \nu^j_{FP}(f,\delta_0,\delta_1) = 0\}$. Then for all $(f,\delta_0,\delta_1) \in \mathcal{H}_{FP}$, u is a solution to an equation of the type $z\partial_x - \partial_{zz}u = g$, where the right-hand side g belongs to $H^{1/3}_x L^2_z$ and satisfies orthogonality conditions. It follows from the interpolation result Proposition 2.33 and Lemma 1.15 that $u \in Z^{1/3} \hookrightarrow H^1_x L^2_z \cap H^{1/3}_x H^2_z$, and therefore $E[u]\partial_z u \in H^{2/3}_x L^2_z$. Bootstrapping twice the same argument, we eventually infer that $u \in Z^1$.

However this argument is based on the existence of Z^0 solutions of the nonlinear problem without any orthogonality condition. For the Burgers equation, the nonlinearity is too strong for such a theory of weak solutions to be available. Therefore, in order to unify the presentation, we have chosen to present a different proof, based on a modification of the iterative scheme.

3.4 Regularity and tangent space of the manifold

We now give another description of the set $\mathcal{M}_{FP}$ defined in (3.35), which we use to prove that it is indeed a Lipschitz submanifold of $\mathcal{H}_{FP}$ of codimension 2, modeled on $\mathcal{H}^\perp_{FP,\mathrm{sg}}$, and we describe its tangent space at the origin. Throughout this paragraph, we denote by $\Xi = (f,\delta_0,\delta_1)$ an element of $\mathcal{H}_{FP}$.

We recall that there exist $\Xi^0, \Xi^1 \in \mathcal{H}_{FP}$ such that $\overline{\ell^j}(\Xi^k) = \mathbf{1}_{j=k}$ (see Lemma 3.5), and such that $\mathcal{H}^\perp_{FP,\mathrm{sg}} := \ker \overline{\ell^0} \cap \ker \overline{\ell^1} \cap \mathcal{H}_{FP} = (\mathbb{R}\Xi^0 + \mathbb{R}\Xi^1)^\perp$. For every $\Xi \in \mathcal{H}_{FP}$, one has the decomposition

$$\Xi = \Xi^\perp + \langle \Xi; \Xi^0\rangle_{\mathcal{H}_{FP}}\Xi^0 + \langle \Xi; \Xi^1\rangle_{\mathcal{H}_{FP}}\Xi^1, \tag{3.41}$$

where $\Xi^\perp \in \mathcal{H}^\perp_{FP,\mathrm{sg}}$ and the linear maps $\Xi \mapsto \Xi^\perp$ and $\Xi \mapsto \langle \Xi^k; \Xi\rangle$ are continuous.

Lemma 3.10. *For $\eta > 0$ small enough, the set $\mathcal{M}_{FP}$ defined in* (3.35) *is equal to*

$$\widetilde{M}_{FP} := \left\{ \widetilde{\Xi} \in \mathcal{H}_{FP};\quad \|\widetilde{\Xi}\|_{\mathcal{H}_{FP}} \le \eta \text{ and } \langle \widetilde{\Xi}; \Xi^j\rangle = \nu^j_{FP}(\widetilde{\Xi}^\perp) \text{ for } j \in \{0,1\} \right\}. \tag{3.42}$$

Proof. We proceed by double inclusion.

• Let $\widetilde{\Xi} \in \widetilde{M}_{FP}$. Consider the solution $(u,\nu^0,\nu^1) \in Z^1(\Omega)\times\mathbb{R}^2$ constructed for the data $\widetilde{\Xi}^\perp$ in Proposition 3.6. Then $u \in Z^1(\Omega)$ is a solution to (3.1)-(3.2) with data $\widetilde{\Xi}^\perp + \nu^0_{FP}(\widetilde{\Xi}^\perp)\Xi^0 + \nu^1_{FP}(\widetilde{\Xi}^\perp)\Xi^1$. Since $\widetilde{\Xi} \in \widetilde{M}_{FP}$, we infer from (3.41) that $u \in Z^1(\Omega)$ is actually a solution with data $\widetilde{\Xi}$. Thus, Proposition 3.8 implies that $\widetilde{\Xi} \in \mathcal{M}_{FP}$.

• Let $\Xi \in \mathcal{M}_{FP}$. We introduce

$$\widetilde{\Xi} := \Xi^\perp + \nu^0_{FP}(\Xi^\perp)\Xi^0 + \nu^1_{FP}(\Xi^\perp)\Xi^1, \tag{3.43}$$

which can be thought of as a good projection of Ξ on $\widetilde{\mathcal{M}}_{FP}$ since $\widetilde{\Xi}^\perp = \Xi^\perp$ and $\widetilde{\Xi} \in \widetilde{\mathcal{M}}_{FP}$. Let $u, \widetilde{u} \in Z^1(\Omega)$ denote the solutions constructed in Proposition 3.6 from Ξ and $\Xi^\perp$. For $k \in \{0,1\}$, we also introduce the coefficients $\mu^k := \nu^k_{FP}(\Xi^\perp) - \langle \Xi; \Xi^k\rangle_{\mathcal{H}_{FP}}$, which characterize how far Ξ is from $\widetilde{\mathcal{M}}_{FP}$. Then $w := \widetilde{u} - u$ belongs to $Z^1(\Omega)$ with

$$\|w\|_{Z^1} \lesssim \eta \tag{3.44}$$

and is a solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u} + \mu^0 f^0 + \mu^1 f^1, \\ w_{|\Sigma_i} = \mu^0 \delta_i^0 + \mu^1 \delta_i^1, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{3.45}$$

By Proposition 2.9, since $w \in Z^1(\Omega)$, the following orthogonality conditions are satisfied for $j = 0, 1$:

$$\begin{aligned} 0 &= \overline{\ell^j}(E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u} + \mu^0 f^0 + \mu^1 f^1, \mu^0 \delta_0^0 + \mu^1 \delta_0^1, \mu^0 \delta_1^0 + \mu^1 \delta_1^1) \\ &= \overline{\ell^j}(E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u}, 0, 0) + \mu^j. \end{aligned} \tag{3.46}$$

Moreover, since

$$\|E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u}\|_{H^1_x L^2_z} \lesssim \eta \|w\|_{Z^1} \tag{3.47}$$

we infer from (3.46) that

$$|\mu^j| \lesssim \eta \|w\|_{Z^1}. \tag{3.48}$$

Applying estimate (1.7) to (3.45), we obtain

$$\|w\|_{Z^1} \lesssim \eta \|w\|_{Z^1} + |\mu^0| + |\mu^1| \lesssim \eta \|w\|_{Z^1} \tag{3.49}$$

For $\eta > 0$ small enough, this entails that $w = 0$ and $\mu^0 = \mu^1 = 0$, so that $\Xi \in \widetilde{\mathcal{M}}_{FP}$. □

Lemma 3.11. *The maps U_{FP} and ν_{FP} of Proposition 3.6 are Lipschitz-continuous.*

Proof. Taking two triplets $\Xi, \Xi' \in \mathcal{H}_{FP}$, one can consider the constructed sequences $u_n, u'_n \in Z^1(\Omega)$ and $\nu_n, \nu'_n \in \mathbb{R}^2$ from (3.22). Then, for $n \geq 1$, $w_n := u_n - u'_n$ is the solution to

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = (f - f') - E[w_{n-1}]\partial_z u_{n-1} - E[u'_{n-1}]\partial_z w_{n-1} \\ \qquad\qquad + (\nu_n^0 - \nu_n'^0) f^0 + (\nu_n^1 - \nu_n'^1) f^1, \\ (w_n)_{|\Sigma_i} = (\delta_i - \delta'_i) + (\nu_n^0 - \nu_n'^0)\delta_i^0 + (\nu_n^1 - \nu_n'^1)\delta_i^1, \\ (w_n)_{|z=\pm 1} = 0, \end{cases} \tag{3.50}$$

where, from (3.23) and Definition 2.10,

$$|\nu_n - \nu'_n| \lesssim \eta \|w_{n-1}\|_{Z^1}. \tag{3.51}$$

Thus, we obtain from Proposition 2.9 that

$$\|w_n\|_{Z^1} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}_{FP}} + \eta \|w_{n-1}\|_{Z^1}. \tag{3.52}$$

For η small enough, we obtain at the limit that

$$\|u - u'\|_{Z^1} + |\nu^0 - \nu'^0| + |\nu^1 - \nu'^2| \lesssim \|\Xi - \Xi'\|_{\mathcal{H}_{FP}}, \tag{3.53}$$

which concludes the proof. □

Remark 3.12. *Since we only proved Lipschitz regularity for the map ν_{FP},* (3.35) *(and equivalently* (3.42)*)* a priori *only defines a Lipschitz manifold. Hence, it is difficult to define tangent spaces to $\mathcal{M}_{FP}$. Nevertheless, one can say that $\mathcal{H}_{FP,\mathrm{sg}}^{\perp}$ is tangent to $\mathcal{M}_{FP}$ at 0 in the following weak senses:*

- *For* $\Xi \in \mathcal{M}_{FP}$, $\mathrm{d}(\Xi, \mathcal{H}^{\perp}_{FP,\mathrm{sg}}) \lesssim \|\Xi\|^2_{\mathcal{H}_{FP}}$.
- *For every* $\Xi^{\perp} \in \mathcal{H}^{\perp}_{FP,\mathrm{sg}}$, *for* $\varepsilon \in \mathbb{R}$ *small enough,* $\mathrm{d}(\varepsilon\Xi^{\perp}, \mathcal{M}_{FP}) \lesssim \varepsilon^2$.

Both facts are straightforward consequences of the equivalent definitions (3.35) *and* (3.42) *and of the estimate*

$$|\nu^j(\Xi) + \overline{\ell^j}(\Xi)| \lesssim \|\Xi\|^2_{\mathcal{H}_{FP}} \tag{3.54}$$

which follows from (3.21) *and* (3.20).

Remark 3.13. *It is likely that similar techniques can be used to prove that* $\mathcal{M}_{FP}$ *has in fact more regularity (say* C^1 *for example) and characterize its tangent spaces in a neighborhood of the origin by computing the orthogonality conditions associated with the linearized problems around small enough solutions* $u \in Z^1(\Omega)$*, but this is not our focus here.*

3.5 A general formalization

The construction deployed in Section 3.3 can be seen as a particular case (see Remark 3.16) of a more general approach to construct solutions to semilinear or quasilinear equations in the presence of orthogonality conditions, in a perturbative regime. We give here a statement in an abstract framework which we will use in the following sections for the Burgers and Prandtl systems.

Our abstract result is related with general results for semilinear problems associated with Fredholm operators with negative index, such as the ones of [59, Chapter 11, Section 4.2.3]. However, the approach in this reference consists in modifying parameters in the nonlinearity to ensure the orthogonality conditions, while we focus on constructing a submanifold of data for which the nonlinear problem has a regular solution.

We intend to construct solutions to problems of the form $Lu = N(\Xi, u)$, where $u \in \mathcal{Z}$ (the space of solutions), $\Xi \in \mathcal{X}$ (the space of data for the nonlinear problem), $N : \mathcal{X} \times \mathcal{Z} \to \mathcal{H}$ is the nonlinearity, with values in $\mathcal{H}$ (the space of source terms $\Theta \in \mathcal{H}$ for the linear problem $Lu = \Theta$).

To avoid investigating the C^1 dependency of the solutions to our nonlinear systems on the data, we use a version of the implicit function theorem for functions which are not C^1 but only "strongly Fréchet-differentiable at a point". We refer the reader to [56, Chapter 25].

Definition 3.14. *Let* E, F *be Banach spaces,* $f : E \to F$ *and* $x^* \in E$*. We say that* f *is* strongly Fréchet-differentiable *at* x^* *when there exists a continuous linear map* $L : E \to F$ *such that*

$$\|f(x_1) - f(x_2) - L(x_1 - x_2)\|_F = \underset{x_1, x_2 \to x^*}{o}(\|x_1 - x_2\|_E)\,. \tag{3.55}$$

The following implicit function theorem is proved in [56, Paragraph 25.13].

Lemma 3.15. *Let* E_1, E_2, F *be Banach spaces and* $f : E_1 \times E_2 \to F$ *such that* $f(0,0) = 0$*. Assume that* f *is strongly Fréchet-differentiable at* $(0,0)$ *and that* $\partial_2 f(0,0) : E_2 \to F$ *is a linear isomorphism. Then there exists a Lipschitz-continous map* g *defined in a neighborhood of* $0 \in E_1$ *such that, for every* (x, y) *in a neighborhood of* $(0,0) \in E_1 \times E_2$*,* $f(x, y) = 0$ *if and only if* $y = g(x)$*. Moreover,* g *is strongly Fréchet-differentiable at* 0 *and* $Dg(0) = -(\partial_2 f(0,0))^{-1}\partial_1 f(0,0)$*.*

Theorem 6. *Let* $\mathcal{H}, \mathcal{X}, \mathcal{Z}$ *be Banach spaces and* $d \in \mathbb{N}$*. Let* $\ell : \mathcal{H} \to \mathbb{R}^d$ *and* $L : \mathcal{Z} \to \mathcal{H}$ *be continuous linear maps. Let* N *be a (nonlinear) map from* $\mathcal{X} \times \mathcal{Z}$ *to* $\mathcal{H}$ *such that* $N(0,0) = 0$*. Assume that*

i) for all $\Theta \in \mathcal{H}$, the equation $Lu = \Theta$ has a unique solution $u \in \mathcal{Z}$ if and only if $\Theta \in \ker \ell$, which moreover satisfies $\|u\|_{\mathcal{Z}} \lesssim \|\Theta\|_{\mathcal{H}}$;

ii) N is strongly Fréchet-differentiable at $(0,0)$ and $\partial_u N(0,0) = 0$, i.e. there exists a continuous linear map $\partial_\Xi N(0,0) : \mathcal{X} \to \mathcal{H}$ such that, as $\Xi, \Xi' \in \mathcal{X}$ and $u, u' \in \mathcal{Z}$ go to 0,

$$\|N(\Xi, u) - N(\Xi', u') - (\partial_\Xi N(0,0))(\Xi - \Xi')\|_{\mathcal{H}} = o\big(\|\Xi - \Xi'\|_{\mathcal{X}} + \|u - u'\|_{\mathcal{Z}}\big); \tag{3.56}$$

iii) $\ell_N := \ell \circ \partial_\Xi N(0,0)$ is onto from $\mathcal{X}$ to $\mathbb{R}^d$.

Then there exists a local Lipschitz submanifold $\mathcal{M}$ of $\mathcal{X}$, modeled on $\ker \ell_N$ (of codimension d) and tangent to it at 0, such that, for any $\Xi \in \mathcal{X}$ small enough, the equation $Lu = N(\Xi, u)$ has a solution $u \in \mathcal{Z}$ if and only if $\Xi \in \mathcal{M}$. Such a solution satisfies $\|u\|_{\mathcal{Z}} \lesssim \|\Xi\|_{\mathcal{X}}$ and is unique.

Proof. Using Item iii), we fix $\Xi^1, \dots, \Xi^d \in \mathcal{X}$ such that $\ell_N^j(\Xi^k) = \mathbf{1}_{j=k}$, and we set $\Theta^k := \partial_\Xi N(0,0)\Xi^k$. We could then mimic the iterative scheme of Section 3.3 by defining sequences $u_n \in \mathcal{Z}$ and $\nu_n \in \mathbb{R}^d$ such that

$$Lu_{n+1} = N(\Xi + \sum_k \nu_{n+1}^k \Xi^k, u_n).$$

Instead, we provide a shorter proof directly relying on the bundled result Lemma 3.15.

Let $\Xi \mapsto \Xi^\perp$ be the linear continuous projection from $\mathcal{X}$ to $\ker \ell_N$ parallel to the space span $(\Xi^1, \dots, \Xi^d)$, i.e. $\Xi^\perp = \Xi - \sum_{j=1}^d \ell_N^j(\Xi)\Xi^j$. Let $f : \ker \ell_N \times (\mathcal{Z} \times \mathbb{R}^d) \to \mathcal{H}$ defined by

$$f(\Xi^\perp, (u, a)) := Lu - N(\Xi^\perp + a_1\Xi^1 + \cdots + a_d\Xi^d, u). \tag{3.57}$$

By Item ii) and continuity of L on $\mathcal{Z}$, f is strongly Fréchet-differentiable at $(0,0)$. Moreover, $\partial_2 f(0,0) : (u, a) \mapsto Lu - a_1\Theta^1 - \cdots - a_d\Theta^d$ is a linear isomorphism from $\mathcal{Z} \times \mathbb{R}^d$ to $\mathcal{H}$ by Item i) and continuity of ℓ on $\mathcal{H}$. Indeed, given $h \in \mathcal{H}$, setting $a^h := -\ell(h)$ and $u^h \in \mathcal{Z}$ the solution to $Lu^h = h + a_1^h\Theta^1 + \cdots + a_d^h\Theta^d$, one has $\partial_2 f(0,0)(u^h, a^h) = h$ and $\|u_h\|_{\mathcal{Z}} \lesssim \|h\|_{\mathcal{H}}$, $|a^h| \lesssim \|h\|_{\mathcal{H}}$.

Hence, the implicit function theorem of Lemma 3.15 yields the existence of Lipschitz-continuous functions $(U, \mu) : \ker \ell_N \to \mathcal{Z} \times \mathbb{R}^d$ such that, for every $\Xi^\perp \in \ker \ell_N$, $u \in \mathcal{Z}$ and $a \in \mathbb{R}^d$ small enough,

$$Lu = N(\Xi^\perp + a_1\Xi^1 + \cdots + a_d\Xi^d, u)$$

if and only if $a = \mu(\Xi^\perp)$ and $u = U(\Xi^\perp)$. From there, we infer that for all $\Xi \in \mathcal{X}$ and $u \in \mathcal{Z}$ small enough, $Lu = N(\Xi, u)$ if and only if $\ell_N(\Xi) = \mu(\Xi^\perp)$ and $u = U(\Xi^\perp)$. Thus the conclusions of the theorem hold provided that we set

$$\mathcal{M} := \left\{\Xi \in \mathcal{X}; \quad \|\Xi\|_{\mathcal{X}} \le \eta \text{ and } \ell_N(\Xi) = \mu(\Xi^\perp)\right\}, \tag{3.58}$$

where Indeed, (3.58) corresponds to the graph characterization of a local Lipschitz submanifold of $\mathcal{X}$ containing 0 and modeled on $\ker \ell_N$; therefore of codimension d by Item iii). Eventually, μ is strongly Fréchet-differentiable at 0 and, since $Dg(0) = -(\partial_2 f(0,0))^{-1}\partial_1 f(0,0)$ with the notation of Lemma 3.15, we obtain that $D\mu(0) = -\ell \circ \partial_\Xi N(0,0) = -\ell_N$ so $D\mu(0) = 0$ on $\ker \ell_N$, which justifies the claim that $\mathcal{M}$ is tangent to $\ker \ell_N$ at 0. □

Remark 3.16. *Item 2 of Theorem 5 can be recovered as a particular case of Theorem 6 with the following setting:*

- $\mathcal{X} = \mathcal{H} = \mathcal{H}_{FP}$ *defined in* (3.4),
- *the solution space*

$$\mathcal{Z} := \Big\{ u \in Z^1(\Omega); \quad u_{|z=\pm 1} = 0, \quad (\partial_z u(x_i, z))/z \in \mathscr{H}^1_z(\Sigma_i), \quad \partial_z u(x_i, (-1)^i) = 0 \Big\}, \tag{3.59}$$

with

$$\|u\|_{\mathcal{Z}} := \|u\|_{Z^1} + \sum_{i \in \{0,1\}} \|(\partial_z u(x_i, z))/z\|_{\mathscr{H}^1_z}, \tag{3.60}$$

- $L : \mathcal{Z} \to \mathcal{X}$ *defined by* $Lu := (z\partial_x u - \partial_{zz} u, u_{|\Sigma_0}, u_{|\Sigma_1})$, *for which one easily checks that the assumption Item i) of Theorem 6 is satisfied thanks to Proposition 2.9.*
- $N : \mathcal{X} \times \mathcal{Z} \to \mathcal{H}$ *defined by* $N(\Xi, u) := (f, \delta_0, \delta_1) - (E[u]\partial_z u, 0, 0)$. *In particular, one has* $\partial_\Xi N(0,0) = \mathrm{Id}$. *To check that* N *takes values in* $\mathcal{H} = \mathcal{H}_{FP} \subset \mathcal{H}_K$, *we must check that* $E[u](x_i)\partial_z u(x_i, (-1)^i) = 0$, *which follows from the fact that, for* $u \in \mathcal{Z}$, $\partial_z u(x_i, (-1)^i) = 0$. *We now check that* N *satisfies Item ii) of Theorem 6.*

  First, for $u, u' \in Z^1(\Omega)$, *by* (3.3),

$$\begin{aligned} \|E[u]\partial_z u - E[u']\partial_z u'\|_{H^1_x L^2_z} &\le \|E[u-u']\partial_z u\|_{H^1_x L^2_z} + \|E[u']\partial_z(u-u')\|_{H^1_x L^2_z} \\ &\lesssim \|u-u'\|_{L^2} \|\partial_z u\|_{H^1_x L^2_z} + \|u'\|_{L^2} \|\partial_z(u-u')\|_{H^1_x L^2_z} \\ &\lesssim (\|u\|_{Z^1} + \|u'\|_{Z^1}) \|u-u'\|_{Z^1}. \end{aligned} \tag{3.61}$$

  Second, one similarly checks that

$$\|(E[u]\partial_z u - E[u']\partial_z u')(x_i, z)/z\|_{\mathscr{H}^1_z(\Sigma_i)} \lesssim (\|u\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}}) \|u-u'\|_{\mathcal{Z}}. \tag{3.62}$$

  Hence, we conclude that

$$\begin{aligned} \|(E[u]\partial_z u - E[u']\partial_z u', 0, 0)\|_{\mathcal{H}_{FP}} &= \|(E[u]\partial_z u - E[u']\partial_z u', 0, 0)\|_{\mathcal{H}_K} \\ &\lesssim (\|u\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}}) \|u-u'\|_{\mathcal{Z}} \end{aligned} \tag{3.63}$$

  so that estimate (3.56) *is satisfied.*
- $d = 2$, $\ell := (\overline{\ell^0}, \overline{\ell^1})_{|\mathcal{H}_{FP}}$ *defined in Definition 2.10, continuous on* $\mathcal{H}_{FP}$ *by Lemma 2.11 and* $\mathcal{H}_{FP} \hookrightarrow \mathcal{H}_K$, *satisfying* $\ell_N(\mathcal{X}) = \ell(\mathcal{X}) = \mathbb{R}^2$ *by Lemma 3.5 and* $\partial_\Xi N(0,0) = \mathrm{Id}$.

4 A viscous Burgers equation

In this section, we consider the following nonlinear parabolic forward-backward system, which can be envisioned as a kind of stationary Burgers equation with transverse viscosity:

$$\begin{cases} u\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \mathfrak{u}_{|\Sigma_i} + \delta_i, \\ u_{|y=\pm 1} = \mathfrak{u}_{|y=\pm 1}. \end{cases} \tag{4.1}$$

As detailed in the introduction, the perturbation (f, δ_0, δ_1) is small and we look for solutions u which are close to the shear flow $\mathfrak{u}(x,y) := y$, which corresponds to $(f, \delta_0, \delta_1) = (0,0,0)$. Thanks to the nonlinear change of variables described in the Introduction and detailed in Section 4.1, the local well-posedness of (4.1) can be proved using the formalism of Section 3.5 (see Sections 4.2 and 4.3).

4.1 A nonlinear change of variables

As is classical for problems with free boundaries, we perform a change of variables which straightens the critical curve $\{u = 0\}$. Heuristically, we swap the roles of the vertical coordinate y and the unknown u, the latter becoming the vertical coordinate, and the former the unknown of the new PDE. Keeping in mind that we are looking for perturbative solutions with u close enough to $\mathfrak{u}$ (in particular $\|u_y - 1\|_{L^\infty} \ll 1$), we change the vertical coordinate y into z, defined as

$$z(x,y) := u(x,y). \tag{4.2}$$

The new unknown $Y(x,z)$ is defined by the implicit relation

$$u(x, Y(x,z)) = z. \tag{4.3}$$

In particular, thanks to the boundary conditions $u_{|y=\pm 1} = \mathfrak{u}_{|y=\pm 1} = \pm 1$, one checks that the domain $(x,y) \in \Omega = [x_0, x_1] \times [-1,1]$ is indeed mapped to $(x,z) \in \Omega$, and one still has $Y_{|z=\pm 1} = \pm 1$. Similarly, if $\delta_i(0) = 0$ and $\delta_i((-1)^i) = 0$, the inflow boundary regions Σ_i are also left invariant by this change of variable.

Remark 4.1. *More rigorously, given u defined on Ω and close enough to $\mathfrak{u}$ (for example in $H^1_x H^2_z$ topology), for each $x \in [x_0, x_1]$, the map $y \mapsto u(x,y)$ is a C^1 monotone increasing bijection from $[-1,1]$ to itself, and the implicit definition (4.3) is equivalent to setting*

$$Y(x,z) := (u(x,\cdot))^{-1}(z). \tag{4.4}$$

From (4.3), we successively derive the relations

$$\begin{aligned} \partial_y u(x, Y(x,z)) &= \frac{1}{\partial_z Y(x,z)}, \\ \partial_x u(x, Y(x,z)) &= -\partial_x Y(x,z) \partial_y u(x, Y(x,z)) = -\frac{\partial_x Y(x,z)}{\partial_z Y(x,z)}, \\ \partial_{yy} u(x, Y(x,z)) &= -\frac{\partial_{zz} Y(x,z)}{(\partial_z Y(x,z))^3}. \end{aligned} \tag{4.5}$$

These identities lead to the following PDE for Y:

$$z\partial_x Y - (\partial_z Y)^{-2}\partial_{zz} Y = -\partial_z Y f(x, Y). \tag{4.6}$$

Moreover, by (4.4), denoting by $(\cdot + \delta_i(\cdot))^{-1}$ the functional inverse of the function $z \mapsto z + \delta_i(z)$ and letting

$$\Upsilon[\delta_i](z) := z - (\cdot + \delta_i(\cdot))^{-1}(z), \tag{4.7}$$

and using $\delta_i(0) = 0$ and $\delta_i((-1)^i) = 0$, we have $Y(x, z) = z - \Upsilon[\delta_i](z)$ for $(x, z) \in \Sigma_i$.

Therefore, we obtain the system

$$\begin{cases} z\partial_x Y - (\partial_z Y)^{-2}\partial_{zz} Y = -\partial_z Y f(x, Y), \\ Y_{|\Sigma_i} = z - \Upsilon[\delta_i](z), \\ Y_{|z=\pm 1} = \pm 1. \end{cases} \tag{4.8}$$

Eventually, to make the perturbative nature of this system explicit, we write $Y(x, z) = z - \widetilde{Y}(x, z)$, which leads to the system

$$\begin{cases} z\partial_x \widetilde{Y} - \partial_{zz}\widetilde{Y} = N_B(f, \widetilde{Y}), \\ \widetilde{Y}_{|\Sigma_i} = \Upsilon[\delta_i], \\ \widetilde{Y}_{|z=\pm 1} = 0 \end{cases} \tag{4.9}$$

where the nonlinearity is given by

$$N_B(f, \widetilde{Y}) := \frac{\partial_z \widetilde{Y}(2 - \partial_z \widetilde{Y})}{(1 - \partial_z \widetilde{Y})^2}\partial_{zz}\widetilde{Y} + (1 - \partial_z \widetilde{Y}) f(x, z - \widetilde{Y}). \tag{4.10}$$

We prove the well-posedness of (4.9) in Section 4.2 and use it to prove Theorem 3 in Section 4.3.

Remark 4.2. *The initial PDE $u\partial_x u - \partial_{yy} u = f$ is quasilinear. After the change of variables described in this paragraph, we obtain system (4.8), which is still a quasilinear one (since the viscosity in front of $\partial_{zz}\widetilde{Y}$ depends on $\widetilde{Y}$). However, we know from Section 2 that, for the linear problem $z\partial_x u - \partial_{zz} u = f$, there is no loss of derivative in the vertical direction. This key point allows us to apprehend (4.8) under the form (4.9), treating this nonlinearity perturbatively as the first term of N_B in (4.10). The fact that there is no loss of vertical derivative explains why we will be able to prove in the following paragraph that the nonlinearity N_B satisfies the mild estimates of Theorem 6. This would not have been possible in the initial form $u\partial_x u - \partial_{yy} u = f$, since the linear theory involves a loss of $\frac{1}{3}$ derivative in the horizontal direction.*

4.2 Well-posedness in the new variables

We now prove the following well-posedness result with $Z^1(\Omega)$ regularity under two orthogonality conditions for system (4.9). Let

$$\mathcal{H}_B := \left\{(f, \delta_0, \delta_1) \in \mathcal{H}_K; \quad \delta_i''(z)/z \in \mathscr{H}^1_z(\Sigma_i), \quad \delta_i''((-1)^i) = 0\right\}, \tag{4.11}$$

$$\mathcal{X}_B := \left\{(f, \delta_0, \delta_1) \in \mathcal{H}_B; \quad f \in H^1_x H^2_z, \quad f_{|\Sigma_i} = 0, \quad \delta_i \in H^5(\Sigma_i), \quad \delta_i(0) = \delta_i''(0) = 0\right\}, \tag{4.12}$$

with the norms

$$\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_B} := \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_K} + \|\delta_0''(z)/z\|_{\mathscr{H}^1_z} + \|\delta_1''(z)/z\|_{\mathscr{H}^1_z}, \tag{4.13}$$

$$\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} := \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_B} + \|f\|_{H^1_x H^2_z} + \|\delta_0\|_{H^5} + \|\delta_1\|_{H^5}. \tag{4.14}$$

The restriction that $f_{|\Sigma_i} = 0$ lightens the exposition but could be partially relaxed. The space $\mathcal{X}_B$ of (4.12) is the same as the one defined in (1.3) in the introduction.

Proposition 4.3. *There exists $\eta > 0$ and a local Lipschitz submanifold $\mathcal{M}_B$ of $\mathcal{X}_B$ included in the ball of radius η, modeled on $\mathcal{X}_B \cap \ker(\overline{\ell^0}, \overline{\ell^1})$ (of codimension 2) and tangent to it at 0 such that, for every $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ such that $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \leq \eta$, (4.9) has a solution $\widetilde{Y} \in Z^1(\Omega)$ if and only if $(f, \delta_0, \delta_1) \in \mathcal{M}_B$. Such solutions are unique and satisfy $\|\widetilde{Y}\|_{Z^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}$.*

Proof. Our strategy is to apply the same nonlinear argument as for our kinetic theory toy model (see Section 3). Before moving on to the formal proof using the abstract Theorem 6, let us give an heuristic overview of the corresponding concrete nonlinear scheme.

Heuristic overview of the nonlinear scheme. We follow the scheme described in Section 3.3. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \leq \eta$ small enough to be chosen later on. We construct a sequence $\widetilde{Y}_n$ of $Z^1(\Omega)$ functions using Proposition 2.9, accommodating for the two orthogonality conditions at each step. We take $\widetilde{Y}_0 := 0$ and, for $n \in \mathbb{N}$, given $\widetilde{Y}_n \in Z^1(\Omega)$, we define $\widetilde{Y}_{n+1} \in Z^1(\Omega)$ as the solution to

$$\begin{cases} z\partial_x \widetilde{Y}_{n+1} - \partial_{zz} \widetilde{Y}_{n+1} = N_B(f, \widetilde{Y}_n) + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ (\widetilde{Y}_{n+1})_{|\Sigma_i} = \Upsilon[\delta_i] + \nu^0_{n+1} \delta^0_i + \nu^1_{n+1} \delta^1_i, \\ (\widetilde{Y}_{n+1})_{|z=\pm 1} = 0, \end{cases} \tag{4.15}$$

where the triplets $(f^k, \delta^k_0, \delta^k_1) \in \mathcal{X}_B$ for $k \in \{0, 1\}$ are such that $\overline{\ell^j}(f^k, \delta^k_0, \delta^k_1) = \mathbf{1}_{j=k}$ and are constructed as in Corollary 2.16 and

$$\nu^j_{n+1} := -\overline{\ell^j}(N_B(f, \widetilde{Y}_n), \Upsilon[\delta_0], \Upsilon[\delta_1]). \tag{4.16}$$

This choice ensures that the two orthogonality conditions

$$\begin{aligned} \overline{\ell^j}\big(&N_B(f, \widetilde{Y}_n) + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ &\Upsilon[\delta_0] + \nu^0_{n+1} \delta^0_0 + \nu^1_{n+1} \delta^1_0, \\ &\Upsilon[\delta_1] + \nu^0_{n+1} \delta^0_1 + \nu^1_{n+1} \delta^1_1\big) = 0 \end{aligned} \tag{4.17}$$

are satisfied. One checks that Proposition 2.9 can be applied, yielding $\widetilde{Y}_{n+1} \in Z^1(\Omega)$. One can then prove that $(\widetilde{Y}_n)_{n \in \mathbb{N}}$ is uniformly bounded by $C\eta$ and is a Cauchy sequence in $Z^1(\Omega)$.

