A nonlinear forward-backward problem

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December 22, 2022

Abstract

We prove the existence and uniqueness of strong solutions to the equation $uu_x - u_{yy} = f$ in the vicinity of the linear shear flow, subject to perturbations of the source term and lateral boundary conditions. Since the solutions we consider have opposite signs in the lower and upper half of the domain, this is a quasilinear forward-backward parabolic problem, which changes type across a critical curved line within the domain. In particular, lateral boundary conditions can be imposed only where the characteristics are inwards.

There are several difficulties associated with this problem. First, the forward-backward geometry depends on the solution itself. This requires to be quite careful with the approximation procedure used to construct solutions. Second, and more importantly, the linearized equations solved at each step of the iterative scheme admit a finite number of singular solutions, of which we provide an explicit construction. This is similar to well-known phenomena in elliptic problems in nonsmooth domains. Hence, the solutions to the equation are regular if and only if the source terms satisfy a finite number of orthogonality conditions. A key difficulty of this work is to cope with these orthogonality conditions during the nonlinear fixed-point scheme. In particular, we are led to prove their stability with respect to the underlying base flow.

To tackle this deceivingly simple problem, we develop a methodology which we believe to be both quite natural and adaptable to other situations in which one wishes to prove the existence of regular solutions to a nonlinear problem for suitable data despite the existence of singular solutions at the linear level.

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1 Introduction

Let $x_0 < x_1$. We investigate the existence and uniqueness of sign-changing solutions to the equation

$$u\partial_x u - \partial_{yy} u = f \tag{1.1}$$

in the rectangular domain $\Omega := (x_0, x_1) \times (-1, 1)$, where f is an external source term.

A natural solution to (1.1) with a null source term $\mathbb{f} = 0$ is the linear shear flow $\mathbb{u}(x, y) := y$, which changes sign across the horizontal line $\{y = 0\}$. We are interested in strong solutions to (1.1) which are close (with respect to an appropriate norm) to this linear shear flow $\mathbb{u}$. Our purpose is to construct such solutions by perturbing the lateral boundary data $\mathbb{u}_{|x=x_0}(y) = y$ and $\mathbb{u}_{|x=x_1}(y) = y$ or the source term $\mathbb{f} = 0$.

Since such solutions will change sign across a line $\{u = 0\}$ lying within Ω, a key feature of this work is that (1.1) must be seen as a quasilinear forward-backward parabolic problem in the horizontal direction. Thus, to ensure the existence of a solution, one must be particularly careful as to how one enforces the lateral perturbations. More precisely, the problem is forward parabolic in the domain above the line $\{u = 0\}$, in which $u > 0$, and therefore we shall prescribe a boundary condition on $\Sigma_0 := \{x = x_0\} \cap \{u > 0\}$; and backward parabolic in the domain below the line $\{u = 0\}$, and we shall prescribe a boundary condition on $\Sigma_1 := \{x = x_1\} \cap \{u < 0\}$.

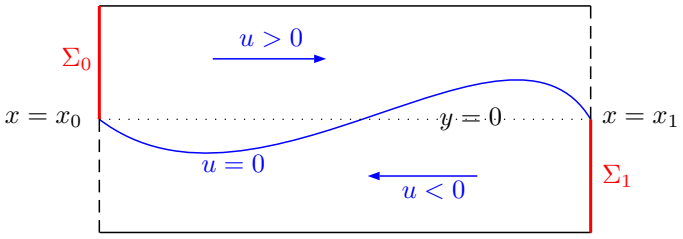


Figure 1: Fluid domain Ω and inflow boundaries $\Sigma_0 \cup \Sigma_1$

We will construct solutions to this problem thanks to an iterative scheme taking into account the geometry of the problem. Because of the nonlinearity, we need to perform this scheme in a high enough regularity space in order to have a suitable control of the derivatives. However, one key difficulty of our work lies in the fact that, even when the source term f is smooth, say in $C_0^\infty(\Omega)$, *solutions to* (1.1) *have singularities in general*. Actually, this feature is already present at the linear level, i.e. for the equation $y\partial_x u - \partial_{xx} u = f$. We prove that if f is smooth, the associated weak solution to the linear system inherits the regularity of f if and only if f satisfies *orthogonality conditions* (i.e. the scalar products of f with some identified profiles must vanish). We also describe the singularities that appear when these orthogonality conditions are not satisfied.

When performing the iterative scheme, we need to make sure that these orthogonality conditions are satisfied at every step of the iteration. Another difficulty then arises: indeed, the orthogonality conditions change at every step! Therefore, a technical but important feature of our work lies in the analysis of the dependency of the orthogonality conditions on the underlying flow. We will come back to this delicate and subtle matter in Section 1.4.

All the features described above (orthogonality conditions for linear forward-backward equations, description of the potential singularities, handling of orthogonality conditions within a quasilinear scheme) appear to be new. We believe that the strategy we use could be extended to other

nonlinear settings in which orthogonality conditions appear (elliptic equations in domains with corners, problems in which the linearized operator is Fredholm with negative index, ...)

All these difficulties stem from the fact that our solutions change sign within the domain. Indeed, if one had $u > 0$ on $\overline{\Omega}$, then (1.1) could merely be seen as a quasilinear parabolic problem of the form $\partial_x u = (1/u)\partial_{yy} u + f$, with initial data at $\{x_0\} \times (-1, 1)$, for which a general well-posedness theory is known (see e.g. [41, Chapter 8]). Similar well-posedness results hold even if u is allowed to vanish on $\partial\Omega$ (for example at $z = -1$), working as in [45, Chapter 2].

1.1 Statement of the main results

Due to the forward-backward nature of the problem, we must choose the lateral perturbations and the source term in a particular product space. We therefore introduce the vector space

$$\begin{aligned}\Big\{(f, \delta_0, \delta_1) \in C_c^\infty(\Omega) \times C^\infty([0,1]) \times C^\infty([-1,0]); \quad \delta_i(0) = \partial_y \delta_i(0) = \partial_y^2 \delta_i(0) = 0 \\ \text{and } \delta_i((-1)^i) = \partial_y^2 \delta_i((-1)^i) = 0 \text{ for } i = 0, 1\Big\}\end{aligned} \tag{1.2}$$

and $\mathcal{H}$, the Hilbert space defined as its completion with respect to the following norm (associated with the corresponding canonical scalar product; see (1.20) for the weighted Sobolev norm $\mathscr{H}_z^1$),

$$\|(f, \delta_0, \delta_1)\|_{\mathcal{H}} := \|f\|_{H_x^1 H_y^1} + \|\partial_y^3 f\|_{L^2} + \sum_{i \in \{0,1\}} \|\delta_i\|_{H^5} + \|(\partial_y^2 \delta_i)/y\|_{\mathscr{H}_z^1}. \tag{1.3}$$

We establish the existence and uniqueness of solutions in the following anisotropic Sobolev space

$$Q^1 := L^2((x_0, x_1); H^5(-1, 1)) \cap H^{5/3}((x_0, x_1); L^2(-1, 1)). \tag{1.4}$$

In particular, for solutions with such regularity, (1.1) holds in a strong sense, almost everywhere and the various boundary conditions hold in the usual sense of traces, almost everywhere. We first state a result concerning the well-posedness in Q^1 of the linear version of (1.1) around the linear shear flow, *up to two orthogonality conditions* (see comments below). Although equation (1.5) below has been thoroughly investigated, as we recall in Section 1.2 below, we could not find this statement in the existing literature.

Theorem 1 (Orthogonality conditions for linear forward-backward parabolic equations)**.** *Let $\Sigma_0 := \{x_0\} \times (0, 1)$ and $\Sigma_1 := \{x_1\} \times (-1, 0)$. There exists a vector subspace $\mathcal{H}_{\mathrm{sg}}^\perp \subset \mathcal{H}$ of codimension two such that, for each $(f, \delta_0, \delta_1) \in \mathcal{H}$, there exists a solution $u \in Q^1$ to the problem*

$$\begin{cases} y\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{1.5}$$

if and only if $(f, \delta_0, \delta_1) \in \mathcal{H}_{\mathrm{sg}}^\perp$. Such a solution is unique and satisfies

$$\|u\|_{Q^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}}. \tag{1.6}$$

We emphasize that this result implies that there exist triplets (f, δ_0, δ_1) that can be chosen arbitrarily smooth and compactly supported, and for which there are no Q^1 solutions to (1.5). Furthermore, the vector space $\mathcal{H}_{\mathrm{sg}}^\perp$ can be fully characterized: classically, $\mathcal{H}_{\mathrm{sg}}^\perp = \ker \overline{\ell^0} \cap \ker \overline{\ell^1}$, where $\overline{\ell^0}$ and $\overline{\ell^1}$ are two linear forms on $\mathcal{H}$ which we shall write explicitly. If the data do not belong to $\mathcal{H}_{\mathrm{sg}}^\perp$, the solution has singularities, which we can describe completely.

Theorem 2 (Decomposition of solutions as a sum of singular profiles and a smooth remainder)**.** *Let $(f, \delta_0, \delta_1) \in \mathcal{H}$. There exists a unique solution $u \in H_x^{2/3} L_y^2 \cap L_x^2 H_y^2$ to equation* (1.5)*. Furthermore, this solution admits the following decomposition: there exists $c_0, c_1 \in \mathbb{R}$, and $u_{\mathrm{reg}} \in Q^1$, such that*

$$u = c_0 \bar{u}^0_{\mathrm{sing}} + c_1 \bar{u}^1_{\mathrm{sing}} + u_{\mathrm{reg}}.$$

Each profile $\bar{u}^i_{\mathrm{sing}}$ is supported in the vicinity of $(x_i, 0)$ and is smooth on $\overline{\Omega} \setminus \{(x_i, 0)\}$. Furthermore, for $|x - x_i| \ll 1$ and $|y| \ll 1$,

$$\bar{u}^i_{\mathrm{sing}}(x, y) = \left(|y|^2 + |x - x_i|^{\frac{2}{3}} \right)^{\frac{1}{4}} \Lambda_0 \left((-1)^i \frac{y}{|x - x_i|^{\frac{1}{3}}} \right),$$

where $\Lambda_0 \in C^\infty(\mathbb{R})$ is such that $\Lambda_0(-\infty) = 1$ and $\Lambda_0(+\infty) = 0$ (see Fig. 2 page 27).

The existence of a weak solution was already known, see in particular [48, 49, 21]. The novelty of the above theorem lies in the identification of the singular profiles $\bar{u}^i_{\mathrm{sing}}$, and in the decomposition of any weak solution. The function Λ_0 is in fact the solution to an ODE, and can be characterized in terms of special functions (namely confluent hypergeometric functions of the second kind, or Tricomi's functions).

We then turn towards the nonlinear problem (1.1). The main result of this paper is the following nonlinear generalization of Theorem 1 for small enough perturbations.

Theorem 3 (Existence and uniqueness of strong solutions to (1.1) under orthogonality conditions)**.** *There exists a Lipschitz submanifold $\mathcal{M}$ of $\mathcal{H}$ of codimension two, containing 0, such that, for every $(f, \delta_0, \delta_1) \in \mathcal{M}$, there exists a strong solution $u \in Q^1$ to*

$$\begin{cases} (y + u)\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0. \end{cases} \tag{1.7}$$

More precisely, $\mathcal{M}$ is modeled on $\mathcal{H}^\perp_{\mathrm{sg}}$ and tangent to it at 0 (see Remark 6.2). Such solutions are unique in a small neighborhood of 0 in Q^1 and satisfy the estimate (1.6)*.*

In the statement above, the condition that the data (f, δ_0, δ_1) belong to the manifold $\mathcal{M}$ is the nonlinear equivalent of the orthogonality conditions from Theorem 1. We emphasize that this is by no means a technical restriction which could be lifted, but actually a necessary condition to solve the equation with smooth solutions, as we state in Proposition 1.1 below. A key difficulty lies in the fact that these orthogonality conditions *depend on the solution itself*. Hence, tracking the dependency of these conditions with respect to the unknown function u is a key part of our result. We will comment more abundantly on these points in the following sections.

Proposition 1.1 (Necessity of the orthogonality conditions)**.** *There exists $\eta > 0$ such that the following result holds. Let $(f, \delta_0, \delta_1) \in \mathcal{H}$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}} \leq \eta$. Let $u \in Q^1$ be a solution to* (1.7) *such that $\|u\|_{Q^1} \leq \eta$. Then $(f, \delta_0, \delta_1) \in \mathcal{M}$.*

Remark 1.2. *By commodity, the above results are stated using the full triplet (f, δ_0, δ_1), and so is the remainder of this paper. Nevertheless, it is possible to obtain similar results either by fixing $\delta_0 = \delta_1 = 0$ and constructing a submanifold of source terms f yielding regular solutions, or by fixing $f = 0$ and constructing a submanifold of boundary data (δ_0, δ_1), see Remark 6.4.*

1.2 Comments and previous results

We start with a few comments on our main results and recall related known results.

Problem (1.5), involving the operator $y\partial_x - \partial_{yy}$, can be seen as a particular case of the class of "degenerate second-order elliptic-parabolic linear equations", also referred to as "second-order equations with nonnegative characteristic form" (as opposed to positive definite ones), "forward-backward" or "mixed type" problems. They date back at least to Gevrey [23].

Problem (1.5) itself, as well as these wide classes of equations, has received a lot of attention and has been investigated under different aspects: with variable coefficients or other geometries [21, 49], higher-order operators [38, Ch. 3, 2.6], abstract operators [10, 50], explicit representation formulas [22, 26] or with a focus on numerical analysis [4].

On weak solutions for the linear problem. It is well-known since the work of Fichera [21] that weak solutions to (1.5) with $L^2_x H^1_y$ regularity exist. For general boundary-value problems for elliptic-parabolic second-order equations, one owes to Fichera the systematic separation of the boundary of the domain into three parts: a "noncharacteristic" part, where one sets either Dirichlet or Neumann boundary conditions (here $y = \pm 1$), an "inflow" part, where one sets a Dirichlet boundary condition (here $\Sigma_0 \cup \Sigma_1$) and an "outflow" part, where one cannot set a boundary condition (here, the two sets $\{x_0\} \times (-1, 0)$ and $\{x_1\} \times (0, 1)$).

Baouendi and Grisvard [9] proved the uniqueness of weak solutions to (1.5) with $L^2_x H^1_y$ regularity, by means of a trace theorem and a Green identity (see Appendix A).

On strong solutions for the linear problem. There is an extensive literature on the regularity of solutions to degenerate elliptic-parabolic linear equations, and whether weak solutions are strong. We refer the reader in particular to the book [44] by Oleĭnik and Radkevič. Generally speaking, depending on the exact setting considered, it is quite often possible to prove that the solutions to such equations are regular far from the boundaries of the domain and/or from the regions where the characteristic form is not positive definite. A nice example is Kohn and Nirenberg's work [34], which proves a very general regularity result. A key assumption of their work is that the "outflow" part of the boundary does not meet the "noncharacteristic" and "inflow" parts (i.e. they are in disjoint connected components of $\partial\Omega$). Hence, it does not apply to (1.5), and hints towards a difficulty near the points $(x_0, 0)$ and $(x_1, 0)$.

In a series of papers [47, 48, 49], Pagani proved the existence of strong solutions to (1.5) (and related equations). More precisely, Pagani proved the existence of solutions such that $y\partial_x u$ and $\partial_{yy} u$ belong to $L^2(\Omega)$. Moreover, he determined the exact regularity of the various traces of such solutions (trace of u at $x = x_i$, at $y = \pm 1$ or $y = 0$, and trace of $\partial_y u$ at $y = 0$). These maximal regularity results play a key role in our analysis and motivate the functional spaces we introduce in Section 1.5.

On orthogonality conditions for higher regularity. As noted by Pyatkov in [51], for such forward-backward problems: "*as a rule, there is no existence theorems for smooth solutions without some additional orthogonality-type conditions on the problem data*". Even for the linear problem (1.5), there have been very few works concerning higher regularity (than the one given by Pagani's framework) in the whole domain. Most of the works focused on higher regularity (such as [51]) involve weighted estimates which entail regularity within the domain but not near the critical

points $(x_i, 0)$. An attempt for global regularity is Goldstein and Mazumdar's work [24, Theorem 4.2] albeit the proof seems incomplete (see Proposition 2.8 below and its proofs for more details).

A misleading aspect is that it is quite easy, assuming the existence of a smooth solution, to prove *a priori* estimates at any order. Such phenomenons are usual in the theory of elliptic problems in domains with corners or mixed Dirichlet-Neumann boundary conditions (see for instance [27]). Let us give an illustration of such a phenomenon in a close context. For a source term $f \in C_c^\infty(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ \partial_x u(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ u(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.8}$$

It is classical that such a system has a unique weak solution $u \in H^1(\Omega)$. Moreover, assuming that u is smooth enough, $v := \partial_x u$ satisfies

$$\begin{cases} -\Delta v = \partial_x f & \text{in } \Omega, \\ \partial_x v(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ v(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ v(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.9}$$

For such systems, one has $\|v\|_{H^1} \lesssim \|\partial_x f\|_{L^2}$. Hence $\|\partial_{xx} u\| \lesssim \|\partial_x f\|_{L^2}$, and, using the equation, $\|u\|_{H^2} \lesssim \|f\|_{H^1}$. So one has an *a priori* estimate. However, it is known that there exist source terms for which the unique weak solution $u \in H^1$ does not enjoy H^2 regularity (see [27, Chapter 4] and Section 2.5). The key point is that, when reconstructing u from the solution v to (1.9), say by setting $u(x, y) := \int_{x_0}^x v(x', y)\, \mathrm{d}x'$ for $y > 0$ and $u(x, y) := \int_{x_1}^x v(x', y)\, \mathrm{d}x'$ for $y < 0$, there might be a discontinuity of u or $\partial_y u$ across the line $y = 0$. Such discontinuities prevent u from solving (1.8). Preventing these discontinuities requires that the source term satisfies appropriate orthogonality conditions.

On orthogonality conditions for nonlinear problems. Of course, such orthogonality conditions make it very difficult to obtain results at a nonlinear level. Generally, one tries to avoid such difficulties when considering nonlinear problems. For instance, for elliptic problems in polygonal domains, the classical textbook [27, Section 8.1] focuses on a nonlinear case where there is no orthogonality condition at the linear level.

Nevertheless, some results are known in the *semilinear* case. For example, for semilinear Fredholm operators with negative index, a theoretical toolbox is known (see e.g. [56, Chapter 11, Section 4.2]) and has been implemented for some reaction-diffusion semilinear systems (see e.g. [57, Chapter 7, Section 2.2], based on [19]).

Outside of the semilinear setting, we are not aware of nonlinear results obtained despite the presence of orthogonality conditions at the linear level prior to our present work (we discuss the very recent preprint [30] by Sameer Iyer and Nader Masmoudi in Section 1.3).

Problem (1.1) is only *quasilinear*, and this makes the analysis harder. In particular, tracking the evolution of the orthogonality conditions during the nonlinear scheme is one of the main difficulties of this work (see Sections 4 and 6.2). At the nonlinear level, these orthogonality conditions are translated in Theorem 3 as the fact that the data must lie within the manifold $\mathcal{M}$, which can be

pictured as a perturbation of the linear subspace $\mathcal{H}^{\perp}_{\mathrm{sg}}$ of data satisfying the orthogonality conditions for the linear problem.

Let us also emphasize that if one wishes to construct solutions with even stronger regularity, say $u \in H^k_x H^1_y$ with $k \geq 1$, then generically, one needs to ensure that $2k$ orthogonality conditions are satisfied by the source terms (see Lemma 2.12). This situation occurs in [30], which concerns the Prandtl problem described below in Section 1.3.

On entropy solutions. An entirely different approach to solve (1.1) is to look directly for weak solutions to the nonlinear problem, for example using an entropy formulation. The regularity for such solutions is $u \in L^\infty_{x,y} \cap L^2_x H^1_y$ and they are typically obtained as limits of solutions u^ε to regularized versions of (1.1), e.g. $u^\varepsilon \partial_x u^\varepsilon - \partial_{yy} u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = 0$. Such solutions satisfy both the equation and the lateral boundary conditions only in the weak sense of appropriate inequalities linked with "entropy pairs". Given $\delta_0, \delta_1 \in L^\infty(-1,1)$, the existence of an entropy solution to

$$\begin{cases} u\partial_x u - \partial_{yy} u = 0, \\ u_{|x=x_i} = \delta_i, \\ u_{|y=\pm 1} = 0 \end{cases} \tag{1.10}$$

was first proved in [12]. More recently, Kuznetsov proved in [35] the uniqueness of the entropy solution to (1.10), determined in which sense the lateral boundary conditions were satisfied and proved a stability estimate of the form

$$\|u - \tilde{u}\|_{L^1(\Omega)} \lesssim \|\delta_0 - \tilde{\delta}_0\|_{L^1(-1,1)} + \|\delta_1 - \tilde{\delta}_1\|_{L^1(-1,1)}. \tag{1.11}$$

In particular, this stability estimate guarantees that one can construct sign-changing solutions in the vicinity of the linear shear flow.

However, an important drawback of the entropy formulation is that the boundary conditions are only satisfied in a very weak sense. Although functions in $L^\infty_{x,y} \cap L^2_x H^1_y$ do not have classical traces at $x = x_i$, one can give a weak sense to the traces using the equation (see [36] for more details). Unfortunately, it is expected that these weak traces do not coincide with the supplied boundary data on sets of positive measure.

In contrast, since the solutions we construct in this work have (at least) $H^1_x L^2_y$ regularity, they have usual traces $u_{|\Sigma_i} \in L^2(\Sigma_i)$ and the equalities $u_{|\Sigma_i} = \delta_i$ hold in $L^2(\Sigma_i)$, so almost everywhere.

On the choice of the linear shear flow. We choose to study the well-posedness of (1.1) in the vicinity of the linear shear flow to lighten the computations. Nonetheless, we expect that our results and proofs can be extended to study the well-posedness of (1.1) in the vicinity of any sufficiently regular reference flow $\mathfrak{u}$ changing sign across a single line $\{\mathfrak{u} = 0\}$, satisfying $\mathfrak{u}_y \geq c_0 > 0$ in Ω (so that (1.5) is the correct toy model) and with $\|\mathfrak{u}_x\|_\infty$ small enough (to ensure *a priori* estimates).

Moreover, taking a step further in the modeling of recirculation problems in fluid mechanics (see Section 1.3), we also expect that our approach could be extended to an unbounded domain of the form $(x_0, x_1) \times (0, +\infty)$, with a reference flow such that $\mathfrak{u}_{|y=0} = 0$, $\mathfrak{u} < 0$ below some critical line and then $\mathfrak{u} > 0$ above, with $\mathfrak{u}$ having some appropriate asymptotic behavior as $y \to +\infty$. In such a setting, the Poincaré inequalities in the vertical direction that we use here should probably be replaced with well-suited Hardy inequalities.

On the conditions $\delta_0(0) = \delta_1(0) = 0$ for fixed end-points. It is an important feature of our work that we are able to enforce precisely the exact endpoints of the (curved) line $\{u = 0\}$ at $x = x_0$ and $x = x_1$. Theorem 3 is stated for perturbations which satisfy $\delta_i(0) = 0$ (see (1.2)), so that the full boundary data $y + \delta_i(y)$ changes sign exactly at $y = 0$. This choice simplifies the definition of the submanifold $\mathcal{M}$ of boundary data for which we are able to solve the problem. Nevertheless, given y_0, y_1 sufficiently close to 0 and δ_0, δ_1 such that $y+\delta_i(y)$ changes sign at $y = y_i$, we expect that a similar existence result holds, provided that the perturbations are chosen in an appropriate modification of $\mathcal{M}$, with suitable modifications to the norm (1.3) and where, in (1.7), the definitions of Σ_i are generalized by setting $\Sigma_i := \{(x_i, y);\ (-1)^i(y + \delta_i(y)) > 0\}$.

On the boundary conditions $u_{|y=\pm 1} = 0$. These boundary conditions are merely chosen to simplify the statements and lighten the computations, since they guarantee that $(x, y) \mapsto (x, y + u(x, y))$ is a well-defined global change of variables mapping Ω to itself (see Section 3.1). Straightforward modifications would ensure the well-posedness of the considered systems with sufficiently regular non-zero boundary data for $u_{|y=\pm 1}$, up to suitable compatibility conditions on $\delta_i((-1)^i)$ and $\delta_i''((-1)^i)$ (see next item).

On the compatibility conditions $\delta_i((-1)^i) = 0$ and $\delta_i''(0) = \delta_i''((-1)^i) = 0$. These conditions are classical compatibility conditions for solutions to elliptic-parabolic equations. For example, the condition $\delta_0(1) = 0$ is intended to match the condition $u_{|y=1} = 0$, and is necessary to have $L^2_x H^2_y$ regularity. The condition $\delta_0''(0) = 0$ comes from the equation. Indeed, if u is a sufficiently regular solution with $f(x_0, 0) = 0$, the equality $\partial_{zz} u = z\partial_x u$ at $(x_0, 0)$ enforces $\partial_{zz} u(x_0, 0) = 0$, so $\delta_0''(0) = 0$. The condition $\delta_0''(1) = 0$ stems similarly from the equation and the fact that $\partial_x u_{|y=1} = 0$. It corresponds to a classical parabolic regularity compatibility condition in order to have $L^2_x H^4_y$ regularity. Note that we actually require the cancellation of two additional boundary conditions, namely $\delta_0'(0) = \delta_1'(0) = 0$. It is possible that these extra assumptions are technical, and could be removed.

1.3 Motivation from recirculation problems in fluid mechanics

Our original motivation stems from fluid mechanics. Indeed, the stationary Prandtl equation, which describes the behavior of a fluid with small viscosity in the vicinity of a wall, reads

$$\begin{cases} u\partial_x u + v\partial_y u - \partial_{yy} u = -\partial_x p_E, \\ u_{|y=0} = v_{|y=0} = 0, \\ \lim_{y\to\infty} u(x, y) = u_E(x), \end{cases} \tag{1.12}$$

where $u_E(x)$ (resp. $p_E(x)$) is the trace of an outer Euler flow (resp. pressure) on the wall, and satisfies $u_E \partial_x u_E = -\partial_x p_E$.

As long as u remains positive, (1.12) can be seen as a nonlocal, nonlinear diffusion type equation, the variable x being the evolution variable. Using this point of view, Oleinik (see e.g. [45, Theorem 2.1.1]) proved the local well-posedness of a solution to (1.12) when the equation (1.12) is supplemented with a boundary data $u_{|x=0} = u_0$, where $u_0(y) > 0$ for $y > 0$ and such that $u_0'(0) > 0$. Let us mention that such positive solutions exist globally when $\partial_x p_E \leq 0$, but are only local when $\partial_x p_E > 0$. More precisely, when $\partial_x p_E = 1$ for instance, for a large class of boundary data u_0, there exists $x^* > 0$ such that $\lim_{x\to x^*} u_y(x, 0) = 0$. Furthermore, the solution may develop a singularity at $x = x^*$, known as Goldstein singularity. The point x^* is called the separation point: intuitively,

if the solution to Prandtl exists beyond x^*, then it must have a negative sign close to the boundary (and therefore change sign). We refer to the seminal works of [25] and Stewartson [54] for formal computations on this problem. A first mathematical statement describing separation was given by Weinan E in [20] in a joint work with Luis Cafarelli, but the complete proof was never published. The first author and Nader Masmoudi then gave a complete description of the formation of the Goldstein singularity [16]. The recent work [53] indicates that this singularity holds for a large class of initial data.

Because of this singularity, it is actually unclear that the Prandtl system is a relevant physical model in the vicinity of the separation point x^*, because the normal velocity v becomes unbounded at $x = x^*$. Consequently, more refined models, such as the triple deck system (see [37] for a presentation of this model, and [32, 17] for a recent mathematical analysis of its time-dependent version), were designed specifically to replace the Prandtl system with a more intricate boundary layer model in the vicinity of the separation point. However, beyond the separation point, i.e. for $x > x^*$, it is expected that the Prandtl system becomes valid again, but with a changing sign solution.

To the best of our knowledge, the well-posedness of (1.12) when the solution u is allowed to change sign has seldom been investigated. Such solutions are called "recirculating solutions", and the zone where $u < 0$ is called a recirculation bubble, the usual convention being that $u_E(x) > 0$, so that the flow is going forward far from the boundary.

Let us mention however the recent preprint [31] by Sameer Iyer and Nader Masmoudi, in which the authors prove *a priori* estimates in high regularity norms for smooth solutions to the Prandtl equation (1.12), in the vicinity of explicit self-similar recirculating flows, called Falkner-Skan profiles. The latter are given by

$$u(x,y) = x^m f'(\zeta), \tag{1.13}$$

$$v(x,y) = -y^{-1}\zeta f(\zeta) - \frac{m-1}{m+1} y^{-1}\zeta^2 f'(\zeta), \tag{1.14}$$

where $\zeta := (\frac{m+1}{2})^{\frac{1}{2}} y x^{\frac{m-1}{2}}$ is the self-similarity variable, m is a real parameter and f is the solution to the Falkner-Skan equation

$$f''' + ff'' + \beta(1 - (f')^2) = 0, \tag{1.15}$$

where $\beta = \frac{2m}{m+1}$, subject to the boundary conditions $f(0) = f'(0) = 0$ and $f'(+\infty) = 1$. Such flows correspond to an outer Euler velocity field $u_E(x) = x^m$. For some particular values of m (or, equivalently, β), these formulas provide physical solutions to (1.12) which exhibit recirculation (see [13]). Obtaining *a priori* estimates for recirculating solutions to the Prandtl system (1.12) is very difficult. This important step was achieved by Sameer Iyer and Nader Masmoudi in [31].

In the present paper, we have chosen to focus on a different type of difficulty, and to consider the toy-model (1.1), which differs from (1.12) through the lack of the nonlinear transport term $v\partial_y u$ and its associated difficulties (nonlocality, loss of derivative) and the exclusion of the zones close to the wall and far from the wall. For the model (1.1), *a priori* estimates are easy to derive, see [52, Chapter 4]. The difficulty lies elsewhere, as explained previously. Indeed, in order to construct a sequence of approximate solutions satisfying the *a priori* estimates, we need to ensure that the orthogonality conditions are satisfied all along the sequence. The core of the proof is to keep track of these orthogonality conditions, and to analyze their dependency on the sequence itself. For the Prandtl system (1.12), this difficulty has very recently been tackled by Sameer Iyer and Nader Masmoudi in [30], building upon their *a priori* estimates of [31] and the ideas developed in the first version of our present work.

1.4 Scheme of proof and plan of the paper

The uniqueness of solutions is fairly easy to prove. For the linear problem (1.5), uniqueness already holds at the level of weak solutions (see Proposition 2.2 and Appendix A). For the nonlinear problem, uniqueness is straightforward since we are considering strong solutions (see Section 6.3). Therefore, the main subject of this paper is the proof of the existence of solutions for the nonlinear problem (1.7).

A first natural idea would be to prove existence thanks to a nonlinear scheme relying on the linear problem (1.5). For example, one could wish to construct a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ by setting $u_0 := 0$ (or any other initial guess) and solving

$$\begin{cases} y\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f - u_n \partial_x u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{1.16}$$

However, this strategy fails. The key point is that the right-hand side contains a full tangential derivative of u_n, whereas the operator $y\partial_x - \partial_{yy}$ only yields a gain of $2/3$ of a derivative in this direction (more precisely, see Proposition 1.4, Remark 1.5 and Proposition 2.4). Hence, this nonlinear scheme would exhibit a "loss of derivative", preventing us from proving a uniform bound on the sequence $(u_n)_{n\in\mathbb{N}}$.

Another drawback of this scheme is that it would not translate well to a setting where one does not assume $\delta_i(0)$. Indeed, in such a case, the inflow boundaries of the problem with the perturbed data $y + \delta_i(y)$ would not match the inflow boundaries of the linear problem (1.5).

Hence, we will rather construct solutions to (1.1) through another iterative scheme, which does not rely directly on (1.5). In a way, the issues stemming from the linear scheme (1.16) come from the following fact: in equation (1.7), the geometry of the problem is dictated by the line where the whole solution $y + u$ changes sign. On the contrary, in (1.5), the geometry of the problem follows the cancellation of y. Keeping this in mind, we will rather rely on the following linearized equation around a base flow perturbation $\bar{u}$, where $\bar{u}$ is a small perturbation of the shear flow y in Q^1

$$\begin{cases} \bar{u}\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0. \end{cases} \tag{1.17}$$

The well-posedness of such linear systems is investigated in Section 3. Exactly as Theorem 1 requires orthogonality conditions to ensure the existence of regular solutions to (1.5), the existence of regular solutions to (1.17) is subject to perturbed orthogonality conditions (see Section 3.4). Of course, these orthogonality conditions *depend on the flow* $\bar{u}$. We will express them as $\ell^j_{\bar{u}}(f, \delta_0, \delta_1) = 0$ for $j = 0, 1$, where $\ell^j_{\bar{u}} : \mathcal{H} \to \mathbb{R}$ are continuous linear forms.

More precisely, we will construct a sequence $(u_n)_{n\in\mathbb{N}}$ solving the following iterative scheme

$$\begin{cases} (y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1}, \\ (u_{n+1})_{|\Sigma_i} = \delta_i^{n+1}, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{1.18}$$

For this scheme, we are able to prove a uniform bound for u_n in an appropriate space Q^1 and the convergence of the sequence in an interpolation space[1] $Q^{1/2}$ (see (1.26) and (1.27)). This scheme

[1] Actually, one could also prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Q^s for s close to $1/2$. We merely chose the space $Q^{1/2}$ in order to fix ideas.

is similar to the one used to construct solutions to quasilinear symmetric hyperbolic systems, see for instance [8, Section 4.3].

In (1.18), $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ are appropriate perturbations of the data (f, δ_0, δ_1) tailored to satisfy the orthogonality conditions associated with the problem (1.17) (for $\bar{u} = y + u_n$), namely $\ell^j_{y+u_n}(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1}) = 0$. This is a key difficulty of this work (see Sections 4 and 6.2). In particular, in order to allow the sequence u_n to converge, we must prove that these perturbations also converge. This amounts to proving that the linear forms $\ell^j_{\bar{u}}$ associated with (1.17) depend continuously (and even in a Lipschitz manner) on $\bar{u}$, *for the same topology as the one within which we will prove the convergence of the sequence* u_n. It turns out that for the spaces Q^1 and $Q^{1/2}$ defined below in (1.26) and (1.27), for any $\bar{u}, \bar{u}' \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$, $\|\bar{u}' - y\|_{Q^1} \ll 1$,

$$\|\ell^j_{\bar{u}} - \ell^j_{\bar{u}'}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.$$

This allows us to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$. This continuity estimate is truly the cornerstone of our strategy to prove nonlinear existence. We also emphasize that this estimate requires identifying quite precisely what the linear forms $\ell^j_{\bar{u}}$ are.

We believe that this methodology is rather robust and can be applied to other nonlinear problems in which orthogonality conditions are present at the linearized level. As an example, following the first version of this paper, some elements of this methodology have very recently been used by Sameer Iyer and Nader Masmoudi in [30] to tackle the Prandtl problem mentioned in Section 1.3.

Given a specific nonlinear problem, one could hope to be able to tweak our nonlinear scheme in order to avoid the computation of the linear forms $\ell^j_{\bar{u}}$ at a varying flow $\bar{u}$, and always rely on the reference orthogonality conditions at some reference flow (e.g. at the shear flow $\mathfrak{u}(x, y) = y$). This idea was suggested by Sameer Iyer and Nader Masmoudi during a personal communication concerning the first version of this paper. We describe an example of such a scheme in Section 6.2.6. We do not follow this "fixed linear forms" idea in this paper. Indeed, we believe that the "varying linear forms" approach is natural and adaptable, and has its own interest (for example, it yields a well-posedness theory for the linearized problems, as in Proposition 3.20). Moreover, we expect that the "fixed linear form" variant does not substantially decrease the overall length or technicality of our method (see Section 6.2.6 for further details).

The plan of this work is as follows. As a preliminary, we introduce in Section 1.5 the functional spaces we will use. First, we study the linear problem (1.5) in Section 2, leading to Theorem 1, and prove that the two orthogonality conditions we expose are indeed nonvoid. We also construct the singular profiles $\bar{u}^i_{\text{sing}}$ and prove Theorem 2. Second, in Section 3, we study linearized problems of the form (1.17). The main task is to derive the modified orthogonality conditions. We also provide a formula for $\ell^j_{\bar{u}}$. Third, we prove the stability with respect to the underlying flow $\bar{u}$ of the orthogonality conditions in Section 4. Note that in order to derive estimates in the intermediate space $Q^{1/2}$, we will need to prove interpolation results on closed subspaces of Hilbert spaces. This rather technical step is performed in Section 5, in which we also generalize the decomposition result of Theorem 2 to the linearized equation (1.17). Then, in Section 6, we turn to the nonlinear problem for which we prove the existence of solutions in Section 6.2 using the scheme mentioned above, then uniqueness in Section 6.3 and the necessity of the nonlinear orthogonality conditions in Section 6.4. This concludes the proof of Theorem 3.

Eventually, in Appendix A, we prove the uniqueness of weak solutions to various linear problems involved in Section 3, by adapting an argument due to Baouendi and Grisvard [9]. In Appendix B,

we prove various technical results of functional analysis that we use throughout the paper. Appendix C and Appendix D contain postponed proofs of some lemmas of Section 3 and Section 5.

As the paper is quite long, a list of notations is provided starting page 116.

1.5 Functional spaces and interpolation results

1.5.1 Notations

Throughout this work, an assumption of the form "$A \ll 1$" will mean that there exists a constant $c > 0$, depending only on Ω such that, if $A \leq c$, the result holds. Similarly, a conclusion of the form "$A \lesssim B$" will mean that there exists a constant $C > 0$, depending only on Ω, such that the estimate $A \leq CB$ holds. For ease of reading, we will not keep track of the value of these constants, mostly linked with embeddings of functional spaces.

We will often use the notations $\Omega_\pm := \Omega \cap \{\pm z > 0\}$.

1.5.2 Trace spaces for the lateral boundaries

For the traces of the solutions to (1.5) or (1.7) at $x = x_0$ and $x = x_1$, we will need the following spaces, due to [48, 49]. We define $\mathscr{L}^2_z(-1,1)$ as the completion of $L^2(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{L}^2_z} := \left(\int_{-1}^{1} |z| \psi^2(z) \, \mathrm{d}z \right)^{\frac{1}{2}} \tag{1.19}$$

and $\mathscr{H}^1_z(-1,1)$ as the completion of $H^1_0(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathscr{H}^1_z} := \|\psi\|_{\mathscr{L}^2_z} + \|\partial_z \psi\|_{\mathscr{L}^2_z}. \tag{1.20}$$

1.5.3 Trace spaces for horizontal cuts

When considering the restriction of a solution to (1.5) or (1.7) at some altitude $z \in (-1,1)$, we will sometimes need the following spaces in the horizontal direction.

The Lions–Magenes space $H^{1/2}_{00}(x_0, x_1)$ is defined as the completion of $H^1_0(x_0, x_1)$ with respect to the following norm

$$\|\phi\|_{H^{1/2}_{00}} := \|\phi\|_{H^{1/2}} + \left(\int_{x_0}^{x_1} \frac{|\phi(x)|^2}{|x - x_0||x_1 - x|} \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{1.21}$$

It is also the interpolation space $[H^1_0(x_0, x_1), L^2(x_0, x_1)]_{\frac{1}{2}}$ (see [42, Theorem 11.7, Chapter 1]), or the subspace of functions of $H^{1/2}(x_0, x_1)$ of which the extension by 0 is in $H^{1/2}(\mathbb{R})$. By [42, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0, x_1)$ to $(H^{1/2}_{00}(x_0, x_1))'$. See also [55, Lecture 33] for another presentation of this space.

We will also need one-sided versions of this space, for functions "vanishing" only on the left side $x = x_0$ (say $H^{1/2}_{00_l}(x_0, x_1)$) or only on the right side $x = x_1$ (say $H^{1/2}_{00_r}(x_0, x_1)$). For example

$$\|\phi\|_{H^{1/2}_{00_r}} := \|\phi\|_{H^{1/2}} + \left(\int_{x_0}^{x_1} \frac{|\phi(x)|^2}{|x_1 - x|} \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{1.22}$$

1.5.4 Pagani's weighted Sobolev spaces

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$. In the works [48, 49] (albeit with swapped variables with respect to our setting), Pagani introduced the space $Z(\mathcal{O})$ of scalar functions ϕ on $\mathcal{O}$ such that ϕ, $\partial_z\phi$, $\partial_{zz}\phi$ and $z\partial_x\phi$ belong to $L^2(\mathcal{O})$ (in the sense of distributions). In this work, we will refer to this space with the notation $Z^0(\mathcal{O})$. It is a Banach space for the following norm

$$\|\phi\|_{Z^0} := \|z\partial_x\phi\|_{L^2} + \|\partial_{zz}\phi\|_{L^2} + \|\partial_z\phi\|_{L^2} + \|\phi\|_{L^2}. \tag{1.23}$$

We will also need the space $Z^1(\mathcal{O})$, which we define as the space of scalar functions ϕ on $\mathcal{O}$ such that ϕ and $\partial_x\phi$ belong to $Z^0(\mathcal{O})$, associated with the following norm

$$\|\phi\|_{Z^1} := \|\phi\|_{Z^0} + \|\partial_x\phi\|_{Z^0}. \tag{1.24}$$

The omitted proofs of the results of this section are postponed to Appendix B. We start with a straightforward extension result, which allows transferring results on $Z^0(\mathbb{R}^2)$ to $Z^0(\Omega)$.

Lemma 1.3. *There exists a continuous extension operator from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.*

The following embedding is the most important result concerning the spaces Z^0. Since solutions to $(z\partial_x - \partial_{zz})u = f$ for $f \in L^2(\Omega)$ belong to $Z^0(\Omega)$ (see Proposition 2.4), the following embedding entails that such solutions belong to $H^{2/3}(\Omega)$.

Proposition 1.4. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $H^{2/3}_x L^2_z$.*

Remark 1.5. *Proposition 1.4 can be seen as an hypoellipticity result for the operator $L = \partial_{zz} - z\partial_x$ in the full space $\mathbb{R}^2$, which is of the form $X_1^2 + X_0$, where $X_1 = \partial_z$, $X_0 = -z\partial_x$ and $[X_0, X_1] = \partial_x$, so the Lie brackets generate the full space and L satisfies Hörmander's sufficient condition of [28] for hypoellipticity. In fact, in the full space $\mathbb{R}^2$, the $H^{2/3}_x L^2_z \cap L^2_x H^2_z$ regularity of solutions to $Lu = f$ for $f \in L^2$ can be derived from the general theory of quadratic operators, which makes a link between the anisotropic gain of regularity and the number of brackets one has to take in order to generate a direction. For instance, this regularity follows from [1, Theorem 2.10] and more precisely Example 2.11 therein applied with*

$$R = 0 \quad \textit{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Lemma 1.6. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $C^0_z(H^{1/2}_x)$.*

Proof. By definition, $Z^0(\mathbb{R}^2) \hookrightarrow H^2_z(L^2_x)$. By Proposition 1.4, $Z^0(\mathbb{R}^2) \hookrightarrow L^2_z(H^{2/3}_x)$. By the "fractional trace theorem" [42, Equation (4.7), Chapter 1], $Z^0(\mathbb{R}^2) \hookrightarrow C^0_z(H^{1/2}_x)$. □

Lemma 1.7. *$Z^0(\Omega)$ is continuously embedded in $C^0([x_0, x_1]; \mathscr{H}^1_z(-1, 1))$.*

Proof. This is contained in the trace result [49, Theorem 2.1]. □

Remark 1.8. *Although it is "almost" the case, there does not hold $Z^0(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$.*

- *Pagani [48, Theorem 2.1] proves that the operator $\phi \mapsto \phi(\cdot, 0)$ is onto from $Z^0(\mathbb{R}^2)$ to $H^{\frac{1}{2}}(\mathbb{R})$. But $H^{\frac{1}{2}}(\mathbb{R})$ contains unbounded functions of x.*
- *Pagani [48, Theorem 2.3] proves that the operator $\phi \mapsto \phi(0, \cdot)$ is onto from $Z^0(\mathbb{R}^2)$ to the space $\mathscr{H}^1_z(\mathbb{R})$. But this space contains unbounded functions, for example $\psi(z) := (-\ln|z|/2)^s \chi(z)$ for $s < \frac{1}{2}$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $z = 0$.*

1.5.5 Anisotropic Sobolev spaces

In the sequel, we will construct solutions to (1.5) and (1.7) in the following anisotropic Sobolev spaces. Within all these spaces, one has heuristically the correspondence $\partial_x \approx \partial_z^3$, which corresponds to the appropriate scaling due to the degeneracy of $z\partial_x$ at $z = 0$. We consider

$$Q^0 := L^2_x(H^2_y) \cap H^{2/3}_x(L^2_y), \tag{1.25}$$

$$Q^1 := L^2_x(H^5_y) \cap H^{5/3}_x(L^2_y), \tag{1.26}$$

in which we have omitted the domain Ω. By Proposition 1.4, $Z^0 \hookrightarrow Q^0$. This is the natural space for strong solutions to our equations. The space Q^1 corresponds to the situation where $\partial_x u$ is itself a strong solution to a similar equation, so $\partial_x u \in Q^0$, which yields the $H^{5/3}_x(L^2_y)$ estimate. The $L^2_x(H^5_y)$ estimate comes from a sort of "hidden regularity" result (see e.g. Section 2.4).

