A nonlinear forward-backward problem

Anne-Laure Dalibard, Frédéric Marbach, Jean Rax

A nonlinear forward-backward problem

Anne-Laure Dalibard[*†], Frédéric Marbach[†‡], Jean Rax[*]

March 21, 2022

Abstract

We prove existence and uniqueness of strong solutions of the equation $uu_x - u_{yy} = f$ in the vicinity of the linear shear flow, subject to perturbations of the source term and lateral boundary conditions. Since the solutions we consider have opposite signs in the lower and upper half of the domain, this is a forward-backward parabolic problem, which changes type across a critical curved line within the domain. In particular, lateral boundary conditions can be imposed only where the characteristics are inwards.

There are several difficulties associated with this problem. First, the forward-backward geometry depends on the solution itself. This requires to be quite careful with the approximation procedure used to construct solutions. Second, and maybe more importantly, the linearized equations solved at each step of the iterative scheme admit a finite number of singular solutions. This is similar to well-known phenomena in elliptic problems in nonsmooth domains. Hence, the solutions of the equation are regular if and only if the source terms satisfy a finite number of orthogonality conditions. A key difficulty of this work is to cope with these orthogonality conditions during the nonlinear fixed-point scheme. In particular, we are led to prove their stability with respect to the underlying base flow.

[*]Sorbonne Université, Université Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris

[†]École Normale Supérieure, Université PSL, Département de Mathématiques et applications, F-75005, Paris, France

[‡]Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

Contents

1 Introduction

We investigate the existence and uniqueness of sign-changing solutions to the equation

$$u\partial_x u - \partial_{yy} u = f \tag{1.1}$$

in the rectangular domain $\Omega := (x_0, x_1) \times (-1, 1)$, where $x_0 < x_1$ are real parameters and f is an external source term.

A natural solution to (1.1) with a null source term $\mathbb{f} = 0$ is the linear shear flow $\mathbb{u}(x, y) := y$, which changes sign across the horizontal line $\{y = 0\}$. We are interested in strong solutions to (1.1) which are close (with respect to an appropriate norm) to this linear shear flow $\mathbb{u}$. Our purpose is to construct such solutions by perturbing the lateral boundary data $\mathbb{u}_{|x=x_0}(y) = y$ and $\mathbb{u}_{|x=x_1}(y) = y$ or the source term $\mathbb{f} = 0$.

Since such solutions will change sign across a line $\{u = 0\}$ lying within Ω, a key feature of this work is that (1.1) must be seen as a nonlinear forward-backward parabolic problem in the horizontal direction. Thus, to ensure the existence of a solution, one must be particularly careful as to how one enforces these lateral perturbations.

1.1 Statement of the main results

Due to the forward-backward nature of the problem, we must choose the lateral perturbations and the source term in a particular product space. We therefore introduce the vector space

$$\begin{aligned}\mathcal{E} := \Big\{ (f, \delta_0, \delta_1) \in C_c^\infty(\Omega) \times C^\infty([0,1]) \times C^\infty([-1,0]); \quad \delta_i(0) = \partial_y \delta_i(0) = \partial_y^2 \delta_i(0) = 0 \\ \text{and } \delta_i((-1)^i) = \partial_y^2 \delta_i((-1)^i) = 0 \text{ for } i = 0, 1 \Big\}\end{aligned} \tag{1.2}$$

and $\mathcal{H}$, the Hilbert space defined as the completion of $\mathcal{E}$ with respect to the following norm (associated with the corresponding canonical scalar product),

$$\begin{aligned}\|(f, \delta_0, \delta_1)\|_{\mathcal{H}}^2 := \|f\|_{H_x^1 H_y^1}^2 &+ \|\partial_y^3 f\|_{L^2}^2 \\ &+ \|\delta_0\|_{H^5}^2 + \|\delta_1\|_{H^5}^2 \\ &+ \int_0^1 \frac{1}{|y|} (\partial_y^2 \delta_0(y))^2 \, \mathrm{d}y + \int_0^1 |y| \left(\partial_y \frac{\partial_y^2 \delta_0(y)}{y} \right)^2 \mathrm{d}y \\ &+ \int_{-1}^0 \frac{1}{|y|} (\partial_y^2 \delta_1(y))^2 \, \mathrm{d}y + \int_{-1}^0 |y| \left(\partial_y \frac{\partial_y^2 \delta_1(y)}{y} \right)^2 \mathrm{d}y.\end{aligned} \tag{1.3}$$

We establish existence and uniqueness of solutions in the following anisotropic Sobolev space

$$Q^1 := L^2((x_0, x_1); H^5(-1, 1)) \cap H^{5/3}((x_0, x_1); L^2(-1, 1)). \tag{1.4}$$

In particular, for solutions with such regularity, (1.1) holds in a strong sense, almost everywhere and the various boundary conditions hold in the usual sense of traces, almost everywhere. We first state a result concerning the well-posedness in Q^1 of the linear version of (1.1) at the linear shear flow, up to two orthogonality conditions (see comments below). Although equation (1.5) below has been thoroughly investigated, as we recall in Section 1.2 below, we could not find this statement in the existing literature.

Theorem 1. *There exists a vector subspace $\mathcal{H}_{\mathrm{sg}}^{\perp} \subset \mathcal{H}$ of codimension two such that, for each $(f, \delta_0, \delta_1) \in \mathcal{H}$, there exists a solution $u \in Q^1$ to the problem*

$$\begin{cases} y\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{1.5}$$

where $\Sigma_0 := \{x_0\} \times (0,1)$ and $\Sigma_1 := \{x_1\} \times (-1,0)$, if and only if $(f, \delta_0, \delta_1) \in \mathcal{H}_{\mathrm{sg}}^{\perp}$.

Such a solution is unique and satisfies

$$\|u\|_{Q^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}}. \tag{1.6}$$

We emphasize that this result implies that there exist triplets (f, δ_0, δ_1) that can be chosen arbitrarily smooth and compactly supported, and for which there are no Q^1 solutions of (1.5).

Our main result is the following nonlinear generalization for small enough perturbations.

Theorem 2. *There exists a Lipschitz submanifold $\mathcal{M}$ of $\mathcal{H}$ of codimension two, containing 0, such that, for every $(f, \delta_0, \delta_1) \in \mathcal{M}$, there exists a strong solution $u \in Q^1$ to*

$$\begin{cases} (y+u)\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0. \end{cases} \tag{1.7}$$

More precisely, $\mathcal{M}$ is modeled on $\mathcal{H}_{\mathrm{sg}}^{\perp}$ and tangent to it at 0. Such solutions are unique in a small neighborhood of 0 in Q^1 and satisfy the estimate (1.6).

The nonlinear orthogonality conditions are necessary in the sense that there exists $\eta > 0$ such that, if $(f, \delta_0, \delta_1) \in \mathcal{H}$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}} < \eta$ and $u \in Q^1 \cap H_x^2(H_y^1)$ with $\|u\|_{Q^1} + \|u\|_{H_x^2 H_y^1} < \eta$ is a solution to (1.7), then $(f, \delta_0, \delta_1) \in \mathcal{M}$.

In the statement above, the condition that the data (f, δ_0, δ_1) belong to the manifold $\mathcal{M}$ is the nonlinear equivalent of the orthogonality conditions from Theorem 1. We emphasize that this is by no means a technical restriction which could be lifted, but actually a necessary condition to solve the equation with smooth solutions, as the second part of Theorem 2 points out. A key difficulty lies in the fact that these orthogonality conditions *depend on the solution itself*. Hence, tracking the dependency of these conditions with respect to the unknown function u is a key part of our result. We will comment more abundantly on these points in the following sections.

1.2 Comments and previous results

We start with a few comments on our main results and recall related known results.

Problem (1.5), involving the operator $y\partial_x - \partial_{yy}$, can be seen as a particular case of the class of "degenerate second-order elliptic-parabolic linear equations", also referred to as "second-order equations with nonnegative characteristic form" (as opposed to positive definite ones), "forward-backward" or "mixed type" problems. They date back at least to Gevrey [16].

Problem (1.5) itself, as well as these wide classes of equations, has received a lot of attention and has been investigated under different aspects: with variable coefficients or other geometries [14, 34], higher-order operators [28, Ch. 3, 2.6], abstract operators [6, 35], explicit representation formulas [15, 19] or with a focus on numerical analysis [2].

On weak solutions for the linear problem. It is well-known since the work of Fichera [14] that weak solutions to (1.5) with $L^2_x H^1_y$ regularity exist. For general boundary-value problems for elliptic-parabolic second-order equations, one owes to Fichera the systematic separation of the boundary of the domain in three parts: a "noncharacteristic" part, where one sets either Dirichlet or Neumann boundary conditions (here $y = \pm 1$), an "inflow" part, where one sets a Dirichlet boundary condition (here $\Sigma_0 \cup \Sigma_1$) and an "outflow" part, where one cannot set a boundary condition (here, the two sets $\{x_0\} \times (-1, 0)$ and $\{x_1\} \times (0, 1)$).

Baouendi and Grisvard [5] proved the uniqueness of weak solutions to (1.5) with $L^2_x H^1_y$ regularity, by means of a trace theorem and a Green identity (see Appendix A).

On strong solutions for the linear problem. There is an extensive literature on the regularity of solutions to degenerate elliptic-parabolic linear equations, and the question of whether weak solutions are strong. We refer the reader in particular to the book [30] by Oleĭnik and Radkevič. Generally speaking, depending on the exact setting considered, it is quite often possible to prove that the solutions to such equations are regular far from the boundaries of the domain and/or from the regions where the characteristic form is not positive definite. A nice example is Kohn and Nirenberg's work [24], which proves a very general regularity result. A key assumption of their work is that the "outflow" part of the boundary does not meet the "noncharacteristic" and "inflow" parts (i.e. they are in disjoint connected components of $\partial\Omega$). Hence, it does not apply to (1.5), and hints towards a difficulty near the points $(x_0, 0)$ and $(x_1, 0)$.

In a series of papers [32, 33, 34], Pagani proved the existence of strong solutions to (1.5) (and related equations). More precisely, Pagani proved the existence of solutions such that $y\partial_x u$ and $\partial_{yy} u$ belong to $L^2(\Omega)$. Moreover, he determined the exact regularity of the various traces of such solutions (trace of u at $x = x_i$, at $y = \pm 1$ or $y = 0$, and trace of $\partial_y u$ at $y = 0$). These maximal regularity results play a key role in our analysis and motivate the functional spaces we introduce in Section 1.5.

On orthogonality conditions for higher regularity. As noted by Pyatkov in [36], for such forward-backward problems: "*as a rule, there is no existence theorems for smooth solutions without some additional orthogonality-type conditions on the problem data*". Even for the linear problem (1.5), there have been very few works concerning higher regularity (than Pagani's framework) in the whole domain. Most of the works focused on higher regularity (such as [36]) involve weighted estimates which entail regularity within the domain but not near the critical points $(x_i, 0)$. An attempt for global regularity is Goldstein and Mazumdar's work [17, Theorem 4.2] albeit the proof seems incomplete (see Proposition 2.7 below and its proofs for more details).

A misleading aspect is that it is quite easy, assuming the existence of a smooth solution, to prove *a priori* estimates at any order. Such phenomenons are usual in the theory of elliptic problems in domains with corners or mixed Dirichlet-Neumann boundary conditions (see for instance [20]). Let us give an illustration of such a phenomenon in a close context. For a source term $f \in C^\infty_c(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ \partial_x u(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ u(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.8}$$

It is classical that such a system has a unique weak solution $u \in H^1(\Omega)$. Moreover, assuming that

u is smooth enough, $v := \partial_x u$ satisfies

$$\begin{cases} -\Delta v = \partial_x f & \text{in } \Omega, \\ \partial_x v(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ v(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ v(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.9}$$

For such systems, one has $\|v\|_{H^1} \lesssim \|\partial_x f\|_{L^2}$. Hence $\|\partial_{xx} u\| \lesssim \|\partial_x f\|_{L^2}$, and, using the equation, $\|u\|_{H^2} \lesssim \|f\|_{H^1}$. So one has an *a priori* estimate. However, it is known that there exist source terms for which the unique weak solution $u \in H^1$ does not enjoy H^2 regularity (see [20, Ch. 4]). The key point is that, when reconstructing u from the solution v to (1.9), say by setting $u(x, y) := \int_{x_0}^{x} v(x', y)\, \mathrm{d}x'$ for $y > 0$ and $u(x, y) := \int_{x_1}^{x} v(x', y)\, \mathrm{d}x'$ for $y < 0$, there might be a discontinuity of u or $\partial_y u$ across the line $y = 0$. Such discontinuities prevent u from solving (1.8). Preventing these discontinuities requires that the source term satisfies appropriate orthogonality conditions.

Of course, such orthogonality conditions make it very difficult to obtain results at a nonlinear level. Even for elliptic problems in polygonal domains, we are not aware of nonlinear results coping with orthogonality conditions. For instance [20, Section 8.1] focuses on a case where there is no orthogonality condition. Tracking the evolution of the orthogonality conditions during the nonlinear scheme is one of the main difficulties of this work (see Sections 4 and 5.1). At the nonlinear level, these orthogonality conditions are translated in Theorem 2 as the fact that the data must lie within the manifold $\mathcal{M}$, which can be pictured as a perturbation of the linear subspace $\mathcal{H}_{\mathrm{sg}}^{\perp}$ of data satisfying the orthogonality conditions for the linear problem.

Let us also emphasize that if one wishes to construct solutions with even stronger regularity, say $u \in H_x^k H_y^1$ with $k \geq 1$, then generically, one needs to ensure that $2k$ orthogonality conditions are satisfied by the source terms.

On entropy solutions. An entirely different approach to solve (1.1) is to look directly for weak solutions to the nonlinear problem, for example using an entropy formulation. The regularity for such solutions is $u \in L^{\infty}_{x,y} \cap L^2_x H^1_y$ and they are typically obtained as limits of solutions u^{ε} to regularized versions of (1.1), e.g. $u^{\varepsilon}\partial_x u^{\varepsilon} - \partial_{yy} u^{\varepsilon} - \varepsilon \partial_{xx} u^{\varepsilon} = 0$. Such solutions satisfy both the equation and the lateral boundary conditions only in the weak sense of appropriate inequalities linked with "entropy pairs". Given $\delta_0, \delta_1 \in L^{\infty}(-1, 1)$, the existence of an entropy solution to

$$\begin{cases} u\partial_x u - \partial_{yy} u = 0, \\ u_{|x=x_i} = \delta_i, \\ u_{|y=\pm 1} = 0 \end{cases} \tag{1.10}$$

was first proved in [8]. More recently, Kuznetsov proved in [25] the uniqueness of the entropy solution to (1.10), determined in which sense the lateral boundary conditions were satisfied and proved a stability estimate of the form

$$\|u - \tilde{u}\|_{L^1(\Omega)} \lesssim \|\delta_0 - \tilde{\delta}_0\|_{L^1(-1,1)} + \|\delta_1 - \tilde{\delta}_1\|_{L^1(-1,1)}. \tag{1.11}$$

In particular, this stability estimate guarantees that one can construct sign-changing solutions in the vicinity of the linear shear flow.

However, an important drawback of the entropy formulation is that the boundary conditions are only satisfied in a very weak sense. Although functions in $L^{\infty}_{x,y} \cap L^2_x H^1_y$ don't have classical

traces at $x = x_i$, one can give a weak sense to the traces using the equation (see [26] for more details). Unfortunately, it is expected that these weak traces do not coincide with the supplied boundary data on sets of positive measure.

In contrast, since the solutions we construct in this work have (at least) $H^1_x L^2_y$ regularity, they have usual traces $u_{|\Sigma_i} \in L^2(\Sigma_i)$ and the equalities $u_{|\Sigma_i} = \delta_i$ hold in $L^2(\Sigma_i)$, so almost everywhere.

On the choice of the linear shear flow. We choose to study the well-posedness of (1.1) in the vicinity of the linear shear flow to lighten the computations. Nonetheless, we expect that our results and proofs can be extended to study the well-posedness of (1.1) in the vicinity of any sufficiently regular reference flow $\mathfrak{u}$ changing sign across a single line $\{\mathfrak{u} = 0\}$, satisfying $\mathfrak{u}_y \geq c_0 > 0$ in Ω (so that (1.5) is the correct toy model) and with $\|\mathfrak{u}_x\|_\infty$ small enough (to ensure *a priori* estimates).

Moreover, taking a step further in the modelization of recirculation problems in fluid mechanics (see Section 1.3), we also expect that our approach could be extended to an unbounded domain of the form $(x_0, x_1) \times (0, +\infty)$, with a reference flow such that $\mathfrak{u}_{|y=0} = 0$, $\mathfrak{u} < 0$ below some critical line and then $\mathfrak{u} > 0$ above, with $\mathfrak{u}$ having some appropriate asymptotic behavior as $y \to +\infty$. In such a setting, the Poincaré inequalities in the vertical direction that we use here should probably be replaced with well-suited Hardy inequalities.

On the conditions $\delta_0(0) = \delta_1(0) = 0$ for fixed end-points. It is an important feature of our work that we are able to enforce precisely the exact endpoints of the (curved) line $\{u = 0\}$ at $x = x_0$ and $x = x_1$. Theorem 2 is stated for perturbations which satisfy $\delta_i(0) = 0$ (see (1.2)), so that the full boundary data $y + \delta_i(y)$ changes sign exactly at $y = 0$. This choice simplifies the definition of the submanifold $\mathcal{M}$ of boundary data for which we are able to solve the problem. Nevertheless, given y_0, y_1 sufficiently close to 0 and δ_0, δ_1 such that $y + \delta_i(y)$ changes sign at $y = y_i$, we expect that a similar existence result holds, provided that the perturbations are chosen in an appropriate modification of $\mathcal{M}$, with suitable modifications to the norm (1.3) and where, in (1.7), the definitions of Σ_i are generalized by setting $\Sigma_i := \{(x_i, y);\ (-1)^i(y + \delta_i(y)) > 0\}$.

On the boundary conditions $u_{|y=\pm 1} = 0$. These boundary conditions are merely chosen to simplify the statements and lighten the computations, since they guarantee that $(x, y) \mapsto (x, y + u(x, y))$ is a well-defined global change of variables mapping Ω to itself (see Section 3.1). Straightforward modifications would ensure the well-posedness of the considered systems with sufficiently regular non-zero boundary data for $u_{|y=\pm 1}$.

On the compatibility conditions $\delta_i((-1)^i) = 0$ and $\delta_i''(0) = \delta_i''((-1)^i) = 0$. These conditions are classical compatibility conditions for solutions to elliptic-parabolic equations. For example, the condition $\delta_0(1) = 0$ is intended to match the condition $u_{|y=1} = 0$. The condition $\delta_0''(0) = 0$ comes from the equation. Indeed, if u is a sufficiently regular solution with $f(x_0, 0) = 0$, the equality $\partial_{zz} u = z \partial_x u$ at $(x_0, 0)$ enforces $\partial_{zz} u(x_0, 0) = 0$, so $\delta_0''(0) = 0$. The condition $\delta_0''(1) = 0$ stems similarly from the equation and the fact that $\partial_x u_{|y=1} = 0$. It corresponds to a classical parabolic regularity compatibility condition. Note that we actually require the cancellation of two additional boundary conditions, namely $\delta_0'(0) = \delta_1'(0) = 0$. We believe that these extra assumptions are technical, and could be removed.

1.3 Motivation from recirculation problems in fluid mechanics

Our original motivation stems from fluid mechanics. Indeed, the stationary Prandtl equation, which describes the behavior of a fluid with small viscosity in the vicinity of a wall, reads

$$\begin{cases} u\partial_x u + v\partial_y u - \partial_{yy} u = -\partial_x p_E, \\ u_{|y=0} = v_{|y=0} = 0, \\ \lim_{y\to\infty} u(x,y) = u_E(x), \end{cases} \tag{1.12}$$

where $u_E(x)$ (resp. $p_E(x)$) is the trace of an outer Euler flow (resp. pressure) on the wall, and satisfies $u_E\partial_x u_E = -\partial_x p_E$.

As long as u remains positive, (1.12) can be seen as a nonlocal, nonlinear diffusion type equation, the variable x being the evolution variable. Using this point of view, Oleinik (see e.g. [31, Theorem 2.1.1]) proved the local well-posedness of a solution of (1.12) when the equation (1.12) is supplemented with a boundary data $u_{|x=0} = u_0$, where $u_0(y) > 0$ for $y > 0$ and such that $u_0'(0) > 0$. Let us mention that such positive solutions exist globally when $\partial_x p_E \leq 0$, but are only local when $\partial_x p_E > 0$. More precisely, when $\partial_x p_E = 1$ for instance, for a large class of boundary data u_0, there exists $x^* > 0$ such that $\lim_{x\to x^*} u_y(x,0) = 0$. Furthermore, the solution may develop a singularity at $x = x^*$, known as Goldstein singularity. The point x^* is called the separation point: intuitively, if the solution of Prandtl exists beyond x^*, then it must have a negative sign close to the boundary (and therefore change sign). We refer to the seminal works of [18] and Stewartson [39] for formal computations on this problem. A first mathematical statement describing separation was given by Weinan E in [13] in a joint work with Luis Cafarelli, but the complete proof was never published. The first author and Nader Masmoudi then gave a complete description of the formation of the Goldstein singularity [10]. The recent work [38] indicates that this singularity holds for a large class of initial data.

Because of this singularity, it is actually unclear that the Prandtl system is a relevant physical model in the vicinity of the separation point x^*, because the normal velocity v becomes unbounded at $x = x^*$. Consequently, more refined models, such as the triple deck system (see [27] for a presentation of this model, and [23, 11] for a recent mathematical analysis of its time-dependent version), were designed specifically to replace the Prandtl system with a more intricate boundary layer model in the vicinity of the separation point. However, beyond the separation point, i.e. for $x > x^*$, it is expected that the Prandtl system becomes valid again, but with a changing sign solution.

To the best of our knowledge, the well-posedness of (1.12) when the solution u is allowed to change sign has seldom been investigated. Such solutions are called "recirculating solutions", and the zone where $u < 0$ is called a recirculation bubble, the usual convention being that $u_E(x) > 0$, so that the flow is going forward far from the boundary.

Let us mention however the very recent preprint [22] by Sameer Iyer and Nader Masmoudi, in which the authors prove *a priori* estimates in high regularity norms for smooth solutions of the Prandtl equation (1.12) in the vicinity of explicit self-similar recirculating flows, called Falkner-Skan profiles. The latter are given by

$$u(x,y) = x^m f'(\zeta), \tag{1.13}$$

$$v(x,y) = -y^{-1}\zeta f(\zeta) - \frac{m-1}{m+1} y^{-1}\zeta^2 f'(\zeta), \tag{1.14}$$

where $\zeta := (\frac{m+1}{2})^{\frac{1}{2}} y x^{\frac{m-1}{2}}$ is the self-similarity variable, m is a real parameter and f is the solution

to the Falkner-Skan equation

$$f''' + ff'' + \beta(1 - (f')^2) = 0, \tag{1.15}$$

where $\beta = \frac{2m}{m+1}$, subject to the boundary conditions $f(0) = f'(0) = 0$ and $f'(+\infty) = 1$. Such flows correspond to an outer Euler velocity field $u_E(x) = x^m$. For some particular values of m (or, equivalently, β), these formulas provide physical solutions to (1.12) which exhibit recirculation (see [9]).

Obtaining *a priori* estimates for recirculating solutions of the Prandtl system (1.12) is very difficult. This important step was achieved by Sameer Iyer and Nader Masmoudi in [22]. In the present paper, we have chosen to focus on a different type of difficulty, and to consider the toy-model (1.1), which differs from (1.12) through the lack of the nonlinear transport term $v\partial_y u$ and its associated difficulties (nonlocality, loss of derivative) and the exclusion of the zones close to the wall and far from the wall. For the model (1.1), *a priori* estimates are easy to derive, see [37, Chapter 4]. The difficulty lies elsewhere, as explained previously. Indeed, in order to construct a sequence of approximate solutions satisfying the *a priori* estimates, we need to ensure that the orthogonality conditions are satisfied all along the sequence. The core of the proof is to keep track of these orthogonality conditions, and to analyze their dependency on the sequence itself.

1.4 Scheme of proof and plan of the paper

Uniqueness of solutions is fairly easy to prove. For the linear problem (1.5), uniqueness already holds at the level of weak solutions (see Proposition 2.2 and Appendix A). For the nonlinear problem, uniqueness is straightforward since we are considering strong solutions (see Section 5.2). Therefore, the main subject of this paper is the proof of the existence of solutions for the nonlinear problem (1.7).

A first natural idea would be to prove existence thanks to a nonlinear scheme relying on the linear problem (1.5). For example, one could wish to construct a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ by setting $u_0 := 0$ (or any other initial guess) and solving

$$\begin{cases} y\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f - u_n \partial_x u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{1.16}$$

However, this strategy fails. The key point is that the right-hand side contains a full tangential derivative of u_n, whereas the operator $y\partial_x - \partial_{yy}$ only yields a gain of $2/3$ of a derivative in this direction (more precisely, see Proposition 1.2 and Proposition 2.4). Hence, this nonlinear scheme would exhibit a "loss of derivative", preventing us to prove a uniform bound on the sequence $(u_n)_{n\in\mathbb{N}}$.

Another drawback of this scheme is that it would not translate well to a setting where one does not assume $\delta_i(0)$. Indeed, in such a case, the inflow boundaries of the problem with the perturbed data $y + \delta_i(y)$ would not match the inflow boundaries of the linear problem (1.5).

Hence we will rather construct solutions of (1.1) through another iterative scheme, which does not rely directly on (1.5). In a way, the issues stemming from the linear scheme (1.16) come from the following fact: in equation (1.7), the geometry of the problem is dictated by the line where the whole solution $y + u$ changes sign. On the contrary, in (1.5), the geometry of the problem follows the cancellation of y. Keeping this in mind, we will rather rely on the following linearized equation

around a base flow perturbation $\bar{u}$, where $\bar{u}$ is a small perturbation of the shear flow y in Q^1

$$\begin{cases} \bar{u}\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0. \end{cases} \tag{1.17}$$

The well-posedness of such linear systems is investigated in Section 3. Exactly as Theorem 1 requires orthogonality conditions to ensure the existence of regular solutions to (1.5), the existence of regular solutions to (1.17) is subject to perturbed orthogonality conditions (see Section 3.4).

More precisely, we will construct a sequence $(u_n)_{n\in\mathbb{N}}$ solving the following iterative scheme

$$\begin{cases} (y+u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1}, \\ (u_{n+1})_{|\Sigma_i} = \delta_i^{n+1}, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{1.18}$$

For this scheme, we are able to prove a uniform bound for u_n in an appropriate space Q^1 and the convergence of the sequence in an interpolation space $Q^{1/2}$ (see (1.29) and (1.30)). This scheme is similar to the one used to construct solutions of quasilinear symmetric hyperbolic systems, see for instance [4, Section 4.3].

In (1.18), $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ are appropriate perturbations of the data (f, δ_0, δ_1) tailored to satisfy the orthogonality conditions associated with the problem (1.17) (for $\bar{u} = u_n$). This is a key difficulty of this work (see Sections 4 and 5.1). In particular, in order to allow the sequence u_n to converge, we must prove that these perturbations also converge, which amounts to prove that the orthogonality conditions for (1.17) depend continuously (and even in a Lipschitz manner) on $\bar{u}$, *for the same topology as the one within which we will prove the convergence of the sequence* u_n.

The plan of this work is as follows. As a preliminary, we introduce in Section 1.5 the functional spaces we will use. First, we study the linear problem (1.5) in Section 2, leading to Theorem 1, and prove that the two orthogonality conditions we expose are indeed nonvoid. Second, in Section 3, we study linearized problems of the form (1.17). The main task is to derive the modified orthogonality conditions and prove their existence. Third, we prove the stability with respect to the underlying flow $\bar{u}$ of the orthogonality conditions in Section 4. Then, in Section 5, we turn to the nonlinear problem for which we prove the existence of solutions in Section 5.1 using the scheme mentioned above, then uniqueness in Section 5.2 and the necessity of the nonlinear orthogonality conditions in Section 5.3. This concludes the proof of Theorem 2.

Eventually, in Appendix A, we prove the uniqueness of weak solutions to various linear problems involved in Section 3, by adapting an argument due to Baouendi and Grisvard [5]. In Appendix B, we prove various technical results of functional analysis that we use throughout the paper.

1.5 Functional spaces and interpolation results

1.5.1 Notations

Throughout this work, an assumption of the form "$A \ll 1$" will mean that there exists a constant $c > 0$, depending only on Ω such that, if $A \leq c$, the result holds. Similarly, a conclusion of the form "$A \lesssim B$" will mean that there exists a constant $C > 0$, depending only on Ω, such that the estimate $A \leq CB$ holds. For ease of reading, we will not keep track of the value of these constants, mostly linked with embeddings of functional spaces.

We will often use the notations $\Omega_\pm := \Omega \cap \{\pm z > 0\}$.

1.5.2 Trace spaces for the lateral boundaries

For the traces of the solutions to (1.5) or (1.7) at $x = x_0$ and $x = x_1$, we will need the following spaces, due to [33, 34]. We define $\mathscr{L}^2_z(-1,1)$ as the completion of $L^2(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{L}^2_z} := \left(\int_{-1}^{1} |z|\psi^2(z)\,\mathrm{d}z\right)^{\frac{1}{2}} \tag{1.19}$$

and $\mathscr{H}^1_z(-1,1)$ as the completion of $H^1_0(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{H}^1_z} := \|\psi\|_{\mathscr{L}^2_z} + \|\partial_z\psi\|_{\mathscr{L}^2_z}. \tag{1.20}$$

1.5.3 Trace spaces for horizontal cuts

When considering the restriction of a solution to (1.5) or (1.7) at some altitude $z \in (-1,1)$, we will sometimes need the following spaces in the horizontal direction.

The Lions-Magenes space $H^{1/2}_{00}(x_0,x_1)$ is defined as the completion of $H^1_0(x_0,x_1)$ with respect to the following norm

$$\|\phi\|_{H^{1/2}_{00}} := \|\phi\|_{H^{1/2}} + \left(\int_{x_0}^{x_1} \frac{|\phi(x)|^2}{|x-x_0||x_1-x|}\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.21}$$

It is also the interpolation space $[H^1_0(x_0,x_1), L^2(x_0,x_1)]_{\frac{1}{2}}$ (see [29, Théorème 11.7, Chapter 1]), or the subspace of functions of $H^{1/2}(x_0,x_1)$ of which the extension by 0 is in $H^{1/2}(\mathbb{R})$. By [29, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0,x_1)$ to $(H^{1/2}_{00}(x_0,x_1))'$.

We will also need one-sided versions of this space, for functions "vanishing" only at the left side $x = x_0$ (say $H^{1/2}_{00_l}(x_0,x_1)$) or only at the right side $x = x_1$ (say $H^{1/2}_{00_r}(x_0,x_1)$). For example

$$\|\phi\|_{H^{1/2}_{00_r}} := \|\phi\|_{H^{1/2}} + \left(\int_{x_0}^{x_1} \frac{|\phi(x)|^2}{|x_1-x|}\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.22}$$

1.5.4 Pagani's weighted Sobolev spaces

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$. In the works [33, 34] (albeit with swapped variables with respect to our setting), Pagani introduced the space $Z(\mathcal{O})$ of scalar functions ϕ on $\mathcal{O}$ such that ϕ, $\partial_z\phi$, $\partial_{zz}\phi$ and $z\partial_x\phi$ belong to $L^2(\mathcal{O})$ (in the sense of distributions). In this work, we will refer to this space with the notation $Z^0(\mathcal{O})$. It is a Banach space for the following norm

$$\|\phi\|_{Z^0} := \|z\partial_x\phi\|_{L^2} + \|\partial_{zz}\phi\|_{L^2} + \|\partial_z\phi\|_{L^2} + \|\phi\|_{L^2}. \tag{1.23}$$

We will also need the space $Z^1(\mathcal{O})$, which we define as the space of scalar functions ϕ on $\mathcal{O}$ such that ϕ and $\partial_x\phi$ belong to $Z^0(\mathcal{O})$, associated with the following norm

$$\|\phi\|_{Z^1} := \|\phi\|_{Z^0} + \|\partial_x\phi\|_{Z^0}. \tag{1.24}$$

The omitted proofs of the results of this section are postponed to Appendix B. We start with a straightforward extension result, which will allows to transfer results on $Z^0(\mathbb{R}^2)$ to $Z^0(\Omega)$.

Lemma 1.1. *There exists a continuous extension operator from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.*

The following embedding is the most important result concerning the space Z^0. Since solutions to $(z\partial_x - \partial_{zz})u = f$ for $f \in L^2(\Omega)$ belong to $Z^0(\Omega)$ (see Proposition 2.4), the following embedding entails that such solutions belong to $H^{2/3}(\Omega)$[1].

Proposition 1.2. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $H^{2/3}_x L^2_z$.*

Proof. Let $\psi \in C^\infty_c(\mathbb{R})$. By Lemma B.10, one has

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{1.25}$$

Using standard dimensional analysis arguments (e.g. by introducing the rescaled function $\psi_\lambda : z \mapsto \psi(\lambda z)$ for $\lambda > 0$ and optimizing the choice of λ), one deduces from (1.25) that

$$\|\psi\|_{L^2} \lesssim \|z\psi\|^{\frac{2}{3}}_{L^2} \|\partial_{zz}\psi\|^{\frac{1}{3}}_{L^2}. \tag{1.26}$$

Let $\phi \in C^\infty_c(\mathbb{R}^2)$. Let $\hat\phi(\xi, z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then using (1.26) and Hölder's inequality,

$$\begin{aligned}
\|\phi\|^2_{H^{2/3}_x L^2_z} &= \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{2}{3}} |\hat\phi(\xi, z)|^2 \,\mathrm{d}\xi \,\mathrm{d}z \\
&\lesssim \|\phi\|^2_{L^2} + \int_{\mathbb{R}} |\xi|^{\frac{4}{3}} \|z\hat\phi(\xi, z)\|^{\frac{4}{3}}_{L^2_z} \|\partial_{zz}\hat\phi(\xi, z)\|^{\frac{2}{3}}_{L^2_z} \,\mathrm{d}\xi \\
&\lesssim \|\phi\|^2_{L^2} + \left(\int_{\mathbb{R}^2} |\xi|^2 z^2 |\hat\phi(\xi, z)|^2 \,\mathrm{d}z \,\mathrm{d}\xi \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |\partial_{zz}\hat\phi(\xi, z)|^2 \,\mathrm{d}z \,\mathrm{d}\xi \right)^{\frac{1}{3}} \\
&\lesssim \|\phi\|^2_{L^2} + \|z\partial_x\phi\|^{\frac{4}{3}}_{L^2} \|\partial_{zz}\phi\|^{\frac{2}{3}}_{L^2}.
\end{aligned} \tag{1.27}$$

Hence $\|\phi\|_{H^{2/3}_x L^2_z} \lesssim \|\phi\|_{Z^0}$. This concludes the proof, by density of $C^\infty_c(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$.