Proof using our abstract toolbox. More precisely, this result follows from Theorem 6, applied with the following setting: $\mathcal{H}_B$ defined in (4.11) and $\mathcal{X}_B$ defined in (4.12),

- the solution space

$$\mathcal{Z}_B := \Big\{ u \in Z^1(\Omega);\ \ u_{|z=\pm 1} = 0,\ \ \partial_{zz} u(x_i, z)/z \in \mathscr{H}^1_z(\Sigma_i),\ \ \partial_{zz} u(x_i, (-1)^i) = 0 \Big\}, \tag{4.18}$$

  with

$$\|u\|_{\mathcal{Z}_B} := \|u\|_{Z^1} + \sum_{i \in \{0,1\}} \|\partial_{zz} u(x_i, z)/z\|_{\mathscr{H}^1_z(\Sigma_i)}; \tag{4.19}$$

- $d = 2$, $\ell := (\overline{\ell^0}, \overline{\ell^1})_{|\mathcal{H}_B}$ defined in Definition 2.10, continuous on $\mathcal{H}_B$ by Lemma 2.11 and $\mathcal{H}_B \hookrightarrow \mathcal{H}_K$, satisfying $\ell(\mathcal{X}_B) = \mathbb{R}^2$ by Proposition 2.13 since $C^\infty_c(\Omega) \times \{0\} \times \{0\} \subset \mathcal{X}_B$;

- $L : \mathcal{Z}_B \to \mathcal{H}_B$ defined by $Lu := (z\partial_x u - \partial_{zz} u, u_{|\Sigma_0}, u_{|\Sigma_1})$, for which one easily checks that the assumption Item i) of Theorem 6 is satisfied thanks to Proposition 2.9;
- $N : \mathcal{X}_B \times \mathcal{Z}_B \to \mathcal{H}_B$ defined by $N(\Xi, \widetilde{Y}) := (N_B(f, \widetilde{Y}), \Upsilon[\delta_0], \Upsilon[\delta_1])$. To prove that N takes values in $\mathcal{H}_B \subset \mathcal{H}_K$, we must check that:
  a) $N_B(f, \widetilde{Y}) \in H^1_x L^2_z$: this follows from Lemma 4.6 and Lemma 4.9 below;
  b) $(N_B(f, \widetilde{Y})/z)|_{\Sigma_i} \in \mathscr{H}^1_z(\Sigma_i)$: this follows from Corollary 4.13 below;
  c) $N_B(f, \widetilde{Y})(x_i, (-1)^i) = 0$: this property follows from the fact that, for $\Xi \in \mathcal{X}_B$, $f_{|\Sigma_i} = 0$ and, for $\widetilde{Y} \in \mathcal{Z}_B$, $\partial_{zz}\widetilde{Y}(x_i, (-1)^i) = 0$.

  Eventually, we claim that N is strongly Fréchet-differentiable at $(0,0)$ in the sense of Definition 3.14 with $\partial_u N(0,0) = 0$ and $\partial_\Xi N(0,0) = \mathrm{Id}$, which corresponds to the following estimate, as $\Xi, \Xi' \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ go to 0,

$$\begin{aligned} \|(N_B(f, \widetilde{Y}), \Upsilon[\delta_0], \Upsilon[\delta_1]) - (N_B(f', \widetilde{Y}'), \Upsilon[\delta_0'], \Upsilon[\delta_1']) - (\Xi - \Xi')\|_{\mathcal{H}_B} \\ = o\left(\|\Xi - \Xi'\|_{\mathcal{X}_B} + \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B}\right). \end{aligned} \tag{4.20}$$

  This follows from Corollary 4.10 for the $H^1_x L^2_z$ estimate of the force, and from Corollary 4.13 and Corollary 4.16 for the estimate of the boundary terms. □

The next subsections are dedicated to the proof of Item a) and Item b) and of estimate (4.20) above. We will repeatedly use the following classical result:

Lemma 4.4. *The pointwise product is (bilinearly) continuous from $H^1_x H^1_z \times H^1_x L^2_z$ to $H^1_x L^2_z$.*

4.2.1 Forcing term

We first derive estimates for the main forcing term $(1 - \partial_z \widetilde{Y}) f(x, z - \widetilde{Y}(x,z))$. We start with an easy one-dimensional lemma:

Lemma 4.5. *For $\phi, \psi \in (H^2 \cap H^1_0)(-1,1)$ small enough (so that the changes of variables $z \mapsto z - \phi(z)$ and $z \mapsto z - \psi(z)$ are well-defined on $[-1,1]$) and $f \in H^1(-1,1)$, one has*

$$\|f(z - \phi(z))\|_{L^2} \lesssim \|f\|_{L^2}, \tag{4.21}$$

$$\|f(z - \phi(z)) - f(z - \psi(z))\|_{L^2} \lesssim \|\partial_z f\|_{L^2} \|\phi - \psi\|_{L^\infty}. \tag{4.22}$$

Proof. First, (4.21) is straight-forward since the Jacobian of the change of variables $z \mapsto z - \phi(z)$ is bounded from below and from above for ϕ small enough in $(H^2 \cap H^1_0)(-1,1)$.

Second, for $z \in [-1,1]$, we write

$$f(z - \phi(z)) - f(z - \psi(z)) = (\psi(z) - \phi(z)) \int_0^1 \partial_z f(z - s\phi(z) - (1-s)\psi(z)) \, \mathrm{d}s. \tag{4.23}$$

Hence, by Cauchy–Schwarz,

$$\|f(z - \phi(z)) - f(z - \psi(z))\|^2_{L^2} \lesssim \|\phi - \psi\|^2_{L^\infty} \int_0^1 \|\partial_z f(z - (s\phi(z) + (1-s)\psi(z)))\|^2_{L^2} \, \mathrm{d}s \tag{4.24}$$

so that (4.22) follows from (4.21) applied to $\partial_z f$ and $s\phi + (1-s)\psi$. □

Lemma 4.6. *For $\phi, \psi \in H^1_x(H^2_z \cap H^1_0)$ small enough and $f \in H^1_x H^2_z$, one has*

$$\begin{aligned} \|(1-\partial_z\phi)f(x,z-\phi)\|_{H^1_x L^2_z} &\lesssim \|f\|_{H^1_x H^2_z}, \\ \|f(x,z-\phi(x,z)) - f(x,z-\psi(x,z))\|_{H^1_x L^2_z} &\lesssim \|f\|_{H^1_x H^2_z}\|\phi-\psi\|_{H^1_x H^2_z}. \end{aligned} \tag{4.25}$$

Proof. First, we observe that $\partial_z\phi, \partial_z\psi \in L^\infty$, and $\partial_z f \in L^\infty$. Since

$$\partial_x(f(x,z-\phi)) = \partial_x f(x,z-\phi) - \partial_x\phi\partial_z f(x,z-\phi)$$

we infer that $f(x,z-\phi) \in H^1_x L^2_z$. From there, we easily deduce the first estimate.

We then turn towards the second estimate. By the chain rule and the triangular inequality, one has

$$\begin{aligned} \|f(x,z-\phi) - f(x,z-\psi)\|_{H^1_x L^2_z} \lesssim{}& \|f(x,z-\phi) - f(x,z-\psi)\|_{L^2_x L^2_z} \\ &+ \|\partial_x f(x,z-\phi) - \partial_x f(x,z-\psi)\|_{L^2_x L^2_z} \\ &+ \|(\partial_z f(x,z-\phi) - \partial_z f(x,z-\psi))\phi_x\|_{L^2_x L^2_z} \\ &+ \|\partial_z f(x,z-\psi)(\phi_x-\psi_x)\|_{L^2_x L^2_z}. \end{aligned} \tag{4.26}$$

By (4.22), the first two terms are bounded by $\|\partial_z f\|_{L^2}\|\phi-\psi\|_{L^\infty}$ and $\|\partial_{xz} f\|_{L^2}\|\phi-\psi\|_{L^\infty}$.

For the third term, using (4.22),

$$\begin{aligned} &\|(\partial_z f(x,z-\phi) - \partial_z f(x,z-\psi))\phi_x\|_{L^2_x L^2_z} \\ &\qquad \le \|\partial_z f(x,z-\phi) - \partial_z f(x,z-\psi)\|_{L^\infty_x L^2_z}\|\phi_x\|_{L^2_x L^\infty_z} \\ &\qquad \lesssim \|\partial_{zz} f\|_{L^\infty_x L^2_z}\|\phi-\psi\|_{L^\infty}\|\phi_x\|_{L^2_x L^\infty_z}. \end{aligned} \tag{4.27}$$

For the fourth term, using (4.21),

$$\begin{aligned} \|\partial_z f(x,z-\psi)(\phi_x-\psi_x)\|_{L^2_x L^2_z} &\le \|\partial_z f(x,z-\psi)\|_{L^\infty_x L^2_z}\|\phi_x-\psi_x\|_{L^2_x L^\infty_z} \\ &\lesssim \|\partial_z f\|_{L^\infty_x L^2_z}\|\phi_x-\psi_x\|_{L^2_x L^\infty_z}. \end{aligned} \tag{4.28}$$

Gathering these inequalities concludes the proof using usual Sobolev embeddings. □

Lemma 4.7. *For $\phi_1, \phi_2 \in H^1_x(H^2_z \cap H^1_0)$ small enough and $f_1, f_2 \in H^1_x H^2_z$,*

$$\begin{aligned} &\|(1-\partial_z\phi_1)f_1(x,z-\phi_1) - (1-\partial_z\phi_2)f_2(x,z-\phi_2) - (f_1-f_2)\|_{H^1_x L^2_z} \\ &\lesssim \big(\|f_1\|_{H^1_x H^2_z} + \|f_2\|_{H^1_x H^2_z} + \|\phi_1\|_{H^1_x H^2_z} + \|\phi_2\|_{H^1_x H^2_z}\big)\big(\|\phi_1-\phi_2\|_{H^1_x H^2_z} + \|f_1-f_2\|_{H^1_x H^2_z}\big). \end{aligned} \tag{4.29}$$

Proof. First, we write

$$\begin{aligned} f_1(x,z-\phi_1) - f_2(x,z-\phi_2) - (f_1-f_2) = {}& (f_1-f_2)(x,z-\phi_1) - (f_1-f_2)(x,z-0) \\ &+ f_2(x,z-\phi_1) - f_2(x,z-\phi_2). \end{aligned} \tag{4.30}$$

Applying Lemma 4.6 to both lines, we have

$$\begin{aligned} &\|f_1(x,z-\phi_1) - f_2(x,z-\phi_2) - (f_1-f_2)\|_{H^1_x L^2_z} \\ &\qquad \lesssim \|f_1-f_2\|_{H^1_x H^2_z}\|\phi_1-0\|_{H^1_x H^2_z} + \|f_2\|_{H^1_x H^2_z}\|\phi_1-\phi_2\|_{H^1_x H^2_z}, \end{aligned} \tag{4.31}$$

which allows to conclude the proof thanks to Lemma 4.4. □

4.2.2 Nonlinear viscous term

We derive estimates for the main nonlinear viscous term $\frac{\partial_z \widetilde{Y}(2-\partial_z \widetilde{Y})}{(1-\partial_z \widetilde{Y})^2}\partial_{zz}\widetilde{Y}$.

Lemma 4.8. *Let $g : \mathbb{R} \to \mathbb{R}$ be a C^3 function in a neighborhood of 0 with $g(0) = 0$. Then the map $G : H^1_x H^2_z \to H^1_x H^1_z$ given by $G(\phi) := g(\phi_z)$ is well-defined and Lipschitz-continuous in a neighborhood of 0, and satisfies $G(0) = 0$.*

Proof. First, for $\phi, \psi \in H^1_x H^2_z$ small enough,

$$\|g(\phi_z) - g(\psi_z)\|_{L^\infty} \leq \|g'\|_{L^\infty} \|\phi_z - \psi_z\|_{L^\infty} \lesssim \|\phi_z - \psi_z\|_{H^1_x H^1_z}. \tag{4.32}$$

Second, for $\phi \in H^1_x H^2_z$ small enough,

$$\partial_{xz}\left(g(\phi_z)\right) = g'(\phi_z)\phi_{xzz} + g''(\phi_z)\phi_{xz}\phi_{zz}. \tag{4.33}$$

Hence, for $\phi, \psi \in H^1_x H^2_z$ small enough, using that $g \in C^3$ and decomposing the difference, one obtains

$$\|\partial_{xz}\left(g(\phi_z) - g(\psi_z)\right)\|_{L^2} \lesssim \|\phi - \psi\|_{H^1_x H^2_z}, \tag{4.34}$$

which concludes the proof. □

Lemma 4.9. *For $\phi, \psi \in H^1_x H^2_z$ small enough, one has*

$$\left\| \frac{\partial_z \phi(2-\partial_z\phi)}{(1-\partial_z\phi)^2}\partial_{zz}\phi - \frac{\partial_z\psi(2-\partial_z\psi)}{(1-\partial_z\psi)^2}\partial_{zz}\psi \right\|_{H^1_x L^2_z} \lesssim \left(\|\phi\|_{H^1_x H^2_z} + \|\psi\|_{H^1_x H^2_z}\right)\|\phi - \psi\|_{H^1_x H^2_z}. \tag{4.35}$$

Proof. Since ∂_{zz} is Lipschitz-continuous from $H^1_x H^2_z$ to $H^1_x L^2_z$, by Lemma 4.4, the result follows from the Lipschitz continuity of $\phi \mapsto \partial_z\phi(2-\partial_z\phi)(1-\partial_z\phi)^{-2}$ from $H^1_x H^2_z$ to $H^1_x H^1_z$, which is a consequence of Lemma 4.8 with $g(s) := s(2-s)(1-s)^{-2}$. □

Gathering Lemma 4.7 (for the part involving f) and Lemma 4.9 (for the quadratic part involving $\widetilde{Y}$ only), we obtain

Corollary 4.10. *For $\Xi = (f, \delta_0, \delta_1)$, $\Xi' = (f', \delta_0', \delta_1') \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ small enough,*

$$\|N_B(f, \widetilde{Y}) - N_B(f', \widetilde{Y}') - (f - f')\|_{H^1_x L^2_z} = o\left(\|\Xi - \Xi'\|_{\mathcal{X}_B} + \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B}\right). \tag{4.36}$$

4.2.3 Boundary contribution of the nonlinearity

We now derive estimates concerning the $\mathscr{H}^1_z(\Sigma_i)$ contribution of $N_B(f, \widetilde{Y})$.

Lemma 4.11. *For $\psi \in \mathscr{H}^1_z(0,1)$, one has $z\psi \in L^\infty(0,1)$ with $\|z\psi\|_{L^\infty} \lesssim \|\psi\|_{\mathscr{H}^1_z}$.*

Proof. Let $\psi \in \mathscr{H}^1_z(0,1)$. First, $\psi \in H^1(1/2, 1)$ and one has $|\psi(1)| \lesssim \|\psi\|_{\mathscr{H}^1_z}$. Thus, for $z_0 \in (0,1)$,

$$|\psi(z_0)| \leq |\psi(1)| + \int_{z_0}^1 |\psi_z| \leq |\psi(1)| + |z_0|^{-\frac{1}{2}}\|z\psi_z\|_{L^2(0,1)} \lesssim |z_0|^{-\frac{1}{2}}\|\psi\|_{\mathscr{H}^1_z}, \tag{4.37}$$

which proves that $|z|^{\frac{1}{2}}\psi \in L^\infty$, so that, in particular, $z\psi \in L^\infty$. □

Lemma 4.12. *Let $g : \mathbb{R} \to \mathbb{R}$ be a C^2 function in a neighborhood of 0 with $g(0) = 0$. Let $\mathcal{E} := \{\psi \in L^2(0,1);\ \psi_{zz}/z \in \mathscr{H}^1_z\}$ with the associated canonical norm. There exists $\eta > 0$ small enough such that, for $\phi, \psi \in \mathcal{E}$ with $\|\phi\|_{\mathcal{E}} \le \eta$ and $\|\psi\|_{\mathcal{E}} \le \eta$,*

$$\left\| \frac{g(\phi_z)\phi_{zz}}{z} - \frac{g(\psi_z)\psi_{zz}}{z} \right\|_{\mathscr{H}^1_z} \lesssim (\|\phi\|_{\mathcal{E}} + \|\psi\|_{\mathcal{E}})\|\phi - \psi\|_{\mathcal{E}}. \tag{4.38}$$

Proof. First, $\mathcal{E} \hookrightarrow H^2 \hookrightarrow W^{1,\infty}$ and, thanks to Lemma 4.11, $\mathcal{E} \hookrightarrow W^{2,\infty}$. We write

$$\frac{g(\phi_z)\phi_{zz}}{z} - \frac{g(\psi_z)\psi_{zz}}{z} = (g(\phi_z) - g(\psi_z))\frac{\phi_{zz}}{z} + g(\psi_z)\frac{\phi_{zz} - \psi_{zz}}{z}. \tag{4.39}$$

For the first term, we have

$$\left\| (g(\phi_z) - g(\psi_z))\frac{\phi_{zz}}{z} \right\|_{\mathscr{H}^1_z} \lesssim \|\phi_z - \psi_z\|_{L^\infty}\|\phi_{zz}/z\|_{\mathscr{H}^1_z} + \|\phi_{zz}/z\|_{\mathscr{L}^2_z}\|\partial_z(g(\phi_z) - g(\psi_z))\|_{L^\infty} \tag{4.40}$$

where

$$\begin{aligned} \|\partial_z(g(\phi_z) - g(\psi_z))\|_{L^\infty} &\le \|(g'(\phi_z) - g'(\psi_z))\phi_{zz}\|_{L^\infty} + \|g'(\phi_z)(\phi_{zz} - \psi_{zz})\|_{L^\infty} \\ &\lesssim \|\phi_z - \psi_z\|_{L^\infty}\|\phi_{zz}\|_{L^\infty} + \|\phi_z\|_{L^\infty}\|\phi_{zz} - \psi_{zz}\|_{L^\infty}. \end{aligned} \tag{4.41}$$

For the second term,

$$\left\| g(\psi_z)\frac{\phi_{zz} - \psi_{zz}}{z} \right\|_{\mathscr{H}^1_z} \lesssim \|g(\psi_z)\|_{L^\infty}\|(\phi_{zz} - \psi_{zz})/z\|_{\mathscr{H}^1_z} + \|(\phi_{zz} - \psi_{zz})/z\|_{\mathscr{L}^2_z}\|g'(\psi_z)\psi_{zz}\|_{L^\infty}. \tag{4.42}$$

Hence, the claimed estimate follows from the embedding $\mathcal{E} \hookrightarrow W^{2,\infty}$. □

Corollary 4.13. *For $\Xi = (f, \delta_0, \delta_1)$, $\Xi' = (f', \delta_0', \delta_1') \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ small enough, $z^{-1}N_B(f, \widetilde{Y})|_{\Sigma_i} \in \mathscr{H}^1_z(\Sigma_i)$ and $z^{-1}N_B(f', \widetilde{Y}')|_{\Sigma_i} \in \mathscr{H}^1_z(\Sigma_i)$ and*

$$\left\| \frac{N_B(f, \widetilde{Y}) - N_B(f', \widetilde{Y}')}{z} \right\|_{\mathscr{H}^1_z(\Sigma_i)} = o\left(\|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B} \right). \tag{4.43}$$

Proof. Since $\Xi \in \mathcal{X}_B$, $f|_{\Sigma_i} = 0$. Thus

$$N_B(f, \widetilde{Y})|_{\Sigma_i} = g(\partial_z \widetilde{Y}|_{\Sigma_i})\partial_{zz}\widetilde{Y}|_{\Sigma_i} \quad \text{with} \quad g(a) = \frac{a(2-a)}{(1-a)^2}. \tag{4.44}$$

The result follows from Lemma 4.12, noting that, for $\widetilde{Y} \in \mathcal{Z}_B$, $\widetilde{Y}|_{\Sigma_i} \in \mathcal{E}$ of Lemma 4.12. □

4.2.4 Contribution of the inversion of the boundary data

We now move on to estimates concerning the Fréchet-differentiability of the map Υ of (4.7).

Lemma 4.14. *For $\phi \in H^2(0,1)$ such that $\phi(0) = 0$,*

$$\|\phi(z)/z\|_{H^1} \lesssim \|\phi\|_{H^2}. \tag{4.45}$$

Proof. Writing a second-order Taylor expansion, one has

$$\phi(z) = z\phi'(0) + \int_0^z (z-s)\phi''(s)\,\mathrm{d}s. \tag{4.46}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi(z)}{z}\right) = \frac{1}{z^2}\int_0^z s\phi''(s)\,\mathrm{d}s, \tag{4.47}$$

from which the conclusion follows by the Hardy inequality of Lemma B.8. □

Lemma 4.15. *Consider the spaces*

$$\mathcal{E}_1 := \{\delta \in \mathscr{H}^1_z(0,1);\ \ \delta''(z)/z \in \mathscr{H}^1_z(0,1),\ \ \delta(0) = \delta(1) = \delta''(0) = \delta''(1) = 0\}, \tag{4.48}$$

$$\mathcal{E}_2 := \{\delta \in H^5(0,1);\ \ \delta(0) = \delta(1) = \delta''(0) = \delta''(1) = 0\}. \tag{4.49}$$

Then the map $\Upsilon[\delta](z) := z - (\cdot + \delta(\cdot))^{-1}(z)$ *as in* (4.7) *is well-defined for* δ *small enough and strongly Fréchet-differentiable at* 0 *from* $\mathcal{E}_2$ *to* $\mathcal{E}_1$*. More precisely, for* $\delta, \eta \in \mathcal{E}_2$ *small enough,*

$$\|\Upsilon[\delta] - \Upsilon[\eta] - (\delta - \eta)\|_{\mathcal{E}_1} \lesssim (\|\delta\|_{\mathcal{E}_2} + \|\eta\|_{\mathcal{E}_2})\,\|\delta - \eta\|_{\mathcal{E}_2}. \tag{4.50}$$

Proof. ***Step 1.*** *We first check that* Υ *is well-defined.* Since $\mathcal{E}_2 \hookrightarrow W^{1,\infty}$, $\widetilde{\delta} := \Upsilon[\delta]$ is well-defined in a neighborhood for $\delta \in \mathcal{E}_2$ small enough, and the boundary conditions $\delta(0) = \delta(1) = 0$ of $\mathcal{E}_2$ entail that $\widetilde{\delta}(0) = \widetilde{\delta}(1) = 0$.

Moreover, one has

$$\widetilde{\delta}(z) = \delta(z - \widetilde{\delta}(z)). \tag{4.51}$$

From this relation, we derive that

$$\widetilde{\delta}'(z) = \frac{\delta'}{1+\delta'}(z - \widetilde{\delta}(z)) \quad \text{and} \quad \widetilde{\delta}''(z) = \frac{\delta''}{(1+\delta')^3}(z - \widetilde{\delta}(z)) \tag{4.52}$$

which ensures that $\widetilde{\delta}''(0) = \widetilde{\delta}''(1) = 0$ since $\delta''(0) = \delta''(1) = 0$.

Step 2. *We prove the strong Fréchet-differentiability at* 0. To control the $\mathcal{E}_1$ norm, it suffices to control the L^2 norm and the $\mathscr{H}^1_z$ norm of the quotient $\partial_{zz}(\cdot)/z$. For $\delta, \eta \in \mathcal{E}_2$ by (4.51),

$$(\widetilde{\delta} - \widetilde{\eta})(z) = (\delta - \eta)(z - \widetilde{\delta}) + (\eta(z - \widetilde{\delta}) - \eta(z - \widetilde{\eta})). \tag{4.53}$$

Hence

$$\|\widetilde{\delta} - \widetilde{\eta}\|_{L^\infty} \le \|\delta - \eta\|_{L^\infty} + \|\partial_z \eta\|_{L^\infty}\|\widetilde{\delta} - \widetilde{\eta}\|_{L^\infty}. \tag{4.54}$$

In particular, for η small enough in $\mathcal{E}_2$,

$$\|\widetilde{\delta} - \widetilde{\eta}\|_{L^\infty} \le 2\|\delta - \eta\|_{L^\infty}. \tag{4.55}$$

Thus, applying estimate (4.22) to (4.53), we obtain

$$\begin{aligned}
\|(\widetilde{\delta} - \widetilde{\eta}) - (\delta - \eta)\|_{L^2} &\le \|(\delta - \eta)(\cdot - \widetilde{\delta}) - (\delta - \eta)(\cdot)\|_{L^2} + \|\eta(\cdot - \widetilde{\delta}) - \eta(\cdot - \widetilde{\eta})\|_{L^2} \\
&\lesssim \|\partial_z(\delta - \eta)\|_{L^2}\|\widetilde{\delta}\|_{L^\infty} + \|\partial_z \eta\|_{L^2}\|\widetilde{\delta} - \widetilde{\eta}\|_{L^\infty} \\
&\lesssim \|\partial_z(\delta - \eta)\|_{L^2}\|\delta\|_{L^\infty} + \|\partial_z \eta\|_{L^2}\|\delta - \eta\|_{L^\infty} \\
&\lesssim (\|\delta\|_{H^1} + \|\eta\|_{H^1})\,\|\delta - \eta\|_{H^1}.
\end{aligned} \tag{4.56}$$

We now move to the estimate of the $\mathscr{H}^1_z$ norm of the quotient $\partial_{zz}(\cdot)/z$. By Lemma 4.14 (which even yields an H^1 estimate, not only $\mathscr{H}^1_z$), since all our functions have null second derivative at 0, it suffices to obtain an H^4 estimate. Differentiating (4.52) twice, we obtain

$$\partial_z^4 \widetilde{\delta}(z) = \frac{\partial_z^4 \delta}{(1+\partial_z \delta)^5}(z - \widetilde{\delta}(z)) + \text{lower order terms.} \tag{4.57}$$

Decomposing the difference in a similar manner as in (4.56) and applying (4.22), one can prove

$$\|(\widetilde{\delta} - \widetilde{\eta}) - (\delta - \eta)\|_{H^4} = (\|\delta\|_{H^5} + \|\eta\|_{H^5}) \, \|\delta - \eta\|_{H^5}. \tag{4.58}$$

Together with Lemma 4.14, this concludes the proof of (4.50). □

Corollary 4.16. *For $\Xi = (f, \delta_0, \delta_1)$, $\Xi' = (f', \delta_0', \delta_1') \in \mathcal{X}_B$ small enough,*

$$\|(0, \Upsilon[\delta_0], \Upsilon[\delta_1]) - (0, \Upsilon[\delta_0'], \Upsilon[\delta_1']) - (0, \delta_0 - \delta_0', \delta_1 - \delta_1')\|_{\mathcal{H}_B} = o\left(\|\Xi - \Xi'\|_{\mathcal{X}_B}\right). \tag{4.59}$$

Proof. Recalling that, for $\Xi = (f, \delta_0, \delta_1) \in \mathcal{X}_B$, $f|_{\Sigma_i} = 0$, this is a direct consequence of Lemma 4.15 and the definitions (4.11) and (4.12) of $\mathcal{H}_B$ and $\mathcal{X}_B$. □

4.3 Reverse change of variables

Proofs of Theorem 3 and Proposition 1.1. It only remains to prove that the change of variables of Section 4.1 is justified in both directions.

First, given $(f, \delta_0, \delta_1) \in \mathcal{M}_B$, let $\widetilde{Y} \in Z^1$ be the solution to (4.9) given by Proposition 4.3 and let $Y(x,z) := z - \widetilde{Y}(x,z)$ the associated solution to (4.6). By Proposition 4.3, $\|Y\|_{Z^1} \lesssim 1 + \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}$. By Lemma 1.15, $\|Y\|_{Q^1} \lesssim 1 + \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}$. Since Y is a solution to (4.6), we have

$$\partial_{zz} Y = (\partial_z Y)^2 (z \partial_x Y) - (\partial_z Y) f(x, Y(x,z)). \tag{4.60}$$

We check that the right-hand side is $L^2_x H^1_z$, from which we deduce that $\partial_z^3 Y \in L^2$. Repeating this argument, we find that the right-hand side of the above equation is in fact $L^2_x H^2_z$ and that

$$\|\partial_z^4 Y\|_{L^2} \lesssim \|Y\|_{Z^1} + \|f\|_{L^2_x H^2_y}. \tag{4.61}$$

Thus, $Y \in Q^1 \cap L^2_x H^4_z$ and

$$\|Y(x,z) - z\|_{Q^1} + \|Y(x,z) - z\|_{L^2_x H^4_z} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}. \tag{4.62}$$

By Corollary B.6, (4.3) defines a $u \in Q^1 \cap L^2_x H^4_y$ such that

$$\|u(x,y) - y\|_{Q^1} + \|u(x,y) - y\|_{L^2_x H^4_y} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}. \tag{4.63}$$

In particular, since both $\partial_y u$ and $\partial_z Y$ are continuous functions on Ω with $\|\partial_y u - 1\|_{L^\infty} \ll 1$ and $\|\partial_z Y - 1\|_{L^\infty} \ll 1$, the computations of Section 4.1 hold. Thus, we have constructed a $u \in Q^1$ solution to (4.1). This proves the existence claim of Theorem 3.

Reciprocally, to prove the claim of Theorem 3 concerning the uniqueness of the solution to (4.1) and the one of Proposition 1.1 concerning the necessity of the nonlinear orthogonality conditions $(f, \delta_0, \delta_1) \in \mathcal{M}_B$, we must perform the reasoning in the other direction. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ small enough, and let $u \in Q^1$ be a solution to (1.10) such that $\|u\|_{Q^1} \ll 1$. Writing the PDE as

$$\partial_y^2 u = u \partial_x u - f \tag{4.64}$$

we obtain that $u \in Q^1 \cap L^2_x H^4_y$. By Corollary B.6, (4.3) defines a function $Y \in Q^1 \cap L^2_x H^4_z$ such that

$$\|Y(x,z) - z\|_{Q^1} + \|Y(x,z) - z\|_{L^2_x H^4_z} \lesssim \|u(x,y) - y\|_{Q^1} + \|u(x,y) - y\|_{L^2_x H^4_z} \ll 1. \tag{4.65}$$

In particular, since both $\partial_y u$ and $\partial_z Y$ are continuous functions on Ω with $\|\partial_y u - 1\|_{L^\infty} \ll 1$ and $\|\partial_z Y - 1\|_{L^\infty} \ll 1$, the computations of Section 4.1 hold. Thus, Y is a solution to (4.6). Since $Y \in Q^1 \cap L^2_x H^4_z$, we have $Y \in H^1_x H^2_z$. From the equation (4.6), we recover that $z\partial_x(\partial_x Y) \in L^2$. Thus $Y \in Z^1(\Omega)$. Hence, the conclusions of Proposition 4.3 apply: Y is unique and $(f, \delta_0, \delta_1) \in \mathcal{M}_B$. □

5 The Prandtl system in the recirculation zone

Let us now continue our analysis of nonlinear parabolic forward-backward systems by considering the Prandtl equation in the vicinity of a recirculating flow $(\mathtt{u}_{\mathrm{P}}, \mathtt{v}_{\mathrm{P}})$, revisiting the results of Iyer and Masmoudi from [34, 35]. Throughout this section, the index t stands for 'top' and the index b for 'bottom'. We refer to Section 1.1.4 of the introduction for the assumptions on $(\mathtt{u}_{\mathrm{P}}, \mathtt{v}_{\mathrm{P}})$.

We consider the system

$$\begin{cases} uu_x + vu_y - u_{yy} = -\frac{dp}{dx} + f & \text{in } \Omega_P, \\ u_x + v_y = 0 & \text{in } \Omega_P, \end{cases} \tag{5.1}$$

where the pressure gradient dp/dx is the one associated with $(\mathtt{u}_{\mathrm{P}}, \mathtt{v}_{\mathrm{P}})$, and where we recall that the domain Ω_P is defined by

$$\Omega_P = \{(x,y) \in (x_0, x_1) \times \mathbb{R}_+; \ \gamma_b(x) < y < \gamma_t(x)\}. \tag{5.2}$$

This system is endowed with the boundary conditions (1.15)-(1.16)-(1.17), which we now recall for the reader's convenience:

$$\begin{cases} u|_{y=\gamma_b} = z_b, \quad \partial_y u|_{y=\gamma_b} = \partial_y \mathtt{u}_{\mathrm{P}}|_{y=\overline{\gamma_b}} + \delta_b, \quad v|_{y=\gamma_b} = \mathtt{v}_{\mathrm{P}}|_{y=\overline{\gamma_b}} + v_b & \text{(bottom BC)}, \\ u|_{y=\gamma_t} = z_t, \quad \partial_y u|_{y=\gamma_t} = \partial_y \mathtt{u}_{\mathrm{P}}|_{y=\overline{\gamma_t}} + \delta_t, & \text{(top BC)}, \\ u|_{\Sigma_i^P} = \mathtt{u}_{\mathrm{P}}|_{\Sigma_i^P} + \delta_i & \text{(lateral BC)}. \end{cases} \tag{5.3}$$

We recall that the lines $\{y = \gamma_j(x)\}$ for $j \in \{t, b\}$, which are level sets of the function u, are free boundaries which are expected to lie in the vicinity of the level sets $\{y = \overline{\gamma_j}(x)\}$ of the function $\mathtt{u}_{\mathrm{P}}$. We refer to the introduction for further comments on these boundary conditions.