Remark 1.9. *If u is a solution to $z\partial_x u - \partial_{zz} u = 0$ say on the whole plane $\mathbb{R}^2$, then the rescaled functions $u_\lambda(x, z) := u(\lambda^3 x, \lambda z)$ are also solutions. This is also consistent with the shape of the singular profiles $\bar{u}^i_{\text{sing}}$ from Theorem 2, and leads to the rule of thumb "one derivative in x equals three derivatives in z" (which is different from the usual parabolic scaling, because of the cancellation on the line $z = 0$). The definitions of the spaces Q^0 and Q^1 are guided by this elementary remark.*

A key argument of our work is that we will prove the Lipschitz-stability of the orthogonality conditions and the convergence of the nonlinear scheme within the following interpolation space:

$$Q^{1/2} := [Q^0, Q^1]_{\frac{1}{2}} = L^2_x(H^{7/2}_y) \cap H^{7/6}_x(L^2_y). \tag{1.27}$$

Lemma 1.10. *By interpolation, we have the following embeddings*

- $Q^1 \hookrightarrow H^\sigma_x H^{\sigma'}_y$ *for all $\sigma, \sigma' \geq 0$ such that $3\sigma + \sigma' = 5$;*
- $Q^{1/2} \hookrightarrow H^\sigma_x H^{\sigma'}_y$ *for all $\sigma, \sigma' \geq 0$ such that $3\sigma + \sigma' = 7/2$;*
- $Q^0 \hookrightarrow H^\sigma_x H^{\sigma'}_y$ *for all $\sigma, \sigma' \geq 0$ such that $3\sigma + \sigma' = 2$.*

In particular, $Q^1 \hookrightarrow L^\infty_x(W^{2,\infty}_y)$ and $Q^{1/2} \hookrightarrow C^0([-1, 1], H^1(x_0, x_1))$.

Proof. These embedding follow from [42, Equation (13.4), Chapter 1] and the fractional trace theorem [42, Equation (4.7), Chapter 1]. □

Eventually, we will use the space

$$\mathcal{B} := \{u \in L^2((x_0, x_1), H^1_0(-1, 1)); \;\; z\partial_x u \in L^2_x(H^{-1}_z)\}. \tag{1.28}$$

This space was introduced by Baouendi and Grisvard in [9], in which the authors proved the uniqueness of solutions to linear degenerate elliptic equations in $\mathcal{B}$. They also proved that functions in $\mathcal{B}$ have traces on $\{x = x_i\}$ in $\mathscr{L}^2_z(-1, 1)$. These results are recalled in Appendix A, and will be used abundantly throughout the paper.

2 The case of the linear shear flow

This section concerns the well-posedness of the linear system (1.5) which we restate here for convenience and by using z as a vertical variable rather than y to prepare for the next sections. We thus consider, in $\Omega = (x_0, x_1) \times (-1, 1)$, the system

$$\begin{cases} z\partial_x u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{2.1}$$

where $\Sigma_0 = \{x_0\} \times (0, 1)$ and $\Sigma_1 = \{x_1\} \times (-1, 0)$.

First, in Section 2.1, we recall the theory of weak solutions, due to Fichera for the existence, and to Baouendi and Grisvard for the uniqueness. Then, in Section 2.2, we recall the theory of strong solutions with maximal regularity, due to Pagani. Our contributions regarding this problem are contained in the following subsections. In Section 2.3, we derive orthogonality conditions which are necessary to obtain higher tangential regularity and prove the existence result of Theorem 1. In Section 2.4, we prove a hidden regularity result for such solutions, which allows controlling five derivatives in the vertical directions, and will be useful in the sequel. Eventually, in Section 2.5, we construct explicit singular solutions and prove the decomposition result of Theorem 2.

2.1 Existence and uniqueness of weak solutions

Definition 2.1 (Weak solution)**.** *Let $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$ and $\delta_0, \delta_1 \in \mathscr{L}^2_z(-1, 1)$. We say that $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ is a* weak solution *to* (2.1) *when, for all $v \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds*

$$-\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v = \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.2}$$

Weak solutions in the above sense are known to exist since the work Fichera [21, Theorem XX] (which concerns generalized versions of (2.1), albeit with vanishing boundary data). Uniqueness dates back to [9, Proposition 2] by Baouendi and Grisvard.

Proposition 2.2. *Let $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$ and $\delta_0, \delta_1 \in \mathscr{L}^2_z(-1, 1)$. There exists a unique weak solution $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ to* (2.1)*. Moreover,*

$$\|u\|_{L^2_x H^1_z} \lesssim \|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathscr{L}^2_z} + \|\delta_1\|_{\mathscr{L}^2_z}. \tag{2.3}$$

Proof. The proof of uniqueness is postponed to Appendix A where we adapt Baouendi and Grisvard's arguments to prove the uniqueness of weak solutions to all the linear problems we encounter in this paper. It relies on the proof of a trace theorem and a Green identity.

Let us prove the existence. We introduce two Hilbert spaces $\mathscr{V} \hookrightarrow \mathscr{U} \hookrightarrow L^2((x_0, x_1); H^1_0(0, 1))$ as follows. Let $\mathscr{V} := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Omega \setminus (\Sigma_0 \cup \Sigma_1)\}$. Let $\mathscr{U}$ be the completion of $H^1(\Omega) \cap L^2((x_0, x_1); H^1_0(-1, 1))$ with respect to the scalar product

$$\langle u, v\rangle_{\mathscr{U}} := \int_\Omega \partial_z u \partial_z v + \int_{\Sigma_0} zuv - \int_{\Sigma_1} zuv. \tag{2.4}$$

For $u, v \in \mathscr{U} \times \mathscr{V}$, let

$$a(u,v) := -\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v, \tag{2.5}$$

$$b(v) := \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.6}$$

In particular, for every $v \in \mathscr{V}$, integration by parts leads to $a(v,v) = \|v\|^2_{\mathscr{U}}$ and

$$|b(v)| \leq \left(\|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathscr{L}^2_z} + \|\delta_1\|_{\mathscr{L}^2_z} \right) \|v\|_{\mathscr{U}}. \tag{2.7}$$

Hence, $b \in \mathcal{L}(\mathscr{V})$ can be extended as a linear form over $\mathscr{U}$[2] and existence follows from the Lax-Milgram type existence principle Lemma B.2 in Appendix B, which also yields the energy estimate (2.3) thanks to (2.7) and Poincaré's inequality. □

Remark 2.3. *Instead of using the weak Lax-Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation* (2.1) *by vanishing viscosity, and to obtain uniform* $L^2_x H^1_z$ *estimates on the approximation.*

2.2 Strong solutions with maximal regularity

We now turn to strong solutions, i.e. solutions for which (2.1) holds almost everywhere. The main result on this topic is due to Pagani.

Proposition 2.4. *Let* $f \in L^2(\Omega)$ *and* $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ *such that* $\delta_0(1) = \delta_1(-1) = 0$. *The unique weak solution* u *to* (2.1) *belongs to* $Z^0(\Omega)$ *and satisfies*

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathscr{H}^1_z} + \|\delta_1\|_{\mathscr{H}^1_z}. \tag{2.8}$$

Moreover, the boundary conditions $u_{|\Sigma_i} = \delta_i$ *hold in the sense of traces in* $\mathscr{H}^1_z(\Sigma_i)$ *(see Lemma 1.7).*

Proof. This is a particular case of [49, Theorem 5.2]. Pagani's proof proceeds by localization. Far from the critical points $(x_0, 0)$ and $(x_1, 0)$, the regularity is rather straightforward. Near these critical points, the regularity stems from the regularity obtained for a similar problem set in a half-space $(0, +\infty) \times \mathbb{R}$ or $\mathbb{R} \times (0, +\infty)$. Pagani studies such half-space problems in [48] where he derives explicit representation formulas for the solutions, using the Mellin transform and the Wiener-Hopf method. We do not reproduce these arguments here for brevity. □

2.3 Orthogonality conditions for higher tangential regularity

We now investigate whether solutions to (2.1) enjoy higher regularity in the horizontal direction. As mentioned in Section 1.2, it is quite easy to obtain *a priori* estimates in the space $Z^1(\Omega)$ (see Proposition 2.5). However, we prove in Proposition 2.8 that the weak solution enjoys such a regularity *if only if the data satisfies appropriate orthogonality conditions*. Eventually, we give statements highlighting the fact that these conditions are non-empty.

[2] Functions in $\mathscr{U}$ *a priori* do not have traces on Σ_i so one could wonder how definition (2.6) makes sense when $v \in \mathscr{U}$. The integrals $\int_{\Sigma_i} z\delta_i v$ make sense precisely because $\mathscr{U}$ is defined as a completion with respect to (2.4). In fact, weak solutions do have traces in a strong sense, as proved in Lemma A.1, thanks to the extra regularity in x provided by the equation.

Proposition 2.5. *Let $f \in H^1((x_0,x_1);H^{-1}(-1,1))$ and $\delta_0,\delta_1 \in \mathscr{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$ and such that $\Delta_0,\Delta_1 \in \mathscr{L}^2_z(-1,1)$, where*

$$\Delta_i(z) := \frac{f(x_i,z) + \partial_z^2\delta_i(z)}{z}. \tag{2.9}$$

If the unique weak solution u to (2.1) belongs to $H^1((x_0,x_1);H^1_0(-1,1))$, then one has the following weak solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{L^2_x H^1_z} \lesssim \|\partial_x f\|_{L^2_x(H^{-1}_z)} + \|\Delta_0\|_{\mathscr{L}^2_z(\Sigma_0)} + \|\Delta_1\|_{\mathscr{L}^2_z(\Sigma_1)}. \tag{2.10}$$

If, moreover, $f \in H^1((x_0,x_1);L^2(-1,1))$, $\Delta_0,\Delta_1 \in \mathscr{H}^1_z(-1,1)$ and $\Delta_0(1) = \Delta_1(-1) = 0$, then $u \in Z^1(\Omega)$ and one has the following strong solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{Z^0} \lesssim \|\partial_x f\|_{L^2} + \|\Delta_0\|_{\mathscr{H}^1_z} + \|\Delta_1\|_{\mathscr{H}^1_z}. \tag{2.11}$$

Proof. The key point is that the information that $\partial_x u$ enjoys $L^2_x H^1_z$ regularity allows us to prove that $\partial_x u$ is the unique weak solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = \partial_x f, \\ w_{|\Sigma_i} = \Delta_i, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{2.12}$$

Then estimate (2.10) follows from (2.3) and estimate (2.11) follows from (2.8).

Hence, let us prove that, if $\partial_x u \in L^2_x H^1_z$, then $\partial_x u$ is a weak solution to (2.12). Let

$$\begin{aligned} \mathscr{V} := \big\{ v \in C^\infty(\overline{\Omega});\quad & v = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1), \\ & \partial_x v = 0 \text{ on } \{x_0\}\times(-1,0) \text{ and } \{x_1\}\times(0,1)\}. \end{aligned} \tag{2.13}$$

Let $v \in \mathscr{V}$. Then $\partial_x v$ is an admissible test function for Definition 2.1. Hence, since u is the weak solution to (2.1), one has

$$-\int_\Omega zu\partial_x(\partial_x v) + \int_\Omega \partial_z u \partial_z(\partial_x v) = \int_\Omega f(\partial_x v) + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v). \tag{2.14}$$

The $H^1_x H^1_z$ regularity of u legitimates integrations by parts in x in the left-hand side. Thus

$$\begin{aligned} \Big[-\int_{-1}^1 zu\partial_x v\Big]_{x_0}^{x_1} + \int_\Omega z(\partial_x u)\partial_x v + \Big[\int_{-1}^1 \partial_z u\partial_z v\Big]_{x_0}^{x_1} - \int_\Omega \partial_z(\partial_x u)\partial_z v \\ = \Big[\int_{-1}^1 fv\Big]_{x_0}^{x_1} - \int_\Omega f_x v + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v), \end{aligned} \tag{2.15}$$

which, after taking the boundary conditions into account, integrating by parts in z in the boundary terms $\int_{-1}^1 \partial_z u\partial_z v$ and recalling (2.9) yields

$$-\int_\Omega z(\partial_x u)\partial_x v + \int_\Omega \partial_z(\partial_x u)\partial_z v = \int_\Omega f_x v + \int_{\Sigma_0} z\Delta_0 v - \int_{\Sigma_1} z\Delta_1 v. \tag{2.16}$$

Since $\mathscr{V}$ is dense in the set of test functions for Definition 2.1, this proves that $\partial_x u$ is the weak solution to (2.12). □

We start by defining "dual profiles" which are necessary to state our orthogonality conditions.

Lemma 2.6 (Dual profiles)**.** *We define* $\overline{\Phi^0}$, $\overline{\Phi^1} \in Z^0(\Omega_\pm)$ *as the unique solutions to*

$$\begin{cases} -z\partial_x \overline{\Phi^j} - \partial_{zz}\overline{\Phi^j} = 0 \quad \text{in } \Omega_\pm, \\ \left[\overline{\Phi^j}\right]_{|z=0} = \mathbf{1}_{j=1}, \\ \left[\partial_z \overline{\Phi^j}\right]_{|z=0} = -\mathbf{1}_{j=0}, \\ \overline{\Phi^j}_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases} \tag{2.17}$$

Proof. Uniqueness is straightforward. Given $j \in \{0,1\}$ and two solutions to (2.17), let ϕ denote their difference. Then $\phi \in Z^0(\Omega_\pm)$ and both ϕ and $\partial_z\phi$ are continuous across the line $\{z=0\}$. Hence, $\phi \in Z^0(\Omega)$ and ϕ is the solution to a problem of the form (2.1) (with reversed tangential direction). So $\phi = 0$ since weak solutions to such problems are unique in Z^0.

We prove the existence of $\overline{\Phi^0}$. We define $\overline{\Phi^0}(x,z) := -z\mathbf{1}_{z>0}\zeta(z) + \Psi^0(x,z)$, where we choose $\zeta \in C_c^\infty(\mathbb{R})$ such that $\zeta \equiv 1$ in a neighborhood of $z=0$ and $\operatorname{supp}\zeta \subset (-1/2, 1/2)$, and where $\Psi^0 \in L^2((x_0,x_1); H_0^1(-1,1))$ is the unique weak solution to

$$\begin{cases} -z\partial_x\Psi^0 - \partial_{zz}\Psi^0 = -2\mathbf{1}_{z>0}\zeta'(z) - z\mathbf{1}_{z>0}\zeta''(z) & \text{in } \Omega, \\ \Psi^0(x_0,z) = 0 & \text{for } z \in (-1,0), \\ \Psi^0(x_1,z) = z\zeta(z) & \text{for } z \in (0,1), \\ \Psi^0_{|z=\pm 1} = 0. \end{cases} \tag{2.18}$$

By Proposition 2.4, $\Psi^0 \in Z^0(\Omega)$. Hence $\partial_{zz}\overline{\Phi^0} \in L^2(\Omega_\pm)$ and $z\partial_x\overline{\Phi^0} \in L^2(\Omega_\pm)$.

The construction of the profile $\overline{\Phi^1}$ is similar and is left to the reader. For example, one can decompose $\overline{\Phi^1}$ as $\overline{\Phi^1}(x,z) = \mathbf{1}_{z>0}\zeta(z) + \Psi^1(x,z)$, where, similarly, $\Psi^1 \in Z^0(\Omega)$. □

Remark 2.7. *The jump conditions in* (2.17) *prevent the dual profiles from enjoying vertical regularity across the line* $\{z=0\}$*. More subtly, even inside each half-domain, neither the* $\overline{\Phi^j}$ *nor their lifted version the* Ψ^j *enjoy tangential regularity. Indeed, formally,* $\partial_x\overline{\Phi^j}$ *and* $\partial_x\Psi^j$ *satisfy systems of the form* (2.1) *(with reversed tangential direction) with zero source term and zero boundary data. Hence, if they were sufficiently regular, they would be zero by the uniqueness results of Appendix A, and so would* $\overline{\Phi^j}$ *and* Ψ^j *by integration, contradicting* (2.17)*. We will see in Corollary 2.28 that these dual profiles indeed do contain an explicit singular part localized near the endpoints* $(x_i, 0)$*.*

We now turn to the main result of this section, which gives a necessary and sufficient condition for the solutions to enjoy the mentioned tangential regularity. Strangely, we could not find a proof of the following result in the literature, although some works mention orthogonality conditions (see [21, Equation (4.2)] or [51]). Hence, we provide here a full proof. This strategy will be extended in the next section to equations with variable coefficients. We prove further that these orthogonality conditions are not empty.

Proposition 2.8. *For* $f \in H^1((x_0,x_1); L^2(-1,1))$*,* $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ *with* $\delta_0(1) = \delta_1(-1) = 0$ *and* $\Delta_0, \Delta_1 \in \mathscr{H}^1_z(-1,1)$ *with* $\Delta_0(1) = \Delta_1(-1) = 0$ *(see* (2.9)*), the unique weak solution* u *to* (2.1) *belongs to* $H^1((x_0,x_1); H_0^1(-1,1))$ *if and only if, for* $j=0$ *and* $j=1$*,*

$$\int_\Omega \partial_x f\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j} = \partial_z^j\delta_1(0) - \partial_z^j\delta_0(0), \tag{2.19}$$

where $\overline{\Phi^0}$ *and* $\overline{\Phi^1}$ *are defined in Lemma 2.6.*

Proof. ***Step 1.*** *We exhibit possible discontinuities.* Let us consider the unique solution $u \in Z^0(\Omega)$ to (2.1). Following the strategy sketched by Goldstein and Mazumdar[3] [24, Theorem 4.2], we introduce the unique strong solution $w \in Z^0(\Omega)$ to (2.12), so that w is a good candidate for $\partial_x u$. The idea is then to introduce the function u_1 defined by

$$u_1(x,z) := \begin{cases} \delta_0(z) + \int_{x_0}^{x} w(x',z)\,\mathrm{d}x' & \text{in } \Omega_+, \\ \delta_1(z) - \int_{x}^{x_1} w(x',z)\,\mathrm{d}x' & \text{in } \Omega_- \end{cases} \tag{2.20}$$

so that $\partial_x u_1 = w$ almost everywhere. Furthermore, it can be easily proved that, in $\mathcal{D}'(\Omega_\pm)$,

$$z\partial_x u_1 - \partial_{zz} u_1 = f. \tag{2.21}$$

However, this does not entail that u_1 is a solution to this equation in the whole domain. Indeed, u_1 and $\partial_z u_1$ may have discontinuities across the line $z = 0$. One checks that u_1 and $\partial_z u_1$ are continuous across $z = 0$ if and only if

$$\begin{aligned} \int_{x_0}^{x_1} w(x,0)\,\mathrm{d}x &= \delta_1(0) - \delta_0(0), \\ \int_{x_0}^{x_1} w_z(x,0)\,\mathrm{d}x &= \partial_z\delta_1(0) - \partial_z\delta_0(0). \end{aligned} \tag{2.22}$$

The two integrals are well-defined since w_z and w_{zz} belong to $L^2(\Omega)$.

Step 2. *We compute the mean value of w and w_z using the dual profiles.* Let $\phi \in Z^0(\Omega_\pm)$ such that $\phi_{|\partial\Omega\setminus(\Sigma_0\cap\Sigma_1)} = 0$. Since $w \in Z^0(\Omega)$, it satisfies (2.12) almost everywhere, so that we can multiply the equation by ϕ and integrate over Ω_+. Hence,

$$\int_{\Omega_+} f_x\phi = \int_{\Omega_+} (z\partial_x w - \partial_{zz} w)\phi, \tag{2.23}$$

where, on the one hand,

$$\int_{\Omega_+} z(\partial_x w)\phi = \int_{\Sigma_1} z\Delta_1\phi - \int_{\Omega_+} zw\partial_x\phi \tag{2.24}$$

and on the other hand,

$$-\int_{\Omega_+} (\partial_{zz} w)\phi = \int_{x_0}^{x_1} (\partial_z w\phi - w\partial_z\phi)(x,0^+)\,\mathrm{d}x - \int_{\Omega_+} w\partial_{zz}\phi. \tag{2.25}$$

Thus, performing the same computation on Ω_- and summing both contributions yields

$$\begin{aligned} \int_{x_0}^{x_1} (\partial_z w[\phi]_{|z=0} - w[\partial_z\phi]_{|z=0})(x,0)\,\mathrm{d}x = \int_\Omega f_x\phi + \int_{\Sigma_0} z\Delta_0\phi - \int_{\Sigma_1} z\Delta_1\phi \\ + \int_{\Omega_\pm} w(z\partial_x\phi + \partial_{zz}\phi). \end{aligned} \tag{2.26}$$

[3] Oddly, Goldstein and Mazumdar do not mention the orthogonality conditions (2.19). They merely state that, "since $\partial_{zz}u_1 = z\partial_x u_1 - f$ in $\mathcal{D}'(\Omega_\pm)$ and since $zu_1, f \in C^0([x_0,x_1];L^2(-1,1))$, consequently $z\partial_x u_1 - \partial_{zz}u_1 = f$ in $L^2(\Omega)$". However, these orthogonality conditions are non-empty, as we show below (see Proposition 2.10).

Hence, for $j \in \{0,1\}$,

$$\int_{x_0}^{x_1} \partial_z^j w(x,0)\,\mathrm{d}x = \int_\Omega f_x \overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0 \overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1 \overline{\Phi^j}, \tag{2.27}$$

where the dual profiles $\overline{\Phi^0}$ and $\overline{\Phi^1}$ are defined in Lemma 2.6.

Step 3. *Conclusion.* Assume that the orthogonality conditions (2.19) are satisfied for $j = 0$ and $j = 1$. Then (2.22) holds, and a consequence, $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$, and $u_1 \in L^2((x_0,x_1); H_0^1(-1,1))$ is a weak solution to (2.1). We infer from the uniqueness of weak solutions that $u = u_1$, and therefore $\partial_x u = w \in L^2((x_0,x_1); H_0^1(-1,1))$. Hence $u \in H^1((x_0,x_1); H_0^1(-1,1))$.

Conversely, if u is a solution to (2.1) with $H^1((x_0,x_1); H_0^1(-1,1))$ regularity, then $\partial_x u$ is a weak solution to (2.12) (see the proof of Proposition 2.5) and u is given in terms of $\partial_x u$ by (2.20) almost everywhere. Thus $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$. Hence $\int_{x_0}^{x_1} u_x(x,0)\,\mathrm{d}x = \delta_1(0) - \delta_0(0)$ and $\int_{x_0}^{x_1} u_{xz}(x,0)\,\mathrm{d}x = \partial_z\delta_1(0) - \partial_z\delta_0(0)$, and thus the orthogonality conditions (2.19) are satisfied. □

Definition 2.9. *In the sequel, we denote by $\overline{\ell^j}$ the linear forms associated with the orthogonality conditions* (2.19) *for the linear shear flow problem, i.e., for $(f,\delta_0,\delta_1) \in \mathcal{H}$, we set*

$$\overline{\ell^j}(f,\delta_0,\delta_1) := \partial_z^j\delta_0(0) - \partial_z^j\delta_1(0) + \int_\Omega \partial_x f\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j}. \tag{2.28}$$

We now prove that the orthogonality conditions (2.19) are non-empty and independent.

Proposition 2.10 (Independence of the orthogonality conditions)**.** *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent over $C_c^\infty(\Omega)\times\{0\}\times\{0\} \subset \mathcal{H}$.*

Proof. Proceeding by contradiction, let $(c_0,c_1) \in \mathbb{R}^2$ such that, for every $f \in C_c^\infty(\Omega)$, there holds $c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0) = 0$. Then $\Phi^c := c_0\overline{\Phi^0} + c_1\overline{\Phi^1}$ satisfies $\int_\Omega \partial_x f\Phi^c = 0$ for every $f \in C_c^\infty(\Omega)$, so $\partial_x\Phi^c = 0$ in $\mathcal{D}'(\Omega_+)$. Since $\Phi^c(x_1,z) = 0$ for $z \in (0,1)$ and $\Phi^c \in Z^0(\Omega_+)$, this implies that $\Phi^c = 0$ in Ω_+ (since Z^0 functions have traces in the usual sense, see Lemma 1.7). The same holds in Ω_-. Hence $[\Phi^c]_{|z=0} = [\partial_z\Phi^c]_{|z=0} = 0$, which implies $c_0 = c_1 = 0$. □

Remark 2.11. *Proposition 2.10 of course implies that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\mathcal{H}$. Although Proposition 2.10 gives a prominent role to the source term f, we will actually also prove that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\{0\}\times C_c^\infty(\Sigma_0)\times C_c^\infty(\Sigma_1) \subset \mathcal{H}$. This property relies on the structure of the dual profiles $\overline{\Phi^j}$ near the points $(x_i,0)$, and will be proved at the end of this section (see Proposition 2.30).*

Similarly, it can be easily checked that the control of k derivatives in x requires the cancellation of $2k$ independent conditions. Although controlling a single x-derivative will be sufficient in the sequel to obtain our nonlinear result, we establish here this short higher-regularity statement as an illustration. More precisely, we have the following result.

Lemma 2.12. *Let $k \geq 1$. Let $f \in C^\infty(\overline{\Omega})$, $\delta_i \in C^\infty(\overline{\Sigma_i})$. Define recursively Δ_i^n for $0 \leq n \leq k$ and $z \in \Sigma_i$ by*

$$\Delta_i^0(z) := \delta_i(z), \tag{2.29}$$

$$\Delta_i^n(z) := \frac{1}{z}\left(\partial_x^{n-1} f(x_i,z) + \partial_{zz}\Delta_i^{n-1}(z)\right). \tag{2.30}$$

Assume that the following compatibility conditions are satisfied:

$$\forall n \in \{0, \cdots, k\}, \quad \Delta_0^n(1) = \Delta_1^n(-1) = 0. \tag{2.31}$$

Assume furthermore that for all $n \in \{0, \cdots k\}$, $\Delta_i^n \in \mathscr{H}_z^1(\Sigma_i)$.

Let u *be the unique solution to* (1.5). *Then* $u \in H_x^k H_z^2$ *if and only if the following orthogonality conditions are satisfied*

$$\overline{\ell^j}(\partial_x^n f, \Delta_0^n, \Delta_1^n) = 0, \quad \forall n \in \{0, \cdots, k-1\}, \quad j \in \{0, 1\}. \tag{2.32}$$

Furthermore, these $2k$ *orthogonality conditions are linearly independent.*

Proof. First, notice that $\partial_x^n u$ satisfies formally

$$\begin{cases} (z\partial_x - \partial_{zz})\partial_x^n u = \partial_x^n f \quad \text{in } \Omega, \\ \partial_x^n u_{|z=\pm 1} = 0, \\ \partial_x^n u_{|\Sigma_i} = \Delta_i^n. \end{cases}$$

The first part of the statement follows easily from Proposition 2.8 and Proposition 2.5 and from an induction argument.

Let us now check the independence of the orthogonality conditions. We extend the methodology used in the proof of Proposition 2.10. Assume that there exist $c_n^j \in \mathbb{R}$, $0 \leq n \leq k-1$, $j = 0, 1$ such that for all (f, δ_0, δ_1) satisfying the assumptions of the lemma,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, \Delta_0^n, \Delta_1^n) = 0.$$

In particular, for any $f \in C_c^\infty(\Omega)$,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, 0, 0) = 0,$$

i.e.

$$\sum_{n=0}^{k-1} \int_\Omega \partial_x^n f \left(\sum_{j=0,1} c_n^j \overline{\Phi^j} \right) = 0.$$

This means that

$$\sum_{j=0,1} \sum_{n=0}^{k-1} (-1)^n c_n^j \partial_x^n \overline{\Phi^j} = 0$$

in the sense of distributions. Since $[\partial_x^n \overline{\Phi^j}]_{|z=0} = [\partial_x^n \partial_z \overline{\Phi^j}]_{|z=0} = 0$ for $n \geq 1$, we infer that

$$\left[c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1} \right]_{|z=0} = \left[\partial_z (c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1}) \right]_{|z=0} = 0.$$

Once again, using the jump conditions on $\overline{\Phi^j}$, we deduce that $c_0^j = 0$, and thus

$$\partial_x \left(\sum_{j=0,1} \sum_{n=1}^{k-1} (-1)^n c_n^j \partial_x^{n-1} \overline{\Phi^j} = 0 \right) = 0.$$

It follows that

$$\sum_{j=0,1}\sum_{n=1}^{k-1}(-1)^n c_n^j \partial_x^{n-1}\overline{\Phi^j} = p(z)$$

for some function p. Note that by parabolic regularity, the profiles $\overline{\Phi^j}$ (and therefore the function p) are smooth away from the line $\{z=0\}$. Taking the trace of the above identity on $\{x_0\}\times(-1,0)\cup\{x_1\}\times(0,1)$, we find that $p=0$. Arguing by induction, we infer eventually that $c_n^j=0$ for all $0\le n\le k-1$, $j=0,1$. □

Corollary 2.13 (Biorthogonal basis)**.** *There exist $\Xi^k=(f^k,\delta_0^k,\delta_1^k)\in\mathcal{H}$ for $k\in\{0,1\}$ such that, for every $j,k\in\{0,1\}$,*

$$\overline{\ell^j}(\Xi^k)=\overline{\ell^j}(f^k,\delta_0^k,\delta_1^k)=\mathbf{1}_{j=k} \tag{2.33}$$

and such that, within $\mathcal{H}$,

$$\mathcal{H}_{\rm sg}^{\perp}:=\big(\mathbb{R}\Xi^0+\mathbb{R}\Xi^1\big)^{\perp}=\ker\overline{\ell^0}\cap\ker\overline{\ell^1} \tag{2.34}$$

is a vector subspace of codimension 2.

Proof. Since $\overline{\ell^0}$ and $\overline{\ell^1}$ are continuous linear forms on $\mathcal{H}$, by the Riesz representation theorem, they can be written as scalar products with two given triplets, say $\overline{\Xi^0},\overline{\Xi^1}\in\mathcal{H}$ which are linearly independent thanks to Proposition 2.10. Then one looks for $\Xi^k=(f^k,\delta_0^k,\delta_1^k)$ as $a_k\overline{\Xi^0}+b_k\overline{\Xi^1}$ where $a_k,b_k\in\mathbb{R}^2$ are such that $a_k\langle\overline{\Xi^j};\overline{\Xi^0}\rangle+b_k\langle\overline{\Xi^j};\overline{\Xi^1}\rangle=\mathbf{1}_{j=k}$. These systems can be solved since $\overline{\Xi^0}$ and $\overline{\Xi^1}$ are free. Moreover, this ensures (2.34) and their independence guarantees that $\mathcal{H}_{\rm sg}^{\perp}$ is of codimension 2 in $\mathcal{H}$. □

2.4 Hidden vertical regularity

The goal of this paragraph is to prove that, if u is a solution to (2.1) such that $\partial_{xzz}u\in L^2(\Omega)$, then one also has $\partial_z^5u\in L^2(\Omega)$, provided that the data is sufficiently regular. There is no additional orthogonality condition. We start with the straightforward claim that $\partial_z^4u\in L^2(\Omega)$.

Lemma 2.14. *Let $f\in L^2_xH^2_z$ and $\delta_0,\delta_1\in\mathscr{L}^2_z(-1,1)$. Let u be the unique weak solution to* (2.1)*. Assume that $u\in H^1_xH^2_z$. Then $u\in L^2_xH^4_z$ and*

$$\|u\|_{L^2_xH^4_z}\lesssim\|u\|_{H^1_xH^2_z}+\|f\|_{L^2_xH^2_z}. \tag{2.35}$$

Proof. Since $u\in H^1_xH^2_z$, one can integrate by parts in the weak formulation (2.2) by testing against smooth compactly supported functions, and prove that $\partial_{zz}u=z\partial_xu-f$ in $L^2(\Omega)$. Hence, in $\mathcal{D}'(\Omega)$, $\partial_z^4u=z\partial_{xzz}u+2\partial_{xz}u-\partial_{zz}f$. Thus $\partial_z^4u\in L^2(\Omega)$ and $\|u\|_{L^2_xH^4_z}\lesssim\|f\|_{L^2_xH^2_z}+\|u\|_{H^1_xH^2_z}$. □

Proposition 2.15. *Let $f\in L^2_xH^3_z$ and $\delta_0,\delta_1\in\mathscr{H}^1_z(-1,1)$. Let u be the unique weak solution to* (2.1)*. Assume that $u\in H^1_xH^2_z$ and $\partial_z^3\delta_i\in\mathscr{H}^1_z(\Sigma_i)$, with $\delta_0(1)=\delta_1(-1)=\Delta_0(1)=\Delta_1(-1)=0$. Assume furthermore that $\partial_x\partial_zf\in L^2((x_0,x_1)\times(1/2,1))\cap L^2((x_0,x_1)\times(-1,-1/2))$ and $\Delta_i\in H^2(\Sigma_i\cap\{|z|\ge1/2\})$. Then $\partial_z^5u\in L^2(\Omega)$ and*

$$\begin{aligned}\|u\|_{L^2_xH^5_z}\lesssim\|u\|_{H^1_xH^2_z}+\|f\|_{L^2_xH^3_z}+\|\partial_x\partial_zf\mathbf{1}_{|z|\ge1/2}\|_{L^2}\\ +\sum_{i\in\{0,1\}}\|\partial_z^3\delta_i\|_{\mathscr{H}^1_z}+\|\Delta_i\|_{H^2(\Sigma_i\cap\{|z|\ge1/2\})}.\end{aligned} \tag{2.36}$$

Proof. In the course of the proof, we will need to distinguish between different regions:

- One "interior" region, close to the line $z = 0$. In this region, we will prove that $\partial_z^3 u$ is such that $(z\partial_x - \partial_z^2)\partial_z^3 u \in L^2$, and use the results of Pagani to deduce that $\partial_z^5 u \in L^2$.
- Two "boundary" regions, in the vicinity of the lines $z = \pm 1$. In these regions, since z is bounded away from zero, we will use classical parabolic regularity arguments.

Step 1. *Interior region.* Let $\phi \in C^\infty(\overline{\Omega})$ such that $\phi = 0$ on $\{x_0\} \times [-1, 0]$, $\phi = 0$ on $\{x_1\} \times [0, 1]$ and ϕ vanishes identically on neighborhoods of $z = \pm 1$. Thanks to the regularity of u, we can multiply the PDE for u by $\partial_z^3 \phi$ and integrate over Ω. Vertical integrations by parts yield vanishing boundary terms because ϕ vanishes identically in neighborhoods of $z = \pm 1$. We proceed with care for the horizontal term so that all manipulations are licit:

$$\begin{aligned}\int_\Omega z\partial_x u\partial_z^3\phi &= \int_\Omega (2\partial_{xz}u + z\partial_{xzz}u)\partial_z\phi \\ &= \int_\Omega 2\partial_{xz}u\partial_z\phi + \int_{\Sigma_1} z\partial_{zz}\delta_1\partial_z\phi - \int_{\Sigma_0} z\partial_{zz}\delta_0\partial_z\phi - \int_\Omega z\partial_{zz}u\partial_{xz}\phi.\end{aligned} \tag{2.37}$$

Recalling that $u \in L^2_x H^4_z$ thanks to Lemma 2.14, we rewrite the last term as

$$\begin{aligned}-\int_\Omega z\partial_{zz}u\partial_{xz}\phi &= \int_\Omega z\partial_z^3 u\partial_x\phi + \int_\Omega \partial_{zz}u\partial_x\phi \\ &= \int_\Omega z\partial_z^3 u\partial_x\phi + \int_{\Sigma_1} \partial_{zz}\delta_1\phi - \int_{\Sigma_0} \partial_{zz}\delta_0\phi - \int_\Omega \partial_{xzz}u\phi.\end{aligned} \tag{2.38}$$

Hence,

$$\int_\Omega z\partial_x u\partial_z^3\phi = -3\int_\Omega \partial_{xzz}u\phi + \int_\Omega z\partial_z^3 u\partial_x\phi - \sum_{i\in\{0,1\}} (-1)^i \int_{\Sigma_i} \partial_{zz}\delta_i(z\partial_z\phi + \phi). \tag{2.39}$$

We also integrate by parts the boundary term. For example, on Σ_0:

$$-\int_{\Sigma_0} \partial_{zz}\delta_0 z\partial_z\phi = -[z\partial_{zz}\delta_0\phi_{|x=x_0}]_0^1 + \int_{\Sigma_0} (z\partial_z^3\delta_0 + \partial_{zz}\delta_0)\phi. \tag{2.40}$$

The pointwise term is null at $z = 1$ because ϕ vanishes identically near $z = 1$ and null at $z = 0$ since ϕ vanishes at 0 and $\partial_{zz}\delta_0 \in \mathscr{H}^1_z(\Sigma_0)$.

Eventually, this proves that

$$-\int_\Omega z(\partial_z^3 u)\partial_x\phi + \int_\Omega \partial_z(\partial_z^3 u)\partial_z\phi = \int_\Omega (\partial_z^3 f - 3\partial_{xzz}u)\phi + \sum_{i\in\{0,1\}} (-1)^i \int_{\Sigma_i} z\partial_z^3\delta_i\phi. \tag{2.41}$$

Since $u \in L^2_x H^4_z$ (by Lemma 2.14), $u \in H^1_x H^2_z$, $f \in L^2_x H^3_z$ and $\partial_z^3\delta_i \in \mathscr{H}^1_z(\Sigma_i)$, by density, this equality still holds for $\phi \in H^1(\Omega)$ such that $\phi = 0$ on $\{x_0\} \times [-1, 0]$, $\{x_1\} \times [0, 1]$ and $z = \pm 1$.

Now, let $\chi_0 \in C_c^\infty((-1, 1))$ such that $\chi_0 \equiv 1$ in a neighborhood of $z = 0$. The above argument shows that $\chi_0\partial_z^3 u \in L^2_x H^1_z$ is the unique weak solution to (2.1) with boundary data $\chi_0\partial_z^3\delta_i$ and source term $\chi_0(\partial_z^3 f - 3\partial_{xzz}u) - \chi_0''\partial_z^3 u - \chi_0'\partial_z^4 u \in L^2(\Omega)$. We infer from Proposition 2.4 that $\chi_0\partial_z^5 u \in L^2$. Note that thanks to the truncation χ_0, the compatibility conditions at $(x_0, 1)$ and $(x_1, -1)$ are automatically satisfied. By (2.8),

$$\|\chi_0\partial_z^5 u\|_{L^2_x H^5_z} \lesssim \|u\|_{H^1_x H^2_z} + \|f\|_{L^2_x H^3_z} + \sum_{i\in\{0,1\}} \|\partial_z^3\delta_i\|_{\mathscr{H}^1_z}. \tag{2.42}$$

Step 2. *Boundary regions.* By symmetry, we only treat the upper boundary region. We consider a function $\chi_1 \in C^\infty(\mathbb{R})$ such that $\chi_1 \equiv 1$ in a neighborhood of $z = 1$, and $\operatorname{supp}\chi_1 \subset (1/2, 1)$.

Then $u_1 := \chi_1 u$ is a solution to

$$\begin{cases} z\partial_x u_1 - \partial_{zz} u_1 = g_1, \quad \text{in } (x_0, x_1) \times (1/2, 1) \\ u_{1|z=1/2} = u_{1|z=1} = 0, \\ u_{1|x=x_0} = \chi_1 \delta_0, \end{cases} \tag{2.43}$$

where $g_1 = f\chi_1 - 2\chi_1' \partial_z u - \chi_1'' u$. This is a standard parabolic equation, for which we can apply classical regularity results. First, the assumptions on u and f ensure that $g_1 \in H^1_x H^1_z((x_0, x_1) \times (1/2, 1))$. Since the compatibility conditions $(\chi_1\delta_0)(1/2) = (\chi_1\delta_0)(1) = 0$ are satisfied, $\partial_x u_1$ is a solution to the parabolic equation

$$\begin{cases} z\partial_x\partial_x u_1 - \partial_{zz}\partial_x u_1 = \partial_x g_1, \quad \text{in } (x_0, x_1) \times (1/2, 1) \\ \partial_x u_{1|z=1/2} = \partial_x u_{1|z=1} = 0, \\ \partial_x u_{1|x=x_0} = z^{-1}(g_1 + \partial_{zz}(\chi_1\delta_0)) = \chi_1\Delta_0. \end{cases}$$

Now $\partial_x g_1 \in L^2$ and the compatibility conditions $(\chi_1\Delta_0)(1/2) = (\chi_1\Delta_0)(1) = 0$ are satisfied. Thus $\partial_x u_1 \in L^\infty_x H^1_z \cap L^2_x H^2_z \cap H^1_x L^2_z$. Differentiating the above equation twice with respect to z, we find that $\partial_x \partial_z^2 u_1$ now solves

$$\begin{cases} z\partial_x\partial_x\partial_z^2 u_1 - \partial_{zz}\partial_x\partial_z^2 u_1 = \partial_x\partial_z^2 g_1 - 2\partial_x^2\partial_z u_1, \quad \text{in } (x_0, x_1) \times (1/2, 1) \\ \partial_x\partial_z^2 u_{1|z=1/2} = 0, \quad \partial_x\partial_z^2 u_{1|z=1} = -\partial_x f_{|z=1}, \\ \partial_x\partial_z^2 u_{1|x=x_0} = \partial_z^2(\chi_1\Delta_0). \end{cases}$$

The right-hand side of the above equation belongs to $L^2_x H^{-1}_z(x_0, x_1) \times (1/2, 1)$, and therefore $\partial_x\partial_z^2 u_1 \in L^\infty_x(L^2_z) \cap L^2_x(H^1_z)$. Therefore, using equation (2.43), we deduce that $\partial_z^5 u_1 \in L^2$. □

2.5 Singular radial solutions in the half-plane and profile decomposition

In this subsection, we give a full description of the singularities that appear when the orthogonality conditions are not satisfied. We start by constructing singular solutions to the homogeneous equation set in the half-plane, using separation of variables in polar-like coordinates. We then localize these solutions near the critical points $(x_i, 0)$ to obtain the decomposition result of Theorem 2.