□

Lemma 1.3. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $C^0_z(H^{1/2}_x)$.*

Proof. By definition, $Z^0(\mathbb{R}^2) \hookrightarrow H^2_z(L^2_x)$. By Proposition 1.2, $Z^0(\mathbb{R}^2) \hookrightarrow L^2_z(H^{2/3}_x)$. By the "fractional trace theorem" [29, Equation (4.7), Chapter 1], $Z^0(\mathbb{R}^2) \hookrightarrow C^0_z(H^{1/2}_x)$. □

Lemma 1.4. *$Z^0(\Omega)$ is continuously embedded in $C^0([x_0, x_1]; \mathscr{H}^1_z(-1, 1))$.*

Proof. This is contained in the trace result [34, Theorem 2.1]. □

Remark 1.5. *Although it is "almost" the case, there does not hold $Z^0(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$.*

- *Pagani [33, Theorem 2.1] proves that the operator $\phi \mapsto \phi(\cdot, 0)$ is onto from $Z^0(\mathbb{R}^2)$ to $H^{\frac{1}{2}}(\mathbb{R})$. But $H^{\frac{1}{2}}(\mathbb{R})$ contains unbounded functions of x.*
- *Pagani [33, Theorem 2.3] proves that the operator $\phi \mapsto \phi(0, \cdot)$ is onto from $Z^0(\mathbb{R}^2)$ to the space $\mathscr{H}^1_z(\mathbb{R})$. But this spaces contains unbounded functions, for example $\psi(z) := (-\ln|z|/2)^s \chi(z)$ for $s < \frac{1}{2}$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $z = 0$.*

[1] This can be seen as an hypoellipticity result for the operator $L = \partial_{zz} - z\partial_x$ in the full space, which is of the form $X_1^2 + X_0$, where $X_1 = \partial_z$, $X_0 = -z\partial_x$ and $[X_0, X_1] = \partial_x$, so the Lie brackets generate the full space and L satisfies Hörmander's sufficient condition of [21] for hypoellipticity.

1.5.5 Anisotropic Sobolev spaces

In the sequel, we will construct solutions of (1.5) and (1.7) in the following anisotropic Sobolev spaces. Within all these spaces, one has heuristically the correspondence $\partial_x \approx \partial_z^3$, which corresponds to the appropriate scaling due to the degeneracy of $z\partial_x$ at $z = 0$. We consider

$$Q^0 := L^2_x(H^2_y) \cap H^{2/3}_x(L^2_y), \tag{1.28}$$

$$Q^1 := L^2_x(H^5_y) \cap H^{5/3}_x(L^2_y). \tag{1.29}$$

By Proposition 1.2, $Z^0 \hookrightarrow Q^0$. This is the natural space for strong solutions to our equations. The space Q^1 corresponds to the situation where $\partial_x u$ is a strong solution to an equation of the same structure, so $\partial_x u \in Q^0$, which yields the $H^{5/3}_x(L^2_y)$ estimate. The $L^2_x(H^5_y)$ estimate comes from a sort of "hidden regularity" result (see e.g. Section 2.4).

Eventually, a key argument of our work is that we will prove the Lipschitz-stability of the orthogonality conditions and the convergence of the nonlinear scheme within the following interpolation space:

$$Q^{1/2} := [Q^0, Q^1]_{\frac{1}{2}} = L^2_x(H^{7/2}_y) \cap H^{7/6}_x(L^2_y). \tag{1.30}$$

Lemma 1.6. *By interpolation, we have the following embeddings*

- $Q^1 \subset H^\sigma_x H^{\sigma'}_y$ *for all* $\sigma, \sigma' \geq 0$ *such that* $3\sigma + \sigma' = 5$*;*
- $Q^{1/2} \subset H^\sigma_x H^{\sigma'}_y$ *for all* $\sigma, \sigma' \geq 0$ *such that* $3\sigma + \sigma' = 7/2$*;*
- $Q^0 \subset H^\sigma_x H^{\sigma'}_y$ *for all* $\sigma, \sigma' \geq 0$ *such that* $3\sigma + \sigma' = 2$.

In particular, $Q^1 \subset L^\infty_x(W^{2,\infty}_y)$.

2 The case of the linear shear flow

This section concerns the well-posedness of the linear system (1.5) which we restate here for convenience and by using z as a vertical variable rather than y to prepare for the next sections. We thus consider, in $\Omega = (x_0, x_1) \times (-1, 1)$, the system

$$\begin{cases} z\partial_x u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{2.1}$$

where $\Sigma_0 = \{x_0\} \times (0, 1)$ and $\Sigma_1 = \{x_1\} \times (-1, 0)$.

First, in Section 2.1, we recall the theory of weak solutions, due to Fichera, Baouendi and Grisvard. Then, in Section 2.2, we recall the theory of strong solutions with maximal regularity, due to Pagani. Our main contribution regarding this problem is contained in Section 2.3, where we derive two orthogonality conditions which are necessary to obtain higher tangential regularity and prove Theorem 1. Eventually, in Section 2.4, we prove a hidden regularity result for such solutions, which allows to control five derivatives in the vertical directions, and will be useful for the sequel.

2.1 Existence and uniqueness of weak solutions

Definition 2.1 (Weak solution)**.** *Let $f \in L^2((x_0,x_1);H^{-1}(-1,1))$ and $\delta_0,\delta_1 \in \mathscr{L}^2_z(-1,1)$. We say that $u \in L^2((x_0,x_1);H^1_0(-1,1))$ is a* weak solution *to* (2.1) *when, for all $v \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds*

$$-\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v = \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.2}$$

Weak solutions in the above sense are known to exist since the work Fichera [14, Theorem XX] (which concerns generalized versions of (2.1), albeit with vanishing boundary data). Uniqueness dates back to [5, Proposition 2] by Baouendi and Grisvard.

Proposition 2.2. *Let $f \in L^2((x_0,x_1);H^{-1}(-1,1))$ and $\delta_0,\delta_1 \in \mathscr{L}^2_z(-1,1)$. There exists a unique weak solution $u \in L^2((x_0,x_1);H^1_0(-1,1))$ to* (2.1)*. Moreover,*

$$\|u\|_{L^2_x(H^1_z)} \lesssim \|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathscr{L}^2_z} + \|\delta_1\|_{\mathscr{L}^2_z}. \tag{2.3}$$

Proof. The proof of uniqueness is postponed to Appendix A were we adapt Baouendi and Grisvard's arguments to prove uniqueness of weak solutions to all the linear problems we encounter in this paper. It relies on the proof of a trace theorem and a Green identity.

Let us prove the existence. We introduce two Hilbert spaces $\mathscr{V} \hookrightarrow \mathscr{U} \hookrightarrow L^2((x_0,x_1);H^1_0(0,1))$ as follows. Let $\mathscr{V} := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Omega \setminus (\Sigma_0 \cup \Sigma_1)\}$. Let $\mathscr{U}$ be the completion of $H^1(\Omega) \cap L^2((x_0,x_1);H^1_0(-1,1))$ with respect to the scalar product

$$\langle u,v\rangle_{\mathscr{U}} := \int_\Omega \partial_z u \partial_z v + \int_{\Sigma_0} zuv - \int_{\Sigma_1} zuv. \tag{2.4}$$

For $u,v \in \mathscr{U}\times\mathscr{V}$, let

$$a(u,v) := -\int_\Omega zu\partial_x v + \int_\Omega \partial_z u\partial_z v, \tag{2.5}$$

$$b(v) := \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.6}$$

In particular, for every $v \in \mathscr{V}$, integration by parts leads to $a(v,v) = \|v\|^2_{\mathscr{U}}$ and

$$|b(v)| \le \left(\|f\|_{L^2(H^{-1})} + \|\delta_0\|_{L^2_z} + \|\delta_1\|_{L^2_z}\right)\|v\|_{\mathscr{U}}. \tag{2.7}$$

Hence, $b \in \mathcal{L}(\mathscr{U})$[2] and existence follows from the Lax-Milgram type existence principle Lemma B.2, which also yields the energy estimate (2.3) thanks to (2.7) and Poincaré's inequality. □

Remark 2.3. *Instead of using the weak Lax-Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation* (2.1) *by vanishing viscosity, and to obtain uniform $L^2_x(H^1_z)$ estimates on the approximation.*

[2]Functions in $\mathscr{U}$ *a priori* do not have traces on Σ_i so one could wonder how definition (2.6) makes sense. The integrals $\int_{\Sigma_i} z\delta_i v$ make sense precisely because $\mathscr{U}$ is defined as a completion with respect to (2.4). In fact, weak solutions do have traces in a strong sense, as proved in Lemma A.1, thanks to the extra regularity in x provided by the equation.

2.2 Strong solutions with maximal regularity

We now turn to strong solutions, i.e. solutions for which (2.1) holds almost everywhere. The main result on this topic are due to Pagani.

Proposition 2.4. *Let $f \in L^2(\Omega)$ and $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$. The unique weak solution u to (2.1) belongs to $Z^0(\Omega)$ and satisfies*

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z}. \tag{2.8}$$

Moreover, the boundary conditions $u_{|\Sigma_i} = \delta_i$ hold in the sense of traces in $\mathscr{H}^1_z(\Sigma_i)$ (see Lemma 1.4).

Proof. This is a particular case of [34, Theorem 5.2]. Pagani's proof proceeds by localization. Far from the critical points $(x_0, 0)$ and $(x_1, 0)$, the regularity is rather straightforward. Near these critical points, the regularity stems from the regularity obtained for a similar problem set in a half-space. Pagani studies such half-space problems in [33] where he derives explicit representation formulas for the solutions, using the Mellin transform and the Wiener-Hopf method. We do not reproduce these arguments here for brevity. □

2.3 Orthogonality conditions for higher tangential regularity

We now investigate the question of whether solutions to (2.1) enjoy higher regularity in the tangential direction. As mentioned in Section 1.2, it is quite easy to obtain *a priori* estimates in the space $Z^1(\Omega)$ (see Proposition 2.5). However, we prove in Proposition 2.7 that the weak solution enjoys such a regularity if only if the data satisfies appropriate orthogonality conditions. Eventually, we give statements highlighting the fact that these conditions are non empty.

Proposition 2.5. *Let $f \in H^1((x_0, x_1); H^{-1}(-1,1))$ and $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$ and such that $\Delta_0, \Delta_1 \in \mathscr{L}^2_z(-1,1)$, where*

$$\Delta_i(z) := \frac{f(x_i, z) + \delta_i''(z)}{z}. \tag{2.9}$$

If the unique weak solution u to (2.1) belongs to $H^1((x_0, x_1); H^1_0(-1,1))$, then one has the following weak solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{L^2_x H^1_z} \lesssim \|\partial_x f\|_{L^2_x H^{-1}_z} + \|\Delta_0\|_{\mathscr{L}^2_z} + \|\Delta_1\|_{\mathscr{L}^2_z}. \tag{2.10}$$

If moreover, $f \in H^1((x_0, x_1); L^2(-1,1))$, $\Delta_0, \Delta_1 \in \mathscr{H}^1_z(-1,1)$ and $\Delta_0(1) = \Delta_1(-1) = 0$, then $u \in Z^1(\Omega)$ and one has the following strong solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{Z^0} \lesssim \|\partial_x f\|_{L^2} + \|\Delta_0\|_{\mathscr{H}^1_z} + \|\Delta_1\|_{\mathscr{H}^1_z}. \tag{2.11}$$

Proof. The key point is that the information that $\partial_x u$ enjoys $L^2_x H^1_z$ regularity allows us to prove that $\partial_x u$ is the unique weak solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = f_x, \\ w_{|\Sigma_i} = \Delta_i, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{2.12}$$

Then estimate (2.10) follows from (2.3) and estimate (2.11) follows from (2.8). Hence, let us prove that $\partial_x u$ is a weak solution to (2.12). Let

$$\begin{aligned}\mathscr{V} := \big\{ v \in C^\infty(\overline{\Omega});\quad & v = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1),\\ & \partial_x v = 0 \text{ on } \{x_0\} \times (-1,0) \text{ and } \{x_1\} \times (0,1)\}.\end{aligned} \tag{2.13}$$

Let $v \in \mathscr{V}$. Then $\partial_x v$ is an admissible test function for Definition 2.1. Hence, since u is the weak solution to (2.1), one has

$$-\int_\Omega zu\partial_x(\partial_x v) + \int_\Omega \partial_z u \partial_z(\partial_x v) = \int_\Omega f(\partial_x v) + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v). \tag{2.14}$$

The $H^1_x H^1_z$ regularity of u legitimates integrations by parts in x in the left-hand side. Thus

$$\begin{aligned}\Big[-\int_{-1}^1 zu\partial_x v\Big]_{x_0}^{x_1} + \int_\Omega z(\partial_x u)\partial_x v + \Big[\int_{-1}^1 \partial_z u \partial_z v\Big]_{x_0}^{x_1} - \int_\Omega \partial_z(\partial_x u)\partial_z v \\ = \Big[\int_{-1}^1 fv\Big]_{x_0}^{x_1} - \int_\Omega f_x v + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v),\end{aligned} \tag{2.15}$$

which, after taking the boundary conditions into account, integrating by parts in z in the boundary terms $\int_{-1}^1 \partial_z u \partial_z v$ and recalling (2.9) yields

$$-\int_\Omega z(\partial_x u)\partial_x v + \int_\Omega \partial_z(\partial_x u)\partial_z v = \int_\Omega f_x v + \int_{\Sigma_0} z\Delta_0 v - \int_{\Sigma_1} z\Delta_1 v. \tag{2.16}$$

Since $\mathscr{V}$ is dense in the set of test functions for Definition 2.1, this proves that $\partial_x u$ is the weak solution to (2.12). □

We start by defining "dual profiles" which are necessary to state our orthogonality conditions.

Lemma 2.6 (Dual profiles)**.** *We define Φ^0, $\Phi^1 \in Z^0(\Omega \setminus \{z=0\})$ as the unique solutions to*

$$\begin{cases} -z\partial_x \Phi^j - \partial_{zz}\Phi^j = 0 \quad \text{in } \Omega \setminus \{z=0\},\\ \big[\Phi^j\big]_{|z=0} = \mathbf{1}_{j=1},\\ \big[\partial_z \Phi^j\big]_{|z=0} = -\mathbf{1}_{j=0},\\ \Phi^j_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0.\end{cases} \tag{2.17}$$

Proof. Uniqueness is straightforward. Given $j \in \{0,1\}$ and two solutions of (2.17), let ϕ denote their difference. Then $\phi \in Z^0(\Omega \setminus \{z=0\})$ and both ϕ and $\partial_z \phi$ are continuous across the line $\{z=0\}$. Hence $\phi \in Z^0(\Omega)$ and is the solution to a problem of the form (2.1) (with reversed tangential direction). So $\phi = 0$ since weak solutions of such problems are unique.

We prove the existence of Φ^0. The profile Φ^1 can be constructed similarly and is left to the reader. We define $\Phi^0(x,z) := -z_+\zeta(z) + \Psi^0(x,z)$, where $\zeta \in C^\infty_c(\mathbb{R})$ is such that $\zeta \equiv 1$ in a neighborhood of $z=0$ and $\operatorname{supp}\zeta \subset (-1/2, 1/2)$ and $\Psi^0 \in L^2((x_0,x_1); H^1(-1,1))$ is the unique weak solution to

$$\begin{cases} -z\partial_x \Psi^0 - \partial_{zz}\Psi^0 = -2\mathbf{1}_{z>0}\zeta'(z) - z_+\zeta''(z) & \text{in } \Omega,\\ \Psi^0(x_0,z) = 0 & \text{for } z \in (-1,0),\\ \Psi^0(x_1,z) = z\zeta(z) & \text{for } z \in (0,1),\\ \Psi^0_{|z=\pm 1} = 0.\end{cases} \tag{2.18}$$

By Proposition 2.4, $\Psi^0 \in Z^0(\Omega)$. Hence $\partial_{zz}\Phi^0 \in L^2(\Omega_\pm)$ and $z\partial_x\Phi^0 \in L^2(\Omega_\pm)$. □

We now turn to the main result of this section, which gives a necessary and sufficient condition for the solutions to enjoy the mentioned tangential regularity. Strangely, we could not find a proof of the above result in the literature, although some works mention orthogonality conditions (see [14, Equation (4.2)] or [36]). Hence we provide here a full proof. This strategy will be extended in the next section to equations with variable coefficients. We prove further that these orthogonality conditions are not empty.

Proposition 2.7. *For $f \in H^1((x_0,x_1);L^2(-1,1))$, $\delta_0,\delta_1 \in \mathscr{H}^1_z(-1,1)$ with $\delta_0(1)=\delta_1(-1)=0$ and $\Delta_0,\Delta_1 \in \mathscr{H}^1_z(-1,1)$ with $\Delta_0(1)=\Delta_1(-1)=0$ (see (2.9)), the unique weak solution u to (2.1) belongs to $H^1((x_0,x_1);H^1_0(-1,1))$ if and only if, for $j=0$ and $j=1$,*

$$\int_\Omega \partial_x f\Phi^j + \int_{\Sigma_0} z\Delta_0\Phi^j - \int_{\Sigma_1} z\Delta_1\Phi_j = \partial_z^j\delta_1(0) - \partial_z^j\delta_0(0). \tag{2.19}$$

Furthermore, under this condition, we actually have $\partial_x u \in Z^0(\Omega)$, so $u \in Z^1(\Omega)$.

Proof. First step: We exhibit possible discontinuities. Let us consider the unique solution $u \in Z^0$ to (2.1). Following the strategy sketched by Goldstein and Mazumdar[3] [17, Theorem 4.2], we introduce the unique strong solution $w \in Z^0(\Omega)$ to (2.12), so that w is a good candidate for $\partial_x u$. The idea is then to introduce the function u_1 defined by

$$\begin{aligned} u_1(x,z) &= \delta_0(z) + \int_{x_0}^{x} w(x',z)\,\mathrm{d}x' \quad \text{in } \Omega_+ = (x_0,x_1)\times(0,1),\\ u_1(x,z) &= \delta_1(z) - \int_{x}^{x_1} w(x',z)\,\mathrm{d}x' \quad \text{in } \Omega_- = (x_0,x_1)\times(-1,0) \end{aligned} \tag{2.20}$$

so that $\partial_x u_1 = w$ almost everywhere. Furthermore it can be easily proved that

$$z\partial_x u_1 - \partial_{zz}u_1 = f \tag{2.21}$$

in $\mathcal{D}'(\Omega_\pm)$. However this does not entail that u_1 is a solution of this equation in the whole domain. Indeed, u_1 and $\partial_z u_1$ may have discontinuities across the line $z=0$. Nevertheless, one checks that u_1 and $\partial_z u_1$ are continuous across $z=0$ if and only if

$$\begin{aligned} \int_{x_0}^{x_1} w(x,0)\,\mathrm{d}x &= \delta_1(0) - \delta_0(0),\\ \int_{x_0}^{x_1} w_z(x,0)\,\mathrm{d}x &= \delta_1'(0) - \delta_0'(0). \end{aligned} \tag{2.22}$$

The two integrals are well defined since w_z and w_{zz} belong to $L^2(\Omega)$.

Second step: We compute the mean value of w and w_z using the dual profiles. Let $\phi \in Z^0(\Omega_\pm)$ such that $\phi_{|\partial\Omega\setminus(\Sigma_0\cap\Sigma_1)} = 0$. Since $w \in Z^0(\Omega)$, it satisfies (2.12) almost everywhere, so that we can multiply the equation by ϕ and integrate over Ω_+. Hence

$$\int_{\Omega_+} f_x\phi = \int_{\Omega_+} (z\partial_x w - \partial_{zz}w)\phi, \tag{2.23}$$

[3]Oddly, Goldstein and Mazumdar do not mention the orthogonality conditions (2.19). They merely state that since $\partial_x u_1 = w$, $u_1 \in H^1((x_0,x_1);H^1_0(-1,1))$. However, these orthogonality conditions are necessary and non-empty, as we show below (see Proposition 2.9).

where, on the one hand,

$$\int_{\Omega_+} z(\partial_x w)\phi = \int_{\Sigma_1} z\Delta_1\phi - \int_{\Omega_+} zw\partial_x\phi \tag{2.24}$$

and on the other hand,

$$-\int_{\Omega_+} (\partial_{zz} w)\phi = \int_{x_0}^{x_1} (\partial_z w\phi - w\partial_z\phi)(x, 0^+)\,\mathrm{d}x - \int_{\Omega_+} w\partial_{zz}\phi. \tag{2.25}$$

Thus, performing the same computation on Ω_- and summing both contributions yields

$$\begin{aligned}\int_{x_0}^{x_1} (\partial_z w[\phi]_{|z=0} - w[\partial_z\phi]_{|z=0})(x,0)\,\mathrm{d}x = \int_\Omega f_x\phi + \int_{\Sigma_0} z\Delta_0\phi - \int_{\Sigma_1} z\Delta_1\phi \\ + \int_{\Omega_\pm} w(z\partial_x\phi + \partial_{zz}\phi).\end{aligned} \tag{2.26}$$

Hence, for $j \in \{0, 1\}$,

$$\int_{x_0}^{x_1} \partial_z^j w(x,0)\,\mathrm{d}x = \int_\Omega f_x\Phi^j + \int_{\Sigma_0} z\Delta_0\Phi^j - \int_{\Sigma_1} z\Delta_1\Phi^j. \tag{2.27}$$

Third step: Conclusion. Assume that the orthogonality conditions (2.19) are satisfied for $j = 0$ and $j = 1$. Then (2.22) holds, and a consequence, $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$, and $u_1 \in L^2((x_0, x_1); H^1_0(-1, 1))$ is a weak solution of (2.1). We infer from the uniqueness of weak solutions solutions that $u = u_1$, and therefore $\partial_x u = w \in L^2((x_0, x_1); H^1_0(-1, 1))$. Hence $u \in H^1((x_0, x_1); H^1_0(-1, 1))$.

Conversely, if u is a solution to (2.1) with $H^1((x_0, x_1); H^1_0(-1, 1))$ regularity, then $\partial_x u$ is a weak solution to (2.12) (see the proof of Proposition 2.5) and u is given in terms of $\partial_x u$ by (2.20) almost everywhere. Since $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$, one has $\int_{x_0}^{x_1} u_x(x,0)\,\mathrm{d}x = \int_{x_0}^{x_1} u_{xz}(x,0)\,\mathrm{d}x = 0$, and thus the orthogonality conditions (2.19) are satisfied. □

Definition 2.8. *In the sequel, we denote by ℓ^j the linear forms associated with the orthogonality conditions* (2.19) *for the linear shear flow problem, i.e. we set*

$$\ell^j(f, \delta_0, \delta_1) := \partial_z^j\delta_0(0) - \partial_z^j\delta_1(0) + \int_{\Sigma_0} z\Delta_0\Phi^j - \int_{\Sigma_1} z\Delta_1\Phi_j + \int_\Omega \partial_x f\Phi^j. \tag{2.28}$$

We now prove that the orthogonality conditions (2.19) are non-empty and independent.

Proposition 2.9. *The linear forms ℓ^0 and ℓ^1 are linearly independent on $C^\infty_c(\Omega) \times \{0\} \times \{0\}$. In particular, this also holds on $C^\infty_c(\Omega) \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1)$.*

Proof. Proceeding by contradiction, let $(c_0, c_1) \in \mathbb{R}^2$ such that, for every $f \in C^\infty_c(\Omega)$, $c_0\ell^0(f, 0, 0) + c_1\ell^1(f, 0, 0) = 0$. Then $\Phi^c := c_0\Phi^0 + c_1\Phi^1$ satisfies $\int_\Omega \partial_x f\Phi^c = 0$ for every $f \in C^\infty_c(\Omega)$, so $\partial_x\Phi^c = 0$ in $\mathcal{D}'(\Omega_+)$. Since $\Phi^c(x_1, z) = 0$ for $z \in (0, 1)$ and $\Phi^c \in Z^0(\Omega_+)$, this implies that $\Phi^c = 0$ in Ω_+ (since Z^0 functions have traces in the usual sense, see Lemma 1.4). The same holds in Ω_-. Hence $[\Phi^c]_{|z=0} = [\partial_z\Phi^c]_{|z=0} = 0$, which implies $c_0 = c_1 = 0$. □

Corollary 2.10 (Biorthogonal basis)**.** *There exist* $\Xi^k = (f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}$ *for* $k \in \{0,1\}$ *such that, for every* $j,k \in \{0,1\}$*,*

$$\ell^j(\Xi^k) = \ell^j(f^k, \delta_0^k, \delta_1^k) = \mathbf{1}_{j=k} \tag{2.29}$$

and such that, within $\mathcal{H}$*,*

$$\mathcal{H}_{\rm sg}^{\perp} := \big(\mathbb{R}\Xi^0 + \mathbb{R}\Xi^1\big)^{\perp} = \ker \ell^0 \cap \ker \ell^1. \tag{2.30}$$

Proof. Since ℓ^0 and ℓ^1 are continuous linear forms on $\mathcal{H}$, by the Riesz representation theorem, they can be written as scalar products with two given triplets, say $\Lambda^0, \Lambda^1 \in \mathcal{H}$ which are linearly independent thanks to Proposition 2.9. Then one looks for $\Xi^k = (f^k, \delta_0^k, \delta_1^k)$ as $a_k\Lambda^0 + b_k\Lambda^1$ where $a_k, b_k \in \mathbb{R}^2$ are such that $a_k\langle\Lambda^j;\Lambda^1\rangle + b_k\langle\Lambda^j;\Lambda^2\rangle = \mathbf{1}_{j=k}$. These systems can be solved since Λ^0 and Λ^1 are free. Moreover, this ensures (2.30) and their independence guarantees that $\mathcal{H}_{\rm sg}^{\perp}$ is of codimension 2 in $\mathcal{H}$. □

2.4 Hidden vertical regularity

The goal of this paragraph is to prove that, if u is a solution to (2.1) such that $\partial_{xzz}u \in L^2(\Omega)$, then one also has $\partial_z^5 u \in L^2(\Omega)$, provided that the data is sufficiently regular. There is no additional orthogonality condition. We start with the straightforward claim that $\partial_z^4 u \in L^2(\Omega)$.

Lemma 2.11. *Let* $f \in L^2_x H^{-1}_z$ *and* $\delta_0, \delta_1 \in \mathscr{L}^2_z$*. Assume that* $\delta_0(1) = \delta_1(-1) = \Delta_0(1) = \Delta_1(-1) = 0$*. Let* u *be the unique weak solution to* (2.1)*. Assume that* $u \in H^1_x H^2_z$ *and* $f \in L^2_x H^2_z$*. Then* $\partial_z^2 u \in Z^0$ *and*

$$\|u\|_{L^2_x H^4_z} + \|z\partial_x\partial_z^2 u\|_{L^2} \lesssim \|u\|_{H^1_x H^2_z} + \|f\|_{L^2_x H^2_z}.$$

Proof. Since u is a strong solution to (2.1), there holds $\partial_{zz}u = z\partial_x u - f$ in $L^2(\Omega)$. Hence, in $\mathcal{D}'(\Omega)$, $\partial_z^4 u = z\partial_{xzz}u + 2\partial_{xz}u - \partial_{zz}f$. Thus $\partial_z^4 u \in L^2(\Omega)$ and $\|u\|_{L^2_x H^4_z} \lesssim \|f\|_{L^2_x H^2_z} + \|u\|_{H^1_x H^2_z}$.

The Z^0 regularity follows from the results of Proposition 2.4, noticing that the compatibility conditions in the corners are satisfied. □

Proposition 2.12. *Let* $f \in L^2_x H^{-1}_z$ *and* $\delta_0, \delta_1 \in \mathscr{L}^2_z$*. Let* u *be the unique weak solution to* (2.1)*. Assume that* $u \in H^1_x H^2_z$*,* $f \in L^2_x H^3_z$ *and* $\partial_z^3\delta_0, \partial_z^3\delta_1 \in \mathscr{H}^1_z$*, with* $\delta_0(1) = \delta_1(-1) = \Delta_0(1) = \Delta_1(-1) = 0$*. Assume furthermore that* $\partial_x\partial_z^2 f \in L^2((x_0,x_1)\times(1/2,1)) \cap L^2((x_0,x_1)\times(-1,-1/2))$*.*

Then $\partial_z^5 u \in L^2(\Omega)$ *and*

$$\|u\|_{L^2_x H^5_z} \lesssim \|u\|_{H^1_x H^2_z} + \|f\|_{L^2_x H^3_z} + \|\partial_x\partial_z^2 f \mathbf{1}_{|z|\geq 1/2}\|_{L^2} + \sum_{i\in\{0,1\}} \|\partial_z^3\delta_i\|_{\mathscr{H}^1_z}. \tag{2.31}$$

Proof. In the course of the proof, we will need to distinguish between different regions:

- One "interior" region, close to the line $z = 0$. In this region, we will prove that $\partial_z^3 u$ is such that $(z\partial_x - \partial_z^2)\partial_z^3 u \in L^2$, and use the results of Pagani to deduce that $\partial_z^5 u \in L^2$.
- Two "boundary" regions, in the vicinity of the lines $z = \pm 1$. In these regions, since z is bounded away from zero, we will use classical parabolic regularity arguments.

• *Interior region:* Let $\phi \in C^\infty(\overline{\Omega})$ such that $\phi = 0$ on $\{x_0\}\times[-1,0]$, $\phi = 0$ on $\{x_1\}\times[0,1]$ and ϕ vanishes identically on neighborhoods of $z = \pm 1$. Thanks to the regularity of u, we can multiply the PDE for u by $\partial_z^3\phi$ and integrate over Ω. Vertical integrations by parts yield vanishing

boundary terms because ϕ vanishes identically in neighborhoods of $z = \pm 1$. We proceed with care for the horizontal term so that all manipulations are licit:

$$\begin{aligned}\int_\Omega z\partial_x u\partial_z^3\phi &= \int_\Omega (2\partial_{xz}u + z\partial_{xzz}u)\partial_z\phi \\ &= \int_\Omega 2\partial_{xz}u\partial_z\phi + \int_{\Sigma_1} z\partial_{zz}\delta_1\partial_z\phi - \int_{\Sigma_0} z\partial_{zz}\delta_0\partial_z\phi - \int_\Omega z\partial_{zz}u\partial_{xz}\phi.\end{aligned} \tag{2.32}$$

We rewrite the last term as

$$\begin{aligned}-\int_\Omega z\partial_{zz}u\partial_{xz}\phi &= \int_\Omega z\partial_{zzz}u\partial_x\phi + \int_\Omega \partial_{zz}u\partial_x\phi \\ &= \int_\Omega z\partial_{zzz}u\partial_x\phi + \int_{\Sigma_1} \partial_{zz}\delta_1\phi - \int_{\Sigma_0} \partial_{zz}\delta_0\phi - \int_\Omega \partial_{xzz}u\phi.\end{aligned} \tag{2.33}$$

Hence,

$$\int_\Omega z\partial_x u\partial_z^3\phi = -3\int_\Omega \partial_{xzz}u\phi + \int_\Omega z\partial_z^3 u\partial_x\phi - \sum_{i\in\{0,1\}} (-1)^i \int_{\Sigma_i} \partial_{zz}\delta_i(z\partial_z\phi + \phi). \tag{2.34}$$

We also integrate by parts the boundary term. For example, on Σ_0:

$$-\int_{\Sigma_0} \partial_{zz}\delta_0 z\partial_z\phi = -[z\partial_{zz}\delta_0\phi_{|x=x_0}]_0^1 + \int_{\Sigma_0} (z\partial_z^3\delta_0 + \partial_{zz}\delta_0)\phi. \tag{2.35}$$

The pointwise term is null at $z = 1$ because ϕ vanishes identically near $z = 1$ and null at $z = 0$ since ϕ vanishes at 0 and $\partial_{zz}\delta_0 \in \mathscr{H}_z^2(\Sigma_0)$.

Eventually, this proves that

$$-\int_\Omega z(\partial_z^3 u)\partial_x\phi + \int_\Omega \partial_z(\partial_z^3 u)\partial_z\phi = \sum_{i\in\{0,1\}} \int_{\Sigma_i} z\partial_z^3\delta_i\phi + \int_\Omega (\partial_z^3 f - 3\partial_{xzz}u)\phi. \tag{2.36}$$

Since $u \in L_x^2 H_z^4$ (by Lemma 2.11), $u \in H_x^1 H_z^2$, $f \in L_x^2 H_z^3$ and $\delta_i \in H_z^3$, by density, this equality still holds for $\phi \in H^1(\Omega)$ such that $\phi = 0$ on $\{x_0\} \times [-1, 0]$, $\{x_1\} \times [0, 1]$ and $z = \pm 1$.

Now, let $\chi_0 \in \mathcal{C}_c^\infty((-1, 1))$ such that $\chi_0 \equiv 1$ in a neighbourhood of $z = 0$. The above argument shows that $\chi_0\partial_z^3 u \in L_x^2 H_z^1$ is the unique weak solution to (2.1) with boundary data $\chi_0\partial_z^3\delta_i$ and source term $\chi_0(\partial_z^3 f - 3\partial_{xzz}u) - \chi_0''\partial_z^3 u - \chi_0'\partial_z^4 u \in L^2(\Omega)$. We infer from Proposition 2.4 that $\chi_0\partial_z^5 u \in L^2$. Note that thanks to the truncation χ_0, the compatibility conditions at $(x_0, 1)$ and $(x_1, -1)$ are automatically satisfied. Furthermore, $z\chi_0\partial_x\partial_z^3 u \in L^2$. As a consequence, using the equation, we infer that $z^2\chi_0\partial_x^2\partial_z u \in L^2$.

• *Boundary regions:*

By symmetry, we only treat the upper boundary region. We consider a function $\chi_1 \in C^\infty(\mathbb{R})$ such that $\chi_1 \equiv 1$ in a neighbourhood of $z = 1$, and $\mathrm{Supp}\chi_1 \subset (1/2, 1)$.

Then $u_1 := \chi_1 u$ is a solution of

$$\begin{aligned} z\partial_x u_1 - \partial_{zz}u_1 = f_1 := f\chi_1 - \chi_1'\partial_z u_1 - \chi_1'' u_1, \quad &\text{in } (x_0, x_1) \times (1/2, 1) \\ u_{1|z=1/2} = u_{1|z=1} &= 0, \\ u_{1|x=x_0} &= \chi_1\delta_0. \end{aligned} \tag{2.37}$$

This is a standard parabolic equation, for which we can apply classical regularity results. Note that the assumptions on f together with the observation that $z\partial_x\partial_z^2 u \in L^2$, $z\chi_0\partial_x\partial_z^3 u \in L^2$, $z^2\partial_x^2\partial_z u\chi_0 \in L^2$ imply that $\partial_z^2 f_1 \in H^1_x(L^2_y)$. The compatibility conditions $\delta_0(1) = \Delta_0(1) = 0$ ensures that $\partial_x\partial_z^2 u_1 \in L^2((x_0, x_1), H^1(-1, 1))$. Therefore, using equation (2.37), we deduce that $\partial_z^5 u_1 \in L^2$. □

3 The linearized problem

The goal of this section is to establish the well-posedness of the linearized problem

$$\begin{cases} \bar{u}\partial_x u - \partial_{yy} u = f \quad \text{in } \Omega, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{3.1}$$

where $\bar{u} \in Q^1$ (see (1.29)) is a given perturbation of the linear shear flow, f is an external source term and (δ_0, δ_1) are lateral boundary data. It is fairly straightforward to adapt the theory of existence and uniqueness of weak solutions depicted in the previous section to the above equation. However, writing the orthogonality conditions for a general shear flow $\bar{u}$ is quite complicated. Indeed, we recall that the strategy is to find the equation solved by u_x in the upper region $\{\bar{u} > 0\}$ and in the lower region $\{\bar{u} < 0\}$, and to glue together these two solutions (provided the orthogonality conditions, which ensure the continuity of u and u_y across the line $\{\bar{u} = 0\}$, are satisfied). When the line $\{\bar{u} = 0\}$ is straight, this is a fairly simple process, which we described in the previous section. However, when $\{\bar{u} = 0\}$ is not a straight line, retrieving u from u_x is not entirely obvious (one needs to integrate u_x on curved lines).