The source term f in (5.1) is a small regular perturbation of the pressure term. From the physical point of view, it is relevant to consider perturbations which depend only on x, since the right-hand side in the Prandtl system is the trace of the pressure gradient of some outer Euler flow on the boundary. However, the analysis is essentially unchanged if we allow f to depend on the vertical variable y, and therefore in the following f will be a smooth function depending on both x and y, for the sake of generality.

Our analysis in this section follows the one from Section 4. We first perform in Section 5.1 a nonlinear change of variables in order to straighten the free boundary $\{(x,y); \ u(x,y) = 0\}$. The whole analysis then takes place in these new variables. One remarkable point lies in the fact that the linear problem associated with the Prandtl system is similar to, but slightly different from the one for the Burgers equation. In fact, the linear problem associated with the *vorticity* studied in Section 5.2 has the same structure as (1.6). Retrieving the velocity from the vorticity in Section 5.3 gives rise to an additional orthogonality condition. Moreover, since the vorticity plays the same role as the function u from Section 4, it turns out that the Prandtl system is actually *more regular* than the Burgers equation (1.1): indeed, there is a gain of one vertical derivative (corresponding to a vertical integration of the velocity) between Burgers and Prandtl. This will allow us to construct solutions with a minimal requirement of regularity, and just one orthogonality condition. We construct solutions to the nonlinear problem in the new variables in Section 5.4, and conclude the proof of Theorem 4 in Section 5.5.

We recall that we focus here on the behavior of the system in the vicinity of the curve $\{u = 0\}$. When studying the system in the whole infinite strip $(x_0, x_1) \times \mathbb{R}_+$, special care must be taken to "glue together" the different zones. As explained in [35], information flows from bottom to

top. The analysis of the system in the vicinity of the lower boundary and for large values of y requires specific tools, which go beyond the scope of the present paper. We refer the interested reader to [34, 35] for the study of the Prandtl system in the whole domain, and for a description of the difficulties associated with the interplay between the different zones. We also present in Section 5.6 a potential strategy to construct a solution to the Prandtl system in the whole infinite strip, stepping on the analysis of the present paper. In particular, we explain why the analysis of the system in an infinite vertical domain may call for an assumption on the horizontal size of the domain $x_1 - x_0$: in [35], the well-posedness of the system holds when $|x_1 - x_0|$ is either small, or outside a countable set (corresponding to the zeros of an analytic function). No such assumption is required when the Prandtl system is studied in the recirculation zone only, see Theorem 4 or Proposition 5.2 below. Let us also recall that our purpose here is merely to present, in a unified framework, different forward-backward problems. Therefore we will put an emphasis on the specific features associated with the Prandtl system in the recirculation zone Ω_P, and on the similarities and differences with the Burgers type system (1.1) studied in Section 4.

5.1 Nonlinear change of variables

At this stage, we assume that a smooth solution to (5.1) exists in order to write the equation in a form that is more amenable to mathematical analysis. We will come back on the justification of the computations below in Section 5.5.

As in Section 4.1, we change variables by setting $(x, z) = (x, u(x, y))$, where u is the unknown tangential velocity. This maps the unknown domain $\Omega_P = \{\gamma_b(x) < y < \gamma_t(x)\}$ depending on the solution u (since the lines γ_b and γ_t are defined by $u(x, \gamma_j(x)) = z_j$ for $j \in \{b, t\}$) to the fixed rectangular domain $(x_0, x_1) \times (z_b, z_t)$.

We denote by $(x, Y(x, z))$ the diffeomorphism such that $u(x, Y(x, z)) = z$. As a consequence, we have the same relations (4.5) between the derivatives of u and Y as for the Burgers case. The top and bottom boundary conditions become $Y(x, z_j) = \gamma_j(x)$ for $j \in \{b, t\}$.

Furthermore, integrating the divergence-free condition and using (1.15),

$$\begin{aligned} v(x, Y(x, z)) &= v|_{\Gamma_b} - \int_{\gamma_b(x)}^{Y(x,z)} \partial_x u(x, y') \, \mathrm{d}y' \\ &= \mathrm{v_P}|_{\overline{\Gamma_b}} + v_b + \int_{z_b}^{z} \frac{\partial_x Y(x, z')}{\partial_z Y(x, z')} \partial_z Y(x, z') \, \mathrm{d}z' \\ &= \mathrm{v_P}|_{\overline{\Gamma_b}} + v_b + \int_{z_b}^{z} \partial_x Y(x, z') \, \mathrm{d}z'. \end{aligned} \tag{5.4}$$

Replacing this expression and (4.5) into (5.1) and evaluating the equation at $y = Y(x, z)$, we find that

$$-\frac{1}{\partial_z Y} \left[z \partial_x Y - \int_{z_b}^{z} \partial_x Y - \mathrm{v_P}|_{y=\overline{\gamma_b}} - v_b \right] + \frac{1}{(\partial_z Y)^3} \partial_z^2 Y = -\partial_x p + f(x, Y(x, z)). \tag{5.5}$$

Let us now denote by $\mathbb{Y}_\mathrm{P}$ the function such that $\mathrm{u_P}(x, \mathbb{Y}_\mathrm{P}(x, z)) = z$. Following the same computations as above, this function satisfies

$$-\frac{1}{\partial_z \mathbb{Y}_\mathrm{P}} \left[z \partial_x \mathbb{Y}_\mathrm{P} - \int_{z_b}^{z} \partial_x \mathbb{Y}_\mathrm{P} - \mathrm{v_P}|_{\overline{\Gamma_b}} \right] + \frac{1}{(\partial_z \mathbb{Y}_\mathrm{P})^3} \partial_z^2 \mathbb{Y}_\mathrm{P} = -\partial_x p. \tag{5.6}$$

Let $\widetilde{Y} := \mathbb{Y}_{\mathrm{P}} - Y$. Then

$$\begin{aligned}\frac{1}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2}\partial_z^2 \mathbb{Y}_{\mathrm{P}} - \frac{1}{(\partial_z Y)^2}\partial_z^2 Y &= \partial_z\left(\frac{1}{\partial_z Y} - \frac{1}{\partial_z \mathbb{Y}_{\mathrm{P}}}\right)\\ &= \partial_z\left(\frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2}\right) + \partial_z\left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y}\right).\end{aligned} \tag{5.7}$$

We obtain eventually the following very simple equation

$$z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y}(x, z')\,\mathrm{d}z' - \partial_x p \partial_z \widetilde{Y} - \partial_z\left(\frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2}\right) = g(x, z) \tag{5.8}$$

where

$$g(x, z) := f(x, Y(x, z))\partial_z(\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - v_b(x) + \partial_z\left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y}\right). \tag{5.9}$$

The top and bottom boundary conditions (1.16) and (1.15) become, for $j \in \{t, b\}$,

$$\begin{aligned}\partial_z \widetilde{Y}(x, z_j) &= \partial_z \mathbb{Y}_{\mathrm{P}}(x, z_j) - \partial_z Y(x, z_j)\\ &= \frac{1}{\partial_y \mathbb{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x))} - \frac{1}{\partial_y \mathbb{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x)) + \delta_j(x)} =: \Upsilon_{\mathrm{P}}^j[\delta_j](x).\end{aligned} \tag{5.10}$$

The unknown function γ_j can be retrieved from $\widetilde{Y}$ by

$$\gamma_j(x) = Y(x, z_j) = \mathbb{Y}_{\mathrm{P}}(x, z_j) - \widetilde{Y}(x, z_j) = \overline{\gamma_j}(x) - \widetilde{Y}(x, z_j). \tag{5.11}$$

We still denote by Σ_0 and Σ_1 the lateral boundaries, i.e. $\Sigma_0 = \{x_0\} \times (0, z_t)$, $\Sigma_1 = \{x_1\} \times (z_b, 0)$. The lateral boundary conditions (1.17) are given by the implicit equation

$$z = \mathbb{u}_{\mathrm{P}}(x_i, Y(x_i, z)) + \delta_i(Y(x_i, z)) \quad \text{on } \Sigma_i, \tag{5.12}$$

which becomes, after noticing that $\mathbb{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i$ is strictly increasing on Σ_i^P and therefore invertible,

$$\widetilde{Y}(x_i, z) = \mathbb{Y}_{\mathrm{P}}(z) - \left(\mathbb{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i\right)^{-1}(z) =: \Upsilon_{\mathrm{P}}^i[\delta_i]. \tag{5.13}$$

For further purposes, we note that the function $\Upsilon_{\mathrm{P}}^j[\delta_j]$ (resp. $\Upsilon_{\mathrm{P}}^i[\delta_i]$) has the same regularity and size as δ_j (resp. δ_i).

Remark 5.1. *When $\mathbb{u}_{\mathrm{P}}(x, y) = y$ (linear shear flow),* (5.8) *simply becomes, at main order*

$$z\partial_x \widetilde{Y} + \widetilde{V} - \partial_z^2 \widetilde{Y} = g, \tag{5.14}$$

where $\widetilde{V} = -\int_{z_b}^{z} \partial_x \widetilde{Y}$. Differentiating this equation with respect to z, and setting $W := \partial_z \widetilde{Y}$ (W is the vorticity in our new variables) we find

$$z\partial_x W - \partial_z^2 W = \partial_z g. \tag{5.15}$$

Therefore, when we consider the Prandtl equation in the vicinity of the linear shear flow, the equation for the vorticity in the new variables is (1.6)*. We retrieve here the following fact, which was already identified by Iyer and Masmoudi in [35]: the Prandtl system in vorticity form is very close to* (1.6)*. This will also be central in our analysis below.*

Let us now state our main result on system (5.8). Since we will state two results within different regularity frameworks, we will work with two different functional spaces for the data. Note that since the boundaries γ_b, γ_t are free, we allow the function f the be defined on a domain that is possibly larger, in the vertical direction, than the reference domain $\{(x,y) \in (x_0,x_1) \times (0,+\infty),\ \overline{\gamma_b}(x) < y < \overline{\gamma_t}(x)\}$. Hence, in order to simplify the statements, we assume that f is defined in the whole infinite strip $(x_0,x_1)\times(0,+\infty)$.

- In the low regularity setting, we choose an index $\sigma \in (0,1/6)$. Our functional space will be

$$\begin{aligned}\mathcal{X}^\sigma =\Big\{&(f,\delta_0,\delta_1,\delta_t,\delta_b,v_b) \in H^\sigma_x H^2_z \times H^4(\Sigma_0^P) \times H^4(\Sigma_1^P) \times H^2(x_0,x_1)^2 \times H^\sigma(x_0,x_1),\\ &f \in L^4_x H^3_z \cap H^{\frac12+\sigma}_x H^1_z \cap L^\infty_x W^{2,\infty}_z,\ (x-x_0)(x-x_1)\partial_x\partial_z f \in L^2,\\ &\Upsilon^t_{\mathrm{P}}[\delta_t](x_0) = \partial_z \Upsilon^0_{\mathrm{P}}[\delta_0](z_t),\ \Upsilon^b_{\mathrm{P}}[\delta_b](x_1) = \partial_z \Upsilon^1_{\mathrm{P}}[\delta_1](z_b)\Big\}\end{aligned} \tag{5.16}$$

  which we endow with its canonical norm.

- In the high regularity setting, our functional space will be

$$\begin{aligned}\mathcal{X}^1 :=\Big\{&(f,\delta_0,\delta_1,\delta_t,\delta_b,v_b) \in H^1_x H^3_z \times H^6(\Sigma_0^P) \times H^6(\Sigma_1^P) \times H^2(x_0,x_1)^2 \times H^1(x_0,x_1),\\ &f|_{\Sigma_i^P} = 0, \quad \partial_z^k \delta_i(0) = 0\ \forall k \in \{0,\cdots 3\},\\ &\Upsilon^t_{\mathrm{P}}[\delta_t](x_0) = \partial_z \Upsilon^0_{\mathrm{P}}[\delta_0](z_t),\ \Upsilon^b_{\mathrm{P}}[\delta_b](x_1) = \partial_z \Upsilon^1_{\mathrm{P}}[\delta_1](z_b),\\ &\Delta_0(z_t) = \partial_x \Upsilon^t_{\mathrm{P}}[\delta_t](x_0),\ \Delta_1(z_b) = \partial_x \Upsilon^b_{\mathrm{P}}[\delta_b](x_1)\Big\},\end{aligned} \tag{5.17}$$

  where

$$\begin{aligned}\Delta_i &:= \frac{1}{z}\partial_z^2\Big[\frac{(\partial_z \Upsilon^i_{\mathrm{P}}[\delta_i])^2}{(\partial_z \mathbb{Y}_{\mathrm{P}}(x_i,\cdot)^2(\partial_z \mathbb{Y}_{\mathrm{P}}(x_i,\cdot) - \partial_z \Upsilon^i_{\mathrm{P}}[\delta_i])} + \alpha(x_i,\cdot)\partial_z \Upsilon^i_{\mathrm{P}}[\delta_i] - \beta(x_i)\Upsilon^i_{\mathrm{P}}[\delta_i]\Big]\\ &= \frac{1}{z}\partial_z^2\Big[\frac{\partial_z \Upsilon^i_{\mathrm{P}}[\delta_i]}{\partial_z \mathbb{Y}_{\mathrm{P}}(x_i,z)(\partial_z \mathbb{Y}_{\mathrm{P}}(x_i,\cdot) - \partial_z \Upsilon^i_{\mathrm{P}}[\delta_i])} - \beta(x_i)\Upsilon^i_{\mathrm{P}}[\delta_i]\Big].\end{aligned}$$

  Once again, we endow $\mathcal{X}^1$ with its canonical norm. The assumptions on f, δ_0 and δ_1 could be relaxed slightly: in particular, it is not compulsory to assume that δ_0 and δ_1 vanish up to order three near $z=0$, or that f vanishes on the lateral boundary. However this simplifies the formulation of some compatibility conditions.

Our result is the following:

Proposition 5.2. *Let $(\mathrm{u_P},\mathrm{v_P})$ be a smooth solution to (5.1) on $(x_0,x_1)\times(0,+\infty)$ such that $\partial_y \mathrm{u_P} > 0$ on $\{\overline{\gamma_b}(x) \le y \le \overline{\gamma_t}(x),\ x \in [x_0,x_1]\}$. Let $\sigma \in (0,1/6)$. There exists $\eta > 0$ and $z_0 > 0$ such that if $|z_b|, z_t \le z_0$, the following result holds.*

- *There exists a manifold $\mathcal{M}_\sigma \subset \mathcal{X}^\sigma$, of codimension 1 within the ball of radius η in $\mathcal{X}^\sigma$, such that (5.8)-(5.13)-(5.10) has a solution in $H^{\frac23+\sigma}_x H^1_z \cap H^\sigma_x H^3_z$ if and only if $(f,\delta_0,\delta_1,\delta_b,\delta_t,v_b) \in \mathcal{M}_\sigma$.*

  This solution, if it exists, is unique.

- *There exists a manifold $\mathcal{M}_1 \subset \mathcal{X}^1$, of codimension 3 within the ball of radius η in $\mathcal{X}^1$, such that* (5.8)-(5.13)-(5.10) *has a solution in $H_x^{5/3}H_z^1 \cap H_x^1 H_z^3$ if and only if $(f, \delta_0, \delta_1, \delta_b, \delta_t, v_b) \in \mathcal{M}_1$.*

The proof of Proposition 5.2 is similar to the one of Theorem 3. We construct a solution to (5.8) thanks to an iterative scheme (or equivalently, thanks to the abstract Theorem 6), relying on several important observations:

- First, the left-hand side of (5.8) depends *linearly* on $\widetilde{Y}$, and the right-hand side depends smoothly on $\widetilde{Y}$. This nice feature stems directly from our change of variables. Note also that our choice of boundary conditions (1.15)-(1.16), which are slightly unusual when we formulate them on the unknown function u, are in fact designed so that they become classical boundary conditions in the variable $\widetilde{Y}$. Indeed, the top and bottom boundaries in the z variable are now fixed (and flat), and the boundary condition for $\widetilde{Y}$ on these boundaries is merely a Neumann condition (so a Dirichlet condition for the vorticity $\partial_z \widetilde{Y}$).

- Second, as mentioned above, the vorticity $\partial_z \widetilde{Y}$ satisfies an equation with a very nice structure. More precisely, setting
$$\begin{aligned} \alpha(x,z) &:= \frac{1}{(\partial_z \mathbb{Y}_{\mathrm{P}}(x,z))^2} > 0, \\ \beta(x) &:= -\partial_x p, \end{aligned} \tag{5.18}$$
and differentiating (5.8) with respect to z, we find that $W := \partial_z \widetilde{Y}$ is a solution to
$$\begin{cases} z\partial_x W + \beta \partial_z W - \partial_z^2(\alpha W) = \partial_z g & \text{in } (x_0, x_1) \times (z_b, z_t), \\ W|_{\Sigma_i} = \partial_z \Upsilon_{\mathrm{P}}^i[\delta_i] & \text{for } i \in \{0,1\}, \\ W|_{z=z_j} = \Upsilon_{\mathrm{P}}^j[\delta_j] & \text{for } j \in \{t,b\}. \end{cases} \tag{5.19}$$
The coefficients α and β are smooth and depend only on the underlying flow $(\mathrm{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$. Furthermore, $\inf \alpha > 0$ in $(x_0, x_1) \times (z_b, z_t)$ by assumption. Hence the structure of system (5.19) is very similar to the one of (1.6), albeit with variable coefficients. The smallness condition on z_b and z_t ensures that we have nice *a priori* estimates for (5.19) (see Lemma 5.3 below).

- Eventually, we observe that, from (5.9),
$$\begin{aligned} \partial_z g =& \partial_y f(x,Y)(\partial_z \mathbb{Y}_{\mathrm{P}} - W)^2 + f(x,Y)(\partial_z^2 \mathbb{Y}_{\mathrm{P}} - \partial_z W) \\ &+ \partial_z^2 \left(\frac{W^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 (\partial_z \mathbb{Y}_{\mathrm{P}} - W)} \right). \end{aligned} \tag{5.20}$$
In order to design a convergent iterative scheme for (5.19), it is necessary to work in a functional space controlling the L^∞ norm of W (for example to ensure that the denominator does not vanish, or that the application $W \mapsto \partial_z^2(W^2) \in L^2$ is Lipschitz continous). Having $W \in Z^0$ is not sufficient as we barely miss the embedding in L^∞ (see Remark 1.12). However, the functional space $W \in H_x^{\frac{2}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^2$, with σ strictly positive and small, will be suitable for our purposes. This is in sharp contrast with the nonlinear scheme for the Burgers system, for which we also needed that $\partial_z \widetilde{Y} = W \in L^\infty$ but for which the function $\widetilde{Y}$ (rather than $\partial_z \widetilde{Y} = W$) was a solution to (1.6). Therefore, having $W \in L^\infty$ required $\widetilde{Y} \in H_x^{\frac{2}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^2$ for some $\sigma > 1/3$. Such a regularity requires two orthogonality conditions

(see Proposition 2.33). This gain of one derivative in the vertical variable (corresponding to a gain of 1/3 of derivative in the horizontal variable) allows us to get rid of two of the orthogonality conditions, leading to the first statement of Proposition 5.2.

5.2 The linearized vorticity equation

This section is devoted to the analysis of system (5.19), for a given source term $\partial_z g \in L^2(\Omega)$. Adapting and stepping on the analysis of Section 2, we prove the existence and uniqueness of solutions in $Z^0(\Omega)$. We also exhibit necessary and sufficient conditions for higher regularity.

For the sake of simplicity, within this section, we denote by Ω the rectangle $(x_0, x_1) \times (z_b, z_t)$, which is a slight abuse of notation since $(z_b, z_t) \neq (-1, 1)$. We still denote by $\Sigma_0 = \{x_0\} \times (0, z_t)$ and $\Sigma_1 = \{x_1\} \times (z_b, 0)$ the lateral boundaries.

Lemma 5.3 (Well-posedness of the linear vorticity equation)**.** *Let $\alpha \in C^2(\overline{\Omega})$ and $\beta \in L^\infty(x_0, x_1)$. Assume that there exists $\lambda > 0$ such that*

$$\forall (x,z) \in \Omega, \quad \frac{1}{\lambda} \leq \alpha(x,z) \leq \lambda. \tag{5.21}$$

There exists $z_0 > 0$, depending only on α, such that if $|z_b|, z_t \leq z_0$, the following result holds.

Let $h \in L^2(\Omega)$, $w_t, w_b \in H^{3/4}(x_0, x_1)$, and $w_i \in \mathscr{H}^1_z(\Sigma_i)$. Assume that the compatibility conditions $w_t(x_0) = w_0(z_t)$, $w_b(x_1) = w_1(z_b)$ are satisfied.

Consider the system

$$\begin{cases} z\partial_x W + \beta \partial_z W - \partial_z^2(\alpha W) = h & \text{in } \Omega, \\ W|_{\Sigma_i} = w_i & \text{for } i \in \{0,1\}, \\ W|_{z=z_j} = w_j & \text{for } j \in \{t,b\}. \end{cases} \tag{5.22}$$

Then (5.22) has a unique solution $W \in Z^0(\Omega)$, which moreover satisfies

$$\|W\|_{Z^0} \leq C \left(\|h\|_{L^2} + \|w_t\|_{H^{3/4}} + \|w_b\|_{H^{3/4}} + \|w_0\|_{\mathscr{H}^1_z(\Sigma_0)} + \|w_1\|_{\mathscr{H}^1_z(\Sigma_1)} \right), \tag{5.23}$$

where the constant C depends only on λ, $\|\beta\|_\infty$ and $\|\partial_z \alpha\|_\infty$.

Proof. According to [52, Theorem 2.1] it is sufficient to prove the result when $w_t = w_b = w_0 = w_1 = 0$, since one could lift these boundary conditions for the given regularity.

In this case, we note that since $\partial_z \beta = 0$, we have the $L^2_x H^1_z$ energy estimate

$$\int_\Omega \alpha (\partial_z W)^2 \leq \|h\|_{L^2} \|W\|_{L^2} + \|\partial_z \alpha\|_{L^\infty} \|W\|_{L^2} \|\partial_z W\|_{L^2}. \tag{5.24}$$

If $|z_b|, z_t \leq z_0$, then $\|W\|_{L^2(\Omega)} \leq z_0 \|\partial_z W\|_{L^2(\Omega)}$. As a consequence, if $z_0 \leq 1/(2\lambda \|\partial_z \alpha\|_\infty)$, we obtain $\|W\|_{L^2_x H^1_z} \lesssim \|h\|_{L^2}$. From there, following the same arguments as in Proposition 2.2, we infer that there exists a solution $W \in \mathcal{B}$ to (5.22) satisfying $\|W\|_{\mathcal{B}} \lesssim \|h\|_{L^2}$. The uniqueness of this solution is proved in Appendix A. Eventually, we see $W \in \mathcal{B}$ as the solution to

$$z\partial_x W - \partial_z(\alpha \partial_z W) = h - \beta \partial_z W + \partial_z(\partial_z \alpha W) \tag{5.25}$$

where the right-hand side belongs to $L^2(\Omega)$ since $\beta \in L^\infty$ and $\alpha \in C^2(\overline{\Omega})$. Since $\alpha \in C^2(\overline{\Omega})$, applying Pagani's result [52, Theorem 5.1] to the operator $z\partial_x - \partial_z(\alpha \partial_z \cdot)$ which is in conservative form, we obtain that $W \in Z^0$ and $\|W\|_{Z^0} \lesssim \|h\|_{L^2} + \|W\|_{\mathcal{B}}$. □

We now rely on the analysis of Section 2 in order to identify two necessary and sufficient orthogonality conditions for higher regularity. Let us first remark that the only potential singular points are $(x_0, 0)$ and $(x_1, 0)$. Indeed, we recall that $z\partial_x W \in L^2(\Omega)$, and therefore $W \in H^1_x L^2_z(\{|z| \geq z_0\})$ for all $z_0 > 0$. Regularity away from the lateral boundaries is ensured by the following lemma.

Lemma 5.4. *Let $\alpha \in C^3(\overline{\Omega})$ satisfying (5.21) and $\beta \in C^1([x_0, x_1])$. There exists $z_0 > 0$, depending only on α, such that if $|z_b|, z_t \leq z_0$, the following result holds.*

Let $h \in L^2(\Omega)$ such that $(x - x_0)(x - x_1)\partial_x h \in L^2$. Let $w_t, w_b \in H^2(x_0, x_1)$ and $w_i \in H^2(\Sigma_i)$ such that the compatibility conditions $w_t(x_0) = w_0(z_t)$, $w_b(x_1) = w_1(z_b)$ are satisfied.

Let $W \in Z^0$ be the unique solution to (5.22). Then $(x - x_0)(x - x_1)\partial_x W \in Z^0$.

The proof is postponed to Appendix C, in order not to burden this section. We are now ready to state our orthogonality conditions for system (5.22). To that end, for $\alpha \in C^4(\overline{\Omega})$, $\beta \in C^2([x_0, x_1])$, $\sigma \in (0, 1)$, we introduce the space

$$\begin{aligned}\mathcal{H}^\sigma_{\alpha,\beta} := \Big\{ &(h, w_0, w_1, w_t, w_b) \in H^\sigma_x L^2_z \times H^2(\Sigma_0) \times H^2(\Sigma_1) \times H^2(x_0, x_1)^2, \\ &(x - x_0)(x - x_1)\partial_x h \in L^2, \quad w_t(x_0) = w_0(z_t),\ w_b(x_1) = w_1(z_b), \\ &\text{and } \Delta_i \in \mathscr{H}^1_z(\Sigma_i) \text{ if } \sigma > 1/2, \\ &\text{and } \Delta_0(z_t) = \partial_x w_t(x_0),\ \Delta_1(z_b) = \partial_x w_b(x_1) \text{ if } \sigma > 1/2, \\ &\text{where } \Delta_i := \frac{1}{z}\left(h(x_i, \cdot) + \partial_z^2(\alpha(x_i, \cdot)w_i) - \beta(x_i)\partial_z w_i\right) \Big\}.\end{aligned} \tag{5.26}$$

We now state a proposition extending the results of Section 2 to equations with smooth variable coefficients:

Proposition 5.5. *Let $\alpha \in C^4(\overline{\Omega})$ satisfying (5.21) and $\beta \in C^1([x_0, x_1])$. There exist two linear forms $\widehat{\ell^0}$, $\widehat{\ell^1}$, continuous on $\mathcal{H}^\sigma_{\alpha,\beta}$ for all $\sigma \in (1/6, 1]$, such that the following result holds.*

- *Let $\sigma \in (0, 1/6)$, and let $(h, w_0, w_1, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$. Let $W \in Z^0$ be the unique solution to (5.22).*

  Then $W \in Z^\sigma = [Z^0, Z^1]_\sigma \hookrightarrow H^{\frac{2}{3}+\sigma}_x L^2_z \cap H^\sigma_x H^2_z \hookrightarrow L^\infty(\Omega)$, and

$$\|W\|_{Z^\sigma} \lesssim \|h\|_{H^\sigma_x L^2_z} + \|(x - x_0)(x - x_1)\partial_x h\|_{L^2} + \sum_{j\in\{b,t\}} \|w_j\|_{H^2(x_0,x_1)} + \sum_{i\in\{0,1\}} \|w_i\|_{H^2(\Sigma_i)}. \tag{5.27}$$

- *Let $\sigma \in (1/6, 1)\backslash\{1/2\}$, and let $(h, w_0, w_1, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$. Let $W \in Z^0$ be the unique solution to (5.22). Then $W \in Z^\sigma$ if and only if*

$$\widehat{\ell^0}(h, w_0, w_1, w_t, w_b) = \widehat{\ell^1}(h, w_t, w_b, w_0, w_1) = 0, \tag{5.28}$$

  and in that case

$$\begin{aligned}\|W\|_{Z^\sigma} \lesssim &\|h\|_{H^\sigma_x L^2_z} + \|(x - x_0)(x - x_1)\partial_x h\|_{L^2} \\ &+ \sum_{j\in\{b,t\}} \|w_j\|_{H^2(x_0,x_1)} + \sum_{i\in\{0,1\}} \|w_i\|_{H^2(\Sigma_i)} + \|\Delta_i\|_{\mathscr{H}^1_z(\Sigma_i)}.\end{aligned} \tag{5.29}$$

Proof. We start with the first statement, and we take $\sigma \in (0, 1/6)$ fixed.

Step 1. *Lifting the top and bottom boundary conditions.* In order to use the theory from Section 2, which is stated with homogeneous Dirichlet boundary conditions at the top and bottom, we first lift the latter. We change W into $W - \rho(z - z_t)w_t - \rho(z - z_b)w_b$, where $\rho \in C_c^\infty(\mathbb{R})$ is such that $\rho \equiv 1$ in a neighbourhood of zero, and $\operatorname{supp}\rho \subset (-r, r)$ for some $r < \min(|z_b|, z_t)/2$. This changes the source term h into

$$h - \sum_{j\in\{t,b\}} (z\partial_x w_j \rho(z - z_j) + \beta w_j \rho'(z - z_j) - w_j \partial_z^2(\alpha\rho(z - z_j)),$$

which belongs to $H_x^\sigma L_z^2$, and the boundary condition w_0 (resp. w_1) into $w_0 - w_0(z_t)\eta(z - z_t)$ (resp. $w_1 - w_1(z_b)\eta(z - z_b)$), which belongs to $H^2(\Sigma_0)$ (resp. $H^2(\Sigma_1)$). With a slight abuse of notation, we still denote by W the unknown function, and by $(h, w_0, w_1, 0, 0)$ the data. Note that this operation does not affect the compatibility conditions in the corners.

Step 2. *Localization in the vicinity of the singular points.* We then localize horizontally the solution in the vicinity of x_0 and x_1. We only treat the localization in the vicinity of x_0 since the other boundary is identical. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be such that $\chi_0 \equiv 1$ in a neighborhood of x_0, and $\operatorname{supp}\chi_0 \subset B(x_0, r)$ for some small $0 < r < (x_1 - x_0)/2$. Then $W_0 := W\chi_0(x)$ is a solution to

$$z\partial_x W_0 + \beta\partial_z W_0 - \partial_z^2(\alpha W_0) = h\chi_0 + zW\partial_x\chi_0. \tag{5.30}$$

Since $W \in Z^0$, $W \in H_x^{2/3}L_z^2$, so the right-hand side belongs to $H_x^\sigma L_z^2$. We then localize the coefficient α. Let $\alpha_0(z) := \alpha(x_0, z)$. Then

$$\begin{aligned} z\partial_x W_0 - \partial_z^2(\alpha_0(z)W_0) = h\chi_0 + zW\partial_x\chi_0 - \beta\partial_z W\chi_0 \\ - \partial_z^2((\alpha_0 - \alpha)W_0). \end{aligned} \tag{5.31}$$

On the support of χ_0, there exists a constant C such that $|\alpha_0 - \alpha| \le C|x - x_0|$ and $(\alpha - \alpha_0)/(x - x_0)$ is a C^3 function of (x, z). According to Lemma 5.4, $(\alpha_0 - \alpha)\partial_z^2 W_0 \in H_x^1 L_z^2$. Hence, the right-hand side of (5.31) belongs to $H_x^\sigma L_z^2$. Note furthermore that W_0 vanishes on $\{z = z_t\}$ and $\{z = z_b\}$ thanks to the first step.

Step 3. *Vertical change of variables to work with constant coefficients.* In order to use the theory from Section 2, we now change the vertical coordinate so that the equation in the new variables is formulated thanks to the Kolmogorov operator. More precisely, we set $W_0(x, z) = \omega_0(x, \zeta)$ where ζ is a function of z such that $\zeta(0) = 0$. We have

$$\partial_z^2(\alpha_0 W_0) = \alpha_0(\zeta')^2\partial_\zeta^2\omega_0 + (\alpha_0\zeta'' + 2\partial_z\alpha_0\zeta')\partial_\zeta\omega_0 + (\partial_z^2\alpha_0)\omega_0. \tag{5.32}$$

We first choose the function ζ so that $\zeta(0) = 0$ and

$$\frac{z}{\alpha_0(z)(\zeta'(z))^2} = \zeta, \quad \text{i.e.} \quad \zeta'(z) = \sqrt{\frac{z}{\alpha_0(z)\zeta(z)}}. \tag{5.33}$$

Explicit resolution for $z > 0$ yields (with a similar formula for $z < 0$):

$$\zeta(z) = \left(\frac{3}{2}\int_0^z \sqrt{\frac{t}{\alpha_0(t)}}\,\mathrm{d}t\right)^{2/3}. \tag{5.34}$$

It can be easily checked that the function ζ thus defined has the same regularity as α_0 on (z_b, z_t) and that $C^{-1} \leq \zeta' \leq C$ for some positive constant C. Moreover, $(\alpha_0(0))^{\frac{1}{3}}\zeta(z) \sim z$ as $z \to 0$.