Our approach is similar to the one developed by Grisvard in [27, Section 4.4] for elliptic problems in polygonal domains (see in particular the singular profiles of equation (4.4.3.7) and the decomposition result of Theorem 4.4.3.7 therein). The main difference is that we cannot use usual polar coordinates and that the construction of the elementary singular profiles is much more technical than, for instance, the classical solution of the form $r^{\frac{1}{2}}\sin(\theta/2)$ which is involved in the resolution of Dirichlet-Neumann junctions as in the elliptic problem (1.8) mentioned in the introduction.

2.5.1 Construction of singular solutions in the half-plane

In this paragraph, we look for elementary singular radial solutions to the following problem without source-term in the half-plane:

$$\begin{cases} z\partial_x u - \partial_{zz} u = 0 & x \geq 0, z \in \mathbb{R}, \\ u(0, z) = 0 & z > 0. \end{cases} \tag{2.44}$$

Remark 2.16. *In [22], Fleming considered the related problem of finding a "fairly explicit formula" for solutions to $z\partial_x u - \partial_{zz} u = 0$ in a strip $(0,1)\times\mathbb{R}$, with prescribed boundary data at $x = 0$, $z > 0$ and $x = 1$, $z < 0$. His proof involves Whittaker functions, which are related to the confluent hypergeometric functions we use below.*

In [26], Gor'kov computes a representation formula for solutions to (2.44) with a non-zero source term and boundary data, and proves uniqueness of such solutions, under a growth assumption of the form $|u(0,z)| \lesssim |z|^\sigma$ for $0 \le \sigma < \frac{1}{2}$ on the line $x = 0$, for which he claims that uniqueness holds. The threshold $\sigma = \frac{1}{2}$ is precisely the scaling (at which uniqueness indeed breaks) of the first fundamental singular solution v_0 which we construct below.

Our setting is a little different from the works mentioned above, as we look for (non-zero) solutions to the homogeneous equation.

Near the point $(0,0)$ which is expected to be singular, balancing the terms $z\partial_x$ and ∂_{zz} leads to the natural scaling $z \sim x^{\frac{1}{3}}$. Thus, we introduce the following polar-like coordinates $(r,t) \in [0,+\infty)\times\mathbb{R}$:

$$r := (z^2 + x^{\frac{2}{3}})^{\frac{1}{2}} \quad \text{and} \quad t := zx^{-\frac{1}{3}} \tag{2.45}$$

The reverse change of coordinates is given by

$$x = \frac{r^3}{(1+t^2)^{\frac{3}{2}}} \quad \text{and} \quad z = \frac{rt}{(1+t^2)^{\frac{1}{2}}}. \tag{2.46}$$

Since it will be convenient to switch from cartesian coordinates (x,z) to the polar-like coordinates (r,t), we compute the Jacobian

$$J(r,t) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{3x^{\frac{1}{3}} r} & \frac{z}{r} \\ -\frac{t}{3x} & \frac{1}{x^{\frac{1}{3}}} \end{pmatrix} = \begin{pmatrix} \frac{(1+t^2)^{\frac{1}{2}}}{3r^2} & \frac{t}{(1+t^2)^{\frac{1}{2}}} \\ -\frac{t(1+t^2)^{\frac{3}{2}}}{3r^3} & \frac{(1+t^2)^{\frac{1}{2}}}{r} \end{pmatrix} \tag{2.47}$$

where we have used the equalities (2.46). In particular,

$$\det J(r,t) = \frac{(1+t^2)^2}{3r^3}, \tag{2.48}$$

which we will use to compute integrals using the (r,t) variables.

By (2.47), for any C^1 function φ,

$$\partial_x \varphi = \frac{(1+t^2)^{\frac{1}{2}}}{3r^2}\partial_r\varphi - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\partial_t\varphi, \tag{2.49}$$

$$\partial_z \varphi = \frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r\varphi + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t\varphi. \tag{2.50}$$

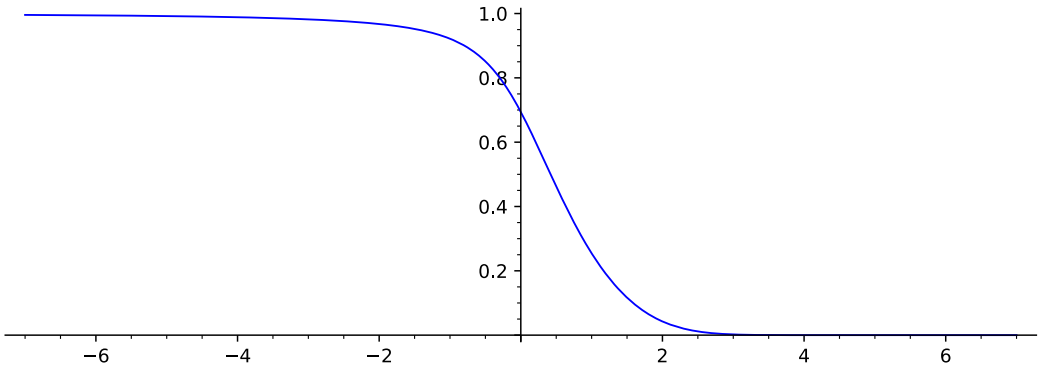


Figure 2: Plot of $t \mapsto \Lambda_0(t)$ for $t \in (-7, 7)$, highlighting the main properties: Λ_0 is a smooth, monotone decreasing function on $\mathbb{R}$, such that $\Lambda_0(-\infty) = 1$ and $\Lambda_0(+\infty) = 0$

In particular, if $u(r,t) = r^\lambda \Lambda(t)$,

$$z\partial_x u = \frac{r^{\lambda-2}}{3}\left[\lambda t\Lambda(t) - t^2(1+t^2)\partial_t\Lambda(t)\right], \tag{2.51}$$

$$\begin{aligned}\partial_{zz} u &= \left[\frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t\right]\left(r^{\lambda-1}\left(\frac{\lambda t}{(1+t^2)^{\frac{1}{2}}}\Lambda(t) + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\right) \\ &= r^{\lambda-2}\left[(\lambda-1)\left(\frac{\lambda t^2}{1+t^2}\Lambda(t) + t\partial_t\Lambda(t)\right) + (1+t^2)^{\frac{1}{2}}\partial_t\left(\frac{\lambda t}{(1+t^2)^{\frac{1}{2}}}\Lambda(t) + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\right].\end{aligned} \tag{2.52}$$

We are now ready to construct solutions to (2.44) using these coordinates.

Proposition 2.17. *For every $k \in \mathbb{Z}$, equation (2.44) has a solution of the form*

$$v_k := r^{\frac{1}{2}+3k}\Lambda_k(t) \tag{2.53}$$

with the variables (r,t) of (2.45) and $\Lambda_k \in C^\infty(\mathbb{R};\mathbb{R})$ is a smooth bounded function satisfying $\Lambda_k(-\infty) = 1$ and $\Lambda_k(+\infty) = 0$. The profile Λ_0 is presented in Fig. 2.

Proof. By separation of variables, we look for a solution to (2.44) under the form $u := r^\lambda\Lambda(t)$ where $\lambda \in \mathbb{R}$ and $\Lambda : \mathbb{R} \to \mathbb{R}$ is a smooth function. The boundary condition $u(0,z) = 0$ for $z > 0$ translates to $\Lambda(+\infty) = 0$. From (2.51) and (2.52) above, one checks that such a u satisfies $z\partial_x u - \partial_{zz} u = 0$ if and only if

$$\partial_t^2\Lambda(t) + \left(\frac{t^2}{3} + \frac{2\lambda t}{1+t^2}\right)\partial_t\Lambda(t) + \lambda\left(-\frac{1}{3}\frac{t}{1+t^2} + \frac{1+(\lambda-1)t^2}{(1+t^2)^2}\right)\Lambda(t) = 0. \tag{2.54}$$

To absorb the $(1+t^2)$ factors, we perform the change of unknown $\Lambda(t) =: (1+t^2)^{-\frac{\lambda}{2}}H(t)$. Then, Λ satisfies (2.54) if and only if H is a solution to

$$\partial_t^2 H(t) + \frac{t^2}{3}\partial_t H(t) - \frac{\lambda t}{3}H(t) = 0. \tag{2.55}$$

Moreover, for $t \neq 0$, using the change of variable $\zeta := -t^3/9$, and looking for $H(t) =: W(-t^3/9)$, we obtain that H solves (2.55) on $\mathbb{R} \setminus \{0\}$ if and only if W is a solution to

$$\zeta \partial_\zeta^2 W(\zeta) + \left(\frac{2}{3} - \zeta\right) \partial_\zeta W(\zeta) - \left(-\frac{\lambda}{3}\right) W(\zeta) = 0 \tag{2.56}$$

which corresponds to Kummer's equation, with $a = -\frac{\lambda}{3}$ and $b = \frac{2}{3}$. It is known (see [46, Section 13.2]) that (2.56) has a unique solution behaving like ζ^{-a} as $\zeta \to \infty$. This (complex valued) solution is usually denoted by $U(a, b, \zeta)$ and called *confluent hypergeometric function of the second kind*, or *Tricomi's function*. In general, U has a branch point at $\zeta = 0$. More precisely, the asymptotic ζ^{-a} holds in the region $|\arg \zeta| < \frac{3\pi}{2}$ and the principal branch of $U(a, b, \zeta)$ corresponds to the principal value of ζ^{-a}. Moreover, when b is not an integer, which is our case, one has (see [46, Equation 13.2.42]),

$$U(a, b, \zeta) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, \zeta) + \frac{\Gamma(b-1)}{\Gamma(a)} \zeta^{1-b} M(a-b+1, 2-b, \zeta), \tag{2.57}$$

where M is the *confluent hypergeometric function of the first kind* or *Kummer's function*,

$$M(a, b, \zeta) := \sum_{n \in \mathbb{N}} \frac{(a)_n}{(b)_n} \frac{\zeta^n}{n!}, \tag{2.58}$$

where $(a)_n$ and $(b)_n$ denote the rising factorial. In particular, M is an entire function of ζ. From (2.57), we see that the singularity in Tricomi's function U stems from the fractional power $\zeta^{1-b} = \zeta^{\frac{1}{3}}$. When $\zeta = -\rho$ (for $\rho > 0$), $\zeta^{\frac{1}{3}} = e^{\frac{i\pi}{3}} \rho^{\frac{1}{3}}$. Since we will evaluate this expression at $\zeta = -t^3/9$ and since we are interested in a real-valued solution to (2.55), we wish to choose another determination of the cubic root. We therefore set

$$W(\zeta) := \Re \left\{ e^{\frac{i\pi}{3}} U\left(-\frac{\lambda}{3}, \frac{2}{3}, \zeta\right) \right\}. \tag{2.59}$$

By linearity, W is still a solution to (2.56). Moreover, by [46, Equation 13.7.3], as $\zeta \to \infty$,

$$W(\zeta) = \Re \left\{ e^{\frac{i\pi}{3}} \zeta^{-a} \left(1 + O\left(\frac{1}{|\zeta|}\right)\right) \right\}. \tag{2.60}$$

In particular, when $\lambda = \frac{1}{2} + 3k$ for $k \in \mathbb{Z}$ (and only in this situation), as $\rho \to +\infty$,

$$W(-\rho) = O(\rho^{-a-1}), \tag{2.61}$$

because $\Re\{e^{i\pi/3} e^{-ia\pi} \rho^{-a}\} = \Re\{(-1)^k e^{i\pi/3} e^{i\pi/6} \rho^{-a}\} = (-1)^k \rho^{-a} \Re\{i\} = 0$. Defining $H(t) := W(-t^3/9)$ for W as in (2.59) and recalling that $\Lambda(t) = (1+t^2)^{-\lambda/2} H(t)$ implies that $\Lambda(+\infty) = 0$. Indeed, as $t \to +\infty$,

$$\Lambda(t) = (1+t^2)^{-\frac{\lambda}{2}} O\left(t^{3(\frac{\lambda}{3}-1)}\right) = O(t^{-3}). \tag{2.62}$$

Moreover, from (2.60), we obtain that Λ is bounded as $t \to -\infty$. Indeed, as $t \to -\infty$,

$$\begin{aligned} \Lambda(t) &= (1+t^2)^{-\frac{\lambda}{2}} \Re \left\{ e^{\frac{i\pi}{3}} \left(-\frac{t^3}{9}\right)^{-a} \left(1 + O\left(\frac{1}{|t|^3}\right)\right) \right\} \\ &= \frac{1}{2} 9^{-\frac{1}{6}-k} (1+t^2)^{-\frac{\lambda}{2}} |t|^{-3a} \left(1 + O(|t|^{-3})\right) = \frac{1}{2} 9^{-\frac{1}{6}-k} + O(|t|^{-2}). \end{aligned} \tag{2.63}$$

Eventually, let us check that H is an entire function of t, which will entail that Λ is smooth. Using (2.59) and (2.57),

$$H(t) = \frac{1}{2}\frac{\Gamma(1-b)}{\Gamma(a-b+1)} M\left(a, b, -\frac{t^3}{9}\right) - \frac{1}{2}\frac{t}{9^{1/3}}\frac{\Gamma(b-1)}{\Gamma(a)} M\left(a-b+1, 2-b, -\frac{t^3}{9}\right), \tag{2.64}$$

so that H is entire because M is. This also entails that H solves (2.55) even across $t = 0$. Moreover, (2.62) and (2.63) imply that Λ is bounded on $\mathbb{R}$. Eventually, using (2.63), we can define Λ_k as $2 \cdot 9^{\frac{1}{6}+k}\Lambda$, which ensures that $\Lambda_k(-\infty) = 1$. For this normalization, one deduces from (2.64) that

$$\Lambda_k(0) = 9^{\frac{1}{6}+k}\frac{\Gamma(1/3)}{\Gamma(1/6-k)}, \tag{2.65}$$

which will be used below. □

If u is a solution to (2.44), then, formally, $\partial_x u$ too (the operator $z\partial_x - \partial_{zz}$ commutes with ∂_x, and the boundary condition at $x = 0$ and $z > 0$ is satisfied thanks to the equation). This property entails that the solutions $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$ are related by a recurrence relation on the profiles Λ_k.

Lemma 2.18 (Recurrence relations)**.** *Let $k \in \mathbb{Z}$ and $c_k := \frac{1}{4} - 9k^2$. One has*

$$\partial_x v_k = c_k v_{k-1}. \tag{2.66}$$

Moreover, for every $t \in \mathbb{R}$,

$$c_k\Lambda_{k-1}(t) = \frac{(1+t^2)^{\frac{1}{2}}}{3}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - t(1+t^2)\Lambda_k'(t)\right), \tag{2.67}$$

or, equivalently,

$$\Lambda_k'(t) = \frac{1}{t(1+t^2)}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - \frac{3c_k\Lambda_{k-1}(t)}{(1+t^2)^{\frac{1}{2}}}\right). \tag{2.68}$$

Proof. By (2.49), one has $\partial_x v_k = r^{\frac{1}{2}+3(k-1)}H_k(t)$, where $H_k(t)$ is the right-hand side of (2.67). Thus $\partial_x v_k$ is a solution to (2.44) of the form studied in Proposition 2.17. Since the proof of Proposition 2.17 proceeds by equivalence, v_{k-1} is the only solution of the form $r^{\frac{1}{2}+3(k-1)}$. This entails that $H_k(t)$ is proportional to $\Lambda_{k-1}(t)$ and the constant can be identified by comparing the values at 0 using (2.65), yielding (2.67), (2.66) and (2.68) (which are all equivalent) with $c_k = \frac{1}{4} - 9k^2$.

Actually, these identities are linked with recurrence relations on Tricomi's function U. Let us give another proof of (2.68) using this approach. By the proof of Proposition 2.17,

$$\Lambda_k(t) = 2 \cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k} \cdot \Re\left\{e^{\frac{i\pi}{3}} U\left(-\frac{1}{6}-k, \frac{2}{3}, -\frac{t^3}{9}\right)\right\}. \tag{2.69}$$

First, using the relation $\partial_\zeta U(a-1, b, \zeta) = (1-a)U(a, b+1, \zeta)$ (see [46, Equation (13.3.22)]),

$$\begin{aligned}\Lambda_k'(t) = &-\left(\frac{1}{2}+3k\right)\frac{t}{1+t^2}\Lambda_k(t) \\ &+ 2 \cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k} \cdot \frac{3}{t}\left(k+\frac{1}{6}\right)\Re\left\{e^{\frac{i\pi}{3}}\left(-\frac{t^3}{9}\right)U\left(-\frac{1}{6}-k+1, \frac{5}{3}, -\frac{t^3}{9}\right)\right\}.\end{aligned} \tag{2.70}$$

Eventually, (2.68) follows from the relation $(b-a)U(a, b, \zeta) + U(a-1, b, \zeta) - \zeta U(a, b+1, \zeta) = 0$ (see [46, Equation (13.3.10)]). □

Remark 2.19. *We will see below that v_0 is linked with a solution to (2.1) which has Z^0 regularity, but does not belong to $H^1_x H^1_z$. Similarly, for each $k \geq 0$, v_k is linked with a solution u such that $\partial_x^k u \in Z^0(\Omega)$ but $u \notin H^{k+1}_x H^1_z$. Conversely, for $k = -1$, one could expect to be able to construct a very weak solution u based on v_{-1} which would entail that uniqueness fails for solutions with less than $L^2_x H^1_z$ regularity.*

Lemma 2.18 entails the following decay estimates, which will be useful in the sequel.

Lemma 2.20. *For every $k \in \mathbb{Z}$, there exists $C_k > 0$ such that, for every $t \in \mathbb{R}$,*

$$|\Lambda_k(t)| + |t^3 \Lambda_k'(t)| + |t^4 \Lambda_k''(t)| + |t^5 \Lambda_k'''(t)| \leq C_k. \tag{2.71}$$

Proof. For all $k \in \mathbb{Z}$, the bound $|\Lambda_k(t)| \leq C_k$ is already contained in Proposition 2.17 which claims that Λ_k is bounded. Since Λ_{k-1} and Λ_k are uniformly bounded over $\mathbb{R}$, we deduce from (2.68) that $t^3 \Lambda_k'(t)$ is also bounded on $\mathbb{R}$. Eventually, differentiating (2.68) with respect to t leads to a uniform bound for $|t^4 \Lambda_k''(t)|$ and $|t^5 \Lambda_k'''(t)|$ over $\mathbb{R}$. □

Moreover, the recurrence relations of Lemma 2.18 also imply that the solutions v_k to (2.44) are smooth, up to the boundary $\{x = 0\}$, except at the origin $(0, 0)$.

Lemma 2.21. *For every $k \in \mathbb{Z}$, $v_k \in C^\infty(P_*)$, where $P_* := ([0, +\infty) \times \mathbb{R}) \setminus \{(0, 0)\}$.*

Proof. The smoothness inside the half-plane $\{x > 0\}$ follows directly from Proposition 2.17 since $\Lambda_k \in C^\infty(\mathbb{R})$ and the function $r \mapsto r^{\frac{1}{2}+3k}$ as well as the change of coordinates of (2.45) are smooth inside this domain.

By Proposition 2.17, since Λ_k is continuous on $\mathbb{R}$ and has limits at $t = \pm\infty$, we obtain that $v_k = r^{\frac{1}{2}+3k} \Lambda_k(t)$ is continuous up to the boundary $\{x = 0\}$, except at the origin: $v_k \in C^0(P_*)$.

We now turn to the continuity of derivatives. Using (2.50),

$$\partial_z v_k = r^{-\frac{1}{2}+3k} \left[\left(\frac{1}{2} + 3k \right) \frac{t}{(1+t^2)^{\frac{1}{2}}} \Lambda_k(t) + (1+t^2)^{\frac{1}{2}} \Lambda_k'(t) \right]. \tag{2.72}$$

Since Λ_k has limits at $t = \pm\infty$ and since, by Lemma 2.20, $t^3 \Lambda_k'(t) = O(1)$, we obtain that $\partial_z v_k$ has limits at $t = \pm\infty$. Hence $\partial_z v_k \in C^0(P_*)$.

Eventually, the $C^\infty(P_*)$ regularity follows from a bootstrap argument. Indeed, by (2.66), $\partial_x v_k = c_k v_{k-1}$, so $\partial_x v_k \in C^0(P_*)$ because $v_{k-1} \in C^0(P_*)$. And, similarly, in the vertical direction, using (2.44), $\partial_{zz} v_k = z \partial_x v_k = z c_k v_{k-1}$ so $\partial_{zz} v_k \in C^0(P_*)$. Iterating the argument concludes the proof. □

2.5.2 Localization and decomposition

We now introduce singular profiles $\bar{u}^i_{\text{sing}}$, for $i = 0, 1$, localized in the vicinity of $(x_i, 0)$ and based on the singular profiles of the previous paragraph. Let $\chi_i \in C^\infty(\overline{\Omega})$ be a cut-off function such that $\chi_i \equiv 1$ in a neighborhood of $(x_i, 0)$, and $\operatorname{supp} \chi \subset B((x_i, 0), \bar{R})$ for some $\bar{R} < \min(1, x_1 - x_0)/2$. These localized profiles are the ones involved in the main decomposition result of Theorem 2.

Definition 2.22. *For $i \in \{0, 1\}$, let*

$$\bar{u}^i_{\text{sing}}(x, z) := r_i^{\frac{1}{2}} \Lambda_0(t_i) \chi_i(x, z), \tag{2.73}$$

where Λ_0 is constructed in Proposition 2.17 and

$$r_i := \left(z^2 + |x - x_i|^{\frac{2}{3}} \right)^{\frac{1}{2}} \quad \text{and} \quad t_i := (-1)^i z |x - x_i|^{-\frac{1}{3}}. \tag{2.74}$$

Lemma 2.23. *For $i \in \{0,1\}$, there exists $\overline{f_i} \in C^\infty(\overline{\Omega})$, with $\overline{f_i} \equiv 0$ in neighborhoods of $(x_i,0)$ and $\{z=\pm 1\}$, such that $\bar{u}^i_{\rm sing}$ is the unique solution with $Z^0(\Omega)$ regularity to*

$$\begin{cases} z\partial_x \bar{u}^i_{\rm sing} - \partial_{zz}\bar{u}^i_{\rm sing} = \overline{f_i}, \\ \bar{u}^i_{{\rm sing}|\Sigma_0\cup\Sigma_1} = 0, \\ \bar{u}^i_{{\rm sing}|z=\pm 1} = 0. \end{cases} \tag{2.75}$$

Moreover, $\bar{u}^i_{\rm sing} \in C^\infty(\overline{\Omega}\setminus\{(x_i,0)\})$ but $\bar{u}^i_{\rm sing} \notin H^1_x(H^1_z)$.

Proof. By symmetry, we only prove the statement for $\overline{f_0}$ and $\bar{u}^0_{\rm sing}$. In order to simplify the computations, we drop the index 0 in r_0, t_0 and χ_0. We introduce positive numbers $0 < r_- < r_+$ such that $\chi \equiv 1$ for $r \le r_-$ and $\chi \equiv 0$ for $r \ge r_+$. In particular, all derivatives of χ are smooth, bounded, and supported in $\mathbf{1}_{r_-<r<r_+}$.

Straightforward computations lead to (2.75), provided that one defines

$$\overline{f_0} := r^{\frac12}\Lambda_0(t)\,(z\partial_x\chi - \partial_{zz}\chi) - 2\partial_z(r^{\frac12}\Lambda_0(t))\partial_z\chi = v_0\,(z\partial_x\chi - \partial_{zz}\chi) - 2\partial_z v_0\partial_z\chi. \tag{2.76}$$

Since the derivatives of χ are supported away from the point $(x_0,0)$, the $C^\infty(\overline{\Omega})$ regularity of $\overline{f_0}$ follows directly from the smoothness of v_0 away from the origin proved in Lemma 2.21. Since $\bar{u}^0_{\rm sing}(x,z) = v_0(x,z)\chi(x,z)$, the $C^\infty(\overline{\Omega}\setminus\{(x_i,0)\})$ regularity of $\bar{u}^0_{\rm sing}$ follows from Lemma 2.21.

Therefore, to prove the lemma, there remains to prove that $\bar{u}^0_{\rm sing}$, $\partial_{zz}\bar{u}^0_{\rm sing}$ and $z\partial_x\bar{u}^0_{\rm sing}$ are in $L^2(\Omega)$ but $\partial_x\partial_z\bar{u}^0_{\rm sing} \notin L^2(\Omega)$. We will use the change of coordinates from cartesian to polar-like ones of Jacobian given by (2.48), so that, for $\varphi : \Omega \to \mathbb{R}$,

$$\|\varphi\|^2_{L^2(\Omega)} = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\varphi(r,t)^2\,{\rm d}t\,{\rm d}r. \tag{2.77}$$

In particular, we have the following integrability criterion. Assume that φ is of the form $r^\mu H(t)\psi$ where $H(t) = O_{t\to\pm\infty}(|t|)$ and $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. If $\mu > -2$ or $\operatorname{supp}\psi \subset \mathbf{1}_{r_-<r}$, then $\varphi \in L^2(\Omega)$.

Step 1. *Preliminary estimates.* Let ψ such that $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. By the previous integrability criterion, since $\Lambda_0(t) = O(1)$, $r^{\frac12}\Lambda_0(t)\psi \in L^2(\Omega)$. Using (2.50),

$$\partial_z\left(r^{\frac12}\Lambda_0(t)\right) = r^{-\frac12}\left[\frac{t}{2(1+t^2)^{\frac12}}\Lambda_0(t) + (1+t^2)^{\frac12}\Lambda_0'(t)\right]. \tag{2.78}$$

By Lemma 2.20, $|t|\Lambda_0'(t) = O(|t|^{-2})$. Thus, $\partial_z(r^{\frac12}\Lambda_0(t))\psi \in L^2(\Omega)$. Using (2.50) again,

$$\begin{aligned}\partial_{zz}\left(r^{\frac12}\Lambda_0(t)\right) = -\frac12 r^{-\frac32}\frac{t}{(1+t^2)^{\frac12}}&\left[\frac{t}{2(1+t^2)^{\frac12}}\Lambda_0(t) + (1+t^2)^{\frac12}\Lambda_0'(t)\right] \\ &+ r^{-\frac32}(1+t^2)^{\frac12}\partial_t\left[\frac{t}{2(1+t^2)^{\frac12}}\Lambda_0(t) + (1+t^2)^{\frac12}\Lambda_0'(t)\right].\end{aligned} \tag{2.79}$$

Using (2.49),

$$\partial_x\left(r^{\frac12}\Lambda_0(t)\right) = r^{-\frac52}\left[\frac{(1+t^2)^{\frac12}}{6}\Lambda_0(t) - \frac{t(1+t^2)^{\frac32}}{3}\Lambda_0'(t)\right]. \tag{2.80}$$

By Lemma 2.20, $\Lambda_0'(t) = O(|t|^{-3})$. Hence, $|t|\Lambda_0(t) = O(|t|)$ and $|t|^4\Lambda_0'(t) = O(|t|)$ so, assuming additionally that $\operatorname{supp}\psi \subset \mathbf{1}_{r_-<r<r_+}$, one concludes that $\partial_x(r^{\frac12}\Lambda_0(t))\psi \in L^2(\Omega)$.

Eventually, using (2.50),

$$\partial_{xz}(r^{\frac{1}{2}}\Lambda_0(t)) = r^{-\frac{7}{2}}\left(-\frac{t}{4}\Lambda_0(t) - \frac{1}{6}\Lambda_0'(t)(1+t^2)(1+3t^2) - \frac{t}{3}(1+t^2)^2\Lambda_0''(t)\right). \tag{2.81}$$

Step 2. *Z_0 estimates on $\bar{u}^0_{\rm sing}$.* By Step 1, $\bar{u}^0_{\rm sing}$ and $\partial_{zz}\bar{u}^0_{\rm sing}$ belong to $L^2(\Omega)$. Since $z\partial_x\bar{u}^0_{\rm sing} = \overline{f_0} + \partial_{zz}\bar{u}^0_{\rm sing}$ and $\overline{f_0} \in L^2(\Omega)$, we infer that $z\partial_x\bar{u}^0_{\rm sing} \in L^2(\Omega)$. Hence $\bar{u}^0_{\rm sing} \in Z^0(\Omega)$.

Step 3. *Lack of $H^1_xH^1_z$ estimate for $\bar{u}^0_{\rm sing}$.* Recalling (2.81),

$$\partial_x\partial_z\bar{u}^0_{\rm sing} = r^{-\frac{7}{2}}H(t)\chi + \partial_x(r^{\frac{1}{2}}\Lambda_0(t))\partial_z\chi + \partial_z(r^{\frac{1}{2}}\Lambda_0(t)\partial_x\chi), \tag{2.82}$$

where, by (2.81), one checks that $H(0) = -\Lambda_0'(0)/6 \neq 0$. The last two terms in the right-hand side belong to $L^2(\Omega)$ according to the previous computations. Since $H(0) \neq 0$, the L^2 norm of the first term is bounded from below by

$$c\int_0^{r_-} r^{-7}r^3\,\mathrm{d}r = +\infty. \tag{2.83}$$

and thus $\partial_x\partial_z\bar{u}^0_{\rm sing} \notin L^2(\Omega)$. □

Actually, we have the following regularity on the profiles $\bar{u}^i_{\rm sing}$, which is slightly better than Z^0.

Lemma 2.24. *For all $\sigma < \frac{1}{2}$, $\bar{u}^i_{\rm sing} \in H_x^{\frac{2+\sigma}{3}}L^2_z \cap L^2_xH^{2+\sigma}_z$.*

Proof. The proof follows from an easy scaling argument. We start with the z derivative and focus on $\bar{u}^0_{\rm sing}$. Dropping the index 0 in r_0 and t_0 as in the previous proof, we have, using (2.45) and Definition 2.22, and setting $\chi(x,z) := \chi_0(x_0+x,z)$,

$$\bar{u}^0_{\rm sing}(x_0+x,z) = x^{\frac{1}{6}}\varphi\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi(x_0,z),$$

where $\varphi(t) = (1+t^2)^{\frac{1}{4}}\Lambda_0(t)$. Therefore,

$$\partial_z^2\bar{u}^0_{\rm sing}(x_0+x,z) = x^{-\frac{1}{2}}\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi + 2x^{-\frac{1}{6}}\varphi'\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi_z + x^{\frac{1}{6}}\varphi\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi_{zz}. \tag{2.84}$$

We focus on the regularity of the first term, which is the most singular. We have, for any $\sigma > 0$,

$$\left\|x^{-\frac{1}{2}}\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi(x,z)\right\|^2_{L^2_xH^\sigma_z} \leq \int \frac{1}{x}\frac{1}{|z-z'|^{1+2\sigma}}\left(\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right) - \varphi''\left(\frac{z'}{x^{\frac{1}{3}}}\right)\right)^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}z'.$$

Changing variables in the above integral, we get

$$\left\|x^{-\frac{1}{2}}\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right)\chi(x,z)\right\|^2_{L^2_xH^\sigma_z} \lesssim \|\varphi''\|^2_{H^\sigma(\mathbb{R})}\int_0^{x_1-x_0}|x|^{-\frac{2}{3}-\frac{2\sigma}{3}}\,\mathrm{d}x.$$

The integral in the right-hand side is finite if and only if $\sigma < \frac{1}{2}$. Moreover, $\|\varphi''\|^2_{H^\sigma(\mathbb{R})} \leq \|\varphi''\|^2_{H^1(\mathbb{R})}$. From the definition of φ and the decay bounds of Lemma 2.20, we infer that $\varphi'' \in H^1(\mathbb{R})$. This shows that $\bar{u}^i_{\rm sing} \in L^2_xH^{2+\sigma}_z$ for $\sigma < \frac{1}{2}$.

The bound in $H_x^{\frac{2+\sigma}{3}}L^2_z$ is obtained similarly and left to the reader. □

Eventually, we introduce the following 2×2 nonsingular matrix which translates the fact that $\bar{u}^0_{\rm sing}$ and $\bar{u}^1_{\rm sing}$ are indeed independent elementary solutions related with the non-satisfaction of the orthogonality constraints $\overline{\ell^0}$ and $\overline{\ell^1}$. We will use this reference matrix multiple times in the sequel for perturbations of this shear flow situation.

Lemma 2.25. *Let $\overline{f_0}$, $\overline{f_1}$ as in Lemma 2.23 and $\overline{\Phi^0}$, $\overline{\Phi^1}$ as in Lemma 2.6. The matrix*

$$\overline{M} := \left(\int_\Omega \partial_x \overline{f_j}\ \overline{\Phi^i} \right)_{0 \le i,j \le 1} \in \mathcal{M}_2(\mathbb{R}) \tag{2.85}$$

is invertible.

Proof. Let $c \in \mathbb{R}^2$ such that $\overline{M}c = 0$. Then, for $j = 0, 1$,

$$\int_\Omega \partial_x (c_0 \overline{f_0} + c_1 \overline{f_1}) \overline{\Phi^j} = 0.$$

Thus, the source term for the function $c_0 \bar{u}^0_{\rm sing} + c_1 \bar{u}^1_{\rm sing}$ satisfies the orthogonality conditions (2.19) (note that in this case, the boundary data are null). It then follows from Proposition 2.8 that $c_0 \bar{u}^0_{\rm sing} + c_1 \bar{u}^1_{\rm sing} \in H^1_x H^1_z$. Localizing in the vicinity of $(x_i, 0)$, we infer that $c_i \bar{u}^i_{\rm sing} \in H^1_x H^1_z$, which, since $\bar{u}^i_{\rm sing} \notin H^1_x H^1_z$ (by Lemma 2.23), implies that $c_i = 0$. Therefore, $c = 0$ and $\overline{M}$ is invertible. □

Corollary 2.26 (Decomposition into singular profiles)**.** *Let $f \in H^1_x(L^2_z) \cap L^2_x H^3_z \cap H^1_x H^1_z(\{|z| \ge 1/2\})$ be arbitrary, and let $\delta_0, \delta_1 \in \mathscr{H}^1_z(-1,1)$ with $\delta_0(1) = \delta_1(-1) = 0$, and $\Delta_0, \Delta_1 \in \mathscr{H}^1_z(-1,1)$ with $\Delta_0(1) = \Delta_1(-1) = 0$. Assume furthermore that $\Delta_i \in H^2(\Sigma_i \cap \{|z| \ge 1/2\})$.*

Let $u \in Z^0(\Omega)$ be the unique weak solution to (2.1). Then there exists two real constants c_0, c_1 and a function $u_{\rm reg} \in Q^1$, as defined in (1.26), such that

$$u = c_0 \bar{u}^0_{\rm sing} + c_1 \bar{u}^1_{\rm sing} + u_{\rm reg}. \tag{2.86}$$

Proof. We recall the definition of the matrix $\overline{M}$ from Lemma 2.25. Since $\overline{M}$ is invertible, we may define $c = (c_0, c_1)$ such that

$$\overline{M}c = \begin{pmatrix} \overline{\ell^0}(f, \delta_0, \delta_1) \\ \overline{\ell^1}(f, \delta_0, \delta_1) \end{pmatrix}. \tag{2.87}$$

Let $\overline{f_0}$ and $\overline{f_1}$ as in Lemma 2.23. By Lemma 2.25 and construction, the triplet $(f - c_0\overline{f_0} - c_1\overline{f_1}, \delta_0, \delta_1)$ satisfies the orthogonality conditions from Proposition 2.8. It follows that the solution $u_{\rm reg}$ to

$$\begin{cases} z\partial_x u_{\rm reg} - \partial_{zz} u_{\rm reg} = f - c_0\overline{f_0} - c_1\overline{f_1} & \text{in } \Omega, \\ u_{{\rm reg}|\Sigma_i} = \delta_i, \\ u_{{\rm reg}|z=\pm 1} = 0 \end{cases} \tag{2.88}$$

satisfies $u_{\rm reg} \in H^1_x(H^1_z)$. Thus, estimate (2.11) of Proposition 2.5 ensures that $\partial_x u_{\rm reg} \in Z^0(\Omega)$. By Proposition 1.4, $u_{\rm reg} \in H^{5/3}_x L^2_z$. Proposition 2.15 entails that $u_{\rm reg} \in L^2_x H^5_z$ (note that with a suitable choice of the truncation, $\overline{f_0}$ and $\overline{f_1}$ vanish identically for $|z| \ge 1/2$). Thus, $u_{\rm reg} \in Q^1$, as defined in (1.26). Now, u and $u_{\rm reg} + c_0 \bar{u}^0_{\rm sing} + c_1 \bar{u}^1_{\rm sing}$ both belong to $Z^0(\Omega)$ and satisfy system (2.1). By the uniqueness result of Proposition 2.2, the result follows. □

Theorem 2 follows easily from Corollary 2.26. Indeed, if $(f, \delta_0, \delta_1) \in \mathcal{H}$, the regularity assumptions of Corollary 2.26 are satisfied (see (1.3)), together with the compatibility conditions in the corners $\delta_0(1) = \delta_1(-1) = \Delta_0(1) = \Delta_1(-1) = 0$ (recall that, in $\mathcal{H}$, $f = 0$ on $\partial\Omega$, thanks to (1.2)).

Remark 2.27. *The constants c_0, c_1 depend (linearly) on u, but do not depend on the choice of the truncation functions χ_i. Indeed, if χ_0', χ_1' is another truncation, associated with constants c_0', c_1', then applying Corollary 2.26 twice yields*

$$c_0\bar{u}^0_{\rm sing} + c_1\bar{u}^1_{\rm sing} - c_0'(\bar{u}^0_{\rm sing})' - c_1'(\bar{u}^1_{\rm sing})' \in Q^1.$$

Therefore, in a small neighborhood $V_i = \chi_i^{-1}(\{1\}) \cap (\chi_i')^{-1}(\{1\})$ of $(x_i, 0)$, we obtain

$$(c_i - c_i')r_i^{\frac{1}{2}}\Lambda_0(t_i) \in H^1_x H^1_z(V_i),$$

and therefore $c_i = c_i'$.

As already claimed in Remark 2.7, we can also prove a related decomposition result for the dual profiles $\overline{\Phi^j}$ defined in Lemma 2.6. Here, the decomposition *always* involves a singular part.

Corollary 2.28. *Let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$. There exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Phi_{\rm reg} \in Q^1$, as defined in (1.26), such that*

$$c_0\overline{\Phi^0} + c_1\overline{\Phi^1} = (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z) + d_0\bar{u}^0_{\rm sing}(x, -z) + d_1\bar{u}^1_{\rm sing}(x, -z) + \Phi_{\rm reg}, \tag{2.89}$$

where ζ is a smooth cut-off function, equal to 1 near $z = 0$ and compactly supported in $(-1, 1)$.

Proof. Using the same decomposition as in Lemma 2.6, set

$$\Psi^c := c_0\overline{\Phi^0} + c_1\overline{\Phi^1} - (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z). \tag{2.90}$$

Then $\widetilde{\Psi^c}(x, z) := \Psi^c(x, -z)$ is the solution to

$$\begin{cases} z\partial_x\widetilde{\Psi^c} - \partial_{zz}\widetilde{\Psi^c} = g_c & \text{in } \Omega, \\ \Psi^c(x_0, z) = 0 & \text{for } z \in (0, 1), \\ \Psi^c(x_1, z) = (-c_0 z - c_1)\zeta(-z) & \text{for } z \in (-1, 0), \\ \Psi^c_{|z=\pm 1} = 0, \end{cases} \tag{2.91}$$

where $g_c = (c_1\zeta''(-z) - 2c_0\zeta'(-z) + c_0 z\zeta''(-z))\mathbf{1}_{z<0}$. Thus, (2.89) follows from Corollary 2.26, applied with $f = g_c \in C^\infty(\overline{\Omega})$, $\delta_0 = \Delta_0 = 0$ and $\delta_1(z) = (-c_0 z - c_1)\zeta(-z)$ and $\Delta_1 = 0$.

It remains to prove that $(d_0, d_1) \neq (0, 0)$. By Proposition 2.8, $\widetilde{\Psi^c} \in H^1_x H^1_z$ if and only if $\overline{\ell^j}(g_c, 0, \delta_1) = 0$ for $j = 0, 1$. By Definition 2.9, since $\partial_x g_c = 0$ and $\Delta_0 = \Delta_1 = 0$,

$$\overline{\ell^j}(g_c, 0, \delta_1) = 0 \Longleftrightarrow \partial_z^j\delta_0(0) - \partial_z^j\delta_1(0) = 0. \tag{2.92}$$

Since $\delta_0 = 0$ and $\delta_1(0) = -c_1$ and $\delta_1'(0) = -c_0$, $(d_0, d_1) = (0, 0)$ if and only if $\widetilde{\Psi^c} \in H^1_x H^1_z$, if and only if $(c_0, c_1) = 0$. □

Remark 2.29. *Using Corollary 2.28 and the regularity result Lemma 2.24 on $\bar{u}^i_{\rm sing}$, we see that $f \mapsto \overline{\ell^j}(f, 0, 0) = \int_\Omega \partial_x f\overline{\Phi^j}$ is not only continuous on $H^1_x L^2_z$ but also on $H^\sigma_x L^2_z$ for every $\sigma > \frac{1}{6}$. We will recover a related threshold of tangential regularity in Section 5.3.*

Using the decomposition of the dual profiles, we can show that the orthogonality conditions are also independent when considering only variations of the inflow boundary data.

Proposition 2.30. *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are independent on $\{0\} \times C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_1)$.*

Proof. By contradiction, let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$ such that, for every $\delta_0 \in C_c^\infty(\Sigma_0)$ and $\delta_1 \in C_c^\infty(\Sigma_1)$,

$$c_0 \overline{\ell^0}(0, \delta_0, \delta_1) + c_1 \overline{\ell^1}(0, \delta_0, \delta_1) = 0. \tag{2.93}$$

Let $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Phi_{\mathrm{reg}} \in Q^1$ be given by Corollary 2.28. By symmetry, assume that $d_0 \neq 0$. Then, by Definition 2.9, for every $\delta_0 \in C_c^\infty(\Sigma_0)$,

$$\begin{aligned} 0 &= c_0 \overline{\ell^0}(0, \delta_0, 0) + c_1 \overline{\ell^1}(0, \delta_0, 0) \\ &= \int_{\Sigma_0} z \Delta_0 \left[(-c_0 z + c_1) \zeta(z) + d_0 \bar{u}^0_{\mathrm{sing}}(x_0, -z) + \Phi_{\mathrm{reg}} \right]. \end{aligned} \tag{2.94}$$

Let $\bar{z} > 0$ small enough, one can ensure that $\zeta \equiv 1$ on $(0, \bar{z})$ and $\bar{u}^0_{\mathrm{sing}}(x_0, -z) = z^{\frac{1}{2}} \Lambda_0(-\infty) = z^{\frac{1}{2}}$ (see Definition 2.22 and Proposition 2.17). Since $Q^1 \hookrightarrow H^1_x H^2_z$ (see Lemma 1.10), $\Phi_{\mathrm{reg}|\Sigma_0} \in H^2(\Sigma_0)$. If $\operatorname{supp} \delta_0 \subset (0, \bar{z})$ for $\bar{z} > 0$, using $z\Delta_0(z) = \delta_0''(z)$ and integrating by parts yields

$$0 = d_0 \int_0^1 \left[-\frac{1}{4} z^{-\frac{3}{2}} + \varphi(z) \right] \delta_0(z), \tag{2.95}$$

where $\varphi(z) := \partial_{zz} \Phi_{\mathrm{reg}}(x_0, z) \in L^2(\Sigma_0)$. Since $z \mapsto z^{-\frac{3}{2}}$ does not belong to $L^2(0, \bar{z})$ but φ does, one easily deduces that there exists $\delta_0 \in C_c^\infty((0, \bar{z}))$ such that the right-hand side is non-zero, reaching a contradiction. □

3 The linearized problem

The goal of this section is to establish the well-posedness of the linearized problem

$$\begin{cases} \bar{u} \partial_x u - \partial_{yy} u = f & \text{in } \Omega, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{3.1}$$

where $\bar{u} \in Q^1$ (see (1.26)) is a given perturbation of the linear shear flow, f is an external source term and (δ_0, δ_1) are lateral boundary data. It is fairly straightforward to adapt the theory of existence and uniqueness of weak solutions depicted in the previous section to the above equation. However, writing the orthogonality conditions for a general shear flow $\bar{u}$ is quite complicated. Indeed, we recall that the strategy is to find the equation solved by u_x in the upper region $\{\bar{u} > 0\}$ and in the lower region $\{\bar{u} < 0\}$, and to glue together these two solutions (provided the orthogonality conditions, which ensure the continuity of u and u_y across the line $\{\bar{u} = 0\}$, are satisfied). When the line $\{\bar{u} = 0\}$ is straight, this is a fairly simple process, which we described in the previous section. However, when $\{\bar{u} = 0\}$ is not a straight line, retrieving u from u_x is not entirely obvious (one needs to integrate u_x on curved lines).