Therefore, we have chosen to first straighten the flow $\bar{u}$ by changing the vertical coordinate. Of course, this introduces variable coefficients in the equation. We then prove existence and uniqueness of weak solutions for the equation in the new coordinates, and exhibit orthogonality conditions, which are necessary and sufficient conditions ensuring that the weak solution has in fact $H^1_x H^1_y$ regularity. Eventually, we go back to the original variables and infer the existence of strong Z^1 solutions of (3.1) under orthogonality conditions.

3.1 A change of vertical coordinate

Throughout this section, we assume that $\bar{u}$ is a Q^1 function such that $\|\bar{u} - y\|_{Q^1}$ is small. In particular, $\|\bar{u}_y - 1\|_{L^\infty} \lesssim \|\bar{u}_y - 1\|_{Q^1} \ll 1$ (see Lemma 1.6). It follows that there exists a line $y = \bar{y}(x)$ on which $\bar{u}$ vanishes, and $\bar{u} \lessgtr 0$ on $y \lessgtr \bar{y}(x)$.

As a consequence, we define an associated change of variables Y such that

$$\forall z \in (-1, 1), \quad \forall x \in (x_0, x_1), \quad \bar{u}(x, Y(x, z)) = z. \tag{3.2}$$

We then look for u under the form

$$u(x, y) = U(x, \bar{u}(x, y)), \tag{3.3}$$

so that $U = U(x, z)$ solves

$$\begin{cases} z\partial_x U + \gamma\partial_z U - \alpha\partial_{zz} U = g \quad \text{in } \Omega, \\ U_{|\Sigma_i} = \widetilde{\delta}_i, \\ U_{|y=\pm 1} = 0, \end{cases} \tag{3.4}$$

where

$$\alpha(x,z) := (\partial_y \bar{u})^2(x, Y(x,z)), \tag{3.5}$$

$$\gamma(x,z) := (z\bar{u}_x - \bar{u}_{yy})(x, Y(x,z)), \tag{3.6}$$

$$g(x,z) := f(x, Y(x,z)) \tag{3.7}$$

and

$$\widetilde{\delta}_i(z) := \delta_i(Y(x_i, z)). \tag{3.8}$$

The next sub-sections are devoted to the analysis of equation (3.4): existence and uniqueness of weak solutions, Z^0 regularity, orthogonality conditions for Z^1 regularity.

3.2 Existence and uniqueness of weak solutions

This section follows exactly the arguments of Section 2.1. The only slight difference lies in the derivation of the *a priori* estimates, in which we use smallness assumptions to treat perturbatively the additional drift term $\gamma\partial_z U$ and the commutator coming from the diffusion.

Definition 3.1 (Weak solution)**.** *Let $g \in L^2((x_0,x_1); H^{-1}(-1,1))$ and $\widetilde{\delta}_0, \widetilde{\delta}_1 \in \mathscr{L}^2_z(-1,1)$. Let $\alpha \in H^1_z(L^\infty_x)$, $\gamma \in L^2_z(L^\infty_x)$.*

We say that $U \in L^2((x_0,x_1); H^1_0(-1,1))$ is a weak solution *to* (3.4) *when, for all $V \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds*

$$-\int_\Omega zU\partial_x V + \int_\Omega (\gamma + \alpha_z)\partial_z UV + \int_\Omega \alpha \partial_z U \partial_z V = \int_\Omega gV - \int_{\Sigma_1} z\widetilde{\delta}_1 V + \int_{\Sigma_0} z\widetilde{\delta}_0 V. \tag{3.9}$$

Proposition 3.2 (Existence and uniqueness of weak solutions)**.** *Assume that α and γ satisfy*

$$\|\alpha - 1\|_{L^\infty} + \|\gamma\|_{L^2_z(L^\infty_x)} + \|\alpha_z\|_{L^2_z(L^\infty_x)} \ll 1. \tag{3.10}$$

Then, for every $g \in L^2((x_0,x_1), H^{-1}(-1,1))$ and $\widetilde{\delta}_1, \widetilde{\delta}_0 \in \mathscr{L}^2_z(-1,1)$, there exists a unique weak solution $U \in L^2((x_0,x_1), H^1_0(-1,1))$ to (3.4)*. Moreover,*

$$\|U\|_{L^2_x(H^1_z)} \lesssim \|g\|_{L^2_x(H^{-1}_z)} + \|\widetilde{\delta}_0\|_{\mathscr{L}^2_z} + + \|\widetilde{\delta}_1\|_{\mathscr{L}^2_z}. \tag{3.11}$$

Proof. We mimic the proof of Proposition 2.2. We take $\mathscr{V} = \{V \in H^1(\Omega), V = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)\}$ and $\mathscr{U}$ the completion of $H^1(\Omega) \cap L^2((x_0,x_1); H^1_0(-1,1))$ with respect to the scalar product

$$\langle U, V\rangle_{\mathscr{U}} := \int_\Omega \alpha \partial_z U \partial_z V + \frac{1}{2}\int_{\Sigma_0} zUV - \frac{1}{2}\int_{\Sigma_1} zUV. \tag{3.12}$$

For $(U,V) \in \mathscr{U} \times \mathscr{V}$, let

$$a(U,V) := -\int_\Omega zU\partial_x V + \int_\Omega (\gamma + \alpha_z)\partial_z UV + \int_\Omega \alpha\partial_z U\partial_z V, \tag{3.13}$$

$$b(V) := \int_\Omega gV - \int_{\Sigma_1} z\widetilde{\delta}_1 V + \int_{\Sigma_0} z\widetilde{\delta}_0 V. \tag{3.14}$$

Now, for any $V \in \mathscr{V}$, using (3.10) and the Poincaré inequality $\|V\|_{L^\infty_z(L^2_x)} \le 2\|\partial_z V\|_{L^2(\Omega)}$,

$$a(V,V) = \|V\|^2_{\mathscr{U}} + \int_\Omega (\gamma + \alpha_z)V\partial_z V \ge \frac{1}{2}\|V\|^2_{\mathscr{U}} \tag{3.15}$$

The linear form b satisfies (2.7). As in Proposition 2.2, the existence follows from the Lax-Milgram type existence principle Lemma B.2, and we obtain the energy estimate (3.11).

As in the proof of Proposition 2.2, uniqueness follows from the result by Baouendi and Grisvard, recalled in Appendix A. □

Remark 3.3. *Instead of using the weak Lax-Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation* (3.4) *by vanishing viscosity, and to obtain uniform* $L^2_x(H^1_z)$ *estimates on the approximation.*

3.3 Strong solutions with maximal regularity

In this paragraph, we adapt the results of [34] to construct solutions to (3.4) with Z^0 regularity, with estimates independent of the coefficients α and γ, provided that (3.10) is satisfied.

Proposition 3.4. *Assume that* α *and* γ *satisfy* (3.10). *Then, for every* $g \in L^2(\Omega)$ *and* $\widetilde{\delta}_0, \widetilde{\delta}_1 \in \mathscr{H}^1_z(-1,1)$, *the unique weak solution* U *to* (3.4) *satisfies* $U \in Z^0(\Omega)$ *with the estimate*

$$\|U\|_{Z^0} \lesssim \|g\|_{L^2} + \|\widetilde{\delta}_0\|_{\mathscr{H}^1_z} + \|\widetilde{\delta}_1\|_{\mathscr{H}^1_z}. \tag{3.16}$$

Proof. Thanks to Proposition 3.2, there exists $c_0 > 0$ such that, if

$$\|\alpha - 1\|_{L^\infty} + \|\gamma\|_{L^2_z(L^\infty_x)} + \|\alpha_z\|_{L^2_z(L^\infty_x)} \leq c_0, \tag{3.17}$$

the problem (3.4) is well-posed at the level of weak solutions. We proceed in four steps.

• *Case of smooth coefficients with a large zero-order term.* We start with coefficients α, γ that are smooth, satisfy (3.17), and we consider the following variant of (3.4):

$$\begin{cases} z\partial_x U + \gamma\partial_z U - \alpha\partial_{zz} U + C_0 U = h \quad \text{in } \Omega, \\ U_{|\Sigma_i} = 0, \\ U_{|y=\pm 1} = 0, \end{cases} \tag{3.18}$$

where $C_0 \geq \frac{1}{2}\alpha_{zz} + \frac{1}{2}\gamma_z$. By Pagani [34, Theorem 5.2] (for the operator $z\partial_x + \gamma\partial_z - \alpha\partial_{zz}$), for every $h \in L^2(\Omega)$, there exists a unique $U \in Z^0(\Omega)$ solution to (3.18) and a constant C (possibly depending on α, γ and C_0 in a way that is not entirely explicit in the work of Pagani) such that

$$\|U\|_{Z^0} \leq C\|h\|_{L^2}. \tag{3.19}$$

Thus, we can define the bounded linear operator $\mathcal{K} : L^2(\Omega) \to Z^0(\Omega) \subset L^2(\Omega)$ which maps h to U, the solution to (3.18). Moreover, $\mathcal{K} \in \mathcal{L}(L^2(\Omega))$ is compact since $Z^0(\Omega) \hookrightarrow H^{2/3}_{x,z}(\Omega)$ by Proposition 1.2.

• *Case of smooth coefficients.* We still consider coefficients α, γ that are smooth, satisfy (3.17), and we consider the equation

$$\begin{cases} z\partial_x U + \gamma\partial_z U - \alpha\partial_{zz} U = h \quad \text{in } \Omega, \\ U_{|\Sigma_i} = 0, \\ U_{|y=\pm 1} = 0. \end{cases} \tag{3.20}$$

Applying Fredholm's alternative to the operator $\mathcal{K}$ we obtain that

- either, for every $h \in L^2$, there exists a unique $U \in Z^0(\Omega)$ solution to (3.20),
- or there exists a nontrivial solution $U \in Z^0(\Omega)$ to (3.20) with $h = 0$.

The second possibility is excluded by the uniqueness of weak solutions stated in Proposition 3.2. Rewriting (3.20) as

$$z\partial_x U - \partial_{zz} U = h - \gamma \partial_z U + (\alpha - 1)\partial_{zz} U \tag{3.21}$$

and applying [34, Estimate (5.13)] (this time to the universal operator $z\partial_x - \partial_{zz}$), we obtain

$$\|U\|_{Z^0} \leq C_{\text{Pagani}} \left(\|h\|_{L^2} + \|\gamma\|_{L^2_z(L^\infty_x)} \|\partial_z U\|_{H^1_z(L^2_x)} + \|\alpha - 1\|_\infty \|\partial_{zz} U\|_{L^2} \right). \tag{3.22}$$

Hence, under condition (3.17) (up to choosing $c_0 < 1/(2C_{\text{Pagani}})$), the last two terms can be treated perturbatively and we obtain

$$\|U\|_{Z^0} \lesssim \|h\|_{L^2}, \tag{3.23}$$

with a constant depending only on the domain Ω.

• *Case of smooth coefficients with boundary data.* We still consider coefficients α, γ that are smooth and satisfy (3.17). By [34, Theorem 2.1], there exists a bounded linear map from $L : \mathscr{H}^1_z(\Sigma_0) \times \mathscr{H}^1_z(\Sigma_1) \to Z^0(\Omega)$ such that $U_\delta := L(\widetilde{\delta}_0, \widetilde{\delta}_1)$ satisfies $(U_\delta)_{|\Sigma_i} = \widetilde{\delta}_i$ and $(U_\delta)_{|y=\pm 1} = 0$. Then we look for a solution U to (3.4) under the form $U = U_\delta + V$, where V is a solution to (3.20) with

$$h := g - z\partial_x U_\delta + \alpha \partial_{zz} U_\delta - \gamma \partial_z U_\delta. \tag{3.24}$$

Under assumption (3.17),

$$\begin{aligned} \|h\|_{L^2} &\leq \|g\|_{L^2} + \|z\partial_x U_\delta\|_{L^2} + \|\alpha\|_\infty \|\partial_{zz} U_\delta\|_{L^2} + \|\gamma\|_{L^2_z(L^\infty_x)} \|\partial_z U\|_{L^\infty_z(L^2_x)} \\ &\lesssim \|g\|_{L^2} + \|U_\delta\|_{Z^0} \end{aligned} \tag{3.25}$$

so we obtain that U has Z^0 regularity and satisfies (3.16) by boundedness of L.

• *Case of general coefficients.* We then address the case of general coefficients satisfying (3.17). We take a smooth approximation sequence (α^n, γ^n) of (α, γ), which satisfies the same smallness assumptions and which converges towards (α, γ) in $L^\infty \cap H^1_z(L^\infty_x) \times L^2_z(L^\infty_x)$. For the sequence (α^n, γ^n), we construct a sequence of solutions $U^n \in Z^0$, which satisfy the estimate (3.16) with uniform bounds. Extracting a subsequence if necessary, we can find a function $U \in Z^0(\Omega)$ such that $U^n \rightharpoonup U$ in Z^0. Passing to the limit in the equation, it can be easily checked that U is a solution to (3.4). By Proposition 3.2, it is in fact the unique weak solution to (3.4), which completes the proof. □

3.4 Orthogonality conditions for higher tangential regularity

As in Section 2.3, we build solutions to (3.4) with higher regularity in the tangential direction, provided that the data satisfy appropriate orthogonality conditions. The main goal of this paragraph is to derive suitable expressions for these orthogonality conditions, analogous to the linear shear flow case.

3.4.1 Derivation of the equations for the dual profiles

We start by formally differentiating (3.4) with respect to x and we find that, if U is regular in the tangential direction, $V := \partial_x U$ is a solution to

$$\begin{cases} z\partial_x V + \gamma\partial_z V - \alpha\partial_{zz} V - \alpha_x \partial_{zz} \int_{x_0}^{x} V + \gamma_x \partial_z \int_{x_0}^{x} V = h_0 & \text{in } \Omega_+, \\ z\partial_x V + \gamma\partial_z V - \alpha\partial_{zz} V + \alpha_x \partial_{zz} \int_{x}^{x_1} V - \gamma_x \partial_z \int_{x}^{x_1} V = h_1 & \text{in } \Omega_-, \\ [V]_{z=0} = [\partial_z V]_{z=0} = 0 & \text{on } (x_0, x_1), \\ V(x_0, z) = \widetilde{\Delta}_0 & \text{for } z \in (0,1), \\ V(x_1, z) = \widetilde{\Delta}_1 & \text{for } z \in (-1,0), \\ V(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1), \end{cases} \tag{3.26}$$

where, for $i \in \{0,1\}$,

$$h_i := \partial_x g + \alpha_x \partial_{zz} \widetilde{\delta}_i - \gamma_x \partial_z \widetilde{\delta}_i, \tag{3.27}$$

and

$$\widetilde{\Delta}_i(z) := \frac{1}{z}\left(g(x_i, z) + \alpha(x_i, z)\partial_{zz}\widetilde{\delta}_i(z) - \gamma(x_i, z)\partial_z \widetilde{\delta}_i(z) \right). \tag{3.28}$$

Reciprocally, if V is a solution of the above system, then U defined by

$$U := \begin{cases} \widetilde{\delta}_0 + \int_{x_0}^{x} V & \text{in } \Omega_+, \\ \widetilde{\delta}_1 + \int_{x_1}^{x} V & \text{in } \Omega_- \end{cases} \tag{3.29}$$

is a solution to (3.4) if and only if V satisfies

$$\begin{aligned} \int_{x_0}^{x_1} V(x,0)\,\mathrm{d}x &= \widetilde{\delta}_1(0) - \widetilde{\delta}_0(0), \\ \int_{x_0}^{x_1} \partial_z V(x,0)\,\mathrm{d}x &= \partial_z \widetilde{\delta}_1(0) - \partial_z \widetilde{\delta}_0(0). \end{aligned} \tag{3.30}$$

For the time being, we do not worry about the regularity of the coefficients, and perform all computations as if the coefficients were smooth. A suitable definition of weak solutions of (3.26), which makes sense at the level of regularity available for the coefficients α and γ, will be given in Definition 3.12. Taking any function φ, sufficiently smooth on Ω_+ and Ω_- (but not necessarily continuous across the line $z = 0$) and vanishing on $\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the weak formulation of the above

system yields

$$
\begin{aligned}
&\int_{\Omega_+} h_0\varphi + \int_{\Omega_-} h_1\varphi - \int_{-1}^{0} z\widetilde{\Delta}_1(z)\varphi(x_1,z)\,\mathrm{d}z + \int_0^1 z\widetilde{\Delta}_0(z)\varphi(x_0,z)\,\mathrm{d}z \\
&\quad = \int_{\Omega_+\cup\Omega_-} V(-z\partial_x\varphi - \partial_{zz}(\alpha\varphi) - \partial_z(\gamma\varphi)) \\
&\quad - \int_{\Omega_+} V\left(\partial_{zz}\int_x^{x_1}\alpha_x\varphi + \partial_z\int_x^{x_1}\gamma_x\varphi\right) + \int_{\Omega_-} V\left(\partial_{zz}\int_{x_0}^{x}\alpha_x\varphi + \partial_z\int_{x_0}^{x}\gamma_x\varphi\right) \\
&\quad + \int_{x_0}^{x_1}\partial_z V(x,0)\Big(\int_x^{x_1}\alpha_x(x',0)\varphi(x',0^+)\,\mathrm{d}x' + \int_{x_0}^{x}\alpha_x(x',0)\varphi(x',0^-)\,\mathrm{d}x' \\
&\qquad\qquad + \alpha(x,0)\left(\varphi(x,0^+) - \varphi(x,0^-)\right)\Big)\,\mathrm{d}x \\
&\quad - \int_{x_0}^{x_1} V(x,0)\Big(\int_x^{x_1}\left(\partial_z(\alpha_x\varphi)(x',0^+) + (\gamma_x\varphi)(x',0^+)\right)\,\mathrm{d}x' \\
&\qquad\qquad + \int_{x_0}^{x}\left(\partial_z(\alpha_x\varphi)(x',0^-) + (\gamma_x\varphi)(x',0^-)\right)\,\mathrm{d}x' \\
&\qquad\qquad + \left(\partial_z(\alpha\varphi)(x,0^+) - \partial_z(\alpha\varphi)(x,0^-)\right) + (\gamma\varphi)(x,0^+) - (\gamma\varphi)(x,0^-)\Big)\,\mathrm{d}x.
\end{aligned}
\tag{3.31}
$$

Following the reasoning of Section 2.3, this leads to the following generalization of Lemma 2.6.

Definition 3.5 (Dual profiles)**.** *We define Φ^0 and Φ^1 as the weak solutions to*

$$
\begin{cases}
-z\partial_x\Phi^j - \partial_z(\gamma\Phi^j) - \partial_{zz}(\alpha\Phi^j) - \partial_{zz}\int_x^{x_1}\alpha_x\Phi^j - \partial_z\int_x^{x_1}\gamma_x\Phi^j = 0 & \text{in } \Omega_+, \\
-z\partial_x\Phi^j - \partial_z(\gamma\Phi^j) - \partial_{zz}(\alpha\Phi^j) + \partial_{zz}\int_{x_0}^{x}\alpha_x\Phi^j + \partial_z\int_{x_0}^{x}\gamma_x\Phi^j = 0 & \text{in } \Omega_-, \\
\Phi^j(x_0,z) = 0 & \text{for } z\in(-1,0), \\
\Phi^j(x_1,z) = 0 & \text{for } z\in(0,1), \\
\Phi^j(x,\pm1) = 0 & \text{on } (x_0,x_1),
\end{cases}
\tag{3.32}
$$

together with the jump conditions

$$
[\alpha\Phi^j]_{z=0}(x) + \int_x^{x_1}\alpha_x(x',0)\Phi^j(x',0^+)\,\mathrm{d}x' + \int_{x_0}^{x}\alpha_x(x',0)\Phi^j(x',0^-)\,\mathrm{d}x' = \mathbf{1}_{j=1}
\tag{3.33}
$$

and

$$
\begin{aligned}
[\partial_z(\alpha\Phi^j) + \gamma\Phi^j]_{z=0}(x) &+ \int_x^{x_1}\left(\partial_z(\alpha_x\Phi^j) + \gamma_x\Phi^j\right)(x',0^+)\,\mathrm{d}x' \\
&+ \int_{x_0}^{x}\left(\partial_z(\alpha_x\Phi^j) + \gamma_x\Phi^j\right)(x',0^-)\,\mathrm{d}x' = -\mathbf{1}_{j=0}.
\end{aligned}
\tag{3.34}
$$

In the next paragraphs, we prove existence and uniqueness of solutions to these dual systems.

3.4.2 Existence and uniqueness of the dual profiles

This paragraph is devoted to the proof of existence and uniqueness of the functions Φ^j, which will then allow us to state the orthogonality conditions for equation (3.4).

Proposition 3.6 (Existence and uniqueness of weak solutions to (3.32)-(3.33)-(3.34))**.** *Assume that α satisfies*

$$\|\alpha_z\|_\infty + \|\alpha - 1\|_\infty + \|\alpha_{xz}\|_{L^2} \ll 1. \tag{3.35}$$

Assume that $\gamma = z\gamma_1 + \gamma_2$, where $\gamma_1 \in L^2_z(L^\infty_x) \cap L^\infty_z(H^{1/2}_x)$, $\gamma_2 \in H^1_x(L^2_z)$ and

$$\|\gamma_1\|_{L^2_z(L^\infty_x)} + \|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\gamma_2\|_{H^1_x(L^2_z)} \ll 1. \tag{3.36}$$

Then the system (3.32)-(3.33)-(3.34) *has a unique weak solution Φ^j such that $\Phi^j \in L^2_x(H^1_z(\Omega_\pm))$ and $z\partial_x\Phi^j \in L^2_x(H^{-1}_z(\Omega_\pm))$ and*

$$\int_0^1 z\Phi^j(x_0,z)^2\,\mathrm{d}z - \int_{-1}^0 z\Phi^j(x_1,z)^2\,\mathrm{d}z < +\infty. \tag{3.37}$$

Furthermore, introducing

$$\Theta^j := \begin{cases} \alpha\Phi^j + \int_x^{x_1} \alpha_x\Phi^j & \text{in } \Omega_+, \\ \alpha\Phi^j - \int_{x_0}^{x} \alpha_x\Phi^j & \text{in } \Omega_-, \end{cases} \tag{3.38}$$

one has

$$\int_0^1 z\Phi^j(x_0,z)^2\,\mathrm{d}z - \int_{-1}^0 z\Phi^j(x_1,z)^2\,\mathrm{d}z + \sum_\pm \|\partial_z\Theta^j\|^2_{L^2(\Omega_\pm)} \lesssim 1. \tag{3.39}$$

Remark 3.7. *Note that $\|\alpha_{xz}\|_{L^2} \ll 1$ implies $\|\alpha_x\|_{L^\infty_z(L^2_x)} \ll 1$. The latter bound will be used several times in the proof.*

Remark 3.8 (Preliminary observations)**.** *Before tackling the proof of Proposition 3.6, we introduce some notations and tools which will be used throughout this section.*

- **Operators F and G, equation in terms of Θ^j:**

  First, let us consider the unknown Θ^j defined in (3.38)*. Formally, $\partial_x\Theta^j = \alpha\partial_x\Phi^j$, so that Φ^j can be retrieved from Θ^j thanks to the inversion formula*

$$\Phi^j := \begin{cases} \dfrac{\Theta^j}{\alpha} - \displaystyle\int_x^{x_1} \frac{\alpha_x}{\alpha^2}\Theta^j & \text{in } \Omega_+, \\ \dfrac{\Theta^j}{\alpha} + \displaystyle\int_{x_0}^{x} \frac{\alpha_x}{\alpha^2}\Theta^j & \text{in } \Omega_-. \end{cases} \tag{3.40}$$

  It follows in particular that $\|\Phi^j\|_{L^2} \lesssim \|\Theta^j\|_{L^2}$ and $\|\partial_z\Phi^j\|_{L^2(\Omega_\pm)} \lesssim \|\partial_z\Theta^j\|_{L^2(\Omega_\pm)}$ thanks to the bounds on α.

  We also define an operator

$$F[\Phi] := \gamma\Phi + \begin{cases} \int_x^{x_1} \gamma_x\Phi & \text{in } \Omega_+, \\ -\int_{x_0}^{x} \gamma_x\Phi & \text{in } \Omega_-. \end{cases} \tag{3.41}$$

  Note that $\partial_x F = \gamma\partial_x\Phi = \frac{\gamma}{\alpha}\partial_x\Theta$. Therefore, setting $\Gamma = \gamma/\alpha$, it will be convenient to define the operator

$$G[\Theta] := F[\Phi] = \Gamma\Theta + \begin{cases} \int_x^{x_1} \Gamma_x\Theta & \text{in } \Omega_+, \\ -\int_{x_0}^{x} \Gamma_x\Theta & \text{in } \Omega_-. \end{cases} \tag{3.42}$$

Under the assumptions of Proposition 3.6, we can define $\Gamma := z\Gamma_1 + \Gamma_2$, *with* $\Gamma_i := \gamma_i/\alpha$, *and*

$$\|\Gamma_1\|_{L^2_z(L^\infty_x)} + \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\Gamma_2\|_{H^1_x(L^2)} \ll 1. \tag{3.43}$$

The $L^\infty_z(H^{1/2}_x)$ *bound for* Γ_1 *stems from the same bound for* γ_1, *the* $L^\infty_z(L^2_x)$ *bound for* α_x *and the pointwise multiplication result Lemma B.3.*

We set $F^j := F[\Phi^j] = G[\Theta^j]$, *so that* (3.32) *becomes, with these functions,*

$$-\frac{z}{\alpha}\partial_x\Theta^j - \partial_z F^j - \partial_{zz}\Theta^j = 0 \quad \text{in } \Omega_+ \cup \Omega_-, \tag{3.44}$$

while the jump conditions (3.33)-(3.34) *ensure that*

$$\begin{aligned} [\Theta^j]_{|z=0} &= \mathbf{1}_{j=1}, \\ [\partial_z\Theta^j + F^j]_{|z=0} &= -\mathbf{1}_{j=0}. \end{aligned} \tag{3.45}$$

- **Lifts for** Θ^j**:**

It will be convenient, in the course of the proof, to introduce a lift for Θ^j *in order to remove the jumps across the line* $\{z = 0\}$. *Actually, we will introduce two such lifts: one lift which will remove the jump of* Θ^j *and of* $F^j + \partial_z\Theta^j$ *(but which will authorize a jump in the* z*-derivative of the lifted function), and one lift such that the lifted function will be* H^2 *in* z.

More precisely, we set

$$\Theta^j = (\delta_{0,j} - z\delta_{1,j})\chi(z)\mathbf{1}_{\Omega_+} + \Theta^j_\sharp,$$

where $\chi \in C^\infty_c((-1,1))$ *is such that* $\chi \equiv 1$ *in a neighbourhood of zero, so that* $\Theta^j_\sharp$ *is such that*

$$[\Theta^j_\sharp]_{|z=0} = 0, \quad [\partial_z\Theta^j_\sharp + F^j]_{|z=0} = 0. \tag{3.46}$$

The lifted function $\Theta^j_\sharp$ *satisfies*

$$\begin{aligned} -\frac{z}{\alpha}\partial_x\Theta^j_\sharp - \partial_z(\partial_z\Theta^j_\sharp + F^j) &= \mathbf{1}_{z>0}\partial_{zz}\left((\delta_{0,j} - z\delta_{1,j})\chi(z)\right), \\ \Theta^j_{\sharp|z=\pm1} &= 0, \\ \Theta^j_\sharp(x_0, z) &= 0 \quad \forall z \in (-1,0), \\ \Theta^j_\sharp(x_1, z) &= -(\delta_{0,j} + z\delta_{1,j})\chi(z) \forall z \in (0,1). \end{aligned} \tag{3.47}$$

We will also consider another lift $\Theta^j_\flat$, *which we define in the following way. First, we set*

$$\Theta^j_\sharp = \Theta^j_l + \Theta^j_\flat,$$

where the lifting term Θ^j_l *is given by*

$$\Theta^j_l = zb_+(x)\chi(z)\mathbf{1}_{\Omega_+} + zb_-(x)\chi(z)\mathbf{1}_{\Omega_-}$$

with some coefficients $b_\pm$ *to be determined. The role of* Θ^j_l *is to ensure that*

$$[\Theta^j_\flat] = 0, \quad [\partial_z\Theta^j_\flat] = 0.$$

Note that the first condition is automatically satisfied with our choice above. Furthermore,

$$\partial_x[F^j(x,\cdot)]_{|z=0} = \gamma(x,0)\partial_x[\Phi^j(x,\cdot)]_{|z=0} = \frac{\gamma(x,0)}{\alpha(x,0)}\partial_x[\Theta^j(x,\cdot)] = 0.$$

Hence the jump of F^j is constant across the line $\{z=0\}$, and is equal to

$$\frac{1}{x_1-x_0}\int_{x_0}^{x_1}[F^j(x,\cdot)]_{|z=0}\,\mathrm{d}x.$$

Consequently, we choose $b_\pm$ to be constant in x, and we define

$$b_+ := -\frac{1}{x_1-x_0}\int_{x_0}^{x_1} F^j(x,0^+)\,\mathrm{d}x,$$
$$b_- := \frac{1}{x_1-x_0}\int_{x_0}^{x_1} F^j(x,0^-)\,\mathrm{d}x.$$

With this choice,

$$\left[F^j + (b_+\mathbf{1}_{z>0} - b_-\mathbf{1}_{z<0})\right]_{|z=0} = 0,$$

and therefore $[\partial_z\Theta^j_\flat]_{|z=0}=0$, and $[F^j+\partial_z\Theta^j_l]_{|z=0}=0$. Note also that $\partial_x\Theta^j_l=0$. It follows that

$$-\frac{z}{\alpha}\partial_x\Theta^j_\flat - \partial_{zz}\Theta^j_\flat = S^j \quad \text{in } \Omega, \tag{3.48}$$

where

$$S^j = \mathbf{1}_{z>0}\partial_{zz}\left((\delta_{0,j}-z\delta_{1,j})\chi(z)\right) + \partial_z\left(\partial_z\Theta^l_j + F^j\right).$$

The boundary conditions for $\Theta^j_\flat$ are

$$\Theta^j_\flat(x,\pm1) = 0,$$
$$\Theta^j_\flat(x_0,z) = -b_-z\chi(z) \quad \forall z\in(-1,0),$$
$$\Theta^j_\flat(x_1,z) = -(\delta_{0j}+z(-\delta_{1,j}+b_+))\chi(z) \quad \forall z\in(0,1).$$

Proof of Proposition 3.6. Throughout the proof, we will actually consider existence and uniqueness in $L^2_x(H^1_z)$ of solutions of the system

$$\begin{aligned} -\partial_z(\partial_z\Theta + G[\Theta]) - \frac{z}{\alpha}\partial_x\Theta &= f \quad \text{in } \Omega,\\ \Theta(x,\pm1) &= 0,\\ \Theta(x_1,z) &= \eta_1 \quad \forall z>0,\\ \Theta(x_0,z) &= \eta_0 \quad \forall z<0, \end{aligned} \tag{3.49}$$

where $\eta_0\in C^2([-1,0])$, $\eta_1\in C^2([0,1])$ and $f\in L^2_x(H^{-1}_z)$. In the above equation, the operator G is defined in (3.42).

According to Remark 3.8, it will then be sufficient to apply the existence and uniqueness result to the case

$$f = \mathbf{1}_{z>0}\partial_{zz}\left((\delta_{0,j}+z\delta_{1,j})\chi(z)\right),$$
$$\eta_0 = 0,$$
$$\eta_1(z) = -(\delta_{0,j}+z\delta_{1,j})\chi(z).$$

Classically, we say that $\Theta \in L^2((x_0,x_1),H^1_0(-1,1))$ is a weak solution of (3.49) if the following weak formulation holds: for any $V \in H^1(\Omega)$ such that $V_{|z=\pm 1}=0$ and $V_{|\Sigma_i}=0$,

$$\int_\Omega \partial_z\Theta\partial_z V + \int_\Omega G[\Theta]\partial_z V + \int_\Omega z\Theta\partial_x\left(\frac{V}{\alpha}\right) \tag{3.50}$$

$$= \int_\Omega fV + \int_0^1 \eta_1 \frac{z}{\alpha(x_1,z)}V(x_1,z)\,\mathrm{d}z - + \int_{-1}^0 \eta_0\frac{z}{\alpha(x_0,z)}V(x_0,z)\,\mathrm{d}z. \tag{3.51}$$

• *First step: Bound on the operator G.*

The purpose of this first step is to prove the following bound: if $\Theta \in L^2_x(H^1_z)$ is such that $z\partial_x\Theta \in L^2_x(H^{-1}_z)$, and if $\Phi(x_0,z)=0$ for all $z<0$, $\Phi(x_1,z)=0$ for all $z>0$, then $G[\Theta]\in L^2(\Omega)$ and

$$\begin{aligned}\|G[\Theta]\|_{L^2} \leq & C\left(\|\Gamma_2\|_{H^1_xL^2_z} + \|\Gamma_1\|_{L^2_z(L^\infty_x)}\right)\|\partial_z\Theta\|_{L^2}\\ &+ C\|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}(\|z\partial_x\Theta\|_{L^2_x(H^{-1}_z)} + \|\partial_z\Theta\|_{L^2}).\end{aligned} \tag{3.52}$$

Concerning the term with Γ_2, since $\partial_x\Gamma_2\in L^2$, we have

$$\left\|\Gamma_2\Theta + \int_x^{x_1}\partial_x\Gamma_2\Theta\right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_2\|_{H^1_xL^2_z}\|\Theta\|_{L^\infty_z(L^2_x)} \lesssim \|\Gamma_2\|_{H^1_xL^2_z}\|\partial_z\Theta\|_{L^2(\Omega_+)}. \tag{3.53}$$

A similar bound holds in Ω_-. The term with γ_1 is more involved. First, we have

$$\|z\Gamma_1\Theta\|_{L^2(\Omega)} \lesssim \|\Gamma_1\|_{L^2_z(L^\infty_x)}\|\Theta\|_{L^\infty_z(L^2_x)} \lesssim \|\Gamma_1\|_{L^2_z(L^\infty_x)}\|\partial_z\Theta\|_{L^2(\Omega_+)}. \tag{3.54}$$

Concerning the integral term, we use Lemma B.15 in the Appendix, from which we deduce that $z\Theta\in L^2((0,1),H^{1/2}_{00r})$. We then observe that by definition of the L^2 norm,

$$\begin{aligned}\left\|\int_x^{x_1} z\partial_x\Gamma_1\Theta\right\|_{L^2(\Omega_+)} &= \sup_{h\in L^2(\Omega_+),\|h\|_{L^2}\leq 1}\int_{\Omega_+} h\int_x^{x_1} z\partial_x\Gamma_1\Theta\\ &= \sup_{h\in L^2(\Omega_+),\|h\|_{L^2}\leq 1}\int_{\Omega_+}\left(\int_{x_0}^x h\right)z\Theta\partial_x\Gamma_1.\end{aligned}$$

Now, using Lemma B.8, for any $z>0$,

$$\left|\int_{x_0}^{x_1}\left(\int_{x_0}^x h\right)z\Theta\partial_x\Gamma_1\right| \leq C\|\Gamma_1\|_{H^{1/2}_x}\|z\Theta\|_{H^{1/2}_{00r}}\|h\|_{L^2((x_0,x_1)}. \tag{3.55}$$

Integrating with respect to z and using a Cauchy-Schwarz inequality, we get

$$\left\|\int_x^{x_1} z\partial_x\Gamma_1\Theta\right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}\|z\Theta\|_{L^2_x(H^{1/2}_{00r})} \tag{3.56}$$

$$\lesssim \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}\left(\|z\partial_x\Theta\|_{L^2(H^{-1})} + \|\partial_z\Theta\|_{L^2(\Omega_+)}\right). \tag{3.57}$$

Gathering (3.53), (3.54) and (3.56), we obtain (3.52).