The function ω_0 then solves

$$\zeta \partial_x \omega_0 - \partial_\zeta^2 \omega_0 = s_0(x, \zeta) \tag{5.35}$$

where $s_0 \in H_x^\sigma L_z^2$. Furthermore, ω_0 is supported in the vicinity of $(x_0, 0)$. We denote by μ_0 the lateral boundary condition on Σ_0 in the new vertical variable, i.e. $\mu_0(\zeta(z)) = \chi_0(x_0, z)w_0(z)$. Note that μ_0 and w_0 enjoy the same regularity, so that $\mu_0 \in H^2(\Sigma_0)$.

Step 4. *Small fractional regularity.* We now consider (5.35), whose right-hand side belongs to $H_x^\sigma L_z^2$. The equation is endowed with homogeneous data on $\{z = z_t\} \cup \{z = z_b\} \cup \Sigma_1$, and with H^2 data on Σ_0 satisfying a compatibility condition at (x_0, z_t). Using Proposition 2.33 and Lemma 1.15, we infer that $\omega_0 \in Z^\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L_\zeta^2 \cap H_x^\sigma H_\zeta^2$, and thus W_0 enjoys the same regularity. Performing a similar change of variables near $(x_1, 0)$, we deduce that $W \in Z^\sigma$. This completes the proof of the first statement from Proposition 5.5.

Step 5. *Identification of the orthogonality conditions.* Let us now assume that $\sigma \in (1/6, 1/3]$ and $h \in H_x^\sigma L_z^2$. The right-hand side of (5.35) now belongs to $H_x^\sigma L_z^2$. Furthermore, in a neighborhood of $\zeta = 0$,

$$s_0(x_0, \zeta) = \frac{1}{\alpha_0(z)(\zeta'(z))^2}\left[h(x_0, z) - \beta(x_0)w_0'(z) + (\alpha_0\zeta'' + 2\alpha_0'\zeta')(z)\partial_\zeta\mu_0(\zeta) + \alpha_0''(z)\mu_0(\zeta)\right],$$

where the primes always denote derivatives with respect to z. Using this equality together with the identity $\partial_\zeta \mu_0(\zeta) = w_0'(z)/\zeta'(z)$, we find, after some tedious but straightforward computations, and for ζ in a neighborhood of zero,

$$\partial_\zeta^2 \mu_0(\zeta) + s_0(x_0, \zeta) = \frac{\zeta(z)}{z}\Delta_0(z) \tag{5.36}$$

Hence $(\partial_\zeta^2\mu_0 + s_0(x_0, \cdot))/\zeta \in \mathscr{H}_\zeta^1(\Sigma_0)$. Note also that the compatibility conditions in the corners are satisfied. We then apply Corollary 2.36 to (5.35) whose right-hand side is in $H_x^\sigma L_\zeta^2$. We infer that if

$$(a_0\overline{\ell^0} + a_1\overline{\ell^1})(s_0, \mu_0, 0) = 0, \tag{5.37}$$

then $\omega_0 \in Z^\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L_\zeta^2 \cap H_x^\sigma H_\zeta^2$ by Lemma 1.15. Similarly, $\omega_1 \in Z^\sigma$, so $W \in Z^\sigma$.

For $\sigma \geq 1/3$ and $\sigma \neq 1/2$, we use a bootstrap argument. Going back to (5.30), we now know that the right-hand side is in $H_x^{\min(\sigma, 2/3)} L_z^2$, so that we can apply Corollary 2.36 to (5.35) whose right-hand side belongs to $H_x^{\min(\sigma, 2/3)} L_\zeta^2$. This implies that $W \in Z^{\min(\sigma, 2/3)}$. We then repeat this procedure one last time if $\sigma \geq 2/3$.

Setting

$$\widehat{\ell^0}(h, w_0, w_1, w_t, w_b) = (a_0\overline{\ell^0} + a_1\overline{\ell^1})(\lambda\omega_0 + s_0, \mu_0, 0), \tag{5.38}$$

and defining in a similar fashion the linear form $\widehat{\ell^1}$ associated with the regularity in the vicinity of $(x_1, 0)$, we obtain the desired result.

Eventually, it follows from the definition of $\widehat{\ell^0}$ in (5.38) and from Remark 2.30 that the linear forms $\widehat{\ell^j}$ are continuous on $\mathcal{H}_{\alpha,\beta}^\sigma$ for all $\sigma > 1/6$. □

Lemma 5.6. *The two linear forms $\widehat{\ell_0}, \widehat{\ell_1} : \mathcal{H}_{\alpha,\beta}^1 \to \mathbb{R}$ defined in Proposition 5.5 are independent. Furthermore, there exists $g^0, g^1 \in C_c^\infty(\Omega)$ such that*

$$\widehat{\ell^j}(g^i, 0, 0, 0, 0) = \delta_{i,j} \quad \forall i, j \in \{0, 1\}.$$

Proof. We begin with the following remark. Following the notations of the proof of Proposition 5.5 above, we set $\alpha_i(z) := \alpha(x_i, z)$,. With the same change of variables as in Step 3 of the proof (see (5.34)), we define

$$U^0(x,z) = \bar{u}^0_{\rm sing}(x, \zeta_0).$$

Then

$$z\partial_x U^0 + \beta\partial_z U^0 - \partial_z^2(\alpha_0 U^0) = \alpha_0(\zeta_0')^2\overline{f^0} + \lambda_0 \bar{u}^0_{\rm sing}(x,\zeta_0) + \gamma_0 \partial_{\zeta_0}\bar{u}^0_{\rm sing}(x,\zeta_0).$$

for some smooth functions λ_0, γ_0 depending on β and α. The right-hand side therefore belongs to $H^{1/3}_x L^2_z \cap H^1_x L^2_z((x-x_0)^2(x-x_1)^2)$. Furthermore U^0 vanishes on $\Sigma_0 \cup \Sigma_1 \cup \{z = z_b\} \cup \{z = z_t\}$.

Of course we may perform the same procedure around $(x_1, 0)$, and we define a function $U^1(x,z)$, localized in a neighborhood of $(x_1, 0)$ and with the same regularity as $\bar{u}^1_{\rm sing}$, such that

$$z\partial_x U^1 + \beta\partial_z U^1 - \partial_z^2(\alpha_1 U^1) \in H^{1/3}_x L^2_z.$$

Note that U^0 and U^1 vanish on $\Sigma_0 \cup \Sigma_1 \cup \{z = z_b\} \cup \{z = z_t\}$. Now, for $i = 0, 1$, let

$$h^i := z\partial_x U^i + \beta\partial_z U^i - \partial_z^2(\alpha U^i).$$

By construction, h_i and U_i are localized in the vicinity of $(x_i, 0)$, and $h^i \in H^{1/3}_x L^2_z$, $(x-x_0)(x-x_1)\partial_x h_i \in L^2$. Furthermore $h_i|_{\Sigma_0\cup\Sigma_1} = 0$. As a consequence, recalling the definition of $\widehat{\ell^0}$ and $\widehat{\ell^1}$ (see (5.38) together with Corollary 2.36), we infer that

$$\widehat{\ell^0}(h_1, 0, 0, 0, 0) = \widehat{\ell^1}(h_0, 0, 0, 0, 0) = 0.$$

Now, assume that $c_0\widehat{\ell^0} + c_1\widehat{\ell^1} = 0$ for some $(c_0, c_1) \in \mathbb{R}^2$. We deduce from the above equalities that

$$\widehat{\ell^0}(c_0 h_0 + c_1 h_1, 0, 0, 0, 0) = c_0\widehat{\ell^0}(h_0, 0, 0, 0, 0) = (c_0\widehat{\ell^0} + c_1\widehat{\ell^1})(h_0, 0, 0, 0, 0) = 0,$$

and similarly $\widehat{\ell^1}(c_0 h_0 + c_1 h_1, 0, 0, 0, 0) = 0$. Using Proposition 5.5, we infer that $c_0 U^0 + c_1 U^1 \in Z^{1/3} \hookrightarrow H^1_x L^2_z \cap H^{1/3}_x H^2_z$. Since U^i has the regularity of $\bar{u}^i_{\rm sing}$ and is localized in the vicinity of $(x_i, 0)$, it follows from Lemma 2.25 that $c_0 = c_1 = 0$.

Note that the above argument also ensures that $\widehat{\ell^i}(h^i, 0, 0, 0, 0) \neq 0$. Hence, up to a multiplication by a constant, we may always assume that $\widehat{\ell^j}(h^i, 0, 0, 0, 0) = \delta_{i,j}$. Let us now take, for $\varepsilon > 0$ small, $h^i_\varepsilon \in C^\infty_c(\Omega)$ such that $\|h^i_\varepsilon - h^i\|_{H^{1/3}_x L^2_z} \leq \varepsilon$ and $\|(x-x_0)(x-x_1)\partial_x(h^i - h^i_\varepsilon)\|_{L^2} \leq \varepsilon$. Then, since the linear forms $\widehat{\ell^j}$ are continuous on $\mathcal{H}^{1/3}_{\alpha,\beta}$, we obtain $|\widehat{\ell^j}(h^i_\varepsilon, 0, 0, 0, 0) - \delta_{i,j}| \lesssim \varepsilon$. As a consequence, there exists $a^0_\varepsilon, a^1_\varepsilon, b^0_\varepsilon, b^1_\varepsilon$ such that

$$\begin{aligned}
\widehat{\ell^0}(a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) &= \widehat{\ell^1}(b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = 1,\\
\widehat{\ell^0}(b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) &= \widehat{\ell^1}(a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = 0,
\end{aligned}$$

and $|a^0_\varepsilon - 1|, |b^1_\varepsilon - 1| \lesssim \varepsilon$, $|a^1_\varepsilon|, |b^0_\varepsilon| \lesssim \varepsilon$. The result follows, taking $g^0 = a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon$ and $g^1 = b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon$. □

5.3 Reconstructing the velocity from the vorticity

Let $(g, \widetilde{\delta}_0, \widetilde{\delta}_1) \in L^2_x H^1_z \times H^3(\Sigma_0) \times H^3(\Sigma_1)$, $w_t, w_b \in H^2(x_0, x_1)$. Assume that $\partial_z \widetilde{\delta}_1(z_b) = w_b(x_1)$, $\partial_z \widetilde{\delta}_0(z_t) = w_t(x_0)$. According to Lemma 5.3, there exists a unique solution $W \in Z^0$ to (5.22) with $h = \partial_z g$ and $w_i = \partial_z \widetilde{\delta}_i$. The purpose of this subsection is to construct a solution to the system

$$\begin{cases} z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y} + \beta \partial_z \widetilde{Y} - \partial_z(\alpha \partial_z \widetilde{Y}) = g & \text{in } \Omega, \\ \widetilde{Y}|_{\Sigma_i} = \widetilde{\delta}_i & \text{for } i \in \{0,1\}, \\ \partial_z \widetilde{Y}|_{z=z_j} = w_j & \text{for } j \in \{t,b\}. \end{cases} \tag{5.39}$$

We therefore set, for $(x,z) \in \Omega$,

$$\widetilde{Y}(x,z) := \widetilde{\gamma}_b(x) + \int_{z_b}^{z} W(x,z')\,\mathrm{d}z', \tag{5.40}$$

where the function $\widetilde{\gamma}_b$ solves the differential equation

$$\begin{aligned} &z_b \partial_x \widetilde{\gamma}_b + (\beta - \partial_z \alpha(\cdot, z_b)) w_b - \alpha(x, z_b)\partial_z W(x, z_b) = g(x, z_b), \\ &\widetilde{\gamma}_b(x_1) = \widetilde{\delta}_1(z_b). \end{aligned} \tag{5.41}$$

Since $W \in Z^0$, the trace $\partial_z W(\cdot, z_b)$ belongs to $H^{1/4}(x_0, x_1)$ by Lemma 1.11. Thus $\widetilde{\gamma}_b \in H^1(x_0, x_1)$, and $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z \subset C^0(\overline{\Omega})$.

By construction, we have

$$\partial_z \left[z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y}(x,z')\,\mathrm{d}z' + \beta \partial_z \widetilde{Y} - \partial_z(\alpha \partial_z \widetilde{Y}) - g \right] = 0 \quad \text{in } \Omega, \tag{5.42}$$

and therefore there exists a function G depending only on x such that

$$z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y}(x,z')\,\mathrm{d}z' + \beta \partial_z \widetilde{Y} - \alpha \partial_z^2 \widetilde{Y} = g(x,z) + G(x). \tag{5.43}$$

The choice of the function $\widetilde{\gamma}_b$ (see (5.41)) then ensures that $G \equiv 0$. By definition of $\widetilde{Y}$, we have $\partial_z \widetilde{Y}|_{z=z_j} = W|_{z=z_j} = w_j$ for $j \in \{t,b\}$.

Let us now investigate the lateral boundary conditions. On Σ_i, we have

$$\partial_z \widetilde{Y}(x_i, z) = W(x_i, z) = \partial_z \widetilde{\delta}_i(z). \tag{5.44}$$

Hence, in order to ensure that $\widetilde{Y}|_{\Sigma_i} = \widetilde{\delta}_i$, it suffices to check that $\widetilde{Y}(x_i, z_i) = \widetilde{\delta}_i(z_i)$ for some $(x_i, z_i) \in \overline{\Sigma_i}$. From there, we treat separately (and differently) the two boundaries Σ_0 and Σ_1.

- On Σ_1, we note that $\widetilde{Y}(x_1, z_b) = \widetilde{\delta}_1(z_b)$ by definition of γ_b. Therefore $\widetilde{Y}|_{\Sigma_1} = \widetilde{\delta}_1$.
- On Σ_0, the situation is different, since $\widetilde{Y}(x_0, 0) \neq \widetilde{\delta}_0(0)$ *a priori*. Indeed,

$$\begin{aligned} \widetilde{Y}(x_0, 0) &= \widetilde{\gamma}_b(x_0) + \int_{z_b}^{0} W(x_0, z')\,\mathrm{d}z' \\ &= \frac{1}{z_b}\left(- \int_{x_0}^{x_1} \left(g(x, z_b) + (\partial_z \alpha(x, z_b) - \beta(x)) w_b(x) + \alpha(x, z_b)\partial_z W(x, z_b) \right) \mathrm{d}x \right) \\ &\quad + \int_{z_b}^{0} W(x_0, z')\,\mathrm{d}z' + \widetilde{\delta}_1(z_b). \end{aligned}$$

The right-hand side of the above equality is a linear form in $(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b)$, which leads to the following definition.

Definition 5.7 (Additional linear form for the solvability of the Prandtl system)**.** *Let $(g, \widetilde{\delta}_0, \widetilde{\delta}_1) \in L^2_x H^1_z \times H^3(\Sigma_0) \times H^3(\Sigma_1)$, $w_t, w_b \in H^2(x_0, x_1)$ such that $w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t)$, $w_b(x_1) = \partial_z \widetilde{\delta}_1(z_b)$. Let $W \in Z^0$ be the unique solution to (5.22) with $h = \partial_z g$ and $w_i = \partial_z \widetilde{\delta}_i$.*

The linear form ℓ^2 is defined by

$$\ell^2\left(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b\right) := -\frac{1}{z_b}\int_{x_0}^{x_1} \left(g(x, z_b) + (\partial_z \alpha(x, z_b) - \beta(x))w_b(x) + \alpha(x, z_b)\partial_z W(x, z_b)\right) \mathrm{d}x + \int_{z_b}^{0} W(x_0, z')\,\mathrm{d}z' + \widetilde{\delta}_1(z_b) - \widetilde{\delta}_0(0). \tag{5.45}$$

The above computations lead to the following result.

Lemma 5.8. *Let $(g, \widetilde{\delta}_0, \widetilde{\delta}_1) \in L^2_x H^1_z \times H^3(\Sigma_0) \times H^3(\Sigma_1)$, $w_t, w_b \in H^2(x_0, x_1)$ such that $w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t)$, $w_b(x_1) = \partial_z \widetilde{\delta}_1(z_b)$.*

Then system (5.39) has a solution $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$ if and only if

$$\ell^2\left(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b\right) = 0.$$

This solution is given by (5.40), and satisfies the estimate

$$\|\widetilde{Y}\|_{H^{2/3}_x H^1_z} + \|\widetilde{Y}\|_{L^2_x H^3_z} + \|\partial_x \widetilde{Y}\|_{L^2_x H^1_z(\{z < z_b/2\})} \lesssim \|g\|_{L^2_x H^1_z} + \|\widetilde{\delta}_i\|_{H^3(\Sigma_i)} + \|w_j\|_{H^2(x_0, x_1)},$$

where we implicitely sum over $i \in \{0, 1\}$ and $j \in \{t, b\}$ in the right-hand side.

Proof. First, assume that (5.39) has a solution $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$. Then, $W = \partial_z \widetilde{Y}$ is an $L^2_x H^1_z$ solution to (5.22) with $h = \partial_z g$ and $w_i = \partial_z \widetilde{\delta}_i$. By uniqueness arguments such as in Lemma A.1, it is equal to the unique Z^0 solution to (5.22) constructed in Lemma 5.3. Furthermore, for $z \neq 0$,

$$z^2 \partial_z \left(\frac{\int_{z_b}^{z} \partial_x \widetilde{Y}}{z}\right) = g + \partial_z(\alpha \partial_z \widetilde{Y}) - \beta \partial_z \widetilde{Y} \in L^2_x H^1_z.$$

It follows that $\partial_z \left(\frac{\int_{z_b}^{z} \partial_x \widetilde{Y}}{z}\right) \in L^2_x H^1_z(\{z < z_b/2\})$, and thus $\partial_x \widetilde{Y} \in L^2_x H^1_z(\{z < z_b/2\})$. In particular, $\partial_x \widetilde{Y}|_{z=z_b} \in L^2(x_0, x_1)$.

Taking the trace of (5.39) at $z = z_b$, we infer that

$$z_b \partial_x \widetilde{Y}|_{z=z_b} + (\beta - \partial_z \alpha(x, z_b))w_b - \alpha \partial_z W(x, z_b) = g(x, z_b) \quad \text{and} \quad \widetilde{Y}(x_1, z_b) = \widetilde{\delta}_1(z_b).$$

Therefore $\widetilde{Y}|_{z=z_b} = \widetilde{\gamma}_b$, where $\widetilde{\gamma}_b$ is defined by (5.41). Since $\widetilde{Y}(x_0, 0) = \widetilde{\delta}_0(0)$, we then deduce that

$$\widetilde{Y}(x_0, z_b) + \int_{z_b}^{0} W(x_0, z)\,\mathrm{d}z = \widetilde{\delta}_0(0),$$

which is precisely the condition $\ell^2(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) = 0$.

Conversely, the above computations ensure that if $\ell^2(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) = 0$, the function defined by (5.40) is a solution to (5.39). □

Assume that $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$, and let $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$ be the unique solution to (5.39). For further purposes, we define the function $\widetilde{\gamma_t}$ by

$$\widetilde{\gamma_t}(x) := \widetilde{Y}(x, z_t).$$

Since $\widetilde{Y} \in C^0(\overline{\Omega})$, we have $\widetilde{\gamma_t}(x_0) = \widetilde{\delta_0}(z_t)$.

Remark 5.9. *As we already mentioned, the nonlocal term $v\partial_y u$ in the Prandtl equation (which becomes $-\int_{z_b}^{z} \partial_x Y$ in our new variables) creates a flow of information upwards, therefore inducing an assymetry between z and $-z$. Because of the forward-backward nature of the equation, this results in an asymmetry between the lateral boundaries Σ_0 and Σ_1. Iyer and Masmoudi [35] deal with this issue by letting the left extremity of the curve $\{(x, y); u(x, y) = 0\}$ as a free parameter, and enforcing data on the vorticity. We choose a different point of view here, introducing an additional orthogonality condition and enforcing data on the inflow velocity.*

In order to simplify the future discussion, it will be useful to modify slightly the definition of the linear forms ℓ^i for $i \in \{0, 1\}$, so that they are defined on the same space as the linear form ℓ^2.

Definition 5.10. *We denote by ℓ^i for $i \in \{0, 1\}$ the linear forms defined by*

$$\ell^i(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) := \widehat{\ell^i}(\partial_z g, \partial_z \widetilde{\delta_0}, \partial_z \widetilde{\delta_1}, w_t, w_b).$$

Remark 5.11. *In spite of their similar appearance, the purpose of the orthogonality conditions $\ell^0(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = \ell^1(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$ on the one hand, and $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$ on the other hand is quite different. The former are necessary and sufficient conditions for the existence of smooth solutions to the vorticity equation (5.22), while the latter is a necessary and sufficient condition for the solvability of system (5.39) at a lower level of regularity, corresponding to Z^0 solutions of the vorticity equation (5.22). In other words, the condition $\ell^2 = 0$ is a necessary and sufficient condition to reconstruct $\widetilde{Y}$ from the vorticity.*

Lemma 5.12. *The linear forms ℓ^0, ℓ^1, ℓ^2 are linearly independent on $C^\infty(\overline{\Omega}) \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1) \times C^\infty_c(x_0, x_1)^2$. There exist Ξ^0, Ξ^1, Ξ^2 such that, for $i, j \in \{0, 1, 2\}$,*

$$\ell^i(\Xi_j) = \delta_{i,j}, \qquad \Xi^j \in C^\infty(\overline{\Omega}) \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1) \times C^\infty_c(x_0, x_1)^2.$$

One may choose $\Xi^j = (f^j, 0, 0, 0, 0)$, with $f^j \in C^\infty(\overline{\Omega})$ such that $f^j|_{\Sigma_0 \cup \Sigma_1} = 0$.

Proof. Assume that there exists $(c_0, c_1, c_2) \in \mathbb{R}^3$ such that

$$c_0\ell^0 + c_1\ell^1 + c_2\ell^2 = 0.$$

Let $W \in C^\infty_c(\overline{\Omega})$ such that $\operatorname{supp} W \subset [x_0, x_0 + \delta] \times [-\delta, -\delta/2]$ for some small $\delta > 0$ such that $\delta < (x_1 - x_0)/2$ and $\delta < |z_b|/2$. We further assume that $\int_{z_b}^0 W(x_0, z)\mathrm{d}z = 1$ and $\int_{z_b}^0 z\partial_x W(x_0, z)\mathrm{d}z = 0$. We set $w_t = w_b = 0$, $\widetilde{\delta_0} = \widetilde{\delta_1} = 0$, and

$$f^2(x, z) := \int_{z_b}^{z} \left(z'\partial_x W(x, z') + \beta(x)\partial_z W(x, z') - \partial_z^2(\alpha(x, z')W(x, z')) \right) \mathrm{d}z'.$$

Then by definition, W is a solution to (5.22) with $h = \partial_z f^2$, and with homogeneous boundary data. Note also that $f^2(x_0, 0) = \int_{z_b}^0 z\partial_x W(x_0, z)\,\mathrm{d}z = 0$. Therefore $f^2|_{\Sigma_0 \cup \Sigma_1} = 0$. The compatibility conditions from Proposition 5.5 are satisfied. Since W is smooth, according to Proposition 5.5,

$$\widehat{\ell^0}(\partial_z f^2, 0, 0, 0, 0) = \widehat{\ell^1}(\partial_z f^2, 0, 0, 0, 0) = 0.$$

Hence

$$c_2 \ell^2(f^2, 0, 0, 0, 0) = 0.$$

Now, by definition of ℓ^2 and f^2, since W and f^2 are identically zero for $z \leq -\delta$,

$$\ell^2(f^2, 0, 0, 0, 0) = \int_{z_b}^{0} W = 1.$$

We infer that $c_2 = 0$. The result then follows from Lemma 5.6, taking $f^i = \int_0^z g^i$ for $i = 0, 1$. □

Gathering the results of Proposition 5.5 and Lemma 5.8, we obtain the following statement:

Corollary 5.13. *Let $\alpha \in C^4(\overline{\Omega})$ satisfying (5.21) and $\beta \in C^1(x_0, x_1)$.*

- *Let $\sigma \in (0, 1/6)$, and let $g \in H^\sigma_x H^1_z$ such that $(\partial_z g, \partial_z \widetilde{\delta}_0, \partial_z \widetilde{\delta}_1, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$ defined in (5.26).*

  Then (5.39) has a solution $\widetilde{Y} \in H_x^{\frac{2}{3}+\sigma} H^1_z \cap H^\sigma_x H^3_z$ if and only if $\ell^2(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) = 0$, and this solution, if it exists, is unique and satisfies the estimate

$$\begin{aligned} &\|\widetilde{Y}\|_{H_x^{\frac{2}{3}+\sigma} H^1_z \cap H^\sigma_x H^3_z} + \|(x - x_0)(x - x_1)\partial_x \partial_z^3 \widetilde{Y}\|_{L^2} + \|\partial_x \partial_z \widetilde{Y}\|_{L^2((x_0,x_1)\times(z_b, z_b/2))} \\ &\quad \lesssim \|g\|_{H^\sigma_x H^1_z} + \|(x - x_0)(x - x_1)\partial_x \partial_z g\|_{L^2} + \|w_j\|_{H^2_x} + \|\widetilde{\delta}_i\|_{H^3(\Sigma_i)}. \end{aligned} \tag{5.46}$$

- *Let $g \in H^1_x H^1_z$, and assume that $(\partial_z g, \partial_z \widetilde{\delta}_0, \partial_z \widetilde{\delta}_1, w_t, w_b) \in \mathcal{H}^1_{\alpha,\beta}$ defined in (5.26).*

  Assume that $\ell^2(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) = 0$, and let $\widetilde{Y} \in H_x^{\frac{2}{3}} H^1_z \cap L^2_x H^3_z$ be the unique solution to (5.39).

  Then $\widetilde{Y} \in H_x^{5/3} H^1_z \cap H^1_x H^3_z$ if and only if $\ell^j(g, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) = 0$ for $j \in \{0, 1\}$, and in this case $\widetilde{Y}$ satisfies the estimate

$$\|\widetilde{Y}\|_{H_x^{5/3} H^1_z} + \|\widetilde{Y}\|_{H^1_x H^3_z} \lesssim \|g\|_{H^1_x H^1_z} + \|(\partial_z g, \partial_z \widetilde{\delta}_0, \partial_z \widetilde{\delta}_1, w_t, w_b)\|_{\mathcal{H}^1_{\alpha,\beta}}. \tag{5.47}$$

Remark 5.14. *The regularity assumptions on g in the first (resp. second) statement of the above corollary can be relaxed into $\partial_z g \in H^\sigma_x L^2_z$, $(x - x_0)(x - x_1)\partial_x \partial_z g \in L^2$ and $g|_{z=z_b} \in L^2(x_0, x_1)$ (resp. $\partial_z g \in H^1_x L^2_z$ and $g|_{z=z_b} \in H^{2/3}(x_0, x_1)$), but we have kept the above assumptions for the sake of simplicity.*

5.4 Local nonlinear well-posedness in the new variables

We are now ready to prove Proposition 5.2. The spirit of the proof is very similar to the one of Section 4. In order to avoid repetition, we do not write the iterative scheme, and we rather apply Theorem 6 directly. We will work with two different settings:

1. *Low regularity setting:* for $\sigma \in (0, 1/6)$ fixed, we take

$$\begin{aligned}
\mathcal{Z}^\sigma =& \big\{ Y \in H_x^{\frac{2}{3}+\sigma} H_z^1 \cap H_x^\sigma H_z^3,\ (x-x_0)(x-x_1)\partial_x \partial_z^3 Y \in L^2(\Omega), \\
& \partial_x \partial_z Y \in L^2((x_0,x_1)\times(z_b, z_b/2)), \\
& Y|_{\Sigma_i} \in H^3(\Sigma_i),\ \partial_z Y|_{z=z_j} \in H^2(x_0,x_1) \big\}, \\
\mathcal{H}^\sigma =& \Big\{ (f, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) \in H_x^\sigma H_z^1 \times H^3(\Sigma_0) \times H^3(\Sigma_1) \times H^2(x_0,x_1) \times H^2(x_0,x_1), \\
& (x-x_0)(x-x_1)\partial_x \partial_z f \in L^2, \\
& w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t),\ w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t) \Big\},
\end{aligned}$$

and our space of data is the space $\mathcal{X}^\sigma$ defined in (5.16). Furthermore, in the low regularity setting, $d = 1$ and the linear form ℓ coincides with the linear form ℓ^2 defined in Definition 5.7.

2. *High regularity setting:* we take

$$\begin{aligned}
\mathcal{Z}^1 =& \big\{ Y \in H_x^{\frac{5}{3}} H_z^1 \cap H_x^1 H_z^3, \quad Y|_{\Sigma_i} \in H^5(\Sigma_i),\ \partial_z Y|_{z=z_j} \in H^2(x_0,x_1) \big\}, \\
\mathcal{H}^1 =& \Big\{ (f, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) \in H_x^1 H_z^1 \times H^5(\Sigma_0) \times H^5(\Sigma_1) \times H^2(x_0,x_1) \times H^2(x_0,x_1), \\
& \partial_z^k \widetilde{\delta}_i(0) = 0 \quad \forall k \in \{0, \cdots, 3\}, \quad z^{-1}\partial_z f(x_i, z) \in \mathscr{H}_z^1(\Sigma_i), \\
& w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t),\ w_t(x_0) = \partial_z \widetilde{\delta}_0(z_t), \\
& \Delta_0(z_t) = \partial_x w_t(x_0),\ \Delta_1(z_b) = \partial_x w_b[\delta_b](x_1), \Big\}
\end{aligned}$$

where

$$\Delta_i(z) = \frac{1}{z}\partial_z \left[f(x_i, z) + \partial_z(\alpha(x_i, z)\partial_z \widetilde{\delta}_i) - \beta(x_i)\partial_z \widetilde{\delta}_i \right].$$

Note that

$$(f, \widetilde{\delta}_0, \widetilde{\delta}_1, w_t, w_b) \in \mathcal{H}^1 \Rightarrow (\partial_z f, \partial_z \widetilde{\delta}_0, \partial_z \widetilde{\delta}_1, w_t, w_b) \in \mathcal{H}^1_{\alpha,\beta},$$

where the space $\mathcal{H}^\sigma_{\alpha,\beta}$ for $\sigma \in (0, 1]$ is defined in (5.26). Our space of data is the space $\mathcal{X}^1$ defined in (5.17). In the high regularity setting, we take $d = 3$ and $\ell = (\ell^0, \ell^1, \ell^2)$ defined in Definition 5.10 and Definition 5.7.

Remark 5.15. *As in the previous sections, in $\mathcal{X}^1$, we could also consider source terms f which do not vanish on Σ_i^P, up to additional technical complications.*

In both settings, the linear operator L is defined as

$$L\widetilde{Y} = \left(z\partial_x \widetilde{Y} - \int_{z_b}^z \partial_x \widetilde{Y} + \beta \partial_z \widetilde{Y} - \partial_z(\alpha \partial_z \widetilde{Y}), \widetilde{Y}|_{\Sigma_0}, \widetilde{Y}|_{\Sigma_1}, \partial_z \widetilde{Y}|_{z=z_t}, \partial_z \widetilde{Y}|_{z=z_b} \right),$$

and the nonlinearity N is defined as

$$\begin{aligned}
N(\Xi, \widetilde{Y}) =& \Big(f(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y})\partial_z(\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - v_b + \partial_z \left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2(\partial_z \mathbb{Y}_{\mathrm{P}} - \partial_z \widetilde{Y})} \right), \\
& \Upsilon^0_{\mathrm{P}}[\delta_0], \Upsilon^1_{\mathrm{P}}[\delta_1], \Upsilon^t_{\mathrm{P}}[\delta_t], \Upsilon^b_{\mathrm{P}}[\delta_b] \Big),
\end{aligned} \tag{5.48}$$

where the operators Υ^i_{P}, Υ^j_{P} for $i \in \{0, 1\}$, $j \in \{t, b\}$ are defined in (5.13) and (5.10) respectively.

Let us now check that the assumptions of Theorem 6 are satisfied in the two settings. The continuity of L from $\mathcal{Z}^\sigma$ to $\mathcal{H}^\sigma$ for $\sigma \in (0,1/6) \cup \{1\}$ is a consequence of the definition of the spaces $\mathcal{Z}^\sigma$. Item i) follows from Corollary 5.13. Furthermore,

$$N(\Xi, 0) = \left(f(x, \mathbb{Y}_{\mathrm{P}})\partial_z \mathbb{Y}_{\mathrm{P}} - v_b, \Upsilon^0_{\mathrm{P}}[\delta_0], \Upsilon^1_{\mathrm{P}}[\delta_1], \Upsilon^t_{\mathrm{P}}[\delta_t], \Upsilon^b_{\mathrm{P}}[\delta_b]\right).$$

Hence it is easily checked that $N(\cdot, 0)$ is differentiable at $\Xi = 0$, and its (partial) differential is given by

$$\partial_\Xi N(0,0)(\Xi) = \Big(f(x, \mathbb{Y}_{\mathrm{P}})\partial_z \mathbb{Y}_{\mathrm{P}} - v_b, \partial_z \mathbb{Y}_{\mathrm{P}}(x_0, z)\delta_0(\mathbb{Y}_{\mathrm{P}}(x_0, z)), \partial_z \mathbb{Y}_{\mathrm{P}}(x_1, z)\delta_1(\mathbb{Y}_{\mathrm{P}}(x_1, z)),$$
$$(\partial_z \mathbb{Y}_{\mathrm{P}}(x, z_t))^2\delta_t(x), (\partial_z \mathbb{Y}_{\mathrm{P}}(x, z_b))^2\delta_b(x)\Big).$$

As a consequence, the first component of $N(\Xi, \widetilde{Y}) - N(\Xi', \widetilde{Y}') - \partial_\Xi N(0,0)(\Xi - \Xi')$ is

$$\partial_z \mathbb{Y}_{\mathrm{P}} \left[\Big(f(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f(\cdot, \mathbb{Y}_{\mathrm{P}})\Big) - \Big(f'(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}') - f'(\cdot, \mathbb{Y}_{\mathrm{P}})\Big)\right] \tag{5.49}$$
$$- \partial_z \widetilde{Y}(f(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f'(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')) - \partial_z(\widetilde{Y} - \widetilde{Y}')f'(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}') \tag{5.50}$$
$$+ \partial_z \left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2(\partial_z \mathbb{Y}_{\mathrm{P}} - \partial_z \widetilde{Y})}\right) - \partial_z \left(\frac{(\partial_z \widetilde{Y}')^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2(\partial_z \mathbb{Y}_{\mathrm{P}} - \partial_z \widetilde{Y}')}\right). \tag{5.51}$$

We therefore turn towards the verification of Item ii) and Item iii) from Theorem 6.