Therefore, we have chosen to first straighten the flow $\bar{u}$ by changing the vertical coordinate. Of course, this introduces variable coefficients in the equation. We then prove the existence and uniqueness of weak solutions for the equation in the new coordinates, and exhibit orthogonality conditions, which are necessary and sufficient conditions ensuring that the weak solution has in fact $H^1_x H^1_y$ regularity. Eventually, we go back to the original variables and infer the existence of strong Z^1 solutions to (3.1) under orthogonality conditions.

3.1 A change of vertical coordinate

Throughout this section, we assume that $\bar{u}$ is a Q^1 function such that $\|\bar{u} - y\|_{Q^1}$ is small. In particular, $\|\bar{u}_y - 1\|_{L^\infty} \lesssim \|\bar{u} - y\|_{Q^1} \ll 1$ (see Lemma 1.10). It follows that there exists a line $y = \bar{y}(x)$ on which $\bar{u}$ vanishes, and $\bar{u} \lesseqgtr 0$ on $y \lesseqgtr \bar{y}(x)$. As a consequence, by the implicit function theorem, we define an associated change of variables Y such that

$$\forall z \in (-1,1), \quad \forall x \in (x_0, x_1), \quad \bar{u}(x, Y(x,z)) = z. \tag{3.2}$$

We then look for u under the form

$$u(x,y) = U(x, \bar{u}(x,y)), \tag{3.3}$$

so that $U = U(x,z)$ solves

$$\begin{cases} z\partial_x U + \gamma \partial_z U - \alpha \partial_{zz} U = g \quad \text{in } \Omega, \\ U_{|\Sigma_i} = \widetilde{\delta}_i, \\ U_{|y=\pm 1} = 0, \end{cases} \tag{3.4}$$

where

$$\alpha(x,z) := (\bar{u}_y)^2(x, Y(x,z)), \tag{3.5}$$
$$\gamma(x,z) := (z\bar{u}_x - \bar{u}_{yy})(x, Y(x,z)) \tag{3.6}$$
$$g(x,z) := f(x, Y(x,z)), \tag{3.7}$$
$$\widetilde{\delta}_i(z) := \delta_i(Y(x_i, z)). \tag{3.8}$$

In the sequel, we will often use the decomposition $\gamma = z\gamma_1 + \gamma_2$, where

$$\gamma_1(x,z) := \bar{u}_x(x, Y(x,z)) \quad \text{and} \quad \gamma_2(x,z) := -\bar{u}_{yy}(x, Y(x,z)). \tag{3.9}$$

The next subsections are devoted to the analysis of equation (3.4): existence and uniqueness of weak solutions, Z^0 regularity, orthogonality conditions for $H^1_x H^1_z$ regularity. More precisely, our results concerning (3.4) can be summarized as below. Eventually, in Section 3.5, we will translate these results in the original variables, and prove the conditional Q^1 regularity of solutions to (3.1).

Proposition 3.1. *Let $\alpha \in H^1_x H^1_z \cap H^{3/5}_x H^2_z$. Assume that there exists $\gamma_1 \in H^{2/3}_x L^2_z \cap L^\infty_z(H^{1/2}_x) \cap W^{1,\infty}_z(L^2_x)$ and $\gamma_2 \in H^1_x L^2_z \cap H^{3/5}_x H^1_z$ such that $\gamma = z\gamma_1 + \gamma_2$ and*

$$\begin{aligned} \|\alpha - 1\|_{H^1_x H^1_z} + \|\partial_{zz}\alpha\|_{H^{3/5}_x L^2_z} &\ll 1, \\ \|\gamma_1\|_{H^{2/3}_x L^2_z} + \|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\partial_z \gamma_1\|_{L^\infty_z(L^2_x)} &\ll 1, \\ \|\gamma_2\|_{H^1_x L^2_z} + \|\partial_z \gamma_2\|_{H^{3/5}_x L^2_z} &\ll 1. \end{aligned} \tag{3.10}$$

Then the following results hold:

- *For any $g \in L^2(\Omega)$, $\widetilde{\delta}_0, \widetilde{\delta}_1 \in \mathscr{H}^1_z(-1,1)$ such that $\widetilde{\delta}_0(1) = \widetilde{\delta}_1(-1) = 0$, equation (3.4) has a unique solution $U \in Z^0(\Omega)$, which satisfies*

$$\|U\|_{Z^0} \lesssim \|g\|_{L^2} + \|\widetilde{\delta}_0\|_{\mathscr{H}^1_z} + \|\widetilde{\delta}_1\|_{\mathscr{H}^1_z}. \tag{3.11}$$

- *There exist two independent linear forms $\ell^0_{\bar u}, \ell^1_{\bar u}$, defined on $\mathcal{H}$, such that the following result holds. Let $(f, \delta_0, \delta_1) \in \mathcal{H}$, and let $g, \widetilde{\delta}_0, \widetilde{\delta}_1$ be given by (3.7) and (3.8). Assume that $\widetilde{\delta}_0(1) = \widetilde{\delta}_1(-1) = 0$, and that*

$$\int_{\Sigma_i} \frac{1}{|z|} \left(f(x_i, Y(x_i, z)) + \delta_i''(Y(x_i, z))\right)^2 \, \mathrm{d}z < +\infty.$$

Let $U \in Z^0(\Omega)$ be the unique solution to (3.4). Then $U \in H^1_x H^1_z$ if and only if

$$\ell^0_{\bar u}(f, \delta_0, \delta_1) = \ell^1_{\bar u}(f, \delta_0, \delta_1) = 0,$$

and in this case

$$\|U\|_{H^1_x H^1_z} \lesssim \|g\|_{H^1_x L^2_z} + \|\widetilde{\delta_i}\|_{H^2} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left(f(x_i, Y(x_i, z)) + \delta_i''(Y(x_i, z))\right)^2 \, \mathrm{d}z \right)^{\frac{1}{2}}.$$

3.2 Existence and uniqueness of weak solutions

This section follows exactly the arguments of Section 2.1. The only slight difference lies in the derivation of the *a priori* estimates, in which we use smallness assumptions to treat perturbatively the additional drift term $\gamma \partial_z U$ and the commutator coming from the diffusion.

Definition 3.2 (Weak solution)**.** *Let $g \in L^2((x_0, x_1); H^{-1}(-1, 1))$ and $\widetilde{\delta}_0, \widetilde{\delta}_1 \in \mathscr{L}^2_z(-1, 1)$. Let $\alpha \in H^1_z(L^\infty_x)$, $\gamma \in L^2_z(L^\infty_x)$. We say that $U \in L^2((x_0, x_1); H^1_0(-1, 1))$ is a* weak solution *to (3.4) when, for all $V \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds:*

$$-\int_\Omega zU\partial_x V + \int_\Omega (\gamma + \alpha_z)\partial_z UV + \int_\Omega \alpha \partial_z U \partial_z V = \int_\Omega gV - \int_{\Sigma_1} z\widetilde{\delta}_1 V + \int_{\Sigma_0} z\widetilde{\delta}_0 V. \tag{3.12}$$

Proposition 3.3 (Existence and uniqueness of weak solutions)**.** *Assume that α and γ satisfy*

$$\|\alpha - 1\|_{L^\infty} + \|\alpha_z\|_{L^2_z(L^\infty_x)} + \|\gamma\|_{L^2_z(L^\infty_x)} \ll 1. \tag{3.13}$$

Then, for every $g \in L^2((x_0, x_1), H^{-1}(-1, 1))$ and $\widetilde{\delta}_1, \widetilde{\delta}_0 \in \mathscr{L}^2_z(-1, 1)$, there exists a unique weak solution $U \in L^2((x_0, x_1), H^1_0(-1, 1))$ to (3.4). Moreover,

$$\|U\|_{L^2_x(H^1_z)} \lesssim \|g\|_{L^2_x(H^{-1}_z)} + \|\widetilde{\delta}_0\|_{\mathscr{L}^2_z} + + \|\widetilde{\delta}_1\|_{\mathscr{L}^2_z}. \tag{3.14}$$

Proof. We mimic the proof of Proposition 2.2. We take $\mathscr{V} = \{V \in H^1(\Omega), V = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)\}$ and $\mathscr{U}$ the completion of $H^1(\Omega) \cap L^2((x_0, x_1); H^1_0(-1, 1))$ with respect to the scalar product

$$\langle U, V \rangle_{\mathscr{U}} := \int_\Omega \alpha \partial_z U \partial_z V + \frac{1}{2} \int_{\Sigma_0} zUV - \frac{1}{2} \int_{\Sigma_1} zUV. \tag{3.15}$$

For $(U, V) \in \mathscr{U} \times \mathscr{V}$, let

$$a(U, V) := -\int_\Omega zU\partial_x V + \int_\Omega (\gamma + \alpha_z)\partial_z UV + \int_\Omega \alpha \partial_z U \partial_z V, \tag{3.16}$$

$$b(V) := \int_\Omega gV - \int_{\Sigma_1} z\widetilde{\delta}_1 V + \int_{\Sigma_0} z\widetilde{\delta}_0 V. \tag{3.17}$$

Now, for any $V \in \mathscr{V}$, assuming that $\|\gamma\|_{L^2_z(L^\infty_x)} + \|\alpha_z\|_{L^2_z(L^\infty_x)} \leq 1/4$ and using the Poincaré inequality $\|V\|_{L^\infty_z(L^2_x)} \leq 2\|\partial_z V\|_{L^2(\Omega)}$,

$$a(V,V) = \|V\|^2_{\mathscr{U}} + \int_\Omega (\gamma + \alpha_z) V \partial_z V \geq \frac{1}{2}\|V\|^2_{\mathscr{U}}. \tag{3.18}$$

The linear form b satisfies (2.7). As in Proposition 2.2, the existence follows from the Lax-Milgram type existence principle Lemma B.2, and we obtain the energy estimate (3.14).

As in the proof of Proposition 2.2, uniqueness follows from the result by Baouendi and Grisvard, recalled in Appendix A. □

Remark 3.4. *Instead of using the weak Lax-Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation* (3.4) *by vanishing viscosity, and to obtain uniform $L^2_x(H^1_z)$ estimates on the approximation.*

3.3 Strong solutions with maximal regularity

In this paragraph, we adapt the results of [49] to construct solutions to (3.4) with Z^0 regularity, with estimates independent of the coefficients α and γ, in the regime (3.13).

Proposition 3.5. *Assume that α and γ satisfy* (3.13). *Then, for every $g \in L^2(\Omega)$ and every $\widetilde{\delta}_0, \widetilde{\delta}_1 \in \mathscr{H}^1_z(-1,1)$, the unique weak solution U to* (3.4) *satisfies $U \in Z^0(\Omega)$ with the estimate*

$$\|U\|_{Z^0} \lesssim \|g\|_{L^2} + \|\widetilde{\delta}_0\|_{\mathscr{H}^1_z} + \|\widetilde{\delta}_1\|_{\mathscr{H}^1_z}. \tag{3.19}$$

Proof. Thanks to Proposition 3.3, there exists $c_0 > 0$ such that, if

$$\|\alpha - 1\|_{L^\infty} + \|\alpha_z\|_{L^2_z(L^\infty_x)} + \|\gamma\|_{L^2_z(L^\infty_x)} \leq c_0, \tag{3.20}$$

the problem (3.4) is well-posed at the level of weak solutions. We proceed in four steps.

Step 1. *Case of smooth coefficients with a large zero-order term.* We start with coefficients α, γ that are smooth, say in $C^2(\overline{\Omega})$, satisfy (3.20), and we consider the following variant of (3.4):

$$\begin{cases} z\partial_x U + \gamma \partial_z U - \alpha \partial_{zz} U + C_0 U = h \quad \text{in } \Omega, \\ U_{|\Sigma_i} = 0, \\ U_{|y=\pm 1} = 0, \end{cases} \tag{3.21}$$

where $C_0 \geq \frac{1}{2}\alpha_{zz} + \frac{1}{2}\gamma_z$. By Pagani [49, Theorem 5.2] (for the operator $z\partial_x + \gamma\partial_z - \alpha\partial_{zz} + C_0$), for every $h \in L^2(\Omega)$, there exists a unique $U \in Z^0(\Omega)$ solution to (3.21) and a constant C (possibly depending on α, γ and C_0 in a way that is not entirely explicit in the work of Pagani) such that

$$\|U\|_{Z^0} \leq C\|h\|_{L^2}. \tag{3.22}$$

Thus, we can define the bounded linear operator $\mathcal{K} : L^2(\Omega) \to Z^0(\Omega) \hookrightarrow L^2(\Omega)$ which maps h to U, the solution to (3.21). Moreover, $\mathcal{K} \in \mathcal{L}(L^2(\Omega))$ is compact since $Z^0(\Omega) \hookrightarrow H^{2/3}_{x,z}(\Omega)$ by Proposition 1.4.

Step 2. *Case of smooth coefficients.* We still consider coefficients α, γ that are smooth, satisfy (3.20), and we consider the equation

$$\begin{cases} z\partial_x U + \gamma \partial_z U - \alpha \partial_{zz} U = h \quad \text{in } \Omega, \\ U_{|\Sigma_i} = 0, \\ U_{|y=\pm 1} = 0. \end{cases} \tag{3.23}$$

Applying Fredholm's alternative to the operator $\mathcal{K}$ we obtain that

- either, for every $h \in L^2$, there exists a unique $U \in Z^0(\Omega)$ solution to (3.23),
- or there exists a nontrivial solution $U \in Z^0(\Omega)$ to (3.23) with $h = 0$.

The second possibility is excluded by the uniqueness of weak solutions stated in Proposition 3.3. Rewriting (3.23) as

$$z\partial_x U - \partial_{zz} U = h - \gamma \partial_z U + (\alpha - 1)\partial_{zz} U \tag{3.24}$$

and applying [49, Estimate (5.13)] (this time to the universal operator $z\partial_x - \partial_{zz}$), we obtain

$$\|U\|_{Z^0} \leq C_{\text{Pagani}} \left(\|h\|_{L^2} + \|\gamma\|_{L^2_z(L^\infty_x)} \|\partial_z U\|_{H^1_z(L^2_x)} + \|\alpha - 1\|_\infty \|\partial_{zz} U\|_{L^2} \right). \tag{3.25}$$

Hence, under condition (3.20) (up to choosing $c_0 < 1/(2C_{\text{Pagani}})$), the last two terms can be treated perturbatively and we obtain

$$\|U\|_{Z^0} \lesssim \|h\|_{L^2}, \tag{3.26}$$

with a constant depending only on the domain Ω.

Step 3. *Case of smooth coefficients with boundary data.* We still consider coefficients α, γ that are smooth and satisfy (3.20). By [49, Theorem 2.1], there exists a bounded linear map $L : \mathscr{H}^1_z(\Sigma_0) \times \mathscr{H}^1_z(\Sigma_1) \to Z^0(\Omega)$ such that $U_\delta := L(\widetilde{\delta}_0, \widetilde{\delta}_1)$ satisfies $(U_\delta)_{|\Sigma_i} = \widetilde{\delta}_i$ and $(U_\delta)_{|y=\pm 1} = 0$. Then we look for a solution U to (3.4) under the form $U = U_\delta + V$, where V is a solution to (3.23) with

$$h := g - z\partial_x U_\delta + \alpha \partial_{zz} U_\delta - \gamma \partial_z U_\delta. \tag{3.27}$$

Under assumption (3.20),

$$\begin{aligned} \|h\|_{L^2} &\leq \|g\|_{L^2} + \|z\partial_x U_\delta\|_{L^2} + \|\alpha\|_\infty \|\partial_{zz} U_\delta\|_{L^2} + \|\gamma\|_{L^2_z(L^\infty_x)} \|\partial_z U\|_{L^\infty_z(L^2_x)} \\ &\lesssim \|g\|_{L^2} + \|U_\delta\|_{Z^0} \end{aligned} \tag{3.28}$$

so we obtain that U has Z^0 regularity and satisfies (3.19) by boundedness of L.

Step 4. *Case of general coefficients.* We then address the case of general (non-smooth) coefficients satisfying (3.20). We take a smooth approximation sequence (α^n, γ^n) of (α, γ), which satisfies the same smallness assumptions and which converges towards (α, γ) in $L^\infty \cap H^1_z(L^\infty_x) \times L^2_z(L^\infty_x)$. For the sequence (α^n, γ^n), we construct a sequence of solutions $U^n \in Z^0(\Omega)$, which satisfy the estimate (3.19) with uniform bounds. Extracting a subsequence if necessary, we can find a function $U \in Z^0(\Omega)$ such that $U^n \rightharpoonup U$ in $Z^0(\Omega)$. Passing to the limit in the equation, it can be easily checked that U is a solution to (3.4). By Proposition 3.3, it is in fact the unique weak solution to (3.4), which completes the proof. □

3.4 Orthogonality conditions for higher tangential regularity

As in Section 2.3, we build solutions to (3.4) with higher regularity in the tangential direction, provided that the data satisfy appropriate orthogonality conditions. The main goal of this paragraph is to derive suitable expressions for these orthogonality conditions, analogous to the linear shear flow case.

3.4.1 Derivation of the equations for the dual profiles

We start by formally differentiating (3.4) with respect to x and we find that, if U is regular in the tangential direction, then $V := \partial_x U$ is a solution to

$$\begin{cases} z\partial_x V + \gamma\partial_z V - \alpha\partial_{zz} V - \alpha_x \partial_{zz}\int_{x_0}^{x} V + \gamma_x \partial_z \int_{x_0}^{x} V = h_0 & \text{in } \Omega_+, \\ z\partial_x V + \gamma\partial_z V - \alpha\partial_{zz} V + \alpha_x \partial_{zz}\int_{x}^{x_1} V - \gamma_x \partial_z \int_{x}^{x_1} V = h_1 & \text{in } \Omega_-, \\ [V]_{z=0} = [\partial_z V]_{z=0} = 0 & \text{on } (x_0, x_1), \\ V(x_0, z) = \widetilde{\Delta}_0 & \text{for } z \in (0,1), \\ V(x_1, z) = \widetilde{\Delta}_1 & \text{for } z \in (-1,0), \\ V(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1), \end{cases} \tag{3.29}$$

where, for $i \in \{0,1\}$,

$$h_i := \partial_x g + \alpha_x \partial_{zz}\widetilde{\delta}_i - \gamma_x \partial_z \widetilde{\delta}_i, \tag{3.30}$$

and

$$\widetilde{\Delta}_i(z) := \frac{1}{z}\left(g(x_i, z) + \alpha(x_i, z)\partial_{zz}\widetilde{\delta}_i(z) - \gamma(x_i, z)\partial_z \widetilde{\delta}_i(z) \right). \tag{3.31}$$

Reciprocally, assuming that the above system has a unique weak solution V (in a sense that will be made explicit later on), then U_1 defined by

$$U_1 := \begin{cases} \widetilde{\delta}_0 + \int_{x_0}^{x} V & \text{in } \Omega_+, \\ \widetilde{\delta}_1 + \int_{x_1}^{x} V & \text{in } \Omega_- \end{cases} \tag{3.32}$$

is a solution to (3.4) if and only if V satisfies

$$\begin{aligned} \int_{x_0}^{x_1} V(x,0)\,\mathrm{d}x &= \widetilde{\delta}_1(0) - \widetilde{\delta}_0(0), \\ \int_{x_0}^{x_1} \partial_z V(x,0)\,\mathrm{d}x &= \partial_z \widetilde{\delta}_1(0) - \partial_z \widetilde{\delta}_0(0). \end{aligned} \tag{3.33}$$

For the time being, we do not worry about the regularity of the coefficients, and perform all computations as if the coefficients were smooth. A suitable definition of weak solutions to (3.29), which makes sense at the level of regularity available for the coefficients α and γ, will be given in Definition 3.13. Taking any function φ, sufficiently smooth on Ω_+ and Ω_- (but not necessarily continuous across the line $z = 0$) and vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the weak formulation of the

above system is

$$\begin{aligned}
&\int_{\Omega_+} h_0\varphi + \int_{\Omega_-} h_1\varphi - \int_{-1}^{0} z\widetilde{\Delta}_1(z)\varphi(x_1,z)\,\mathrm{d}z + \int_0^1 z\widetilde{\Delta}_0(z)\varphi(x_0,z)\,\mathrm{d}z \\
&\quad = \int_{\Omega_+\cup\Omega_-} V(-z\partial_x\varphi - \partial_z(\gamma\varphi) - \partial_{zz}(\alpha\varphi)) \\
&\quad - \int_{\Omega_+} V\left(\partial_{zz}\int_x^{x_1}\alpha_x\varphi + \partial_z\int_x^{x_1}\gamma_x\varphi\right) + \int_{\Omega_-} V\left(\partial_{zz}\int_{x_0}^{x}\alpha_x\varphi + \partial_z\int_{x_0}^{x}\gamma_x\varphi\right) \\
&\quad + \int_{x_0}^{x_1}\partial_z V(x,0)\Big(\int_x^{x_1}\alpha_x(x',0)\varphi(x',0^+)\,\mathrm{d}x' + \int_{x_0}^{x}\alpha_x(x',0)\varphi(x',0^-)\,\mathrm{d}x' \\
&\qquad\qquad + \alpha(x,0)\left(\varphi(x,0^+) - \varphi(x,0^-)\right)\Big)\,\mathrm{d}x \\
&\quad - \int_{x_0}^{x_1} V(x,0)\Big(\int_x^{x_1}\left(\partial_z(\alpha_x\varphi)(x',0^+) + (\gamma_x\varphi)(x',0^+)\right)\,\mathrm{d}x' \\
&\qquad\qquad + \int_{x_0}^{x}\left(\partial_z(\alpha_x\varphi)(x',0^-) + (\gamma_x\varphi)(x',0^-)\right)\,\mathrm{d}x' \\
&\qquad\qquad + \left(\partial_z(\alpha\varphi)(x,0^+) - \partial_z(\alpha\varphi)(x,0^-)\right) + (\gamma\varphi)(x,0^+) - (\gamma\varphi)(x,0^-)\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.34}$$

Following the reasoning of Section 2.3, this leads to the following generalization of Lemma 2.6.

Definition 3.6 (Dual profiles)**.** *We define Φ^0 and Φ^1 as the weak solutions to*

$$\begin{cases}
-z\partial_x\Phi^j - \partial_z(\gamma\Phi^j) - \partial_{zz}(\alpha\Phi^j) - \partial_{zz}\int_x^{x_1}\alpha_x\Phi^j - \partial_z\int_x^{x_1}\gamma_x\Phi^j = 0 & \text{in } \Omega_+, \\
-z\partial_x\Phi^j - \partial_z(\gamma\Phi^j) - \partial_{zz}(\alpha\Phi^j) + \partial_{zz}\int_{x_0}^{x}\alpha_x\Phi^j + \partial_z\int_{x_0}^{x}\gamma_x\Phi^j = 0 & \text{in } \Omega_-, \\
\Phi^j(x_0,z) = 0 & \text{for } z\in(-1,0), \\
\Phi^j(x_1,z) = 0 & \text{for } z\in(0,1), \\
\Phi^j(x,\pm1) = 0 & \text{on } (x_0,x_1),
\end{cases} \tag{3.35}$$

together with the jump conditions

$$[\alpha\Phi^j]_{z=0}(x) + \int_x^{x_1}\alpha_x(x',0)\Phi^j(x',0^+)\,\mathrm{d}x' + \int_{x_0}^{x}\alpha_x(x',0)\Phi^j(x',0^-)\,\mathrm{d}x' = \mathbf{1}_{j=1} \tag{3.36}$$

and

$$\begin{aligned}
[\partial_z(\alpha\Phi^j) + \gamma\Phi^j]_{z=0}(x) &+ \int_x^{x_1}\left(\partial_z(\alpha_x\Phi^j) + \gamma_x\Phi^j\right)(x',0^+)\,\mathrm{d}x' \\
&+ \int_{x_0}^{x}\left(\partial_z(\alpha_x\Phi^j) + \gamma_x\Phi^j\right)(x',0^-)\,\mathrm{d}x' = -\mathbf{1}_{j=0}.
\end{aligned} \tag{3.37}$$

In the next paragraphs, we prove the existence and uniqueness of solutions to these dual systems, in a suitable sense. We emphasize in particular that our assumptions on the coefficient γ_1 do not ensure that $\partial_x\gamma_1 \in L^1_{\mathrm{loc}}(\Omega)$, for instance. Therefore, we will take special care to define rigorously the integral $\int_x^{x_i}\gamma_x\Phi^j$.

3.4.2 Preliminary observations

As emphasized above, since the coefficient γ_1 has low regularity, the meaning of equation (3.35) is not so clear at this stage. Therefore, we shall start with some preliminary remarks that will allow us to define the profiles Φ^j is a rigorous fashion. We will introduce in particular some notations and tools which will be used throughout this section.

- **Unknown Θ^j:**

  First, let us consider the unknown Θ^j defined by

$$\Theta^j := \begin{cases} \alpha\Phi^j + \int_x^{x_1} \alpha_x \Phi^j & \text{in } \Omega_+, \\ \alpha\Phi^j - \int_{x_0}^{x} \alpha_x \Phi^j & \text{in } \Omega_-, \end{cases} \tag{3.38}$$

  Formally, $\partial_x \Theta^j = \alpha \partial_x \Phi^j$, so that Φ^j can be retrieved from Θ^j thanks to the inversion formula

$$\Phi^j := \begin{cases} \dfrac{\Theta^j}{\alpha} - \displaystyle\int_x^{x_1} \frac{\alpha_x}{\alpha^2}\Theta^j & \text{in } \Omega_+, \\ \dfrac{\Theta^j}{\alpha} + \displaystyle\int_{x_0}^{x} \frac{\alpha_x}{\alpha^2}\Theta^j & \text{in } \Omega_-. \end{cases} \tag{3.39}$$

  It follows in particular that $\|\Phi^j\|_{L^2} \lesssim \|\Theta^j\|_{L^2}$ and $\|\partial_z \Phi^j\|_{L^2(\Omega_\pm)} \lesssim \|\partial_z \Theta^j\|_{L^2(\Omega_\pm)}$ thanks to the $L^\infty \cap H^1_x H^1_z$ bound on $\alpha - 1$ of (3.10).

- **Operator G:**

  When $\gamma \in H^1_x(L^2_z)$, setting $\Gamma := \gamma/\alpha$, it will be convenient to define the operator

$$G[\Theta] := \Gamma\Theta + \begin{cases} \int_x^{x_1} \Gamma_x \Theta & \text{in } \Omega_+, \\ -\int_{x_0}^{x} \Gamma_x \Theta & \text{in } \Omega_-. \end{cases} \tag{3.40}$$

  We will generalize in Lemma 3.8 the definition of the operator G when γ merely satisfies the assumptions of Proposition 3.1. Formally, $\partial_x G[\Theta^j] = \gamma \partial_x \Phi^j$ and $G[\Theta^j] = \gamma\Phi^j + \int_x^{x_i} \gamma_x \Phi^j$.

  Therefore, equation (3.35) can be written as

$$\begin{cases} -\frac{z}{\alpha}\partial_x \Theta^j - \partial_z G[\Theta^j] - \partial_{zz}\Theta^j = 0 & \text{in } \Omega_+ \cup \Omega_-, \\ \Theta^j(x_0, z) = 0 & \text{for } z \in (-1, 0), \\ \Theta^j(x_1, z) = 0 & \text{for } z \in (0, 1), \end{cases} \tag{3.41}$$

  while the jump conditions (3.36)-(3.37) become

$$\begin{aligned} [\Theta^j]_{|z=0} &= \mathbf{1}_{j=1}, \\ \left[\partial_z \Theta^j + G[\Theta^j]\right]_{|z=0} &= -\mathbf{1}_{j=0}. \end{aligned} \tag{3.42}$$

Remark 3.7. *Under the assumptions of Proposition 3.1, we can define $\Gamma := z\Gamma_1 + \Gamma_2$, with $\Gamma_k := \gamma_k/\alpha$, and Γ_1, Γ_2 satisfy the same bounds (3.10) as γ_1, γ_2. For instance, the $L^\infty_z(H^{1/2}_x)$ bound for Γ_1 stems from the same bound for γ_1, the $L^\infty_z(L^2_x)$ bound for α_x and the pointwise multiplication result Lemma B.3.*

- **Lifts for Θ^j:**

  It will be convenient, in the course of the proof, to introduce a lift for Θ^j in order to remove some jumps across the line $\{z=0\}$. More precisely, we introduce a lift which will remove the jump of Θ^j and of $G[\Theta^j]+\partial_z\Theta^j$ (but which will authorize a jump in the z-derivative of the lifted function). Other lifts are also considered in Appendix C.1. We write

$$\Theta^j = (\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\mathbf{1}_{\Omega_+} + \Theta^j_\sharp,$$

  where $\chi \in C_c^\infty((-1,1))$ is such that $\chi \equiv 1$ in a neighborhood of zero, so that $\Theta^j_\sharp$ is such that

$$[\Theta^j_\sharp]_{|z=0} = 0, \quad [\partial_z\Theta^j_\sharp + G[\Theta^j]]_{|z=0} = 0. \tag{3.43}$$

  Note that

$$\begin{aligned} G[\Theta^j] &= G\left[(\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\mathbf{1}_{\Omega_+}\right] + G[\Theta^j_\sharp] \\ &= \mathbf{1}_{\Omega_+}\Gamma(x_1,z)(\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z) + G[\Theta^j_\sharp]. \end{aligned}$$

  Therefore, the lifted function $\Theta^j_\sharp$ satisfies, in $\Omega_\pm$,

$$\begin{aligned} -\frac{z}{\alpha}\partial_x\Theta^j_\sharp - \partial_z(\partial_z\Theta^j_\sharp + G[\Theta^j_\sharp]) &= \mathbf{1}_{z>0}\partial_{zz}\left((\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\right) \\ &\quad + \partial_z\left(\mathbf{1}_{z>0}(\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\Gamma(x_1,z)\right), \\ \Theta^j_{\sharp|z=\pm1} &= 0, \\ \Theta^j_\sharp(x_0,z) &= 0 \quad \forall z \in (-1,0), \\ \Theta^j_\sharp(x_1,z) &= -\left(\mathbf{1}_{j=1} - z\mathbf{1}_{j=0}\right)\chi(z) \quad \forall z \in (0,1). \end{aligned} \tag{3.44}$$

We conclude this preliminary step by a result providing some useful bounds on the operator G. We recall the definition of the functional space $\mathcal{B} := \{u \in L^2_x(H^1_0);\ z\partial_x u \in L^2_x(H^{-1}_z)\}$ (see (1.28)), whose properties are detailed in Appendix A. We set

$$\|\Theta\|_{\mathcal{B}} := \|\partial_z\Theta\|_{L^2} + \|z\partial_x\Theta\|_{L^2_x(H^{-1}_z)}.$$

Lemma 3.8. *Let $\Theta \in \mathcal{B}$, and let $\eta_0, \eta_1 \in H^1_0(-1,1)$. Assume that $\Theta(x_0,z) = \eta_0(z)$ for all $z<0$, $\Theta(x_1,z) = \eta_1(z)$ for all $z>0$. Let $\alpha \in H^1_xH^1_z$ such that $\|\alpha - 1\|_{H^1_xH^1_z} \ll 1$. Let $\gamma \in H^1$. Then*

$$\begin{aligned} \|G[\Theta]\|_{L^2} \lesssim \left(\|\Gamma_2\|_{H^1_xL^2_z} + \|\Gamma_1\|_{L^2_z(L^\infty_x)}\right)\left(\|\partial_z\Theta\|_{L^2} + \|\eta_0'\|_{L^2} + \|\eta_1'\|_{L^2}\right) \\ + \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}(\|\Theta\|_{\mathcal{B}} + \|\eta_0\|_{H^1} + \|\eta_1\|_{H^1}) \\ + (\|\Gamma(x_1,\cdot)\|_{L^2(0,1)} + \|\Gamma\|_{L^2(\Omega_+)})\|\eta_1\|_{H^1} \\ + (\|\Gamma(x_0,\cdot)\|_{L^2(-1,0)} + \|\Gamma\|_{L^2(\Omega_-)})\|\eta_0\|_{H^1}. \end{aligned} \tag{3.45}$$

As a consequence, the linear application

$$\gamma \in H^1 \mapsto G[\Theta] \in L^2$$

can be extended in a unique way to the subspace

$$\{\gamma \in L^2(\Omega), \quad \gamma = z\gamma_1 + \gamma_2, \quad \gamma_1 \in L^2_z(L^\infty_x) \cap L^\infty_z(H^{1/2}_x),\ \gamma_2 \in H^1_xL^2_z\},$$

and the extension still satisfies (3.45).

Proof. ***Step 1.*** *We first treat the case* $\eta_0 = \eta_1 = 0$. We start with the part with Γ_2 which is easier. Using $\partial_x \Gamma_2 \in L^2$, we have

$$\left\| \Gamma_2 \Theta + \int_x^{x_1} \partial_x \Gamma_2 \Theta \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_2\|_{H^1_x L^2_z} \|\Theta\|_{L^\infty_z(L^2_x)} \lesssim \|\Gamma_2\|_{H^1_x L^2_z} \|\partial_z \Theta\|_{L^2(\Omega_+)}. \tag{3.46}$$

A similar bound holds in Ω_-. The term with Γ_1 is more involved. First, we have

$$\|z\Gamma_1 \Theta\|_{L^2(\Omega)} \lesssim \|\Gamma_1\|_{L^2_z(L^\infty_x)} \|\Theta\|_{L^\infty_z(L^2_x)} \lesssim \|\Gamma_1\|_{L^2_z(L^\infty_x)} \|\partial_z \Theta\|_{L^2(\Omega_+)}. \tag{3.47}$$

Concerning the integral term, we use Lemma B.16 in the Appendix, from which we deduce that $z\Theta \in L^2((0,1), H^{1/2}_{00_r})$. We then observe that by definition of the L^2 norm,

$$\begin{aligned} \left\| \int_x^{x_1} z \partial_x \Gamma_1 \Theta \right\|_{L^2(\Omega_+)} &= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \le 1} \int_{\Omega_+} h \int_x^{x_1} z \partial_x \Gamma_1 \Theta \\ &= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \le 1} \int_{\Omega_+} \left(\int_{x_0}^{x} h \right) z \Theta \partial_x \Gamma_1. \end{aligned}$$

Now, using Lemma B.9, for any $z > 0$,

$$\left| \int_{x_0}^{x_1} \left(\int_{x_0}^{x} h \right) z \Theta \partial_x \Gamma_1 \right| \lesssim \|\Gamma_1\|_{H^{1/2}_x} \|z\Theta\|_{H^{1/2}_{00_r}} \|h\|_{L^2(x_0,x_1)}. \tag{3.48}$$

Integrating with respect to z and using a Cauchy–Schwarz inequality, we get

$$\left\| \int_x^{x_1} z \partial_x \Gamma_1 \Theta \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)} \|z\Theta\|_{L^2_x(H^{1/2}_{00_r})} \tag{3.49}$$

$$\lesssim \|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)} \left(\|z \partial_x \Theta\|_{L^2(H^{-1})} + \|\partial_z \Theta\|_{L^2(\Omega_+)} \right). \tag{3.50}$$

Gathering (3.46), (3.47) and (3.49), we obtain (3.45) in the case $\eta_0 = \eta_1 = 0$.

Step 2. *The case of non-vanishing* η_i. We write

$$\Theta = \widetilde{\Theta} + \eta_1(z)\chi(x - x_1) + \eta_0(z)\chi(x - x_0),$$

where $\chi \in C^\infty_c(\mathbb{R})$ is such that $\chi \equiv 1$ in a neighborhood of zero and supp $\chi \subset B(0, (x_1 - x_0)/2)$. According to the first step, $G[\widetilde{\Theta}]$ satisfies (3.45) with $\eta_0 = \eta_1 = 0$. We then note that

$$G[\eta_1 \chi(x - x_1)] = \eta_1(z) \begin{cases} \mathbf{1}_{z>0} \Gamma(x_1, z) - \int_x^{x_1} \Gamma(x', z) \chi'(x' - x_1) \, \mathrm{d}x' & \text{if } z > 0, \\ \int_{x_0}^{x} \Gamma(x', z) \chi'(x' - x_1) \, \mathrm{d}x' & \text{if } z < 0. \end{cases}$$

The estimate follows easily.

Step 3. *Conclusion.* Now, assume merely that $\gamma_1 \in L^2_z(L^\infty_x) \cap L^\infty_z(H^{1/2}_x)$ and $\gamma_2 \in H^1_x L^2_z$. Let $\gamma_{i,n} \in H^1$ such that

$$\|\gamma_1 - \gamma_{1,n}\|_{L^2_z(L^\infty_x)} + \|\gamma_1 - \gamma_{1,n}\|_{L^\infty_z(H^{1/2}_x)} + \|\gamma_2 - \gamma_{2,n}\|_{H^1_x(L^2_z)} \to 0.$$

Let G_n be the operator associated with $\gamma_n = z\gamma_{1,n} + \gamma_{2,n} \in H^1$. Then, according to the previous estimates, $G_n[\Theta]$ is uniformly bounded in L^2. In fact, since the operator G depends linearly on γ, the sequence $G_n[\Theta]$ is a Cauchy sequence in L^2. Hence, it has a unique limit in $L^2(\Omega)$, which we still denote $G[\Theta]$. It follows from standard arguments that the limit does not depend on the choice of the approximating sequence γ_n, and thus $G[\Theta]$ is well-defined. □

This lemma allows us to define a notion of weak solution for the following system

$$\begin{cases} -\frac{z}{\alpha}\partial_x\Theta - \partial_z G[\Theta] - \partial_{zz}\Theta = f, & \text{in } \Omega_\pm \\ \Theta(\cdot, \pm 1) = 0, \\ \Theta(x_1, z) = \eta_1 \quad \forall z > 0, \\ \Theta(x_0, z) = \eta_0 \quad \forall z < 0. \end{cases} \tag{3.51}$$

Definition 3.9 (Weak solution to (3.51))**.** *Assume that the coefficients α, γ satisfy the assumptions of Proposition 3.1. Let $\eta_0, \eta_1 \in H^1_0(-1,1)$, $f \in L^2_x H^{-1}_z$*

We say that $\Theta \in \mathcal{B}$ is a weak solution *to* (3.51) *when*[4] *$\Theta(x_i, z) = \eta_i$ for $(-1)^{i+1} z > 0$ and, for every $V \in L^2_x(H^1_0(-1,1))$,*

$$\int_\Omega \partial_z\Theta\partial_z V + \int_\Omega G[\Theta]\partial_z V - \left\langle z\partial_x\Theta, \frac{V}{\alpha} \right\rangle_{L^2_x H^{-1}_z, L^2_x H^1_0} = \langle f, V\rangle_{L^2_x H^{-1}_z, L^2_x H^1_0}. \tag{3.52}$$

3.4.3 Existence and uniqueness of the dual profiles Φ^j

Let us now state our main result concerning the profiles Φ^j:

Proposition 3.10 (Existence, uniqueness and regularity of the dual profiles)**.** *Assume that α and γ satisfy the assumptions of Proposition 3.1. Then system* (3.35)-(3.36)-(3.37) *has a unique solution Φ^j such that $\Phi^j \in Z^0(\Omega_\pm)$. Equivalently, system* (3.41)-(3.42) *has a unique solution Θ^j such that $\Theta^j \in Z^0(\Omega_\pm)$. Furthermore,*

$$\|\Theta^j\|_{Z^0(\Omega_+)} + \|\Theta^j\|_{Z^0(\Omega_-)} \lesssim 1. \tag{3.53}$$

The above proposition will rely on two separate lemmas, whose proofs are postponed to Appendix C. The first result is the existence and uniqueness of weak solutions to equation (3.51), proved in Appendix C.2.

Lemma 3.11. *Assume that $\gamma = z\gamma_1 + \gamma_2$ and that $\alpha, \gamma_1, \gamma_2$ satisfy*

$$\begin{aligned} \|\alpha_z\|_\infty + \|\alpha - 1\|_\infty + \|\alpha_{xz}\|_{L^2} &\ll 1, \\ \|\gamma_1\|_{L^2_z(L^\infty_x)} + \|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\gamma_2\|_{H^1_x(L^2_z)} &\ll 1. \end{aligned} \tag{3.54}$$

Let $\eta_0, \eta_1 \in H^1_0(-1,1)$, $f \in L^2_x H^{-1}_z$.

Then system (3.51) *has a unique weak solution in the sense of Definition 3.9. Furthermore, this solution satisfies the estimate*

$$\begin{aligned} \int_0^1 z\Theta(x_0, z)^2 \,\mathrm{d}z - \int_{-1}^0 z\Theta(x_1, z)^2 \,\mathrm{d}z + \|\Theta\|^2_{L^2_x H^1_z} + \|z\partial_x\Theta\|^2_{L^2_x H^{-1}_z} \\ \lesssim \|f\|^2_{L^2_x H^{-1}_z} + \|\eta_0\|^2_{H^1(-1,1)} + \|\eta_1\|^2_{H^1(-1,1)}. \end{aligned} \tag{3.55}$$

[4]Recall that if $\Theta \in \mathcal{B}$, the traces on $x = x_i$ are well-defined in $\mathscr{L}^2_z(-1,1)$ by Lemma A.1

Note that this implies the existence and uniqueness of weak solutions $\Theta^j \in L^2_x H^1_z(\Omega_\pm)$ to (3.41)-(3.42). Indeed, according to Section 3.4.2, it is sufficient to apply Lemma 3.11 to the case

$$\begin{aligned} f(x,z) &= \mathbf{1}_{z>0}\partial_{zz}\left((\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\right) + \partial_z\left(\mathbf{1}_{z>0}\left((\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\right)\Gamma(x_1,z)\right), \\ \eta_0(z) &= 0, \\ \eta_1(z) &= -(\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z). \end{aligned}$$

The existence and uniqueness of $\Theta^j_\sharp$ which solves (3.44) follows. From there, we obtain the existence and uniqueness of Θ^j in $L^2_x H^1_z(\Omega_\pm)$, and eventually of Φ^j given by (3.39).

The next result is the Z^0 regularity of these dual profiles, which holds under slightly more stringent conditions on the coefficients α and γ, and is proved in Appendix C.3.

Lemma 3.12. *Assume that α and γ satisfy the assumptions of Proposition 3.1. Let $\Theta^j \in \mathcal{B}$ be the unique weak solution to* (3.41)-(3.42)*. Then $\Theta^j \in Z^0(\Omega_\pm)$.*

3.4.4 Well-posedness of equation (3.29)

We now turn towards the existence and uniqueness of solutions to (3.29). The first step is to exhibit a definition of weak solutions for this equation. This definition follows from the computations from (3.34) and Section 3.4.2.