In the rest of the proof, to lighten the notation, we set

$$\mu_0 := \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\Gamma_1\|_{L^2_z(L^\infty_x)} + \|\Gamma_2\|_{H^1_xL^2_z} + \|\alpha-1\|_\infty + \|\alpha_z\|_\infty + \|\alpha_x\|_{L^\infty_z(L^2_x)}.$$

• *Second step: uniqueness:*

Let $\Theta \in L^2((x_0,x_1),H_0^1(-1,1))$ such that $\alpha^{-1}z\partial_x\Theta \in L^2(H^{-1})$ be a weak solution of (3.49) with $f=0$, $\eta_i=0$. Note that since $\alpha_z \in L^\infty(\Omega)$, $z\partial_x\Theta \in L^2(H^{-1})$.

We adapt the arguments of Baouendi and Grisvard (see Appendix A). For any $V \in L^2_x(H^1_0)$ such that $z\partial_x V \in L^2_x(H_z^{-1})$, the trace $V_{|x=x_i}$ is well-defined in the sense of functions in $\mathscr{L}^2(-1,1)$ thanks to Lemma A.1, and we have

$$\langle \frac{z}{\alpha}\partial_x\Theta, V\rangle_{L^2(H^{-1},L^2(H^1)} = \int_\Omega G[\Theta]\partial_z V + \int_\Omega \partial_z\Theta\partial_z V. \tag{3.58}$$

Furthermore,

$$\begin{aligned}
\left\langle \frac{z}{\alpha}\partial_x\Theta, V\right\rangle_{L^2(H^{-1}),L^2(H^1_0)} &= \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \langle z\left(\frac{1}{\alpha}-1\right)\partial_x\Theta, V\rangle_{L^2(H^{-1},L^2(H^1_0)} \\
&= \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \left\langle \frac{z}{\alpha}\partial_x\Theta, (\alpha-1)V\right\rangle_{L^2(H^{-1}),L^2(H^1_0)} \\
&= \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H^1_0)} \qquad (3.59)\\
&\quad + \int_\Omega G[\Theta]\partial_z((\alpha-1)V) + \int_\Omega \partial_z\Theta\partial_z((\alpha-1)V). \qquad (3.60)
\end{aligned}$$

From there, it follows that

$$\begin{aligned}
\|z\partial_x\Theta\|_{L^2(H^{-1})} &= \sup_{V\in L^2(H^1_0),\|V\|_{L^2H^1}\le 1} \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H^1)} \qquad (3.61)\\
&\le (\|G[\Theta]\|_{L^2} + \|\partial_z\Theta\|_{L^2})\,(1+\|\alpha-1\|_\infty + \|\alpha_z\|_{L^2_z(L^\infty_x)}). \qquad (3.62)
\end{aligned}$$

Using (3.52) together with the smallness assumption on Γ_1, we obtain

$$\begin{aligned}
\|z\partial_x\Theta\|_{L^2(H^{-1})} &\lesssim \|\partial_z\Theta\|_{L^2},\\
\|G[\Theta]\|_{L^2} &\le \mu_0\|\partial_z\Theta\|_{L^2}.
\end{aligned} \tag{3.63}$$

Eventually, thanks to Corollary A.2, we observe that

$$\begin{aligned}
\langle z\partial_x\Theta,\Theta\rangle_{L^2(H^{-1}),L^2(H^1)} &= \frac{1}{2}\left(\int_{-1}^1 z\Theta(x_1,z)^2\,\mathrm{d}z - \int_{-1}^1 z\Theta(x_0,z)^2\,\mathrm{d}z\right) \qquad (3.64)\\
&= \frac{1}{2}\left(\int_{-1}^0 z\Theta(x_1,z)^2\,\mathrm{d}z - \int_0^1 z\Theta(x_0,z)^2\,\mathrm{d}z\right). \qquad (3.65)
\end{aligned}$$

Now, take $V=\Theta$ in (3.58). Using (3.60), (3.63) and (3.65), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\int_0^1 z\Theta(x_0,z)^2\,\mathrm{d}z - \int_{-1}^0 z\Theta(x_1,z)^2\,\mathrm{d}z\right) + \int_\Omega \partial_z\Theta^2\\
\le\ & \|G[\Theta]\|_{L^2}\left(\|\partial_z\Theta\|_{L^2} + \|\partial_z((\alpha-1)\Theta)\|_{L^2}\right) + \|\partial_z\Theta\|_{L^2}\|\partial_z((\alpha-1)\Theta\|_{L^2}\\
\lesssim\ & \mu_0\|\partial_z\Theta\|^2_{L^2}.
\end{aligned}$$

Hence, for $\mu_0<1$, we infer that $\Theta=0$.

• *Third step: Energy estimates for equation* (3.49)

First, since we have assumed that the boundary terms η_0, η_1 are smooth, note that we can always lift them with a function $\Theta_\eta(x,z) = \chi_0(x)\eta_0(z) + \chi_1(x)\eta_1(z)$ for some cut-off functions

$\chi_i \in C^\infty([x_0, x_1])$ such that $\chi_i \equiv 1$ in a neighborhood of x_i and $\mathrm{Supp}\chi_i \subset \{|x - x_i| \leq (x_1 - x_0)/2\}$. This will add a smooth additional source term to the equation. Hence, in the following, we will consider the case $\eta_i = 0$, without loss of generality.

Let us multiply formally (3.49) by Θ and integrate on Ω. We obtain

$$\int_\Omega (\partial_z \Theta)^2 = \left\langle \frac{z}{\alpha} \partial_x \Theta, \Theta \right\rangle_{L^2(H^{-1}), L^2(H^1)} + \langle f, \Theta \rangle_{L^2(H^{-1}), L^2(H^1)} - \int_\Omega G[\Theta] \partial_z \Theta. \tag{3.66}$$

As in the previous step, we decompose the first term in the right-hand side as follows

$$\begin{aligned} \left\langle \frac{z}{\alpha} \partial_x \Theta, \Theta \right\rangle_{L^2(H^{-1}), L^2(H^1)} &= \langle z \partial_x \Theta, \Theta \rangle_{L^2(H^{-1}), L^2(H^1)} & (3.67) \\ &\quad + \left\langle \frac{z}{\alpha} \partial_x \Theta, (\alpha - 1)\Theta \right\rangle_{L^2(H^{-1}), L^2(H^1)}. & (3.68) \end{aligned}$$

The term (3.67) yields a positive contribution on $\Sigma_0 \cup \Sigma_1$, namely

$$- \langle z \partial_x \Theta, \Theta \rangle_{L^2(H^{-1}), L^2(H^1)} = \frac{1}{2} \int_0^1 z(\Theta(x_0, z))^2 \, \mathrm{d}z - \frac{1}{2} \int_{-1}^0 z(\Theta(x_1, z))^2 \, \mathrm{d}z. \tag{3.69}$$

We bound (3.68) as follows

$$\left| \left\langle \frac{z}{\alpha} \partial_x \Theta, (\alpha - 1)\Theta \right\rangle_{L^2(H^{-1}), L^2(H^1)} \right| \leq \left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2_x(H^{-1}_z)} \|(\alpha - 1)\Theta\|_{L^2_x(H^1_z)}. \tag{3.70}$$

Using the equation (3.47) together with (3.52),

$$\begin{aligned} \left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2(H^{-1})} &\leq C \left(\|f\|_{L^2(H^{-1})} + \|\partial_z \Theta\|_{L^2} + \|G[\Theta]\|_{L^2} \right) \\ &\leq C \left(\|f\|_{L^2(H^{-1})} + \|\partial_z \Theta\|_{L^2} \right) + \mu_0 \|z \partial_x \Theta\|_{L^2(H^{-1})} \end{aligned}, \tag{3.71}$$

while

$$\|(\alpha - 1)\Theta\|_{L^2_x(H^1_z)} \lesssim \left(\|\alpha_z\|_{L^2_z(L^\infty_x)} + \|\alpha - 1\|_\infty \right) \|\partial_z \Theta\|_{L^2(\Omega_+)} \lesssim \mu_0 \|\partial_z \Theta\|_{L^2(\Omega_+)}.$$

Using the same type of computation as in (3.60), we also obtain that

$$\|z \partial_x \Theta\|_{L^2(H^{-1})} \leq C(\|\partial_z \Theta\|_{L^2} + \|f\|_{L^2(H^{-1})}),$$

for some universal constant C, provided $\mu_0 < 1$. Hence

$$\|G[\Theta]\|_{L^2} \lesssim \mu_0(\|\partial_z \Theta\|_{L^2} + \|f\|_{L^2(H^{-1})}).$$

Gathering the previous estimates, we obtain, if μ_0 is small enough,

$$\begin{aligned} \|\partial_z \Theta\|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^1 z(\Theta(x_0, z))^2 \, \mathrm{d}z - \frac{1}{2} \int_{-1}^0 z(\Theta(x_1, z))^2 \, \mathrm{d}z \leq C \|f\|^2_{L^2(H^{-1})}, \\ \|z \partial_x \Theta\|_{L^2_x(H^{-1}_z)} \leq C \|f\|_{L^2(H^{-1})}. \end{aligned} \tag{3.72}$$

• *Fourth step: Existence of weak solutions for smooth coefficients, with a large zero order term.* We assume in this paragraph that the coefficients are smooth, say $\alpha, \gamma_i \in C^2_b(\Omega)$.

The purpose is to prove that for $C_0 > 0$ sufficiently large, depending on some norms of the coefficients, the equation

$$\begin{aligned} -\frac{z}{\alpha}\partial_x\Theta - \partial_z(G[\Theta] + \partial_z\Theta) + C_0\Theta = f \quad \text{in } \Omega, \\ \Theta(x,\pm 1) = 0, \\ \Theta(x_1, z) = 0 \quad \forall z > 0, \\ \Theta(x_0, z) = 0 \quad \forall z < 0 \end{aligned} \tag{3.73}$$

has a unique solution in $L^2_x(H^1_z)$.

We proceed by viscous regularization and consider, for every $\varepsilon > 0$, the system

$$\begin{aligned} -\frac{z}{\alpha}\partial_x\Theta_\varepsilon - \partial_z G[\Theta_\varepsilon] + \partial_{zz}\Theta_\varepsilon - \varepsilon\partial_{xx}\Theta^j_\varepsilon + C_0\Theta_\varepsilon = f \quad \text{in } \Omega_\pm, \\ \Theta_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.74}$$

We then perform the same type of estimates as before. Since we allow the coefficients to be smooth and the constant C_0 to be large, the estimates are somewhat simpler. It can be easily proved that if

$$C_0 \gtrsim \|\alpha_x\|_\infty + \|\gamma_x\|_\infty,$$

the equation (3.74) has a unique solution in $H^1_0(\Omega)$, which satisfies uniform in ε bounds in $L^2_x(H^1_z)$. Passing to the limit as $\varepsilon \to 0$, we deduce that for the above choice of C_0, there exists a solution $\Theta \in L^2(H^1)$ of (3.73). Using the equation, we infer that this solution is such that $z\partial_x\Theta \in L^2_x(H^{-1}_z)$. Using the second step (or a variant including the term C_0), we deduce that this solution is unique.

• *Fifth step: Existence of weak solutions of* (3.49) *for smooth coefficients and* L^2 *source.*

Now, for any $f \in L^2(\Omega)$, consider the solution of (3.73). As in Remark 3.8, we introduce coefficients $b_\pm$ defined by

$$b_\pm := \mp\frac{1}{x_1 - x_0}\int_{x_0}^{x_1} G[\Theta](x, 0^\pm)\,\mathrm{d}x.$$

and a function $\Theta_\flat$ such that

$$\Theta = \sum_\pm z b_\pm \chi(z)\mathbf{1}_{\pm z > 0} + \Theta_\flat.$$

Note that

$$|b_\pm| \lesssim \|\partial_x\gamma_2\|_\infty \|\partial_z\Theta\|_{L^2},$$

so that $\|\partial_z\Theta_\flat\|_{L^2} \lesssim \|\partial_z\Theta\|_{L^2}$. The function $\Theta_\flat$ is a solution of

$$-\frac{z}{\alpha}\partial_x\Theta_\flat - \partial_{zz}\Theta_\flat = S_\flat,$$

where

$$S_\flat = f + \partial_z\left(G[\Theta] - \partial_z\left(\sum_\pm z b_\pm\chi(z)\mathbf{1}_{\pm z>0}\right)\right) \in L^2(\Omega).$$

According to the results of Pagani [34], $\Theta_\flat \in Z^0$. In particular, $\Theta_\flat \in H^{2/3}_x(L^2_z)$. Since $\Theta_\flat$ is constant in x, it follows that $\Theta \in H^{2/3}_x(L^2_z)$. As a consequence, the operator $\mathcal{K} : f \in L^2 \mapsto \Theta \in H^{2/3}_x(L^2_z) \cap L^2_x(H^1_z)$, where Θ is the solution of (3.73), is compact.

We now apply the Fredholm alternative to the operator $\mathcal{K}$, which implies the following:

(i) either, for any $f \in L^2(\Omega)$, there exists a unique solution in $H^{2/3}_x(L^2_z) \cap L^2_x(H^1_z)$ of (3.49);

(ii) or there exists a non-trivial solution $\bar{\Theta}$ of

$$\begin{aligned} -\partial_z(\partial_z\bar{\Theta} + G[\bar{\Theta}]) - \frac{z}{\alpha}\partial_x\bar{\Theta} = 0 \quad \text{in } \Omega, \\ \bar{\Theta}(x, \pm 1) = 0, \\ \bar{\Theta}(x_1, z) = 0 \quad \forall z > 0, \\ \bar{\Theta}(x_0, z) = 0 \quad \forall z < 0; \end{aligned}$$

According to the second step (uniqueness for (3.49)), the second case never arises.

We deduce that when the coefficients α, γ are smooth and satisfy the smallness assumptions above, for any source term $f \in L^2$, there exists a unique solution of (3.49).

• *Sixth step: Existence of weak solutions of* (3.49) *for general coefficients and source terms.*

We argue by density and consider sequences of smooth coefficients $\alpha^n, \gamma_1^n, \gamma_2^n$ converging towards $\alpha, \gamma_1, \gamma_2$ in the relevant norms (i.e. in the norms in which we stated the smallness assumptions.) We also consider a sequence $(f_n)_{n\in\mathbb{N}}$ of L^2 functions such that $f_n \to f$ in $L^2(H^{-1})$.

For any $n \in \mathbb{N}$, there exists a solution Θ^n of (3.49) with the coefficients $\alpha^n, \gamma_1^n, \gamma_2^n$ and source term f_n. Furthermore, the third step shows that this solution is uniformly bounded in $L^2(H^1)$, and $z\partial_x\Theta^n$ is uniformly bounded in $L^2(H^{-1})$. Hence we can extract a subsequence such that

$$\Theta^n \rightharpoonup \Theta \quad \text{in } L^2_x(H^1_z).$$

The limit is a weak solution of (3.49). This concludes the proof.

□

During the proof, we actually obtained the following result, which will be used hereafter.

Corollary 3.9. *Assume that the coefficients α and γ satisfy the assumptions of Proposition 3.6. For every $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$, there exists a unique solution $\Theta \in L^2_x(H^1_z)$ to*

$$\begin{aligned} -\frac{z}{\alpha}\partial_x\Theta - \partial_z(G[\Theta] + \partial_z\Theta) = f, \\ \Theta(\cdot, \pm 1) = 0, \\ \Theta(x_1, z) = 0 \quad \forall z > 0, \\ \Theta(x_0, z) = 0 \quad \forall z < 0. \end{aligned} \tag{3.75}$$

Furthermore, this solution satisfies

$$\|\Theta\|_{L^2(H^1)} + \|z\partial_x\Theta\|_{L^2(H^{-1})} \lesssim \|f\|_{L^2(H^{-1})}. \tag{3.76}$$

3.4.3 Orthogonality conditions for regular solutions

Going back to conditions (3.30) and using the definition of the functions Φ^j, we infer that the orthogonality conditions for (3.4) in order to have $H^1_x H^1_z$ regularity can be formally written as

$$\begin{aligned} \int_\Omega (\mathbf{1}_{z>0}h_0 + \mathbf{1}_{z<0}h_1)\Phi^0 + \int_{\Sigma_0} z\widetilde{\Delta}_0\Phi^0 - \int_{\Sigma_1} z\widetilde{\Delta}_1\Phi^0 = \delta_1'(0) - \delta_0'(0), \\ \int_\Omega (\mathbf{1}_{z>0}h_0 + \mathbf{1}_{z<0}h_1)\Phi^1 + \int_{\Sigma_0} z\widetilde{\Delta}_0\Phi^1 - \int_{\Sigma_1} z\widetilde{\Delta}_1\Phi^1 = \delta_1(0) - \delta_0(0). \end{aligned} \tag{3.77}$$

However, since $\partial_x \gamma_1$ does not belong to L^2, the source terms h_i do not belong to L^2 in general. Therefore we first perform formal transformations in order to re-write the terms involving γ_1 in a weaker form. We recall that

$$z\widetilde{\Delta}_i(z) = g(x_i, z) + \alpha_x(x_i, z)\widetilde{\delta}_i''(z) - \gamma(x_i, z)\widetilde{\delta}_i'(z).$$

When γ_1 is smooth, we have

$$\begin{aligned}
&-\int_0^1 z\gamma_1(x_0, z)\widetilde{\delta}_0'(z)\Phi^j(x_0, z)\, dz - \int_{\Omega_+} z\partial_x\gamma_1(x, z)\widetilde{\delta}_0'(z)\Phi^j(x, z)\, dx\, dz \\
&= \int_{\Omega_+} \gamma_1(x, z)\widetilde{\delta}_0'(z) z\partial_x\Phi^j(x, z)\, dx\, dz \\
&= \int_{\Omega_+} \Gamma_1(x, z)\widetilde{\delta}_0'(z) z\partial_x\Theta^j(x, z)\, dx\, dz.
\end{aligned} \tag{3.78}$$

Similarly,

$$\begin{aligned}
&\int_{-1}^0 z\gamma_1(x_1, z)\widetilde{\delta}_1'(z)\Phi^j(x_1, z)\, dz - \int_{\Omega_-} z\partial_x\gamma_1(x, z)\widetilde{\delta}_1'(z)\Phi^j(x, z)\, dx\, dz \\
&= \int_{\Omega_-} \Gamma_1(x, z)\widetilde{\delta}_1'(z) z\partial_x\Theta^j(x, z)\, dx\, dz.
\end{aligned} \tag{3.79}$$

Now, assume that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$ and $\partial_z\gamma_1 \in L^2$. Then $\Gamma_1(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0') \in L^2_x(H^1_0)$. And

$$\begin{aligned}
&\int_{\Omega_+} \Gamma_1(x, z)\widetilde{\delta}_0'(z) z\partial_x\Theta^j(x, z)\, dx\, dz + \int_{\Omega_-} \Gamma_1(x, z)\widetilde{\delta}_1'(z) z\partial_x\Theta^j(x, z)\, dx\, dz \\
&= \left\langle z\partial_x\Theta^j, \Gamma_1(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0')\right\rangle_{L^2(H^{-1}), L^2(H^1_0)}.
\end{aligned} \tag{3.80}$$

Eventually, recalling the definition of Θ^j and h_i, we re-write the orthogonality conditions (3.77) as

$$\begin{aligned}
&\int_\Omega \partial_x g\Phi^j + \int_{\Sigma_0} (g\Phi^j)_{|\Sigma_0} + \int_{\Sigma_0} (\widetilde{\delta}_0'' - \Gamma_{2|\Sigma_0}\widetilde{\delta}_0')\Theta^j_{|\Sigma_0} \\
&- \int_{\Sigma_1} (g\Phi^j)_{|\Sigma_1} - \int_{\Sigma_1} (\widetilde{\delta}_1'' - \Gamma_{2|\Sigma_1}\widetilde{\delta}_1')\Theta^j_{|\Sigma_1} + \left\langle z\partial_x\Theta^j, \Gamma_1(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0')\right\rangle_{L^2(H^{-1}), L^2(H^1_0)} \\
&= \partial_z^j\delta_1(0) - \partial_z^j\delta_0(0).
\end{aligned} \tag{3.81}$$

Definition 3.10 (Linear forms associated with equation (3.4))**.** *Let $(\delta_0, \delta_1, f) \in \mathcal{H}$. We define $g = f(x, Y(x, z))$, $\widetilde{\delta}_i(z) = \delta_i(Y(x_i, z))$. Assume that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$ and $\partial_z\gamma_1 \in L^2$. Assume furthermore that*

$$\int_{\Sigma_i} \frac{1}{|z|}\left[\frac{g}{\alpha} + \left(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2\right)\right]^2 < +\infty.$$

We define, for $j = 0, 1$,

$$\begin{aligned}\ell^j_{\bar u}(f,\delta_0,\delta_1) := &\int_\Omega \partial_x g\frac{\Theta^j}{\alpha} + \int_{\Sigma_0}\Big(\frac{g}{\alpha} + \widetilde{\delta}_0'' - \Gamma_{2|\Sigma_0}\widetilde{\delta}_0'\Big))\Theta^j_{|\Sigma_0}\\ &- \int_{\Sigma_1}\Big(\frac{g}{\alpha} + \widetilde{\delta}_1'' - \Gamma_{2|\Sigma_1}\widetilde{\delta}_1'\Big)\Theta^j_{|\Sigma_1}\\ &+ \Big\langle z\partial_x\Theta^j, \Gamma_1(\mathbf{1}_{z<0}\widetilde{\delta}_1' + \mathbf{1}_{z>0}\widetilde{\delta}_0')\Big\rangle_{L^2(H^{-1}),L^2(H^1_0)}\\ &- \partial_z^j\widetilde{\delta}_1(0) + \partial_z^j\widetilde{\delta}_0(0).\end{aligned} \tag{3.82}$$

Remark 3.11. *Assume that Y is such that $\bar u(x, Y(x,z)) = z$ for all $(x,z) \in \Omega$, and that $\alpha(x,Y(x,z)) = \bar u_y^2(x,Y(x,z))$, $\gamma_2 = -\bar u_{yy}(x,Y(x,z))$.*

Then it is easily checked that

$$\begin{aligned}\widetilde{\delta}_i'(z) &= \frac{1}{\bar u_y(x_i, Y(x_i,z))}\delta_i'(Y(x_i,z)),\\ \widetilde{\delta}_i'(z) &= \Gamma_2(x_i,z)\widetilde{\delta}_i'(z) + \frac{1}{\alpha(x_i,z)}\delta_i''(Y(x_i,z)).\end{aligned} \tag{3.83}$$

In this case, the boundary terms in the definition of $\ell^j_{\bar u}$ can be written as

$$\int_{\Sigma_i}\Big(\frac{g}{\alpha} + \widetilde{\delta}_i'' - \Gamma_{2|\Sigma_i}\widetilde{\delta}_i'\Big)\Theta^j_{|\Sigma_i} = \int_{\Sigma_i}\frac{1}{\alpha(x_i,z)}(f(x_i,Y(x_i,z)) + \delta_i''(Y(x_i,z))\Theta^j(x_i,z)\,dz. \tag{3.84}$$

We are now ready to formulate our regularity results for solutions of (3.4). To that end, we first give a definition of weak solutions of (3.26). This definition follows the computations from (3.31) and Remark 3.8.

Definition 3.12. *Assume that the coefficients α, γ_i satisfy the assumptions of Proposition 3.6. Assume furthermore that that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$, $\partial_z\gamma_1 \in L^2$, and*

$$\int_{\Sigma_i}\frac{1}{|z|}\Big[\frac{g}{\alpha} + \Big(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2\Big)\Big]^2 < +\infty.$$

Let $V \in L^2_x(H^1_0(-1,1))$. We say that V is a weak solution of (3.26) if and only if, for any $\Theta \in L^2_x(H^1_0(-1,1))$ such that $z\partial_x\Theta \in L^2_x(H^{-1}_z)$ and such that $\Phi = 0$ on $\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)$,

$$\begin{aligned}&-\Big\langle\frac{z}{\alpha}\partial_x\Theta, V\Big\rangle_{L^2H^{-1},L^2H^1_0} + \int_\Omega G[\Theta]\partial_z V + \int_\Omega\partial_z\Theta\partial_z V\\ &= \int_\Omega\partial_x g\frac{\Theta}{\alpha}\\ &- \int_{-1}^0\Big[\frac{g}{\alpha}(x_1,z) + \Big(\widetilde{\delta}_0''(z) - \widetilde{\delta}_0'(z)\Gamma_2(x_1,z)\Big)\Big]\Theta(x_1,z)\,dz\\ &+ \int_0^1\Big[\frac{g}{\alpha}(x_0,z) + \Big(\widetilde{\delta}_1''(z) - \widetilde{\delta}_1'(z)\Gamma_2(x_0,z)\Big)\Big]\Theta(x_0,z)\,dz\\ &+ \Big\langle\frac{z}{\alpha}\partial_x\Theta, \gamma_1(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0')\Big\rangle_{L^2(H^{-1}),L^2(H^1_0)}.\end{aligned} \tag{3.85}$$

Lemma 3.13. *Assume that the coefficients α, γ_i satisfy the assumptions of Proposition 3.6. Assume furthermore that that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$, $\widetilde{\delta}_0(1) = \widetilde{\delta}_1(-1) = 0$, $\partial_z \gamma_1 \in L^2$, and*

$$\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \left(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right) \right]^2 < +\infty.$$

Consider the unique solution $U \in Z^0(\Omega)$ of (3.4). Then $U \in H_x^1 H_z^1(\Omega)$ if and only if the orthogonality conditions (3.81) are satisfied, and in this case

$$\|U\|_{H_x^1 H_z^1}^2 \lesssim \|g\|_{H_x^1 L_z^2}^2 + \sum_{i=0,1} \int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \left(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right) \right]^2 + \|\widetilde{\delta}_i'\|_{H^1}^2. \tag{3.86}$$

Proof. • We first prove that under the assumptions of Proposition 3.6, equation (3.26) has a unique solution in $L^2(H_0^1)$. We start with uniqueness. Assume that $g = 0$, $\widetilde{\delta}_i$. Let $f \in L^2(H^{-1})$ be arbitrary and let Θ be the unique solution of (3.49). It follows from Definition 3.12 that

$$\langle f, V \rangle_{L^2 H^{-1}, L^2 H_0^1} = 0$$

Since the function f is arbitrary, we infer that $V = 0$.

The same argument allows us to prove *a priori* estimates. Indeed, assume that $V \in L^2(H_0^1)$ is a solution of (3.26) in the sense of Definition 3.12. Consider the unique solution $\Theta \in L^2(H_0^1)$ of (3.49) with $f = -\partial_{zz} V \in L^2(H^{-1})$. Then

$$\left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2(H^{-1})} + \|\Theta\|_{L^2(H_0^1)} \lesssim \|V\|_{L^2(H_0^1)}.$$

Furthermore, combining the weak formulation from Definition 3.12 together with the one of equation (3.49), we deduce that

$$\begin{aligned} \int_\Omega (\partial_z V)^2 = & \int_\Omega \partial_x g \frac{\Theta}{\alpha} \\ & - \int_{-1}^0 \left(\frac{g}{\alpha}(x_1, z) + \widetilde{\delta}_0''(z) - \widetilde{\delta}_0'(z) \Gamma_2(x_1, z) \right) \Theta(x_1, z)\, dz \\ & + \int_0^1 \left(\frac{g}{\alpha}(x_0, z) + \widetilde{\delta}_1''(z) - \widetilde{\delta}_1'(z) \Gamma_2(x_0, z) \right) \Theta(x_0, z)\, dz \\ & + \left\langle \frac{z}{\alpha} \partial_x \Theta, \gamma_1 (\mathbf{1}_{z<0} \widetilde{\delta}_1' + \mathbf{1}_{z>0} \widetilde{\delta}_0') \right\rangle_{L^2(H^{-1}), L^2(H_0^1)}. \end{aligned} \tag{3.87}$$

Using the assumptions on the coefficients α, γ_i together with Lemma A.1, we see that the right-hand side is lower than

$$\begin{aligned} C \left(\|\partial_x g\|_{L^2} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \left(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right) \right]^2 \right)^{1/2} + \|\widetilde{\delta}_i'\|_{H^1} \right) \\ \times \left(\left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2(H^{-1})} + \|\Theta\|_{L^2(H_0^1)} \right). \end{aligned}$$

Combining the two estimates and using a Cauchy-Schwarz inequality, we deduce that

$$\|V\|_{L^2(H_0^1)} \lesssim \|\partial_x g\|_{L^2} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \left(\widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right) \right]^2 \right)^{1/2} + \|\widetilde{\delta}_i'\|_{H^1}.$$

Once *a priori* estimates are available, we can adapt the arguments of Proposition 3.6 to prove existence of solutions in $L^2(H^1)$.

• The rest of the proof goes along the same lines as the one of Proposition 2.7. Assuming that the orthogonality conditions (3.81) are satisfied, and defining $\tilde{U}$ by

$$\tilde{U} := \begin{cases} \widetilde{\delta}_0 + \int_{x_0}^x V & \text{in } \Omega_+, \\ \widetilde{\delta}_1 + \int_{x_1}^x V & \text{in } \Omega_- \end{cases} \tag{3.88}$$

we see that $\tilde{U}$ is a solution of (3.4), and belongs to $L^2(H^1)$. Hence $\tilde{U} = U$, and therefore $\partial_x U \in L^2_x(H^1_z)$. Conversely, if $U \in H^1_x H^1_z$, then $\partial_x U$ is the only solution of (3.26). Since the jumps of U and $\partial_z U$ across the line $z = 0$ is zero, the orthogonality conditions (3.81) are satisfied.

□

3.5 Well-posedness results for the linearized problem

Proposition 3.14. *Let $\bar{u} \in Q^1$ such that the coefficients α, γ defined by*

$$\alpha(x,z) = \bar{u}_y^2(x, Y(x,z)), \quad \gamma_1(x,z) = \bar{u}_x(x, Y(x,z)), \quad \gamma_2(x,z) = -\bar{u}_{yy}(x, Y(x,z))$$

satisfy the assumptions of Proposition 3.6. Assume also that $\partial_z \gamma_1 \in L^2$, $\bar{u}(x, \pm 1) = \pm 1$, and $\|\bar{u} - y\|_{Q^1} \ll 1$.

Let $\delta_0, \delta_1 \in H^4([-1,1])$, $f \in H^1_x(L^2_y)$, and define $\widetilde{\delta}_i$ by (3.8).

Assume that $\delta_0(1) = \delta_0(-1) = 0$. Then there exists a unique solution $u \in Q^0$ to (3.1). It satisfies

$$\|u\|_{Q^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{H^1} + \|\delta_1\|_{H^1}. \tag{3.89}$$

Moreover, when additionally $\widetilde{\delta}_i'(0) = 0$ for $i = 0, 1$ and $(f(x_i) + \delta_i'')(Y(x_i, z))/z \in \mathscr{H}^1_z(\Sigma_i)$, this solution has $H^1_x H^1_y$ regularity if and only if $\ell^j_{\bar{u}}(f, \delta_0, \delta_1) = 0$ for $j \in \{0, 1\}$ (recall Definition 3.10). In this case, and if $f(x_0, 1) + \delta_0''(1) = f(x_1, -1) + \delta_0''(-1) = 0$ and $\partial_x \partial_y f \in L^2((-1, -1/4) \cup (1/4, 1))$, u actually enjoys Q^1 regularity and one has the estimates

$$\|u\|_{Q^1} \lesssim \|f\|_{H^1_x L^2_y} + \|f\|_{L^2_x H^3_y} + \|\partial_x \partial_y f\|_{L^2((-1,-1/4)\cup(1/4,1))} \tag{3.90}$$

$$+ \sum_{i=0,1} \|\delta_i\|_{H^4} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i, z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)} \tag{3.91}$$

$$\|u\|_{Q^{1/2}} \lesssim \|f\|_{H^{1/2}_x L^2_y} + \|f\|_{L^2_x H^{3/2}_y} + \|f\|_{H^{1/2}_x H^{1/2}_y((-1,-1/4)\cup(1/4,1))} \tag{3.92}$$

$$+ \sum_{i=0,1} \|\delta_i\|_{H^{5/2}} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i, z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}. \tag{3.93}$$

Proof. The first part of the statement is an immediate consequence of the equivalence between u and U: u is a solution of (3.1) if and only if U is a solution of (3.4). Furthermore, using Lemma B.4, we see that

$$\|u\|_{H^{2/3}_x L^2_y} \lesssim \|U\|_{H^{2/3}_x L^2_z} + \|U\|_{L^2_x H^{4/3}_z},$$
$$\|U\|_{H^{2/3}_x L^2_z} \lesssim \|u\|_{H^{2/3}_x L^2_y} + \|u\|_{L^2_x H^{4/3}_y}.$$

Additionally, since $u(x,y) = U(x, \bar{u}(x,y))$,

$$u_y(x,y) = \bar{u}_y U_z(x, \bar{u}(x,y)), \quad u_{yy}(x,y) = \bar{u}_{yy} U_z(x, \bar{u}(x,y)) + \bar{u}_y^2 U_{zz}(x, \bar{u}(x,y)),$$

and since $\bar{u}_y, \bar{u}_{yy} \in L^\infty$ (see Lemma 1.6)

$$\|u\|_{L^2_x(H^2_y)} \lesssim \|U\|_{L^2_x(H^2_z)},$$

and conversely,

$$\|U\|_{L^2_x(H^2_z)} \lesssim \|u\|_{L^2_x(H^2_y)}.$$

We infer that

$$\|u\|_{Q^0} \lesssim \|U\|_{Q^0} \lesssim \|u\|_{Q^0}.$$

Hence Proposition 3.4 implies that equation (3.1) has a unique solution $u \in Q^0$.