Verification of Item ii) in the low regularity setting. For $\sigma \in (0, 1/6)$, let $\Xi, \Xi' \in \mathcal{X}^\sigma$, and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}^\sigma$ small enough. We need to estimate (5.49)-(5.51) in $H^\sigma_x H^1_z \cap H^1_x H^1_z((x-x_0)^2(x-x_1)^2)$.

In order not to burden the proof, we only estimate some of the norms above, and leave the other estimates to the reader. We focus for instance on

$$\left\|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \left[\Big(f(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f(\cdot, \mathbb{Y}_{\mathrm{P}})\Big) - \Big(f'(\cdot, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}') - f'(\cdot, \mathbb{Y}_{\mathrm{P}})\Big)\right]\right\|_{H^\sigma_x L^2_z}.$$

Using Lemma B.3 in the Appendix, we bound this term by

$$\|(\partial_z \mathbb{Y}_{\mathrm{P}})^2\|_{L^\infty_z\left(H^{\frac{1}{2}+\sigma}_x\right)} \Big(\|\partial_y(f - f')(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - \partial_y(f - f')(x, \mathbb{Y}_{\mathrm{P}})\|_{H^\sigma_x L^2_z} \tag{5.52}$$
$$+ \|\partial_y f'(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - \partial_y f'(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')\|_{H^\sigma_x L^2_z}\Big) \tag{5.53}$$
$$+ \|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac{1}{2}+\sigma}_x)} \|\partial_y(f - f')(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y})\|_{H^\sigma_x L^2_z} \tag{5.54}$$
$$+ \|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac{1}{2}+\sigma}_x)} \|\partial_y f'(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - \partial_y f'(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')\|_{H^\sigma_x L^2_z} \tag{5.55}$$
$$+ \|\partial_z \mathbb{Y}_{\mathrm{P}}(\partial_z \widetilde{Y} - \partial_z \widetilde{Y}')\|_{L^\infty_z(H^{\frac{1}{2}+\sigma}_x)} \|\partial_y f'(x, \mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')\|_{H^\sigma_x L^2_z}. \tag{5.56}$$

Using the fractional trace theorem [42, Equation (4.7), Chapter 1], $\mathcal{Z}^\sigma \hookrightarrow C^1_z(H^{\frac{1}{2}+\sigma}_x)$. Furthermore, since $L^\infty_z(H^{\frac{1}{2}+\sigma}_x)$ is an algebra,

$$\|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac{1}{2}+\sigma}_x)} \lesssim \|\widetilde{Y}\|_{\mathcal{Z}^\sigma},$$
$$\|\partial_z \mathbb{Y}_{\mathrm{P}}(\partial_z \widetilde{Y} - \partial_z \widetilde{Y}')\|_{L^\infty_z(H^{\frac{1}{2}+\sigma}_x)} \lesssim \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}^\sigma}.$$

There remains to estimate the norms involving f and f'. Using Lemma B.4 in the Appendix, we infer that

$$\begin{aligned}\|\partial_y(f-f')(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-\partial_y(f-f')(x,\mathbb{Y}_{\mathrm{P}})\|_{H^\sigma_x L^2_z} &= \left\|\widetilde{Y}\int_0^1 \partial_y^2(f-f')(x,\mathbb{Y}_{\mathrm{P}}-\tau\widetilde{Y})d\tau\right\|_{H^\sigma_x L^2_z}\\ &\lesssim \|\widetilde{Y}\|_{H^{2/3}_x H^1_z}\left(\|\partial_y^2(f-f')\|_{H^\sigma_x L^2_z}+\|\partial_y^3(f-f')\|_{L^4_x L^2_y}\right)\\ &\lesssim \|\widetilde{Y}\|_{\mathcal{Z}^\sigma}\|\Xi-\Xi'\|_{\mathcal{X}^\sigma}.\end{aligned} \tag{5.57}$$

In a similar fashion,

$$\begin{aligned}\|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z} &\lesssim \|\Xi'\|_{\mathcal{X}^\sigma},\\ \|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z} &\lesssim \|\widetilde{Y}-\widetilde{Y}'\|_{\mathcal{Z}^\sigma}\|\Xi'\|_{\mathcal{X}^\sigma}.\end{aligned}$$

The other terms are evaluated in a similar way. For instance, using again Lemma B.3 and the embedding $Z^\sigma \hookrightarrow C^1_z(H^{\frac12+\sigma}_x)$,

$$\begin{aligned}\left\|\partial_z^3(\widetilde{Y}-\widetilde{Y}')\frac{\partial_z\widetilde{Y}}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2\partial_z(\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})}\right\|_{H^\sigma_x L^2_z} &\lesssim \|\partial_z^3(\widetilde{Y}-\widetilde{Y}')\|_{H^\sigma_x L^2_z}\left\|\frac{\partial_z\widetilde{Y}}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2\partial_z(\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})}\right\|_{L^\infty_z H^{\frac12+\sigma}_x}\\ &\lesssim \|\widetilde{Y}-\widetilde{Y}'\|_{H^\sigma_x H^3_z}\|\partial_z\widetilde{Y}\|_{L^\infty_z H^{\frac12+\sigma}_x}\\ &\lesssim \|\widetilde{Y}-\widetilde{Y}'\|_{\mathcal{Z}^\sigma}\|\widetilde{Y}\|_{\mathcal{Z}^\sigma},\end{aligned}$$

and, using once again Lemma B.4,

$$\begin{aligned}&\left\|\partial_z^2\widetilde{Y}(f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\right\|_{H^\sigma_x L^2_z}\\ \lesssim&\|\partial_z^2\widetilde{Y}\|_{H^\sigma_x H^1_z}\left\|(f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\right\|_{H^{\frac12+\sigma}_x L^2_z}\\ \lesssim&\|\partial_z^2\widetilde{Y}\|_{H^\sigma_x H^1_z}\|\widetilde{Y}-\widetilde{Y}'\|_{H^{\frac12+\sigma}_x H^1_z}\left(\|\partial_y f\|_{H^{\frac12+\sigma}_x L^2_z}+\|\partial_y^2 f\|_{L^\infty}\right).\end{aligned}$$

The estimate on the $H^1_x H^1_z((x-x_0)^2(x-x_1)^2)$ norm follows from similar arguments and is left to the reader.

We then turn towards the estimation of the boundary terms.

- For $i\in\{0,1\}$ and $\delta_i,\eta_i\in H^4(\Sigma_i^P)$, we obtain that

$$\left\|\Upsilon^i_{\mathrm{P}}[\delta_i]-\Upsilon^i_{\mathrm{P}}[\eta_i]-\partial_z\mathbb{Y}_{\mathrm{P}}(x_i,z)(\delta_i-\eta_i)(\mathbb{Y}_{\mathrm{P}}(x_i,z))\right\|_{H^3(\Sigma_i)}=o\left(\|\delta_i-\eta_i\|_{H^4(\Sigma_i^P)}\right). \tag{5.58}$$

  The proof is similar to the one of Lemma 4.15 for the Burgers case, although slightly less technical because we only need a standard Sobolev estimate here, and slightly more technical because the reference flow is now $\mathfrak{u}_{\mathrm{P}}$ instead of the linear shear flow.

- For $j\in\{t,b\}$ and $\delta_j,\eta_j\in H^2(x_0,x_1)$, we obtain that

$$\left\|\Upsilon^j_{\mathrm{P}}[\delta_j]-\Upsilon^j_{\mathrm{P}}[\eta_j]-(\partial_z\mathbb{Y}_{\mathrm{P}}(x,z_j))^2(\delta_j-\eta_j)(x)\right\|_{H^2(x_0,x_1)}=o\left(\|\delta_j-\eta_j\|_{H^2(x_0,x_1)}\right). \tag{5.59}$$

  The proof is immediate because the maps Υ^j_{P} defined in (5.10) are in fact of the form $\Upsilon^j_{\mathrm{P}}[\delta^j](x)=h_j(\delta_j(x))$ where $h_j:\mathbb{R}\to\mathbb{R}$ is a smooth function with $h_j(0)=0$.

Eventually, we conclude that

$$\|N(\Xi,\widetilde{Y})-N(\Xi',\widetilde{Y}')-\partial_\Xi N(0,0)(\Xi-\Xi')\|_{\mathcal{H}^\sigma}=o\left(|\Xi-\Xi'\|_{\mathcal{X}^\sigma}+\|\widetilde{Y}-\widetilde{Y}'\|_{\mathcal{Z}^\sigma}\right). \tag{5.60}$$

Verification of Item ii) in the high regularity setting. The estimates in this case are similar to the low regularity setting and left to the reader. They are actually slightly easier since $H^1(x_0,x_1)$ is an algebra, and close to the ones performed for the Burgers system.

The only new estimate bears on the boundary term. More precisely, taking two data tuples $\Xi=(f,\delta_0,\delta_1,\delta_t,\delta_b,v_b)$ and $\Xi'=(f',\eta_0,\eta_1,\eta_t,\eta_b,v_b')$, and denoting by N_1 the first component of the nonlinearity N defined in (5.48), we need to bound in $\mathscr{H}^1_z(\Sigma_i)$ the quantity

$$\begin{aligned}z^{-1}\Big[&\partial_z(N_1(\Xi,\widetilde{Y})-N_1(\Xi',\widetilde{Y}')-\partial_\Xi N_1(0,0)(\Xi-\Xi'))|_{\Sigma_i}\\&+\left(\partial_z^2(\alpha(x_i,z)\cdot)-\beta(x_i)\partial_z\right)\partial_z\left(\Upsilon^i_{\mathrm{P}}[\delta_i]-\Upsilon^i_{\mathrm{P}}[\eta_i]-\partial_z\mathbb{Y}_{\mathrm{P}}(x_i,z)(\delta_i-\eta_i)\right)\Big].\end{aligned}$$

We recall that $f|_{\Sigma_i^P}=f'|_{\Sigma_i^P}=0$, so that the terms stemming from f and f' in N_1 vanish on the boundary. Let us focus for instance on

$$\left\|z^{-1}\partial_z\Big[\left(\partial_z^2(\alpha(x_i,z)\cdot)-\beta(x_i)\partial_z\right)\left(\Upsilon^i_{\mathrm{P}}[\delta_i]-\Upsilon^i_{\mathrm{P}}[\eta_i]-\partial_z\mathbb{Y}_{\mathrm{P}}(x_i,z)(\delta_i-\eta_i)(\mathbb{Y}_{\mathrm{P}}(x_i,z))\right)\Big]\right\|_{\mathscr{H}^1_z(\Sigma_i)}. \tag{5.61}$$

Since $\partial_z^k\delta_i(0)=0$ for $k\in\{0,\cdots,3\}$, we have $\partial_z^k\Upsilon^i_{\mathrm{P}}[\delta_i](z=0)=0$. We also note that there exists a constant C such that

$$C^{-1}\leq\frac{(\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})(x_i,z)}{z}\leq C\qquad\forall z\in(z_b,z_t)\setminus\{0\}.$$

As a consequence,

$$(5.61)\lesssim\left(\|\delta_i\|_{H^6(\Sigma_i)}+\|\eta_i\|_{H^6(\Sigma_i)}\right)\|\eta_i-\delta_i\|_{H^6(\Sigma_i)}. \tag{5.62}$$

The other term is treated in a similar fashion, and we obtain

$$\|N(\Xi,\widetilde{Y})-N(\Xi',\widetilde{Y}')-\partial_\Xi N(0,0)(\Xi-\Xi')\|_{\mathcal{H}^1}=o\left(\|\Xi-\Xi'\|_{\mathcal{X}^1}+\|\widetilde{Y}-\widetilde{Y}'\|_{\mathcal{Z}^1}\right) \tag{5.63}$$

Verification of Item iii) in the low regularity setting. We just need to check that the application $\ell^2\circ\partial_\Xi N(0,0)$ is not identically zero. This is actually trivial: take $\Xi=(0,0,0,0,0,v_b)$. Then the solution to the vorticity equation is zero, and recalling Definition 5.7,

$$\ell^2\circ\partial_\Xi N(0,0)=-\frac{1}{z_b}\int_{x_0}^{x_1}v_b(x)\,\mathrm{d}x.$$

Therefore it suffices to choose v_b such that the above integral is non-zero.

Verification of Item iii) in the high regularity setting. Using Lemma 5.12, we take $\Theta^j=(f^j,0,0,0,0)$ such that $\ell^i(\Theta^j)=\delta_{i,j}$ for $0\leq i,j\leq 2$, with $f^i\in C^\infty(\overline{\Omega})$ such that $f|_{\Sigma_0\cup\Sigma_1}=0$. We then set $\Xi^j:=(g^j,0,0,0,0,0)$, where

$$g^j(x,y):=\partial_y\mathfrak{u}_{\mathrm{P}}(x,y)f^j(x,\mathfrak{u}_{\mathrm{P}}(x,y)).$$

Then, by design, $\partial_\Xi N(0,0)(\Xi^j)=\Theta^j$, so that $\ell^j\circ\partial_\Xi N(0,0)(\Xi^j)=\delta_{i,j}$. Furthermore $\Xi^j\in\mathcal{X}^1$. The result follows.

Conclusion. We have checked the assumptions of Theorem 6 both in the low regularity case $\sigma \in (0, 1/6)$ and in the high regularity case $\sigma = 1$. Proposition 5.2 is now a straightforward consequence of our abstract framework.

5.5 Well-posedness of the Prandtl system

We conclude this section with the proof of Theorem 4, which follows from Proposition 5.2.

High regularity case. The proof of Theorem 4 in the high regularity case corresponding to $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_1$ is very similar to the proof for Burgers carried out in Section 4.3. We leave it to the reader. As in the Burgers case, one uses the equations satisfied by u and $\widetilde{Y}$ to check that they actually also enjoy $L^2_x H^5_y$ regularity and one can prove a lemma similar to Corollary B.6 to prove that the formula $u(x, Y(x, z)) = z$ allows to transfer such a regularity back and forth.

Low regularity case. We focus on the case when $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_\sigma$ with $\sigma \in (0, 1/6)$, and we consider the unique solution $\widetilde{Y} \in \mathcal{Z}^\sigma$ of (5.8). Let $\Omega_P = \{(x, y) \in (x_0, x_1) \times \mathbb{R},\ \gamma_b(x) < y < \gamma_t(x)\}$, where $\gamma_j(x) = Y(x, z_j)$. For almost every $x \in (x_0, x_1)$, $z \mapsto \mathbb{Y}_{\mathrm{P}}(x, z) + \widetilde{Y}(x, z)$ is an H^3 diffeomorphism. We note that there exists a constant $\lambda > 0$ such that $\lambda^{-1} \leq \partial_z Y \leq \lambda$ in Ω. Let us define the reverse change of variables u such that $u(x, (\mathbb{Y}_{\mathrm{P}} + \widetilde{Y})(x, z)) = z$. Classical results ensure that for a.e. x, $u(x, \cdot) \in H^3_y$. Furthermore, differentiating the formula (4.5), we obtain

$$\partial_y^3 u(x, Y(x, z)) = -\frac{\partial_z^3 Y(x, z)}{(\partial_z Y(x, z))^4} + 3\frac{(\partial_z^2 Y(x, z))^2}{(\partial_z Y(x, z))^5},$$

which ensures that $\partial_y^3 u \in L^2(\Omega_P)$. Since

$$\partial_y u(x, y) = \frac{1}{\partial_z Y(x, u(x, y))},$$

we also infer that $\partial_y u \in L^\infty$ and $\lambda^{-1} \leq \partial_y u \leq \lambda$ for some $\lambda > 0$.

Additionally, since $(x-x_0)(x-x_1)Y \in H^1_x H^3_z$, we also infer that $(x-x_0)(x-x_1)\partial_y^k u(x, Y(x, z)) \in H^1_x L^2_y(\Omega_P)$ for $0 \leq k \leq 3$. From there, we deduce that $(x - x_0)(x - x_1)\partial_x \partial_y^k u \in L^2(\Omega_P)$ for $0 \leq k \leq 3$. Furthermore, since $z\partial_x \partial_z Y \in L^2$, we also deduce that $u\partial_x \partial_y u \in L^2$. Tracing back the computations at the beginning of Section 5.1, and noticing that $u \in H^1_x H^3_y(\omega)$ for all $\omega \Subset \Omega_P$ as well as in the vicinity of $\overline{\Gamma_b}$, we infer that u is a weak solution to the Prandtl system (5.1). This proves the existence of a solution to (5.1)-(5.3).

Let us now prove the uniqueness of this solution within the regularity class

$$u \in L^2_x H^3_y(\Omega_P), \quad \partial_y u \in L^\infty, \quad (x - x_0)(x - x_1)u \in H^1_x H^3_y(\Omega_P), \quad u\partial_x \partial_y u \in L^2(\Omega_P),$$

and assuming that u is close to $\mathfrak{u}_{\mathrm{P}}$ in the associated norm. Note that this implies in particular that $\partial_y u$ is bounded pointwise from above and below. The associated function Y is such that $Y \in L^2_x H^3_z$, $\partial_z Y \in L^\infty$, $(x-x_0)(x-x_1)Y \in H^1_x H^3_y$, and $z\partial_x \partial_z Y \in L^2$. In particular, $\partial_z Y \in Z^0$. This regularity is sufficient to justify the computations of Section 5.1, and thus $\widetilde{Y} = Y - \mathbb{Y}_{\mathrm{P}}$ is a solution to (5.8) in the sense of distributions. It follows that $\partial_z \widetilde{Y}$ is a solution to (5.19), and $\partial_z Y$ is bounded pointwise from above and below by positive constants. From there, we deduce that $\partial_z Y \in \mathcal{Z}^\sigma$. Applying the first statement of Proposition 5.2, we deduce that $(f, \delta_0, \delta_1, \delta_b, \delta_t, v_b) \in \mathcal{M}_\sigma$.

Now, let u_1, u_2 be two solutions of (5.1) within the above regularity class, corresponding to solutions $\widetilde{Y}_1, \widetilde{Y}_2$ of (5.8). Let

$$g_i := f(x, Y_i)\partial_z Y_i - v_b + \partial_z \left(\frac{(\partial_z \widetilde{Y}_i)^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y_i} \right).$$

Then $W := \partial_z(\widetilde{Y}_1 - \widetilde{Y}_2) \in Z^0$ is a solution to (5.22) with homogeneous boundary data and with a source term $h = \partial_z g_1 - \partial_z g_2$. Therefore, multiplying the equation by W and integrating by parts, we obtain

$$\int \alpha |\partial_z W|^2 \leq C(\|g_1 - g_2\|_{L^2}^2 + \|W\|_{L^2}^2),$$

where the constant C depends only on the underlying flow $\mathfrak{u}_{\mathrm{P}}$. As in the proof of Lemma 5.3, for $|z_b|, z_t \leq z_0$, we infer that

$$\|W\|_{L^2_x H^1_z} \lesssim \|g_1 - g_2\|_{L^2}.$$

From there, using equation (5.22), we obtain

$$\|W\|_{\mathcal{B}} \lesssim \|g_1 - g_2\|_{L^2}.$$

Using the formula for g_i above, we deduce that

$$\begin{aligned} \|g_1 - g_2\|_{L^2} \lesssim & \|\partial_y f\|_\infty \|Y_1 - Y_2\|_{L^2} \|\partial_z Y_1\|_\infty + \|f\|_\infty \|\partial_z (Y_1 - Y_2)\|_{L^2} \\ & + \|\partial_z \widetilde{Y}_1\|_\infty \|\partial_z^2 (Y_1 - Y_2)\|_{L^2} + \|\partial_z (Y_1 - Y_2)\|_{L^\infty_z (L^3_x)} \|\partial_z^2 \widetilde{Y}_2\|_{L^2_z (L^6_x)} \end{aligned}$$

Setting

$$\eta := \|\partial_z \widetilde{Y}_1\|_\infty + \|\partial_z \widetilde{Y}_2\|_{Z^0} + \|f\|_{L^\infty_x W^{1,\infty}_y},$$

and using the embeddings $Z^0 \hookrightarrow L^2_z H^{1/3}_x \hookrightarrow L^2_z(L^6_x)$, $\mathcal{B} \hookrightarrow C^0_z([z_b, z_t]; H^{1/6}_x) \hookrightarrow L^\infty_z(L^3_x)$ (see Lemma 1.14), we infer

$$\|g_1 - g_2\|_{L^2} \lesssim \eta \|W\|_{\mathcal{B}}.$$

Hence we obtain $\|W\|_{\mathcal{B}} \lesssim \eta \|W\|_{\mathcal{B}}$, and provided η is small enough, $W = 0$.

Remark 5.16. *Note that in the case $\sigma \in (0, 1/6)$, we are not able to transfer completely the fractional horizontal regularity from Y to u. Indeed, one can easily check from the formulas in (4.5) that $\partial_y^k u(x, Y(x,z)) \in H_x^{\frac{3-k}{3}+\sigma} L^2_z \cap H^\sigma_x H^{3-k}_z$ for $k \in \{1, \cdots, 3\}$. Then, one may try to get some regularity on u by computing*

$$\|\partial_y u\|^2_{H_x^{\frac{2}{3}+\sigma} L^2_y} = \|\partial_y u\|^2_{L^2} + \int_{x_0}^{x_1} \int_{x_0}^{x_1} \int_{\mathbb{R}} \mathbf{1}_{(x,y)\in\Omega_P} \mathbf{1}_{(x',y)\in\Omega_P} \frac{|u(x,y) - u(x',y)|^2}{|x - x'|^{\frac{7}{3}+2\sigma}} \, \mathrm{d}x \, \mathrm{d}x' \, \mathrm{d}y.$$

It is quite natural to change variables in the second integral in the right-hand side by setting $y = Y(x,z)$, the associated jacobian being bounded from above and below, and to split the resulting integral into

$$\begin{aligned} & \int_{x_0}^{x_1} \int_{x_0}^{x_1} \int_{z_b}^{z_t} \frac{|\partial_y u(x, Y(x,z)) - \partial_y u(x', Y(x',z))|^2}{|x - x'|^{\frac{7}{3}+2\sigma}} \, \mathrm{d}x \, \mathrm{d}x' \, \mathrm{d}z \\ & + \int_{x_0}^{x_1} \int_{x_0}^{x_1} \int_{z_b}^{z_t} \frac{|\partial_y u(x', Y(x',z)) - \partial_y u(x', Y(x,z))|^2}{|x - x'|^{\frac{7}{3}+2\sigma}} \, \mathrm{d}x \, \mathrm{d}x' \, \mathrm{d}z. \end{aligned}$$

The first integral above is bounded by $\|\partial_y u(x, Y(x,z))\|^2_{H^{2/3+\sigma}_x L^2_z}$. *As for the second integral, if* $\partial_y u$ *were Lipschitz continuous with respect to* y *(or even Hölder continuous with some suitable exponent), we would bound this integral by* $\|Y\|^2_{H^{2/3+\sigma}_x L^2_z}$. *But unfortunately, this Lipschitz regularity does not hold in general. However, thanks to the regularity result far from the lateral boundaries from Lemma 5.4, we have sufficient regularity on* u *to ensure uniqueness.*

5.6 Potential strategy in a whole infinite strip

In this paragraph, we sketch a potential strategy to solve the Prandtl equation (5.1) in the whole infinite strip $(x_0, x_1) \times (0, +\infty)$, based on the previous analysis. To that end, we first propose a scheme to solve a system with a modified source term (and without any orthogonality condition). Once the solvability of this modified system is understood, the solvability of the original system follows for data within a finite codimensional manifold.

As above, we start from a smooth solution $(\mathtt{u}_\mathrm{P}, \mathtt{v}_\mathrm{P})$ of (5.1) such that $\mathtt{u}_\mathrm{P}(x, \gamma_\mathrm{P}(x)) = 0$ for some smooth function γ_P, and $\mathtt{u}_\mathrm{P}(x,y) < 0$ (resp. $\mathtt{u}_\mathrm{P}(x,y) > 0$) for $y \in (0, \gamma_\mathrm{P}(x))$ (resp. for $y > \gamma_\mathrm{P}(x)$). We also have the boundary conditions $\mathtt{u}_\mathrm{P}|_{y=0} = \mathtt{v}_\mathrm{P}|_{y=0} = 0$, and $\mathtt{u}_\mathrm{P}(x,y) \to u_\infty(x)$ as $y \to \infty$, where $u_\infty u'_\infty = -dp/dx$. As before, we fix two small numbers $z_b < 0 < z_t$, such that there exist smooth lines $\{y = \overline{\gamma_j}(x)\}$ with $\mathtt{u}_\mathrm{P}(x, \overline{\gamma_j}(x)) = z_j$. We consider perturbations $\Theta := (\delta_0, \delta_1, f) \in H^k(0,+\infty) \times H^k(0,+\infty) \times H^k((x_0,x_1)\times(0,+\infty))$ for some sufficiently large k, and for simplicity, we also assume that δ_i vanishes at $\gamma_\mathrm{P}(x_i)$. We then define an application $\mathcal{A} : (\Theta; \widetilde{\gamma_b}, \delta_t) \mapsto (\gamma_b, \delta'_t)$ in the following way:

1. We solve the Prandtl system in the domain $\{(x,y) \in (x_0,x_1)\times(0,+\infty),\ y < \widetilde{\gamma_b}(x)\}$ in the vicinity of the flow $(\mathtt{u}_\mathrm{P}, \mathtt{v}_\mathrm{P})$, with source term $-dp/dx + f$ and boundary data

$$\begin{aligned} u|_{x=x_1} &= \mathtt{u}_\mathrm{P}|_{x=x_1} + \delta_1, \\ u|_{y=0} = v|_{y=0} &= 0, \\ u|_{y=\widetilde{\gamma_b}(x)} &= z_b. \end{aligned}$$

   In (interior of) this domain, $\mathtt{u}_\mathrm{P} < 0$, and therefore the system is backward parabolic. Hence we expect that it is solvable (see [48]). A possible way to solve it could be to introduce the "von Mise type good unknown" from [35].

   Assuming that the above system is solvable, we set $v_b := v|_{y=\widetilde{\gamma_b}} - \mathtt{v}_\mathrm{P}|_{y=\gamma_\mathrm{P}(x)}$, and $\delta_b := \partial_y u|_{y=\widetilde{\gamma_b}} - \partial_y \mathtt{u}_\mathrm{P}|_{y=\widetilde{\gamma_b}}$. Note that there are typically compatibility conditions which are necessary to ensure the existence of smooth solutions of this system. We leave this issue aside in the present discussion. The compatibility conditions are automatically ensured if f is supported in $(x_0 + \delta_x, x_1 - \delta_x)$ for $|\delta_x| \ll 1$, and if δ_0 (resp. δ_1) is compactly supported in $(0, \overline{\gamma_t}(x_0))$ (resp. $(\overline{\gamma_b}(x_0), 0)$).

2. We then consider the Prandtl system in the recirculating zone. More precisely, using the analysis of the previous subsections, we construct a solution to

$$\begin{aligned} uu_x + vu_y - \partial_{yy} u &= -\frac{dp}{dx} + f - \left(\nu^0 f^0 + \nu^1 f^1 + \nu^2 f^2\right)(x, u(x,y))\partial_y u(x,y) \\ u_x + v_y &= 0, \end{aligned}$$

   together with the boundary conditions (1.15), (1.16), (1.17), in which the bottom data v_b, δ_b are provided by the first step. Note that the free boundary $\{y = \gamma_b(x)\}$ is different from the

boundary $\{y = \widetilde{\gamma_b}(x)\}$ *a priori*. The coefficients (ν^0, ν^1, ν^2) are Lipschitz functions of the data (f, δ_0, δ_1) and ensure that the associated solution u belongs to $H^{5/3}_x H^1_y \cap H^1_x H^3_y$. Note that the structure of the right-hand side is designed so that the equation in the variables (x, z) is

$$z\partial_x Y - \int_{z_b}^{z} \partial_x Y - \frac{1}{(\partial_z Y)^2}\partial_z^2 Y = \left(\frac{dp}{dx} - f(x, Y)\right)\partial_z Y + \mathtt{v}_{\mathrm{P}}|_{y=\overline{\gamma_b}} + v_b + \nu^0 f^0 + \nu^1 f^1 + \nu^2 f^2.$$

We denote by $\mathcal{V}(f, \delta_0, \delta_1; \widetilde{\gamma_b}, \delta_t)$ the quantity $v|_{y=\gamma_t(x)}$, where $\gamma_t(x) = Y(x, z_t)$. The boundary $\{y = \gamma_t(x)\}$ will be the lower boundary of the upper domain considered in the next step, but is not a variable of the implicit function argument.

3. Eventually, we solve the Prandtl system in $\{(x, y) \in (x_0, x_1) \times (0, +\infty),\ y > \gamma_t(x)\}$ in the vicinity of the flow $(\mathtt{u}_{\mathrm{P}}, \mathtt{v}_{\mathrm{P}})$, with source term $-dp/dx + f$ and boundary data

$$\begin{aligned} u|_{x=x_0} &= \mathtt{u}_{\mathrm{P}}|_{x=x_0} + \delta_0, \\ u|_{y=\gamma_t(x)} &= z_t, \\ v|_{y=\gamma_t(x)} &= \mathcal{V}(f, \delta_0, \delta_1; \widetilde{\gamma_b}, w_t), \\ \lim_{y\to\infty} u(x, y) &= u_\infty(x). \end{aligned}$$

This system is now forward parabolic. It can be solved with the tools of [48]. Note that $\inf \mathtt{u}_{\mathrm{P}} > 0$ in the upper domain, so that the system is in fact non degenerate after a suitable change of variables. We then define $w_t' = \partial_y u|_{y=\gamma_t(x)}$.

Eventually, we set $\mathcal{A}(\Theta; \widetilde{\gamma_b}, w_t) = (\gamma_b, w_t')$, where γ_b is the lower free boundary in the recirculating zone, and w_t' is the vertical derivative of the velocity on the upper free boundary $\{y = \gamma_t(x)\}$. The first question which needs to be solved is the following:

For every $\Theta \in H^k(0, +\infty) \times H^k(0, +\infty) \times H^k((x_0, x_1) \times (0, +\infty))$ *such that* $\|\Theta\| \le \delta$,
Find $(\widetilde{\gamma_b}, w_t)$ *such that* $\mathcal{A}(\Xi; \widetilde{\gamma_b}, w_t) = (\widetilde{\gamma_b}, w_t)$.