Let us start with the case when $\partial_x\gamma \in L^2$, so that $h_0 \in L^2(\Omega_+)$, $h_1 \in L^2(\Omega_-)$. In this case, we say that V is a weak solution to (3.29) when, for every $\Phi \in \mathcal{B}$ such that $\Phi = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$\begin{aligned} &-\langle z\partial_x\Phi, V\rangle_{L^2_x H^{-1}_z, L^2_x H^1_0} + \int_\Omega G[\Theta]\partial_z V + \int_\Omega \partial_z\Theta\partial_z V \\ &\qquad = \int_{\Omega_+} h_0\Phi + \int_{\Omega_-} h_1\Phi - \int_{\Sigma_1} z\widetilde{\Delta}_1(z)\Phi(x_1,z)\,\mathrm{d}z + \int_{\Sigma_0} z\widetilde{\Delta}_0(z)\Phi(x_0,z)\,\mathrm{d}z, \end{aligned} \tag{3.56}$$

where Θ is defined by (3.38) (replacing Φ^j by Φ).

We now perform transformations of the source and boundary terms, whose purpose is to rewrite the terms involving γ_1 in the right-hand side in a weaker form. We also force the appearance of the function Θ, which will be convenient in the rest of the paper. We recall that

$$z\widetilde{\Delta}_i(z) = g(x_i,z) + \alpha(x_i,z)\widetilde{\delta}_i''(z) - \gamma(x_i,z)\widetilde{\delta}_i'(z).$$

and that $h_i = \partial_x g + \alpha_x\partial_{zz}\widetilde{\delta}_i - \gamma_x\partial_z\widetilde{\delta}_i$. As a consequence, we have

$$\int_{\Omega_+}\partial_x g\Phi + \int_{\Sigma_0}(g\Phi)_{|\Sigma_0} = -\int_{\Omega_+} g\partial_x\Phi = -\int_{\Omega_+}\frac{g}{\alpha}\partial_x\Theta = \int_{\Omega_+}\partial_x\left(\frac{g}{\alpha}\right)\Theta + \int_{\Sigma_0}\left(\frac{g}{\alpha}\Theta\right)_{|\Sigma_0}.$$

Similarly, for any $z > 0$,

$$\int_{x_0}^{x_1}\alpha_x(x,z)\Phi(x,z)\,\mathrm{d}x + \alpha(x_0,z)\Phi(x_0,z) = \Theta(x_0,z)$$

and

$$\int_{x_0}^{x_1}\gamma_x(x,z)\Phi(x,z)\,\mathrm{d}x + \gamma(x_0,z)\Phi(x_0,z) = G[\Theta](x_0,z).$$

Therefore,

$$\int_{\Omega_+} h_0\Phi + \int_{\Sigma_0} z\widetilde{\Delta}_0(z)\Phi(x_0,z)\,\mathrm{d}z$$
$$= \int_{\Omega_+} \partial_x\left(\frac{g}{\alpha}\right)\Theta + \int_{\Sigma_0}\left[\left(\frac{g}{\alpha}(x_0,z) + \widetilde{\delta}_0''(z)\right)\Theta(x_0,z) - \widetilde{\delta}_0'(z)G[\Theta](x_0,z)\right]\,\mathrm{d}z.$$

The next step is to isolate the terms involving Γ_1. To that end, we set $G = zG_1 + G_2$, following the decomposition $\Gamma = z\Gamma_1 + \Gamma_2$. Then, for a.e. $z > 0$,

$$zG_1[\Theta](x_0,z) = z(\Gamma_1\Theta)(x_0,z) + \int_{x_0}^{x_1} z\partial_x\Gamma_1\Theta = -\int_{x_0}^{x_1}\Gamma_1 z\partial_x\Theta,$$
$$G_2[\Theta](x_0,z) = (\Gamma_2\Theta)(x_0,z) + \int_{x_0}^{x_1}\partial_x\Gamma_2\Theta$$

It follows that

$$\int_{\Omega_+} h_0\Phi + \int_{\Sigma_0} z\widetilde{\Delta}_0(z)\Phi(x_0,z)\,\mathrm{d}z = \int_{\Omega_+}\partial_x\left(\frac{g}{\alpha}\right)\Theta - \int_{\Omega_+}\widetilde{\delta}_0'(z)\partial_x\Gamma_2\Theta + \int_{\Omega_+}\widetilde{\delta}_0'(z)\Gamma_1 z\partial_x\Theta$$
$$+ \int_{\Sigma_0}\left[\frac{g}{\alpha} + \widetilde{\delta}_0''(z) - \widetilde{\delta}_0'(z)\Gamma_2\right](x_0,z)\Theta(x_0,z)\,\mathrm{d}z.$$

Similar computations in Ω_- lead to

$$\int_{\Omega_-} h_1\Phi - \int_{\Sigma_1} z\widetilde{\Delta}_1(z)\Phi(x_1,z)\,\mathrm{d}z = \int_{\Omega_-}\partial_x\left(\frac{g}{\alpha}\right)\Theta - \int_{\Omega_-}\widetilde{\delta}_1'(z)\partial_x\Gamma_2\Theta + \int_{\Omega_-}\widetilde{\delta}_1'(z)\Gamma_1 z\partial_x\Theta$$
$$- \int_{\Sigma_1}\left(\frac{g}{\alpha} + \widetilde{\delta}_1''(z) - \widetilde{\delta}_1'(z)\Gamma_2\right)(x_1,z)\Theta(x_1,z)\,\mathrm{d}z.$$

Gathering all the terms, we are led to the following definition.

Definition 3.13 (Weak solutions to (3.29))**.** *Let $g \in H^1_x L^2_z$. Assume that the coefficients α and γ satisfy the assumptions of Proposition 3.1. Assume furthermore that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$ and*

$$\int_{\Sigma_i}\frac{1}{|z|}\left[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2\right]^2 < +\infty.$$

We say that $V \in L^2_x(H^1_0(-1,1))$ is a weak solution *to* (3.29) *when, for any $\Theta \in L^2_x(H^1_0(-1,1))$ such that $z\partial_x\Theta \in L^2_x(H^{-1}_z)$ and such that $\Theta = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,*

$$\begin{aligned}
-\left\langle z\partial_x\Theta, \frac{V}{\alpha}\right\rangle_{L^2H^{-1},L^2H^1_0} &+ \int_\Omega G[\Theta]\partial_z V + \int_\Omega \partial_z\Theta\partial_z V = \int_\Omega \partial_x\left(\frac{g}{\alpha}\right)\Theta \\
&- \int_{-1}^0\left[\frac{g}{\alpha}(x_1,z) + \widetilde{\delta}_1''(z) - \widetilde{\delta}_1'(z)\Gamma_2(x_1,z)\right]\Theta(x_1,z)\,\mathrm{d}z \\
&+ \int_0^1\left[\frac{g}{\alpha}(x_0,z) + \widetilde{\delta}_0''(z) - \widetilde{\delta}_0'(z)\Gamma_2(x_0,z)\right]\Theta(x_0,z)\,\mathrm{d}z \\
&+ \langle z\partial_x\Theta, \Gamma_1(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0')\rangle_{L^2(H^{-1}),L^2(H^1_0)} \\
&- \int_\Omega(\mathbf{1}_{z<0}\delta_1' + \mathbf{1}_{z>0}\delta_0')\partial_x\Gamma_2\Theta.
\end{aligned} \tag{3.57}$$

Let us now state the existence and uniqueness of weak solutions to equation (3.29). Once again, the proof of this result is postponed to Appendix C.4 for the sake of readability.

Lemma 3.14. *Let $g \in H^1_x L^2_z$. Assume that the coefficients α and γ satisfy the assumptions of Proposition 3.1. Assume furthermore that $\widetilde{\delta}_0'(0) = \widetilde{\delta}_1'(0) = 0$, $\widetilde{\delta}_0(1) = \widetilde{\delta}_1(-1) = 0$, and*

$$\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2 \right]^2 < +\infty.$$

Then equation (3.29) *has a unique weak solution in the sense of Definition 3.13. Furthermore, this solution satisfies the estimate*

$$\|V\|_{L^2(H^1_0)} \lesssim \|g\|_{H^1_x L^2_z} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2 \right]^2 \right)^{\frac{1}{2}} + \|\widetilde{\delta}_i'\|_{H^1}. \tag{3.58}$$

3.4.5 Orthogonality conditions for regular solutions

We are now ready to formulate the orthogonality conditions for the existence of $H^1_x H^1_z$ solutions to (3.4). The computations are identical to the ones preceding Definition 3.13, and are therefore, for the most part, left to the reader. We first assume that $g \in H^1$ and $\gamma_1 \in H^1$. In that case, the source terms h_i defined in (3.30) belong to $L^2(\Omega)$. During this first step, we also assume that $g_{|\Sigma_i} z^{-1/2} \in L^2(\Sigma_i)$, and that $\Gamma_{|\Sigma_i} z^{-1/2} \in L^2(\Sigma_i)$.

Going back to conditions (3.33) and using the definition of the functions Φ^j, we infer that the orthogonality conditions for (3.4) in order to have $H^1_x H^1_z$ regularity can be written as

$$\int_\Omega (\mathbf{1}_{z>0} h_0 + \mathbf{1}_{z<0} h_1)\Phi^j + \int_{\Sigma_0} z\widetilde{\Delta}_0 \Phi^j - \int_{\Sigma_1} z\widetilde{\Delta}_1 \Phi^j = \partial_z^j \widetilde{\delta}_1(0) - \partial_z^j \widetilde{\delta}_0(0). \tag{3.59}$$

We then transform the above equalities. As in the previous paragraph, we have

$$\begin{aligned}\int_{\Omega_+} h_0 \Phi^j + \int_{\Sigma_0} z\widetilde{\Delta}_0(z)\Phi^j(x_0,z)\,\mathrm{d}z = \int_{\Omega_+} \partial_x \left(\frac{g}{\alpha}\right)\Theta^j - \int_{\Omega_+} \widetilde{\delta_0}'(z)\partial_x \Gamma_2 \Theta + \int_{\Omega_+} \widetilde{\delta_0}'(z)\Gamma_1 z \partial_x \Theta^j \\ + \int_{\Sigma_0} \left[\frac{g}{\alpha} + \widetilde{\delta_0}''(z) - \widetilde{\delta_0}'(z)\Gamma_2 \right](x_0,z)\Theta^j(x_0,z)\,\mathrm{d}z.\end{aligned}$$

and a similar equality holds on Ω_-. Grouping together the integrals on Ω_+ and Ω_-, we arrive at the following definition.

Definition 3.15 (Linear forms associated with equation (3.4))**.** *Let $(\delta_0, \delta_1, f) \in \mathcal{H}$. We define $g = f(x, Y(x,z))$ and $\widetilde{\delta}_i(z) = \delta_i(Y(x_i, z))$. We define, for $j = 0, 1$,*

$$\begin{aligned}\ell^j_{\widetilde{u}}(f, \delta_0, \delta_1) := &\int_\Omega \partial_x \left(\frac{g}{\alpha}\right)\Theta^j + \int_\Omega \left(z\partial_x\Theta^j\Gamma_1 - \partial_x\Gamma_2\Theta^j\right)\left(\mathbf{1}_{z<0}\widetilde{\delta}_1' + \mathbf{1}_{z>0}\widetilde{\delta}_0'\right) \\ &+ \int_{\Sigma_0} \left(\frac{g}{\alpha} + \widetilde{\delta}_0'' - \Gamma_2\widetilde{\delta}_0'\right)\Theta^j - \int_{\Sigma_1} \left(\frac{g}{\alpha} + \widetilde{\delta}_1'' - \Gamma_2\widetilde{\delta}_1'\right)\Theta^j \\ &- \partial_z^j\widetilde{\delta}_1(0) + \partial_z^j\widetilde{\delta}_0(0).\end{aligned} \tag{3.60}$$

Remark 3.16. *In fact, when $(\delta_0, \delta_1, f) \in \mathcal{H}$, then $\widetilde{\delta}_i(0) = \widetilde{\delta}_i'(0) = 0$ for $i = 0, 1$. Therefore, the last term in the right-hand side of the definition can be removed. We deliberately chose to leave the terms $\partial_z^j\widetilde{\delta}_i(0)$ in order to help the reader follow the computations.*

Remark 3.17. *Assume that Y is such that $\bar{u}(x, Y(x,z)) = z$ for all $(x,z) \in \Omega$, and that $\alpha(x, Y(x,z)) = \bar{u}_y^2(x, Y(x,z))$, $\gamma_2 = -\bar{u}_{yy}(x, Y(x,z))$. Then it is easily checked from* (3.8) *that*

$$\widetilde{\delta}_i'(z) = \frac{1}{\bar{u}_y(x_i, Y(x_i,z))}\delta_i'(Y(x_i,z)), \tag{3.61}$$

$$\widetilde{\delta}_i''(z) = \Gamma_2(x_i,z)\widetilde{\delta}_i'(z) + \frac{1}{\alpha(x_i,z)}\delta_i''(Y(x_i,z)). \tag{3.62}$$

In this case, the boundary terms in the definition of $\ell_{\bar{u}}^j$ can be written as

$$\int_{\Sigma_i}\Big(\frac{g}{\alpha} + \widetilde{\delta}_i'' - \Gamma_2\widetilde{\delta}_i'\Big)\Theta^j = \int_{\Sigma_i}\frac{1}{\alpha(x_i,z)}(f(x_i,Y(x_i,z)) + \delta_i''(Y(x_i,z))\Theta^j(x_i,z)\,\mathrm{d}z. \tag{3.63}$$

Remark 3.18. *By Proposition 3.10, $\Theta^j \in Z^0(\Omega_\pm)$, so $\Theta^j \in H_x^{2/3}L_z^2(\Omega_\pm)$ by Proposition 1.4. A consequence of this regularity is the fact that the linear forms $\ell_{\bar{u}}^j$ can also be extended to the space*

$$H_x^{1/3}L_y^2 \cap L_x^2 H_y^{2/3} \times H^2(\Sigma_0) \times H^2(\Sigma_1).$$

Indeed, setting $g = f(x, Y(x,z))$ and $\widetilde{\delta}_i(z) = \delta_i(Y(x_i,z))$, we have, using Definition 3.15,

$$\begin{aligned}\ell_{\bar{u}}^j(f,\delta_0,\delta_1) = &-\Big\langle\frac{g}{\alpha}, \partial_x\Theta^j\Big\rangle_{H_x^{1/3}L_z^2, H_x^{-1/3}L_z^2} + \int_{\Sigma_0}\Big(\widetilde{\delta}_0'' - \Gamma_2\widetilde{\delta}_0'\Big)\Theta^j - \int_{\Sigma_1}\Big(\widetilde{\delta}_1'' - \Gamma_2\widetilde{\delta}_1'\Big)\Theta^j \\ &+ \int_\Omega z\partial_x\Theta^j\Gamma_1(\mathbf{1}_{z<0}\widetilde{\delta}_1' + \mathbf{1}_{z>0}\widetilde{\delta}_0') - \int_\Omega(\mathbf{1}_{z<0}\widetilde{\delta}_1' + \mathbf{1}_{z>0}\widetilde{\delta}_0')\partial_x\Gamma_2\Theta^j \\ &- \partial_z^j\widetilde{\delta}_1(0) + \partial_z^j\widetilde{\delta}_0(0).\end{aligned}$$

In particular, using Lemma B.4,

$$|\ell_{\bar{u}}^j(f,0,0)| \lesssim \Big\|\frac{g}{\alpha}\Big\|_{H_x^{1/3}L_z^2} \lesssim \|f\|_{H_x^{1/3}L_y^2} + \|f\|_{L_x^2 H_y^{2/3}}.$$

We are now ready to formulate our regularity results for solutions to (3.4).

Lemma 3.19. *Let $f \in H_x^1L_z^2 \cap L_x^2H_z^2$, $\delta_i \in H^2(\Sigma_i)$, and let $\bar{u} \in Q^1$. Let $g(x,z) = f(x, Y(x,z))$, $\widetilde{\delta}_i(z) = \delta_i(Y(x_i,z))$. Assume that the coefficients α and γ satisfy the assumptions of Proposition 3.1. Assume furthermore that*

$$\int_{\Sigma_i}\frac{1}{|z|}\Big[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2\Big]^2 < +\infty.$$

Consider the unique solution $U \in Z^0(\Omega)$ to (3.4)*. Then $U \in H_x^1H_z^1(\Omega)$ if and only if*

$$\ell_{\bar{u}}^j(f,\delta_0,\delta_1) = 0, \quad j = 0,1. \tag{3.64}$$

Furthermore, in this case

$$\|U\|_{H_x^1H_z^1}^2 \lesssim \|f\|_{H_x^1L_z^2}^2 + \|f\|_{L_x^2H_z^2}^2 + \sum_{i=0,1}\int_{\Sigma_i}\frac{1}{|z|}\Big[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z)\Gamma_2\Big]^2 + \|\widetilde{\delta}_i'\|_{H^1}^2. \tag{3.65}$$

Proof. The proof goes along the same lines as the one of Proposition 2.8. Assuming that the orthogonality conditions (3.64) are satisfied, and defining $\tilde{U}$ by

$$\tilde{U} := \begin{cases} \widetilde{\delta}_0 + \int_{x_0}^x V & \text{in } \Omega_+, \\ \widetilde{\delta}_1 + \int_{x_1}^x V & \text{in } \Omega_- \end{cases} \tag{3.66}$$

we see that $\tilde{U}$ is a solution to (3.4), and belongs to $L^2(H^1)$. Hence $\tilde{U} = U$, and therefore $\partial_x U \in L^2_x(H^1_z)$. Conversely, if $U \in H^1_x H^1_z$, then $\partial_x U$ is the only solution to (3.29). Since the jumps of U and $\partial_z U$ across the line $z = 0$ are zero, the orthogonality conditions (3.64) are satisfied. Eventually, using Lemma B.4, note that $\|g\|_{H^1_x L^2_z} \lesssim \|f\|_{H^1_x L^2_z} + \|f\|_{L^2_x H^2_z}$. □

3.5 Well-posedness results for the linearized problem

Proposition 3.20. *Let $\bar{u} \in Q^1$ such that the coefficients α, γ defined by*

$$\alpha(x,z) = \bar{u}_y^2(x, Y(x,z)), \quad \gamma_1(x,z) = \bar{u}_x(x, Y(x,z)), \quad \gamma_2(x,z) = -\bar{u}_{yy}(x, Y(x,z))$$

satisfy the assumptions of Proposition 3.1. Assume also that $\bar{u}(x, \pm 1) = \pm 1$, and $\|\bar{u} - y\|_{Q^1} \ll 1$. Let $\delta_0, \delta_1 \in H^4([-1,1])$ such that $\delta_0(1) = \delta_0(-1) = 0$, $f \in H^1_x(L^2_y) \cap L^2_x H^3_z$, and define $\widetilde{\delta}_i$ by (3.8).

Then there exists a unique solution $u \in Q^0$ to (3.1). It satisfies

$$\|u\|_{Q^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{H^1} + \|\delta_1\|_{H^1}. \tag{3.67}$$

Moreover, when additionally $\widetilde{\delta}_i'(0) = 0$ for $i = 0, 1$ and $(f(x_i) + \delta_i'')(Y(x_i, z))/z \in \mathscr{H}^1_z(\Sigma_i)$, this solution has $H^1_x H^1_y$ regularity if and only if $\ell^j_{\bar{u}}(f, \delta_0, \delta_1) = 0$ for $j \in \{0, 1\}$ (recall Definition 3.15). In this case, and if $f(x_0, 1) + \delta_0''(1) = f(x_1, -1) + \delta_0''(-1) = 0$ and $\partial_x \partial_y f \in L^2(\{|y| \geq 1/4\})$, u actually enjoys Q^1 regularity and one has the estimates

$$\begin{aligned} \|u\|_{Q^1} \lesssim \|f\|_{H^1_x L^2_y} + \|f\|_{L^2_x H^3_y} + \|\partial_x \partial_y f\|_{L^2(\{|y| \geq 1/4\})} \\ + \sum_{i=0,1} \|\delta_i\|_{H^4} + \left\| \frac{1}{z}(f(x_i) + \delta_i'')(Y(x_i, z)) \right\|_{\mathscr{H}^1_z(\Sigma_i)}. \end{aligned} \tag{3.68}$$

Remark 3.21. *It is quite tempting, in view of the above estimates, to infer by interpolation (at least when $f_{|\partial\Omega} = 0$ and $\delta_i = 0$) that*

$$\|\bar{u}\|_{Q^{1/2}} \lesssim \|f\|_{H^{1/2}_{00} L^2_y} + \|f\|_{L^2_x H^{3/2}_y} + \|f\|_{H^{1/2}_{00} H^{1/2}_y (\{|y| \geq 1/4\})}. \tag{3.69}$$

However, because of the linear forms $\ell^j_{\bar{u}}$, such a result is not obvious. Indeed, we need to interpolate between $L^2(\Omega)$ and the closed subspace of source terms in $H^1_x L^2_y \cap L^2_x H^3_y \cap H^1_x H^1_y(\{|y| \geq 1/4\})$ satisfying the orthogonality conditions. This important, but technical step will be the purpose of Section 5.3 (see Lemma 5.2). As a consequence, we will deduce that the above inequality is true (see Proposition 5.1).

Another approach to prove (3.69) would be to rely on the characterization of fractional Sobolev norms by finite differences to establish (3.69) directly, without relying on interpolation theory. However, while this approach works quite straightforwardly in the full plane $\mathbb{R}^2$, we have not been able to adapt it to handle the boundary conditions on $\Sigma_0 \cup \Sigma_1$.

Proof. The first part of the statement is an immediate consequence of the equivalence between u and U: u is a solution to (3.1) if and only if U is a solution to (3.4).

Step 1. *Necessity of the orthogonality conditions.* Using Lemma B.4, we see that

$$\|u\|_{H^{2/3}_x L^2_y} \lesssim \|U\|_{H^{2/3}_x L^2_z} + \|U\|_{L^2_x H^{4/3}_z},$$
$$\|U\|_{H^{2/3}_x L^2_z} \lesssim \|u\|_{H^{2/3}_x L^2_y} + \|u\|_{L^2_x H^{4/3}_y}.$$

Additionally, since $u(x,y) = U(x, \bar{u}(x,y))$,

$$u_y(x,y) = \bar{u}_y U_z(x, \bar{u}(x,y)), \quad u_{yy}(x,y) = \bar{u}_{yy} U_z(x, \bar{u}(x,y)) + \bar{u}_y^2 U_{zz}(x, \bar{u}(x,y)),$$

and since $\bar{u}_y, \bar{u}_{yy} \in L^\infty$ (see Lemma 1.10)

$$\|u\|_{L^2_x(H^2_y)} \lesssim \|U\|_{L^2_x(H^2_z)},$$

and conversely,

$$\|U\|_{L^2_x(H^2_z)} \lesssim \|u\|_{L^2_x(H^2_y)}.$$

We infer that

$$\|u\|_{Q^0} \lesssim \|U\|_{Q^0} \lesssim \|u\|_{Q^0}.$$

Hence Proposition 3.5 implies that equation (3.1) has a unique solution $u \in Q^0$.

Now, assume that $\widetilde{\delta_i}'(0) = 0$, $|z|^{-1}(f(x_i) + \delta_i'')(Y(x_i,z)) \in \mathscr{H}^1_z(\Sigma_i)$. Assume that the orthogonality conditions $\ell^j_{\bar{u}}(f, \delta_0, \delta_1) = 0$ are satisfied. Then according to Lemma 3.19, $U \in H^1_x H^1_z$. Using the equation and using the fact that $\gamma_2 \in L^\infty$, $\gamma_1 \in L^\infty_z(H^{1/2}_x) \hookrightarrow L^p$ for all $p < \infty$ and $\partial_z U \in H^1(\Omega) \hookrightarrow L^p$ for all $p < \infty$, it follows that

$$\alpha \partial_{zz} U = \underbrace{z\partial_x U}_{\in L^2_x H^1_z} + \underbrace{\gamma \partial_z U}_{\in L^p \ \forall p < \infty} - \underbrace{g}_{\in H^1_x L^2_z} \in L^2_x(L^\infty_z) + L^4 + L^2_z(L^\infty_x).$$

Using the identities

$$u_x(x,y) = U_x(x, \bar{u}(x,y)) + \bar{u}_x U_z(x, \bar{u}(x,y)),$$
$$u_{xy}(x,y) = \bar{u}_y U_{xz}(x, \bar{u}(x,y)) + \bar{u}_{xy} U_z(x, \bar{u}(x,y)) + \bar{u}_x \bar{u}_y U_{zz}(x, \bar{u}(x,y)),$$

and recalling that $\bar{u}_x \in L^\infty_x(L^2_z) \cap L^4 \cap L^\infty_z(L^2_x)$, we see that $u_x \in L^2_x(H^1_y)$. Conversely, if $u_x \in L^2_x(H^1_y)$, using the same line of argument, we infer that $u_{yy} \in L^2_x(L^\infty_z) + L^2_z(L^\infty_x)$. We deduce, using the change of variables Y, that $U_{xz} \in L^2$ and therefore $U \in H^1_x H^1_z$. As a consequence, the orthogonality conditions (3.64) are satisfied.

Step 2. Q^1 *regularity.* Let us now prove that in this case, we have $u \in Q^1$. Note that we do not try to prove that $U \in Q^1$. Indeed, U_x is a solution to

$$(z\partial_x + \gamma \partial_z - \alpha \partial_{zz}) U_x = g - \gamma_x U_z + \alpha_x U_{zz}.$$

In the right-hand side of the above formula, there is a term $-z\partial_x \gamma_1 U_z$, which does not belong to L^2 *a priori*. Instead, we go back to the equation in u and we notice that $u_x \in L^2(H^1)$ is the unique weak solution to

$$(\bar{u}\partial_x - \partial_{yy}) u_x = -\bar{u}_x u_x + \partial_x f =: g_1,$$
$$u_{x|\Sigma_i} = \frac{f(x_i, y) + \delta_i''(y)}{\bar{u}},$$
$$u_{x|\pm 1} = 0.$$

Furthermore, since $\bar{u}_x \in Q^0$, we have $\bar{u}_x \in L^2_x(L^\infty_y)$, so that the right-hand side g_1 belongs to L^2. Hence, we can write

$$u_x(x,y) = W(x, \bar{u}(x,y)).$$

Note that W and U_x are slightly different: indeed,

$$W(x,z) - U_x(x,z) = \gamma_1 U_z(x,z).$$

Since the term $\partial_x \gamma_1 U_z$ is precisely the one preventing us that U_x belongs to $Z^0(\Omega)$ when the coefficients are not smooth, we see that the purpose of this new change of variables is to remove a potential singular part in U.

Then W is a solution to

$$\begin{aligned} z\partial_x W + \gamma \partial_z W - \alpha \partial_{zz} W &= g_1(x, Y(x,z)) \in L^2(\Omega), \\ W_{z=\pm 1} &= 0, \\ W_{|\Sigma_i} &= \frac{g(x_i,z) + \delta_i''(Y(x_i,z))}{z}. \end{aligned} \tag{3.70}$$

Hence we can apply the results of Proposition 3.5. Note that the compatibility conditions at $(x_0, 1)$ and at $(x_1, -1)$ are satisfied. We infer that $W \in Z^0(\Omega)$, and thus $u_x \in Z^0(\Omega)$. It follows that $u \in H^{5/3}_x L^2_y$.

We now prove that $\partial^5_y u \in L^2$. As in the proof of Proposition 2.15, we will need to distinguish between an "interior regularity", close to the line $z = 0$, and a "boundary regularity", close to the lines $z = \pm 1$.

Step 3. *Interior regularity.* Assume that the orthogonality conditions (3.64) are satisfied, so that $u_x \in Q^0$, and that $\partial^3_y f \in L^2$. In that case, writing

$$u_{yy} = \bar{u} u_x - f,$$

we immediately infer that $u \in L^2_x(H^4_y)$. Indeed, since $u_x \in Q^0$, $u_x \in L^2_y(H^{2/3}_x) \hookrightarrow L^2_y(L^\infty_x)$, and $\bar{u}_{yy} \in L^2_x(H^1_y) \hookrightarrow L^\infty_y(L^2_x)$. Note also that $\bar{u}, \bar{u}_y \in L^\infty$.

In order to prove that $\partial^5_y u \in L^2((x_0,x_1) \times (-1+\delta, 1+\delta)$ for any $\delta > 0$, we mimick the proof of Proposition 2.15. Multiplying (3.1) by a test function $\partial^3_y \phi$, for some $\phi \in C^\infty(\bar{\Omega})$ such that ϕ identically vanishes in a neighborhood of ± 1 and on $\{x_0\} \times (-1,0) \cup \{x_1\} \times (0,1)$, we obtain

$$\begin{aligned} \int \bar{u} u_x \partial^3_y \phi &= \int \partial^2_y (\bar{u} u_x) \partial_y \phi \\ &= \int (\bar{u}_{yy} u_x + 2 \bar{u}_y u_{xy}) \partial_y \phi + \int \bar{u} u_{xyy} \partial_y \phi \\ &= \int (\bar{u}_{yy} u_x + 2 \bar{u}_y u_{xy}) \partial_y \phi - \int \partial_x (\bar{u} \partial_y \phi) u_{yy} - \sum_{i=0,1} (-1)^i \int_{\Sigma_i} (\bar{u} \partial_y \phi)(x_i, y) \delta_i'' \, \mathrm{d}y. \end{aligned}$$

We integrate by parts both the boundary terms and the interior terms and we are led to

$$\begin{aligned}\int \bar{u}u_x\partial_y^3\phi &= \int(-\partial_y^3\bar{u}u_x - 3\bar{u}_{yy}u_{xy} - 2\bar{u}_y u_{xyy} + \bar{u}_{xy}u_{yy} + \bar{u}_x\partial_y^3 u)\phi + \int \partial_x\phi\partial_y(\bar{u}u_{yy}) \\ &\quad + \sum_{i=0,1}(-1)^i\int_{\Sigma_i}\phi(x_i,y)\partial_y(\bar{u}(x_i,y)\delta_i''(y))\,\mathrm{d}y \\ &= \int(-\partial_y^3\bar{u}u_x - 3\bar{u}_{yy}u_{xy} - 2\bar{u}_y u_{xyy} + \bar{u}_{xy}u_{yy} + \bar{u}_x\partial_y^3 u)\phi - \int \phi\partial_x(\bar{u}_y u_{yy}) \\ &\quad + \int \partial_x\phi\bar{u}\partial_y^3 u \\ &\quad + \sum_{i=0,1}(-1)^i\int_{\Sigma_i}\phi(x_i,y)\left[\partial_y(\bar{u}(x_i,y)\delta_i''(y)) - \bar{u}_y\delta_i''(y)\right]\,\mathrm{d}y.\end{aligned}$$

But we also have

$$\int \bar{u}u_x\partial_y^3\phi = \int(f+\partial_y^2)u\partial_y^3\phi = -\int\partial_y^3 f\phi + \int\partial_y^4 u\partial_y\phi.$$

Gathering all the terms and using a density argument, we deduce that for any function $\phi\in H^1(\Omega)$ vanishing on $\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)$,

$$\begin{aligned}&-\int_\Omega(\bar{u}\partial_y^3 u\partial_x\phi + \bar{u}_x\partial_y^3 u\phi) + \int\partial_y^4 u\partial_y\phi \\ &= \int\left[\partial_y^3 f - \left(\partial_y^3\bar{u}u_x + 3\partial_y^2\bar{u}u_{xy} + 3\partial_y u u_{xyy}\right)\right]\phi \\ &\quad + \sum_{i=0,1}(-1)^i\int_{\Sigma_i}\phi(x_i,y)\bar{u}(x_i,y)\delta_i^{(3)}(y)).\end{aligned}$$

This proves that for any test function $\chi_0\in C_c^\infty(]-1,1[)$ such that $\chi\equiv 1$ in a neighborhood of $z=0$, say $\chi_0\equiv 1$ on $(-1/2,1/2)$ and $\operatorname{supp}\chi_0\subset(-3/4,3/4)$, $\chi_0\partial_y^3 u$ is the unique weak solution to

$$\bar{u}\partial_x(\chi_0\partial_y^3 u) - \partial_y^2(\chi_0\partial_y^3 u) = \chi_0\left(\partial_y^3 f - \partial_y^3\bar{u}u_x - 3\partial_y^2\bar{u}u_{xy} - 3\partial_y u u_{xyy}\right) - \chi_0''\partial_y^3 u - 2\chi_0'\partial_y^4 u.$$

The right-hand side belongs to $L^2(\Omega)$, and the compatibility conditions at the corners $(x_0,1)$ and $(x_1,-1)$ are automatically satisfied because of the truncation χ_0. Hence, Proposition 3.5 ensures that $\chi_0(Y(x,z))\partial_y^3 u(x,Y(x,z))\in Z^0(\Omega)$. We infer that $\chi_0\partial_y^5 u\in L^2$, and $\bar{u}\chi_0\partial_x\partial_y^3 u\in L^2$. Using the equation satisfied by u, it follows that $\chi_0\bar{u}^2\partial_x^2\partial_y u\in L^2$.

In conclusion, we find that the following estimate holds: there exists a constant C, depending only on Ω and on χ_0, such that for any $f\in H^1_x L^2_y\cap L^2_x H^3_y$, and for any $(\delta_0,\delta_1)\in H^4(-1,1)$ satisfying the assumptions of Proposition 3.20 and such that $\ell_{\bar{u}}^j(f,\delta_0,\delta_1)=0$,

$$\|\chi_0 u\|_{Q^1}\le C\left(\|f\|_{H^1_x L^2_y} + \|f\|_{L^2_x H^3_y} + \sum_{i=0,1}\|\delta_i\|_{H^4} + \left\|\frac{1}{z}(f(x_i)+\delta_i'')(Y(x_i,z))\right\|_{\mathscr{H}^1_z(\Sigma_i)}\right). \quad (3.71)$$

Step 4. *Boundary regularity.* We now take $\chi_1\in C_c^\infty(\mathbb{R})$, with $\chi\equiv 1$ on $[1/2,1]$, and $\operatorname{supp}\chi_1\subset[1/4,1]$. Note that with the choice above, $\operatorname{supp}\chi_1'\subset\chi_0^{-1}(\{1\})$. Let $u_1=u\chi_1$. Then u_1 satisfies

$$\begin{aligned}\bar{u}\partial_x u_1 - \partial_{yy}u_1 = S_1 := f\chi_1 - 2\chi_1'\partial_y u - \chi_1'' u \quad &\text{in } (x_0,x_1)\times(1/4,1), \\ u_1(\cdot,1/4) = u_1(\cdot,1) &= 0, \\ u_1(x_0,y) &= \chi_1\delta_0.\end{aligned} \quad (3.72)$$

By assumption, $\|\bar{u}-y\|_\infty \le \bar{c}$ for some $\bar{c} \ll 1$. Without loss of generality, we assume that $\bar{c} \le 1/8$ so that $\bar{u} \ge 1/8$ on $(x_0, x_1) \times (1/4, 1)$. Hence the equation is parabolic. Note also that

$$\bar{u}\partial_x^2 u_1 = \partial_x S_1 + \partial_y^2 \partial_x u_1 - \partial_x \bar{u} \partial_x u_1,$$

and the right-hand side is in L^2. Thus $\partial_x^2 u_1 \in L^2, \partial_y^2 \partial_x u_1 \in L^2$, from where it follows that $\partial_x u_1 \in L^\infty$, with a bound $\|\partial_x u_1\|_\infty \lesssim (\|u_x\|_{Q^0} + \|u\|_{Q^0} + \|\partial_x f \chi_1\|_{L^2})$.

We now differentiate (3.72) twice with respect to y. Since $(\chi_0\delta_0)(1) = (\chi_0\delta_0)(1/4) = 0$, $\partial_y^2 u_1$ is a solution to

$$\begin{aligned} (\bar{u}\partial_x - \partial_{yy})\partial_y^2 u_1 &= \partial_y^2 S_1 - \bar{u}_{yy}\partial_x u_1 - 2\bar{u}_y \partial_{xy} u_1, \\ \partial_y^2 u_1(\cdot, 1/4) = \partial_y^2 u_1(\cdot, 1) &= 0, \\ \partial_y^2 u_1(x_0, y) &= \partial_y^2(\chi_1 \delta_0). \end{aligned} \tag{3.73}$$

And since $\partial_y^2(\chi_1\delta_0)(1/4) = \partial_y^2(\chi_1\delta_0)(1) = 0$, we have

$$\begin{aligned} (\bar{u}\partial_x - \partial_{yy})\partial_x\partial_y^2 u_1 &= \partial_x\partial_y^2 S_1 - \bar{u}_{xyy}\partial_x u_1 - \bar{u}_{yy}\partial_x^2 u_1 - 2\partial_y(\bar{u}_x \partial_{xy} u_1), \\ \partial_x\partial_y^2 u_1(\cdot, 1/4) = \partial_x\partial_y^2 u_1(\cdot, 1) &= 0, \\ \partial_x\partial_y^2 u_1(x_0, y) &= \partial_y^2(\chi_1 \Delta_0). \end{aligned} \tag{3.74}$$

Now, let us perform energy estimates for the above parabolic system. We multiply by $\bar{u}^{-1}\partial_x\partial_y^2 u_1$ and integrate over $(x_0, x_1) \times (1/4, 1)$. According to the previous regularity estimates on u and u_1, we know that $\partial_x\partial_y^2 S_1 \in L^2(H^{-1})$, $\bar{u}_{xyy}\partial_x u_1 \in L^2$, $\bar{u}_{yy}\partial_x^2 u_1 \in L^2$. We estimate the terms $\partial_y(\bar{u}_x\partial_{xy}u_1)$ in $L^2((x_0, x_1), H^{-1}((1/4, 1))$. It is easily proved that it is bounded in this space. We infer that $\partial_x\partial_y^3 u_1 \in L^2((x_0, x_1), (1/4, 1))$. Eventually, writing

$$\partial_y^2 u_1 = \bar{u}\partial_x u_1 - S_1,$$

we deduce that $\partial_y^5 u_1 \in L^2((x_0, x_1), (1/4, 1))$. □

4 Local stability of the orthogonality conditions

This section is devoted to the derivation of some key estimates for the nonlinear scheme we use in Section 6. Indeed, as explained in Section 1.4, a crucial point of our proof lies in the fact that the linear forms associated with the orthogonality conditions depend continuously on the data $\bar{u}$, in a suitable norm. This is stated in Proposition 4.1 below.

In this section, we consider two flows $\bar{u}$, $\bar{u}'$ in a Q^1 neighborhood of the linear shear flow profile $\mathfrak{u}(x, y) = y$. We define two changes of variables Y, Y' such that

$$\forall x \in (x_0, x_1), \quad \forall z \in (-1, 1), \quad \bar{u}(x, Y(x, z)) = \bar{u}'(x, Y'(x, z)) = z. \tag{4.1}$$

We define, as in Section 3.1,

$$\alpha(x, z) := \bar{u}_y^2(x, Y(x, z)), \tag{4.2}$$

$$\gamma_1(x, z) := \bar{u}_x(x, Y(x, z)), \tag{4.3}$$

$$\gamma_2(x, z) := -\bar{u}_{yy}(x, Y(x, z)), \tag{4.4}$$

and analogously, we define α', γ_1' and γ_2' from $\bar{u}'$. We set $\gamma := z\gamma_1 + \gamma_2$ and $\gamma' := z\gamma_1' + \gamma_2'$.

We then consider the profiles Φ^j, $(\Phi^j)'$ constructed in Proposition 3.10, and the associated linear forms $\ell_{\bar{u}}^j$, $\ell_{\bar{u}'}^j$ introduced in Definition 3.15. The main result of this section is the following.

Proposition 4.1. *There exists $\bar{c} > 0$ such that, for every $\bar{u}, \bar{u}' \in Q^1$ with $\|\bar{u} - y\|_{Q^1} \leq \bar{c}$, $\bar{u}(x_i, 0) = 0$, $\bar{u}(x, \pm 1) = \pm 1$ (and similarly for $\bar{u}'$), there holds*

$$\|\ell^j_{\bar{u}} - \ell^j_{\bar{u}'}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \tag{4.5}$$

We decompose the proof of the proposition into several lemmas. We first investigate bounds on the coefficients $\alpha - \alpha'$, $\gamma - \gamma'$ in terms of $\|\bar{u} - \bar{u}'\|_{Q^{1/2}}$. Then, we prove that the solutions Φ^j to (3.35) depend continuously on the coefficients. Putting together these two results leads to Proposition 4.1.

Remark 4.2. *According to Corollary B.5, the changes of variables Y, Y' satisfy the assumptions of Lemma B.4, which will be used abundantly throughout this section.*

4.1 Stability of the change of variables

We start by estimating the difference between the two changes of variables.

Lemma 4.3. *There exists $\bar{c} \in (0, 1)$ and $C > 0$ such that, for any $\bar{u}, \bar{u}' \in Q^1$ such that $\bar{u}(\pm 1) = \bar{u}'(\pm 1) = \pm 1$ and $\|\bar{u} - y\|_{Q^1} \leq \bar{c}$ and $\|\bar{u}' - y\|_{Q^1} \leq \bar{c}$, if Y, Y' are defined by (4.1),*

$$\|Y - Y'\|_{L^\infty(\Omega)} \lesssim \|Y - Y'\|_{H^{7/12}_x H^1_z} \leq C\|\bar{u} - \bar{u}'\|_{Q^{1/2}}. \tag{4.6}$$

Proof. From the definition of Y, one infers that

$$Y(x, z) = -1 + \int_{-1}^{z} \frac{\mathrm{d}s}{\bar{u}_y(x, Y(x, s))}. \tag{4.7}$$

Hence, combined with the corresponding relation for Y', one has

$$Y(x, z) - Y'(x, z) = \int_{-1}^{z} \frac{\bar{u}'_y(Y'(x, s)) - \bar{u}_y(x, Y(x, s))}{\bar{u}_y(x, Y(x, s))\bar{u}'_y(x, Y'(x, s))} \, \mathrm{d}s. \tag{4.8}$$

From there, it follows that for a.e. $x \in (x_0, x_1)$,

$$\|Y(x, \cdot) - Y'(x, \cdot)\|_{H^1_z} \leq \left\| \frac{\bar{u}'_y(Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{L^2_z}. \tag{4.9}$$

We decompose the right-hand side as

$$\frac{\bar{u}'_y(x, Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} = \frac{(\bar{u}'_y - \bar{u}_y)(x, Y'(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} + \frac{\bar{u}_y(x, Y'(x, \cdot)) - \bar{u}_y(x, Y(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))}. \tag{4.10}$$

The first term is bounded in $H^{7/12}_x L^2_z$ as follows

$$\begin{aligned} &\left\| \frac{(\bar{u}'_y - \bar{u}_y)(x, Y'(x, \cdot))}{\bar{u}_y(x, Y(x, \cdot))\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{H^{7/12}_x L^2_z} \\ &\leq \|(\bar{u}'_y - \bar{u}_y)(x, Y(x, \cdot))\|_{H^{7/12}_x L^2_z} \left\| \frac{1}{\bar{u}_y(x, Y(x, \cdot))} \right\|_{L^\infty_z(H^1_x)} \left\| \frac{1}{\bar{u}'_y(x, Y'(x, \cdot))} \right\|_{L^\infty_z(H^1_x)}. \end{aligned} \tag{4.11}$$

Recalling that $Y_x(x,z) = -(\bar u_x/\bar u_y)(x,Y(x,z))$, we infer, using Lemma B.4

$$\begin{aligned}
&\left\| \frac{(\bar u'_y - \bar u_y)(x,Y'(x,\cdot))}{\bar u_y(x,Y(x,\cdot))\bar u'_y(x,Y'(x,\cdot))} \right\|_{H^{7/12}_x L^2_z} \\
&\lesssim \left(\|\bar u_y - \bar u'_y\|_{H^{7/12}_x L^2_y} + \|\bar u_y - \bar u'_y\|_{L^2_x H^{7/6}_y} \right) \\
&\qquad \times \left(1 + \|\bar u_{xy}\|_{L^\infty_z(L^2_x)} + \|\bar u_x \bar u_{yy}\|_{L^\infty_z(L^2_x)}\right) \left(1 + \|\bar u'_{xy}\|_{L^\infty_z(L^2_x)} + \|\bar u'_x \bar u'_{yy}\|_{L^\infty_z(L^2_x)}\right) \\
&\lesssim \|\bar u - \bar u'\|_{Q^{1/2}} (1 + \|\bar u\|^2_{Q^1})(1 + \|\bar u'\|^2_{Q^1}).
\end{aligned} \tag{4.12}$$

Concerning the second term in the right-hand side of (4.10), we write a Taylor formula, namely

$$\bar u_y(x,Y'(x,z)) - \bar u_y(x,Y(x,z)) = (Y'(x,z) - Y(x,z)) \int_0^1 \partial^2_y \bar u(x, \tau Y'(x,z) + (1-\tau)Y(x,z))\, \mathrm{d}\tau. \tag{4.13}$$

Hence,

$$\begin{aligned}
&\left\| \frac{\bar u_y(x,Y'(x,\cdot)) - \bar u_y(x,Y(x,\cdot))}{\bar u_y(x,Y(x,\cdot))\bar u'_y(x,Y'(x,\cdot))} \right\|_{H^{7/12}_x L^2_z} \\
&\le \|Y' - Y\|_{L^\infty_z(H^{7/12}_x)} \left\| \int_0^1 \partial^2_y \bar u(x, \tau Y'(x,z) + (1-\tau)Y(x,z))\, \mathrm{d}\tau \right\|_{H^{7/12}_x L^2_z} \\
&\qquad \times \left\| \frac{1}{\bar u_y(x,Y(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} \left\| \frac{1}{\bar u'_y(x,Y'(x,\cdot))} \right\|_{L^\infty_z(H^1_x)}.
\end{aligned} \tag{4.14}$$

As previously, we have

$$\left\| \frac{1}{\bar u_y(x,Y(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} \lesssim 1 + \|\bar u\|^2_{Q^1},$$
$$\left\| \frac{1}{\bar u'_y(x,Y'(x,\cdot))} \right\|_{L^\infty_z(H^1_x)} \lesssim 1 + \|\bar u'\|^2_{Q^1}.$$

Furthermore,

$$\|Y' - Y\|_{L^\infty_z(H^{7/12}_x)} \lesssim \|Y - Y'\|_{H^{7/12}_x H^1_z}. \tag{4.15}$$

And using Lemma B.4,

$$\left\| \int_0^1 \partial^2_y \bar u(x, \tau Y'(x,z) + (1-\tau)Y(x,z))\, \mathrm{d}\tau \right\|_{H^{7/12}_x L^2_z} \lesssim \|\partial^2_y \bar u\|_{H^{7/12}_x L^2_z} + \|\partial^2_y \bar u\|_{L^2_x H^{7/6}_z} \lesssim \|\bar u - y\|_{Q^1}. \tag{4.16}$$

Therefore, since $\bar c \le 1$, we infer that there exists a universal constant $\bar C$ such that

$$\|Y - Y'\|_{H^{7/12}_x H^1_z} \le \bar C \left(\|\bar u - \bar u'\|_{Q^{1/2}} + \bar c \|Y - Y'\|_{H^{7/12}_x H^1_z} \right). \tag{4.17}$$

For $\bar c$ sufficiently small, we can absorb the second term in the right-hand side into the left-hand side, and we obtain the result announced in the lemma. □

4.2 Boundedness and stability of the coefficients

We now state two lemmas allowing us to estimate the coefficients α and γ, as well as their difference in terms of the Q^1 and $Q^{1/2}$ norms of the functions $\bar{u}, \bar{u}'$.