Furthermore, assuming that $\widetilde{\delta}_i'(0) = 0$ and using the identities

$$\begin{aligned} u_x(x,y) &= U_x(x,\bar{u}(x,y)) + \bar{u}_x U_z(x,\bar{u}(x,y)), \\ u_{xy}(x,y) &= \bar{u}_y U_{xz}(x,\bar{u}(x,y)) + \bar{u}_{xy} U_z(x,\bar{u}(x,y)) + \bar{u}_x \bar{u}_y U_{zz}(x,\bar{u}(x,y)), \end{aligned}$$

we see that $u_x \in L^2(H^1_y)$ if and only if $U_x \in L^2(H^1_z)$, which occurs if and only if the two orthogonality conditions (3.81) are satisfied.

Let us now prove that in this case, we have $\bar{u} \in Q^1$. Note that we do not try to prove that $U \in Q^1$. Indeed, U_x is a solution of

$$(z\partial_x + \gamma\partial_z - \alpha\partial_{zz})U = g - \gamma_x U_z + \alpha_x U_{zz}.$$

In the right-hand side of the above formula, there is a term $-z\partial_x\gamma_1 U_z$, which does not belong to L^2 *a priori*. Instead, we go back to the equation in u and we notice that $u_x \in L^2(H^1)$ is the unique weak solution of

$$\begin{aligned} (\bar{u}\partial_x - \partial_{yy})u_x = -\bar{u}_x u_x + \partial_x f =: g_1, \\ u_{x|\Sigma_i} = \frac{f(x_i,y) + \delta_i''(y)}{\bar{u}}, \\ u_{x|\pm 1} = 0. \end{aligned}$$

Furthermore, since $\bar{u}_x \in Q^0$, we have $\bar{u}_x \in L^2_x(L^\infty_y)$, so that the right-hand side g_1 belongs to L^2. Hence we can write

$$u_x(x,y) = W(x,\bar{u}(x,y)).$$

Note that W and U_x are slightly different: indeed,

$$W(x,z) - U_x(x,z) = \gamma_1 U_z(x,z).$$

Since the term $\partial_x\gamma_1 U_z$ is precisely the one preventing us that U belongs to L^2 when the coefficients are not smooth, we see that the purpose of this new change of variables is to remove a potential singular part in U.

Then W is a solution of

$$\begin{aligned} z\partial_x W + \gamma\partial_z W - \alpha\partial_{zz} W = g_1(x, Y(x,z)) \in L^2(\Omega), \\ W_{z=\pm 1} = 0, \\ W_{|\Sigma_i} = \frac{g(x_i,z) + \delta_i''(Y(x_i,z))}{z}. \end{aligned}$$

Hence we can apply the results of Proposition 3.4. Note that the compatibility conditions at $(x_0, 1)$ and at $(x_1, -1)$ are satisfied. We infer that $W \in Z^0$, and thus $u_x \in Z^0$. It follows that $u \in H^{5/3}_x L^2_y$.

We now prove that $\partial_y^5 u \in L^2$. As in the proof of Proposition 2.12, we will need to distinguish between an "interior regularity", close to the line $z = 0$, and a "boundary regularity", close to the lines $z = \pm 1$.

• *Interior regularity.*

Assume that the orthogonality conditions (3.81) are satisfied, so that $u_x \in Q^0$, and that $\partial_y^3 f \in L^2$. In that case, writing

$$u_{yy} = \bar{u} u_x - f,$$

we immediately infer that $u \in L^2_x(H^4_y)$. Indeed, since $u_x \in Q^0$, $u_x \in L^2_y(H^{2/3}_x) \subset L^2_y(L^\infty_x)$, and $\bar{u}_{yy} \in L^2_x(H^1_y) \subset L^\infty_y(L^2_x)$. Note also that $\bar{u}, \bar{u}_y \in L^\infty$.

In order to prove that $\partial_y^5 u \in L^2((x_0, x_1) \times (-1+\delta, 1+\delta)$ for any $\delta > 0$, we mimick the proof of Proposition 2.12. Multiplying (3.1) by a test function $\partial_y^3 \phi$, for some $\phi \in C^\infty(\bar{\Omega})$ such that ϕ identically vanishes in a neighbourhood of ± 1 and on $\{x_0\} \times (-1, 0) \cup \{x_1\} \times (0, 1)$, we obtain

$$\begin{aligned}
\int \bar{u} u_x \partial_y^3 \phi &= \int \partial_y^2(\bar{u} u_x) \partial_y \phi \\
&= \int (\bar{u}_{yy} u_x + 2 \bar{u}_y u_{xy}) \partial_y \phi + \int \bar{u} u_{xyy} \partial_y \phi \\
&= \int (\bar{u}_{yy} u_x + 2 \bar{u}_y u_{xy}) \partial_y \phi - \int \partial_x(\bar{u} \partial_y \phi) u_{yy} - \sum_{i=0,1} (-1)^i \int_{\Sigma_i} (\bar{u} \partial_y \phi)(x_i, y) \delta_i'' \, \mathrm{d}y.
\end{aligned}$$

We integrate by parts both the boundary terms and the interior terms and we are led to

$$\begin{aligned}
\int \bar{u} u_x \partial_y^3 \phi &= \int (-\partial_y^3 \bar{u} u_x - 3 \bar{u}_{yy} u_{xy} - 2 \bar{u}_y u_{xyy} + \bar{u}_{xy} u_{yy} + \bar{u}_x \partial_y^3 u) \phi + \int \partial_x \phi \partial_y(\bar{u} u_{yy}) \\
&\quad + \sum_{i=0,1} (-1)^i \int_{\Sigma_i} \phi(x_i, y) \partial_y(\bar{u}(x_i, y) \delta_i''(y)) \, \mathrm{d}y \\
&= \int (-\partial_y^3 \bar{u} u_x - 3 \bar{u}_{yy} u_{xy} - 2 \bar{u}_y u_{xyy} + \bar{u}_{xy} u_{yy} + \bar{u}_x \partial_y^3 u) \phi - \int \phi \partial_x(\bar{u}_y u_{yy}) \\
&\quad + \int \partial_x \phi \bar{u} \partial_y^3 u \\
&\quad + \sum_{i=0,1} (-1)^i \int_{\Sigma_i} \phi(x_i, y) \left[\partial_y(\bar{u}(x_i, y) \delta_i''(y)) - \bar{u}_y \delta_i''(y) \right] \mathrm{d}y.
\end{aligned}$$

But we also have

$$\int \bar{u} u_x \partial_y^3 \phi = \int (f + \partial_y^2) u \partial_y^3 \phi = - \int \partial_y^3 f \phi + \int \partial_y^4 u \partial_y \phi.$$

Gathering all the terms and using a density argument, we deduce that for any function $\phi \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$\begin{aligned}
&- \int_\Omega (\bar{u} \partial_y^3 u \partial_x \phi + \bar{u}_x \partial_y^3 u \phi) + \int \partial_y^4 u \partial_y \phi \\
&= \int \left[\partial_y^3 f - \left(\partial_y^3 \bar{u} u_x + 3 \partial_y^2 \bar{u} u_{xy} + 3 \partial_y u u_{xyy} \right) \right] \phi \\
&\quad + \sum_{i=0,1} (-1)^i \int_{\Sigma_i} \phi(x_i, y) \bar{u}(x_i, y) \delta_i^{(3)}(y)).
\end{aligned}$$

This proves that for any test function $\chi_0 \in C_c^\infty(]-1,1[)$ such that $\chi \equiv 1$ in a neighbourhood of $z=0$, say $\chi_0 \equiv 1$ on $(-1/2,1/2)$ and $\mathrm{Supp}\chi_0 \subset (-3/4,3/4)$, $\chi_0 \partial_y^3 u$ is the unique weak solution of

$$\bar{u}\partial_x(\chi_0\partial_y^3 u) - \partial_y^2(\chi_0\partial_y^3 u) = \chi_0\left(\partial_y^3 f - \partial_y^3\bar{u}u_x - 3\partial_y^2\bar{u}u_{xy} - 3\partial_y u u_{xyy}\right) - \chi_0''\partial_y^3 u - 2\chi_0'\partial_y^4 u.$$

The right-hand side belongs to $L^2(\Omega)$, and the compatibility conditions at the corners $(x_0,1)$ and $(x_1,-1)$ are automatically satisfied because of the truncation χ_0. Hence Proposition 3.4 ensures that $\chi_0(Y(x,z))\partial_y^3 u(x,Y(x,z)) \in Z^0$. We infer that $\chi_0\partial_y^5 u \in L^2$, and $\bar{u}\chi_0\partial_x\partial_y^3 u \in L^2$. Using the equation satisfied by u, it follows that $\chi_0\bar{u}^2\partial_x^2\partial_y u \in L^2$.

• *Boundary regularity:*

We now take $\chi_1 \in C_c^\infty(\mathbb{R})$, with $\chi \equiv 1$ on $[1/2,1]$, and $\mathrm{Supp}\chi_1 \subset [1/4,1]$. Note that with the choice above, $\mathrm{Supp}\chi_1' \subset \chi_0^{-1}(\{1\})$. Let $u_1 = u\chi_1$. Then u_1 satisfies

$$\begin{aligned} \bar{u}\partial_x u_1 - \partial_{yy}u_1 = S_1 := f\chi_1 - 2\chi_1'\partial_y u - \chi_1'' u \quad \text{in } (x_0,x_1)\times(1/4,1),\\ u_1(\cdot,1/4) = u_1(\cdot,1) = 0,\\ u_1(x_0,y) = \chi_1\delta_0. \end{aligned} \tag{3.94}$$

By assumption, $\|\bar{u}-y\|_\infty \le m$ for some $m \ll 1$. Without loss of generality, we assume that $m \le 1/8$ so that $\bar{u} \ge 1/8$ on $(x_0,x_1)\times(1/4,1)$. Hence the equation is parabolic. Note also that

$$\bar{u}\partial_x^2 u_1 = \partial_x S_1 + \partial_y^2\partial_x u_1 - \partial_x\bar{u}\partial_x u_1,$$

and the right-hand side is in L^2. Thus $\partial_x^2 u_1 \in L^2, \partial_y^2\partial_x u_1 \in L^2$, from where it follows that $\partial_x u_1 \in L^\infty$, with a bound $\|\partial_x u_1\|_\infty \lesssim (\|u_x\|_{Q^0} + \|u\|_{Q^0} + \|\partial_x f\chi_1\|_{L^2})$.

We now differentiate (3.94) twice with respect to y. Since $(\chi_0\delta_0)(1) = (\chi_0\delta_0)(1/4) = 0$, $\partial_y^2 u_1$ is a solution of

$$\begin{aligned} (\bar{u}\partial_x - \partial_{yy})\partial_y^2 u_1 = \partial_y^2 S_1 - \bar{u}_{yy}\partial_x u_1 - 2\bar{u}_y\partial_{xy}u_1,\\ \partial_y^2 u_1(\cdot,1/4) = \partial_y^2 u_1(\cdot,1) = 0,\\ \partial_y^2 u_1(x_0,y) = \partial_y^2(\chi_1\delta_0). \end{aligned} \tag{3.95}$$

And since $\partial_y^2(\chi_1\delta_0)(1/4) = \partial_y^2(\chi_1\delta_0)(1) = 0$, we have

$$\begin{aligned} (\bar{u}\partial_x - \partial_{yy})\partial_x\partial_y^2 u_1 = \partial_x\partial_y^2 S_1 - \bar{u}_{xyy}\partial_x u_1 - \bar{u}_{yy}\partial_x^2 u_1 - 2\partial_y(\bar{u}_x\partial_{xy}u_1),\\ \partial_x\partial_y^2 u_1(\cdot,1/4) = \partial_x\partial_y^2 u_1(\cdot,1) = 0,\\ \partial_x\partial_y^2 u_1(x_0,y) = \partial_y^2(\chi_1\Delta_0). \end{aligned} \tag{3.96}$$

Now, let us perform energy estimates for the above parabolic system. We multiply by $\bar{u}^{-1}\partial_x\partial_y^2 u_1$ and integrate over $(x_0,x_1)\times(1/4,1)$. According to the previous regularity estimates on u and u_1, we know that $\partial_x\partial_y^2 S_1 \in L^2(H^{-1})$, $\bar{u}_{xyy}\partial_x u_1 \in L^2$, $\bar{u}_{yy}\partial_x^2 u_1 \in L^2$. We estimate the terms $\partial_y(\bar{u}_x\partial_{xy}u_1)$ in $L^2((x_0,x_1),H^{-1}((1/4,1)))$. It is easily proved that it is bounded in this space. We infer that $\partial_x\partial_y^3 u_1 \in L^2((x_0,x_1),(1/4,1))$. Eventually, writing

$$\partial_y^2 u_1 = \bar{u}\partial_x u_1 - S_1,$$

we deduce that $\partial_y^5 u_1 \in L^2((x_0,x_1),(1/4,1))$.

• *Interpolation:*

We have proved that

$$\begin{aligned}\|\chi_0 u\|_{Q^0} \lesssim & \|f\|_{L^2} + \|\delta_0\|_{H^1} + \|\delta_1\|_{H^1},\\ \|\chi_0 u\|_{Q^1} \lesssim & \|f\|_{H^1_x L^2_y} + \|f\|_{L^2_x H^3_y}\\ & + \sum_{i=0,1} \|\delta_i\|_{H^4} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i,z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}.\end{aligned}$$

By interpolation, it follows that

$$\|\chi_0 u\|_{Q^{1/2}} \lesssim \|f\|_{H^{1/2}_x L^2_y} + \|f\|_{L^2_x H^{3/2}_y} + \sum_{i=0,1} \|\delta_i\|_{H^{5/2}} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i,z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}.$$

Similarly,

$$\begin{aligned}\|\chi_1 u\|_{Q^0} \lesssim & \|f\|_{L^2} + \|\delta_0\|_{H^1} + \|\delta_1\|_{H^1},\\ \|\chi_1 u\|_{Q^1} \lesssim & \|f\|_{H^1_x L^2_y} + \|f\|_{L^2_x H^3_y} + \|\partial_x \partial_y f\|_{L^2((x_0,x_1)\times(1/4,1))}\\ & + \sum_{i=0,1} \|\delta_i\|_{H^4} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i,z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}.\end{aligned}$$

By interpolation,

$$\begin{aligned}\|\chi_1 u\|_{Q^{1/2}} \lesssim & \|f\|_{H^{1/2}_x L^2_y} + \|f\|_{L^2_x H^{3/2}_y} + \|f\|_{H^{1/2}_x H^{1/2}_y((x_0,x_1)\times(1/4,1))}\\ & + \sum_{i=0,1} \|\delta_i\|_{H^{5/2}} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i,z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}.\end{aligned}$$

Gathering the two estimates, we obtain the desired result. □

4 Local stability of the orthogonality conditions

This section is devoted to the derivation of some key estimates for the nonlinear scheme we will perform in Section 5. Indeed, as explained in Section 1.4, a crucial point of our proof lies in the fact that the linear forms associated with the orthogonality conditions depend continuously on the data $\bar{u}$, in a suitable norm. This is proved in Proposition 4.1 below.

In this section, we consider two flows $\bar{u}$, $\bar{u}'$ in a Z^1 neighborhood of the linear profile $(x,y) \mapsto y$. We define changes of variables Y, Y' such that

$$\forall z \in (-1,1) \quad \forall x \in (x_0,x_1), \quad \bar{u}(x, Y(x,z)) = \bar{u}'(x, Y'(x,z)) = z. \tag{4.1}$$

We define

$$\alpha(x,z) = (\partial_y \bar{u})^2(x, Y(x,z)), \tag{4.2}$$

$$\gamma_1(x,z) = \bar{u}_x(x, Y(x,z)), \tag{4.3}$$

$$\gamma_2(x,z) = \bar{u}_{yy}(x, Y(x,z)), \tag{4.4}$$

and analogously, we define α', γ_i'. We set $\gamma = z\gamma_1 + \gamma_2$.

We then consider the profiles $\Phi^j, (\Phi^j)'$ constructed in Proposition 3.6, and the associated linear forms $\ell_{\bar{u}}$, $\ell_{\bar{u}'}$ introduced in Definition 3.10. The main result of this section is the following.

Proposition 4.1. *There exists $\bar{c} > 0$ such that, for every $\bar{u}, \bar{u}' \in Q^1$ with $\|\bar{u} - y\|_{Q^1} \leq \bar{c}, \|\bar{u}' - y\|_{Q^1} \leq \bar{c}$ and $\bar{u}(x_i, 0) = 0$, $\bar{u}(x, \pm 1) = \pm 1$, there holds*

$$\|\ell^j_{\bar{u}} - \ell^j_{\bar{u}'}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \tag{4.5}$$

We decompose the proof of the Proposition into several Lemmas. We first investigate bounds on the coefficients $\alpha - \alpha'$, $\gamma - \gamma'$ in terms of $\|\bar{u} - \bar{u}'\|_{Q^{1/2}}$. Then, we prove that the solutions Φ^j of (3.32) depend continuously on the coefficients. Putting together these two results leads to Proposition 4.1.

4.1 Stability of the change of variables

We start with a technical lemma which will be used abundantly throughout this section.

Lemma 4.2. *There exists $m \in (0, 1)$ and $C > 0$ such that, for any $\bar{u}, \bar{u}' \in Q^1$ such that $\bar{u}(\pm 1) = \bar{u}'(\pm 1) = \pm 1$ and $\|\bar{u} - y\|_{Q^1} \leq m$ and $\|\bar{u}' - y\|_{Q^1} \leq m$, if Y, Y' are defined by (4.1),*

$$\|Y - Y'\|_{L^\infty(\Omega)} \lesssim \|Y - Y'\|_{H^{7/12}_x H^{3/4}_z} \leq C\|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \tag{4.6}$$

Proof. From the definition of Y, one infers that

$$Y(x, z) = -1 + \int_{-1}^{z} \frac{\mathrm{d}s}{\bar{u}_y(x, Y(x, s))}. \tag{4.7}$$

Hence, combined with the corresponding relation for Y', one has

$$Y(x, z) - Y'(x, z) = \int_{-1}^{z} \frac{\bar{u}'_y(Y'(x, s)) - \bar{u}_y(x, Y(x, s))}{\bar{u}_y(x, Y(x, s))\bar{u}'_y(x, Y'(x, s))} \, \mathrm{d}s. \tag{4.8}$$

From there, it follows that for a.e. $x \in (x_0, x_1)$,

$$\|Y(x, \cdot) - Y'(x, \cdot)\|_{H^1_z} \leq \left\| \frac{\bar{u}'_y(Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{L^2_z}. \tag{4.9}$$

We decompose the right-hand side as

$$\frac{\bar{u}'_y(x, Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} = \frac{(\bar{u}'_y - \bar{u}_y)(x, Y'(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} + \frac{\bar{u}_y(x, Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))}. \tag{4.10}$$

The first term is bounded in $H^{7/12}_x L^2_z$ as follows, using Lemma B.4 together with the Sobolev embeddings $H^{2/5}_x \subset L^{10}_x$, $H^{1/10}_x \subset L^{5/2}_x$,

$$\begin{aligned}
&\left\| \frac{(\bar{u}'_y - \bar{u}_y)(x, Y'(x, \cdot)}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{H^{7/12}_x L^2_z} \\
&\leq \|(\bar{u}'_y - \bar{u}_y)(x, Y(x, \cdot))\|_{H^{7/12}_x L^2_z} \left\| \frac{1}{\bar{u}_y(x, Y(x, \cdot))} \right\|_{L^\infty_z(H^1_x)} \left\| \frac{1}{\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{L^\infty_z(H^1_x)} \\
&\lesssim \left(\|\bar{u}_y - \bar{u}'_y\|_{H^{7/12}_x L^2_y} + \|\bar{u}_y - \bar{u}'_y\|_{L^2_x H^{7/6}_y} \right) \left(1 + \|\bar{u}_x \bar{u}_{xy}\|_{L^\infty_z(L^2_x)}\right) \left(1 + \|\bar{u}_x \bar{u}_{xy}\|_{L^\infty_z(L^2_x)}\right) \\
&\lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}} \left(1 + \|\bar{u}_x\|_{L^\infty_y(H^{2/5}_x)} \|\bar{u}_{xy}\|_{L^\infty_y(H^{1/10}_x)}\right) \left(1 + \|\bar{u}'_x\|_{L^\infty_y(H^{2/5}_x)} \|\bar{u}'_{xy}\|_{L^\infty_y(H^{1/10}_x)}\right) \\
&\lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}} (1 + \|\bar{u}\|^2_{Q^1})(1 + \|\bar{u}'\|^2_{Q^1}).
\end{aligned} \tag{4.11}$$

Concerning the second term in the right-hand side of (4.10), we write a Taylor formula, namely

$$\bar{u}_y(x,Y'(x,z)) - \bar{u}_y(x,Y(x,z)) = (Y'(x,z)-Y(x,z)) \int_0^1 \partial_y^2 \bar{u}(x,\tau Y'(x,z)+(1-\tau)Y(x,z))\,\mathrm{d}\tau. \tag{4.12}$$

Hence

$$\begin{aligned}
&\left\| \frac{\bar{u}_y(x,Y'(x,\cdot)-\bar{u}_y(x,Y(x,\cdot)}{\bar{u}_y(x,Y(x,\cdot))\bar{u}'_y(x,Y'(x,\cdot))} \right\|_{H^{7/12}_x L^2_z} \\
&\leq \|Y'-Y\|_{L^\infty_z(H^{7/12}_x)} \left\| \int_0^1 \partial_y^2 \bar{u}(x,\tau Y'(x,z)+(1-\tau)Y(x,z))\,\mathrm{d}\tau \right\|_{H^{7/12}_x L^2_z} \\
&\qquad \times \left\| \frac{1}{\bar{u}_y(x,Y(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} \left\| \frac{1}{\bar{u}'_y(x,Y'(x,\cdot))} \right\|_{L^\infty_z(H^1_x)}.
\end{aligned} \tag{4.13}$$

As previously, we have

$$\begin{aligned}
\left\| \frac{1}{\bar{u}_y(x,Y(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} &\lesssim 1+\|\bar{u}\|^2_{Q^1}, \\
\left\| \frac{1}{\bar{u}'_y(x,Y'(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} &\lesssim 1+\|\bar{u}'\|^2_{Q^1}.
\end{aligned}$$

Furthermore,

$$\|Y'-Y\|_{L^\infty_z(H^{7/12}_x)} \lesssim \|Y-Y'\|_{H^{7/12}_x H^{3/4}_z}. \tag{4.14}$$

And using Lemma B.4,

$$\left\| \int_0^1 \partial_y^2 \bar{u}(x,\tau Y'(x,z)+(1-\tau)Y(x,z))\,\mathrm{d}\tau \right\|_{H^{7/12}_x L^2_z} \lesssim \|\partial_y^2 \bar{u}\|_{H^{7/12}_x L^2_z} + \|\partial_y^2 \bar{u}\|_{L^2_x H^{7/6}_z} \lesssim \|\bar{u}-y\|_{Q^1}. \tag{4.15}$$

Therefore, since $m \leq 1$, we infer that there exists a universal constant $\bar{C}$ such that

$$\|Y-Y'\|_{H^{7/12}_x H^{3/4}_z} \leq \bar{C}\left(\|\bar{u}-\bar{u}'\|_{Q^{1/2}} + m\|Y-Y'\|_{H^{7/12}_x H^{3/4}_z} \right). \tag{4.16}$$

For m sufficiently small, we can absorb the second term in the right-hand side into the left-hand side, and we obtain the result announced in the Lemma. □

4.2 Bounds and stability of the coefficients α and γ

We now state two lemmas allowing us to estimate the coefficients and their difference in terms of the Q^1 and $Q^{1/2}$ norms of the functions $\bar{u}, \bar{u}'$:

Lemma 4.3 (Bounds on the coefficients in terms of the Q^1 norm)**.** *Let $\bar{u} \in Q^1$ such that $\|\bar{u}-y\|_{Q^1} \leq m$. Let $\alpha, \gamma_1, \gamma_2$ be given by* (4.2), (4.3), (4.4).

Then the following estimates hold:

$$\begin{aligned}
\|\alpha_z\|_\infty + \|\alpha-1\|_\infty + \|\alpha_x\|_{L^\infty_z(L^2_x)} &\lesssim \|\bar{u}-y\|_{Q^1}, \\
\|\gamma_1\|_{L^2_z(H^{2/3}_x)} + \|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\partial_z\gamma_1\|_{L^\infty_z(L^2_x)} &\lesssim \|\bar{u}-y\|_{Q^1}, \\
\|\gamma_2\|_{L^2_z(H^1_x)} + \|\gamma_2\|_{L^2_z(H^{3/5}_x)} &\lesssim \|\bar{u}-y\|_{Q^1}.
\end{aligned} \tag{4.17}$$

Lemma 4.4 (Bounds on differences between coefficients in terms of the $Q^{1/2}$ norm)**.** *Let $\bar{u}, \bar{u}' \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \leq m$, $\|\bar{u}' - y\|_{Q^1} \leq m$. Let $\alpha, \alpha', \gamma_i, \gamma_i'$ be given by* (4.2)*,* (4.3)*,* (4.4)*.*

Then the following estimates hold:

$$\|\alpha - \alpha'\|_{L_z^\infty(H_x^{7/12})} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}, \tag{4.18}$$

$$\|\gamma_1 - \gamma_1'\|_{L_z^\infty(L_x^2)} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}, \tag{4.19}$$

$$\|\gamma_2 - \gamma_2'\|_{L_z^2(H_x^{1/2})} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}} \tag{4.20}$$

Proof of Lemma 4.3. Concerning the bounds on α, we recall that $\alpha(x,z) = \bar{u}_y^2(x, Y(x,z))$, so that

$$\alpha_z = 2\bar{u}_{yy}(x, Y(x,z))\bar{u}_y(x, Y(x,z))\frac{\partial Y}{\partial_z} = 2\bar{u}_{yy}(x, Y(x,z)).$$

Hence $\|\alpha_z\|_\infty = 2\|\bar{u}_{yy}\|_\infty$, and $\bar{u}_{yy} \in H_x^{2/3}H_y^1$. Thus $\|\bar{u}_{yy}\|_\infty \lesssim \|\bar{u} - y\|_{Q^1}$. Furthermore, $\|\alpha - 1\|_\infty \lesssim \|\bar{u}_y - 1\|_\infty$, and since $\int_{-1}^1 (\bar{u}_y(x,y) - y)\,\mathrm{d}y = 0$, $\|\bar{u}_y - 1\|_\infty \leq \|\bar{u}_{yy}\|_\infty$.

In a similar fashion,

$$\alpha_x(x,z) = 2\bar{u}_{xy}(x,Y)\bar{u}_y(x,Y) + 2\bar{u}_{yy}(x,Y)\bar{u}_y(x,Y)\frac{\partial Y}{\partial x} = 2\bar{u}_{xy}(x,Y)\bar{u}_y(x,Y) - 2\bar{u}_{yy}(x,Y)\bar{u}_x(x,Y).$$

Hence

$$\begin{aligned}\|\alpha_x\|_{L_z^\infty(L_x^2)} &\lesssim \|\bar{u}_y\|_\infty \|\bar{u}_{xy}\|_{L_x^2(L_y^\infty)} + \|\bar{u}_{yy}\|_\infty \|\bar{u}_x\|_{L_x^2(L_y^\infty)} \\ &\lesssim \|\bar{u}\|_{Q^1}\|\bar{u}_{xyy}\|_{L^2} \\ &\lesssim \|\bar{u}\|_{Q^1}\|\bar{u} - y\|_{Q^1}.\end{aligned}$$

We now address the bounds on $\gamma_1 = \bar{u}_x(x, Y(x,z))$. We have, using Lemma B.4

$$\|\gamma_1\|_{H_x^{2/3}L_z^2} \lesssim \|\bar{u}_x\|_{H_x^{2/3}L_y^2} + \|\bar{u}_x\|_{L_x^2(H_y^{4/3})} \lesssim \|\bar{u} - y\|_{Q^1}.$$

Furthermore,

$$\partial_z \gamma_1 = \frac{\partial Y}{\partial_z}\bar{u}_{xy}(x, Y(x,z)),$$

so that

$$\|\partial_z\gamma_1\|_{L_z^\infty(L_x^2)} \leq \|\partial_z\gamma_1\|_{L_x^2(L_z^\infty)} \lesssim \|\bar{u}_{xy}\|_{L_x^2(L_z^\infty)} \lesssim \|\bar{u}_{xy}\|_{L_x^2(H_y^1)}.$$

Hence $\|\partial_z\gamma_1\|_{L_z^\infty(L_x^2)} \lesssim \|\bar{u} - y\|_{Q^1}$.

Concerning the $L_z^\infty(H_x^{1/2})$ bound on γ_1, we use Lemma B.16 in the Appendix, which yields

$$\|\gamma_1\|_{L_z^\infty(H_x^{1/2})} \lesssim \|\gamma_1\|_{H_x^{2/3}L_z^2} + \|\partial_z\gamma_1\|_{H_x^{1/3}L_z^2}. \tag{4.21}$$

We then bound the two terms in the right-hand side, using Lemma B.4. We have

$$\begin{aligned}\|\gamma_1\|_{H_x^{2/3}L_z^2} &\lesssim \|\bar{u}_x\|_{H_x^{2/3}L_y^2} + \|\bar{u}_x\|_{L_x^2 H_y^{4/3}} \\ &\lesssim \|\bar{u} - y\|_{Q^1}.\end{aligned}$$

Furthermore,

$$\partial_z\gamma_1(x,z) = \frac{\partial Y}{\partial z}(x,z)\bar{u}_{xy}(x, Y(x,z)) = \frac{\bar{u}_{xy}}{\bar{u}_y}(x, Y(x,z)), \tag{4.22}$$

so that, using Lemma B.4,

$$\|\partial_z \gamma_1\|_{H_x^{1/3} L_z^2} \lesssim \left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{H_x^{1/3} L_y^2} + \left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{L_x^2 H_y^{2/3}}. \tag{4.23}$$

Concerning the first term, we write

$$\left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{H_x^{1/3} L_y^2} \lesssim \|\bar{u}_{xy}\|_{H_x^{1/3} L_y^2} \left\| \frac{1}{\bar{u}_y} \right\|_{L_y^\infty (H_x^1)} \lesssim \|\bar{u} - y\|_{Q^1} \|\bar{u}_y\|_{H_y^1 (H_x^1)}.$$

Concerning the second term, we have

$$\left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{L_x^2 H_y^{2/3}} \lesssim \|\bar{u}_{xy}\|_{L_x^2 H_y^{2/3}} \left\| \frac{1}{\bar{u}_y} \right\|_{L_x^\infty (H_y^1)} \lesssim \|\bar{u} - y\|_{Q^1} \|\bar{u}_{yy}\|_\infty.$$

Eventually, we get

$$\|\gamma_1\|_{L_z^\infty (H_x^{1/2})} \lesssim \|\bar{u}\|_{Q^1} \|\bar{u} - y\|_{Q^1}.$$

Concerning the term $\gamma_2 = -\bar{u}_{yy}(x, Y(x,z))$, we write

$$-\partial_x \gamma_2(x,z) = \bar{u}_{xyy}(x, Y(x,z)) + \frac{\partial Y}{\partial x} \partial_y^3 \bar{u}(x, Y(x,z)).$$

The first term is bounded in L^2 by $\|\bar{u}_{xyy}\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1}$ (see Lemma B.4). Concerning the second one, we recall that $\partial_x Y = -(\bar{u}_x / \bar{u}_y)(x, Y(x,z))$, and therefore $\partial_x Y \in L^p(\Omega)$ for all $p < \infty$ (note that the jacobian of the change of variables $y = Y(x,z)$ is bounded from above and below by a uniform constant). And since $\partial_y^3 \bar{u} \in H^{2/3}(\Omega) \subset L^6(\Omega)$, we obtain, thanks to the Hölder inequality,

$$\|\partial_x \gamma_2\|_{L^2} \lesssim (1 + \|\bar{u}\|_{Q^1}) \|\bar{u} - y\|_{Q^1}.$$

Eventually, using once again Lemma B.4,

$$\|\gamma_2\|_{L_z^2 (H_x^{3/5})} \lesssim \|\bar{u}_{yy}\|_{L_z^2 (H_x^{3/5})} + \|\bar{u}_{yy}\|_{L_x^2 (H_y^{6/5})} \lesssim \|\bar{u} - y\|_{Q^1}.$$

□

Proof of Lemma 4.4. We use the same type of techniques as in the previous Lemma. Recalling the definition of α, α', we write

$$\begin{aligned} \alpha(x,z) - \alpha'(x,z) &= \bar{u}_y^2(x, Y(x,z)) - (\bar{u}_y')^2(x, Y(x,z)) && (4.24) \\ &\quad + (\bar{u}_y')^2(x, Y(x,z)) - (\bar{u}_y')^2(x, Y'(x,z)). && (4.25) \end{aligned}$$

Using the results of Lemma B.4, the term (4.24) is bounded as follows

$$\begin{aligned} \|\bar{u}_y^2(x, Y(x,z)) - (\bar{u}_y')^2(x, Y(x,z))\|_{L_z^\infty (H_x^{7/12})} &\lesssim \|\bar{u}_y^2(x, Y(x,z)) - (\bar{u}_y')^2(x, Y(x,z))\|_{H_z^{3/4} (H_x^{7/12})} \\ &\lesssim \|\bar{u}_y^2 - (\bar{u}_y')^2\|_{H_x^{7/12} H_y^{3/4}} + \|\bar{u}_y^2 - (\bar{u}_y')^2\|_{L_x^2 H_y^{23/12}}. \end{aligned}$$

Note that $H_x^{7/12} H_y^{3/4} \subset L^\infty$, and therefore it is an algebra. Hence

$$\|\bar{u}_y^2 - (\bar{u}_y')^2\|_{H_x^{7/12} H_y^{3/4}} \lesssim \|\bar{u}_y - \bar{u}_y'\|_{H_x^{7/12} H_y^{3/4})} \left(\|\bar{u}_y\|_{H_x^{7/12} H_y^{3/4})} + \|\bar{u}_y'\|_{H_x^{7/12} H_y^{3/4})} \right) \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \tag{4.26}$$

Concerning the other term, we have

$$\begin{aligned}\|\bar{u}_y^2-(\bar{u}'_y)^2\|_{H_y^{23/12}} &\leq \|\bar{u}_y\bar{u}_{yy}-\bar{u}'_y\bar{u}'_{yy}\|_{H_y^{11/12}}\\ &\leq \|\bar{u}_y-\bar{u}'_y\|_{H_y^{11/12}}\|\bar{u}_{yy}\|_{H_y^{11/12}}+\|\bar{u}_{yy}-\bar{u}'_{yy}\|_{H_y^{11/12}}\|\bar{u}'_y\|_{H_y^{11/12}}.\end{aligned}$$