For $\Theta = 0$, by definition of the application $\mathcal{A}$, one has $\mathcal{A}(0; \overline{\gamma_b}, \partial_y \mathtt{u}_{\mathrm{P}}|_{y=\overline{\gamma_t}}) = (\overline{\gamma_b}, \partial_y \mathtt{u}_{\mathrm{P}}|_{y=\overline{\gamma_t}})$. Hence a possible strategy could be to apply an implicit function theorem, in the spirit of [15] or Lemma 3.15. This requires to prove the invertibility of the function $d_{(\widetilde{\gamma_b}, w_t)}\mathcal{A}(0; \overline{\gamma_b}, \partial_y \mathtt{u}_{\mathrm{P}}|_{y=\overline{\gamma_t}}) - \mathrm{Id}$. In turn, this requires to prove the well-posedness of a linearized type Prandtl system (or of three coupled linearized Prandtl systems) in the infinite strip $(x_0, x_1) \times (0, +\infty)$. Such a result may typically involve restrictions on the size of the domain, as the following toy example demonstrates. Let $a \in L^\infty((x_0, x_1) \times \mathbb{R})$. Consider the forward-backward system

$$\begin{cases} z\partial_x u - \partial_{zz} u - au = 0 & \text{in } (x_0, x_1) \times \mathbb{R}, \\ u(x_0, z) = 0 & \text{for } z > 0, \\ u(x_1, z) = 0 & \text{for } z < 0. \end{cases}$$

Let us assume that there exists a solution with high enough decay for $|z| \gg 1$; our purpose is to prove that such a solution is identically zero. To that end, we multiply the above system by $u \exp(-(x - x_0)z/(x_1 - x_0))$, and perform integrations by parts. We obtain

$$\begin{aligned} &\frac{1}{2(x_1 - x_0)} \int_{x_0}^{x_1} \int_{\mathbb{R}} z^2 u^2 \exp\left(-\frac{x - x_0}{x_1 - x_0} z\right) \mathrm{d}x\, \mathrm{d}z + \int_{x_0}^{x_1} \int_{\mathbb{R}} (\partial_z u)^2 \exp\left(-\frac{x - x_0}{x_1 - x_0} z\right) \mathrm{d}x\, \mathrm{d}z \\ &= -\int_{x_0}^{x_1} \int_{\mathbb{R}} a u^2 \exp\left(-\frac{x - x_0}{x_1 - x_0} z\right) \mathrm{d}x\, \mathrm{d}z + \frac{1}{2} \int_{x_0}^{x_1} \int_{\mathbb{R}} \left(\frac{x - x_0}{x_1 - x_0}\right)^2 u^2 \exp\left(-\frac{x - x_0}{x_1 - x_0} z\right) \mathrm{d}x\, \mathrm{d}z. \end{aligned}$$

For $|z| \geq 2(\|a\|_\infty + 1)^{1/2}(x_1 - x_0)^{1/2}$, the two terms in the right-hand side can be absorbed in the left-hand side. On the other hand, for $|z| \leq 2(\|a\|_\infty + 1)^{1/2}(x_1 - x_0)^{1/2}$ and $|x_1 - x_0| \leq 1$, the weight $\exp\left(-\frac{x-x_0}{x_1-x_0}z\right)$ is bounded from above and below. We then use the inequality

$$\|\phi\|_{L^2_z} \lesssim \|z\phi\|_{L^2_z} + \|\partial_z\phi\|_{L^2_z}$$

for any $\phi \in H^1(\mathbb{R})$ such that $z\phi \in L^2(\mathbb{R})$. The proof of the inequality follows from arguments similar to the ones of Lemma B.7 and is left to the reader. We infer that for $x_1 - x_0$ small enough, the only decaying solution to the above system is $\phi \equiv 0$. For $x_1 - x_0$ large, the situation is not so clear. These considerations could be seen as a toy example of why Iyer and Masmoudi in [35] need to exclude a "resonant set" of lengths $x_1 - x_0$ for which non trivial solutions of a system similar to the one above may exist.

6 Interpolation estimate for the linear shear flow problem

This section is devoted to the proof of Proposition 2.33, which is used in particular in the construction of weak solutions for the Prandtl system (see Proposition 5.5). The idea is to interpolate between the Z^0 estimate from Proposition 2.4, and the Z^1 estimate from Proposition 2.9. However, because of the orthogonality conditions, justifying that the interpolate space for the source terms is the expected one turns out to be quite complicated.

We introduce the following spaces for the source terms

$$\mathcal{Y}_0 := \{f \in L^2(\Omega)\}, \tag{6.1}$$

$$\mathcal{Y}_1 := \left\{f \in H^1_x L^2_y;\ \ f_{|\Sigma_0 \cup \Sigma_1} = 0\right\} \tag{6.2}$$

endowed with their usual norms and

$$\mathcal{Y}_1^{\overline{\ell}} := \left\{f \in \mathcal{Y}_1;\ \ \overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0\right\}, \tag{6.3}$$

endowed with the norm of $\mathcal{Y}_1$, where $\overline{\ell^0}$ and $\overline{\ell^1}$ are defined in Definition 2.10. Since $\overline{\ell^0}$ and $\overline{\ell^1}$ are continuous for the $H^1_x L^2_y$ norm, $\mathcal{Y}_1^{\overline{\ell}}$ is a closed subspace of $\mathcal{Y}_1$.

We wish to interpolate between $\mathcal{Y}_0$ and $\mathcal{Y}_1^{\overline{\ell}}$. Using classical interpolation theory, one can determine $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$ quite easily (see Lemma 6.4 below). Nevertheless, there is a difficulty in the determination of the space $[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma$. This corresponds to the well-known problem of "subspace interpolation", for which we give a short survey in Section 6.1.

The proof of Proposition 2.33 relies on a careful analysis of the dual profiles $\overline{\Phi^j}$, and in particular on a decomposition of the latter into an explicit singular part and a regular part. This decomposition allows us to have quantitative upper and lower bounds on the functions $\tau \mapsto I(\tau, \overline{\ell^j})$, which play a paramount role in interpolation theory (see [43] and Section 6.1.2 below).

The organization of this section is as follows. We start by introducing the theory of subspace interpolation, and associated notations in Section 6.1. We then turn towards the proof of Proposition 2.33 in Section 6.2, illustrating how the general theory can be applied for our problem, thanks to the knowledge of the singular profiles of Section 2.4.

6.1 A primer on subspace interpolation

Using interpolation theory in a context where constraints are enforced on the data comes with a specific difficulty, known as "subspace interpolation". In this subsection, we give a short introduction and set up notations and a lemma that will be used in the next subsections.

6.1.1 An introduction to subspace interpolation

Let us start by a short introduction to the topic of subspace interpolation and the associated difficulty. This difficulty is *not* linked with the difference between complex and real interpolation methods. Indeed, it occurs even in the case of "quadratic" interpolation between separable Hilbert spaces, for which all methods construct the same interpolation spaces (see [13, Remark 3.6] and [14, Section 3.3, item (4)] based on the initial geometric argument of [46]).

Setting of the problem Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Banach spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$. Let $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for $\sigma \in (0,1)$, say for the complex method to fix

ideas. Let ℓ be a continuous linear form on $\mathcal{Y}_1$, which is however unbounded on $\mathcal{Y}_0$[8], and define its kernel $\mathcal{Y}_1^\ell := \{f \in \mathcal{Y}_1;\ \ell(f) = 0\}$, which is a closed subspace of $\mathcal{Y}_1$. The question of "subspace interpolation" consists in determining the relation between $\mathcal{Y}_\sigma$ and $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$. This question of course admits a straightforward generalization to the case of a finite number of orthogonality conditions.

Generally, one checks that the closure of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ in $\mathcal{Y}_\sigma$ is either a subspace of codimension 1, when ℓ is continuous on $\mathcal{Y}_\sigma$, or the whole of $\mathcal{Y}_\sigma$, when ℓ is unbounded on $\mathcal{Y}_\sigma$. In the former case, there is no guarantee that $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ itself is closed in $\mathcal{Y}_\sigma$ (or, equivalently, that the associated norms are equivalent on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$). The first systematic occurrence of this question seems to date back to [45, Problem 18.5, Chapter 1], which claims that a major difficulty to use interpolation theory is that "*l'interpolé de sous-espaces fermés n'est pas nécessairement un sous-espace fermé dans l'interpolé*" (the interpolation space between closed subspaces is not necessarily a closed subspace in the interpolation space), and asks for sufficient conditions for $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ to be closed in $\mathcal{Y}_\sigma$.

Some examples The best known and most simple example of such a phenomenon, introduced in [45, Theorem 11.7, Chapter 1] concerns the construction of the space $H_{00}^{1/2}(0,1) = [L^2(0,1), H_0^1(0,1)]_{1/2}$. It is known that $H_{00}^{1/2}(0,1)$ is not closed in $H^{1/2}(0,1)$ and that the associated norm involves a non-equivalent "additional term".

In [61], using real interpolation between L^1 and L^∞, Wallstén constructed examples illustrating that this pathological behavior is not limited to exceptional values of the interpolation parameter, since there exist constraints for which it occurs for every $\sigma \in (0,1)$.

Short survey of known results Precising earlier results of Löfström [43, 44], Ivanov and Kalton proved in [33] that, in the general case, there exist two thresholds $0 \le \sigma_0 \le \sigma_1 \le 1$ such that:

- when $0 < \sigma < \sigma_0$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \mathcal{Y}_\sigma$, with equivalent norms,
- when $\sigma_0 \le \sigma \le \sigma_1$, the norm on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ is not equivalent to the one on $\mathcal{Y}_\sigma$,
- when $\sigma_1 < \sigma < 1$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ is a closed subspace of codimension 1 in $\mathcal{Y}_\sigma$.

In the first case, ℓ is unbounded on $\mathcal{Y}_\sigma$ (the constraint does not make sense). In the second and third cases, ℓ admits a continuous extension to $\mathcal{Y}_\sigma$ and the closure of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ in $\mathcal{Y}_\sigma$ is of codimension 1.

This classification has generalizations to the case of multiple constraints (see [2]), potentially involving multiple pathological intervals, associated with each constraint.

In the difficult regime $\sigma_0 \le \sigma \le \sigma_1$, more precise results [5, 6] allow the computation of the "additional norm" stemming from the presence of the constraints.

The recent work [62] considers a kind of dual problem, by computing interpolation spaces between $\mathcal{Y}_0$ and $\mathcal{Y}_1 \oplus \mathbb{R}\omega$, where ω is a singular function of $\mathcal{Y}_0 \setminus \mathcal{Y}_1$, whose singularity is expressed in polar coordinates. In this work, $\sigma_0 = \sigma_1$. This is also our case below, and our dual profiles also involve singular parts which are expressed in radial-like coordinates, as constructed in Section 2.4.

6.1.2 A variant of a criterium due to Löfström

To prove Proposition 2.33, we will rely on an abstract interpolation result proved by Löfström in [43]. Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$.

[8] When ℓ is continuous for the topology of $\mathcal{Y}_0$, there is no difficulty. Indeed, one checks that, for every $\sigma \in (0,1)$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \{f \in \mathcal{Y}_\sigma;\ \ell(f) = 0\}$, endowed with the topology of $\mathcal{Y}_\sigma$, for which ℓ is continuous (see e.g. the related result [45, Theorem 13.3, Chapter 1]).

For $f \in \mathcal{Y}_1$ and $\tau \in (0,1)$, let

$$\|f\|_\tau^2 := \|f\|_{\mathcal{Y}_0}^2 + \tau^2 \|f\|_{\mathcal{Y}_1}^2. \tag{6.4}$$

This notation is actually inspired by [33], as [43] uses instead the quantity $\max(\|f\|_{\mathcal{Y}_0}, \tau\|f\|_{\mathcal{Y}_1})$. Since $\|f\|_\tau/\sqrt{2} \leq \max(\|f\|_{\mathcal{Y}_0}, \tau\|f\|_{\mathcal{Y}_1}) \leq \|f\|_\tau$, both can be used equivalently.

Given a linear form ℓ on $\mathcal{Y}_1$, one defines, for $\tau \in (0,1)$,

$$I(\tau, \ell) := \sup_{f \in \mathcal{Y}_1 \setminus \{0\}} \frac{\ell(f)}{\|f\|_\tau}. \tag{6.5}$$

As $\tau \to 0$, upper bounds on $I(\tau, \ell)$ are linked with the boundedness of ℓ on intermediate spaces between $\mathcal{Y}_0$ and $\mathcal{Y}_1$, while lower bounds on $I(\tau, \ell)$ are linked with the non-degeneracy of ℓ on these spaces. In particular, one has the following result, which is a reformulation of [43, Theorem 2] in the particular case of two constraints having the same "order".

Lemma 6.1. *Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$. Let ℓ^0, ℓ^1 be two linear forms on $\mathcal{Y}_1$. Assume that there exists $C_\pm > 0$ and $\bar\sigma \in (0,1)$ such that, for every $(c_0, c_1) \in \mathbb{S}^1$ and every $\tau \in (0,1)$,*

$$C_- \tau^{-\bar\sigma} \leq I(\tau, c_0\ell^0 + c_1\ell^1) \leq C_+ \tau^{-\bar\sigma}. \tag{6.6}$$

As in Section 6.1.1, let $\mathcal{Y}_1^\ell := \{f \in \mathcal{Y}_1;\ \ell^0(f) = \ell^1(f) = 0\}$ and, for $\sigma \in (0,1)$, $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for the complex interpolation method. Then,

- *for every $\sigma \in (0, \bar\sigma)$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \mathcal{Y}_\sigma$, with equivalent norms,*
- *for every $\sigma \in (\bar\sigma, 1)$, the linear forms ℓ^0 and ℓ^1 have continuous extensions to $\mathcal{Y}_\sigma$ and $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \{f \in \mathcal{Y}_\sigma;\ \ell^0(f) = \ell^1(f) = 0\}$, endowed with the norm of $\mathcal{Y}_\sigma$.*

Remark 6.2. *Lemma 6.1 does not say anything on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ for the critical value $\sigma = \bar\sigma$. In fact, with the notations of [33] mentioned above, one has $\sigma_0 = \sigma_1 = \bar\sigma$, so the norm of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_{\bar\sigma}$ is not equivalent to the norm of $\mathcal{Y}_{\bar\sigma}$.*

Remark 6.3. *In assumption (6.6), it is important to consider arbitrary linear combinations of the two linear forms ℓ^0 and ℓ^1. It would not be sufficient to assume (6.6) with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$. Indeed, the lower bound of this condition ensures that the two linear forms remain sufficiently independent on the intermediate spaces. We state here a formulation giving a symmetrical role to ℓ^0 and ℓ^1, whereas [43] uses a hierarchical formulation. We prove below that our formulation indeed implies Löfström's one.*

Proof of Lemma 6.1. This is an application of [43, Theorem 2]. By (6.6) with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, both ℓ^0 and ℓ^1 have "order" $\bar\sigma$ in Löfström's vocabulary. Therefore, there only remains to check that they form a "strongly independent basis", i.e. that there exists $C > 0$ such that, for every $\tau \in (0,1)$,

$$I(\tau, \ell^1) \leq C I_0(\tau, \ell^1), \tag{6.7}$$

where

$$I_0(\tau, \ell^1) := \sup\left\{ \frac{\ell^1(f)}{\|f\|_\tau};\ \ f \in \mathcal{Y}_1 \setminus \{0\} \text{ and } \ell^0(f) = 0 \right\}. \tag{6.8}$$

Let $\tau \in (0,1)$. Denote by $\langle \cdot, \cdot \rangle_\tau$ the scalar product associated with the norm $\|\cdot\|_\tau$ on $\mathcal{Y}_1$. By the Riesz representation theorem, there exists $g_\tau^0, g_\tau^1 \in \mathcal{Y}_1$ such that $\ell^j = \langle g_\tau^j, \cdot \rangle_\tau$. In particular

$I(\tau,\ell^j)=\|g_\tau^j\|_\tau$. Moreover, by (6.8), $I_0(\tau,\ell^1)$ is the supremum of ℓ^1 on the intersection of $\ker\ell^0$ with the unit ball in $\mathcal{Y}_1$ for the norm $\|\cdot\|_\tau$. Thus, a natural candidate to bound $I_0(\tau,\ell^1)$ from below is the orthogonal projection of $g_\tau^1/\|g_\tau^1\|_\tau$ on $\ker\ell^0$, namely,

$$f_\tau^1:=\frac{g_\tau^1}{\|g_\tau^1\|_\tau}-R_\tau\frac{g_\tau^0}{\|g_\tau^0\|_\tau}\quad\text{where}\quad R_\tau:=\left\langle\frac{g_\tau^1}{\|g_\tau^1\|_\tau},\frac{g_\tau^0}{\|g_\tau^0\|_\tau}\right\rangle_\tau. \tag{6.9}$$

In particular $\|f_\tau^1\|_\tau=(1-R_\tau^2)^{\frac12}$ and $\ell^0(f_\tau^1)=\langle g_\tau^0,f_\tau^1\rangle=0$. Thus

$$I_0(\tau,\ell^1)\geq\frac{\langle f_\tau^1,g_\tau^1\rangle_\tau}{\|f_\tau^1\|_\tau}=(1-R_\tau^2)^{\frac12}\|g_\tau^1\|_\tau=(1-R_\tau^2)^{\frac12}I(\tau,\ell^1). \tag{6.10}$$

Thus, to prove (6.7), it is sufficient to prove that the ratio R_τ^2 is bounded away from 1. By (6.6), for every $(c_0,c_1)\in\mathbb{S}^1$,

$$C_-\tau^{-\bar\sigma}\leq\|c_0g_\tau^0+c_1g_\tau^1\|_\tau\leq C_+\tau^{-\bar\sigma}. \tag{6.11}$$

In particular,

$$C_-\tau^{-\bar\sigma}\leq\|g_\tau^j\|_\tau\leq C_+\tau^{-\bar\sigma}. \tag{6.12}$$

By homogeneity, from (6.11), for every $(c_0,c_1)\in\mathbb{R}^2$,

$$C_-^2\tau^{-2\bar\sigma}(c_0^2+c_1^2)\leq c_0^2\|g_\tau^0\|_\tau^2+c_1^2\|g_\tau^1\|_\tau^2+2c_0c_1\langle g_\tau^0,g_\tau^1\rangle_\tau\leq C_+^2\tau^{-2\bar\sigma}(c_0^2+c_1^2). \tag{6.13}$$

Substituting $c_j\leftarrow c_j/\|g_\tau^j\|_\tau$ and using (6.12) leads to the fact that, for every $(c_0,c_1)\in\mathbb{R}^2$,

$$\rho^2(c_0^2+c_1^2)\leq c_0^2+c_1^2+2R_\tau c_0c_1\leq\rho^{-2}(c_0^2+c_1^2), \tag{6.14}$$

where $\rho:=C_-/C_+$. In particular, using $(c_0,c_1)=(1,1)$ and $(1,-1)$ yields $\rho^2\leq1+R_\tau$ and $\rho^2\leq1-R_\tau$. Hence, (6.10) proves that

$$I_0(\tau,\ell^1)\geq\rho^2I(\tau,\ell^1), \tag{6.15}$$

which implies (6.7) with $C=\rho^{-2}$. So ℓ^0 and ℓ^1 form a "strongly independent basis" and Lemma 6.1 follows from [43, Theorem 2]. □

6.2 Interpolated theory in the case of the linear shear flow

In this subsection, we consider the problem (2.1) at the linear shear flow, with vanishing boundary data. We proved in Section 2.2 (see Proposition 2.4) that, when $f\in L_x^2L_z^2$, the solutions to this problem have Z^0 regularity, and in Section 2.3 (see Proposition 2.9) that they have Z^1 regularity when $f\in H_x^1L_z^2$ and the two orthogonality conditions (2.22) are satisfied. Here, we establish an interpolated theory for the problem (2.1) with source terms $f\in H_x^\sigma L_z^2$, $\sigma\in(0,1)$, see Proposition 2.33. This interpolated theory involves the difficulty exposed in Section 6.1. We define $\mathcal{Y}_0$, $\mathcal{Y}_1$ and $\mathcal{Y}_1^{\bar\ell}$ by (6.1), (6.2) and (6.3) respectively, endowed with their usual norms. For $\sigma\in(0,1)$, let $\mathcal{Y}_\sigma:=[\mathcal{Y}_0,\mathcal{Y}_1]_\sigma$. The identification of the space $\mathcal{Y}_\sigma$ is classical and provided by the following lemma:

Lemma 6.4. *Let $\sigma\in(0,1)$. Let $\mathcal{Y}_\sigma:=[\mathcal{Y}_0,\mathcal{Y}_1]_\sigma$, for the complex interpolation method.*

- *When $\sigma\in(0,1/2)$, $\mathcal{Y}_\sigma=H_x^\sigma L_y^2$.*

- *When $\sigma = 1/2$, recalling that $\Omega_\pm = \Omega \cap \{\pm y > 0\}$*

$$\|f\|^2_{\mathcal{Y}_{1/2}} \approx \|f\|^2_{H^{1/2}_x L^2_y} + \int_{\Omega_+} \frac{f^2(x,y)}{|x-x_0|}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_-} \frac{f^2(x,y)}{|x-x_1|}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.16}$$

- *When $\sigma \in (1/2,1)$, $\mathcal{Y}_\sigma = \{f \in H^\sigma_x L^2_y;\ f_{|\Sigma_0 \cup \Sigma_1 = 0}\}$, with the usual norm.*

Proof. This follows from classical interpolation theory for intersections (see [45, Theorem 13.1 and Equation (13.4), Chapter 1]), and from (one-sided versions of) the equality $H^{1/2}_{00}(x_0,x_1) = [H^1_0(x_0,x_1), L^2(x_0,x_1)]_{\frac{1}{2}}$ (see [45, Theorem 11.7, Chapter 1]) . □

In order to extend the theory of Section 2 to fractional tangential regularity, we start by identifying the spaces $[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma$. More precisely, we prove the following characterization.

Lemma 6.5. *Let $\mathcal{Y}_0$, $\mathcal{Y}_1$ and $\mathcal{Y}_1^{\overline{\ell}}$ be given by* (6.1), (6.2) *and* (6.3) *respectively. Then,*

- *For every $\sigma \in (0,1/6)$, $[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma = \mathcal{Y}_\sigma$ with equivalent norms.*
- *For every $\sigma \in (1/6,1)$, the linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ admit continuous extensions to $\mathcal{Y}_\sigma$ and*

$$[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma = \left\{ f \in \mathcal{Y}_\sigma;\ \overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0 \right\}, \tag{6.17}$$

  endowed with the norm of $\mathcal{Y}_\sigma$.

Remark 6.6. *The threshold at $\sigma = 1/6$ is consistent with the observation of Remark 2.30 that the maps $\overline{\ell^j}(\cdot,0,0)$ are bounded on $H^\sigma_x L^2_z$ for every $\sigma > 1/6$.*

For $\tau \in (0,1)$, we use the notations of the previous paragraph, in particular the norm $\|\cdot\|_\tau$ of (6.4) and the function $I(\tau,\cdot)$ of (6.5), with $\mathcal{Y}_0$ and $\mathcal{Y}_1$ defined as above.

To derive the estimates required to apply Lemma 6.1, two strategies would be possible. Both rely on the explicit knowledge of the singular radial solutions constructed in Section 2.4, which are involved in the orthogonality conditions. First, one could impose periodic boundary conditions on f, compute a 2D Fourier-series representation of (an extension by parity of) the singular profiles and estimate the functions I working in the Fourier space. Such a frequency-domain approach is carried out in [5], assuming some appropriate asymptotic decay of the Fourier transform of the profile defining the orthogonality condition. We choose a second strategy, which stays in the spatial domain, and involves estimates using cut-off functions whose space-scale are linked with the parameter τ. This strategy is related to the one used in [62] and inspired by the links between the K functional of real interpolation theory and the notions of modulus of continuity and modulus of smoothness of functions (see e.g. [37]).

To prove Lemma 6.5, we intend to apply Lemma 6.1. Hence, we need to bound from below and from above the functions $I(\tau,\overline{\ell^j})$. By Definition 2.10, $\overline{\ell^j}(f,0,0) = \int_\Omega \partial_x f \overline{\Phi^j}$. As highlighted in Corollary 2.29, the profiles $\overline{\Phi^j}$ can be decomposed as the sum of a singular radial part, an x-independent part, and a regular part. The singular radial part is the one that will be dominating the behavior of the orthogonality conditions. Thus, we start by two lemmas concerning estimates from above and from below for integrals of the form $\int_\Omega (\partial_x f)\bar{u}^i_{\mathrm{sing}}$, before moving to the general case.

Lemma 6.7. *Let $h \in H^1_x L^2_z$ such that $h = 0$ on $\Sigma_0 \cup \Sigma_1$. Then, for $\tau \in (0,1)$,*

$$\left| \int_\Omega (\partial_x h(x,z)) \bar{u}^i_{\mathrm{sing}}(x,-z)\, \mathrm{d}x\, \mathrm{d}z \right| \lesssim \tau^{-1/6} \left(\|h\|_{L^2} + \tau \|\partial_x h\|_{L^2} \right), \tag{6.18}$$

where $\bar{u}^i_{\mathrm{sing}}$ is defined in Definition 2.23.

Proof. By symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.23. We also let $\chi(x,z) := \chi_i(x,-z)$ of Definition 2.23 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.18. With these notations

$$\bar{u}^0_{\mathrm{sing}}(x,-z) = r^{\frac{1}{2}} \Lambda(t) \chi(x,z). \tag{6.19}$$

In particular, since $\Lambda_0(+\infty) = 0$, $\bar{u}^0_{\mathrm{sing}}(x_0,-z) = 0$ for $z \in (-1,0)$. We split the integral to be estimated depending on whether $r \le \tau^\alpha$ or $r \ge \tau^\alpha$, where $\alpha > 0$ is to be chosen later. Let $\eta \in C^\infty(\mathbb{R};[0,1])$ such that $\eta(s) \equiv 1$ for $s \le 1$ and $\eta(s) \equiv 0$ for $s \ge 2$.

Step 1. *Estimate in the region: $r \le \tau^\alpha$.* By Cauchy–Schwarz,

$$\left| \int_\Omega \partial_x h \cdot r^{\frac{1}{2}} \Lambda(t) \chi \cdot \eta(r/\tau^\alpha) \right| \le \|\chi\|_\infty \|\Lambda\|_\infty \|\partial_x h\|_{L^2} \left(\int_\Omega r \eta^2(r/\tau^\alpha) \right)^{\frac{1}{2}}. \tag{6.20}$$

Using the polar-like change of coordinates of (2.40) and (2.43), one has

$$\int_\Omega r\eta^2(r/\tau^\alpha) = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2} r\eta^2(r/\tau^\alpha)\, \mathrm{d}r\, \mathrm{d}t \lesssim (\tau^\alpha)^5. \tag{6.21}$$

Hence, in this region,

$$\left| \int_\Omega \partial_x h \cdot r^{\frac{1}{2}} \Lambda(t) \chi \cdot \eta(r/\tau^\alpha) \right| \lesssim (\tau^\alpha)^{5/2} \|\partial_x h\|_{L^2}. \tag{6.22}$$

Step 2. *Estimate in the region: $r \ge \tau^\alpha$.* We intend to integrate by parts in x. At $x = x_1$, $\bar{u}^0_{\mathrm{sing}}(x,-z) = 0$ for $z \in (-1,1)$ because $\chi = 0$. At $x = x_0$, when $z > 0$, $h = 0$ by assumption, and, when $z < 0$, $\bar{u}^0_{\mathrm{sing}}(x,-z) = 0$ as recalled above. Hence, there is no boundary term and

$$\int_\Omega \partial_x h \cdot r^{\frac{1}{2}} \Lambda(t) \chi \cdot (1 - \eta(r/\tau^\alpha)) = -\int_\Omega h \partial_x \left(\chi \cdot r^{\frac{1}{2}} \Lambda(t) (1 - \eta(r/\tau^\alpha)) \right). \tag{6.23}$$

First, one easily bounds

$$\left| \int_\Omega h \partial_x \chi \cdot r^{\frac{1}{2}} \Lambda(t)(1 - \eta(r/\tau^\alpha)) \right| \le \|h\|_{L^2} \|\Lambda\|_\infty \|\partial_x \chi\|_{L^2} \max_\Omega r^{\frac{1}{2}} \lesssim \|h\|_{L^2}. \tag{6.24}$$

For the second term, when ∂_x hits on the function expressed in (r,t) coordinates, we use the derivative formula (2.44):

$$\begin{aligned} \int_\Omega h\chi \partial_x \left(r^{\frac{1}{2}} \Lambda(t)(1 - \eta(r/\tau^\alpha)) \right) &= \int_\Omega h\chi \frac{(1+t^2)^{\frac{1}{2}}}{3r^2} \Lambda(t) \partial_r \left(r^{\frac{1}{2}} (1 - \eta(r/\tau^\alpha)) \right) \\ &\quad - \int_\Omega h\chi \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3} r^{\frac{1}{2}} (1 - \eta(r/\tau^\alpha)) \partial_t \Lambda(t). \end{aligned} \tag{6.25}$$

We bound both terms using the Cauchy–Schwarz inequality and the polar-like change of coordinates (2.40) with Jacobian determinant (2.43). In particular, on the one hand,

$$\begin{aligned}
&\int_\Omega \frac{1+t^2}{9r^4}\Lambda^2(t)\left(\partial_r\left(r^{\frac12}(1-\eta(r/\tau^\alpha))\right)\right)^2 \\
&\qquad = \int_0^\infty\int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\frac{1+t^2}{9r^4}\Lambda^2\left(\partial_r\left(r^{\frac12}(1-\eta(r/\tau^\alpha))\right)\right)^2 \,\mathrm{d}t\,\mathrm{d}r \\
&\qquad \lesssim \int_0^\infty \left(r^{-1}(1-\eta(r/\tau^\alpha))^2 + r(\eta'(r/\tau^\alpha))^2/(\tau^\alpha)^2\right)\frac{\mathrm{d}r}{r} \\
&\qquad = (\tau^\alpha)^{-1}\int_1^\infty \left(s^{-1}(1-\eta(s))^2 + s(\eta'(s))^2\right)\frac{\mathrm{d}s}{s} \lesssim (\tau^\alpha)^{-1}.
\end{aligned} \tag{6.26}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \frac{t^2(1+t^2)^3}{9r^6} r(1-\eta(r/\tau^\alpha))^2(\partial_t\Lambda(t))^2 \\
&\qquad = \int_0^\infty\int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\frac{t^2(1+t^2)^3}{9r^6} r(1-\eta(r/\tau^\alpha))^2(\partial_t\Lambda(t))^2 \,\mathrm{d}t\,\mathrm{d}r \\
&\qquad \lesssim \left(\int_{\mathbb{R}} t^2(1+t^2)(\partial_t\Lambda(t))^2\,\mathrm{d}t\right)\int_0^\infty \frac{1}{3r^2}(1-\eta(r/\tau^\alpha))\,\mathrm{d}r \\
&\qquad \lesssim (\tau^\alpha)^{-1}
\end{aligned} \tag{6.27}$$

by the integrability property $t^3\partial_t\Lambda(t) = O(1)$ of Lemma 2.21.

Thus, gathering the estimates in this region proves that

$$\left|\int_\Omega \partial_x h\cdot r^{\frac12}\Lambda(t)\chi\cdot(1-\eta(r/\tau^\alpha))\right| \lesssim (\tau^\alpha)^{-\frac12}\|h\|_{L^2}. \tag{6.28}$$

Gathering the estimates in both regions and choosing $\alpha = 1/3$ concludes the proof. □

Lemma 6.8. *There exists a family $(h_\tau^i)_{\tau\in(0,1)}$ of non-zero, smooth, compactly supported functions on Ω such that, as $\tau\to 0$,*

$$\left|\int_\Omega (\partial_x h_\tau^i(x,z))\bar{u}^i_{\mathrm{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z\right| \gtrsim \tau^{-1/6}\left(\|h_\tau^i\|_{L^2} + \tau\|\partial_x h_\tau^i\|_{L^2}\right), \tag{6.29}$$

and $\int_\Omega \partial_x h_\tau^j \bar{u}^i_{\mathrm{sing}} = 0$ for $j\neq i$, where $\bar{u}^i_{\mathrm{sing}}$ is defined in Definition 2.23.

Proof. As in the previous lemma, by symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.23. We also let $\chi(x,z) := \chi_i(x,-z)$ of Definition 2.23 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.18. With these notations, one has (6.19).

Let $\alpha > 0$. Let $H\in C_c^\infty(\mathbb{R};[-1,1])$ and $\eta\in C_c^\infty(\mathbb{R};[-1,1])$ such that $\operatorname{supp}\eta\subset(1/2,3/2)$. For $\tau\in(0,1)$, we define

$$h_\tau := \eta(r/\tau^\alpha)H(t). \tag{6.30}$$

By the support properties of H and η, one checks that h_τ is both smooth and compactly supported in Ω. Moreover, it is non-zero if $H\neq 0$ and $\eta\neq 0$.

Let $\tau > 0$ be sufficiently small such that the support of h_τ is included in the region where $\chi \equiv 1$. Note that with this choice, we also have $\int_\Omega \partial_x h_\tau \bar{u}^1_{\text{sing}} = 0$. Then, using the formula (2.44) for ∂_x and the determinant (2.43),

$$\begin{aligned}
&\int_\Omega \partial_x h_\tau \bar{u}_{\text{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z \\
&\quad= \int_0^\infty \int_{\mathbb{R}} r^{\frac{1}{2}}\Lambda(t)\left(\frac{(1+t^2)^{\frac{1}{2}}}{3r^2}(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\eta(r/\tau^\alpha)H'(t)\right)\frac{3r^3}{(1+t^2)^2}\,\mathrm{d}t\;\mathrm{d}r \\
&\quad= \int_0^\infty \int_{\mathbb{R}} \frac{r^{\frac{1}{2}}\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\left(r(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) - t(1+t^2)\eta(r/\tau^\alpha)H'(t)\right)\,\mathrm{d}t\;\mathrm{d}r \\
&\quad= (\tau^\alpha)^{3/2}\int_{\mathbb{R}} \frac{H(t)\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t \int_0^\infty s^{3/2}\eta'(s)\,\mathrm{d}s - (\tau^\alpha)^{3/2}\int_{\mathbb{R}} \frac{tH'(t)\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\,\mathrm{d}t\int_0^\infty s^{1/2}\eta(s)\,\mathrm{d}s.
\end{aligned} \tag{6.31}$$

Note that since $\eta \in C_c^\infty((0,+\infty))$,

$$\int_0^\infty s^{3/2}\eta'(s)\,\mathrm{d}s = -\frac{3}{2}\int_0^\infty s^{1/2}\eta(s)\,\mathrm{d}s.$$

We claim that we may choose η and H such that

$$\int_0^\infty s^{1/2}\eta(s)\,\mathrm{d}s = \frac{3}{2}\int_{\mathbb{R}} \frac{H(t)\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t + \int_{\mathbb{R}} \frac{tH'(t)\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\,\mathrm{d}t = 1.$$

The claim for η is obvious. As for H, we assume that $\operatorname{supp} H \subset (0,+\infty)$ and we write the sum of integrals as

$$\begin{aligned}
&\int_{\mathbb{R}} H(t)\left[\frac{3}{2}\frac{\Lambda(t)}{(1+t^2)^{\frac{3}{2}}} - \frac{d}{dt}\left(\frac{t\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\right)\right]\,\mathrm{d}t \\
&= -\int_0^\infty H(t)\frac{d}{dt}\left(\frac{t\Lambda(t)}{(1+t^2)^{1/2}}t^{-3/2}(1+t^2)^{3/4}\right)t^{3/2}(1+t^2)^{-3/4}\,\mathrm{d}t.
\end{aligned}$$

Since $\Lambda(t) \neq Ct^{1/2}(1+t^2)^{-1/4}$ on $\mathbb{R}_+$, the claim for H follows.