Lemma 4.4 (Bounds on the coefficients in terms of the Q^1 norm)**.** *Let $\bar{u} \in Q^1$ such that $\|\bar{u}-y\|_{Q^1} \leq \bar{c} \ll 1$. Let $\alpha, \gamma_1, \gamma_2$ be given by* (4.2)*,* (4.3)*,* (4.4)*.*

Then the following estimates hold (uniformly in $\bar{c}$):

$$\begin{aligned} \|\alpha - 1\|_{H^1_x H^1_z} + \|\partial_{zz}\alpha\|_{H^{3/5}_x L^2_z} &\lesssim \|\bar{u} - y\|_{Q^1}, \\ \|\gamma_1\|_{H^{2/3}_x L^2_z} + \|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} + \|\partial_z \gamma_1\|_{L^\infty_z(L^2_x)} &\lesssim \|\bar{u} - y\|_{Q^1}, \\ \|\gamma_2\|_{H^1_x L^2_z} + \|\partial_z \gamma_2\|_{H^{3/5}_x L^2_z} &\lesssim \|\bar{u} - y\|_{Q^1}. \end{aligned} \tag{4.18}$$

Remark 4.5. *Thanks to the above lemma, if $\|\bar{u}-y\|_{Q^1} \leq \bar{c} \ll 1$, then α and γ satisfy the smallness assumptions of Proposition 3.1.*

Lemma 4.6 (Bounds on differences between coefficients in terms of the $Q^{1/2}$ norm)**.** *Let $\bar{u}, \bar{u}' \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \leq \bar{c} \ll 1$, $\|\bar{u}' - y\|_{Q^1} \leq \bar{c} \ll 1$. Let $\alpha, \alpha', \gamma_1, \gamma_1', \gamma_2, \gamma_2'$ be given by* (4.2)*,* (4.3)*,* (4.4)*. Then the following estimates hold (uniformly in $\bar{c}$):*

$$\|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}, \tag{4.19}$$

$$\|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}, \tag{4.20}$$

$$\|\gamma_2 - \gamma_2'\|_{L^2_z(H^{1/2}_x)} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}} \tag{4.21}$$

Proof of Lemma 4.4. Concerning the bounds on α, we recall that $\alpha(x,z) = \bar{u}_y^2(x, Y(x,z))$, so that

$$\alpha_z = 2\bar{u}_{yy}(x, Y(x,z))\bar{u}_y(x, Y(x,z))\frac{\partial Y}{\partial z} = 2\bar{u}_{yy}(x, Y(x,z))$$

and, recalling that $Y_x(x,z) = -(\bar{u}_x/\bar{u}_y)(x, Y(x,z))$,

$$\alpha_{xz} = 2\bar{u}_{xyy}(x, Y(x,z)) - 2\frac{\partial_y^3 \bar{u}\partial_x \bar{u}}{\partial_y \bar{u}}(x, Y(x,z)).$$

Noticing that $\bar{u}_x, \partial_y^3 \bar{u} \in Q^0 \hookrightarrow L^\infty_x(L^2_y) \cap L^2_x(L^\infty_y)$, we infer that $\|\alpha_{xz}\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1} + \|\bar{u} - y\|^2_{Q^1}$.

In a similar fashion,

$$\alpha_{zz}(x,z) = 2\frac{\partial_y^3 \bar{u}}{\partial_y \bar{u}}(x, Y(x,z)),$$

so that according to Lemma B.4,

$$\|\alpha_{zz}\|_{H^{3/5}_x L^2_z} \lesssim \left\|\frac{\partial_y^3 \bar{u}}{\partial_y \bar{u}}\right\|_{H^{3/5}_x L^2_z} + \left\|\frac{\partial_y^3 \bar{u}}{\partial_y \bar{u}}\right\|_{L^2_x H^{6/5}_z} \lesssim \|\bar{u} - y\|_{Q^1}.$$

We now address the bounds on $\gamma_1 = \bar{u}_x(x, Y(x,z))$. We have, using Lemma B.4

$$\|\gamma_1\|_{H^{2/3}_x L^2_z} \lesssim \|\bar{u}_x\|_{H^{2/3}_x L^2_y} + \|\bar{u}_x\|_{L^2_x(H^{4/3}_y)} \lesssim \|\bar{u} - y\|_{Q^1}.$$

Furthermore,

$$\partial_z \gamma_1 = \frac{\partial Y}{\partial z} \bar{u}_{xy}(x, Y(x,z)),$$

so that

$$\|\partial_z \gamma_1\|_{L^\infty_z(L^2_x)} \leq \|\partial_z \gamma_1\|_{L^2_x(L^\infty_z)} \lesssim \|\bar{u}_{xy}\|_{L^2_x(L^\infty_z)} \lesssim \|\bar{u}_{xy}\|_{L^2_x(H^1_y)}.$$

Hence $\|\partial_z \gamma_1\|_{L^\infty_z(L^2_x)} \lesssim \|\bar{u} - y\|_{Q^1}$.

Concerning the $L^\infty_z(H^{1/2}_x)$ bound on γ_1, we note that $H^{2/3}_x L^2_z \cap H^{1/3}_x H^1_z \hookrightarrow C((-1,1), H^{1/2}_x)$ according to the "fractional trace theorem" [42, Equation (4.7), Chapter 1], so that

$$\|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} \lesssim \|\gamma_1\|_{H^{2/3}_x L^2_z} + \|\partial_z \gamma_1\|_{H^{1/3}_x L^2_z}. \tag{4.22}$$

We then bound the two terms in the right-hand side, using Lemma B.4. We have

$$\|\gamma_1\|_{H^{2/3}_x L^2_z} \lesssim \|\bar{u}_x\|_{H^{2/3}_x L^2_y} + \|\bar{u}_x\|_{L^2_x H^{4/3}_y} \lesssim \|\bar{u} - y\|_{Q^1}.$$

Furthermore,

$$\partial_z \gamma_1(x,z) = \frac{\partial Y}{\partial z}(x,z) \bar{u}_{xy}(x, Y(x,z)) = \frac{\bar{u}_{xy}}{\bar{u}_y}(x, Y(x,z)), \tag{4.23}$$

so that, using Lemma B.4,

$$\|\partial_z \gamma_1\|_{H^{1/3}_x L^2_z} \lesssim \left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{H^{1/3}_x L^2_y} + \left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{L^2_x H^{2/3}_y}. \tag{4.24}$$

We bound the first term as follows

$$\left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{H^{1/3}_x L^2_y} \lesssim \|\bar{u}_{xy}\|_{H^{1/3}_x L^2_y} \left\| \frac{1}{\bar{u}_y} \right\|_{L^\infty_y(H^1_x)} \lesssim \|\bar{u} - y\|_{Q^1} \|\bar{u}_y\|_{H^1_y(H^1_x)}.$$

The second term is bounded similarly

$$\left\| \frac{\bar{u}_{xy}}{\bar{u}_y} \right\|_{L^2_x H^{2/3}_y} \lesssim \|\bar{u}_{xy}\|_{L^2_x H^{2/3}_y} \left\| \frac{1}{\bar{u}_y} \right\|_{L^\infty_x(H^1_y)} \lesssim \|\bar{u} - y\|_{Q^1} \|\bar{u}_{yy}\|_\infty.$$

Eventually, we get

$$\|\gamma_1\|_{L^\infty_z(H^{1/2}_x)} \lesssim \|\bar{u}\|_{Q^1} \|\bar{u} - y\|_{Q^1}.$$

Concerning the term $\gamma_2 = -\bar{u}_{yy}(x, Y(x,z))$, we write

$$-\partial_x \gamma_2(x,z) = \bar{u}_{xyy}(x, Y(x,z)) + \frac{\partial Y}{\partial x} \partial^3_y \bar{u}(x, Y(x,z)).$$

The first term is bounded in L^2 by $\|\bar{u}_{xyy}\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1}$ (see Lemma B.4). Concerning the second one, we recall that $\partial_x Y = -(\bar{u}_x/\bar{u}_y)(x, Y(x,z))$, and therefore $\partial_x Y \in L^p(\Omega)$ for all $p < \infty$ (note that the Jacobian of the change of variables $y = Y(x,z)$ is bounded from above and below by a uniform constant). And since $\partial^3_y \bar{u} \in H^{2/3}(\Omega) \hookrightarrow L^6(\Omega)$, we obtain, thanks to the Hölder inequality,

$$\|\partial_x \gamma_2\|_{L^2} \lesssim (1 + \|\bar{u}\|_{Q^1}) \|\bar{u} - y\|_{Q^1}.$$

Eventually, $\partial_z \gamma_2 = -\partial^3_y \bar{u} / \partial_y \bar{u}(x, Y(x,z))$, so that, using one last time Lemma B.4, we obtain the desired estimate on $\partial_z \gamma_2$. □

Proof of Lemma 4.6. We use the same type of techniques as in the previous lemma. Recalling the definition of α, α', we write

$$\begin{aligned}\alpha(x,z)-\alpha'(x,z) &= \bar{u}_y^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y(x,z)) &(4.25)\\ &\quad +(\bar{u}'_y)^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y'(x,z)). &(4.26)\end{aligned}$$

Using the results of Lemma B.4, the term (4.25) is bounded as follows

$$\begin{aligned}\|\bar{u}_y^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y(x,z))\|_{L^\infty_z(H^{7/12}_x)} &\lesssim \|\bar{u}_y^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y(x,z))\|_{H^{3/4}_z(H^{7/12}_x)}\\ &\lesssim \|\bar{u}_y^2-(\bar{u}'_y)^2\|_{H^{7/12}_x H^{3/4}_y}+\|\bar{u}_y^2-(\bar{u}'_y)^2\|_{L^2_x H^{23/12}_y}.\end{aligned}$$

Note that $H^{7/12}_x H^{3/4}_y \hookrightarrow L^\infty$, and therefore it is an algebra. Hence

$$\|\bar{u}_y^2-(\bar{u}'_y)^2\|_{H^{7/12}_x H^{3/4}_y} \lesssim \|\bar{u}_y-\bar{u}'_y\|_{H^{7/12}_x H^{3/4}_y}\left(\|\bar{u}_y\|_{H^{7/12}_x H^{3/4}_y}+\|\bar{u}'_y\|_{H^{7/12}_x H^{3/4}_y}\right) \lesssim \|\bar{u}-\bar{u}'\|_{Q^{1/2}}. \tag{4.27}$$

Concerning the other term, we have

$$\begin{aligned}\|\bar{u}_y^2-(\bar{u}'_y)^2\|_{H^{23/12}_y} &\lesssim \|\bar{u}_y\bar{u}_{yy}-\bar{u}'_y\bar{u}'_{yy}\|_{H^{11/12}_y}\\ &\lesssim \|\bar{u}_y-\bar{u}'_y\|_{H^{11/12}_y}\|\bar{u}_{yy}\|_{H^{11/12}_y}+\|\bar{u}_{yy}-\bar{u}'_{yy}\|_{H^{11/12}_y}\|\bar{u}'_y\|_{H^{11/12}_y}.\end{aligned}$$

Note that

$$\|\bar{u}_{yy}\|_{L^\infty_x(H^{11/12}_y)} \lesssim \|\bar{u}_{yy}\|_{H^{2/3}_x(H^{11/12}_y)} \lesssim \|\bar{u}-y\|_{Q^1},$$

and

$$\|\bar{u}_{yy}-\bar{u}'_{yy}\|_{L^2_x H^{11/12}_y} \lesssim \|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We deduce that

$$\|\bar{u}_y^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y(x,z))\|_{L^\infty_z(H^{7/12}_x)} \lesssim \|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We now address the term (4.26), which we write as

$$(\bar{u}'_y)^2(x,Y(x,z))-(\bar{u}'_y)^2(x,Y'(x,z)) = 2(Y(x,z)-Y'(x,z))\int_0^1 (\bar{u}'_y\bar{u}'_{yy})(x,\tau Y+(1-\tau)Y')\,\mathrm{d}\tau.$$

Hence, using Lemma B.4 and Lemma 4.3

$$\begin{aligned}\|(4.26)\|_{H^{3/4}_y H^{7/12}_x} &\lesssim \left(\|Y-Y'\|_{H^{3/4}_y H^{7/12}_x}+\|Y-Y'\|_{L^2_x H^{23/12}_y}\right)\\ &\quad\times\left(\|\bar{u}_y\|_{H^{3/4}_y H^{7/12}_x}+\|\bar{u}_y\|_{L^2_x H^{23/12}_y}\right)\\ &\quad\times\left(\|\bar{u}_{yy}\|_{H^{3/4}_y H^{7/12}_x}+\|\bar{u}_{yy}\|_{L^2_x H^{23/12}_y}\right)\\ &\lesssim \|\bar{u}-\bar{u}'\|_{Q^{1/2}}.\end{aligned} \tag{4.28}$$

Gathering (4.27) and (4.28), we infer that

$$\|\alpha-\alpha'\|_{L^\infty_z(H^{7/12}_x)} \lesssim \|\bar{u}-\bar{u}'\|_{Q^{1/2}}.$$

We then address the bounds on $\gamma_1 - \gamma_1'$. As previously, we write

$$(\gamma_1 - \gamma_1')(x,z) = (\bar{u}_x(x,Y(x,z)) - \bar{u}_x(x,Y'(x,z))) + (\bar{u}_x(x,Y'(x,z)) - \bar{u}_x'(x,Y'(x,z)))\,.$$

Using Lemma 4.3, the first term is bounded in the following way

$$\|\bar{u}_x(x,Y(x,z)) - \bar{u}_x(x,Y'(x,z))\|_{L^\infty_z(L^2_x)} \le \|\bar{u}_{xy}\|_{L^2_x(L^\infty_y)}\|Y - Y'\|_\infty \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.$$

For the second term, recall that $Q^{1/2} \hookrightarrow C^0([-1,1], H^1_x)$. Therefore, using once again Lemma B.4,

$$\begin{aligned}\|\partial_x((\bar{u} - \bar{u}')(x,Y'(x,z)))\|_{L^\infty_z(L^2_x)} \lesssim & \|(\bar{u} - \bar{u}')(x,Y'(x,z)))\|_{H^{7/6}_x L^2_z} + \|(\bar{u} - \bar{u}')(x,Y'(x,z)))\|_{L^2_x H^{7/2}_z} \\ \lesssim & \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.\end{aligned}$$

Since

$$\bar{u}_x(x,Y'(x,z)) - \bar{u}_x'(x,Y'(x,z)) = \partial_x((\bar{u} - \bar{u}')(x,Y'(x,z))) - \partial_x Y' \partial_y(\bar{u} - \bar{u}')(x,Y'(x,z))),$$

we infer, recalling that $Y_x' = -\bar{u}_x'/\bar{u}_y'(x,Y')$,

$$\begin{aligned}& \|\bar{u}_x(x,Y'(x,z)) - \bar{u}_x'(x,Y'(x,z))\|_{L^\infty_z(L^2_x)} \\ \lesssim\ & \|\bar{u} - \bar{u}'\|_{Q^{1/2}} + \|\partial_x Y'\|_{L^\infty_z(L^2_x)}\|\bar{u}_- \bar{u}_y'\|_{L^\infty} \\ \lesssim\ & \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.\end{aligned}$$

Consequently,

$$\|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} \lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.$$

Eventually, we address the bounds on $\gamma_2 - \gamma_2'$, which we decompose as previously as

$$(\gamma_2' - \gamma_2)(x,z) = \big(\bar{u}_{yy}(x,Y(x,z)) - \bar{u}_{yy}'(x,Y(x,z))\big) + \big(\bar{u}_{yy}'(x,Y(x,z)) - \bar{u}_{yy}'(x,Y'(x,z))\big)\,. \quad (4.29)$$

Concerning the second term, we use a Taylor formula

$$\bar{u}_{yy}'(x,Y) - \bar{u}_{yy}'(x,Y') = (Y - Y')\int_0^1 \partial_y^3 \bar{u}'(x,\tau Y + (1-\tau)Y')d\tau. \quad (4.30)$$

From there, it follows that for some $\sigma > 1/2$, using Lemma 4.3 and Lemma B.3,

$$\begin{aligned}\|\bar{u}_{yy}'(x,Y) - \bar{u}_{yy}'(x,Y')\|_{L^2_z(H^{1/2}_x)} &\le \|Y - Y'\|_{L^\infty_z(H^{1/2}_x)}\left\|\int_0^1 \partial_y^3 \bar{u}'(x,\tau Y + (1-\tau)Y')d\tau\right\|_{L^2_z(H^\sigma_x)} \\ &\le \|Y - Y'\|_{L^\infty_z(H^{1/2}_x)}\left(\|\partial_y^3 \bar{u}'\|_{L^2_z(H^\sigma_x)} + \|\partial_y^3 \bar{u}'\|_{H^{2\sigma}_z(L^2_x)}\right) \\ &\lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}\|\bar{u}'\|_{Q^1}.\end{aligned} \quad (4.31)$$

We then address the first term in (4.29). Using Lemma B.4, we obtain

$$\begin{aligned}\|(\bar{u}_{yy} - \bar{u}_{yy}')(x,Y(x,z))\|_{L^2_z(H^{1/2}_x)} &\lesssim \left(\|\bar{u}_{yy} - \bar{u}_{yy}'\|_{L^2_z(H^{1/2}_x)} + \|\bar{u}_{yy} - \bar{u}_{yy}'\|_{L^2_x H^1_y}\right) \\ &\lesssim \|\bar{u} - \bar{u}'\|_{Q^{1/2}}.\end{aligned} \quad (4.32)$$

This completes the proof. □

4.3 Stability of the dual profiles

The penultimate step of this section consists in evaluating the dependency of the profiles Θ^j in terms of the coefficients α, γ. This is crucial to have a continuity estimate of the linear forms $\ell^j_{\bar{u}}$ in terms of $\bar{u}$. Combined with the estimates of Lemma 4.6, the following lemma will lead immediately to Proposition 4.1 in Section 4.4.

We consider the profiles Θ^j and $(\Theta^j)'$ constructed in Proposition 3.10.

Lemma 4.7. *Let (α, γ) and (α', γ') be two sets of coefficients satisfying the assumptions of Proposition 3.1. Let $\Theta^j, (\Theta^j)'$ be the associated solutions to* (3.41)-(3.42).

Then

$$\begin{aligned}\|\Theta^j - (\Theta^j)'\|_{L^2_x(H^1_z)} + \|z\partial_x(\Theta^j - (\Theta^j)')\|_{L^2_x(H_z^{-1})} \\ \lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)} + \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_2 - \gamma_2'\|_{H^{1/2}_x L^2_z}.\end{aligned} \tag{4.33}$$

As a consequence,

$$\begin{aligned}\|\Phi^j - (\Phi^j)'\|_{L^2(\Omega)} + \int_0^1 z(\Phi^j - (\Phi^j)')^2(x_0, z)\,\mathrm{d}z + \int_{-1}^0 |z|(\Phi^j - (\Phi^j)')^2(x_1, z)\,\mathrm{d}z \\ \lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)} + \|\gamma_1 - \gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_2 - \gamma_2'\|_{H^{1/2}_x L^2_z}.\end{aligned} \tag{4.34}$$

Proof. In order to alleviate the notation, we drop the superscripts j, choosing one of the cases $j = 0$ or $j = 1$ (the two cases are similar.) Following Proposition 3.10, we introduce Θ, Θ' defined by (3.38), and we define $\theta = \Theta - \Theta'$. Note that $[\theta]_{|z=0} = 0$, so that $\theta \in L^2_x(H^1_z)$. We also denote by G, G' the operators defined in (3.40) associated with $(\alpha, \gamma), (\alpha', \gamma')$.

It follows that θ satisfies

$$\begin{aligned}-\partial_z(\partial_z\theta + G[\theta]) - \frac{z}{\alpha}\partial_x\theta = \left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right) z\partial_x\Theta' + \partial_z(G - G')[\Theta'], \\ \theta(\cdot, \pm 1) = 0, \\ \theta(x_1, z) = 0 \quad \forall z > 0, \\ \theta(x_0, z) = 0 \quad \forall z < 0.\end{aligned} \tag{4.35}$$

Then, according to Lemma 3.11, it suffices to estimate the right-hand side in $L^2(H^{-1})$.

Step 1. *Estimate of $(1/\alpha - 1/\alpha')z\partial_x\Theta'$ in $L^2(H^{-1})$.* Using the Sobolev embedding $L^q(-1,1) \hookrightarrow H^{-1}(-1,1)$ for all $q > 1$, we obtain

$$\begin{aligned}\left\| z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta' \right\|_{L^2_x(H_z^{-1})} &\leq \left\| z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta' \right\|_{L^2_x(L_z^{4/3})} \\ &\leq C\|\alpha - \alpha'\|_{L^\infty_x(L^4_z)}\|z\partial_x\Theta'\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the Z^0 bounds on Θ' from Lemma 3.12, we get

$$\left\| z\left(\frac{1}{\alpha} - \frac{1}{\alpha'}\right)\partial_x\Theta' \right\|_{L^2_x(H_z^{-1})} \lesssim \|\alpha - \alpha'\|_{L^\infty} \lesssim \|\alpha - \alpha'\|_{L^\infty_z(H^{7/12}_x)}. \tag{4.36}$$

Step 2. *Estimate of* $(G-G')[\Theta']$ *in* $L^2(\Omega)$. As we already did several times, we write $\Gamma = z\Gamma_1+\Gamma_2$. We focus on the bound of $(G-G')[\Theta']$ in $L^2(\Omega_+)$, since the bound in Ω_- is identical. Note that

$$(\Gamma-\Gamma')\Theta' + \int_x^{x_1} \partial_x(\Gamma-\Gamma')\Theta' = -\int_x^{x_1} (\Gamma-\Gamma')\partial_x\Theta'.$$

We start with the terms involving Γ_1 and Γ_1'. We have

$$\left\| \int_x^{x_1} z(\Gamma_1-\Gamma_1')\partial_x\Theta' \right\|_{L^2(\Omega_+)} \leq \|z\partial_x\Theta'\|_{L^2(\Omega_+)} \|\Gamma_1-\Gamma_1'\|_{L^\infty_z(L^2_x)}. \tag{4.37}$$

Since

$$\Gamma_1-\Gamma_1' = \frac{\gamma_1-\gamma_1'}{\alpha} + \gamma_1'\frac{\alpha'-\alpha}{\alpha\alpha'},$$

we get

$$\begin{aligned}\|\Gamma_1-\Gamma_1'\|_{L^\infty_z(L^2_x)} &\lesssim \|\gamma_1-\gamma_1'\|_{L^\infty_z(L^2_x)} + \|\gamma_1'\|_{L^\infty_z(L^2_x)}\|\alpha-\alpha'\|_\infty \\ &\lesssim \|\gamma_1-\gamma_1'\|_{L^\infty_z(L^2_x)} + \|\alpha-\alpha'\|_{L^\infty_z(H^{7/12}_x)}.\end{aligned}$$

Hence

$$\left\| \int_x^{x_1} z(\Gamma_1-\Gamma_1')\partial_x\Theta' \right\|_{L^2(\Omega_+)} \lesssim \|\gamma_1-\gamma_1'\|_{L^\infty_z(L^2_x)} + \|\alpha-\alpha'\|_{L^\infty_z(H^{7/12}_x)}. \tag{4.38}$$

We now address the terms with Γ_2, Γ_2'. Integrating by parts, we have

$$\begin{aligned}\left\| \int_x^{x_1} (\Gamma_2-\Gamma_2')\partial_x\Theta' \right\|_{L^2(\Omega_+)} &= \sup_{h\in L^2(\Omega_+), \|h\|_{L^2}\leq 1} \int_{\Omega_+} h \int_x^{x_1} (\Gamma_2-\Gamma_2')\partial_x\Theta' \\ &= \sup_{h\in L^2(\Omega_+), \|h\|_{L^2}\leq 1} \int_{\Omega_+} \left(\int_{x_0}^x h\right)(\Gamma_2-\Gamma_2')\partial_x\Theta'.\end{aligned}$$

Then, for any $z\in(0,1)$, using Lemma B.10, we have

$$\left| \int_{x_0}^{x_1} \left(\int_{x_0}^x h\right)(\Gamma_2-\Gamma_2')\partial_x\Theta' \right| \lesssim \|h\|_{L^2_x} \|\Gamma_2-\Gamma_2'\|_{H^{1/2}_x} \|\Theta'\|_{H^{1/2}_{00_r}(x_0,x_1)}. \tag{4.39}$$

Integrating over $z\in(0,1)$ and using the embedding of Lemma B.14, we obtain, for any $h\in L^2(\Omega_+)$,

$$\begin{aligned}\left| \int_{\Omega_+} h \int_x^{x_1} (\Gamma_2-\Gamma_2')\partial_x\Theta' \right| &\lesssim \|h\|_{L^2_{x,z}} \|\Gamma_2-\Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{L^\infty_z H^{1/2}_{00_r}(x_0,x_1)} \\ &\lesssim \|h\|_{L^2_{x,z}} \|\Gamma_2-\Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{Z^0(\Omega_+)}.\end{aligned} \tag{4.40}$$

Hence

$$\left\| \int_x^{x_1} (\Gamma_2-\Gamma_2')\partial_x\Theta' \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_2-\Gamma_2'\|_{L^2_z H^{1/2}_x} \|\Theta'\|_{Z^0(\Omega_+)}. \tag{4.41}$$

Now, writing

$$\Gamma_2-\Gamma_2' = \frac{\gamma_2-\gamma_2'}{\alpha} + \gamma_2'\frac{\alpha'-\alpha}{\alpha\alpha'},$$

we obtain

$$\begin{aligned}\|\Gamma_2-\Gamma_2'\|_{L^2_zH^{1/2}_x}\lesssim&\|\gamma_2-\gamma_2'\|_{L^2_zH^{1/2}_x}\|\alpha\|_{L^\infty_z(H^1_x)}\\&+\|\gamma_2'\|_{L^\infty_z(H^{3/5}_x)}(\|\alpha\|_{L^\infty_z(H^1_x)}+\|\alpha'\|_{L^\infty_z(H^1_x)})\|\alpha-\alpha'\|_{L^\infty_z(H^{1/2}_x)}.\end{aligned}\tag{4.42}$$

Therefore

$$\left\|\int_x^{x_1}(\Gamma_2-\Gamma_2')\partial_x\Theta'\right\|_{L^2(\Omega_+)}\lesssim\|\gamma_2-\gamma_2'\|_{L^2_zH^{1/2}_x}+\|\alpha-\alpha'\|_{L^\infty_z(H^{1/2}_x)}.\tag{4.43}$$

Gathering (4.36), (4.38) and (4.43), we obtain the result announced in the lemma. □

4.4 Proof of the stability of the orthogonality conditions

Let us now say a few words about the proof of Proposition 4.1. Let $\bar u,\bar u'$ be two profiles such that $\|\bar u-y\|_{Q^1}\le\bar c$, $\|\bar u'-y\|_{Q^1}\le\bar c$. According to Lemma 4.4, the coefficients α,α' $\gamma_1,\gamma_1',\gamma_2,\gamma_2'$ satisfy the smallness assumptions of Proposition 3.1. Hence, by Proposition 3.10, we can construct the profiles Θ^j, $(\Theta^j)'\in Z^0(\Omega_\pm)$ associated with α,γ and α',γ' respectively.

We now use Lemma 4.6 together with Lemma 4.7, from which we infer that

$$\begin{aligned}\|\Theta^j-(\Theta^j)'\|^2_{L^2(\Omega)}+\int_0^1 z(\Theta^j-(\Theta^j)')^2(x_0,z)\,\mathrm{d}z+\int_{-1}^0|z|(\Theta^j-(\Theta^j)')^2(x_1,z)\,\mathrm{d}z\\\lesssim\|\bar u-\bar u'\|^2_{Q^{1/2}}.\end{aligned}\tag{4.44}$$

Now, let $\Xi=(f,\delta_0,\delta_1)\in\mathcal{H}$. In this case, note that $f(x_i,\cdot)=0$. Furthermore, since $\bar u(x_i,0)=0$, we have $Y(x_i,0)=0$ and thus $\widetilde{\delta_i}(0)=\widetilde{\delta_i}'(0)=0$. Additionally, there exists a universal constant $\bar C$ such that $\bar C^{-1}z\le Y(x_i,z)\le\bar Cz$.

Therefore, the linear forms $\ell^j_{\bar u}(\Xi)$ can be written as

$$\begin{aligned}\ell^j_{\bar u}(\Xi):=&\int_\Omega\partial_x\left(\frac{f(x,Y(x,z))}{\alpha}\right)\Theta^j+\int_0^1\frac{1}{\alpha(x_0,z)}\delta_0''(Y(x_0,z))\Theta^j(x_0,z)\,\mathrm{d}z\\&-\int_{-1}^0\frac{1}{\alpha(x_1,z)}\delta_1''(Y(x_1,z))\Theta^j(x_1,z)\,\mathrm{d}z\\&+\int\left(z\partial_x\Theta^j\Gamma_1-\Theta^j\partial_x\Gamma_2\right)\left(\mathbf{1}_{z<0}\frac{\delta_1'(Y(x_1,z))}{\bar u_y(x_1,Y(x_1,z))}+\mathbf{1}_{z>0}\frac{\delta_0'(Y(x_0,z))}{\bar u_y(x_0,Y(x_0,z))}\right).\end{aligned}\tag{4.45}$$

Using the definition of the $\mathcal{H}$ norm together with (4.44) and with the estimates from Lemma 4.3 and Lemma 4.6, we infer that

$$\left|\ell^j_{\bar u}(\Xi)-\ell^j_{\bar u'}(\Xi)\right|\lesssim\|\bar u-\bar u'\|_{Q^{1/2}}\|\Xi\|_{\mathcal{H}}.\tag{4.46}$$

This completes the proof of Proposition 4.1.

5 Interpolation estimate for the linearized problem

This section is devoted to a rather technical step of the proof, whose necessity we now justify. As explained in the introduction, the proof of Theorem 3 will rely on an iterative scheme. More precisely, we will construct a sequence $(u_n)_{n\in\mathbb{N}}$, and we will use the estimates of Proposition 3.20 to

prove that, under orthogonality conditions (that change at every step!), u_n is uniformly bounded in Q^1. However, in order to prove the convergence, we will need to show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Because of the nonlinear term uu_x in the equation, it seems too difficult to show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Q^1. In view of the Lipschitz continuity of the applications $\bar{u} \in Q^{1/2} \mapsto \ell^j_{\bar{u}}$ (see Proposition 4.1), it seems more reasonable to try to prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$. To that end, it is natural to interpolate the Q^0 and Q^1 estimates from Proposition 3.20. This step is precisely the purpose of this section. However, because of the orthogonality conditions, justifying that the interpolate space for the source terms is the expected one turns out to be quite complicated. In fact, the proof of this result relies strongly on the shape of the singular profiles $\bar{u}^i_{\text{sing}}$ (see Definition 2.22).

The main result of this section is the following estimate in the intermediate space $Q^{1/2}$. We recall that the Lions–Magenes spaces $H^{1/2}_{00}$, $H^{1/2}_{00_l}$ and $H^{1/2}_{00_r}$ are defined in Section 1.5.3. We denote by $H^{3/2}_0(-1,1)$ the space $\{f \in H^{3/2}(-1,1), f(\pm 1) = 0\}$.

Proposition 5.1. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$ and such that $\partial^k_y(\bar{u} - y)(x_i, 0) = 0$ for $k = 0, 1, 2$ and $i = 0, 1$, and $\bar{u}(x, \pm 1) = \pm 1$. Let f such that*

$$f \in L^2_x H^{3/2}_0 \cap H^{1/2}_{00_l} L^2_y(\Omega_+) \cap H^{1/2}_{00_r} L^2_y(\Omega_-), \tag{5.1}$$

$$f \in H^{1/2}_{00_l}((x_0, x_1); H^{1/2}_{00_r}(1/4, 1)) \cap H^{1/2}_{00_r}((x_0, x_1); H^{1/2}_{00_l}(-1, -1/4)) \tag{5.2}$$

Assume that $\ell^0_{\bar{u}}(f, 0, 0) = \ell^1_{\bar{u}}(f, 0, 0) = 0$. Define the coefficients α and γ by (3.5) and (3.6). Let u be the unique solution to (3.1) with $\delta_0 = \delta_1 = 0$, i.e.

$$\begin{cases} \bar{u}\partial_x u - \partial_{yy} u = f \quad \text{in } \Omega, \\ u_{|\Sigma_i} = 0, \\ u_{|y=\pm 1} = 0. \end{cases} \tag{5.3}$$

Then $u \in Q^{1/2}$ and

$$\begin{aligned} \|u\|_{Q^{1/2}} \lesssim \|f\|_{L^2_x H^{3/2}_0} + \|f\|_{H^{1/2}_{00_l} L^2_y(\Omega_+)} + \|f\|_{H^{1/2}_{00_r} L^2_y(\Omega_-)} \\ + \|f\|_{H^{1/2}_{00_l}((x_0,x_1);H^{1/2}_{00_r}(1/4,1))} + \|f\|_{H^{1/2}_{00_r}((x_0,x_1);H^{1/2}_{00_l}(-1,-1/4))}. \end{aligned} \tag{5.4}$$

As explained above, Proposition 5.1 stems from the Q^0 and Q^1 estimates from Proposition 3.20, which we shall interpolate. However, because of the orthogonality conditions, interpolating is not immediate. We introduce the following spaces for the source terms

$$\mathcal{Y}_0 := L^2(\Omega), \tag{5.5}$$

$$\mathcal{Y}_1 := \left\{ f \in H^1_x L^2_y \cap L^2_x H^3_y; \;\; f_{|\Sigma_0 \cup \Sigma_1} = 0 \right\} \tag{5.6}$$

endowed with their usual norms and

$$\mathcal{Y}_1^{\bar{u}} := \left\{ f \in \mathcal{Y}_1; \;\; \ell^0_{\bar{u}}(f, 0, 0) = \ell^1_{\bar{u}}(f, 0, 0) = 0 \right\}, \tag{5.7}$$

endowed with the norm of $\mathcal{Y}_1$, where $\ell^0_{\bar{u}}$ and $\ell^1_{\bar{u}}$ are defined in Definition 3.15. Since $\ell^0_{\bar{u}}(\cdot, 0, 0)$ and $\ell^1_{\bar{u}}(\cdot, 0, 0)$ are continuous for the $H^1_x L^2_y$ norm, $\mathcal{Y}_1^{\bar{u}}$ is a closed subspace of $\mathcal{Y}_1$.

In order to obtain the desired regularity of our solutions in the interior of the domain (i.e. close to the line $y = 0$) we need to interpolate between $\mathcal{Y}_0$ and $\mathcal{Y}_1^{\bar{u}}$. Using classical interpolation theory (see Lemma 5.17), one can determine $\mathcal{Y}_{1/2} := [\mathcal{Y}_0, \mathcal{Y}_1]_{1/2}$ quite easily[5].

Nevertheless, there is a difficulty in the determination of the space $[\mathcal{Y}_0, \mathcal{Y}_1^{\bar{u}}]_{1/2}$. This corresponds to the well-known problem of "subspace interpolation", for which we give a short survey in Section 5.3. An occurrence of this difficulty was already encountered in Section 1.5.3, since the space $H_{00}^{1/2}$ illustrates that the half-interpolate of $L^2(x_0, x_1)$ and $H_0^1(x_0, x_1)$ is neither $H^{1/2}(x_0, x_1)$, nor a closed subspace of codimension 2 of $H^{1/2}(x_0, x_1)$, but involves an "added norm" (see the integral term in (1.21)), which is reminiscent of the orthogonality conditions $f(x_0) = f(x_1) = 0$.

Therefore, one crucial result of this section is the following lemma.

Lemma 5.2. *Let $\bar{u} \in Q^1$ satisfying the assumptions of Proposition 5.1. The linear forms $\ell^0_{\bar{u}}(\cdot, 0, 0)$ and $\ell^1_{\bar{u}}(\cdot, 0, 0)$ admit continuous extensions to $\mathcal{Y}_{1/2}$. Moreover, the half-interpolate between $\mathcal{Y}_0$ and $\mathcal{Y}_1^{\bar{u}}$ of (5.7) is*

$$\mathcal{Y}_{1/2}^{\bar{u}} := \left[\mathcal{Y}_0, \mathcal{Y}_1^{\bar{u}}\right]_{1/2} = \left\{ f \in \mathcal{Y}_{1/2};\ \ \ell^0_{\bar{u}}(f, 0, 0) = \ell^1_{\bar{u}}(f, 0, 0) = 0 \right\}, \tag{5.8}$$

endowed with the norm of $\mathcal{Y}_{1/2}$ (see Lemma 5.17).

This lemma relies on a careful analysis of the dual profiles Θ^j, and in particular on a decomposition of the latter into an explicit singular part and a regular part. This decomposition allows us to have quantitative upper and lower bounds on the functions $\tau \mapsto N(\tau, \ell^j_{\bar{u}})$, which play a paramount role in interpolation theory (see [39] and Section 5.1.2 below).

The organization of this section is as follows. We start by introducing the theory of subspace interpolation, and associated notations in Section 5.1. Then, as a first step, we prove an interpolated well-posedness theory for the case of the linear shear flow in Section 5.2, illustrating how the general theory can be applied for our problem, thanks to the knowledge of the singular profiles of Section 2.5. Using this first step, we will deduce in Appendix D a decomposition result for the profiles Θ^j into a regular part and a singular part involving the profiles $\bar{u}^i_{\text{sing}}$. This decomposition allows us to identify the interpolation space for the source terms and prove Lemma 5.2 in Section 5.3. Eventually, we prove the $Q^{1/2}$ estimate of Proposition 5.1 in Section 5.4.

5.1 A primer on subspace interpolation

Using interpolation theory in a context where one needs to enforce constraints on the data comes with a specific difficulty, known as "subspace interpolation". In this subsection, we give a short introduction and set up notations and a lemma that will be used in the next subsections.

5.1.1 An introduction to subspace interpolation

Let us start by a short introduction to the topic of subspace interpolation and the associated difficulty. This difficulty is *not* linked with the difference between complex and real interpolation methods. Indeed, it occurs even in the case of "quadratic" interpolation between separable Hilbert spaces, for which all methods construct the same interpolation spaces (see [14, Remark 3.6] and [15, Section 3.3, item (4)] based on the initial geometric argument of [43]).

[5]Due to the boundary condition $f_{|\Sigma_0 \cup \Sigma_1} = 0$, it is not exactly true that $\mathcal{Y}_{1/2} = H_x^{1/2} L_y^2 \cap L_x^2 H_y^{3/2}$ (see (5.67))

Setting of the problem Let $\mathcal{X}_0$ and $\mathcal{X}_1$ denote two Banach spaces with a dense continuous embedding $\mathcal{X}_1 \hookrightarrow \mathcal{X}_0$. Let $\mathcal{X}_\sigma := [\mathcal{X}_0, \mathcal{X}_1]_\sigma$, for $\sigma \in (0,1)$, say, for the complex method to fix ideas. Let ℓ be a continuous linear form on $\mathcal{X}_1$, which is however unbounded on $\mathcal{X}_0$[6], and define its kernel $\mathcal{X}_1^\ell := \{f \in \mathcal{X}_1;\ \ell(f) = 0\}$, which is a closed subspace of $\mathcal{X}_1$. The question of "subspace interpolation" consists in the determining the relation between $\mathcal{X}_\sigma$ and $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$. This question of course admits a straightforward generalization to the case of a finite number of orthogonality conditions.

Generally, one checks that the closure of $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ in $\mathcal{X}_\sigma$ is either a subspace of codimension one, when ℓ is continuous on $\mathcal{X}_\sigma$, or the whole of $\mathcal{X}_\sigma$, when ℓ is unbounded on $\mathcal{X}_\sigma$. In the former case, there is no guarantee that $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ itself is closed in $\mathcal{X}_\sigma$ (or, equivalently, that the associated norms are equivalent on $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$). The first systematic occurrence of this question seems to date back to [42, Problem 18.5, Chapter 1], which claims that a major difficulty to use interpolation theory is that "*l'interpolé de sous-espaces fermés n'est pas nécessairement un sous-espace fermé dans l'interpolé*" (the interpolation space between closed subspaces is not necessarily a closed subspace in the interpolation space), and asks for sufficient conditions for $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ to be closed in $\mathcal{X}_\sigma$.