Note that

$$\|\bar{u}_{yy}\|_{L_x^\infty(H_y^{11/12})}\lesssim\|\bar{u}_{yy}\|_{H_x^{2/3}(H_y^{11/12})}\lesssim\|\bar{u}-y\|_{Q^1},$$

and

$$\|\bar{u}_{yy}-\bar{u}'_{yy}\|_{L_x^2H_y^{11/12}}\lesssim\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We deduce that

$$\|\bar{u}_y^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y(x,z))\|_{L_z^\infty(H_x^{7/12})}\lesssim\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We now address the term (4.25), which we write as

$$(\bar{u}'_y)^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y'(x,z))=2(Y(x,z)-Y'(x,z))\int_0^1(\bar{u}'_y\bar{u}'_{yy})(x,\tau Y+(1-\tau)Y')\,\mathrm{d}\tau.$$

Hence, using Lemma B.4

$$\begin{aligned}\|(4.25)\|_{H_y^{3/4}H_x^{7/12}}\lesssim&\left(\|Y-Y'\|_{H_y^{3/4}H_x^{7/12}}+\|Y-Y'\|_{L_x^2H_y^{23/12}}\right)\\ &\times\left(\|\bar{u}_y\|_{H_y^{3/4}H_x^{7/12}}+\|\bar{u}_y\|_{L_x^2H_y^{23/12}}\right)\\ &\times\left(\|\bar{u}_{yy}\|_{H_y^{3/4}H_x^{7/12}}+\|\bar{u}_{yy}\|_{L_x^2H_y^{23/12}}\right)\\ &\lesssim\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.\end{aligned}\tag{4.27}$$

Gathering (4.26) and (4.27), we infer that

$$\|\alpha-\alpha'\|_{L_z^\infty(H_x^{7/12})}\lesssim\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We then address the bounds on $\gamma_1-\gamma'_1$. As previously, we write

$$(\gamma_1-\gamma'_1)(x,z)=(\bar{u}_x(x,Y(x,z))-\bar{u}_x(x,Y'(x,z)))+(\bar{u}_x(x,Y'(x,z))-\bar{u}'_x(x,Y'(x,z)))\,.$$

Using Lemma 4.2, the first term is bounded in the following way

$$\|\bar{u}_x(x,Y(x,z))-\bar{u}_x(x,Y'(x,z))\|_{L_z^\infty(L_x^2)}\leq\|\bar{u}_{xy}\|_{L_x^2(L_y^\infty)}\|Y-Y'\|_\infty\leq C_m\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

As for the second term,

$$\begin{aligned}\|\bar{u}_x(x,Y'(x,z))-\bar{u}'_x(x,Y'(x,z))\|_{L_z^\infty(L_x^2)} &\leq \|\bar{u}_x(x,Y'(x,z))-\bar{u}'_x(x,Y'(x,z))\|_{L_x^2(L_z^\infty)}\\ &\leq \|\bar{u}_x-\bar{u}'_x\|_{L_x^2(L_y^\infty)}\\ &\leq \|\bar{u}-\bar{u}'\|_{Q^{1/2}}.\end{aligned}$$

Consequently,

$$\|\gamma_1-\gamma'_1\|_{L_z^\infty(L_x^2)}\leq C_m\|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

Eventually, we address the bounds on $\gamma_2 - \gamma_2'$, which we decompose as previously as

$$(\gamma_2 - \gamma_2')(x,z) = \big(\bar u_{yy}(x,Y(x,z)) - \bar u'_{yy}(x,Y(x,z))\big) + \big(\bar u'_{yy}(x,Y(x,z)) - \bar u'_{yy}(x,Y'(x,z))\big)\,. \quad (4.28)$$

Concerning the second term, we use a Taylor formula

$$\bar u'_{yy}(x,Y) - \bar u'_{yy}(x,Y') = (Y-Y')\int_0^1 \partial_y^3 \bar u'(x,\tau Y + (1-\tau)Y')d\tau. \quad (4.29)$$

From there, it follows that for some $\sigma > 1/2$,

$$\begin{aligned}\|\bar u'_{yy}(x,Y) - \bar u'_{yy}(x,Y')\|_{L^2_z(H^{1/2}_x)} &\le \|Y-Y'\|_{L^\infty_z(H^{1/2}_x)} \left\| \int_0^1 \partial_y^3 \bar u'(x,\tau Y + (1-\tau)Y')d\tau \right\|_{L^2_z(H^\sigma_x)} \\ &\le \|Y-Y'\|_{L^\infty_z(H^{1/2}_x)} \big(\|\partial_y^3 \bar u'\|_{L^2_z(H^\sigma_x)} + \|\partial_y^3 \bar u'\|_{H^{2\sigma}_z(L^2_x)}\big) \\ &\lesssim \|\bar u - \bar u'\|_{Q^{1/2}} \|\bar u'\|_{Q^1}. \end{aligned} \quad (4.30)$$

We then address the first term in (4.28). Using Lemma B.4, we obtain

$$\begin{aligned}\|(\bar u_{yy} - \bar u'_{yy})(x,Y(x,z))\|_{L^2_z(H^{1/2}_x)} &\le C_m \left(\|\bar u_{yy} - \bar u'_{yy}\|_{L^2_z(H^{1/2}_x)} + \|\bar u_{yy} - \bar u'_{yy}\|_{L^2_x H^1_y} \right) \\ &\le C_m \|\bar u - \bar u'\|_{Q^{1/2}}. \end{aligned} \quad (4.31)$$

□

4.3 Uniform regular bounds on the dual profiles

In Proposition 3.6, we proved uniform bounds in $L^2_x H^1_z$ for the profiles Θ^j (defined in (3.38)). We now prove uniform bounds in Z^0 which will be useful to prove the stability of the dual profiles.

Lemma 4.5 (Z^0 estimates for Θ^j)**.** *Assume that the coefficients α, γ satisfy the hypotheses of Proposition 3.6. Assume furthermore that $\alpha \in H^1_x(H^1_z)$ and $\partial_{zz}\alpha \in L^2_z(H^{3/5}_x)$, and that $\gamma_1 \in L^2_z(H^{2/3}_x) \cap L^\infty_z(H^{1/3}_x) \cap W^{1,\infty}_z(L^2_x)$, $\gamma_2 \in L^2_z(L^\infty_x) \cap H^1_x(L^2_z) \cap H^1_z(H^{3/5}_x)$.*

There exists a constant $\mu_0 > 0$, depending only on Ω, such that if

$$\begin{aligned}\|\alpha - 1\|_{H^1_x(H^1_z)} + \|\partial_{zz}\alpha\|_{L^2_z(H^{3/5}_x)} &\le \mu_0, \\ \|\gamma_1\|_{L^2_z(H^{2/3}_x)} + \|\gamma_1\|_{L^\infty_z(H^{1/3}_x)} + \|\partial_z \gamma_1\|_{L^\infty_z(L^2_x)} &\le \mu_0, \\ \|\gamma_2\|_{L^2_z(L^\infty_x)} + \|\partial_x \gamma_2\|_{L^2} + \|\partial_z \gamma_2\|_{L^2_z(H^{3/5}_x)} &\le \mu_0 \end{aligned}$$

then the solution constructed in Proposition 3.6 belongs to $Z^0(\Omega_\pm)$, and satisfies the estimate

$$\|\Theta^j\|_{Z^0(\Omega_+)} + \|\Theta^j\|_{Z^0(\Omega_-)} \lesssim 1.$$

Proof. For the proof of the Z^0 estimates, it will be convenient to work with the lift $\Theta^j_\flat$ defined in Remark 3.8. We recall that $\Theta^j_\flat$ satisfies equation (3.48).

• *First step: existence of a solution in Z^0 for smooth coefficients.*

We still assume that α and γ are smooth, and we now derive Z^0 estimates for our $L^2(H^1)$ solution. In this case, we notice that S^j belongs to $L^2(\Omega)$.

It follows from the work of Pagani that

$$\|z\partial_x\Theta_\flat^j\|_{L^2(\Omega)} + \|\partial_{zz}\Theta_\flat^j\|_{L^2(\Omega)} + \|\Theta_\flat^j\|_{L_z^\infty(H_x^{1/2})} \tag{4.32}$$

$$\lesssim \quad \|S^j\|_{L^2(\Omega)} + 1 + |b_+| + |b_-| + \|\Theta_\flat^j\|_{L^2(\Omega)}. \tag{4.33}$$

Thus there remains to evaluate each of the terms in the right-hand side. We start with $\|S^j\|_{L^2(\Omega)}$. According to the definition of Θ_l^j, we have

$$\begin{aligned}\|S^j\|_{L^2(\Omega_\pm)} \quad \leq \quad & C\,(1+|b_+|+|b_-|) \\ & + \left(\|\gamma\|_\infty + \|\partial_z\gamma\|_{L_z^2(L_x^\infty)}\right)\|\partial_z\Theta^j\|_{L^2(\Omega_\pm)} \\ & + \left(\|\gamma_x\|_\infty + \|\partial_{xz}\gamma\|_{L_z^2(L_x^\infty)}\right)\|\partial_z\Theta^j\|_{L^2(\Omega_\pm)}.\end{aligned}$$

Note also that

$$|b_\pm| \leq C\left(\|\Gamma\|_\infty + \|\Gamma_x\|_\infty\right)\|\Theta^j(\cdot,0^+)\|_{L^2(x_0,x_1)} \leq \left(\|\gamma\|_\infty + \|\Gamma_x\|_\infty\right)\|\partial_z\Theta^j\|_{L^2(\Omega_+)},$$

and

$$\|\Theta_\flat^j\|_{L^2} \lesssim \|\Theta_\sharp^j\|_{L^2} + |b_+| + |b_-|$$

Since Proposition 3.6 ensures that $\|\partial_z\Theta^j\|_{L^2(\Omega_\pm)} \lesssim 1$, we obtain a bound on $\|S^j\|_{L^2(\Omega_+)}$ and $b_\pm$ (depending on some high order norms of the coefficients). We therefore obtain a first Z^0 estimate on $\tilde\Theta^j$.

• *Second step: Z^0 estimates for smooth coefficients.*

We still assume that the coefficients α, γ are smooth, but our purpose is now to derive a Z^0 bound on the solution that only depends on norms of the coefficients γ_i, α in lower order norms.

More precisely, we now assume that

$$\gamma = z\gamma_1 + \gamma_2,$$

where $\gamma_i \in W^{2,\infty}$, $\alpha \in W^{2,\infty}$, and with

$$\begin{aligned}\|\gamma_1\|_{L_z^2(H_x^{2/3})} + \|\gamma_1\|_{L_z^\infty(H_x^{1/3})} + \|\partial_z\gamma_1\|_{L_z^\infty(L_x^2)} &\ll 1, \\ \|\gamma_2\|_{L_z^2(L_x^\infty)} + \|\partial_x\gamma_2\|_{L^2} + \|\partial_z\gamma_2\|_{L_z^2(H_x^{3/5})} &\ll 1.\end{aligned}$$

We also assume that

$$\partial_z\alpha \in L_z^2(H_x^1), \quad \partial_{zz}\alpha \in L_z^2(H_x^{3/5}).$$

It can be easily checked that these assumptions, together with the ones of Proposition 3.6, ensure that the coefficients Γ_i satisfy the same smallness assumptions as the coefficients γ_i. We first rewrite the equation for $\Theta_\flat^j$ as

$$-z\partial_x\Theta_\sharp^j - \partial_{zz}\Theta_\sharp^j = (\alpha-1)\partial_{zz}\Theta_\sharp^j + S^j,$$

so that there exists a universal constant $\bar C$ such that

$$\|z\partial_x\Theta_\sharp^j\|_{L^2} + \|\partial_{zz}\Theta_\sharp^j\|_{L^2} \leq \bar C\left(\|\alpha-1\|_\infty\|\partial_{zz}\Theta_\sharp^j\|_{L^2} + \|S^j\|_{L^2} + |b_+| + |b_-| + \|\Theta_\sharp^j\|_{L^2}\right).$$

For $\|\alpha-1\|_\infty \ll 1$, we absorb the first term into the left-hand side. Furthermore,

$$\|S^j\|_{L^2(\Omega)} \lesssim |b_+| + |b_-| + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_+)} + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_-)}.$$

Note also that $b_\pm$ are the (integrals of the) traces of $G[\Theta^j]$ at $z = 0^\pm$. It follows that if the coefficients α, γ satisfy the smallness assumptions of Proposition 3.6, there exists a universal constant $\bar{C}$ such that

$$\|z\partial_x\Theta^j_\sharp\|_{L^2} + \|\partial_{zz}\Theta^j_\sharp\|_{L^2} \leq \bar{C}\left(1 + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_+)} + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_-)}\right). \tag{4.34}$$

Therefore we now focus on the derivation of a bound for $\partial_z G[\Theta^j]$ in $L^2(\Omega_\pm)$. By symmetry, we only treat the case of Ω_+.

Differentiating the definition of $G[\Theta^j]$ (3.42) with respect to z, we have, in Ω_+,

$$\begin{aligned}
\partial_z G[\Theta^j] &= \Gamma_2\partial_z\Theta^j + \int_x^{x_1} \partial_x\Gamma_2\partial_z\Theta^j && (4.35)\\
&\quad +\partial_z\Gamma_2\Theta^j + \int_x^{x_1} \partial_{xz}\Gamma_2\Theta^j && (4.36)\\
&\quad +z\Gamma_1\partial_z\Theta^j + \int_x^{x_1} \partial_x\Gamma_1 z\partial_z\Theta^j && (4.37)\\
&\quad +\Gamma_1\Theta^j + \int_x^{x_1} \partial_x\Gamma_1\Theta^j && (4.38)\\
&\quad -\int_x^{x_1} z\partial_z\Gamma_1\partial_x\Theta^j. && (4.39)
\end{aligned}$$

We then evaluate each term of the right-hand side separately.

- The term (4.35) is the easiest. Recalling that $\|\partial_z\Theta^j\|_{L^\infty(0,1)(L^2_x)} \lesssim \|\Theta^j\|_{Z^0(\Omega_+)}$, we have

$$\|(4.35)\|_{L^2(\Omega_+)} \leq C\|\Theta^j\|_{Z^0(\Omega_+)}\left(\|\Gamma_2\|_{L^2_z(L^\infty_x)} + \|\partial_x\Gamma_2\|_{L^2(\Omega)}\right)$$

- For the term (4.36), we recall that $\|\Theta^j\|_{L^\infty_z(H^{1/2}_x)} \lesssim \|\Theta^j\|_{Z^0}$. Using a $H^{-2/5}_x - H^{2/5}_x$ duality, we obtain

$$\|(4.36)\|_{L^2(\Omega_+)} \leq C\|\Theta^j\|_{L^\infty_z(H^{2/5}_x)}\|\partial_z\Gamma_2\|_{L^2_z(H^{3/5}_x)} \leq C\|\Theta^j\|_{Z^0(\Omega_+)}\|\partial_z\Gamma_2\|_{L^2_z(H^{3/5}_x)}.$$

  The term (4.38) is treated in a similar fashion, using a $H^{-1/3} - H^{1/3}$ duality

$$\|(4.38)\|_{L^2(\Omega_+)} \leq C\|\Theta^j\|_{L^\infty_z(H^{1/3}_x)}\|\Gamma_1\|_{L^2_z(H^{2/3}_x)}.$$

- For the term (4.37), the first part is easily bounded, using the embeddings $H^{1/2} \hookrightarrow L^4$ in 1D and Lemma B.12, as

$$\|\Gamma_1 z\partial_z\Theta^j\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x}\left\||z|^{\frac{1}{2}}\partial_z\Theta^j\right\|_{L^2_z H^{1/2}_x} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x}\|\Theta^j\|_{Z^0(\Omega_+)}. \tag{4.40}$$

  We write the L^2 norm of the second part in the following way:

$$\begin{aligned}
\left\|\int_x^{x_1} \partial_x\Gamma_1 z\partial_z\Theta^j\right\|_{L^2(\Omega_+)} &= \sup_{h\in L^2(\Omega_+),\|h\|_{L^2}\leq 1}\int_{\Omega_+} h\int_x^{x_1}\partial_x\Gamma_1 z\partial_z\Theta^j\\
&= \sup_{h\in L^2(\Omega_+),\|h\|_{L^2}\leq 1}\int_0^1\int_{x_0}^{x_1}\left(\int_{x_0}^x h\right)\partial_x\Gamma_1 z\partial_z\Theta^j.
\end{aligned} \tag{4.41}$$

By Lemma B.8, for each $z \in (0,1)$,

$$\left| \int_{x_0}^{x_1} \left(\int_{x_0}^{x} h \right) \partial_x \Gamma_1 z \partial_z \Theta^j \right| \lesssim \|h(\cdot,z)\|_{L^2_x} \|\Gamma_1(\cdot,z)\|_{H^{1/2}_x} \|z\partial_z \Theta^j(\cdot,z)\|_{H^{1/2}_{00_r}(x_0,x_1)}. \tag{4.42}$$

Integrating over $z \in (0,1)$ yields

$$\int_0^1 \left| \int_{x_0}^{x_1} \left(\int_{x_0}^{x} h \right) \partial_x \Gamma_1 z \partial_z \Theta^j \right| \lesssim \|h\|_{L^2_{x,z}} \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|z\partial_z \Theta^j\|_{L^2_z H^{1/2}_{00_r}(x_0,x_1)}. \tag{4.43}$$

Thus, using the embedding of Lemma B.14,

$$\left\| \int_x^{x_1} \partial_x \Gamma_1 z \partial_z \Theta^j \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|\Theta^j\|_{Z^0(\Omega_+)}. \tag{4.44}$$

Combining (4.40) and (4.44) yields

$$\|(4.37)\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|\Theta^j\|_{Z^0(\Omega_+)}. \tag{4.45}$$

- At last, we bound (4.39) by

$$\|(4.37)\|_{L^2(\Omega_+)} \le C\|z\partial_x \Theta^j\|_{L^2(\Omega_+)} \|\partial_z \Gamma_1\|_{L^\infty_z(L^2_z)} \le C\|\Theta^j\|_{Z^0(\Omega_+)} \|\partial_z \Gamma_1\|_{L^\infty_z(L^2_z)}.$$

In conclusion, setting

$$\mu_0 := \|\Gamma_1\|_{L^\infty_z H^{1/2}_x}\| + \|\partial_z \Gamma_1\|_{L^\infty_z(L^2_z)} + \|\Gamma_1\|_{L^2_z(H^{2/3}_x)} + \|\Gamma_2\|_{L^2_z(H^1_x)} + \|\partial_z \Gamma_2\|_{L^2_z(H^{3/5}_x)}$$

we infer that there exists a universal constant $\bar{C}$ such that

$$\|\partial_z G[\Theta^j]\|_{L^2(\Omega_\pm)} \le \bar{C}\mu_0 \|\Theta^j\|_{Z^0(\Omega_\pm)}.$$

Hence

$$|b_\pm| \bar{C}\mu_0 \|\Theta^j\|_{Z^0(\Omega_\pm)},$$

and

$$\|\Theta^j\|_{Z^0(\Omega_\pm)} \le \bar{C}\|\Theta^j_\flat\|_{Z^0(\Omega_\pm)}.$$

Plugging these estimates into (4.34), we get

$$\|\Theta^j_\flat\|_{Z^0} \le \bar{C}(1 + \mu_0 \|\Theta^j_\flat\|_{Z^0}),$$

and thus

$$\|\Theta^j_\flat\|_{Z^0} \le \bar{C}. \tag{4.46}$$

- *Third step: Z^0 estimates for general coefficients.*

We take a sequence α_n, γ^n of smooth coefficients such that $\gamma^n = z\gamma^n_1 + \gamma^n_2$ and γ^n_i (resp. α^n) converges towards γ_i (resp. α) in the relevant norms, namely

$$\begin{aligned} \|\gamma_1 - \gamma^n_1\|_{L^2_z(H^{2/3}_x)} + \|\gamma_1 - \gamma^n_1\|_{L^\infty_z(H^{1/3}_x)} + \|\partial_z(\gamma_1 - \gamma^n_1)\|_{L^\infty_z(L^2_x)} &\to 0, \\ \|\gamma_2 - \gamma^n_2\|_{L^2_z(L^\infty_x)} + \|\partial_x(\gamma_2 - \gamma^n_2)\|_{L^2} + \|\partial_z(\gamma_2 - \gamma^n_2)\|_{L^2_z(H^{3/5}_x)} &\to 0, \\ \|\alpha - \alpha^n\|_{H^1_z(H^1_x)} + \|\partial_{zz}(\alpha - \alpha^n)\|_{L^2_z(H^{3/5}_x)} &\to 0. \end{aligned}$$

We consider the profiles Θ^j_n associated with α^n, γ^n. According to the previous step, for all $n \in \mathbb{N}$, $\Theta^j_{n\flat} \in Z^0$, and we have the estimate (4.46). Therefore we can extract a subsequence and pass to the limit in the equation. It can be easily checked that the limit is a solution of (3.47), and satisfies (4.46). This concludes the proof of the Lemma. □

4.4 Stability of the dual profiles

We consider the profiles Θ^j and $(\Theta^j)'$ constructed in Proposition 3.6.

Lemma 4.6. *Assume that the hypotheses of Proposition 3.6 and Lemma 4.5 are satisfied for the coefficients $\alpha, \alpha', \gamma, \gamma'$.*

Then

$$\begin{aligned}&\|\Theta^j - (\Theta^j)'\|_{L^2_x(H^1_z)} + \|z\partial_x(\Theta^j - (\Theta^j)')\|_{L^2_x(H^{-1}_z)}\\ &\lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)} + \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_2 - \gamma_2'\|_{L^2_z(H^{1/2}_x)}.\end{aligned}$$

As a consequence,

$$\begin{aligned}&\|\Phi^j - (\Phi^j)'\|_{L^2(\Omega)} + \int_0^1 z(\Phi^j - (\Phi^j)')^2(x_0, z)\,\mathrm{d}z + \int_{-1}^0 |z|(\Phi^j - (\Phi^j)')^2(x_1, z)\,\mathrm{d}z\\ &\lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)} + \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_2 - \gamma_2'\|_{L^2_z(H^{1/2}_x)}.\end{aligned}$$

Proof. In order to alleviate the notation, we drop the superscripts j, choosing one of the cases $j = 0$ or $j = 1$ (the two cases are similar.) Following Proposition 3.6, we introduce Θ, Θ' defined by (3.38), and we define $\theta = \Theta - \Theta'$. Note that $[\theta]_{|z=0} = 0$, so that $\theta \in L^2_x(H^1_z)$. We also denote by G, G' the operators defined in (3.42) associated with $(\alpha, \gamma), (\alpha', \gamma')$.

It follows that θ satisfies

$$\begin{aligned}-\partial_z(\partial_z\theta + G[\theta]) - \frac{z}{\alpha}\partial_x\theta = \left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right) z\partial_x\Theta' + \partial_z(G - G')[\Theta'],\\ \theta(\cdot, \pm 1) = 0,\\ \theta(x_1, z) = 0 \quad \forall z > 0,\\ \theta(x_0, z) = 0 \quad \forall z < 0.\end{aligned} \tag{4.47}$$

Then, according to Corollary 3.9, it suffices to estimate the right-hand side in $L^2(H^{-1})$.

• *Estimate of $(1/\alpha - 1/\alpha')z\partial_x\Theta'$ in $L^2(H^{-1})$:*

Using the Sobolev embedding $L^q(-1, 1) \subset H^{-1}(-1, 1)$ for all $q > 1$, we obtain

$$\begin{aligned}\left\|z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta'\right\|_{L^2_x(H^{-1}_z)} &\leq \left\|z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta'\right\|_{L^2_x(L^{4/3}_z)}\\ &\leq C\|\alpha - \alpha'\|_{L^\infty_x(L^4_z)}\|z\partial_x\Theta'\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the Z^0 bounds on Θ' from Lemma 4.5, we get

$$\left\|z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta'\right\|_{L^2_x(H^{-1}_z)} \lesssim \|\alpha - \alpha'\|_{L^\infty} \lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)}. \tag{4.48}$$

• *Estimate of $(G - G')[\Theta']$ in $L^2(\Omega)$:*

As in the proof of Proposition 3.6, we write $\Gamma = z\Gamma_1 + \Gamma_2$. We focus on the bound of $(G - G')[\Theta']$ in $L^2(\Omega_+)$, since the bound in Ω_- is identical. Note that

$$(\Gamma - \Gamma')\Theta' + \int_x^{x_1} \partial_x(\Gamma - \Gamma')\Theta' = -\int_x^{x_1} (\Gamma - \Gamma')\partial_x\Theta'.$$

We start with the terms involving Γ_1 and Γ_1'. We have

$$\left\| \int_x^{x_1} z(\Gamma_1 - \Gamma_1')\partial_x \Theta' \right\|_{L^2(\Omega_+)} \leq \|z\partial_x\Theta'\|_{L^2(\Omega_+)} \|\Gamma_1 - \Gamma_1'\|_{L^\infty_z(L^2_x)}. \tag{4.49}$$

Since

$$\Gamma_1 - \Gamma_1' = \frac{\gamma_1 - \gamma_1'}{\alpha} + \gamma_1' \frac{\alpha' - \alpha}{\alpha\alpha'},$$

we get

$$\begin{aligned} \|\Gamma_1 - \Gamma_1'\|_{L^\infty_z(L^2_x)} &\lesssim \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_1'\|_{L^\infty_z(L^2_x)} \|\alpha - \alpha'\|_\infty \\ &\lesssim \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)}. \end{aligned}$$

Hence

$$\left\| \int_x^{x_1} z(\Gamma_1 - \Gamma_1')\partial_x \Theta' \right\|_{L^2(\Omega_+)} \lesssim \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)}. \tag{4.50}$$

We now address the terms with Γ_2, Γ_2'. Integrating by parts, we have

$$\begin{aligned} \left\| \int_x^{x_1} (\Gamma_2 - \Gamma_2')\partial_x \Theta' \right\|_{L^2(\Omega_+)} &= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \leq 1} \int_{\Omega_+} h \int_x^{x_1} (\Gamma_2 - \Gamma_2')\partial_x \Theta' \\ &= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \leq 1} \int_{\Omega_+} \left(\int_{x_0}^x h \right) (\Gamma_2 - \Gamma_2')\partial_x \Theta'. \end{aligned}$$

Then, for any $z \in (0,1)$, using Lemma B.9, we have

$$\left| \int_{x_0}^{x_1} \left(\int_{x_0}^x h \right) (\Gamma_2 - \Gamma_2')\partial_x \Theta' \right| \lesssim \|h\|_{L^2_x} \|\Gamma_2 - \Gamma_2'\|_{H^{1/2}_x} \|\Theta'\|_{H^{1/2}_{00_r}(x_0,x_1)}. \tag{4.51}$$

Integrating over $z \in (0,1)$ and using the embedding of Lemma B.13, we obtain, for any $h \in L^2(\Omega_+)$,

$$\begin{aligned} \left| \int_{\Omega_+} h \int_x^{x_1} (\Gamma_2 - \Gamma_2')\partial_x \Theta' \right| &\lesssim \|h\|_{L^2_{x,z}} \|\Gamma_2 - \Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{L^\infty_z H^{1/2}_{00_r}(x_0,x_1)} \\ &\lesssim \|h\|_{L^2_{x,z}} \|\Gamma_2 - \Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{Z^0(\Omega_+)}. \end{aligned} \tag{4.52}$$

Hence

$$\left\| \int_x^{x_1} (\Gamma_2 - \Gamma_2')\partial_x \Theta' \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_2 - \Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{Z^0(\Omega_+)}. \tag{4.53}$$

Now, writing

$$\Gamma_2 - \Gamma_2' = \frac{\gamma_2 - \gamma_2'}{\alpha} + \gamma_2' \frac{\alpha' - \alpha}{\alpha\alpha'},$$

we obtain

$$\begin{aligned} \|\Gamma_2 - \Gamma_2'\|_{L^2_z H^{1/2}_x} \lesssim & \|\gamma_2 - \gamma_2'\|_{L^2_z H^{1/2}_x} \|\alpha\|_{L^\infty_z(H^1_x)} \\ & + \|\gamma_2'\|_{L^\infty_z(H^{3/5}_x)} (\|\alpha\|_{L^\infty_z(H^1_x)} + \|\alpha'\|_{L^\infty_z(H^1_x)}) \|\alpha - \alpha'\|_{L^\infty_z(H^{1/2}_x)}. \end{aligned} \tag{4.54}$$

Therefore

$$\left\| \int_x^{x_1} (\Gamma_2 - \Gamma_2')\partial_x \Theta' \right\|_{L^2(\Omega_+)} \lesssim \|\gamma_2 - \gamma_2'\|_{L^2_z H^{1/2}_x} + \|\alpha - \alpha'\|_{L^\infty_z(H^{1/2}_x)}. \tag{4.55}$$

Gathering (4.48), (4.50) and (4.55), we obtain the result announced in the Lemma.

□

4.5 Proof of the stability of the orthogonality conditions

Let us now say a few words about the proof of Proposition 4.1. Let $\bar{u}, \bar{u}'$ be two profiles such that $\|\bar{u} - y\|_{Q^1} \leq m$, $\|\bar{u}' - y\|_{Q^1} \leq m$. According to Lemma 4.3, the coefficients α, α' γ_i, γ_i' satisfy the smallness assumptions of Proposition 3.6 and Lemma 4.5. Hence we can construct the profiles Θ^j, (Θ^j) associated with α, γ α', γ' respectively. Furthermore, according to Lemma 4.5, these profiles belong to $Z^0(\Omega_\pm)$.

We now use Lemma 4.4 together with Lemma 4.6, from which we infer that

$$\begin{aligned} \|\Theta^j - (\Theta^j)'\|_{L^2(\Omega)} + \int_0^1 z(\Theta^j - (\Theta^j)')^2(x_0, z)\,\mathrm{d}z + \int_{-1}^0 |z|(\Theta^j - (\Theta^j)')^2(x_1, z)\,\mathrm{d}z \\ \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \end{aligned} \tag{4.56}$$

Now, let $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}$. In this case, note that $f(x_i, \cdot) = 0$. Furthermore, since $\bar{u}(x_i, 0) = 0$, we have $Y(x_i, 0) = 0$ and thus $\widetilde{\delta_i}(0) = \widetilde{\delta_i}'(0) = 0$. Additionally, $z \lesssim Y(x_i, z) \lesssim z$.

Therefore the linear forms $\ell^j_{\bar{u}}(\Xi)$ can be written as

$$\begin{aligned} \ell^j_{\bar{u}}(\Xi) := &\int_\Omega \partial_x(f(x, Y(x, z)))\frac{\Theta^j}{\alpha} + \int_0^1 \frac{1}{\alpha(x_0, z)}\delta_0''(Y(x_0, z))\Theta^j(x_0, z)\,dz \\ &- \int_{-1}^0 \frac{1}{\alpha(x_1, z)}\delta_1''(Y(x_1, z))\Theta^j(x_1, z)\,dz \\ &+ \int z\partial_x\Theta^j\Gamma_1\left(\mathbf{1}_{z<0}\frac{\delta_1'(Y(x_1, z))}{\bar{u}_y(x_1, Y(x_1, z))} + \mathbf{1}_{z>0}\frac{\delta_0'(Y(x_0, z))}{\bar{u}_y(x_0, Y(x_0, z))}\right). \end{aligned} \tag{4.57}$$

Using the definition of the $\mathcal{H}$ norm together with (4.56) and with the estimates from Lemma 4.2 and Lemma 4.4, we infer that

$$\left|\ell^j_{\bar{u}}(\Xi) - \ell^j_{\bar{u}'}(\Xi)\right| \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}\|\Xi\|_{\mathcal{H}}. \tag{4.58}$$

This completes the proof of Proposition 4.1.

5 The nonlinear problem

In this section, we prove Theorem 2. The most difficult part is the existence statement, which relies on the scheme described in Section 1.4.

5.1 Execution of the nonlinear scheme

For $\eta > 0$ denote by B_η the open ball of radius η and centered at 0 in $\mathcal{H}^\perp_{\mathrm{sg}}$ (the subspace of $\mathcal{H}$ of data for which the model problem (2.1) can be solved with Z^1 regularity, see (2.30)). For every $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}$, one has the decomposition

$$\Xi = \Xi^\perp + \langle \Xi^0; \Xi\rangle_{\mathcal{H}}\Xi^0 + \langle \Xi^1; \Xi\rangle_{\mathcal{H}}\Xi^1, \tag{5.1}$$

where $\Xi^\perp \in \mathcal{H}^\perp_{\mathrm{sg}}$, Ξ^0, Ξ^1 are defined in Corollary 2.10, and the linear maps $\Xi \to \Xi^\perp$ and $\langle \Xi^k; \Xi\rangle$ are continuous. We will deduce the existence statement of Theorem 2 from the following proposition.

Proposition 5.1. *There exist $\eta > 0$, two Lipschitz maps ν^k for $k \in \{0,1\}$ from B_η to $\mathbb{R}$ with $\nu^0(0) = \nu^1(0) = 0$ and a map $\mathcal{U}_\perp : B_\eta \to Q^1$, which is Lipschitz from B_η to $Q^{1/2}$, such that, for every triplet $\Xi \in B_\eta$, $\mathcal{U}_\perp(\Xi)$ is a solution to (1.7) with data $\Xi + \nu^0(\Xi)\Xi^0 + \nu^1(\Xi)\Xi^1$ (where Ξ^0, Ξ^1 are defined in Corollary 2.10).*

Proof of the existence statement of Theorem 2. The existence statement of Theorem 2 is a direct consequence of Proposition 5.1. Indeed, with the notations of Proposition 5.1, define

$$\mathcal{M} := \left\{\Xi \in \mathcal{H};\ \ \|\Xi\|_{\mathcal{H}} < \eta \text{ and } \langle \Xi^k; \Xi\rangle_{\mathcal{H}} = \nu^k(\Xi^\perp) \text{ for } k = 0,1\right\} \tag{5.2}$$

and set, for $\Xi \in \mathcal{M}$,

$$\mathcal{U}(\Xi) := \mathcal{U}_\perp(\Xi^\perp). \tag{5.3}$$

Then $\mathcal{M}$ is a Lipschitz manifold modeled on $\mathcal{H}^\perp_{\mathrm{sg}}$ since ν^0 and ν^1 are Lipschitz maps. It contains $0_{\mathcal{H}}$ since $\nu^0(0_{\mathcal{H}^\perp_{\mathrm{sg}}}) = \nu^1(0_{\mathcal{H}^\perp_{\mathrm{sg}}}) = 0_{\mathbb{R}}$. Moreover, $\mathcal{M}$ is "tangent" to $\mathcal{H}^\perp_{\mathrm{sg}}$ at 0 in the sense of Remark 5.2.