The above choice of η and H ensures that

$$\int_\Omega \partial_x h_\tau \cdot r^{\frac{1}{2}}\Lambda(t)\chi = -(\tau^\alpha)^{3/2}. \tag{6.32}$$

Using once again the formula (2.44) for ∂_x and the change of coordinates of Jacobian (2.43), one obtains that $\|h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2$ and $\|\partial_x h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Similarly, using (2.45) to compute $\partial_z^3 h_\tau$ and the same technique, $\|\partial_z^3 h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Thus, choosing $\alpha = 1/3$ leads to

$$\|h_\tau\|_{L^2} + \tau\|\partial_x h_\tau\|_{L^2} + \tau\|\partial_z^3 h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2, \tag{6.33}$$

which concludes the proof. □

We are now ready to prove Lemma 6.5.

Proof of Lemma 6.5. This is an application of Lemma 6.1 with $\bar{\sigma} = 1/6$. Therefore, we need to find constants $C_\pm > 0$ such that, for every $\tau \in (0,1)$ and $(c_0, c_1) \in \mathbb{S}^1$,

$$C_- \tau^{-1/6} \leq I(\tau, c_0\overline{\ell^0} + c_1\overline{\ell^1}) \leq C_+ \tau^{-1/6}. \tag{6.34}$$

Let $(c_0, c_1) \in \mathbb{S}^1$ and $f \in \mathcal{Y}_1$. By Definition 2.10,

$$\overline{\ell^j}(f,0,0) = \int_\Omega \partial_x f \overline{\Phi^j}, \tag{6.35}$$

where $\overline{\Phi^j}$ is the solution to (2.17).

By Corollary 2.29, there exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\begin{aligned} c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0) = &\int_\Omega \partial_x f (d_0 \bar{u}^0_{\mathrm{sing}}(x,-z) + d_1 \bar{u}^1_{\mathrm{sing}}(x,-z)) \\ &+ \int_\Omega \partial_x f \left(\Phi_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right), \end{aligned} \tag{6.36}$$

where $\Phi_{\mathrm{reg}} \in Z^1$. By linearity, d_0, d_1 and Φ_{reg} are uniformly bounded for $(c_0, c_1) \in \mathbb{S}^1$. The first term corresponds to the one studied in Lemma 6.7 and Lemma 6.8. We want to integrate by parts in the second term. Since $f \in \mathcal{Y}_1$, $f_{|\Sigma_0 \cup \Sigma_1} = 0$. At $x = x_0$ and $z \in (-1, 0)$, $\overline{\Phi^j} = 0$ by (2.17). Moreover, $\bar{u}^0_{\mathrm{sing}}(x_0, -z) = 0$ because $\Lambda_0(+\infty) = 0$ and $\bar{u}^1_{\mathrm{sing}}(x_0, -z) = 0$ because $\bar{u}^1_{\mathrm{sing}}$ is compactly supported near $(x_1, 0)$. Hence, $\Phi_{\mathrm{reg}}(x_0, z) + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0} \equiv 0$ on $(-1, 0)$. The same conclusion holds at $x = x_1$ and $z \in (0, 1)$. Thus, we can integrate by parts with no boundary term and the second term is estimated as

$$\left| \int_\Omega \partial_x f \left(\Phi_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right) \right| \leq \|f\|_{L^2} \|\partial_x \Phi_{\mathrm{reg}}\|_{L^2}. \tag{6.37}$$

Step 1. *Bound from above.* For $\tau \in (0,1)$, using Lemma 6.7 and (6.37),

$$|c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0)| \lesssim \tau^{-1/6} \left(\|f\|_{L^2} + \tau \|f\|_{\mathcal{Y}_1}\right). \tag{6.38}$$

Step 2. *Bound from below.* For $\tau \in (0,1)$, let $f_\tau := h_\tau$, where h_τ is constructed in Lemma 6.8, which ensures that f_τ is compactly supported in Ω so satisfies $(f_\tau)_{|\Sigma_0 \cup \Sigma_1} = 0$. Substituting in (6.36) and integrating by parts yields

$$c_0\overline{\ell^0}(f_\tau,0,0) + c_1\overline{\ell^1}(f_\tau,0,0) = -\int_\Omega h_\tau \partial_x \Phi_{\mathrm{reg}} + \sum_{i \in \{0,1\}} d_i \int_\Omega (\partial_x h_\tau) \bar{u}^i_{\mathrm{sing}}(x,-z). \tag{6.39}$$

By Corollary 2.29 and linearity, $\min(|d_0|, |d_1|)$ is uniformly bounded from below. We choose h_τ as either h^0_τ or h^1_τ of Lemma 6.8 accordingly. Thus, by Lemma 6.8, as $\tau \to 0$,

$$\begin{aligned} |c_0\overline{\ell^0}(f_\tau,0,0) + c_1\overline{\ell^1}(f_\tau,0,0)| &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau\|h_\tau\|_{\mathcal{Y}_1}\right) - C\|h_\tau\|_{L^2}\|\partial_x \Phi_{\mathrm{reg}}\|_{L^2} \\ &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau\|h_\tau\|_{\mathcal{Y}_1}\right) \\ &= \tau^{-1/6} \left(\|f_\tau\|_{L^2} + \tau\|f_\tau\|_{\mathcal{Y}_1}\right) \end{aligned} \tag{6.40}$$

for $\tau > 0$ sufficiently small. This concludes the proof. □

To conclude this section, we turn towards the proof of Proposition 2.33.

Proof of Proposition 2.33. ***Step 1.*** *Case* $\delta_0 = \delta_1 = 0$ *and* $\mathbf{1}_{\sigma>1/2} f_{|\Sigma_i} = 0$. By Proposition 2.4, for every $f \in L^2(\Omega)$, there exists a unique solution $u \in Z^0(\Omega)$ to (2.1) with $\delta_0 = \delta_1 = 0$ and $\|u\|_{Z^0} \lesssim \|f\|_{L^2}$. By Proposition 2.9 and Proposition 2.5, for every $f \in H^1_x L^2_z$ such that $f_{|\Sigma_0 \cup \Sigma_1} = 0$ (so that $\Delta_0 = \Delta_1 = 0$) and $\overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0$, this solution satisfies $u \in Z^1(\Omega)$ with $\|u\|_{Z_1} \lesssim \|f\|_{H^1_x L^2_z}$. Hence, by interpolation, the mapping $f \mapsto u$ is bounded from $[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma$ to $Z^\sigma(\Omega)$. Moreover, by Lemma 6.4 and Lemma 6.5, when $\sigma \in (0,1) \setminus \{1/6, 1/2\}$, $[\mathcal{Y}_0, \mathcal{Y}_1^{\overline{\ell}}]_\sigma = H^\sigma_x L^2_z$ (with null boundary conditions on $\Sigma_0 \cup \Sigma_1$ when $\sigma > 1/2$, and null linear forms constraints when $\sigma > 1/6$). This proves estimate (2.94) in the case of vanishing boundary data.

Step 2. *Arbitrary boundary data.* When δ_0 and δ_1 are arbitrary, we extend them to $(-1,1)$ in such a way that the extension belongs to $H^2_0(-1,1)$. We then lift the boundary data by setting $u_l(x,z) = \chi(x-x_0)\delta_0 + \chi(x-x_1)\delta_1$, with $\chi \in C^\infty_c(\mathbb{R})$, supported in $B(0,(x_1-x_0)/2)$, and equal to 1 in a neighborhood of zero. This introduces a source term $f_l = z\partial_x u_l - \partial_{zz} u_l \in H^1_x L^2_z$ in the equation, whose trace on Σ_i is $-\delta_i''$, so that the trace of $f - f_l$ on Σ_i is $z\Delta_i$. When $\sigma < 1/2$, we immediately obtain the desired result thanks to the previous step.

For $\sigma > 1/2$, we first note that, since $u, u_l \in Z^1(\Omega)$, by Proposition 2.9,

$$\overline{\ell^j}(f - f_l, 0, 0) = 0. \tag{6.41}$$

We further decompose $f - f_l$ into $f - f_l = z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + g_l$, where $g_l \in H^\sigma_x L^2_z$ is such that $g_{l|\Sigma_0\cup\Sigma_1} = 0$. Using Proposition 2.13 we construct $h_l \in C^\infty_c(\Omega)$ such that

$$\|h_l\|_{H^1_x L^2_z} \lesssim \|\Delta_0\|_{\mathscr{H}^1(\Sigma_0)} + \|\Delta_1\|_{\mathscr{H}^1(\Sigma_1)} \tag{6.42}$$

and

$$\overline{\ell^j}(z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + h_l, 0, 0) = \overline{\ell^j}(g_l - h_l, 0, 0) = 0. \tag{6.43}$$

We then apply the result of the first step to the system with source term $g_l - h_l$ (which vanishes on $\Sigma_0 \cup \Sigma_1$) and homogeneous boundary data, and the result of Proposition 2.9 to the system with source term $z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + h_l$ and homogeneous boundary data, using the conditions $\Delta_0(1) = \Delta_1(-1) = 0$. This concludes the proof. □

A Uniqueness of weak solutions for linear problems

The purpose of this section is to prove the following uniqueness result, which is a slight generalization to the case of variable coefficients of the uniqueness result of [8, Section 5] for (2.1).

Lemma A.1. *Let $\Omega = (x_0, x_1) \times (z_b, z_t)$, where $x_0 < x_1$ and $z_b < 0 < z_t$. Let $\alpha \in C^2(\overline{\Omega})$ such that $\inf \alpha > 0$ and $\beta \in L^\infty(\Omega)$. Assume that one of the two following conditions is satisfied:*

- *either $\|\beta\|_\infty \ll 1$ and $\|\partial_z \alpha\|_\infty \ll 1$,*
- *or $|z_b|, z_t \leq z_0$, for some small constant z_0 depending only on α.*

Let $g \in L^2_x H^{-1}_z$, $\delta_0 \in \mathscr{L}^2_z(\Sigma_0)$, $\delta_1 \in \mathscr{L}^2_z(\Sigma_1)$. There exists at most one weak solution $U \in L^2_x H^1_0$ to

$$\begin{cases} z\partial_x U + \beta \partial_z U - \partial_{zz}(\alpha U) = g, \\ U|_{\Sigma_0} = \delta_0, \\ U|_{\Sigma_1} = \delta_1, \\ U|_{z=z_t} = U|_{z=z_b} = 0. \end{cases} \tag{A.1}$$

The proof follows the arguments of Baouendi and Grisvard in [8], which concern the case of the model equation (2.1). For the reader's convenience, we recall the main steps of the proof here, and adapt them to the present (slightly different) context. The proof involves the spaces $\mathcal{B}$ defined in (1.35) and $\mathcal{A} := \mathcal{B} \cap H^1(\Omega)$.

Note that if $U \in L^2((x_0, x_1), H^1_0(z_b, z_t))$ is a weak solution to (A.1), then $U \in \mathcal{B}$. Indeed, it follows from the weak formulation that for any $V \in H^1_0(\Omega)$,

$$\langle z\partial_x U, V \rangle_{L^2(H^{-1}), L^2(H^1_0)} = -\int_\Omega \partial_z(\alpha U)\partial_z V - \int_\Omega \beta \partial_z U V + \langle g, V \rangle_{L^2(H^{-1}), L^2(H^1_0)}. \tag{A.2}$$

By density, this formula still holds for $V \in L^2_x(H^1_0)$, and therefore $z\partial_x U \in L^2_x(H^{-1}_z)$.

We then recall the following result from [8]:

Lemma A.2. *The set $\mathcal{A}$ is dense in $\mathcal{B}$. Furthermore, there exists a constant C depending only on Ω, such that for $i \in \{0, 1\}$,*

$$\forall v \in \mathcal{A}, \quad \int_{z_b}^{z_t} |z| \, |v(x_i, y)|^2 \, \mathrm{d}y \leq C \|v\|^2_{\mathcal{B}}. \tag{A.3}$$

As a consequence, the applications

$$v \in \mathcal{A} \mapsto v_{|x=x_i} \in \mathscr{L}^2_z(z_b, z_t) \tag{A.4}$$

can be uniquely extended into continuous applications on $\mathcal{B}$.

As a consequence, Baouendi and Grisvard [8] obtain the following corollary:

Corollary A.3. *For all $u, v \in \mathcal{B}$,*

$$\langle z\partial_x u, v \rangle_{L^2(H^{-1}), L^2(H^1_0)} + \langle z\partial_x v, u \rangle_{L^2(H^{-1}), L^2(H^1_0)} = \int_{z_b}^{(} z_t (zuv)|_{x=x_1} - \int_{z_b}^{(} z_t (zuv)|_{x=x_0}. \tag{A.5}$$

Proof. Thanks to Lemma A.2, it suffices to prove the identity when $u, v \in \mathcal{A}$. In that case, the left-hand side is simply

$$\int_\Omega z\partial_x uv + zu\partial_x v = \int_\Omega \partial_x(zuv). \tag{A.6}$$

The result follows by integration. □

Proof of Lemma A.1. Let $U \in L^2_x(H^1_0)$ be a weak solution to (A.1) with $g = 0$ and $\delta_i = 0$. As mentioned above, $U \in \mathcal{B}$. According to Corollary A.3, for any $V \in \mathcal{B}$ such that $V = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$-\langle z\partial_x V, U\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \int_\Omega (\beta\partial_z UV + \alpha_z U\partial_z V + \alpha\partial_z U\partial_z V) = 0. \tag{A.7}$$

Now, let $h \in C^\infty_c(\Omega)$ be arbitrary, and let $V \in L^2(H^1_0)$ be a weak solution to

$$\begin{cases} -z\partial_x V - \partial_z(\beta V) - \alpha\partial_{zz} V = h, \\ V_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases} \tag{A.8}$$

(The existence of weak solutions for this adjoint problem is proved in the same way as existence for the direct problem in Proposition 2.2 in the case $\|\beta\|_\infty \ll 1$, $\|\alpha_z\|_\infty \ll 1$, and Lemma 5.3 in the case $|z_b|, z_t$ small).

Then $V \in \mathcal{B}$, and choosing U as a test function in the variational formulation for V, we obtain

$$\int_\Omega hU = 0. \tag{A.9}$$

Thus $U = 0$. Uniqueness of weak solutions to (A.1) follows. □

B Proofs of functional analysis results

B.1 An abstract existence principle

As Fichera in [22], we use the following abstract existence principle (see [19, Theorem 1]), which allows skipping a viscous regularization scheme.

Lemma B.1. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}$ be three Hilbert spaces. Let $F_i \in \mathcal{L}(\mathscr{H}_i;\mathscr{H})$ for $i \in \{1,2\}$. Then the following statements are equivalent:*

- $\operatorname{range} F_1 \subset \operatorname{range} F_2$,
- *There exists a constant $C > 0$ such that*

$$\forall h \in \mathscr{H}', \quad \|F_1^* h\|_{\mathscr{H}_1'} \le C\|F_2^* h\|_{\mathscr{H}_2'}. \tag{B.1}$$

- *There exists $G \in \mathcal{L}(\mathscr{H}_1;\mathscr{H}_2)$ such that $F_1 = F_2 G$.*

Moreover, when these hold, there exists a unique $G \in \mathcal{L}(\mathscr{H}_1;\mathscr{H}_2)$ such that $\ker G = \ker F_1$, $\operatorname{range} G \subset (\operatorname{range} F_2^*)^\perp$ and $\|G\| = \inf\{C > 0;\ \text{(B.1) holds}\}$.

Indeed, this yields the following weak Lax-Migram result, where the linear right-hand side is assumed to be continuous for the weaker norm.

Lemma B.2. *Let $\mathscr{U}$ and $\mathscr{V}$ be two Hilbert spaces with $\mathscr{V}$ continuously embedded in $\mathscr{U}$. Let a be a continuous bilinear form on $\mathscr{U} \times \mathscr{V}$ and b be a continuous linear form on $\mathscr{U}$. Assume that there exists a constant $c > 0$ such that, for every $v \in \mathscr{V}$,*

$$a(v, v) \geq c\|v\|_{\mathscr{U}}^2. \tag{B.2}$$

Then, there exists $u \in \mathscr{U}$ such that $\|u\|_{\mathscr{U}} \leq \frac{1}{c}\|b\|_{\mathcal{L}(\mathscr{U})}$ and, for every $v \in \mathscr{V}$, $a(u, v) = b(v)$.

Proof. Set $\mathscr{H} := \mathcal{L}(\mathscr{V})$, $\mathscr{H}_1 := \mathcal{L}(\mathscr{U})$, $F_1 := \mathrm{Id}$ (from $\mathcal{L}(\mathscr{U})$ to $\mathcal{L}(\mathscr{V})$), $\mathscr{H}_2 := \mathscr{U}$ and $F_2 : \mathscr{U} \to \mathcal{L}(\mathscr{V})$ defined by $F_2 u := a(u, \cdot)$. Then $F_1^* = \mathrm{Id}$ (from $\mathscr{V}$ to $\mathscr{U}$) and $F_2^* v = a(\cdot, v)$. Moreover

$$\|F_2^* v\|_{\mathcal{L}(\mathscr{U})} \geq |a(v, v)|/\|v\|_{\mathscr{U}} \geq c\|v\|_{\mathscr{U}} = c\|F_1^* v\|_{\mathscr{U}}. \tag{B.3}$$

So (B.1) holds with $C = 1/c$ and Lemma B.1 yields the existence of $G \in \mathcal{L}(\mathcal{L}(\mathscr{U}); \mathscr{U})$ such that $F_1 = F_2 G$ and $\|G\| \leq \frac{1}{c}$. The conclusions follow by setting $u := Gb$. □

B.2 Product and composition rules in Sobolev spaces

Lemma B.3 (Pointwise multiplication)**.** *Pointwise multiplication is a continuous bilinear map*

- *from $H^{3/2}(-1, 1) \times H^{3/2}(-1, 1)$ to $H^{3/2}(-1, 1)$,*
- *from $H^{1/2}(x_0, x_1) \times H^s(x_0, x_1)$ to $H^{1/2}(x_0, x_1)$ for any $s > 1/2$,*
- *from $H^{1/2}(x_0, x_1) \times H^s(x_0, x_1)$ to $H^{s'}(x_0, x_1)$ for any $s' < \min(s, 1/2)$.*
- *from $H^s(x_0, x_1) \times H^{s'}(x_0, x_1)$ to $H^{s'}(x_0, x_1)$ for any $s > 1/2$, $s \geq s'$.*

Proof. These are particular cases of [10, Theorem 7.4]. □

Lemma B.4 (Composition of H^σ functions)**.** *Let $\sigma \in (0, 1/6)$, and let $\Omega_y = (x_0, x_1) \times \mathbb{R}$, $\Omega_z = (x_0, x_1) \times (z_b, z_t)$. For $f \in H_x^\sigma H_y^1 \cap L_x^4 H_y^1(\Omega_y)$ and $Y \in H_x^{\frac{2}{3}+\sigma} H_z^1(\Omega_z)$, such that $\lambda \leq \partial_z Y \leq \lambda^{-1}$ for some positive constant λ,*

$$\|f(x, Y(x, z))\|_{H_x^\sigma L_z^2(\Omega_z)} \lesssim C_{\|Y\|} \left(\|f\|_{H_x^\sigma L_y^2} + \|f\|_{L_x^4 H_y^1} \right). \tag{B.4}$$

and

$$\|f(x, Y(x, z))\|_{L^\infty((z_b, z_t), H^\sigma(x_0, x_1))} \lesssim C_{\|Y\|} \left(\|f\|_{H_x^\sigma H_z^1} + \|f\|_{L_x^4 H_y^1} \right). \tag{B.5}$$

In a similar way, if $f \in H_x^{\frac{1}{2}+\sigma} L_y^2 \cap L_x^\infty W_y^{1,\infty}(\Omega_y)$,

$$\|f(x, Y(x, z))\|_{H_x^{\frac{1}{2}+\sigma} L_z^2(\Omega_z)} \lesssim C_{\|Y\|} \left(\|f\|_{H_x^{\frac{1}{2}+\sigma} L_z^2} + \|f\|_{L_x^\infty W_z^{1,\infty}} \right).$$

Proof. First, using the classical definition of fractional Sobolev spaces, for all $z \in (z_b, z_t)$,

$$\|f(\cdot, Y(\cdot, z))\|_{H^\sigma(x_0, x_1)}^2 = \|f(\cdot, Y(\cdot, z))\|_{L^2(x_0, x_1)}^2 + \int_{x_0}^{x_1} \int_{x_0}^{x_1} \frac{|f(x, Y(x, z)) - f(x', Y(x', z))|^2}{|x - x'|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}x'. \tag{B.6}$$

We start with the $H^\sigma_x L^2_z$ estimate. Integrating with respect to z, the norm of the first term is bounded by the square of the L^2 norm of f after a change of variable with bounded jacobian. We then decompose the second integral into

$$\int_{x_0}^{x_1}\int_{x_0}^{x_1} \frac{|f(x,Y(x,z))-f(x',Y(x,z))|^2}{|x-x'|^{1+2\sigma}}\mathrm{d}x\,\mathrm{d}x' + \int_{x_0}^{x_1}\int_{x_0}^{x_1} \frac{|f(x',Y(x,z))-f(x',Y(x',z))|^2}{|x-x'|^{1+2\sigma}}\mathrm{d}x\,\mathrm{d}x'.$$

Once again, the first integral is bounded by $\|f\|^2_{H^\sigma_x L^2_z}$ after vertical integration. As for the second one, using the embedding $H^1(-1,1)\hookrightarrow C^{1/2}$, we have

$$|f(x',Y(x,z))-f(x',Y(x',z))|^2 \lesssim \|\partial_y f(x',\cdot)\|^2_{L^2_y}|Y(x,z)-Y(x',z)|.$$

Since $Y\in H^{\frac{2}{3}+\sigma}_x H^1_z$,

$$\begin{aligned}
&\int_{x_0}^{x_1}\int_{x_0}^{x_1} \|\partial_y f(x',\cdot)\|^2_{L^2_y}\frac{|Y(x,z)-Y(x',z)|}{|x-x'|^{1+2\sigma}}\mathrm{d}x\mathrm{d}x'\\
\lesssim&\|Y(\cdot,z)\|_{H^{\frac{2}{3}+\sigma}_x}\left(\int_{x_0}^{x_1}\int_{x_0}^{x_1}\|\partial_y f(x',\cdot)\|^4_{L^2_y}|x-x'|^{\frac{1}{3}-3\sigma}\,\mathrm{d}x\,\mathrm{d}x'\right)^{1/2}\\
\lesssim&\|Y\|_{H^{\frac{2}{3}+\sigma}_x H^1_z}\|f\|^2_{L^4_x H^1_y}.
\end{aligned}$$

The first estimate follows. The other ones go along the same lines and are left to the reader. □

Lemma B.5 (Composition with a Q^1 function)**.** *Let $\phi\in Q^1(\Omega)$ such that $\phi(x,\pm1)=\pm1$. Assume that there exists $m>0$ such that $\partial_z\phi(x,z)\in[m^{-1},m]$. Let $\sigma\in[0,1]$. There exists $C(m,\sigma)$ such that, for any $g\in H^\sigma_x L^2_y\cap L^2_x(W^{\sigma,4}_y)$,*

$$\|g(x,\phi(x,z))\|_{H^\sigma_x L^2_z}\le C\left(\|g\|_{H^\sigma_x L^2_z}+(1+\|\phi\|^\sigma_{Q^1})\|g\|_{L^2_x(W^{\sigma,4}_y)}\right). \tag{B.7}$$

Proof. Throughout the proof, we set $G(x,z):=g(x,\phi(x,z))$. First, note that, since the Jacobian of the change of variable $z\mapsto\phi(x,z)$ is bounded from below, for any $p,q\in[1,\infty]$,

$$\|G\|_{L^p_x L^q_z}\le m^{\frac{1}{q}}\|g\|_{L^p_x L^q_z}. \tag{B.8}$$

In particular $\|G\|_{L^2}\le m^{\frac{1}{2}}\|g\|_{L^2}$. Furthermore, for $\sigma=1$,

$$\partial_x G(x,z)=\partial_x g(x,\phi(x,z))+\partial_x\phi(x,z)\partial_y g(x,\phi(x,z)). \tag{B.9}$$

Hence,

$$\|\partial_x G\|_{L^2}\le\|(\partial_x g)\circ\phi\|_{L^2}+\|\partial_x\phi\|_{L^\infty_x L^4_z}\|(\partial_y g)\circ\phi\|_{L^2_x L^4_z}. \tag{B.10}$$

By the "fractional trace theorem" [45, Equation (4.7), Chapter 1],

$$\|\partial_x\phi\|_{L^\infty_x H^{1/2}_z}\lesssim\|\partial_x\phi\|_{H^{2/3}_x L^2_z}+\|\partial_x\phi\|_{L^2_x H^2_z}\lesssim\|\phi\|_{Q^1}. \tag{B.11}$$

Hence, we obtain from (B.8) and (B.10) that

$$\|\partial_x G\|_{L^2}\lesssim\|\partial_x g\|_{L^2}+\|\phi\|_{Q^1}\|g\|_{L^2_x(W^{1,4}_y)}. \tag{B.12}$$

Now, note that the application $g\mapsto G$ is linear. By interpolation, we obtain, for any $\sigma\in(0,1)$,

$$\|G\|_{H^\sigma_x L^2_z}\lesssim\|g\|_{H^\sigma_x L^2_z}+(1+\|\phi\|^\sigma_{Q^1})\|g\|_{L^2_x(W^{\sigma,4}_y)}, \tag{B.13}$$

which concludes the proof. □

Corollary B.6. *Let $\phi \in Q^1 \cap L^2_x H^4_z$ such that $\phi(x,\pm 1) = \pm 1$ and $\|\phi - z\|_{Q^1} + \|\phi - z\|_{L^2_x H^4_z} \ll 1$. Let $\psi(x,y)$ be such that $\psi(x,\phi(x,z)) = z$ for all $(x,z) \in \Omega$. Then $\psi \in Q^1 \cap L^2_x H^4_y$ and*

$$\|\psi - y\|_{Q^1} + \|\psi - y\|_{L^2_x H^4_y} \lesssim \|\phi - z\|_{Q^1} + \|\phi - z\|_{L^2_x H^4_z}. \tag{B.14}$$

Proof. In this statement and this proof, we use the variable y as second argument for ψ and z as second argument for ϕ. First, observe that $\partial_y \psi(x,y) = 1/(\partial_z \phi(x,\psi(x,y)))$, so that $\|\partial_y \psi - 1\|_{L^\infty} \ll \|\partial_z \phi - 1\|_{L^\infty}$. In particular the associated changes of vertical variables are well-defined and bounded so that estimates such as (B.8) hold and will be used abundantly.

Step 1. *Vertical regularity of ψ.* By the "fractional trace theorem" [45, Equation (4.7), Chapter 1], for $\phi \in Q^1 \cap L^2_x H^4_z$, $\phi \in H^1_x H^2_z \cap L^2_x H^4_z \hookrightarrow C^0_x(H^3_z)$. In particular $\partial_z^2 \phi \in L^\infty$. Differentiating the definition $\psi(x,\phi(x,z)) = z$, we obtain the following relations and estimates. First, we already said that $\partial_y \psi \in L^\infty$. Second, $\partial_y^2 \psi \in L^2$ since

$$-(\partial_z \phi)^2 \partial_y^2 \psi \circ \phi = \underbrace{(\partial_y \psi \circ \phi)}_{L^\infty} \underbrace{(\partial_{zz} \phi)}_{L^\infty}. \tag{B.15}$$

Third, $\partial_y^3 \psi \in L^2$ since

$$-(\partial_z \phi)^3 \partial_y^3 \psi \circ \phi = 3 \underbrace{\partial_y^2 \psi \circ \phi}_{L^2} \underbrace{\partial_z \phi \partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_y \psi \circ \phi}_{L^\infty} \underbrace{\partial_z^3 \phi}_{L^2}. \tag{B.16}$$

Fourth, omitting the composition with ϕ in every occurence of ψ in order to alleviate the notation,

$$-(\partial_z \phi)^4 \partial_y^4 \psi = 6 \underbrace{\partial_y^3 \psi}_{L^2} \underbrace{(\partial_z \phi)^2 \partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_y^2 \psi}_{L^2_x H^1_y} \big(3 \underbrace{(\partial_z^2 \phi)^2}_{L^\infty} + 4 \underbrace{(\partial_z \phi)}_{L^\infty} \underbrace{(\partial_z^3 \phi)}_{L^\infty_x L^2_z} \big) + \underbrace{\partial_y \psi}_{L^\infty} \underbrace{\partial_z^4 \phi}_{L^2}. \tag{B.17}$$

Remembering that $1/(\partial_z \phi) \in L^\infty$, we conclude that

$$\|\psi(x,y) - y\|_{L^2_x H^4_y} \lesssim \|\phi(x,z) - z\|_{Q^1} + \|\phi(x,z) - z\|_{L^2_x H^4_z}. \tag{B.18}$$

Step 2. *Integer horizontal regularity of ψ.* This step uses that $\partial_x \phi \in L^\infty_x L^2_z$ which follows from $\phi \in H^{5/3}_x L^2_z$. Note however that, even for $\phi \in Q^1 \cap L^2_x H^4_y$, one does not have $\partial_x \phi \in L^\infty$. We proceed similarly for the integer horizontal regularity. First,

$$-\partial_x \psi \circ \phi = \underbrace{\partial_y \psi \circ \phi}_{L^\infty} \underbrace{\partial_x \phi}_{L^2}. \tag{B.19}$$

Second,

$$-(\partial_z \phi) \partial_{xy} \psi \circ \phi = \underbrace{\partial_y^2 \psi \circ \phi}_{L^2_x H^1_y} \underbrace{\partial_z \phi}_{L^\infty} \underbrace{\partial_x \phi}_{L^\infty_x L^2_z} + \underbrace{\partial_y \psi \circ \phi}_{L^\infty} \underbrace{\partial_{xz} \phi}_{L^2}. \tag{B.20}$$

Third,

$$-(\partial_z \phi)^2 \partial_{xyy} \psi \circ \phi = 2 \partial_y^2 \psi \circ \phi \, \partial_z \phi \, \partial_{xz} \phi + h \tag{B.21}$$

where, omitting once again the composition with ϕ in the derivatives of ψ,

$$h = \underbrace{\partial_y^3 \psi}_{L^2_x H^1_y} \underbrace{\partial_x \phi}_{L^\infty_x L^2_z} \underbrace{(\partial_z \phi)^2}_{L^\infty} + \underbrace{\partial_{xy} \psi}_{L^2} \underbrace{\partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_y^2 \psi}_{L^2_x H^1_y} \underbrace{\partial_x \phi}_{L^\infty_x L^2_z} \underbrace{\partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_z \psi}_{L^\infty} \underbrace{\partial_{xzz} \phi}_{L^2}. \tag{B.22}$$

From (B.21) and $h \in L^2$, we obtain that

$$\begin{aligned}\|\partial_{xyy}\psi\|_{L^2} &\lesssim \|h\|_{L^2} + \|\partial_{xz}\phi\|_{L^2_x H^1_z}\|\partial_{yy}\psi\|_{L^\infty_x L^2_z} \\ &\lesssim \|h\|_{L^2} + \|\phi(x,z) - z\|_{H^1_x H^2_z}(\|\partial_{yy}\psi\|_{L^2} + \|\partial_{xyy}\psi\|_{L^2}).\end{aligned} \tag{B.23}$$

Hence, using the smallness of $\|\phi(x,z) - z\|_{H^1_x H^2_z}$, we conclude that

$$\|\psi(x,y) - y\|_{H^1_x H^2_y} \lesssim \|\phi(x,z) - z\|_{Q^1} + \|\phi(x,z) - z\|_{L^2_x H^4_z}. \tag{B.24}$$

Step 3. *Fractional horizontal regularity of ψ.* Eventually, to obtain the $H^{5/3}_x L^2_y$ regularity, we write

$$\partial_x \psi(x,y) = -\frac{\partial_x \phi}{\partial_z \phi}(x, \psi(x,y)) \tag{B.25}$$

and we apply Lemma B.5 with $\sigma = 2/3$. Let us first assume that ϕ is smooth (so that ψ is smooth as well by usual results) and then argue by density. Estimate (B.7) yields

$$\|\psi_x\|_{H^{2/3}_x L^2_y} \lesssim \|\partial_x\phi/\partial_z\phi\|_{H^{2/3}_x L^2_z} + (1 + \|\psi\|^{2/3}_{Q^1})\|\partial_x\phi/\partial_z\phi\|_{L^2_x H^2_z}. \tag{B.26}$$

Since we already know that ψ can be estimated in $H^1_x H^2_y$, we can use (the Peter–Paul version) of Young's inequality to obtain

$$\|\psi_x\|_{H^{2/3}_x L^2_y} \lesssim \|\partial_x\phi/\partial_z\phi\|_{H^{2/3}_x L^2_z} + \|\partial_x\phi/\partial_z\phi\|_{L^2_x H^2_z} + \|\partial_x\phi/\partial_z\phi\|^{3/2}_{L^2_x H^2_z}. \tag{B.27}$$

Moreover, one easily proves, using standard product rules, that

$$\|\partial_x\phi/\partial_z\phi\|_{H^{2/3}_x L^2_z} + \|\partial_x\phi/\partial_z\phi\|_{L^2_x H^2_z} \lesssim \|\phi(x,z) - z\|_{Q^1} + \|\phi(x,z) - z\|_{L^2_x H^4_z} \ll 1. \tag{B.28}$$

Hence we obtain

$$\|\psi_x\|_{H^{2/3}_x L^2_y} \lesssim \|\phi(x,z) - z\|_{Q^1} + \|\phi(x,z) - z\|_{L^2_x H^4_z} \tag{B.29}$$

when ϕ is smooth and $\|\phi(x,z) - z\|_{Q^1} + \|\phi(x,z) - z\|_{L^2_x H^4_z} \ll 1$. We conclude by density. □

B.3 Extension operators

We start with Lemma 1.6, which allows extending functions from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.