Some examples The best known and most simple example of such a phenomenon, introduced in [42, Theorem 11.7, Chapter 1], was already discussed in Section 1.5.3, with the construction of the space $H_{00}^{1/2}(x_0, x_1) = [L^2(x_0, x_1), H_0^1(x_0, x_1)]_{1/2}$. As already noted, $H_{00}^{1/2}(x_0, x_1)$ is not closed in $H^{1/2}(x_0, x_1)$, since its norm involves a non-equivalent "additional term" (see (1.21)).

In [58], using real interpolation between L^1 and L^∞, Wallstén constructed examples illustrating that this pathological behavior is not limited to exceptional values of the interpolation parameter, since there exist constraints for which it occurs for every $\sigma \in (0,1)$.

Short survey of known results Precising earlier results of Löfström [39, 40], Ivanov and Kalton proved in [29] that, in the general case, there exist two thresholds $0 \le \sigma_0 \le \sigma_1 \le 1$ such that:

- when $0 < \sigma < \sigma_0$, $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma = \mathcal{X}_\sigma$, with equivalent norms,
- when $\sigma_0 \le \sigma \le \sigma_1$, the norm on $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ is not equivalent to the one on $\mathcal{X}_\sigma$,
- when $\sigma_1 < \sigma < 1$, $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ is a closed subspace of codimension 1 in $\mathcal{X}_\sigma$.

In the first case, ℓ is unbounded on $\mathcal{X}_\sigma$ (the constraint does not make sense). In the second and third cases, ℓ admits a continuous extension to $\mathcal{X}_\sigma$ and the closure of $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ in $\mathcal{X}_\sigma$ is of codimension one.

This classification has generalizations to the case of multiple constraints (see [3]), potentially involving multiple pathological intervals, associated with each constraint.

In the difficult regime $\sigma_0 \le \sigma \le \sigma_1$, more precise results [6, 7] allow computing the "additional norm" stemming from the presence of the constraints.

The recent work [59] considers a kind of dual problem, by computing interpolation spaces between $\mathcal{X}_0$ and $\mathcal{X}_1 \oplus \mathbb{R}\omega$, where ω is a singular function of $\mathcal{X}_0 \setminus \mathcal{X}_1$, whose singularity is expressed in polar coordinates. In this work, $\sigma_0 = \sigma_1$. This is also our case below, and our dual profiles also involve singular parts which are expressed in radial-like coordinates, as constructed in Section 2.5.

[6] When ℓ is continuous for the topology of $\mathcal{X}_0$, there is no difficulty. Indeed, one checks that, for every $\sigma \in (0,1)$, $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma = \{f \in \mathcal{X}_\sigma;\ \ell(f) = 0\}$, endowed with the topology of $\mathcal{X}_\sigma$, for which ℓ is continuous (see e.g. the related result [42, Theorem 13.3, Chapter 1]).

5.1.2 A variant of a criterium due to Löfström

To prove Lemma 5.2, we will rely on an abstract interpolation result proved by Löfström in [39]. Let $\mathcal{X}_0$ and $\mathcal{X}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{X}_1 \hookrightarrow \mathcal{X}_0$.

For $f \in \mathcal{X}_1$ and $\tau \in (0,1)$, let

$$\|f\|_\tau^2 := \|f\|_{\mathcal{X}_0}^2 + \tau^2 \|f\|_{\mathcal{X}_1}^2. \tag{5.9}$$

This notation is actually inspired by [29], as [39] uses instead the quantity

$$J(\tau, f) := \max\left(\|f\|_{\mathcal{X}_0}, \tau\|f\|_{\mathcal{X}_1}\right). \tag{5.10}$$

One checks however that $J(\tau,f) \le \|f\|_\tau \le \sqrt{2} J(\tau,f)$, so they can be used equivalently.

Given a linear form ℓ on $\mathcal{X}_1$, one defines, for $\tau \in (0,1)$,

$$N(\tau, \ell) := \sup_{f \in \mathcal{X}_1 \setminus \{0\}} \frac{\ell(f)}{\|f\|_\tau}. \tag{5.11}$$

As $\tau \to 0$, upper bounds on $N(\tau,\ell)$ are linked with the boundedness of ℓ on intermediate spaces between $\mathcal{X}_0$ and $\mathcal{X}_1$, while lower bounds on $N(\tau,\ell)$ are linked with the non-degeneracy of ℓ on these spaces. In particular, one has the following result, which is a reformulation of [39, Theorem 2] in the particular case of two constraints having the same "order".

Lemma 5.3. *Let $\mathcal{X}_0$ and $\mathcal{X}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{X}_1 \hookrightarrow \mathcal{X}_0$. Let ℓ^0, ℓ^1 be two linear forms on $\mathcal{X}_1$. Assume that there exists $C_\pm > 0$ and $\bar\sigma \in (0,1)$ such that, for every $(c_0,c_1) \in \mathbb{S}^1$ and every $\tau \in (0,1)$,*

$$C_- \tau^{-\bar\sigma} \le N(\tau, c_0\ell^0 + c_1\ell^1) \le C_+ \tau^{-\bar\sigma}. \tag{5.12}$$

As in Section 5.1.1, let $\mathcal{X}_1^\ell := \{f \in \mathcal{X}_1;\ \ell^0(f) = \ell^1(f) = 0\}$ and, for $\sigma \in (0,1)$, $\mathcal{X}_\sigma := [\mathcal{X}_0, \mathcal{X}_1]_\sigma$, for the complex interpolation method. Then,

- *for every $\sigma \in (0, \bar\sigma)$, $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma = \mathcal{X}_\sigma$, with equivalent norms,*
- *for every $\sigma \in (\bar\sigma, 1)$, the linear forms ℓ^0 and ℓ^1 have continuous extensions to $\mathcal{X}_\sigma$ and $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma = \{f \in \mathcal{X}_\sigma;\ \ell^0(f) = \ell^1(f) = 0\}$, endowed with the norm of $\mathcal{X}_\sigma$.*

Remark 5.4. *Lemma 5.3 does not say anything on $[\mathcal{X}_0, \mathcal{X}_1^\ell]_\sigma$ for the critical value $\sigma = \bar\sigma$. In fact, with the notations of [29] mentioned above, one has $\sigma_0 = \sigma_1 = \bar\sigma$, so the norm of $[\mathcal{X}_0, \mathcal{X}_1^\ell]_{\bar\sigma}$ is not equivalent to the norm of $\mathcal{X}_{\bar\sigma}$.*

Remark 5.5. *In assumption* (5.12)*, it is important to consider arbitrary linear combinations of the two linear forms ℓ^0 and ℓ^1. It would not be sufficient to assume* (5.12) *with $(c_0,c_1) = (1,0)$ and $(c_0,c_1) = (0,1)$. Indeed, the lower bound of this condition ensures that the two linear forms remain sufficiently independent on the intermediate spaces. We state here a formulation giving a symmetrical role to ℓ^0 and ℓ^1, whereas [39] uses a hierarchical formulation. We prove below that our formulation indeed implies Löfström's one.*

Proof of Lemma 5.3. This is an application of [39, Theorem 2]. By (5.12) with $(c_0,c_1) = (1,0)$ and $(c_0,c_1) = (0,1)$, both ℓ^0 and ℓ^1 have "order" $\bar\sigma$ in Löfström's vocabulary. Therefore, there only remains to check that they form a "strongly independent basis", i.e. that there exists $C > 0$ such that, for every $\tau \in (0,1)$,

$$N(\tau, \ell^1) \le C N_0(\tau, \ell^1), \tag{5.13}$$

where

$$N_0(\tau,\ell^1) := \sup\left\{\frac{\ell^1(f)}{\|f\|_\tau};\ \ f\in\mathcal{X}_1\setminus\{0\} \text{ and } \ell^0(f)=0\right\}. \tag{5.14}$$

Let $\tau\in(0,1)$. Denote by $\langle\cdot,\cdot\rangle_\tau$ the scalar product associated with the norm $\|\cdot\|_\tau$ on $\mathcal{X}_1$. By the Riesz representation theorem, there exists $g^0_\tau, g^1_\tau\in\mathcal{X}_1$ such that $\ell^j = \langle g^j_\tau,\cdot\rangle_\tau$. In particular $N(\tau,\ell^j) = \|g^j_\tau\|_\tau$. Moreover, by (5.14), $N_0(\tau,\ell^1)$ is the supremum of ℓ^1 on the intersection of $\ker\ell^0$ with the unit ball in $\mathcal{X}_1$ for the norm $\|\cdot\|_\tau$. Thus, a natural candidate to bound $N_0(\tau,\ell^1)$ from below is the orthogonal projection of $g^1_\tau/\|g^1_\tau\|_\tau$ on $\ker\ell^0$, namely,

$$f^1_\tau := \frac{g^1_\tau}{\|g^1_\tau\|_\tau} - R_\tau\frac{g^0_\tau}{\|g^0_\tau\|_\tau} \quad\text{where}\quad R_\tau := \left\langle\frac{g^1_\tau}{\|g^1_\tau\|_\tau},\frac{g^0_\tau}{\|g^0_\tau\|_\tau}\right\rangle_\tau. \tag{5.15}$$

In particular $\|f^1_\tau\|_\tau = (1-R^2_\tau)^{\frac12}$ and $\ell^0(f^1_\tau) = \langle g^0_\tau, f^1_\tau\rangle = 0$. Thus

$$N_0(\tau,\ell^1) \geq \frac{\langle f^1_\tau, g^1_\tau\rangle_\tau}{\|f^1_\tau\|_\tau} = (1-R^2_\tau)^{\frac12}\|g^1_\tau\|_\tau = (1-R^2_\tau)^{\frac12}N(\tau,\ell^1). \tag{5.16}$$

Thus, to prove (5.13), it is sufficient to prove that the ratio R^2_τ is bounded away from 1. By (5.12), for every $(c_0,c_1)\in\mathbb{S}^1$,

$$C_-\tau^{-\bar\sigma} \leq \|c_0g^0_\tau + c_1g^1_\tau\|_\tau \leq C_+\tau^{-\bar\sigma}. \tag{5.17}$$

In particular,

$$C_-\tau^{-\bar\sigma} \leq \|g^j_\tau\|_\tau \leq C_+\tau^{-\bar\sigma}. \tag{5.18}$$

By homogeneity, for every $(c_0,c_1)\in\mathbb{R}^2$,

$$C^2_-\tau^{-2\bar\sigma}(c^2_0+c^2_1) \leq c^2_0\|g^0_\tau\|^2_\tau + c^2_1\|g^1_\tau\|^2_\tau + 2c_0c_1\langle g^0_\tau,g^1_\tau\rangle_\tau \leq C^2_+\tau^{-2\bar\sigma}(c^2_0+c^2_1). \tag{5.19}$$

Substituting $c_j \leftarrow c_j/\|g^j_\tau\|_\tau$ and using (5.18) leads to the fact that, for every $(c_0,c_1)\in\mathbb{R}^2$,

$$\rho^2(c^2_0+c^2_1) \leq c^2_0+c^2_1+2R_\tau c_0c_1 \leq \rho^{-2}(c^2_0+c^2_1), \tag{5.20}$$

where $\rho := C_-/C_+$. In particular, using $(c_0,c_1) = (1,1)$ and $(1,-1)$ yields $\rho^2\leq 1+R_\tau$ and $\rho^2\leq 1-R_\tau$. Hence, (5.16) proves that

$$N_0(\tau,\ell^1) \geq \rho^2 N(\tau,\ell^1), \tag{5.21}$$

which implies (5.13) with $C=\rho^{-2}$. So ℓ^0 and ℓ^1 form a "strongly independent basis" and Lemma 5.3 follows from [39, Theorem 2]. □

5.2 Interpolated theory in the case of the linear shear flow

In this subsection, we consider the problem (2.1) at the linear shear flow, with vanishing boundary data. We proved in Section 2.2 that, when $f\in L^2_xL^2_z$, the solutions to this problem have Z^0 regularity, and in Section 2.3 that they have Z^1 regularity when $f\in H^1_xL^2_z$ and the two orthogonality conditions (2.19) are satisfied. Here, we establish an interpolated theory for the problem (2.1) with source terms $f\in H^\sigma_xL^2_z$, $\sigma\in(0,1)$, see Lemma 5.11. This interpolated theory involves the difficulty exposed in Section 5.1. We define

$$\mathcal{X}_0 := L^2(\Omega), \tag{5.22}$$

$$\mathcal{X}_1 := \left\{f\in H^1_xL^2_y;\ \ f_{|\Sigma_0\cup\Sigma_1}=0\right\}, \tag{5.23}$$

endowed with their usual norms and

$$\overline{\mathcal{X}_1} := \{f \in \mathcal{X}_1;\ \ \overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0\}. \tag{5.24}$$

For $\sigma \in (0,1)$, let $\mathcal{X}_\sigma := [\mathcal{X}_0, \mathcal{X}_1]_\sigma$. In order to extend the theory of Section 2 to fractional tangential regularity, we start by identifying the spaces $[\mathcal{X}_0, \overline{\mathcal{X}_1}]_\sigma$. More precisely, we prove the following characterization.

Lemma 5.6. *Let $\mathcal{X}_0$, $\mathcal{X}_1$ and $\overline{\mathcal{X}_1}$ as above. Then,*

- *For every $\sigma \in (0,1/6)$, $[\mathcal{X}_0, \overline{\mathcal{X}_1}]_\sigma = \mathcal{X}_\sigma$ with equivalent norms.*
- *For every $\sigma \in (1/6,1)$, the linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ admit continuous extensions to $\mathcal{X}_\sigma$ and*

$$[\mathcal{X}_0, \overline{\mathcal{X}_1}]_\sigma = \left\{ f \in \mathcal{X}_\sigma;\ \ \overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0 \right\}, \tag{5.25}$$

  endowed with the norm of $\mathcal{X}_\sigma$.

Remark 5.7. *The threshold at $\sigma = 1/6$ is consistent with the observation of Remark 2.29 that the maps $\overline{\ell^j}(\cdot,0,0)$ are bounded on $H^\sigma_x L^2_z$ for every $\sigma > 1/6$.*

For $\tau \in (0,1)$, we use the notations of the previous paragraph, in particular the norm $\|\cdot\|_\tau$ of (5.9) and the function $N(\tau,\cdot)$ of (5.11), with $\mathcal{X}_0$ and $\mathcal{X}_1$ defined as above.

To derive the estimates required to apply Lemma 5.3, two strategies would be possible. Both rely on the explicit knowledge of the singular radial solutions constructed in Section 2.5, which are involved in the orthogonality conditions. First, one could impose periodic boundary conditions on f, compute a 2D Fourier-series representation of (an extension by parity of) the singular profiles and estimate the functions N working in the Fourier space. Such a frequency-domain approach is carried out in [6], assuming some appropriate asymptotic decay of the Fourier transform of the profile defining the orthogonality condition. We choose a second strategy, which stays in the spatial domain, and involves estimates using cut-off functions whose space-scale are linked with the parameter τ. This strategy is related to the one used in [59] and inspired by the links between the K functional of real interpolation theory and the notions of modulus of continuity and modulus of smoothness of functions (see e.g. [33]).

To prove Lemma 5.6, we intend to apply Lemma 5.3. Hence, we need to bound from below and from above the functions $N(\tau, \overline{\ell^j})$. By Definition 2.9, $\overline{\ell^j}(f,0,0) = \int_\Omega \partial_x f \overline{\Phi^j}$. As highlighted more generally in Corollary 2.28, the profiles $\overline{\Phi^j}$ can be decomposed as the sum of a singular radial part, an x-independent part, and a regular part. The singular radial part is the one that will be dominating the behavior of the orthogonality conditions. Thus, we start by two lemmas concerning estimates from above and from below for integrals of the form $\int_\Omega (\partial_x f)\bar{u}^i_{\mathrm{sing}}$, before moving to the general case.

Lemma 5.8. *Let $h \in H^1_x L^2_z$ such that $h = 0$ on $\Sigma_0 \cup \Sigma_1$. Then, for $\tau \in (0,1)$,*

$$\left| \int_\Omega (\partial_x h(x,z)) \bar{u}^i_{\mathrm{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z \right| \lesssim \tau^{-1/6} \left(\|h\|_{L^2} + \tau \|\partial_x h\|_{L^2} \right), \tag{5.26}$$

where $\bar{u}^i_{\mathrm{sing}}$ is defined in Definition 2.22.

Proof. By symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.22. We also let $\chi(x, z) := \chi_i(x, -z)$ of Definition 2.22 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.17. With these notations

$$\bar{u}^0_{\text{sing}}(x, -z) = r^{\frac{1}{2}}\Lambda(t)\chi(x, z). \tag{5.27}$$

In particular, since $\Lambda_0(+\infty) = 0$, $\bar{u}^0_{\text{sing}}(x_0, -z) = 0$ for $z \in (-1, 0)$. We split the integral to be estimated depending on whether $r \leq \tau^\alpha$ or $r \geq \tau^\alpha$, where $\alpha > 0$ is to be chosen later. Let $\eta \in C^\infty(\mathbb{R}; [0, 1])$ such that $\eta(s) \equiv 1$ for $s \leq 1$ and $\eta(s) \equiv 0$ for $s \geq 2$.

Step 1. *Estimate in the region:* $r \leq \tau^\alpha$. By Cauchy–Schwarz,

$$\left|\int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi \cdot \eta(r/\tau^\alpha)\right| \leq \|\chi\|_\infty \|\Lambda\|_\infty \|\partial_x h\|_{L^2} \left(\int_\Omega r\eta^2(r/\tau^\alpha)\right)^{\frac{1}{2}}. \tag{5.28}$$

Using the polar-like change of coordinates of (2.45) and (2.48), one has

$$\int_\Omega r\eta^2(r/\tau^\alpha) = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1 + t^2)^2} r\eta^2(r/\tau^\alpha)\, \mathrm{d}r\, \mathrm{d}t \lesssim (\tau^\alpha)^5. \tag{5.29}$$

Hence, in this region,

$$\left|\int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi \cdot \eta(r/\tau^\alpha)\right| \lesssim (\tau^\alpha)^{5/2} \|\partial_x h\|_{L^2}. \tag{5.30}$$

Step 2. *Estimate in the region:* $r \geq \tau^\alpha$. We intend to integrate by parts in x. At $x = x_1$, $\bar{u}^0_{\text{sing}}(x, -z) = 0$ for $z \in (-1, 1)$ because $\chi = 0$. At $x = x_0$, when $z > 0$, $h = 0$ by assumption, and, when $z < 0$, $\bar{u}^0_{\text{sing}}(x, -z) = 0$ as recalled above. Hence, there is no boundary term and

$$\int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi \cdot (1 - \eta(r/\tau^\alpha)) = -\int_\Omega h\partial_x \left(\chi \cdot r^{\frac{1}{2}}\Lambda(t)(1 - \eta(r/\tau^\alpha))\right). \tag{5.31}$$

First, one easily bounds

$$\left|\int_\Omega h\partial_x\chi \cdot r^{\frac{1}{2}}\Lambda(t)(1 - \eta(r/\tau^\alpha))\right| \leq \|h\|_{L^2}\|\Lambda\|_\infty \|\partial_x\chi\|_\infty \max_\Omega r^{\frac{1}{2}} \lesssim \|h\|_{L^2}. \tag{5.32}$$

For the second term, when ∂_x hits on the function expressed in (r, t) coordinates, we use the derivative formula (2.49):

$$\begin{aligned}\int_\Omega h\chi\partial_x \left(r^{\frac{1}{2}}\Lambda(t)(1 - \eta(r/\tau^\alpha))\right) &= \int_0^\infty \int_{\mathbb{R}} h\chi \frac{(1 + t^2)^{\frac{1}{2}}}{3r^2}\Lambda(t)\partial_r \left(r^{\frac{1}{2}}(1 - \eta(r/\tau^\alpha))\right) \mathrm{d}t\, \mathrm{d}r \\ &\quad - \int_0^\infty \int_{\mathbb{R}} h\chi \frac{t(1 + t^2)^{\frac{3}{2}}}{3r^3} r^{\frac{1}{2}}(1 - \eta(r/\tau^\alpha))\partial_t\Lambda(t).\end{aligned} \tag{5.33}$$

We bound both terms using the Cauchy–Schwarz inequality and the polar-like change of coordinates

(2.45) with Jacobian determinant (2.48). In particular, on the one hand,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \frac{1+t^2}{9r^4}\Lambda^2(t)\,\mathrm{d}t\,\mathrm{d}r \left(\partial_r\left(r^{\frac{1}{2}}(1-\eta(r/\tau^\alpha))\right)\right)^2 \\ &= \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\frac{1+t^2}{9r^4}\Lambda^2\left(\partial_r\left(r^{\frac{1}{2}}(1-\eta(r/\tau^\alpha))\right)\right)^2 \,\mathrm{d}t\,\mathrm{d}r \\ &\lesssim \int_0^\infty \left(r^{-1}(1-\eta(r/\tau^\alpha))^2 + r(\eta'(r/\tau^\alpha))^2/(\tau^\alpha)^2\right)\frac{\mathrm{d}r}{r} \\ &= (\tau^\alpha)^{-1}\int_1^\infty \left(s^{-1}(1-\eta(s))^2 + s(\eta'(s))^2\right)\frac{\mathrm{d}s}{s} \lesssim (\tau^\alpha)^{-1}.\end{aligned} \tag{5.34}$$

On the other hand,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \frac{t^2(1+t^2)^3}{9r^6} r(1-\eta(r/\tau^\alpha))^2(\partial_t\Lambda(t))^2\,\mathrm{d}t\,\mathrm{d}r \\ &= \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\frac{t^2(1+t^2)^3}{9r^6} r(1-\eta(r/\tau^\alpha))^2(\partial_t\Lambda(t))^2\,\mathrm{d}t\,\mathrm{d}r \\ &\lesssim \left(\int_{\mathbb{R}} t^2(1+t^2)(\partial_t\Lambda(t))^2\,\mathrm{d}t\right)\int_0^\infty \frac{1}{3r^2}(1-\eta(r/\tau^\alpha))\,\mathrm{d}r \\ &\lesssim (\tau^\alpha)^{-1}\end{aligned} \tag{5.35}$$

by the integrability property $t^3\partial_t\Lambda(t) = O(1)$ of Lemma 2.20.

Thus, gathering the estimates in this region proves that

$$\left|\int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi\cdot(1-\eta(r/\tau^\alpha))\right| \lesssim (\tau^\alpha)^{-\frac{1}{2}}\|h\|_{L^2}. \tag{5.36}$$

Gathering the estimates in both regions and choosing $\alpha = 1/3$ concludes the proof. □

Lemma 5.9. *There exists a family $(h^i_\tau)_{\tau\in(0,1)}$ of non-zero, smooth, compactly supported, functions on Ω such that, as $\tau\to 0$,*

$$\left|\int_\Omega (\partial_x h^i_\tau(x,z))\bar{u}^i_{\mathrm{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z\right| \gtrsim \tau^{-1/6}\left(\|h^i_\tau\|_{L^2} + \tau\|\partial_x h^i_\tau\|_{L^2} + \tau\|\partial_z^3 h^i_\tau\|_{L^2}\right), \tag{5.37}$$

and $\int_\Omega \partial_x h^j_\tau \bar{u}^i_{\mathrm{sing}} = 0$ for $j\neq i$, where $\bar{u}^i_{\mathrm{sing}}$ is defined in Definition 2.22.

Remark 5.10. *The bound from below* (5.37) *involves a term $\|\partial_z^3 h^i_\tau\|_{L^2}$ which is not required at this stage since the space $\mathcal{X}_1$ of* (5.23) *does not involve regularity in the vertical direction. However, it will be used in the proof of Lemma 5.16 (see more precisely* (5.6) *and* (5.62)*).*

Proof. As in the previous lemma, by symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.22. We also let $\chi(x,z) := \chi_i(x,-z)$ of Definition 2.22 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.17. With these notations, one has (5.27).

Let $\alpha > 0$. Let $H \in C^\infty_c(\mathbb{R};[-1,1])$ and $\eta\in C^\infty_c(\mathbb{R};[-1,1])$ such that $\operatorname{supp}\eta\subset(1/2,3/2)$. For $\tau\in(0,1)$, we define

$$h_\tau := \eta(r/\tau^\alpha)H(t). \tag{5.38}$$

By the support properties of H and η, one checks that h_τ is both smooth and compactly supported in Ω. Moreover, it is non-zero if $H \neq 0$ and $\eta \neq 0$.

Let $\tau > 0$ be sufficiently small such that the support of h_τ is included in the region where $\chi \equiv 1$. Note that with this choice, we also have $\int_\Omega \partial_x h_\tau \bar{u}^1_{\mathrm{sing}} = 0$. Then, using the formula (2.49) for ∂_x and the determinant (2.48),

$$\begin{aligned}
&-\int_\Omega \partial_x h_\tau \bar{u}_{\mathrm{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z \\
&\quad= \int_0^\infty \int_{\mathbb{R}} r^{\frac{1}{2}}\Lambda(t)\left(\frac{(1+t^2)^{\frac{1}{2}}}{3r^2}(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\eta(r/\tau^\alpha)H'(t)\right)\frac{3r^3}{(1+t^2)^2}\,\mathrm{d}t\ \mathrm{d}r \\
&\quad= \int_0^\infty \int_{\mathbb{R}} \frac{r^{\frac{1}{2}}\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\left(r(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) - t(1+t^2)\eta(r/\tau^\alpha)H'(t)\right)\,\mathrm{d}t\ \mathrm{d}r \\
&\quad= (\tau^\alpha)^{3/2}\int_{\mathbb{R}} \frac{H(t)\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t \int_0^\infty s^{3/2}\eta'(s)\,\mathrm{d}s - (\tau^\alpha)^{3/2}\int_{\mathbb{R}} \frac{tH'(t)\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\,\mathrm{d}t\int_0^\infty s^{1/2}\eta(s)\,\mathrm{d}s.
\end{aligned} \tag{5.39}$$

In particular, one can choose H and η such that the two first integrals are equal to 1 and such that $\int s^{1/2}\eta(s)\,\mathrm{d}s = 0$, which ensures that

$$\int_\Omega \partial_x h_\tau \cdot r^{\frac{1}{2}}\Lambda(t)\chi = -(\tau^\alpha)^{3/2}. \tag{5.40}$$

Using once again the formula (2.49) for ∂_x and the change of coordinates of Jacobian (2.48), one obtains that $\|h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2$ and $\|\partial_x h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Similarly, using (2.50) to compute $\partial_z^3 h_\tau$ and the same technique, $\|\partial_z^3 h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Thus, choosing $\alpha = 1/3$ leads to

$$\|h_\tau\|_{L^2} + \tau\|\partial_x h_\tau\|_{L^2} + \tau\|\partial_z^3 h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2, \tag{5.41}$$

which concludes the proof. □

We are now ready to prove Lemma 5.6.

Proof of Lemma 5.6. This is an application of Lemma 5.3 with $\bar{\sigma} = 1/6$. Therefore, we need to find constants $C_\pm > 0$ such that, for every $\tau \in (0,1)$ and $(c_0, c_1) \in \mathbb{S}^1$,

$$C_-\tau^{-1/6} \leq N(\tau, c_0\overline{\ell^0} + c_1\overline{\ell^1}) \leq C_+\tau^{-1/6}. \tag{5.42}$$

Let $(c_0, c_1) \in \mathbb{S}^1$ and $f \in \mathcal{X}_1$. By Definition 3.15,

$$\overline{\ell^j}(f,0,0) = \int_\Omega \partial_x f\overline{\Phi^j}, \tag{5.43}$$

where $\overline{\Phi^j}$ is the solution to (2.17).

By Corollary 2.28, there exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\begin{aligned}
c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0) = &\int_\Omega \partial_x f(d_0\bar{u}^0_{\mathrm{sing}}(x,-z) + d_1\bar{u}^1_{\mathrm{sing}}(x,-z)) \\
&+ \int_\Omega \partial_x f\left(\Phi_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right),
\end{aligned} \tag{5.44}$$

where $\Phi_{\mathrm{reg}} \in Q^1$. By linearity, d_0, d_1 and Φ_{reg} are uniformly bounded for $(c_0, c_1) \in \mathbb{S}^1$. The first term corresponds to the one studied in Lemma 5.8 and Lemma 5.9. We want to integrate by parts in the second term. Since $f \in \mathcal{X}_1$, $f_{|\Sigma_0 \cup \Sigma_1} = 0$. At $x = x_0$ and $z \in (-1, 0)$, $\overline{\Phi^j} = 0$ by (2.17). Moreover, $\bar{u}^0_{\mathrm{sing}}(x_0, -z) = 0$ because $\Lambda_0(+\infty) = 0$ and $\bar{u}^1_{\mathrm{sing}}(x_0, -z) = 0$ because $\bar{u}^1_{\mathrm{sing}}$ is compactly supported near $(x_1, 0)$. Hence, $\Phi_{\mathrm{reg}}(x_0, z) + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0} \equiv 0$ on $(-1, 0)$. The same conclusion holds at $x = x_1$ and $z \in (0, 1)$. Thus, we can integrate by parts with no boundary term and the second term is estimated as

$$\left| \int_\Omega \partial_x f \left(\Phi_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0} \right) \right| \leq \|f\|_{L^2} \|\partial_x \Phi_{\mathrm{reg}}\|_{L^2}. \tag{5.45}$$

Step 1. *Bound from above.* Thus, for $\tau \in (0, 1)$, using Lemma 5.8,

$$|c_0 \overline{\ell^0}(f, 0, 0) + c_1 \overline{\ell^1}(f, 0, 0)| \lesssim \tau^{-1/6} \left(\|f\|_{L^2} + \tau \|f\|_{\mathcal{X}_1} \right). \tag{5.46}$$

Step 2. *Bound from below.* For $\tau \in (0, 1)$ let $f_\tau := h_\tau$, where h_τ is constructed in Lemma 5.9, which ensures that f_τ is compactly supported in Ω so satisfies $(f_\tau)_{|\Sigma_0 \cup \Sigma_1} = 0$. Substituting in (5.44) and integrating by parts yields

$$c_0 \overline{\ell^0}(f_\tau, 0, 0) + c_1 \overline{\ell^1}(f_\tau, 0, 0) = -\int_\Omega h_\tau \partial_x \Phi_{\mathrm{reg}} + \sum_{i \in \{0,1\}} d_i \int_\Omega (\partial_x h_\tau) \bar{u}^i_{\mathrm{sing}}(x, -z). \tag{5.47}$$

By Corollary 2.28 and linearity, $\min |d_0|, |d_1|$ is uniformly bounded below. We choose h_τ as either h^0_τ or h^1_τ of Lemma 5.9 accordingly. Thus, by Lemma 5.9, as $\tau \to 0$,

$$\begin{aligned} |c_0 \overline{\ell^0}(f_\tau, 0, 0) + c_1 \overline{\ell^1}(f_\tau, 0, 0)| &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau \|h_\tau\|_{\mathcal{X}_1} \right) - C \|h_\tau\|_{L^2} \|\partial_x \Phi_{\mathrm{reg}}\|_{L^2} \\ &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau \|h_\tau\|_{\mathcal{X}_1} \right) \\ &= \tau^{-1/6} \left(\|f_\tau\|_{L^2} + \tau \|f_\tau\|_{\mathcal{X}_1} \right) \end{aligned} \tag{5.48}$$

for $\tau > 0$ sufficiently small. This concludes the proof. □

Thus, we are now ready to state our well-posedness result with fractional tangential regularity for the problem (2.1).

Lemma 5.11. *Let $\sigma \in (0, 1/6) \cup (1/6, 1/2) \cup (1/2, 1)$. Let $f \in H^\sigma_x L^2_y$, $\delta_0 \in H^2(\Sigma_0)$, $\delta_1 \in H^2(\Sigma_1)$ such that $\delta_0(1) = \delta_1(-1) = 0$.*

- *If $\sigma > 1/6$, assume that $\overline{\ell^0}(f, \delta_0, \delta_1) = \overline{\ell^1}(f, \delta_0, \delta_1) = 0$.*
- *If $\sigma > 1/2$, assume that also that $f_{|\Sigma_0 \cup \Sigma_1 = 0}$.*

The unique strong solution $u \in Z^0(\Omega)$ to (2.1) satisfies $u \in Z^\sigma(\Omega)$ where $Z^\sigma(\Omega) := [Z^0(\Omega), Z^1(\Omega)]_\sigma$ (and Z^1 is defined in (1.24)), with the estimate

$$\|u\|_{Z^\sigma} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2} + \|\delta_1\|_{H^2}. \tag{5.49}$$

In particular,

$$\|u\|_{H^{2/3+\sigma}_x L^2_z} + \|u\|_{H^\sigma_x H^2_z} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2} + \|\delta_1\|_{H^2}. \tag{5.50}$$

Remark 5.12. *The case $\sigma = 1/6$ is not covered in the above result, since this critical level of regularity corresponding to the maximal continuity of the orthogonality conditions was already excluded from Lemma 5.3. In this case, one would expect a similar result to hold, but with a supplementary norm on f, in the spirit of [6, 7]. The case $\sigma = 1/2$ is also excluded, but it would be possible to include it provided one introduces the appropriate $H^{1/2}_{00}$ norms on f corresponding to $f_{|\Sigma_0 \cup \Sigma_1} = 0$ at the $H^{1/2}$ level.*

Remark 5.13. *Lemma 5.11 is a tangential result, and does not take into account the hidden vertical regularity gains of Section 2.4. Of course, one could adapt it to obtain a fractional result also in the vertical direction, up to requiring more vertical regularity on f. This is essentially what we do in Proposition 5.1 in the general linearized case. Here, we stick to this tangential version because it is the one we need in Appendix D to prove Proposition 5.15 below.*

Remark 5.14. *The regularity assumptions on the δ_i's are not optimal and could be enhanced, but this is not our concern here.*

Proof. ***Step 1.*** *Case $\delta_0 = \delta_1 = 0$.* By Proposition 2.4, for every $f \in L^2(\Omega)$, there exists a unique solution $u \in Z^0(\Omega)$ to (2.1) with $\delta_0 = \delta_1 = 0$ and $\|u\|_{Z^0} \lesssim \|f\|_{L^2}$. By Proposition 2.8 and Proposition 2.5, for every $f \in H^1_x L^2_z$ such that $f_{|\Sigma_0 \cup \Sigma_1} = 0$ (so that $\Delta_0 = \Delta_1 = 0$) and $\overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0$, this solution satisfies $u \in Z^1(\Omega)$ with $\|u\|_{Z_1} \lesssim \|f\|_{H^1_x L^2_z}$. Hence, by interpolation, the mapping $f \mapsto u$ is bounded from $[\mathcal{X}_0, \overline{\mathcal{X}_1}]_\sigma$ to $Z^\sigma(\Omega)$. Moreover, from standard interpolation theory (see e.g. [42, Equation (13.4), Chapter 1]), when $\sigma \neq 1/2$, $[\mathcal{X}_0, \mathcal{X}_1]_\sigma = H^\sigma_x L^2_z$ (with null boundary conditions on $\Sigma_0 \cup \Sigma_1$ when $\sigma > 1/2$). This proves the first part of the statement and estimate (5.49). By Proposition 1.4, $Z^0(\Omega) \hookrightarrow H^{2/3}_x L^2_z$ and $Z^1(\Omega) \hookrightarrow H^{5/3}_x L^2_z$. Hence $Z^\sigma(\Omega) \hookrightarrow [H^{2/3}_x L^2_z, H^{5/3}_x L^2_z]_\sigma = H^{2/3+\sigma}_x L^2_z$ (again, using e.g. [42, Equation (13.4), Chapter 1]). By definition, $Z^0(\Omega) \hookrightarrow L^2_x H^2_z$ and $Z^1(\Omega) \hookrightarrow H^1_x H^2_z$, so $Z^\sigma(\Omega) \hookrightarrow H^\sigma_x H^2_z$. This concludes the proof of (5.50) when $\delta_0 = \delta_1 = 0$.

Step 2. *Arbitrary boundary data.* When δ_0 and δ_1 are arbitrary, we extend them to $(-1,1)$ in such a way that the extension belongs to $H^2_0(-1,1)$. We then lift the boundary data by setting $u_l(x,z) = \chi(x - x_0)\delta_0 + \chi(x - x_1)\delta_1$, with $\chi \in C^\infty_c(\mathbb{R})$, supported in $B(0,(x_1 - x_0)/2)$, and equal to 1 in a neighborhood of zero. This introduces a source term $f_l = z\partial_x u_l - \partial_{zz} u_l \in H^1_x L^2_z$ in the equation, whose trace on Σ_i is δ_i''. When $\sigma < 1/2$, we immediately obtain the desired result. For $\sigma > 1/2$, we first note that

$$\overline{\ell^j}(f - f_l, 0, 0) = 0.$$

We therefore introduce a function $h_l \in C^\infty_c(\Omega)$ such that

$$\|h_l\|_{H^1_x L^2_z} \lesssim \|f\|_{H^{1/2}_x L^2_z} + \|\delta_0\|_{H^2} + \|\delta_1\|_{H^2}$$

and

$$\overline{\ell^j}(f - z\partial_x u_l + h_l, 0, 0) = \overline{\ell^j}(\partial_{zz} u_l - h_l, 0, 0) = 0.$$

We then apply the result of the first step to the system with source term $f - z\partial_x u_l + h_l$ (which vanishes on $\Sigma_0 \cup \Sigma_1$), and the result of Proposition 2.8 to the system with source term $\partial_{zz} u_l - h_l$. This concludes the proof. □

5.3 Identification of the interpolation space for source terms

In this subsection, we prove Lemma 5.16, which is a generalized version of Lemma 5.2. The key point is the following decomposition result.

Proposition 5.15. *Let $\bar{u} \in Q^1$ satisfying the assumptions of Proposition 5.1. Define α, γ by* (3.5), (3.6). *Let Θ^0, Θ^1 be the profiles defined in Proposition 3.10, and let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$. There exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Theta_{\mathrm{reg}} \in H^1_x L^2_z$ such that*

$$c_0\Theta^0 + c_1\Theta^1 = (-c_0 z + c_1)\mathbf{1}_{z>0}\chi(z) + d_0\bar{u}^0_{\mathrm{sing}}(x,-z) + d_1\bar{u}^1_{\mathrm{sing}}(x,-z) + \Theta_{\mathrm{reg}}. \tag{5.51}$$

This decomposition result is reminiscent of Corollary 2.26 and Corollary 2.28 in the case of the shear flow, and its proof relies on the same key observations. Nevertheless, the proof is quite technical, so we postpone it to Appendix D. The main idea is to write the equation satisfied by Θ_{reg} under the form $z\partial_x\Theta_{\mathrm{reg}} - \partial_{zz}\Theta_{\mathrm{reg}} = S$, where S is a source term which can be estimated in $H^{1/3}_x L^2_z$. Moreover, we prove that we can find $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ such that S satisfies the orthogonality conditions $\overline{\ell^j} = 0$ associated with the shear flow problem. Then, relying on the interpolated well-posedness theory of Lemma 5.11 for the shear flow (with $\sigma = 1/3$), we infer that $\Theta_{\mathrm{reg}} \in H^1_x L^2_z$ (see (5.50)). Although it is rather easy to close an energy estimate with this strategy, its proper justification turns out to be unfortunately quite technical, because we do not *a priori* know that $S \in H^{1/3}_x L^2_z$.

Thanks to the decomposition result of Proposition 5.15, the following result is a rather straightforward generalization of Lemma 5.6 to the linearized case.

Lemma 5.16. *Let $\bar{u} \in Q^1$ satisfying the assumptions of Proposition 5.1. Let $\mathcal{Y}_0$, $\mathcal{Y}_1$ and $\mathcal{Y}_1^{\bar{u}}$ as defined in* (5.5), (5.6) *and* (5.7). *Then,*

- *For every $\sigma \in (0, 1/6)$, $[\mathcal{Y}_0, \mathcal{Y}_1^{\bar{u}}]_\sigma = [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$ with equivalent norms.*
- *For every $\sigma \in (1/6, 1)$, the linear forms $\ell^0_{\bar{u}}(\cdot, 0, 0)$ and $\ell^1_{\bar{u}}(\cdot, 0, 0)$ admit continuous extensions to $\mathcal{Y}_\sigma$ and*

$$\left[\mathcal{Y}_0, \mathcal{Y}_1^{\bar{u}}\right]_\sigma = \left\{ f \in \mathcal{Y}_\sigma;\ \ \ell^0_{\bar{u}}(f,0,0) = \ell^1_{\bar{u}}(f,0,0) = 0 \right\}, \tag{5.52}$$

  endowed with the norm of $\mathcal{Y}_\sigma$.