Eventually, for every $\Xi \in \mathcal{M}$, $\mathcal{U}(\Xi) \in Q^1$ is a strong solution to (1.7). So the conclusions of the existence statement of Theorem 2 are satisfied. □

Remark 5.2. *Since we only proved Lipschitz regularity for the maps ν^1 and ν^2,* (5.2) a priori *only defines a Lipschitz manifold. Hence, it is difficult to define tangent spaces to $\mathcal{M}$. Nevertheless, one can say that $\mathcal{H}^\perp_{\mathrm{sg}}$ is tangent to $\mathcal{M}$ at 0 in the following senses:*

- *For $\Xi \in \mathcal{M}$, $\mathrm{d}(\Xi, \mathcal{H}^\perp_{\mathrm{sg}}) \lesssim \|\Xi\|^2_{\mathcal{H}}$.*
- *For every $\Xi \in \mathcal{H}^\perp_{\mathrm{sg}}$, for $t \in \mathbb{R}$ small enough, $t\Xi + \mathcal{O}(t^2) \in \mathcal{M}$.*

Proof of Proposition 5.1. **Construction of the sequence and uniform Q^1 bound.**

Let $\eta > 0$ small enough to be chosen later. Let $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}^\perp_{\mathrm{sg}}$ with $\|\Xi\|_{\mathcal{H}} < \eta$.

Let $\chi \in C^\infty(\mathbb{R}, [0,1])$, identically equal to one on $[-\frac{1}{3}, \frac{1}{3}]$ and compactly supported in $[-\frac{1}{2}, \frac{1}{2}]$. We define the initialization profile of our iterative scheme as

$$u_0(x,y) := \delta_0(y)\chi\left(\frac{x - x_0}{x_1 - x_0}\right) + \delta_1(y)\chi\left(\frac{x_1 - x}{x_1 - x_0}\right). \tag{5.4}$$

Hence, there exists $C_\chi > 0$ such that $u_0 \in Q^1$ and $\|u_0\|_{Q^1} \le C_\chi(\|\delta_0\|_{\mathcal{H}} + \|\delta_1\|_{\mathcal{H}}) \le 2\eta C_\chi$. Furthermore, $\partial_x u_0$ identically vanishes on the boundaries.

For each $n \in \mathbb{N}$, we let u_{n+1} be the solution to

$$\begin{cases} (y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ (u_{n+1})_{|\Sigma_i} = \delta_i + \nu^0_{n+1}\delta^0_i + \nu^1_{n+1}\delta^1_i, \\ (u_{n+1})_{|y=\pm 1} = 0, \end{cases} \tag{5.5}$$

where the coefficients ν^k_{n+1} are defined below, and where the triplets $(f^k, \delta^k_0, \delta^k_1)$ are defined in Corollary 2.10. Let us assume that $u_n \in Q^1$ is such that $\|u_n\|_{Q^1} \le \bar{C}\eta$ for some universal constant $\bar{C}$, and that $\partial^k_y u_n(x_i, 0) = 0$ for $k = 0, 1, 2$, so that $y \lesssim y + u_n(x_i, y) \lesssim y$. Assume furthermore that $\|\partial^3_y u_n\|_{L^\infty(\Sigma_i)} \lesssim \eta$.

As a consequence, denoting by Y_n the change of variables associated with $y + u_n$, we have

$$\left\| \frac{\delta_i''(Y_n(x_i, z))}{z} \right\|_{\mathscr{L}^2(\Sigma_i)} \lesssim \left\| \frac{\delta_i''(z)}{z} \right\|_{\mathscr{L}^2(\Sigma_i)}, \tag{5.6}$$

and

$$\begin{aligned}\partial_z \frac{\delta_i''(Y_n(x_i,z))}{z} &= \frac{Y_n(x_i,z)\delta_i''(Y_n(x_i,z)) - \delta_i^{(3)}(Y_n(x_i,z))}{z^2} \\ &\quad + (z - Y_n(x_i,z))\frac{\delta_i''(Y_n(x_i,z))}{z^2} \\ &\quad - \frac{\partial_y u_n(x_i, Y_n(x_i,z))}{z^2(1+\partial_y u_n(x_i,Y_n(x_i,z)))}\delta_i^{(3)}(Y_n(x_i,z)).\end{aligned}$$

The first two terms are bounded in $\mathscr{L}^2(\Sigma_i)$ by $\|\delta_i''(z)/z\|_{\mathscr{H}^1(\Sigma_i)}$. As for the last one, since $\partial_y u_n(x_i,0) = \partial_y^2 u_n(x_i,0) = 0$, it satisfies

$$\left|\frac{\partial_y u_n(x_i, Y_n(x_i,z))}{z^2(1+\partial_y u_n(x_i,Y_n(x_i,z)))}\delta_i^{(3)}(Y_n(x_i,z))\right| \lesssim \|\partial_y^3 u_n\|_{L^\infty(\Sigma_i)}|\delta_i^{(3)}(Y_n(x_i,z))|$$

and thus its $\mathscr{L}^2(\Sigma_i)$ norm is bounded by $C\eta\|\delta_i^{(3)}\|_{L^2}$.

We let ℓ_n^j denote the boundary linear forms associated with the flow $\bar{u}_n := y + u_n$ (see Definition 3.10). For $u \in Q^1$, we define the following 2 by 2 matrix:

$$M_u := \left(\ell_{y+u}^j(f^k, \delta_0^k, \delta_1^k)\right)_{0\le j,k\le 1}. \tag{5.7}$$

In particular, since $M_0 = \mathrm{Id}$ and $u \mapsto \ell_{y+u}$ is locally Lipschitz (by Proposition 4.1) from $Q^{1/2}$ to $\mathcal{L}(\mathcal{H})$, M_u is invertible for u small enough, and $u \in Q^{1/2} \mapsto M_u^{-1}$ is Lipschitz. We set $M_n := M_{u_n}$. Then the coefficients ν_{n+1} are defined by

$$\nu_{n+1} := -M_n^{-1}\begin{pmatrix}\ell_n^0(f,\delta_0,\delta_1)\\ \ell_n^1(f,\delta_0,\delta_1)\end{pmatrix}. \tag{5.8}$$

It is easily checked that this choice ensures that

$$\ell_n^j\left(f + \nu_{n+1}^0 f^0 + \nu_{n+1}^1 f^1, \delta_0 + \nu_{n+1}^0\delta_0^0 + \nu_{n+1}^1\delta_0^1, \delta_1 + \nu_{n+1}^0\delta_1^0 + \nu_{n+1}^1\delta_1^1\right) = 0 \quad i = 0,1. \tag{5.9}$$

Furthermore, since $\|u_n\|_{Q^1} \lesssim \eta$, $|\nu_{n+1}| \le \bar{C}'\|\Xi\|_{\mathscr{H}}$ for some universal constant $\bar{C}'$.

By Proposition 3.14 and Lemma 4.3, this choice ensures that the system has a solution in Q^1 and the existence of a constant $C > 0$ such that

$$\|u_{n+1}\|_{Q^1} \le C\|\Xi\|_{\mathcal{H}} + C|\nu_{n+1}| \lesssim \|\Xi\|_{\mathcal{H}} \le C_1\eta \tag{5.10}$$

for some universal constant C_1. In order to complete the induction, there only remains to check that $\partial_y^{(k)} u_{n+1}(x_i,0) = 0$ and that $\|\partial_y^3 u_{n+1|\Sigma_i}\|_{L^\infty(\Sigma_i} \lesssim \eta$. These properties follow from the fact that $(u_{n+1})_{|\Sigma_i} = \delta_i + \nu_{n+1}^0\delta_i^0 + \nu_{n+1}^1\delta_i^1$ and from the properties $\delta_i^{(k)}(0) = 0$ for $k = 0,1,2$.

We conclude that the sequence u_n is uniformly bounded in Q^1 by a small constant $m \le C_1\eta$.

Convergence of the sequence in $Q^{1/2}$. We now turn to the convergence of the sequence. Let $w_n := u_{n+1} - u_n$ for $n \ge 1$. Thanks to the previous paragraph, the sequence w_n is uniformly bounded in Q^1 by $2C_1\eta$. Moreover, for each $n \in \mathbb{N}^*$, w_n is the strong solution to

$$\begin{cases}(y+u_n)\partial_x w_n - \partial_{yy} w_n = -w_{n-1}\partial_x u_n + (\nu_{n+1}^0 - \nu_n^0)f^0 + (\nu_{n+1}^1 - \nu_n^1)f^1,\\ (w_n)_{|\Sigma_i} = (\nu_{n+1}^0 - \nu_n^0)\delta_i^0 + (\nu_{n+1}^1 - \nu_n^1)\delta_i^1,\\ (w_n)_{|y=\pm 1} = 0.\end{cases} \tag{5.11}$$

We already know that the solution w_n belongs to Q^1, as the difference between two Q^1 functions. Hence there is no need to check that the orthogonality conditions are satisfied, and we can apply Proposition 3.14, recalling that $\|u_n\|_{Q^1}$ is small thanks to (5.10). Note that the source terms f^j vanish on the lateral boundaries. Moreover, using the boundary conditions and the assumptions on δ_i, it can be easily proved that

$$\left\| \frac{1}{y}(w_{n-1}\partial_x u_n)_{|\Sigma_i} \right\|_{\mathscr{H}^1(\Sigma_i)} \lesssim \eta|\nu_n - \nu_{n-1}|.$$

We obtain, by (3.92),

$$\begin{aligned} \|w_n\|_{Q^{1/2}} \lesssim & \|w_{n-1}\partial_x u_n\|_{H_x^{1/2}L_y^2} + \|w_{n-1}\partial_x u_n\|_{L_x^2 H_y^{3/2}} + \|\chi_1 w_{n-1}\partial_x u_n\|_{H_x^{1/2}H_y^{1/2}} \\ & + |\nu_{n+1} - \nu_n| + \eta|\nu_n - \nu_{n-1}|. \end{aligned} \tag{5.12}$$

Indeed, the elementary functions f^j and δ_i^j of Corollary 2.10 have fixed norms.

By (5.8) and Proposition 4.1,

$$\begin{aligned} |\nu_{n+1} - \nu_n| &= |M_n^{-1}\ell_n(\Xi) - M_{n-1}^{-1}\ell_{n-1}(\Xi)| \\ &\le \|M_n^{-1} - M_{n-1}^{-1}\||\ell_n(\Xi)| + \|M_{n-1}^{-1}\||\ell_n(\Xi) - \ell_{n-1}(\Xi)| \\ &\lesssim \|u_n - u_{n-1}\|_{Q^{1/2}}\|\Xi\|_{\mathcal{H}}. \end{aligned} \tag{5.13}$$

Let us now derive bounds on $w_{n-1}\partial_x u_n$.

$$\|w_{n-1}\partial_x u_n\|_{H_x^{1/2}L_y^2} \lesssim \|w_{n-1}\|_{L_y^\infty H_x^{1/2}}\|\partial_x u_n\|_{L_y^2 H_x^{2/3}} \lesssim \|u_n\|_{Q^1}\|w_{n-1}\|_{Q^{1/2}}, \tag{5.14}$$

and

$$\|w_{n-1}\partial_x u_n\|_{L_x^2 H_y^{3/2}} \lesssim \|w_{n-1}\|_{L_x^\infty(H_y^{3/2})}\|\partial_x u_n\|_{L_x^2 H_y^{3/2}} \lesssim \|u_n\|_{Q^1}\|w_{n-1}\|_{Q^{1/2}},. \tag{5.15}$$

The bound on $\|\chi_1 w_{n-1}\partial_x u_n\|_{H_x^{1/2}H_y^{1/2}}$ is slightly more involved. Note that the Q^1 bound on u_n, in itself, does not allow us to bound $\partial_x u_n$ in $H_x^{1/2}H_y^{1/2}$. However, we can use the same arguments as in Proposition 3.14, and observe that on the support of χ_1, u_n satisfies a classical parabolic equation. More precisely, recall that

$$(y + u_{n-1})\partial_x u_n = f + \nu_n^0 f^0 + \nu_n^1 f^1 + \partial_y^2 u_n,$$

where f, f^0, f^1 are smooth, and $\partial_y^3 u_n \in L^2$. Differentiating the equation with respect to y and using the estimates $\|\partial_y^k u_{n-1}\|_\infty \lesssim \|u_n\|_{Q^1}$ for $k = 1, 2$, we infer that $(y + u_n)\partial_x\partial_y^3 u_n \in L^2$, and

$$\|(y + u_n)\partial_x\partial_y^3 u_n\|_{L^2} \lesssim |\nu_n| + (1 + \|u_{n-1}\|_{Q^1})\|u_n\|_{Q^1} \lesssim \eta.$$

Hence

$$\|\chi_1\partial_x\partial_y^3 u_n\|_{L^2} \le C\eta,$$

uniformly in n. Using once again the same method, we infer that

$$\|\chi_1\partial_x^2 u_n\|_{L^2} \le C\eta, \quad \|\chi_1\partial_x^2\partial_y u_n\|_{L^2} \le C\eta,$$

uniformly in n. Hence $\chi_1 \partial_x u_n$ is bounded by $C\eta$ in $H^1_x H^1_y \cap L^2_x H^3_y$, uniformly in n. We deduce that

$$\|\chi_1 w_{n-1} \partial_x u_n\|_{H^{1/2}_x H^{1/2}_y} \lesssim \|w_{n-1}\|_{H^{1/2}_x H^{1/2}_y} \|\chi_1 \partial_x u_n\|_{H^1_x H^1_y} \lesssim C\eta \|w_{n-1}\|_{Q^{1/2}}. \tag{5.16}$$

Gathering (5.14), (5.15) and (5.16), we obtain

$$\|w_n\|_{Q^{1/2}} \lesssim \eta \|w_{n-1}\|_{Q^{1/2}} + \eta^2 \|w_{n-2}\|_{Q^{1/2}}. \tag{5.17}$$

Classically, for η small enough, we infer that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$. We recall that it is also uniformly bounded in Q^1. Hence there exists $u = \mathcal{U}_\perp(\Xi) \in Q^1$ such that

$$\begin{aligned} u_n &\to u \quad \text{strongly in } Q^{1/2}, \\ u_n &\rightharpoonup u \quad \text{weakly in } Q^1. \end{aligned}$$

The strong convergence is sufficient to pass to the limit in (5.5). Furthermore, thanks to the continuity of the linear forms $u \in Q^{1/2} \mapsto \ell^i_u$, we can also pass to the limit in (5.8). We denote by $\nu(f, \delta_0, \delta_1) = \nu(\Xi)$ the limit of the sequence ν_n.

Lipschitz regularity of the constructed maps. In this paragraph, we prove that the maps $\Xi \mapsto \mathcal{U}_\perp(\Xi)$ and $\Xi \mapsto \nu(\Xi)$ have Lipschitz regularity. Let $\Xi, \Xi' \in B_\eta$. We use the prime notation to denote all the quantities associated with Ξ' during the nonlinear scheme.

In particular, one has

$$\begin{cases} (y+u_n)\partial_x(u_{n+1} - u'_{n+1}) - \partial_{yy}(u_{n+1} - u'_{n+1}) \\ \qquad = f - f' - (u_n - u'_n)\partial_x u'_{n+1} + (\nu^0_{n+1} - {\nu'}^0_{n+1}) f^0 + (\nu^1_{n+1} - {\nu'}^1_{n+1}) f^1 \\ (u_{n+1} - u'_{n+1})_{|\Sigma_i} = \delta_i - \delta'_i + (\nu^0_{n+1} - {\nu'}^0_{n+1})\delta^0_i + (\nu^1_{n+1} - {\nu'}^1_{n+1})\delta^1_i, \\ (u_{n+1} - u'_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{5.18}$$

Using the same estimates as previously,

$$\|u_{n+1} - u'_{n+1}\|_{Q^{1/2}} \lesssim \eta \|(u_n - u'_n)\|_{Q^{1/2}} + \|\Xi - \Xi'\|_{\mathcal{H}} + |\nu_{n+1} - \nu'_{n+1}|. \tag{5.19}$$

And, using one again Proposition 4.1 together with the definition (5.8),

$$|\nu_{n+1} - \nu'_{n+1}| \lesssim \|\Xi - \Xi'\|_{\mathcal{H}} + \eta \|u_n - u'_n\|_{Q^{1/2}} \tag{5.20}$$

Summing recursively these estimates this leads to the uniform estimates

$$\|u_{n+1} - u'_{n+1}\|_{Q^{1/2}} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}}, \tag{5.21}$$

$$\|\nu_{n+1} - \nu'_{n+1}\|_{\mathbb{R}^2} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}}. \tag{5.22}$$

This proves that $\Xi \mapsto \mathcal{U}_\perp(\Xi)$ is Lipschitz from $\mathcal{H}^\perp_{\mathrm{sg}}$ to $Q^{1/2}$ and ν is Lipschitz from $\mathcal{H}^\perp_{\mathrm{sg}}$ to $\mathbb{R}^2$.

Value of ν^0 and ν^1 at zero. One checks that, for $\Xi = 0 = (0,0,0)$, the constructed initialization u_0 defined in (5.4) is null. Since $0 \in \mathcal{H}^\perp_{\mathrm{sg}}$ and $\ell^j_0 = \ell^j$, this leads to $\nu_1 = 0$ (by (5.8)). Hence, in (5.5) for $n = 0$, the system solved by u_1 has vanishing boundary data and vanishing source term. Hence $u_1 = 0$. This property propagates for every $n \geq 0$. Hence $\nu^j(0) = \lim \nu^j_n(0) = 0$. □

5.2 Local uniqueness of solutions to the nonlinear problem

We prove the local uniqueness statement in Theorem 2.

The argument is straightforward: if two strong solutions u_1 and u_2 exist and are small in Q^1, their difference $w := u_1 - u_2$ is the solution to a degenerate elliptic linear equation with null source term and boundary values, so it vanishes identically.

More precisely, let $u, u' \in Q^1$ be two solutions to (1.7) satisfying $\|u\|_{Q^1} \leq \eta$ and $\|u'\|_{Q^1} \leq \eta$ for some small η. Then $w := u - u' \in Q^1$ and solves the system

$$\begin{cases} (y+u)\partial_x w + (\partial_x u')w - \partial_y^2 w = 0, \\ w_{|y=\pm 1} = 0, \\ w_{|\Sigma_i} = 0. \end{cases} \tag{5.23}$$

We follow the arguments of Section 3 and introduce a change of variables Y such that $\forall (x,z) \in \Omega$,

$$Y(x,z) + u(x, Y(x,z)) = z.$$

Let $\alpha(x,z) = (1+u_y)^2(x, Y(x,z))$, $\gamma = (zu_x - u_{yy})(x, Y(x,z))$, $\gamma_1' = \bar{u}_x'(x, Y(x,z))$. Then $W(x,z) = w(x, Y(x,z))$ is a solution of

$$\begin{cases} z\partial_x W + \gamma \partial_z W + \gamma_1' W - \alpha \partial_{zz} W = 0, \\ W_{|z=\pm 1} = 0, \\ W_{|\Sigma_i} = 0. \end{cases} \tag{5.24}$$

Furthermore, α, γ, γ_1' satisfy the estimates of Lemma 4.3. Using the results of Appendix A, we infer that $W = 0$. This concludes the proof of the uniqueness within the ball of radius η in Q^1.

5.3 Necessity of the orthogonality conditions

At the linear level, Theorem 1 states that $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}_{\mathrm{sg}}^{\perp}$ is a necessary condition to solve (2.1) with tangential regularity Q^1. Our purpose in this paragraph is to prove that, at the nonlinear level, if a solution has $Q^1 \cap H^2$ regularity, this necessary condition generalizes to $\Xi \in \mathcal{M}$. In particular, the nonlinear phenomena do not eliminate the need for orthogonality conditions, at least when one tries to obtain solutions with such regularity.

More precisely, we prove the following claim (which is part of Theorem 2 but which we recall here for the reader's convenience).

Proposition 5.3. *There exists $\eta > 0$ such that, if $\Xi \in \mathcal{H}$ with $\|\Xi\|_{\mathcal{H}} < \eta$ and $u \in Q^1 \cap H^2_x H^1_y(\Omega)$ with $\|u\|_{Q^1} + \|u\|_{H^2} < \eta$ is a solution to (1.7), then $\Xi \in \mathcal{M}$ and $u = \mathcal{U}(\Xi)$.*

Proof. Let $\eta > 0$ to be chosen small enough later on in the proof. Let $\Xi \in \mathcal{H}$ with $\|\Xi\|_{\mathcal{H}} < \eta$ and assume that there exists $u \in Q^1 \cap H^2(\Omega)$ with $\|u\|_{Q^1} + \|u\|_{H^2_x H^1_y} < \eta$ such that u is a solution to (1.7). We introduce

$$\widetilde{\Xi} := \Xi^{\perp} + \nu^0(\Xi^{\perp})\Xi^0 + \nu^1(\Xi^{\perp})\Xi^1, \tag{5.25}$$

which can be thought of as a good projection of Ξ on $\mathcal{M}$. Thanks to Proposition 5.1, we introduce $\widetilde{u} := \mathcal{U}_{\perp}(\Xi^{\perp}) \in Q^1$, which is a solution to (1.7) with data $\widetilde{\Xi}$ and $\|\widetilde{u}\|_{Q^1} \lesssim \eta$ (by Lipschitz regularity of the solution operator $\mathcal{U}_{\perp}$)

For $k \in \{0,1\}$, we also introduce the coefficients $\alpha^k := \nu^k(\Xi^\perp) - \langle \Xi^k; \Xi \rangle_{\mathcal{H}}$, which characterize how far Ξ is from $\mathcal{M}$. Then $w := \widetilde{u} - u$ belongs to Q^1 with

$$\|w\|_{Q^1} \lesssim \eta \tag{5.26}$$

and is a solution to

$$\begin{cases} (y + \widetilde{u})\partial_x w - \partial_{yy} w = -w(\partial_x u) + \alpha^0 f^0 + \alpha^1 f^1, \\ w_{|\Sigma_i} = \alpha^0 \delta_i^0 + \alpha^1 \delta_i^1, \\ w_{|y=\pm 1} = 0. \end{cases} \tag{5.27}$$

Note that since we have assumed that $u \in H^2_x(H^1_y)$, the right-hand side belongs to $H^1_x H^1_y \cap L^2_x(H^3_y)$. Furthermore, since we know that $w \in Q^1$, we can use the estimates of Proposition 3.14, from which we infer that

$$\begin{aligned} \|w\|_{Q^{1/2}} &\lesssim |\alpha^j| + \|w\partial_x u\|_{H^{1/2}_x H^{1/2}_y} + \|w\partial_x u\|_{L^2_x H^{3/2}_y} && (5.28)\\ &\lesssim |\alpha^j| + \|w\|_{Q^{1/2}}(\|u\|_{Q^1} + \|u\|_{H^2_x(H^1_y)}) && (5.29)\\ &\lesssim |\alpha^j| + \eta \|w\|_{Q^{1/2}}. && (5.30) \end{aligned}$$

Thus, for η small enough, we obtain

$$\|w\|_{Q^{1/2}} \lesssim |\alpha^j| \tag{5.31}$$

By Proposition 3.14, the fact that $w \in Q^1$ implies that, for $j \in \{0,1\}$, the following orthogonality conditions are satisfied

$$\begin{aligned} 0 &= \ell^j_{\widetilde{u}}(-w\partial_x u + \alpha^0 f^0 + \alpha^1 f^1, \alpha^0\delta^0_0 + \alpha^1\delta^1_0, \alpha^0\delta^0_1 + \alpha^1\delta^1_1) \\ &= \ell^j_{\widetilde{u}}(-w\partial_x u, 0, 0) + \alpha^0 \ell^j_{\widetilde{u}}(\Xi^0) + \alpha^1 \ell^j_{\widetilde{u}}(\Xi^1). \end{aligned} \tag{5.32}$$

By Proposition 4.1, for η small enough

$$\|\ell^j_{\widetilde{u}} - \ell^j\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\widetilde{u}\|_{Q^{1/2}} \leq \|\widetilde{u}\|_{Q^1} \lesssim \eta, \tag{5.33}$$

where ℓ^j are the orthogonality conditions for the linear shear flow (see Definition 2.8). Hence, recalling Corollary 2.10, we obtain from (5.32),

$$|\alpha^j| \lesssim \eta(|\alpha^0| + |\alpha^1|) + \left| \int_\Omega \Phi^j_{\widetilde{u}}(x,z)\partial_x F(x,z) \right|, \tag{5.34}$$

using Definition 3.10 and where

$$F(x,z) := (w\partial_x u)(x, \widetilde{Y}(x,z)), \tag{5.35}$$

where $\widetilde{Y}$ is the change of variable associated with $\widetilde{u}$ (see (3.2)). Then, using Lemma B.4,

$$\begin{aligned} \|\partial_x F\|_{L^2} &\lesssim \|\partial_x(w\partial_x u)\|_{L^2} + \|\partial^2_y(w\partial_x u)\|_{L^2} \\ &\lesssim \|w\|_{Q^{1/2}} \|u\|_{Q^1 \cap H^2} \\ &\lesssim \eta \|w\|_{Q^{1/2}}. \end{aligned} \tag{5.36}$$

Hence, since $\|\Phi^j_{\widetilde{u}}\|_{L^2} \lesssim 1$ by Proposition 3.6, we obtain, using (5.31),

$$|\alpha^j| \lesssim \eta(|\alpha^0| + |\alpha^1|). \tag{5.37}$$

We infer that $\alpha^j = 0$, and thus $w = 0$. □

A Uniqueness of weak solutions for linear problems

The purpose of this section is to prove the uniqueness of weak solutions in $L^2_x(H^1_0(-1,1))$ to (3.4), which is stated in Proposition 3.2. Such a uniqueness result is also proved in [17, Section 5].

The proof follows the arguments of Baouendi and Grisvard in [5], which concerns the case of the model equation (2.1). For the reader's convenience, we recall the main steps of the proof here, and adapt them to the present (slightly different) context.

We assume that the coefficients α, γ satisfy the assumptions of Proposition 3.2, and we introduce the sets

$$\mathcal{B} := \{u \in L^2_x(H^1_0(-1,1)),\ \ z\partial_x u \in L^2_x(H^{-1}_y)\},$$
$$\mathcal{A} := \mathcal{B} \cap H^1(\Omega).$$

Note that if $u \in L^2_x(H^1_0(-1,1))$ is a weak solution of (3.4), then $u \in \mathcal{B}$. Indeed, it follows from the weak formulation (3.9) that for any $V \in H^1_0(\Omega)$,

$$\langle z\partial_x U, V\rangle_{L^2H^{-1},L^2H^1} = -\int_\Omega (\gamma+\alpha_z)V\partial_z U - \int_\Omega \alpha\partial_z U\partial_z V + \int_\Omega gV.$$

By density, this formula still holds for $V \in L^2(H^1)$, and therefore $z\partial_x U \in L^2(H^{-1})$.

We then recall the following result from [5]:

Lemma A.1. *The set $\mathcal{A}$ is dense in $\mathcal{B}$. Furthermore, there exists a constant C depending only on Ω, such that for $i \in \{0,1\}$,*

$$\forall v \in \mathcal{A}, \quad \int_{-1}^{1} |z|\ |v(x_i,y)|^2\,\mathrm{d}y \le C\|v\|^2_{\mathcal{B}}.$$

As a consequence, the applications

$$v \in \mathcal{A} \mapsto v_{|x=x_i} \in \mathscr{L}^2_z(-1,1)$$

can be uniquely extended into continuous applications on $\mathcal{B}$.

As a consequence, Baouendi and Grisvard [5] obtain the following corollary:

Corollary A.2. *For all $u, v \in \mathcal{B}$,*

$$\langle z\partial_x u, v\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \langle z\partial_x v, u\rangle_{L^2(H^{-1}),L^2(H^1_0)} = \int_0^H (zuv)_{x=x_1} - \int_0^H (zuv)_{x=x_0}. \tag{A.1}$$

Proof. Thanks to Lemma A.1, it suffices to prove the identity when $u, v \in \mathcal{A}$. In that case, the left-hand side is simply

$$\int_\Omega z\partial_x uv + zu\partial_x v = \int_\Omega \partial_x(zuv).$$

The result follows by integration. □

Proof of uniqueness of weak solutions to (3.4). Let $U \in L^2_x(H^1_0)$ be a weak solution of (3.4) with $f = 0$ and $\delta_i = 0$. As mentioned above, $U \in \mathcal{B}$. According to Corollary A.2, for any $V \in \mathcal{B}$ such that $V = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$-\langle z\partial_x V, U\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \int_\Omega (\gamma+\alpha_z)\partial_z UV + \int_\Omega \alpha\partial_z U\partial_z V = 0.$$

Now, let $g \in C_c^\infty(\Omega)$ be arbitrary, and let $V \in L^2(H_0^1)$ be a weak solution of

$$\begin{cases} -z\partial_x V - \partial_z(\gamma V) - \partial_{zz}(\alpha V) = g, \\ V_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases}$$

(The existence of weak solutions for this adjoint problem is proved in the same way as existence for the direct problem in Proposition 3.2).

Then $V \in \mathcal{B}$, and choosing U as a test function in the variational formulation for V, we obtain

$$\int_\Omega gu = 0.$$

Thus $u = 0$. Uniqueness of weak solutions of (3.1) follows. □

B Proofs of functional analysis results

B.1 An abstract existence principle

As Fichera in [14], we use the following abstract existence principle (see [12, Theorem 1]), which allows to skip a viscous regularization scheme.

Lemma B.1. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}$ be three Hilbert spaces. Let $F_i \in \mathcal{L}(\mathscr{H}_i;\mathscr{H})$ for $i \in \{1,2\}$. Then the following statements are equivalent:*

- $\operatorname{range} F_1 \subset \operatorname{range} F_2$,
- *There exists a constant $C > 0$ such that*

$$\forall h \in \mathscr{H}', \quad \|F_1^* h\|_{\mathscr{H}_1'} \le C\|F_2^* h\|_{\mathscr{H}_2'}. \tag{B.1}$$

- *There exists $G \in \mathcal{L}(\mathscr{H}_1;\mathscr{H}_2)$ such that $F_1 = F_2 G$.*

Moreover, when these hold, there exists a unique $G \in \mathcal{L}(\mathscr{H}_1;\mathscr{H}_2)$ such that $\ker G = \ker F_1$, $\operatorname{range} G \subset (\operatorname{range} F_2^*)^\perp$ and $\|G\| = \inf\{C > 0;\ \text{(B.1) holds}\}$.

Indeed, this yields the following weak Lax-Migram result, where the linear right-hand side is assumed to be continuous for the weaker norm.

Lemma B.2. *Let $\mathscr{U}$ and $\mathscr{V}$ be two Hilbert spaces with $\mathscr{V}$ continuously embedded in $\mathscr{U}$. Let a be a continuous bilinear form on $\mathscr{U} \times \mathscr{V}$ and b be a continuous linear form on $\mathscr{U}$. Assume that there exists a constant $c > 0$ such that, for every $v \in \mathscr{V}$,*

$$a(v,v) \ge c\|v\|_{\mathscr{U}}^2. \tag{B.2}$$

Then, there exists $u \in \mathscr{U}$ such that $\|u\|_{\mathscr{U}} \le \frac{1}{c}\|b\|_{\mathcal{L}(\mathscr{U})}$ and, for every $v \in \mathscr{V}$, $a(u,v) = b(v)$.

Proof. Set $\mathscr{H} := \mathcal{L}(\mathscr{V})$, $\mathscr{H}_1 := \mathcal{L}(\mathscr{U})$, $F_1 := \mathrm{Id}$ (from $\mathcal{L}(\mathscr{U})$ to $\mathcal{L}(\mathscr{V})$), $\mathscr{H}_2 := \mathscr{U}$ and $F_2 : \mathscr{U} \to \mathcal{L}(\mathscr{V})$ defined by $F_2 u := a(u,\cdot)$. Then $F_1^* = \mathrm{Id}$ (from $\mathscr{V}$ to $\mathscr{U}$) and $F_2^* v = a(\cdot, v)$. Moreover

$$\|F_2^* v\|_{\mathcal{L}(\mathscr{U})} \ge |a(v,v)|/\|v\|_{\mathscr{U}} \ge c\|v\|_{\mathscr{U}} = c\|F_1^* v\|_{\mathscr{U}}. \tag{B.3}$$

So (B.1) holds with $C = 1/c$ and Lemma B.1 yields the existence of $G \in \mathcal{L}(\mathcal{L}(\mathscr{U});\mathscr{U})$ such that $F_1 = F_2 G$ and $\|G\| \le \frac{1}{c}$. The conclusions follow by setting $u := Gb$. □

B.2 Product and composition rules in Sobolev spaces

Lemma B.3 (Pointwise multiplication)**.** *Pointwise multiplication is a continuous bilinear map*

- *from* $H^{3/2}(-1,1) \times H^{3/2}(-1,1)$ *to* $H^{3/2}(-1,1)$,
- *from* $H^{1/2}(x_0,x_1) \times H^s(x_0,x_1)$ *to* $H^{1/2}(x_0,x_1)$ *for any* $s > 1/2$.

Proof. These are particular cases of [7, Theorem 7.4]. □

Lemma B.4 (Composition of H^σ functions)**.** *Let* $Y \in H^1(\Omega)$ *such that* $\partial_z Y \in L^\infty(\Omega)$, $\partial_x Y \in L^2_z(H^{2/3}_x)$.

Assume that there exists a constant $m > 0$ *such that* $\partial_z Y(x,z) \in [m, m^{-1}]$ *a.e. and* $Y(x,\pm 1) = \pm 1$.

Then for any $\sigma, \sigma' \in (0,1)$ *such that* $\sigma + \sigma' \leq 1$, *for any* $g \in H^\sigma_x(H^{\sigma'}_y) \cap L^2_x(H^{\sigma+2\sigma'}_y)$

$$\|g(x,Y(x,z))\|_{H^\sigma_x(H^{\sigma'}_z)} \leq C\left(\|g\|_{H^\sigma_x(H^{\sigma'}_y)} + \|g\|_{L^2_x(H^{\sigma+2\sigma'}_y)}\right).$$

Proof. Throughout the proof, we set

$$G(x,z) = g(x,Y(x,z)).$$

First, note that for all $g \in L^2(\Omega)$,

$$\|G\|^2_{L^2(\Omega)} = \int_{x_0}^{x_1}\int_{-1}^{1} g^2(x,Y(x,z))\,\mathrm{d}x\,\mathrm{d}z \leq m^{-1}\int_{x_0}^{x_1}\int_{-1}^{1} g^2(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

so that $\|G\|_{L^2} \leq \|g\|_{L^2}$.