Proof of Lemma 1.6. Up to translation and rescaling, we can assume that $(x_0, x_1) = (0,1)$.

We start by constructing a continuous horizontal extension operator P_x from $Z^0((0,1)\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,1))$. Let $\chi \in C^\infty(\mathbb{R};[0,1])$ such that $\chi \equiv 1$ on $(0,1)$ and $\operatorname{supp}\chi \subset (-1,2)$. Let $\phi \in Z^0((0,1)\times(-1,1))$. For $x \in (-1,2)$ and $z \in (-1,1)$, let

$$(Q_x\phi)(x,z) := \begin{cases} \phi(-x,z) & \text{if } x \in (-1,0) \\ \phi(x,z) & \text{if } x \in (0,1), \\ \phi(2-x,z) & \text{if } x \in (1,2), \end{cases} \tag{B.30}$$

$$(P_x\phi)(x,z) := \chi(x)(Q_x\phi)(x,z). \tag{B.31}$$

First, $\|P_x\|_{L^2_{x,z}\to L^2_{x,z}} \leq 3$. Moreover, $\partial_z^k(P_x\phi) = P_x\partial_z^k\phi$ for $k \in \{1,2\}$. Hence $\|P_x\|_{L^2_xH^2_z\to L^2_xH^2_z} \leq 3$. Eventually, $\|z\partial_x(Q_x\phi)\|_{L^2((-1,2)\times(-1,1)} \leq 3\|z\partial_x\phi\|_{L^2((0,1)\times(-1,1)}$, so that

$$\|z\partial_x(P_x\phi)\|_{L^2} \leq 3\|z\partial_x\phi\|_{L^2} + 2\|\chi'\|_{L^\infty}\|\phi\|_{L^2}.$$

Thus P_x defines a continuous extension operator from $Z^0((0,1)\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,1))$.

We now construct a continuous upwards vertical extension operator P_+ from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$. We proceed, as classical (see e.g. [4]), by considering a weighted linear combination of rescaled reflections. For $\phi \in Z^0(\mathbb{R}\times(-1,1))$, $x\in\mathbb{R}$ and $z\in(-1,\infty)$, let

$$(Q_+\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z\in(-1,1),\\ 3\phi(x,2-z) - 2\phi(x,3-2z) & \text{if } z\in(1,2),\end{cases} \tag{B.32}$$

$$(P_+\phi)(x,z) := \chi_+(z)(Q_+\phi)(x,z), \tag{B.33}$$

where $\chi_+ \in C^\infty(\mathbb{R};[0,1])$ is such that $\chi_+\equiv 1$ on $(-1,1)$ and $\operatorname{supp}\chi_+ \subset (-2,1+\frac14)$. The chosen coefficients ensure that both $Q_+\phi$ and $\partial_z(Q_+\phi)$ are continuous at $z=1$. Hence $P_+\phi \in L^2_xH^2_z$ and

$$\|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,+\infty))} = \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,1))} + \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(1,+\infty))} \leq C_+\|\phi\|_{L^2_xH^2_z}, \tag{B.34}$$

for some constant C_+ depending only on $\|\chi_+\|_{W^{2,\infty}}$. Moreover, using that $\chi(z)=0$ for $z>1+\frac14$,

$$\begin{aligned}\|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,+\infty))} &= \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,1+\frac14))}\\ &\lesssim \|\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}\\ &\lesssim \|z\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}.\end{aligned} \tag{B.35}$$

Hence P_+ is a continuous extension operator from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$.

The extension for $z<-1$ is performed in a similar fashion and left to the reader. □

B.4 Embeddings

We collect in this paragraph various embedding results used throughout the paper.

B.4.1 Embedding of the Pagani space Z^0 in $H^{2/3}_xL^2_z$

We start with an easy one dimensional inequality.

Lemma B.7. *For $\psi\in C^\infty_c(\mathbb{R})$,*

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.36}$$

Proof. On the one hand, for $|z|\geq 1$,

$$\int_{|z|\geq1}\psi^2 \leq \|z\psi\|^2_{L^2}. \tag{B.37}$$

On the other hand, for every $(z_0,z)\in(-2,2)$,

$$|\partial_z\psi(z)| \leq |\partial_z\psi(z_0)| + 2\|\partial_{zz}\psi\|_{L^2}. \tag{B.38}$$

Moreover, by classical Sobolev embeddings,

$$\|\partial_z\psi\|_{L^2(1,2)} \lesssim \|\psi\|_{L^2(1,2)} + \|\partial_{zz}\psi\|_{L^2(1,2)} \leq \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.39}$$

Thus, integrating (B.38) for $z_0 \in (1,2)$,

$$\|\partial_z\psi\|_{L^\infty(-2,2)} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.40}$$

Now, writing $\psi(z) = \psi(z_0) + \int_{z_0}^z \psi'$ and integrating for $z_0 \in (1,2)$ yields

$$\|\psi\|_{L^2(-1,1)} \lesssim \|\psi\|_{L^2(1,2)} + \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}, \tag{B.41}$$

which concludes the proof. □

We then turn towards the proof of the key result $Z^0(\mathbb{R}^2) \hookrightarrow H^{2/3}_x L^2_z(\mathbb{R}^2)$.

Proof of Proposition 1.7. Let $\psi \in C^\infty_c(\mathbb{R})$. By Lemma B.7, one has

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.42}$$

Using standard dimensional analysis arguments (e.g. by introducing the rescaled function $\psi_\lambda : z \mapsto \psi(\lambda z)$ for $\lambda > 0$ and optimizing the choice of λ), one deduces from (B.42) that

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2}^{\frac{2}{3}} \|\partial_{zz}\psi\|_{L^2}^{\frac{1}{3}}. \tag{B.43}$$

Let $\phi \in C^\infty_c(\mathbb{R}^2)$. Let $\hat\phi(\xi, z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then using (B.43) and Hölder's inequality,

$$\begin{aligned}
\|\phi\|^2_{H^{2/3}_x L^2_z} &= \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{2}{3}} |\hat\phi(\xi,z)|^2 \,\mathrm{d}\xi\,\mathrm{d}z \\
&\lesssim \|\phi\|^2_{L^2} + \int_{\mathbb{R}} |\xi|^{\frac{4}{3}} \|z\hat\phi(\xi,z)\|^{\frac{4}{3}}_{L^2_z} \|\partial_{zz}\hat\phi(\xi,z)\|^{\frac{2}{3}}_{L^2_z} \,\mathrm{d}\xi \\
&\lesssim \|\phi\|^2_{L^2} + \left(\int_{\mathbb{R}^2} |\xi|^2 z^2 |\hat\phi(\xi,z)|^2 \,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |\partial_{zz}\hat\phi(\xi,z)|^2 \,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac{1}{3}} \\
&\lesssim \|\phi\|^2_{L^2} + \|z\partial_x\phi\|^{\frac{4}{3}}_{L^2} \|\partial_{zz}\phi\|^{\frac{2}{3}}_{L^2}.
\end{aligned} \tag{B.44}$$

Hence $\|\phi\|_{H^{2/3}_x L^2_z} \lesssim \|\phi\|_{Z^0}$. This concludes the proof, by density of $C^\infty_c(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$. □

B.4.2 Embedding of the Baouendi–Grisvard space $\mathcal{B}$ in $H^{1/3}_x L^2_z$

Once again, we start with a one-dimensional inequality of Hardy type.

Lemma B.8. *For $\phi \in L^2(0,1)$,*

$$\int_0^1 \left(\frac{1}{z^2}\int_0^z s\phi(s)\,\mathrm{d}s\right)^2 \mathrm{d}z \leq \frac{4}{5}\|\phi\|^2_{L^2}. \tag{B.45}$$

Proof. For $z \in (0,1)$, by the Cauchy–Schwarz inequality

$$\left(\int_0^z s\phi(s)\,\mathrm{d}s\right)^2 \le \left(\int_0^z \phi^2(s)s^{\frac{1}{2}}\,\mathrm{d}s\right)\left(\int_0^z s^{2-\frac{1}{2}}\,\mathrm{d}s\right) = \frac{2}{5}z^{\frac{5}{2}}\left(\int_0^z \phi^2(s)s^{\frac{1}{2}}\,\mathrm{d}s\right). \tag{B.46}$$

Hence, by Fubini,

$$\begin{aligned}\int_0^1 \left(\frac{1}{z^2}\int_0^z s\phi(s)\,\mathrm{d}s\right)^2 \mathrm{d}z &\le \frac{2}{5}\int_0^1 z^{-4+\frac{5}{2}}\left(\int_0^z \phi^2(s)s^{\frac{1}{2}}\,\mathrm{d}s\right)\mathrm{d}z\\ &= \frac{2}{5}\int_0^1 \phi^2(s)s^{\frac{1}{2}}\left(\int_s^1 z^{-\frac{3}{2}}\,\mathrm{d}z\right)\mathrm{d}s\\ &= \frac{2}{5}\int_0^1 \phi^2(s)s^{\frac{1}{2}}\left(2(s^{-\frac{1}{2}}-1)\right)\mathrm{d}s\\ &= \frac{4}{5}\int_0^1 \phi^2(s)(1-s^{\frac{1}{2}})\,\mathrm{d}s,\end{aligned} \tag{B.47}$$

which implies (B.45). □

We then turn towards the proof of the embedding.

Proof of Lemma 1.13. ***Step 1.*** *Extension to a compactly supported function in* $(x_0,x_1)\times\mathbb{R}$.

Let $u \in L^2((x_0,x_1), H_0^1(-1,1))$ such that $z\partial_x u \in L^2((x_0,x_1)), H^{-1}(-1,1))$. We first extend u to $(x_0,x_1)\times(-3,3)$ by setting, for all $x \in (x_0,x_1)$ and $z' \in (0,2)$,

$$\begin{aligned}u(x,1+z') &= \quad -u(x,1-z'),\\ u(x,-1-z') &= -u(x,-1+z').\end{aligned}$$

It is clear that the above extension belongs to $L^2((x_0,x_1),H_0^1(-3,3))$, and we further extend u by zero on $(x_0,x_1)\times\{z\in\mathbb{R}, |z|\ge 3\}$. We then take $\chi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset (-3/2,3/2)$, and $\chi \equiv 1$ on $(-1,1)$, and we prove that $u\chi \in \mathcal{B}((x_0,x_1)\times\mathbb{R})$. Using a partition of unity, we write $\chi = \chi_{-1}+\chi_0+\chi_1$, where $\operatorname{supp}\chi_{\pm 1} \subset (\pm 1/2, \pm 3/2)$, and $\operatorname{supp}\chi_0 \subset (-3/4,3/4)$. It is clear that $\chi_0 u \in \mathcal{B}((x_0,x_1)\times\mathbb{R})$, and therefore by symmetry is is sufficient to prove the result for $\chi_1 u$.

Let us take $\phi \in H_0^1((x_0,x_1)\times\mathbb{R})$ be arbitrary, and compute

$$-\int_{x_0}^{x_1}\int_{\mathbb{R}} z\chi_1 u\partial_x\phi.$$

By definition of u on $(x_0,x_1)\times(1,2)$,

$$\begin{aligned}-\int_{x_0}^{x_1}\int_{\mathbb{R}} z\chi_1 u\partial_x\phi &= -\int_{x_0}^{x_1}\int_0^1 zu(x,z)\chi_1(z)\partial_x\phi(x,z)\,\mathrm{d}x\,\mathrm{d}z\\ &\quad + \int_{x_0}^{x_1}\int_0^1 (1+z')u(x,1-z')\chi_1(1+z')\partial_x\phi(x,1+z')\,\mathrm{d}x\,\mathrm{d}z'\\ &= -\int_{x_0}^{x_1}\int_0^1 zu(x,z)\chi_1(z)\partial_x\phi(x,z)\,\mathrm{d}x\,\mathrm{d}z\\ &\quad + \int_{x_0}^{x_1}\int_0^1 \frac{(1+z')}{1-z'}(1-z')u(x,1-z')\chi_1(1+z')\partial_x\phi(x,1+z')\,\mathrm{d}x\,\mathrm{d}z'.\end{aligned}$$

Since $z\partial_x u \in L^2((x_0,x_1),H^{-1}(-1,1))$, we may write $z\partial_x u = f + \partial_z g$, with $f,g \in L^2((x_0,x_1)\times(-1,1))$. Then

$$\begin{aligned}&-\int_{x_0}^{x_1}\int_{\mathbb{R}} z\chi_1 u\partial_x\phi\\ &=\int_{x_0}^{x_1}\int_0^1 (f+\partial_z g)(x,z)\left[\chi_1(z)\phi(x,z)-\frac{2-z}{z}\chi_1(2-z)\phi(x,2-z)\right]\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

The assumptions on $\operatorname{supp}\chi_1$ ensure that the function in brackets belongs to $L^2((x_0,x_1),H^1_0(0,1))$. We conclude that for all $\phi\in H^1_0((x_0,x_1)\times\mathbb{R})$,

$$\left|-\int_{x_0}^{x_1}\int_{\mathbb{R}} z\chi_1 u\partial_x\phi\right| \lesssim \|u\|_{\mathcal{B}((x_0,x_1)\times(0,1))}\|\phi\|_{L^2_x H^1_z}.$$

It follows that $z\chi_1\partial_x u\in L^2((x_0,x_1),H^1(\mathbb{R}))$, and

$$\|\chi u\|_{\mathcal{B}((x_0,x_1)\times\mathbb{R})}\lesssim \|u\|_{\mathcal{B}((x_0,x_1)\times(0,1))}.$$

Step 2. *The vertical anti-derivative of χu belongs to Z^0.*

We now work with the extension of the previous step, and we set $U:=-\int_z^\infty \chi u$. Let us prove that $U\in Z^0((x_0,x_1)\times\mathbb{R}_+)$. Since $\partial_z^2 U=\partial_z(\chi u)\in L^2((x_0,x_1)\times\mathbb{R}_+)$, it suffices to prove that $z\partial_x U\in L^2((x_0,x_1)\times\mathbb{R}_+)$. Hence we take $\phi\in L^2((x_0,x_1)\times\mathbb{R}_+)$ arbitrary, and we compute, after observing that U is supported in $\{z\le 3/2\}$,

$$\begin{aligned}\int_{x_0}^{x_1}\int_0^\infty s\partial_x U(x,s)\phi(x,s)\,\mathrm{d}x\;\mathrm{d}s &= -\int_{x_0}^{x_1}\int_0^\infty s\left(\int_s^{3/2}\partial_x\chi u(x,z)\,\mathrm{d}z\right)\phi(x,s)\,\mathrm{d}s\,\mathrm{d}x\\ &= -\int_{x_0}^{x_1}\int_0^\infty \frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<3/2}\phi(x,s)\,\mathrm{d}s\right)z\partial_x\chi u(x,z)\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

Therefore

$$\|s\partial_x U\|_{L^2((x_0,x_1)\times\mathbb{R}_+)}\lesssim \|\chi u\|_{\mathcal{B}}\sup_{\substack{\phi\in L^2,\\ \|\phi\|_{L^2}\le 1}}\left\|\frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<3/2}\phi(x,s)\,\mathrm{d}s\right)\right\|_{L^2((x_0,x_1),H^1_0(0,+\infty))}.$$

The claim therefore follows from the following result, which is postponed to the third and last step:

Lemma B.9. *For all $z_0>0$, there exists a constant C_{z_0} such that for all $\psi\in L^2(\mathbb{R})$,*

$$\left\|\frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<z_0}\psi(s)\,\mathrm{d}s\right)\right\|_{H^1_0(0,+\infty)}\le C_{z_0}\|\psi\|_{L^2(\mathbb{R})}. \tag{B.48}$$

From there, we infer that $U\in Z^0((x_0,x_1)\times\mathbb{R}_+)$, and $\|U\|_{Z^0}\lesssim\|u\|_{\mathcal{B}}$. Using the embedding $Z^0\hookrightarrow H^{1/3}_x H^1_z$, we deduce that $\partial_z U=\chi u\in H^{1/3}_x L^2_z((x_0,x_1)\times\mathbb{R}_+)$. Since $\chi\equiv 1$ on $(-1,1)$, we obtain the desired result.

Step 3. *Proof of* (B.48).

First, note that for all $s\in(0,+\infty)$,

$$\left|\int_0^z s\mathbf{1}_{0<s<z_0}\psi(s)\,\mathrm{d}s\right|\le C_{z_0}\inf(s^{3/2},1)\|\psi\|_{L^2}.$$

Thus we only need to prove that

$$\int_0^{z_0} \left(\frac{1}{z^2} \int_0^z s\psi(s)\,\mathrm{d}s \right)^2 \mathrm{d}z \leq C_{z_0} \|\psi\|_{L^2}^2.$$

This is a rescaling of inequality (B.45) of Lemma B.8. □

C Unconditional regularity away from lateral boundaries

Proof of Lemma 5.4. We proceed by (horizontal) viscous regularization to obtain uniform estimates which pass to the limit.

Let us extend the functions w_0, w_1 into H^2 functions on the whole interval (z_b, z_t), such that $w_0(z_b) = w_b(x_0)$ and $w_1(z_t) = w_t(x_1)$. For $\varepsilon > 0$, consider the solution to the elliptic equation

$$\begin{cases} -\varepsilon \partial_x^2 W^\varepsilon - \partial_z^2(\alpha W^\varepsilon) + z\partial_x W^\varepsilon + \beta \partial_z W^\varepsilon = h, \\ W^\varepsilon|_{x=x_i} = w_i, \\ W^\varepsilon|_{z=z_j} = w_j. \end{cases} \tag{C.1}$$

Let us recall that $|z_b|, z_t \leq z_0$ for some small constant z_0 depending only on α. Classical results on elliptic equations ensure that if z_0 is small enough, (C.1) has a unique solution in $H^1(\Omega)$ for all $\varepsilon > 0$, which satisfies the energy estimate

$$\sqrt{\varepsilon}\|\partial_x W^\varepsilon\|_{L^2} + \|\partial_z W^\varepsilon\|_{L^2} \lesssim \|h\|_{L^2} + \|w_0\|_{H^2_z} + \|w_1\|_{H^2_z} + \|w_t\|_{H^2_x} + \|w_b\|_{H^2_x}. \tag{C.2}$$

Hence W^ε is uniformly bounded in $L^2_x H^1_z$. It follows that $W^\varepsilon \rightharpoonup W$ in $L^2_x H^1_z$, where $W \in Z^0$ is the unique solution to (5.22).

Furthermore, since $h \in L^2(\Omega)$, the compatibility conditions in the corners of the domain and the fact that Ω is a rectangle ensure that $W^\varepsilon \in H^2(\Omega)$ (see [28, Chapter 4]). Hence $\partial_x W^\varepsilon \in H^1(\Omega)$ is a weak solution to

$$-\varepsilon \partial_x^2 \partial_x W^\varepsilon - \partial_z^2(\alpha \partial_x W^\varepsilon) + z\partial_x^2 W^\varepsilon + \beta \partial_z \partial_x W^\varepsilon = \partial_x h + \partial_z^2(\partial_x \alpha W^\varepsilon) - \partial_x \beta \partial_z W^\varepsilon. \tag{C.3}$$

Without loss of generality, we assume from now on that $w_t = w_b = 0$ in order to simplify the computations. This condition can always be satisfied up to a lift of the boundary conditions. Let $\rho(x) := (x - x_0)(x_1 - x)$. We multiply (C.3) by $\rho^2 \partial_x W^\varepsilon$, integrate by parts and obtain

$$\begin{aligned} \varepsilon \int_\Omega \rho^2 (\partial_x^2 W^\varepsilon)^2 + \int_\Omega \alpha \rho^2 (\partial_{xz} W^\varepsilon)^2 \leq \varepsilon \int_\Omega \partial_x(\rho \partial_x \rho)(\partial_x W^\varepsilon)^2 - \int_\Omega \rho^2 \partial_x^2 W^\varepsilon (z \partial_x W^\varepsilon) \\ + (\|\rho \partial_x h\|_{L^2} + \|\rho \partial_x \beta \partial_z W^\varepsilon\|_{L^2}) \|\rho \partial_x W^\varepsilon\|_{L^2} \\ + \|\rho \partial_z(\partial_x \alpha W^\varepsilon)\|_{L^2} \|\rho \partial_{xz} W^\varepsilon\|_{L^2} \\ + \|\partial_z \alpha\|_\infty \|\rho \partial_x W^\varepsilon\|_{L^2} \|\rho \partial_{xz} W^\varepsilon\|_{L^2}. \end{aligned} \tag{C.4}$$

The first integral in the right-hand side is uniformly bounded thanks to (C.2). Let us focus momentarily on the second integral in the right-hand side. Using the equation satisfied by W^ε, we infer

$$\int_\Omega \rho^2 \partial_x^2 W^\varepsilon (z \partial_x W^\varepsilon) = \int_\Omega \rho^2 \partial_x^2 W^\varepsilon \left[h - \beta \partial_z W^\varepsilon + \partial_z^2(\alpha W^\varepsilon) + \varepsilon \partial_x^2 W^\varepsilon \right]. \tag{C.5}$$

We then perform integration by parts in the right-hand side, which is equal to

$$\begin{aligned}\varepsilon\int_\Omega \rho^2(\partial_x^2 W^\varepsilon)^2 + \int_\Omega \rho^2\alpha(\partial_{xz}W^\varepsilon)^2 \\ - 2\int \rho\partial_x\rho\partial_x W^\varepsilon(h-\beta\partial_z W^\varepsilon) - \int \rho^2\partial_x W^\varepsilon\partial_x(h-\beta\partial_z W^\varepsilon) \\ + 2\int \rho\partial_x\rho\partial_{xz}W^\varepsilon\partial_z(\alpha W^\varepsilon) + \int \rho^2\partial_{xz}W^\varepsilon\left(\partial_x(\partial_z\alpha W^\varepsilon) + \partial_x\alpha\partial_z W^\varepsilon\right).\end{aligned} \tag{C.6}$$

Since $\partial_z\beta = 0$ and $w_t = w_b = 0$, we have

$$\int \rho^2\partial_x W^\varepsilon\beta\partial_{xz}W^\varepsilon = 0. \tag{C.7}$$

We also recall that $\|\rho\partial_x W^\varepsilon\|_{L^2} \le z_0\|\rho\partial_x\partial_z W^\varepsilon\|_{L^2}$. Therefore, provided that z_0 is small enough, there exists $C > 0$ such that

$$\begin{aligned}\int_\Omega \rho^2\partial_x^2 W^\varepsilon(z\partial_x W^\varepsilon) \ge& \varepsilon\int_\Omega \rho^2(\partial_x^2 W^\varepsilon)^2 + \frac{1}{2}\int_\Omega \rho^2\alpha(\partial_{xz}W^\varepsilon)^2 \\ &- C\left(\|h\|_{L^2}^2 + \|W^\varepsilon\|_{L^2_xH^1_z}^2 + \|\rho\partial_x h\|_{L^2}\right).\end{aligned} \tag{C.8}$$

Gathering all the terms and using the $L^2_xH^1_z$ estimate on W^ε of (C.2), for z_0 small enough,

$$\varepsilon\int_\Omega \rho^2(\partial_x^2 W^\varepsilon)^2 + \int_\Omega \alpha\rho^2(\partial_x\partial_z W^\varepsilon)^2 \lesssim \|h\|_{L^2}^2 + \|\rho\partial_x h\|_{L^2}^2 + \sum_{i\in\{0,1\}}\|w_i\|_{H^2_z}^2 + \sum_{j\in\{t,b\}}\|w_j\|_{H^2_x}^2. \tag{C.9}$$

Hence $\rho\partial_{xz}W^\varepsilon$ is uniformly bounded in L^2. Passing to the limit, we obtain

$$\|\rho\partial_x W\|_{L^2_xH^1_z} \lesssim \|h\|_{L^2} + \|\rho\partial_x h\|_{L^2} + \sum_{i\in\{0,1\}}\|w_i\|_{H^2_z} + \sum_{j\in\{t,b\}}\|w_j\|_{H^2_x}. \tag{C.10}$$

It follows that $\rho\partial_x W$ is a weak $L^2_xH^1_z$ solution to

$$\begin{cases} z\partial_x(\rho\partial_x W) + \beta\partial_z(\rho\partial_x W) - \partial_z^2(\alpha\rho\partial_x W) = g, \\ \rho\partial_x W|_{\Sigma_i} = 0, \\ \rho\partial_x W|_{z=z_j} = \rho\partial_x w_j \end{cases} \tag{C.11}$$

where

$$g := \rho\partial_x h - \rho\partial_x\beta\partial_z W + \rho\partial_{zz}(\partial_x\alpha W) + (x_0 + x_1 - 2x)z\partial_x W. \tag{C.12}$$

Since $\alpha \in C^3(\overline{\Omega})$ and $W \in Z^0$, $g \in L^2$. Hence, according to Lemma 5.3, we obtain $\rho\partial_x W \in Z^0$. □

List of notations

Functional spaces

$\mathcal{B}$ Baouendi–Grisvard solution space of (1.35), used in Appendix A, *p. 19*

$\mathscr{H}^1_z$ Weighted H^1 space for boundary data with norm (1.32), *p. 17*

$\mathcal{H}_B$ Hilbert space of data triplets for the solvability of the linearized Burgers system, *p. 52*

$\mathcal{H}_{FP}$ Hilbert space with norm (3.5) of data triplets (f, δ_0, δ_1) for the Vlasov–Poisson–Fokker–Planck toy model, *p. 40*

$\mathcal{H}_K$ Hilbert space with norm (2.20) of data triplets (f, δ_0, δ_1) for the shear flow problem, *p. 23*

$\mathcal{H}^\sigma_{\alpha,\beta}$ Space of data tuples (5.26) for the vorticity equation (5.22) at regularity σ, *p. 67*

$\mathscr{L}^2_z$ Weighted L^2 space for boundary data with norm (1.31), *p. 17*

Q^1 Solution space $H^{5/3}_x L^2_y \cap L^2_x H^2_y$, *p. 4*

$\mathcal{X}_B$ Hilbert space with norm (1.4) of data triplets (f, δ_0, δ_1) for nonlinear Burgers, *p. 4*

$\mathcal{X}^\sigma$ Banach space of data $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b)$ with low regularity for the Prandtl problem, defined in (5.16), *p. 64*

$\mathcal{X}^1$ Banach space of data $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b)$ with higher regularity for the Prandtl problem, defined in (5.17), *p. 64*

$\mathcal{Y}_0$ Notation for $L^2(\Omega)$ during discussions on interpolation, *p. 84*

$\mathcal{Y}_1$ Notation for $H^1_x L^2_y$ with $f_{|\Sigma_0 \cup \Sigma_1} = 0$ during discussions on interpolation, *p. 84*

Z^0 Pagani solution space such that u, $z\partial_x u$ and $\partial_{zz} u$ are L^2, with norm (1.33), *p. 17*

Z^1 Solution space such that $u, \partial_x u \in Z^0$, with norm (1.34), *p. 17*

$\mathcal{Z}_B$ Solution space for the Burgers system, *p. 53*

Z^σ Interpolation space $[Z^0, Z^1]$ for fractional Pagani regularity, *p. 39*

Other

δ_i Boundary data at the inflow boundary Σ_i, *p. 4*

Δ_i Boundary data for $\partial_x u$, given by $\Delta_i = (f + \delta_i'')/z$, see (2.9), *p. 22*

Λ_k Angular profile of the k-th explicit singular solution in the half-plane, *p. 29*

Ξ Shorthand for a data triplet $\Xi = (f, \delta_0, \delta_1)$., *p. 46*

Σ_0 Left inflow boundary $\{x_0\} \times (0, 1)$, see Fig. 1, *p. 3*

Σ_1 Right inflow boundary $\{x_1\} \times (-1, 0)$, see Fig. 1, *p. 3*

Σ^P_i Lateral inflow boundaries for the Prandtl system, *p. 7*

$\overline{\Phi^j}$ Dual profiles of Lemma 2.6 involved in orthogonality conditions for the shear flow, *p. 23*

χ_i Cut-off function localized near $(x_i, 0)$, *p. 33*

Ω Physical rectangular domain $(x_0, x_1) \times (-1, 1)$, see Fig. 1, *p. 3*

$\Omega_\pm$ Upper and lower halves of the domain Ω, *p. 17*

Ω_P Physical domain for the resolution of the Prandtl system, *p. 7*

δ_b, δ_t Boundary data for the vorticity on the bottom and top boundaries of Ω_P, *p. 7*

$\overline{\gamma_b}, \overline{\gamma_t}$ Level sets of the function $\mathfrak{u}_{\mathrm{P}}$, *p. 6*

γ_b, γ_t Level sets of the unknown solution u of the Prandtl system, *p. 7*

$\overline{f_i}$ Smooth source term associated with the singular solution $\bar{u}^i_{\mathrm{sing}}$, *p. 33*

ℓ^0, ℓ^1 Linear orthogonality conditions of Definition 5.10 for the solvability of Prandtl at high regularity, *p. 73*

$\overline{\ell^j}$ Linear forms on $\mathcal{H}_K$ giving the orthogonality conditions for the shear flow, *p. 25*

ℓ^2 Additional linear form of Definition 5.7 to reconstruct the velocity from the vorticity, *p. 72*

$\overline{M}$ Invertible matrix relating the singular solutions $\bar{u}^i_{\mathrm{sing}}$ with the dual profiles $\overline{\Phi^j}$, *p. 36*

r Radial-like variable given by $r = (z^2 + x^{\frac{2}{3}})^{\frac{1}{2}}$, *p. 29*

r_i Radial-like variable near $(x_i, 0)$ given by $r_i = (z^2 + |x - x_i|^{\frac{2}{3}})^{\frac{1}{2}}$, *p. 33*

t Angular-like variable given by $t = zx^{-\frac{1}{3}}$, *p. 29*

t_i Angular-like variable near $(x_i, 0)$, given by $t_i = (-1)^i z|x - x_i|^{-\frac{1}{3}}$, *p. 33*

$\bar{u}^i_{\mathrm{sing}}$ Reference singular solution localized near $(x_i, 0)$, *p. 33*

v_k k-th explicit singular solution in the half plane, $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$, *p. 29*

N_B Nonlinearity associated with the Burgers-type system, *p. 52*

$\mathfrak{u}_{\mathrm{P}}$ Reference recirculationg flow for the Prandtl system, *p. 6*

v_b Boundary datum for the vertical velocity on the bottom boundary of Ω_P, *p. 7*

$\mathbb{Y}_{\mathrm{P}}$ Inverse function of the reference flow $\mathfrak{u}_{\mathrm{P}}$, *p. 62*

$\Upsilon[\delta_i]$ Lateral boundary data for the Burgers system after the change of variables, *p. 52*

$\Upsilon^i_{\mathrm{P}}[\delta_i]$ Boundary data on Σ_i for the Prandtl system in the new variables, *p. 63*

$\Upsilon^b_{\mathrm{P}}[\delta_b]$ Boundary data at the bottom for the Prandtl system in the new variables, *p. 63*

$\Upsilon^t_{\mathrm{P}}[\delta_t]$ Boundary data at the top for the Prandtl system in the new variables, *p. 63*

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