Furthermore,

$$\partial_x G(x,z) = \partial_x g(x,Y(x,z)) + \partial_x Y(x,z)\partial_y g(x,Y(x,z)).$$

Hence

$$\|\partial_x G\|_{L^2(\Omega)} \leq C_m\left(\|\partial_x g\|_{L^2(\Omega)} + \|\partial_x Y\|_{L^2(H^{2/3}_x)}\|\partial^2_y g\|_{L^2}\right).$$

Now, note that the application $g \mapsto G$ is linear. By interpolation, we obtain, for any $\sigma \in (0,1)$,

$$\|G\|_{H^\sigma_x(L^2_z)} \leq C_m\left(\|g\|_{H^\sigma_x(L^2_z)} + \|g\|_{L^2(H^{2\sigma}_y)}\right).$$

We now prove the same type of estimates for the z derivatives. We have

$$\partial_z G(x,z) = \partial_z Y(x,z)\partial_y g(x,Y(x,z)),$$

and thus $\|G\|_{L^2_x H^1_z} \lesssim \|g\|_{L^2_x H^1_y}$. By interpolation, we infer that for any $\sigma \in (0,1)$,

$$\|G\|_{L^2_x H^\sigma_z} \leq C_m \|g\|_{L^2_x H^\sigma_y}.$$

Combining the two estimates and interpolating once again, we infer that for any $\sigma, \sigma' \in (0,1)$ such that $\sigma + \sigma' \leq 1$,

$$\|G\|_{H^\sigma_x(H^{\sigma'}_z)} \leq C_{\alpha,m}\left(\|g\|_{H^\sigma_x(H^{\sigma'}_y)} + \|g\|_{L^2_x(H^{\sigma+2\sigma'}_y)}\right).$$

□

Corollary B.5. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$, and $\bar{u}(x, \pm 1) = \pm 1$. Let $Y = Y(x, z)$ such that $\bar{u}(x, Y(x, z)) = z$ for all $(x, z) \in \Omega$.*

Then $\|Y_z - 1\|_\infty \ll 1$, $Y_x \in L^2_z(H^{2/3}_x)$, and $\|Y_x\|_{L^2_z(H^{2/3}_x)} \ll 1$.

Proof. First, observe that $Y_z = 1/\bar{u}_y(x, Y(x, z))$, so that the first estimate follows from Lemma 1.6.

Concerning the estimate on Y_x, we observe that

$$Y_x = -\frac{\bar{u}_x}{\bar{u}_y}(x, Y(x, z)).$$

Let us first assume that $\bar{u}$ is smooth (say C^∞), and then argue by density. If $\bar{u}$ is smooth, then the above formula first shows that $Y_x \in L^\infty$. Differentiating the identity with respect to x once again, we infer that $Y \in W^{2,\infty}$. Furthermore, according to Lemma B.4,

$$\|Y_x\|_{H^{2/3}_x L^2_z} \lesssim \left\| \frac{\bar{u}_x}{\bar{u}_y} \right\|_{H^{2/3}_x L^2_z} + \|Y_x\|_{H^{2/3}_x L^2_z} \left\| \partial_y^2 \left(\frac{\bar{u}_x}{\bar{u}_y} \right) \right\|_{L^2}.$$

When $\|\bar{u} - y\|_{Q^1} \ll 1$,

$$\left\| \partial_y^2 \left(\frac{\bar{u}_x}{\bar{u}_y} \right) \right\|_{L^2} \ll 1.$$

Treating the last term in the right-hand side perturbatively, we obtain

$$\|Y_x\|_{H^{2/3}_x L^2_z} \lesssim \left\| \frac{\bar{u}_x}{\bar{u}_y} \right\|_{H^{2/3}_x L^2_z} \leq \|\bar{u}_x\|_{H^{2/3}_x L^2_z} \|\bar{u}_y\|_{L^\infty_z(H^{2/3}_x)} \ll 1.$$

Hence we obtain the desired result when $\bar{u}$ is smooth and $\|\bar{u} - y\|_{Q^1} \ll 1$. We then conclude by density. □

B.3 Extension operators

We start with Lemma 1.1, which allows to extend functions from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.

Proof of Lemma 1.1. Up to translation and rescaling, we can assume that $(x_0, x_1) = (0, 1)$.

We start by constructing a continuous horizontal extension operator P_x from $Z^0((0,1)\times(-1,1))$ to $Z^0(\mathbb{R} \times (-1, 1))$. Let $\chi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\chi \equiv 1$ on $(0, 1)$ and $\operatorname{supp} \chi \subset (-1, 2)$. Let $\phi \in Z^0((0, 1) \times (-1, 1))$. For $x \in (-1, 2)$ and $z \in (-1, 1)$, let

$$(Q_x\phi)(x, z) := \begin{cases} \phi(-x, z) & \text{if } x \in (-1, 0) \\ \phi(x, z) & \text{if } x \in (0, 1), \\ \phi(2 - x, z) & \text{if } x \in (1, 2), \end{cases} \tag{B.4}$$

$$(P_x\phi)(x, z) := \chi(x)(Q_x\phi)(x, z). \tag{B.5}$$

First, $\|P_x\|_{L^2_{x,z} \to L^2_{x,z}} \leq 3$. Moreover, $\partial_z^k(P_x\phi) = P_x\partial_z^k\phi$ for $k \in \{1, 2\}$. Hence $\|P_x\|_{L^2_x H^2_z \to L^2_x H^2_z} \leq 3$. Eventually, $z\partial_x(P_x\phi) = P_x(z\partial_x\phi) + \chi' Q_x\phi$. Hence $\|z\partial_x(P_x\phi)\|_{L^2} \leq 3\|z\partial_x\phi\|_{L^2} + 2\|\chi'\|_{L^\infty}\|\phi\|_{L^2}$. Thus P_x defines a continuous extension operator from $Z^0((0, 1) \times (-1, 1))$ to $Z^0(\mathbb{R} \times (-1, 1))$.

We now construct a continuous upwards vertical extension operator P_+ from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$. We proceed, as classical (see e.g. [3]), by considering a weighted linear combination of rescaled reflections. For $\phi\in Z^0(\mathbb{R}\times(-1,1))$, $x\in\mathbb{R}$ and $z\in(-1,\infty)$, let

$$(Q_+\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z\in(-1,1),\\ 3\phi(x,2-z)-2\phi(x,3-2z) & \text{if } z\in(1,2), \end{cases} \tag{B.6}$$

$$(P_+\phi)(x,z) := \chi_+(z)(Q_+\phi)(x,z), \tag{B.7}$$

where $\chi_+\in C^\infty(\mathbb{R};[0,1])$ is such that $\chi_+\equiv 1$ on $(-1,1)$ and $\operatorname{supp}\chi_+\subset(-2,1+\frac14)$. The chosen coefficients ensure that both $Q_+\phi$ and $\partial_z(Q_+\phi)$ are continuous at $z=1$. Hence $P_+\phi\in L^2_xH^2_z$ and

$$\|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,+\infty))} = \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,1))} + \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(1,+\infty))} \le C_+\|\phi\|_{L^2_xH^2_z}, \tag{B.8}$$

for some constant C_+ depending only on $\|\chi_+\|_{W^{2,\infty}}$. Moreover, using that $\chi(z)=0$ for $z>1+\frac14$,

$$\begin{aligned} \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,+\infty))} &= \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,1+\frac14))}\\ &\lesssim \|\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}\\ &\lesssim \|z\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}. \end{aligned} \tag{B.9}$$

Hence P_+ is a continuous extension operator from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$.

The extension for $z<-1$ is performed in a similar fashion and left to the reader. □

We will also need the following extension result in this appendix.

Lemma B.6. *There exists a continuous extension operator P from $Z^0(\Omega_+)$ to $Z^0((x_0,x_1)\times\mathbb{R})$ such that, if $\phi_{|x=x_1}=0$ on $(0,1)$, $(P\phi)_{|x=x_1}=0$.*

Proof. We proceed, as in the proof of Lemma 1.1, by extension by reflections and truncation. The reflection at $z=1$ is done in the proof of Lemma 1.1. The truncation is left to the reader. We only check here the reflection at $z=0$ due to the degeneracy of the Z^0 norm at $z=0$.

Let $\phi\in Z^0(\Omega_+)$. We define an extension $Q\phi$ on Ω by

$$(Q\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z\in(0,1),\\ 3\phi(x,-z/2)-2\phi(x,-z) & \text{if } z\in(-1,0). \end{cases} \tag{B.10}$$

In particular $(Q\phi)(x,0^-)=(Q\phi)(x,0^+)$ and $\partial_z(Q\phi)(x,0^-)=\partial_z(Q\phi)(x,0^+)$, so

$$\|Q\phi\|_{L^2_xH^2_z(-1,1)} \lesssim \|Q\phi\|_{L^2_xH^2_z(-1,0)} + \|Q\phi\|_{L^2_xH^2_z(0,1)} \lesssim \|\phi\|_{L^2_xH^2_z} \le \|\phi\|_{Z^0}. \tag{B.11}$$

Moreover,

$$\|z\partial_x(Q\phi)\|_{L^2_xL^2_z(-1,0)} \le 3\|2(z/2)\partial_x\phi(x,z/2)\|_{L^2_xL^2_z(0,1)} + 2\|z\partial_x\phi(x,z)\|_{L^2_xL^2_z(0,1)} \lesssim \|\phi\|_{Z^0}. \tag{B.12}$$

Eventually, if $\phi_{|x=x_1}=0$ on $(0,1)$ (B.10) implies $(Q\phi)_{|x=x_1}=0$ on $(-1,1)$. □

B.4 Critical dualities

Lemma B.7. *For $a, b \in H^1(x_0, x_1)$ such that $a(x_0) = b(x_1) = 0$,*

$$\|ab\|_{H^{1/2}_{00}} \lesssim \|a'\|_{L^2} \|b\|_{H^{1/2}_{00_r}}. \tag{B.13}$$

Proof. On the one hand, by Lemma B.3,

$$\|ab\|_{H^{1/2}} \lesssim \|a\|_{H^1} \|b\|_{H^{1/2}} \lesssim \|a'\|_{L^2} \|b\|_{H^{1/2}_{00_r}}. \tag{B.14}$$

On the other hand, since for every $x \in (x_0, x_1)$, $|a(x)| \leq |x - x_0|^{\frac{1}{2}} \|a'\|_{L^2}$,

$$\int_{x_0}^{x_1} \frac{|a(x)b(x)|^2}{|x - x_0||x - x_1|} \, \mathrm{d}x \leq \|a'\|_{L^2}^2 \int_{x_0}^{x_1} \frac{|b(x)|^2}{|x - x_1|} \lesssim \|a'\|_{L^2}^2 \|b\|_{H^{1/2}_{00_r}}^2. \tag{B.15}$$

Gathering both estimates concludes the proof. □

Lemma B.8. *For $q, v, w \in H^1(x_0, x_1)$ such that $q(x_0) = 0$ and $w(x_1) = 0$,*

$$\left| \int_{x_0}^{x_1} q(\partial_x v) w \right| \lesssim \|q'\|_{L^2} \|v\|_{H^{1/2}} \|w\|_{H^{1/2}_{00_r}}. \tag{B.16}$$

Proof. By [29, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0, x_1)$ to $(H^{1/2}_{00}(x_0, x_1))'$. Hence

$$\left| \int_{x_0}^{x_1} q(\partial_x v) w \right| \lesssim \|v\|_{H^{1/2}} \|qw\|_{H^{1/2}_{00}}. \tag{B.17}$$

The conclusion follows from Lemma B.7. □

Lemma B.9. *For $q, v, w \in H^1(x_0, x_1)$ such that $q(x_0) = 0$ and $w(x_1) = 0$,*

$$\left| \int_{x_0}^{x_1} qv \partial_x w \right| \lesssim \|q'\|_{L^2} \|v\|_{H^{1/2}} \|w\|_{H^{1/2}_{00_r}}. \tag{B.18}$$

Proof. Let $\chi \in C^\infty([x_0, x_1]; [0, 1])$ such that $\chi \equiv 1$ in a neighborhood of x_0 and $\chi \equiv 0$ in a neighborhood of x_i. Let us write

$$\int_{x_0}^{x_1} qv \partial_x w = \int_{x_0}^{x_1} (\chi qv) \partial_x w - \int_{x_0}^{x_1} (1 - \chi) w \partial_x (qv) + \int_{x_0}^{x_1} \chi' wqv. \tag{B.19}$$

By [29, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0, x_1)$ to $(H^{1/2}_{00}(x_0, x_1))'$. Thus

$$\left| \int_{x_0}^{x_1} qv \partial_x w \right| \lesssim \|\chi qv\|_{H^{1/2}_{00}} \|w\|_{H^{1/2}} + \|(1 - \chi)w\|_{H^{1/2}_{00}} \|qv\|_{H^{1/2}} + \|\chi' q\|_{L^\infty} \|v\|_{L^2} \|w\|_{L^2}. \tag{B.20}$$

By Lemma B.3, $\|\chi qv\|_{H^{1/2}} \lesssim \|\chi\|_{H^1} \|q\|_{H^1} \|v\|_{H^{1/2}}$, $\|(1 - \chi)w\|_{H^{1/2}} \lesssim \|(1 - \chi)\|_{H^1} \|w\|_{H^{1/2}}$ and $\|qv\|_{H^{1/2}} \lesssim \|q\|_{H^1} \|v\|_{H^{1/2}}$. First, since $\chi \equiv 1$ near x_0, $(1 - \chi(x))^2 \lesssim |x - x_0|$. Thus

$$\int_{x_0}^{x_1} \frac{(1 - \chi(x))^2 |w(x)|^2}{|x - x_0||x - x_1|} \, \mathrm{d}x \lesssim \int_{x_0}^{x_1} \frac{|w(x)|^2}{|x - x_1|} \, \mathrm{d}x \lesssim \|w\|_{H^{1/2}_{00_r}}^2. \tag{B.21}$$

Similarly, since $\chi^2(x) \lesssim |x - x_1|$ and $q^2(x) \lesssim |x - x_0| \|q'\|_{L^2}^2$ by Cauchy-Schwarz, there holds

$$\int_{x_0}^{x_1} \frac{|\chi(x) q(x) v(x)|^2}{|x - x_0||x - x_1|} \, \mathrm{d}x \lesssim \|q'\|_{L^2}^2 \|v\|_{L^2}^2. \tag{B.22}$$

This concludes the proof, since $\|q\|_{H^1} \lesssim \|q'\|_{L^2}$ thanks to the condition $q(x_0) = 0$. □

B.5 Embeddings

We collect in this paragraph various embedding results used throughout the paper.

B.5.1 Full domain embeddings

The following inequality is used in the proof of the key result Proposition 1.2.

Lemma B.10. *For $\psi \in C_c^\infty(\mathbb{R})$,*

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.23}$$

Proof. On the one hand, for $|z| \geq 1$,

$$\int_{|z|\geq 1} \psi^2 \leq \|z\psi\|_{L^2}^2. \tag{B.24}$$

On the other hand, for every $(z_0, z) \in (-2, 2)$,

$$|\partial_z\psi(z)| \leq |\partial_z\psi(z_0)| + 2\|\partial_{zz}\psi\|_{L^2}. \tag{B.25}$$

Moreover, by classical Sobolev embeddings,

$$\|\partial_z\psi\|_{L^2(1,2)} \lesssim \|\psi\|_{L^2(1,2)} + \|\partial_{zz}\psi\|_{L^2(1,2)} \leq \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.26}$$

Thus, integrating (B.25) for $z_0 \in (1, 2)$,

$$\|\partial_z\psi\|_{L^\infty(-2,2)} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.27}$$

Now, writing $\psi(z) = \psi(z_0) + \int_{z_0}^z \psi'$ and integrating for $z_0 \in (1, 2)$ yields

$$\|\psi\|_{L^2(-1,1)} \lesssim \|\psi\|_{L^2(1,2)} + \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}, \tag{B.28}$$

which concludes the proof. □

Lemma B.11. *For $\psi \in C_c^\infty(\mathbb{R})$,*

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.29}$$

Proof. For $|z| \leq 2$, (B.27) yields directly

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2(-2,2)} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.30}$$

Let $\chi \in C^\infty(\mathbb{R}_+; [0, 1])$ with $\chi \equiv 0$ on $[0, 1]$, $\chi \equiv 1$ on $[2; +\infty)$ and $|\partial_z\chi| \leq 2$. Then

$$\begin{aligned}\int_2^{+\infty} z|\partial_z\psi(z)|^2 \,\mathrm{d}z &\leq \int_0^{+\infty} z\chi(z)|\partial_z\psi(z)|^2 \,\mathrm{d}z \\ &= -\int_0^{+\infty} z\chi(z)\psi(z)\partial_{zz}\psi(z) \,\mathrm{d}z - \int_0^{+\infty} z\partial_z\chi(z)\psi(z)\partial_z\psi(z) \,\mathrm{d}z \\ &\leq \|z\psi\|_{L^2}\|\partial_{zz}\psi\|_{L^2} + 2\|z\psi\|_{L^2}(\|\partial_z\psi\|_{L^2(2;+\infty)} + \|\partial_z\psi\|_{L^2(0,2)})\end{aligned} \tag{B.31}$$

The $\|\partial_z\psi\|_{L^2(0,2)}$ term can be bounded by (B.27) and the $\|\partial_z\psi\|_{L^2(2;+\infty)}$ term can be treated perturbatively via the Peter-Paul inequality. This yields

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2(2,+\infty)} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.32}$$

By symmetry, the same holds on $(-\infty, -2)$, which concludes the proof. □

Lemma B.12. *For $\phi \in Z^0(\Omega)$,*

$$\||z|^{\frac{1}{2}}\partial_z\phi\|_{L^2_z(H^{1/2}_x)} \lesssim \|\phi\|_{Z^0} \tag{B.33}$$

Proof. Thanks to Lemma 1.1, it is sufficient to prove the embedding and (B.33) for $\phi \in Z^0(\mathbb{R}^2)$ with bounded support in the vertical direction, say $\operatorname{supp}\phi \subset \mathbb{R}\times(-5,5)$ (as one can always take a smooth truncation of the extended function). We proceed as in the proof of Proposition 1.2. Let $\psi \in C^\infty_c(\mathbb{R})$. By Lemma B.11, and using standard dimensional analysis arguments, one deduces that

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2} \lesssim \|z\psi\|^{\frac{1}{2}}_{L^2}\|\partial_{zz}\psi\|^{\frac{1}{2}}_{L^2}. \tag{B.34}$$

Let $\phi \in C^\infty_c(\mathbb{R}^2)$. Let $\hat\phi(\xi,z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then using (B.34) and Hölder's inequality,

$$\begin{aligned}
\||z|^{\frac{1}{2}}\partial_z\phi\|^2_{L^2_z(H^{1/2}_x)} &= \int_{\mathbb{R}^2}(1+|\xi|^2)^{\frac{1}{2}}|z||\partial_z\hat\phi(\xi,z)|^2\,\mathrm{d}\xi\,\mathrm{d}z\\
&\lesssim \||z|^{\frac{1}{2}}\partial_z\phi\|^2_{L^2} + \int_{\mathbb{R}}|\xi|\|z\hat\phi(\xi,z)\|_{L^2_z}\|\partial_{zz}\hat\phi(\xi,z)\|_{L^2_z}\,\mathrm{d}\xi\\
&\lesssim \||z|^{\frac{1}{2}}\partial_z\phi\|^2_{L^2} + \left(\int_{\mathbb{R}^2}|\xi|^2z^2|\hat\phi(\xi,z)|^2\,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|\partial_{zz}\hat\phi(\xi,z)|^2\,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\lesssim \||z|^{\frac{1}{2}}\partial_z\phi\|^2_{L^2} + \|z\partial_x\phi\|^{\frac{1}{2}}_{L^2}\|\partial_{zz}\phi\|^{\frac{1}{2}}_{L^2}.
\end{aligned} \tag{B.35}$$

Moreover, since $\phi(\cdot,z)=0$ for $|z|\geq 5$,

$$\||z|^{\frac{1}{2}}\partial_z\phi\|_{L^2} \lesssim \|\partial_{zz}\phi\|_{L^2} \tag{B.36}$$

Hence gathering both inequalities proves that $\||z|^{\frac{1}{2}}\partial_z\phi\|^2_{L^2_z(H^{1/2}_x)} \lesssim \|\phi\|_{Z^0}$. This concludes the proof, by density of $C^\infty_c(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$. □

B.5.2 Embeddings involving the Lions-Magenes space

Lemma B.13. *Let $\phi \in Z^0(\Omega_+)$ such that $\phi_{|x=x_1}=0$. Then $\phi \in C^0_z([0,1];H^{1/2}_{00_r}(x_0,x_1))$ and*

$$\|\phi\|_{L^\infty_z H^{1/2}_{00_r}} \lesssim \|\phi\|_{Z^0}. \tag{B.37}$$

Proof. Thanks to the extension result Lemma B.6, it is sufficient to prove this result with Ω_+ replaced by $\mathcal{O} := (x_0,x_1)\times\mathbb{R}$. Therefore let $\phi \in Z^0(\mathcal{O})$ such that $\phi_{|x=x_1}=0$ on $\mathbb{R}$. By Proposition 1.2, $\phi \in L^2_z(\mathbb{R};H^1_{0_r}(x_0,x_1)) \cap H^2_z(\mathbb{R};L^2(x_0,x_1))$, where $H^1_{0_r}(x_0,x_1)$ denotes H^1 functions vanishing at $x=x_1$. By the fractional trace theorem [29, Chapter 1, Theorem 4.2 and equation (4.7)], this implies that $\phi \in C^0_z(\mathbb{R};G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0,x_1),L^2(x_0,x_1)]_{\frac{1}{4}}$ in this reference. By [29, Chapter 1, Theorem 11.7][4], $G = H^{1/2}_{00_r}(x_0,x_1)$. The claimed norm estimate readily follows. □

[4] This reference considers the case of vanishing conditions on the full boundary of the domain (so at $x=x_0$ and at $x=x_1$), but the adaptation to functions vanishing only at x_1 is straightforward.

Lemma B.14. *Let $\phi \in Z^0(\Omega_+)$ such that $\phi_{|x=x_1} = 0$. Then $z\partial_z\phi \in L^2_z H^{1/2}_{00_r}$ and*

$$\|z\partial_z\phi\|_{L^2_z H^{1/2}_{00_r}} \lesssim \|\phi\|_{Z^0}. \tag{B.38}$$

Proof. Let $\phi \in Z^0(\Omega_+)$ such that $\phi_{|x=x_1} = 0$. We extend ϕ to $Z^0(\mathcal{O})$, where $\mathcal{O} := (x_0, x_1) \times \mathbb{R}$, thanks to the extension result Lemma B.6. We then truncate the extension for $|z| \geq 2$ thanks to a C^∞_c function, so that the extension is now supported in $(x_0, x_1) \times (-3, 3)$, coincides with ϕ on Ω, and belongs to $Z^0(\mathcal{O})$. With a slight abuse of notation, we still denote the extension by ϕ. Note that $\phi_{|x=x_1} = 0$ on $\mathbb{R}$.

Let $\psi := z\phi$. Then $\psi \in L^2_z(\mathbb{R}; H^1_{0_r}(x_0, x_1)) \cap H^2_z(\mathbb{R}; L^2(x_0, x_1))$ with $\|\psi\| \lesssim \|\phi\|_{Z^0}$ for the associated norm. By the intermediate derivative theorem [29, Chapter 1, Theorem 2.3], $\psi \in H^1_z(\mathbb{R}; G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0, x_1), L^2(x_0, x_1)]_{\frac{1}{2}}$ in this reference. By [29, Chapter 1, Theorem 11.7], $G = H^{1/2}_{00_r}(x_0, x_1)$ (see Footnote 4, Page 68). This yields

$$\|\partial_z\psi\|_{L^2_z H^{1/2}_{00_r}} \lesssim \|\phi\|_{Z^0}. \tag{B.39}$$

The conclusion follows since $\partial_z\psi = z\partial_z\phi + \phi$ and $\phi \in L^2_z H^{2/3}_x$ by Proposition 1.2 (with trace $\phi_{|x=x_1} = 0$, which makes sense in $H^{2/3}_x$). □

Lemma B.15. *Let $\phi \in L^2((x_0, x_1); H^1_0(-1, 1))$ such that $z\partial_x\phi \in L^2((x_0, x_1); H^{-1}(-1, 1))$. Assume that $\phi = 0$ on $\{x_1\} \times (0, 1) \cup \{x_0\} \times (-1, 0)$ (in the sense of traces in $\mathscr{L}^2_z(-1, 1)$, see Lemma A.1). Then $z\phi \in L^2((0, 1); H^{1/2}_{00_r}) \cap L^2((-1, 0); H^{1/2}_{00_l})$.*

Proof. Let $\psi := z\phi$. Then $\psi \in H^1_0((-1, 1); L^2(x_0, x_1)) \cap H^{-1}((-1, 1); H^1(x_0, x_1))$ with $\psi = 0$ on $\{x_1\} \times (0, 1) \cup \{x_0\} \times (-1, 0)$ (in the same sense). Moreover, letting $\bar\psi$ denote the restriction to $(x_0, x_1) \times (0, 1)$ of ψ, we have $\bar\psi \in H^1((0, 1); L^2(x_0, x_1)) \cap H^{-1}((0, 1); H^1_{0_r}(x_0, x_1))$. We then construct an extension to $(x_0, x_1) \times \mathbb{R}$, still denoted by $\bar\psi$ which satisfies $\bar\psi \in H^1(\mathbb{R}; L^2(x_0, x_1)) \cap H^{-1}(\mathbb{R}; H^1_{0_r}(x_0, x_1))$. By [1, Theorem 4.5.5] (which applies since both $L^2(x_0, x_1)$ and $H^1_{0_r}(x_0, x_1)$ are Hilbert spaces so enjoy the UMD property), $\bar\psi \in L^2(\mathbb{R}; G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0, x_1), L^2(x_0, x_1)]_{\frac{1}{2}}$. By [29, Chapter 1, Theorem 11.7], $G = H^{1/2}_{00_r}(x_0, x_1)$ (see Footnote 4, Page 68). This proves the right part of the statement. The left part is proved similarly. □

B.5.3 A derivative estimate

Lemma B.16. *Let $\phi \in H^{2/3}_x L^2_z(\Omega)$ with $\partial_z\phi \in H^{1/3}_x L^2_z(\Omega)$. Then $\phi \in C^0_z([-1, 1]; H^{1/2}(x_0, x_1))$. Furthermore, for a.e. $z \in (-1, 1)$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\|\phi(z)\|^2_{H^{1/2}(x_0,x_1)} &= 2\int_{x_0}^{x_1} \phi(x, z)\partial_z\phi(x, z)\,\mathrm{d}x \\ &+ 2\int_{(x_0,x_1)^2} \frac{(\phi(x, z) - \phi(x', z))(\phi_z(x, z) - \phi_z(x', z))}{|x - x'|^2}\,\mathrm{d}x\,\mathrm{d}x' \ \in L^1(-1, 1).\end{aligned}$$

Proof. First, we observe that, if $f \in H^{2/3}(x_0, x_1)$ and $g \in H^{1/3}(x_0, x_1)$, then

$$\begin{aligned}\int_{(x_0,x_1)^2} \frac{|f(x) - f(x')|\,|g(x) - g(x')|}{|x - x'|^2}\,\mathrm{d}x\,\mathrm{d}x' &= \int_{(x_0,x_1)^2} \frac{|f(x) - f(x')|}{|x - x'|^{7/6}} \frac{|g(x) - g(x')|}{|x - x'|^{5/6}}\,\mathrm{d}x\,\mathrm{d}x' \\ &\leq \|f\|_{H^{2/3}}\|g\|_{H^{1/3}}.\end{aligned} \tag{B.40}$$

Then, we prove the result for $\phi \in C^2(\overline{\Omega})$. In that case, differentiating under the integral and using the definition of the $H^{1/2}$ norm, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\|\phi(z)\|^2_{H^{1/2}(x_0,x_1)} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\int_{x_0}^{x_1}\phi(x,z)^2\,\mathrm{d}x + \int_{(x_0,x_1)^2}\frac{|\phi(x,z)-\phi(x',z)|^2}{|x-x'|^2}\,\mathrm{d}x\,\mathrm{d}x'\right)\\ &= 2\int_{x_0}^{x_1}\phi(x,z)\partial_z\phi(x,z)\,\mathrm{d}x\\ &\quad +2\int_{(x_0,x_1)^2}\frac{(\phi(x,z)-\phi(x',z))(\phi_z(x,z)-\phi_z(x',z))}{|x-x'|^2}\,\mathrm{d}x\,\mathrm{d}x'.\end{aligned}$$

Using (B.40), we infer that the right-hand side is bounded by

$$2\|\phi(z)\|_{H^{2/3}(x_0,x_1)}\|\phi_z(z)\|_{H^{1/3}(x_0,x_1)},$$

and, by the Cauchy-Schwarz inequality,

$$\int_{-1}^{1}\|\phi(z)\|_{H^{2/3}(x_0,x_1)}\|\phi_z(z)\|_{H^{1/3}(x_0,x_1)}\,\mathrm{d}z \le \|\phi\|_{H^{2/3}_x L^2_z}\|\phi_z\|_{H^{1/3}_x L^2_z}.$$

Arguing by density, we eventually obtain the result announced in the Lemma. □

Acknowledgements

The authors thank Jérémie Szeftel and Frédéric Hérau for nice discussions around this problem.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program Grant agreement No 637653, project BLOC "Mathematical Study of Boundary Layers in Oceanic Motion". This work was supported by the SingFlows project, grants ANR-18-CE40-0027 of the French National Research Agency (ANR). A.-L. D. acknowledges the support of the Institut Universitaire de France. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while A.-L. D. participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

References

[1] Herbert Amann. *Linear and quasilinear parabolic problems. Volume II: Function spaces*, volume 106. Cham: Birkhäuser, 2019.

[2] Kadir Aziz, Donald French, Soren Jensen, and Bruce Kellogg. Origins, analysis, numerical analysis, and numerical approximation of a forward-backward parabolic problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(5):895–922, 1999.

[3] Vasilii Mikhailovich Babich. On the extension of functions. *Uspekhi Matematicheskikh Nauk*, 8(2):111–113, 1953.

[4] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Fundamental Principles of Mathematical Sciences*. Springer, Heidelberg, 2011.

[5] Mohammed Baouendi and Pierre Grisvard. Sur une équation d'évolution changeant de type. *Journal of Functional Analysis*, 2:352–367, 1968.

[6] Richard Beals. An abstract treatment of some forward-backward problems of transport and scattering. *Journal of Functional Analysis*, 34(1):1–20, 1979.

[7] Ali Behzadan and Michael Holst. Multiplication in Sobolev spaces, revisited. *Ark. Mat.*, 59(2):275–306, 2021.

[8] Oleg Bocharov. On the first boundary value problem for the heat equation with an alternating coefficient. *Dinamika Sploshn. Sredy (Novosibirsk)*, 37:27–39, 1978.

[9] Susan Brown and Keith Stewartson. On the reversed flow solutions of the Falkner-Skan equation. *Mathematika*, 13:1–6, 1966.

[10] Anne-Laure Dalibard and Nader Masmoudi. Separation for the stationary prandtl equation. *Publications mathématiques de l'IHÉS*, 130(1):187–297, 2019.

[11] Helge Dietert and David Gérard-Varet. On the ill-posedness of the triple deck model. *arXiv preprint arXiv:2105.02053*, 2021.

[12] Ronald Douglas. On majorization, factorization, and range inclusion of operators on hilbert space. *Proceedings of the American Mathematical Society*, 17(2):413–415, 1966.

[13] Weinan E. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)*, 16(2):207–218, 2000.

[14] Gaetano Fichera. On a unified theory of boundary value problems for elliptic-parabolic equations of second order. In *Boundary problems in differential equations*, pages 97–120. Univ. of Wisconsin Press, Madison, 1960.

[15] Wendell Fleming. A problem of random accelerations. Technical report, Wisconsin Univ Madison Mathematics Research Center, 1963.

[16] Maurice Gevrey. Sur les équations aux dérivées partielles du type parabolique (suite). *Journ. de Math. (6)*, 10:105–148, 1914.

[17] Jerome Goldstein and Tapas Mazumdar. A heat equation in which the diffusion coefficient changes sign. *Journal of mathematical analysis and applications*, 103(2):533–564, 1984.

[18] Sydney Goldstein. On laminar boundary-layer flow near a position of separation. *Quart. J. Mech. Appl. Math.*, 1:43–69, 1948.

[19] Yurii Pavlovich Gor'kov. A formula for the solution of a boundary value problem for the stationary equation of brownian motion. In *Doklady Akademii Nauk*, volume 223(3), pages 525–528. Russian Academy of Sciences, 1975.

[20] Pierre Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[21] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967.

[22] Sameer Iyer and Nader Masmoudi. Reversal in the Stationary Prandtl Equations. *arXiv e-prints*, page arXiv:2203.02845, March 2022.

[23] Sameer Iyer and Vlad Vicol. Real analytic local well-posedness for the triple deck. *Communications on Pure and Applied Mathematics*, 74(8):1641–1684, 2021.

[24] Joseph John Kohn and Louis Nirenberg. Degenerate elliptic-parabolic equations of second order. *Communications on pure and applied mathematics*, 20(4):797–872, 1967.

[25] Ivan Vladimirovich Kuznetsov. Entropy solutions to a second order forward-backward parabolic differential equation. *Siberian Mathematical Journal*, 46(3):467–488, 2005.

[26] Ivan Vladimirovich Kuznetsov. Traces of entropy solutions to second order forward-backward parabolic equations. *Journal of Mathematical Sciences*, 211(6), 2015.

[27] Pierre-Yves Lagrée. Interactive boundary layer (IBL). In *Asymptotic methods in fluid mechanics: Survey and recent advances. Papers from the course and workshop, Udine, Italy, September 21–25, 2009.*, pages 247–286. Wien: Springer, 2010.

[28] Jacques-Louis Lions. *Quelques méthodes de résolution des problemes aux limites non linéaires.* Dunod, 1969.

[29] Enrico Magenes and Jacques-Louis Lions. *Problèmes aux limites non homogènes et applications, Volume I.* Dunod, 1968.

[30] Olga Oleĭnik and Evgenii Radkevič. *Second order equations with nonnegative characteristic form.* Plenum Press, New York-London, 1973. Translated from the Russian by Paul C. Fife.

[31] Olga Oleĭnik and Vyacheslav Samokhin. *Mathematical models in boundary layer theory*, volume 15 of *Applied Mathematics and Mathematical Computation.* Chapman & Hall/CRC, Boca Raton, FL, 1999.

[32] Carlo Domenico Pagani. On the parabolic equation $sgn(x)|x|^p u_y - u_{xx} = 0$ and a related one. *Ann. Mat. Pura Appl. (4)*, 99:333–399, 1974.

[33] Carlo Domenico Pagani. On an initial-boundary value problem for the equation $w_t = w_{xx} - xw_y$. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 2:219–263, 1975.

[34] Carlo Domenico Pagani. On forward-backward parabolic equations in bounded domains. *Boll. Unione Mat. Ital., V. Ser., B*, 13:336–354, 1976.

[35] Fabio Paronetto. Existence results for a class of evolution equations of mixed type. *Journal of Functional Analysis*, 212(2):324–356, 2004.

[36] Sergey Grigor'evich Pyatkov. On some classes of nonlocal boundary-value problems for singular parabolic equations. *Mathematical Notes*, 106(3):602–615, 2019.

[37] Jean Rax. Fluid boundary layers and degenerate elliptic equations. PhD manuscript, Sorbonne université, 2020.

[38] Weiming Shen, Yue Wang, and Zhifei Zhang. Boundary layer separation and local behavior for the steady prandtl equation. *Advances in Mathematics*, 389:107896, 2021.

[39] Keith Stewartson. On Goldstein's theory of laminar separation. *Quart. J. Mech. Appl. Math.*, 11, 1958.