Proof. The proof is very similar to the one of Lemma 5.6, substituting the linear form $\overline{\ell^j}$ with $\ell^j_{\bar{u}}$. By Definition 3.15,

$$\ell^j_{\bar{u}}(f,0,0) = \int_\Omega \partial_x\left(\frac{g}{\alpha}\right)\Theta^j, \tag{5.53}$$

where $g = f \circ Y$ and $\alpha = \bar{u}^2_y \circ Y$ (see (3.5) and (3.7)), for the change of variable Y defined in (3.2), and Θ^j is the solution to (3.41)-(3.42). Let $(c_0, c_1) \in \mathbb{S}^1$. By Proposition 5.15, there exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ and $\Theta_{\mathrm{reg}} \in H^1_x L^2_z$ such that, for $f \in \mathcal{Y}_1$,

$$\begin{aligned} c_0\ell^0_{\bar{u}}(f,0,0) + c_1\ell^1_{\bar{u}}(f,0,0) = &\int_\Omega \partial_x\left(\frac{g}{\alpha}\right)\left(d_0\bar{u}^0_{\mathrm{sing}}(x,-z) + d_1\bar{u}^1_{\mathrm{sing}}(x,-z)\right) \\ &+ \int_\Omega \partial_x\left(\frac{g}{\alpha}\right)\left(\Theta_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right). \end{aligned} \tag{5.54}$$

As in the proof of Lemma 5.6, we can integrate by parts with no boundary term and the second term is estimated as

$$\left| \int_\Omega \partial_x\left(\frac{g}{\alpha}\right)\left(\Theta_{\mathrm{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right) \right| \leq \|g/\alpha\|_{L^2}\|\partial_x\Theta_{\mathrm{reg}}\|_{L^2}. \tag{5.55}$$

Step 1. *Bound from above.* Thus, for $\tau \in (0,1)$, using Lemma 5.8,

$$|c_0 \ell^0_{\bar u}(f,0,0) + c_1 \ell^1_{\bar u}(f,0,0)| \lesssim \|g/\alpha\|_{L^2} + \tau^{-1/6} \left(\|g/\alpha\|_{L^2} + \tau \|\partial_x (g/\alpha)\|_{L^2} \right). \tag{5.56}$$

By the estimate of Lemma B.4 on Sobolev norms of compositions, whose assumptions are satisfied thanks to Corollary B.5, there holds

$$\|g/\alpha\|_{L^2} \lesssim \|f/\bar u_y^2\|_{L^2} \quad \text{and} \quad \|\partial_x (g/\alpha)\|_{L^2} \lesssim \|f/\bar u_y^2\|_{H^1_x L^2_y} + \|f/\bar u_y^2\|_{L^2_x H^2_y}. \tag{5.57}$$

Thanks to the assumption $\bar u \in Q^1$ and $\|\bar u - y\|_{Q^1} \ll 1$, and using the embeddings of Lemma 1.10, we can estimate all terms. Since the computations are very similar to those of Section 4, we leave them to the reader. Using the embeddings $Q^1 \hookrightarrow L^\infty_x W^{2,\infty}_y$, $Q^1 \hookrightarrow H^1_x H^2_y$ and $Q^1 \hookrightarrow H^{2/3}_x H^3_y$, we infer

$$\left\| \frac{f}{\bar u_y^2} \right\|_{H^1_x L^2_y} \lesssim \|f\|_{H^1_x L^2_y} \quad \text{and} \quad \left\| \frac{f}{\bar u_y^2} \right\|_{L^2_x H^2_y} \lesssim \|f\|_{L^2_x H^2_y}. \tag{5.58}$$

We conclude that

$$|c_0 \ell^0_{\bar u}(f,0,0) + c_1 \ell^1_{\bar u}(f,0,0)| \lesssim \tau^{-1/6} \left(\|f\|_{L^2} + \tau \|f\|_{\mathcal{Y}_1} \right). \tag{5.59}$$

Step 2. *Bound from below.* For $\tau \in (0,1)$ and $(x,y) \in \Omega$, define

$$f_\tau(x,y) := (\bar u_y^2 h_\tau)(x, \bar u(x,y)), \tag{5.60}$$

where h_τ is constructed in Lemma 5.9, which ensures that f_τ is compactly supported in Ω so satisfies $(f_\tau)_{|\Sigma_0 \cup \Sigma_1} = 0$. Letting $g_\tau := f_\tau \circ Y$, one has $g_\tau/\alpha = h_\tau$. Substituting in (5.54) and integrating by parts yields

$$c_0 \ell^0_{\bar u}(f_\tau,0,0) + c_1 \ell^1_{\bar u}(f_\tau,0,0) = \int_\Omega h_\tau \partial_x \Theta_{\mathrm{reg}} + \sum_{i \in \{0,1\}} d_i \int_\Omega (\partial_x h_\tau) \bar u^i_{\mathrm{sing}}(x,-z). \tag{5.61}$$

By Proposition 5.15 and linearity, $\min |d_0|, |d_1|$ is uniformly bounded below. We choose h_τ as either h^0_τ or h^1_τ of Lemma 5.9 accordingly. Thus, by Lemma 5.9, as $\tau \to 0$,

$$\begin{aligned} |c_0 \ell^0_{\bar u}(f_\tau,0,0) + c_1 \ell^1_{\bar u}(f_\tau,0,0)| &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau \|h_\tau\|_{\mathcal{Y}_1} \right) - C \|h_\tau\|_{L^2} \|\partial_x \Theta_{\mathrm{reg}}\|_{L^2} \\ &\gtrsim \tau^{-1/6} \left(\|h_\tau\|_{L^2} + \tau \|h_\tau\|_{\mathcal{Y}_1} \right) \end{aligned} \tag{5.62}$$

for $\tau > 0$ sufficiently small. By Lemma B.4,

$$\|f_\tau\|_{L^2} \lesssim \|\bar u_y^2 h_\tau\|_{L^2} \lesssim \|h_\tau\|_{L^2}. \tag{5.63}$$

By Corollary B.6 about estimates for compositions,

$$\|f_\tau\|_{\mathcal{Y}_1} \lesssim \|\bar u_y^2 h_\tau\|_{\mathcal{Y}_1}. \tag{5.64}$$

Using the embeddings $Q^1 \hookrightarrow H^1_x H^2_y$ and $Q^1 \hookrightarrow L^\infty_x W^{2,\infty}_y$, we infer

$$\|\bar u_y^2 h_\tau\|_{H^1_x L^2_y} \lesssim \|h_\tau\|_{H^1_x L^2_y}, \quad \|\partial_y^3 (\bar u_y^2 h_\tau)\|_{L^2} \lesssim \|h_\tau\|_{\mathcal{X}_1}. \tag{5.65}$$

Thus $\|f_\tau\|_{\mathcal{Y}_1} \lesssim \|h_\tau\|_{\mathcal{Y}_1}$ and, recalling (5.63) and (5.62),

$$|c_0 \ell^0_{\bar u}(f_\tau,0,0) + c_1 \ell^1_{\bar u}(f_\tau,0,0)| \gtrsim \tau^{-1/6} \left(\|f_\tau\|_{L^2} + \tau \|f_\tau\|_{\mathcal{Y}_1} \right). \tag{5.66}$$

This concludes the proof. □

Lemma 5.17. *Let $\sigma \in (0,1)$. Let $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for the complex interpolation method.*

- *When $\sigma \in (0,1/2)$, $\mathcal{Y}_\sigma = H^\sigma_x L^2_y \cap L^2_x H^{3\sigma}_y$.*
- *When $\sigma = 1/2$,*

$$\|f\|^2_{\mathcal{Y}_{1/2}} \approx \|f\|^2_{H^{1/2}_x L^2_y} + \|f\|^2_{L^2_x H^{3/2}_y} + \int_{\Omega_+} \frac{f^2(x,y)}{|x-x_0|}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_-} \frac{f^2(x,y)}{|x-x_1|}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.67}$$

- *When $\sigma \in (1/2,1)$, $\mathcal{Y}_\sigma = \{f \in H^\sigma_x L^2_y \cap L^2_x H^{3\sigma}_y;\ f_{|\Sigma_0 \cup \Sigma_1 = 0}\}$, with the usual norm.*

Proof. This follows from classical interpolation theory for intersections (see [42, Theorem 13.1 and Equation (13.4), Chapter 1]), and from (one-sided versions of) the equality $H^{1/2}_{00}(x_0,x_1) = [H^1_0(x_0,x_1), L^2(x_0,x_1)]_{\frac{1}{2}}$ (see [42, Theorem 11.7, Chapter 1]) . □

5.4 Proof of the interpolation estimate

Proof of Proposition 5.1. We distinguish between the regularity of the solution to (3.1) close to the line $z = 0$, for which we use the interpolation result of Lemma 5.2, and the regularity close to the lines $z = \pm 1$, for which we use the parabolic nature of the equation together with classical interpolation results.

Step 1. *$Q^{1/2}$ regularity close to the line $z = 0$.* Let $\chi_0 \in C^\infty_c(\mathbb{R})$ such that $\chi_0 \equiv 1$ in $(-1/2, 1/2)$, and supp $\chi_0 \subset (-3/4, 3/4)$. Consider the application

$$\mathcal{L}_0 : f \in L^2 \mapsto \chi_0 u \in Q^0, \tag{5.68}$$

where $u \in Q^0$ is the unique solution to (3.1) with $\delta_0 = \delta_1 = 0$.

Then, according to the proof of Proposition 3.20 (see in particular estimate (3.71)), $\mathcal{L}_0 : \mathcal{Y}_0 \to Q^0$, $\mathcal{L}_0 : \mathcal{Y}^{\bar{u}}_1 \to Q^1$, and there exist finite constants C_0, C_1 such that

$$\|\mathcal{L}_0\|_{\mathcal{L}(\mathcal{Y}_0, Q^0)} \le C_0, \quad \|\mathcal{L}_0\|_{\mathcal{L}(\mathcal{Y}^{\bar{u}}_1, Q^1)} \le C_1. \tag{5.69}$$

The constants C_i depend only on Ω and on χ_0. By the interpolation result of Lemma 5.2, it follows that $\mathcal{L}_0 : \mathcal{Y}^{\bar{u}}_{1/2} \to Q^{1/2}$.

Therefore,

$$\|\chi_0 u\|_{Q^{1/2}} \lesssim \|f\|_{H^{1/2}_{00_l} L^2_y(\Omega_+)} + \|f\|_{H^{1/2}_{00_r} L^2_y(\Omega_-)} + \|f\|_{L^2 H^{3/2}}. \tag{5.70}$$

Step 2. *$Q^{1/2}$ regularity close to the lines $z = \pm 1$:*

We focus on the regularity close to the line $z = 1$. As in the proof of Proposition 3.20, we introduce a cut-off function $\chi_1 \in C^\infty(\mathbb{R})$ such that $\chi_1 \equiv 1$ on $[1/2, 1]$ and supp $\chi_1 \subset [1/4, 1]$, and we set $u_1 = u\chi_1$. Note that with this choice, supp $\chi_1' \subset \{\chi_0 = 1\}$. We recall that u_1 satisfies the parabolic equation (3.72), with $\delta_0 = 0$. Hence, we will prove the interpolation result directly on the parabolic equation (3.72). One remarkable point lies in the fact that the orthogonality conditions are not necessary to have regularity close to the lines $z = \pm 1$.

Let $\Omega_1 := (x_0, x_1) \times (1/4, 1)$. For any $S \in L^2(\Omega_1)$, consider the solution v of

$$\begin{aligned} \bar{u}\partial_x v - \partial_{yy} v &= S \quad \text{in } \Omega_1, \\ v_{|y=1/4} = v_{|y=1} &= 0, \\ v_{|x=x_0} &= 0. \end{aligned} \tag{5.71}$$

and we consider the application

$$\mathcal{L}_1 : S \in L^2(\Omega_1) \mapsto v \in L^\infty((x_0,x_1),L^2(1/4,1)) \cap L^2((x_0,x_1),H^1_0(1/4,1)). \tag{5.72}$$

Note that since v vanishes on $\{y=1/4\}\cup\{y=1\}\cup\{x=x_0\}$, compatibility conditions at first order are automatically satisfied in the corners $(x_0,1/4)$ and $(x_0,1)$. From classical parabolic arguments, it follows that

$$\mathcal{L}_1(L^2(\Omega_1)) \hookrightarrow L^2((x_0,x_1),H^2(1/4,1)) \cap H^1((x_0,x_1),L^2(1/4,1)). \tag{5.73}$$

Now, let

$$\mathcal{K}_1 := \{S \in H^1((x_0,x_1),H^1_0(1/4,1)) \cap L^2((x_0,x_1),H^3_0(1/4,1)), \quad S(x_0,\cdot)=0\}. \tag{5.74}$$

Let $S \in \mathcal{K}_1$ be arbitrary, and let $v = \mathcal{L}_1(S)$. Then v_x is the unique weak solution to

$$\begin{aligned} \bar{u}\partial_x v_x + \bar{u}_x v_x - \partial_{yy} v_x = S_x \quad \text{in } \Omega_1, \\ v_{x|y=1/4} = v_{x|y=1} = 0, \\ v_{x|x=x_0} = 0. \end{aligned} \tag{5.75}$$

Note that the initial data for v_x at $x=x_0$ is obtained by taking the trace of (5.71) and noticing that the compatibility conditions in the corners $(x_0,1/4)$ and $(x_0,1)$ are satisfied.

As a consequence, since $\|\bar{u}_x\|_\infty \ll 1$,

$$\|v_x\|_{L^\infty((x_0,x_1),H^1_0(1/4,1))} + \|v_{xyy}\|_{L^2(\Omega_1)} + \|v_{xx}\|_{L^2(\Omega_1)} \lesssim \|S_x\|_{L^2(\Omega_1)}.$$

We then follow the same path as in the proof of Proposition 3.20. Differentiating (5.71) twice with respect to y and then once with respect to x, we find first that

$$\|v_{yy}\|_{L^\infty((x_0,x_1),H^1_0(1/4,1))} + \|\partial_y^4 v\|_{L^2(\Omega_1)} \lesssim \|S_x\|_{L^2(\Omega_1)} + \|\partial_y^2 S\|_{L^2(\Omega_1)}$$

and then

$$\|\partial_x \partial_y^2 v\|_{L^\infty((x_0,x_1),L^2(1/4,1))} + \|\partial_x \partial_y^3 v\|_{L^2(\Omega_1)} \lesssim \|\partial_y S\|_{H^1(\Omega_1)} + \|\partial_x S\|_{L^2(\Omega_1)}.$$

Using equation (5.71), we infer eventually that

$$\|\partial_y^5 v\|_{L^2(\Omega_1)} \lesssim \|S\|_{\mathcal{K}_1}.$$

Therefore

$$\mathcal{L}_1(\mathcal{K}_1) \subset L^2_x H^5_y \cap H^2_x L^2_y. \tag{5.76}$$

Let us now interpolate between (5.73) and (5.76). We set

$$\mathcal{K}_{1/2} = \left[L^2(\Omega_1),\mathcal{K}_1\right]_{1/2} = H^{1/2}_{00_l}((x_0,x_1),H^{1/2}_{00}(1/4,1)) \cap L^2((x_0,x_1),H^{3/2}_0(1/4,1)).$$

We find that

$$\mathcal{L}_1(\mathcal{K}_{1/2}) \subset L^2((x_0,x_1),H^{7/2}(1/4,1)) \cap H^{3/2}((x_0,x_1),L^2(1/4,1)).$$

Going back to (3.72), we infer that

$$\|\chi_1 u\|_{Q^{1/2}} \le \|\chi_1 u\|_{L^2_x H^{7/2}_y} + \|\chi_1 u\|_{H^{3/2}_x L^2_y} \lesssim \|S_1\|_{\mathcal{K}_{1/2}},$$

where $S_1 = f\chi_1 - 2\chi_1'\partial_y u - \chi_1'' u$.

Recall that supp χ_1', supp $\chi_1'' \subset \{\chi_0 = 1\}$. Therefore, using the estimates on $\chi_0 u$,

$$\|S_1\|_{\mathcal{K}_{1/2}} \lesssim \|f\|_{H^{1/2}_{00_l}((x_0,x_1),H^{1/2}_{00_r}(1/4,1))} + \|f\|_{L^2_x H^{3/2}_y} + \|\chi_0 u\|_{H^{1/2}_{00} H^{3/2}_y} + \|\chi_0 u\|_{L^2_x H^{5/2}_y}, \tag{5.77}$$

where, by Lemma 1.10 and the estimates on $\chi_0 u$,

$$\|\chi_0 u\|_{H^{1/2}_{00} H^{3/2}_y} + \|\chi_0 u\|_{L^2_x H^{5/2}_y} \lesssim \|\chi_0 u\|_{Q^{1/2}} \lesssim \|f\|_{H^{1/2}_{00_l} L^2_y(\Omega_+)} + \|f\|_{H^{1/2}_{00_r} L^2_y(\Omega_-)} + \|f\|_{L^2 H^{3/2}}. \tag{5.78}$$

Gathering the results on $\chi_0 u$, $\chi_1 u$ (and, symmetrically, on $\chi_{-1} u$), we obtain the estimate (5.4). □

6 The nonlinear problem

In this section, we prove Theorem 3. The most difficult part is the existence statement, which relies on the scheme described in Section 1.4. Let us recall its main steps, and their connections with the intermediate results of the preceding sections. Let $(f, \delta_0, \delta_1) \in \mathcal{H}$, with $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}} \leq \eta \ll 1$.

- We define iteratively a sequence $(u_n)_{n\in\mathbb{N}}$ by solving the equation

$$\begin{cases} (y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1 =: f_{n+1}, \\ (u_{n+1})_{|\Sigma_i} = \delta_i + \nu^0_{n+1}\delta^0_i + \nu^1_{n+1}\delta^1_i =: \delta_{i,n+1}, \\ (u_{n+1})_{|y=\pm 1} = 0, \end{cases} \tag{6.1}$$

  where the triplets $(f^k, \delta^k_0, \delta^k_1)$ are defined in Corollary 2.13. The coefficients ν^k_{n+1} ensure that the orthogonality conditions

$$\ell^j_{y+u_n}(f_{n+1}, \delta_{0,n+1}, \delta_{1,n+1}) = 0$$

  are satisfied. It then follows from the analysis of Section 3 (see in particular Proposition 3.20) that $u_n \in Q^1$ for all $n \in \mathbb{N}$.

- We also prove uniform bounds on the sequence $(u_n)_{n\in\mathbb{N}}$ in Q^1, and on the sequences $(\nu^k_n)_{n\in\mathbb{N}}$. More precisely, we prove that there exists a constant C such that for all $n \in \mathbb{N}$,

$$\|u_n\|_{Q^1} + |\nu^0_n| + |\nu^1_n| \leq C\eta.$$

  These bounds follow from Proposition 3.20 and from the Lipschitz continuity of the maps $u \in Q^{1/2} \mapsto \ell^j_{y+u} \in \mathcal{L}(\mathcal{H})$ (see Proposition 4.1).

- The next step is to prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$. To that end, we write an equation for $w_n := u_{n+1} - u_n$, namely

$$\begin{cases} (y + u_n)\partial_x w_n - \partial_{yy} w_n = -w_{n-1}\partial_x u_n + f_{n+1} - f_n, \\ w_{n|\Sigma_i} = \delta_{i,n+1} - \delta_{i,n}, \\ w_{n|\pm 1} = 0. \end{cases}$$

  Note that the nonlinear term $w_{n-1}\partial_x u_n$ in the right-hand side does not belong to $H^1_x L^2_y \cap L^2_x H^3_y$, and therefore we cannot use our Q^1 estimate. However, the term $f_{n+1} - f_n$ involves

$\nu_{n+1} - \nu_n$, and therefore the difference $\ell^j_{y+u_n} - \ell^j_{y+u_{n-1}}$, which is controlled by $\|w_{n-1}\|_{Q^{1/2}}$. This dictates our choice of the $Q^{1/2}$ norm to control w_n. Fortunately, the nonlinear term $w_{n-1}\partial_x u_n$ is bounded in $H^{1/2}_x L^2_y \cap L^2_x H^{3/2}_y$ by $\|w_{n-1}\|_{Q^{1/2}}\|u_n\|_{Q^1}$.

The estimate on w_n therefore relies on two main ingredients: (i) the estimate from Proposition 5.1, which itself uses crucially the interpolation result of Lemma 5.2; and (ii) the Lipschitz regularity of $\bar{u} \in Q^{1/2} \mapsto \ell^j_{\bar{u}}$, which will lead to an estimate of the type

$$|\nu_{n+1} - \nu_n| \lesssim \eta \|w_{n-1}\|_{Q^{1/2}}.$$

We therefore obtain a recursive estimate of the type

$$\|w_n\|_{Q^{1/2}} \lesssim \eta \|w_{n-1}\|_{Q^{1/2}} + \eta^2 \|w_{n-2}\|_{Q^{1/2}},$$

from which we deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$.

- We then pass to the limit in the equation for u_n, and obtain a solution in Q^1 of an equation of the type

$$\begin{cases} (y+u)\partial_x u - \partial_{yy} u = f + \nu^0 f^0 + \nu^1 f^1 \\ u_{|\Sigma_i} = \delta_i + \nu^0 \delta_i^0 + \nu^1 \delta_i^1, \\ u_{|y=\pm 1} = 0, \end{cases}$$

where the numbers ν^0, ν^1 are such that

$$\ell^j_{y+u}(f + \nu^0 f^0 + \nu^1 f^1, \delta_0 + \nu^0 \delta_0^0 + \nu^1 \delta_0^1, \delta_1 + \nu^0 \delta_1^0 + \nu^1 \delta_1^1) = 0.$$

This leads us to study the nonlinear maps $(f, \delta_0, \delta_1) \in \mathcal{H} \mapsto (\nu^0, \nu^1)$. We prove that these maps are Lipschitz continuous. The definition of the manifold $\mathcal{M}$ from Theorem 3 will then rely on these two maps, accounting for some nonlinear orthogonality conditions.

The organization of this section is as follows. We first state a proposition giving the existence of the two maps ν^0, ν^1, and explain how Theorem 3 follows from that proposition. The rest of the section is then mostly devoted to the construction of the maps ν^0 and ν^1, following the steps outlined above. We end the section with a proof of uniqueness, and of the necessity of the orthogonality conditions.

6.1 Definition of the manifold and reformulation of the existence result

Throughout this section, we denote by $\Xi = (f, \delta_0, \delta_1)$ an element of $\mathcal{H}$. We recall that there exists $\Xi^0, \Xi^1 \in \mathcal{H}$ such that $\overline{\ell^j}(\Xi^k) = \mathbf{1}_{j=k}$ (see Corollary 2.13), and such that $\ker \overline{\ell^0} \cap \ker \overline{\ell^1} = (\mathbb{R}\Xi^0 + \mathbb{R}\Xi^1)^\perp$. We set $\mathcal{H}^\perp_{\mathrm{sg}} = \ker \overline{\ell^0} \cap \ker \overline{\ell^1}$, and we recall that if $\Xi \in \mathcal{H}^\perp_{\mathrm{sg}}$, the model problem (2.1) can be solved with Q^1 regularity, see Proposition 2.8 and Proposition 2.15.

For $\eta > 0$ denote by B_η the open ball of radius η and centered at 0 in $\mathcal{H}^\perp_{\mathrm{sg}}$. For every $\Xi \in \mathcal{H}$, one has the decomposition

$$\Xi = \Xi^\perp + \langle \Xi^0; \Xi \rangle_{\mathcal{H}} \Xi^0 + \langle \Xi^1; \Xi \rangle_{\mathcal{H}} \Xi^1, \tag{6.2}$$

where $\Xi^\perp \in \mathcal{H}^\perp_{\mathrm{sg}}$ and the linear maps $\Xi \mapsto \Xi^\perp$ and $\Xi \mapsto \langle \Xi^k; \Xi \rangle$ are continuous. We will deduce the existence statement of Theorem 3 from the following proposition.

Proposition 6.1. *There exist $\eta > 0$, two Lipschitz maps ν^0 and ν^1 from B_η to $\mathbb{R}$ with $\nu^0(0) = \nu^1(0) = 0$ and a map $\mathcal{U}_\perp : B_\eta \to Q^1$, which is Lipschitz from B_η to $Q^{1/2}$, such that, for every triplet $\Xi \in B_\eta$, $\mathcal{U}_\perp(\Xi)$ is a solution to (1.7) with data $\Xi + \nu^0(\Xi)\Xi^0 + \nu^1(\Xi)\Xi^1$ (where Ξ^0, Ξ^1 are defined in Corollary 2.13).*

Proof of the existence statement of Theorem 3. The existence statement of Theorem 3 is a direct consequence of Proposition 6.1. Indeed, with the notations of Proposition 6.1, define

$$\mathcal{M} := \left\{\Xi \in \mathcal{H};\ \ \|\Xi\|_{\mathcal{H}} < \eta \text{ and } \langle \Xi^k; \Xi\rangle_{\mathcal{H}} = \nu^k(\Xi^\perp) \text{ for } k = 0, 1\right\} \tag{6.3}$$

and set, for $\Xi \in \mathcal{M}$,

$$\mathcal{U}(\Xi) := \mathcal{U}_\perp(\Xi^\perp). \tag{6.4}$$

Then $\mathcal{M}$ is a Lipschitz manifold modeled on $\mathcal{H}^\perp_{\mathrm{sg}}$ since ν^0 and ν^1 are Lipschitz maps. It contains $0_{\mathcal{H}}$ since $\nu^0(0_{\mathcal{H}^\perp_{\mathrm{sg}}}) = \nu^1(0_{\mathcal{H}^\perp_{\mathrm{sg}}}) = 0_{\mathbb{R}}$. Moreover, $\mathcal{M}$ is "tangent" to $\mathcal{H}^\perp_{\mathrm{sg}}$ at 0 in the sense of Remark 6.2 below.

Eventually, for every $\Xi \in \mathcal{M}, \mathcal{U}(\Xi) \in Q^1$ is a strong solution to (1.7) with data $\Xi^\perp + \nu^0(\Xi^\perp)\Xi^0 + \nu^1(\Xi^\perp)\Xi^1 = \Xi$. So the conclusions of the existence statement of Theorem 3 are satisfied. □

Remark 6.2. *Since we only proved Lipschitz regularity for the maps ν^0 and ν^1,* (6.3) a priori *only defines a Lipschitz manifold. Hence, it is difficult to define tangent spaces to $\mathcal{M}$. Nevertheless, one can say that $\mathcal{H}^\perp_{\mathrm{sg}}$ is tangent to $\mathcal{M}$ at 0 in the following weak senses:*

- *For $\Xi \in \mathcal{M}$, $\mathrm{d}(\Xi, \mathcal{H}^\perp_{\mathrm{sg}}) \lesssim \|\Xi\|^2_{\mathcal{H}}$.*
- *For every $\Xi^\perp \in \mathcal{H}^\perp_{\mathrm{sg}}$, for $\varepsilon \in \mathbb{R}$ small enough, $\mathrm{d}(\varepsilon\Xi^\perp, \mathcal{M}) \lesssim \varepsilon^2$.*

Both facts are straightforward consequences of the definition (6.3) *and the estimate* (6.25)*, which we prove below. For the second item, we note that for all $\Xi^\perp \in \mathcal{H}^\perp_{\mathrm{sg}}$, $\Xi^\perp + \nu^0(\Xi^\perp)\Xi^0 + \nu^1(\Xi^\perp)\Xi^1 \in \mathcal{M}$ by definition of $\mathcal{M}$, so that $\mathrm{d}(\varepsilon\Xi^\perp, \mathcal{M}) \lesssim |\nu^0(\varepsilon\Xi^\perp)| + |\nu^1(\varepsilon\Xi^\perp)|$.*

6.2 Execution of the nonlinear scheme

We prove Proposition 6.1.

6.2.1 Construction of the sequence and uniform Q^1 bound

Let $\eta > 0$ small enough to be chosen later. Let $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}^\perp_{\mathrm{sg}}$ with $\|\Xi\|_{\mathcal{H}} \leq \eta$.

Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$, identically equal to one on $[-\frac{1}{3}, \frac{1}{3}]$ and compactly supported in $[-\frac{1}{2}, \frac{1}{2}]$. We define the initialization profile of our iterative scheme as

$$u_0(x, y) := \delta_0(y)\chi\left(\frac{x - x_0}{x_1 - x_0}\right) + \delta_1(y)\chi\left(\frac{x_1 - x}{x_1 - x_0}\right). \tag{6.5}$$

Hence, there exists $C_\chi > 0$ such that $u_0 \in Q^1$ and $\|u_0\|_{Q^1} \leq C_\chi(\|\delta_0\|_{\mathcal{H}} + \|\delta_1\|_{\mathcal{H}}) \leq 2\eta C_\chi$. In particular, this stems from the fact that $\|\delta_i\|_{H^5_y} \leq \|(f, \delta_0, \delta_1)\|_{\mathcal{H}}$, by definition (1.3). Furthermore, $\partial_x u_0$ identically vanishes on the boundaries.

For each $n \in \mathbb{N}$, we let u_{n+1} be the solution to

$$\begin{cases} (y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ (u_{n+1})_{|\Sigma_i} = \delta_i + \nu^0_{n+1}\delta^0_i + \nu^1_{n+1}\delta^1_i, \\ (u_{n+1})_{|y=\pm 1} = 0, \end{cases} \tag{6.6}$$

where the coefficients ν_{n+1}^k are defined below, and where the triplets $(f^k, \delta_0^k, \delta_1^k)$ are defined in Corollary 2.13.

We prove the following result:

Lemma 6.3. *There exists a constant $\bar{c}$ depending only on Ω such that if $\eta \leq \bar{c}$, there exist two sequences $(\nu_n^0)_{n\geq 1}$ and $(\nu_n^1)_{n\geq 1}$ so that the following properties are satisfied:*

- *For all $n \in \mathbb{N}$, $u_n \in Q^1$;*
- *There exists a constant $\bar{C}$ depending only on Ω such that for all $n \in \mathbb{N}^*$,*

$$\|u_n\|_{Q^1} + |\nu_n^0| + |\nu_n^1| \leq \bar{C}\eta;$$

- *$\partial_y^k u_n(x_i, 0) = 0$ for $k = 0, 1, 2$ and $u_n(x, \pm 1) = \pm 1$.*

Proof. We argue by induction. The properties stated in the lemma clearly hold for u_0. Assume that u_j, ν_j^k are constructed for $j \leq n$ and satisfy the assumptions stated in the lemma. Then $y \lesssim y + u_n(x_i, y) \lesssim y$.

Our purpose is now to apply Proposition 3.20. Hence we first check that the boundary data have sufficient regularity. As a consequence, denoting by Y_n the change of variables associated with $y + u_n$, i.e.

$$Y_n(x, z) + u_n(x, Y_n(x, z)) = z \quad \forall (x, z) \in \overline{\Omega},$$

we have

$$\left\| \frac{\delta_i''(Y_n(x_i, z))}{z} \right\|_{\mathscr{L}_z^2(\Sigma_i)} \lesssim \left\| \frac{\delta_i''(z)}{z} \right\|_{\mathscr{L}_z^2(\Sigma_i)}, \tag{6.7}$$

and, recalling that $\partial_z Y_n = (1 + \partial_y u_n(x, Y_n))^{-1}$,

$$\begin{aligned} \partial_z \frac{\delta_i''(Y_n(x_i, z))}{z} &= \frac{Y_n(x_i, z)\delta_i^{(3)}(Y_n(x_i, z)) - \delta_i''(Y_n(x_i, z))}{z^2} \\ &\quad + (z - Y_n(x_i, z))\frac{\delta_i^{(3)}(Y_n(x_i, z))}{z^2} \\ &\quad - \frac{\partial_y u_n(x_i, Y_n(x_i, z))}{z(1 + \partial_y u_n(x_i, Y_n(x_i, z)))}\delta_i^{(3)}(Y_n(x_i, z)). \end{aligned}$$

The first term is bounded in $\mathscr{L}_z^2(\Sigma_i)$ by $\|\delta_i''(z)/z\|_{\mathscr{H}_z^1(\Sigma_i)}$. The second one is also bounded, noticing that $|z - Y_n(x_i, z)| = O(z^2)$ because of the cancellations $u_n(x_i, 0) = \partial_y u_n(x_i, 0) = 0$. As for the last one, using once again the property $\partial_y u_n(x_i, 0) = 0$, it satisfies

$$\left| \frac{\partial_y u_n(x_i, Y_n(x_i, z))}{z(1 + \partial_y u_n(x_i, Y_n(x_i, z)))}\delta_i^{(3)}(Y_n(x_i, z)) \right| \lesssim \|\partial_y^2 u_n\|_{L^\infty(\Sigma_i)} |\delta_i^{(3)}(Y_n(x_i, z))|$$

and thus its $\mathscr{L}_z^2(\Sigma_i)$ norm is bounded by $C\eta\|\delta_i^{(3)}\|_{L^2}$. (Here we use the fact that $\|\partial_y^2 u_n\|_{L^\infty} \lesssim \|u_n\|_{Q^1} \lesssim \eta$.) We conclude that $\delta_i''(Y_n(x_i, z))/z \in \mathscr{H}(\Sigma_i)$.

We now choose ν_{n+1}^0, ν_{n+1}^1 such that the orthogonality conditions are satisfied. For brevity and with a slight abuse of notation, we denote by ℓ_n^j the linear forms associated with the flow $\bar{u}_n := y + u_n$ (see Definition 3.15). For $u \in Q^1$, we define the following 2 by 2 matrix:

$$M_u := \left(\ell_{y+u}^j(f^k, \delta_0^k, \delta_1^k) \right)_{0 \leq j,k \leq 1}. \tag{6.8}$$

In particular, since $M_0 = \mathrm{Id}$ and $u \mapsto \ell_{y+u}$ is locally Lipschitz (by Proposition 4.1) from $Q^{1/2}$ to $\mathcal{L}(\mathcal{H})$, M_u is invertible for u small enough in Q^1, and $u \in Q^{1/2} \mapsto M_u^{-1}$ is Lipschitz. We set $M_n := M_{u_n}$. Then the coefficients ν_{n+1} are defined by

$$\nu_{n+1} := -M_n^{-1} \begin{pmatrix} \ell_n^0(f, \delta_0, \delta_1) \\ \ell_n^1(f, \delta_0, \delta_1) \end{pmatrix}. \tag{6.9}$$

It is easily checked that this choice ensures that, for $j = 0, 1$,

$$\ell_n^j \left(f + \nu_{n+1}^0 f^0 + \nu_{n+1}^1 f^1, \delta_0 + \nu_{n+1}^0 \delta_0^0 + \nu_{n+1}^1 \delta_0^1, \delta_1 + \nu_{n+1}^0 \delta_1^0 + \nu_{n+1}^1 \delta_1^1 \right) = 0. \tag{6.10}$$

Furthermore, since $\|u_n\|_{Q^1} \leq \bar{C}\eta$, $|\nu_{n+1}| \leq \bar{C}'\|\Xi\|_{\mathcal{H}}$ for some universal constant $\bar{C}'$ involving only the norms of the linear forms $\overline{\ell^j}$.

By Proposition 3.20 and Lemma 4.4, this choice ensures that the system has a solution in Q^1 and the existence of a constant $C > 0$ such that

$$\|u_{n+1}\|_{Q^1} \leq C\|\Xi\|_{\mathcal{H}} + C|\nu_{n+1}| \lesssim \|\Xi\|_{\mathcal{H}} \leq C_1\eta \tag{6.11}$$

for some universal constant C_1. In order to complete the induction, there only remains to check that $\partial_y^l u_{n+1}(x_i, 0) = 0$ for $l = 0, 1, 2$. These cancellations follow from the fact that $(u_{n+1})_{|\Sigma_i} = \delta_i + \nu_{n+1}^0 \delta_i^0 + \nu_{n+1}^1 \delta_i^1$ and from the properties $\partial_y^l \delta_i(0) = \partial_y^l \delta_i^k(0) = 0$ for $l = 0, 1, 2$ since $\Xi, \Xi^k \in \mathcal{H}$.

This completes the proof of the lemma. □

Remark 6.4. *At each step, we solve* (6.1) *with data* $\Xi_{n+1} = \Xi + \nu_{n+1}^0 \Xi^0 + \nu_{n+1}^1 \Xi^1$*, designed to satisfy the orthogonality conditions at this step. The important property of the correctors* Ξ^0 *and* Ξ^1 *is that they satisfy* $\overline{\ell^j}(\Xi^k) = \mathbf{1}_{j=k}$ *(see Corollary 2.13). Their existence stems from the independence of the orthogonality conditions on* $\mathcal{H}$*. By Proposition 2.10 and Proposition 2.30, they are also independent on* $C_c^\infty(\Omega) \times \{0\} \times \{0\}$ *and on* $\{0\} \times C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_1)$*. Hence, it would be possible to construct correctors* Ξ^k *involving only the source term* f *(respectively, only boundary data* (δ_0, δ_1)*). Using such correctors, as claimed in Remark 2.11, our nonlinear scheme would prove the existence of a submanifold of codimension 2 of source terms (resp. a manifold of codimension 2 of boundary data) for which the nonlinear problem* (1.7) *is well posed in* Q^1 *with null boundary data (resp. with null source term). Of course, similar claims hold at the linear level.*

6.2.2 Convergence of the sequence in $Q^{1/2}$.

We now prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$, using the following result:

Lemma 6.5. *There exist uniform constants* $\bar{C}, \bar{c} > 0$*, such that if* $\eta \leq \bar{c}$*, for all* $n \in \mathbb{N}$*,*

$$|\nu_{n+1} - \nu_n| + \|u_{n+1} - u_n\|_{Q^{1/2}} \leq \bar{C}\eta^n.$$

Proof. Let $w_n := u_{n+1} - u_n$ for $n \geq 1$. Thanks to Lemma 6.3, the sequence w_n is uniformly bounded in Q^1 by $2C_1\eta$. Moreover, for each $n \in \mathbb{N}^*$, w_n is the strong solution to

$$\begin{cases} (y + u_n)\partial_x w_n - \partial_{yy} w_n = -w_{n-1}\partial_x u_n + (\nu_{n+1}^0 - \nu_n^0) f^0 + (\nu_{n+1}^1 - \nu_n^1) f^1, \\ (w_n)_{|\Sigma_i} = (\nu_{n+1}^0 - \nu_n^0)\delta_i^0 + (\nu_{n+1}^1 - \nu_n^1)\delta_i^1, \\ (w_n)_{|y=\pm 1} = 0. \end{cases} \tag{6.12}$$

We already know that the solution w_n belongs to Q^1, as the difference between two Q^1 functions. Hence, there is no need to check that the orthogonality conditions are satisfied, and we can apply

Proposition 5.1, recalling that $\|u_n\|_{Q^1}$ is small thanks to (6.11). There is just one technicality: Proposition 5.1 requires that the source term vanishes on $\Sigma_0 \cup \Sigma_1$, which is not the case of the term $-w_{n-1}\partial_x u_n$. Hence, we first set

$$D_i^n := -(w_{n-1}\partial_x u_n)_{|\Sigma_i}.$$

Note that D_i^n can be expressed in terms of δ_i, δ_i^k, namely

$$D_i^n = -\left((\nu_n^0 - \nu_{n-1}^0)\delta_i^0 + (\nu_n^1 - \nu_{n-1}^1)\delta_i^1\right) \frac{\partial_y^2(\delta_i + \nu_n^0\delta_i^0 + \nu_n^1\delta_i^1)}{y + \delta_i + \nu_{n-1}^0\delta_i^0 + \nu_{n-1}^1\delta_i^1}.$$

Since $(f, \delta_0, \delta_1) \in \mathcal{H}$ and $(f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}$, $D_i^n \in H^2(\Sigma_i)$. Moreover, extending D_i^n by zero on $(-1,1) \setminus \Sigma_i$, the extension remains in H^2.

We then lift these boundary conditions by setting

$$f_l^n = D_0^n(y)\chi\left(\frac{x - x_0}{x_1 - x_0}\right) + D_1^n(y)\chi\left(\frac{x_1 - x}{x_1 - x_0}\right),$$

where $\chi \in C^\infty(\mathbb{R})$ is the same cut-off function as in (6.5). Then $f_l^n \in H_x^k H_y^2$ for any $k \in \mathbb{N}$, and $f_l^n(y = \pm 1) = 0$. Furthermore $\|f_l^n\|_{H^2} \lesssim \eta|\nu_n - \nu_{n-1}|$. Of course introducing this lift in the source term will perturb the orthogonality conditions. Hence we also introduce a smooth function $h_n \in C_0^\infty(\Omega)$ such that

$$\ell_n^j(-w_n\partial_x u_n - f_l^n + (\nu_{n+1}^0 - \nu_n^0)f^0 + (\nu_{n+1}^1 - \nu_n^1)f^1 + h_n, 0, 0) = 0 \quad \text{for } j = 0, 1.$$

In view of Remark 3.18, we can choose h_n (say, involving regularized versions of the biorthogonal basis of Corollary 2.13) such that

$$\begin{aligned} \|h_n\|_{H^2} &\le \bar{C}\left(|\nu_{n+1} - \nu_n| + \|f_l^n\|_{H^2} + \|w_n\partial_x u_n\|_{H_x^{1/3}L_y^2} + \|w_n\partial_x u_n\|_{L_x^2 H_y^{2/3}}\right) \\ &\le \bar{C}\left(|\nu_{n+1} - \nu_n| + \eta|\nu_n - \nu_{n-1}| + \|u_n\|_{Q^1}\|w_{n-1}\|_{Q^{1/2}}\right). \end{aligned} \tag{6.13}$$

We then write w_n as $w_n = \bar{w}_n + \tilde{w}_n$, where $\bar{w}_n$ and $\tilde{w}_n$ solve the respective equations

$$\begin{cases} \bar{u}_n\partial_x\bar{w}_n - \partial_{yy}\bar{w}_n = -w_{n-1}\partial_x u_n + (\nu_{n+1}^0 - \nu_n^0)f^0 + (\nu_{n+1}^1 - \nu_n^1)f^1 - f_n^l + h_n, \\ \bar{w}_{n|\Sigma_i} = 0, \\ \bar{w}_{n|y=\pm 1} = 0 \end{cases} \tag{6.14}$$

and

$$\begin{cases} \bar{u}_n\partial_x\tilde{w}_n - \partial_{yy}\tilde{w}_n = f_n^l - h_n, \\ \tilde{w}_{n|\Sigma_i} = (\nu_{n+1}^0 - \nu_n^0)\delta_i^0 + (\nu_{n+1}^1 - \nu_n^1)\delta_i^1, \\ \tilde{w}_{n|y=\pm 1} = 0. \end{cases} \tag{6.15}$$

Note that the orthogonality conditions are satisfied for both systems by choice of h_n, and that the source term for $\bar{w}_n$ vanishes on the lateral boundaries thanks to f_n^l.

Therefore, we are now ready to apply Proposition 5.1 to $\bar{w}_n$. We obtain, by (5.4),

$$\begin{aligned} \|\bar{w}_n\|_{Q^{1/2}} \lesssim & \|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}L_y^2} + \|w_{n-1}\partial_x u_n\|_{L_x^2 H_y^{3/2}} \\ & + \|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}((x_0,x_1),H_{00_r}^{1/2}(1/4,1)} + \|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}((x_0,x_1),H_{00_l}^{1/2}(-1,-1/4)} \\ & + |\nu_{n+1} - \nu_n| + \eta|\nu_n - \nu_{n-1}| + \|h_n\|_{H^2}. \end{aligned} \tag{6.16}$$

Indeed, the elementary functions f^j and δ_i^j of Corollary 2.13 have fixed norms.

By (6.9) and Proposition 4.1,

$$\begin{aligned}|\nu_{n+1}-\nu_n| &= |M_n^{-1}\ell_n(\Xi)-M_{n-1}^{-1}\ell_{n-1}(\Xi)|\\ &\le \|M_n^{-1}-M_{n-1}^{-1}\||\ell_n(\Xi)|+\|M_{n-1}^{-1}\||\ell_n(\Xi)-\ell_{n-1}(\Xi)|\\ &\lesssim \|u_n-u_{n-1}\|_{Q^{1/2}}\|\Xi\|_{\mathcal{H}} \lesssim \eta\|w_{n-1}\|_{Q^{1/2}}.\end{aligned} \tag{6.17}$$

Let us now derive bounds on $w_{n-1}\partial_x u_n - f_n^l$. For any $\sigma\in]0,1/6[$,

$$\begin{aligned}\|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}L_y^2} &\lesssim \|w_{n-1}\partial_x u_n - f_n^l\|_{H^{\sigma+\frac12}L_y^2}\\ &\lesssim \|w_{n-1}\|_{L_y^\infty H_x^{\sigma+\frac12}}\|\partial_x u_n\|_{L_y^2H_x^{2/3}} + \|f_n^l\|_{H^{\sigma+\frac12}L_y^2}\\ &\lesssim \|u_n\|_{Q^1}\|w_{n-1}\|_{Q^{1/2}} + \eta|\nu_n-\nu_{n-1}|,\end{aligned} \tag{6.18}$$

and

$$\|w_{n-1}\partial_x u_n\|_{L_x^2H_y^{3/2}} \lesssim \|w_{n-1}\|_{L_x^\infty(H_y^{3/2})}\|\partial_x u_n\|_{L_x^2H_y^{3/2}} \lesssim \|u_n\|_{Q^1}\|w_{n-1}\|_{Q^{1/2}}. \tag{6.19}$$

The bound on $\|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}((x_0,x_1),H_{00_r}^{1/2}(1/4,1)}$ is slightly more involved. Note that the Q^1 bound on u_n, in itself, does not allow us to bound $\partial_x u_n$ in $H_x^{1/2}H_y^{1/2}$. However, we can use the same arguments as in Proposition 3.20, and observe that on $(x_0,x_1)\times(1/4,1)$, u_n satisfies a classical parabolic equation. More precisely, recall that

$$(y+u_{n-1})\partial_x u_n = f+\nu_n^0 f^0+\nu_n^1 f^1+\partial_y^2 u_n,$$

where f, f^0, f^1 are smooth, and $\partial_y^3 u_n\in L^2$. Differentiating the equation with respect to y and using the estimates $\|\partial_y^k u_{n-1}\|_\infty \lesssim \|u_n\|_{Q^1}$ for $k=1,2$, we infer that $(y+u_n)\partial_x\partial_y^3 u_n\in L^2$, and

$$\|(y+u_n)\partial_x\partial_y^3 u_n\|_{L^2} \lesssim \eta+|\nu_n|+(1+\|u_{n-1}\|_{Q^1})\|u_n\|_{Q^1}\lesssim\eta.$$

Hence

$$\|\partial_x\partial_y^3 u_n\|_{L^2(\{y\ge1/4\})}\le C\eta,$$

uniformly in n. Using once again the same method, we infer that

$$\|\partial_x^2 u_n\|_{L^2(\{y\ge1/4\})}\le C\eta,\quad \|\partial_x^2\partial_y u_n\|_{L^2(\{y\ge1/4\})}\le C\eta,$$

uniformly in n. Hence, $\partial_x u_n$ is bounded by $C\eta$ in $H_x^1H_y^1\cap L_x^2H_y^3((x_0,x_1)\times(1/4,1))$, uniformly in n. We deduce that for $\sigma>0$ sufficiently small,

$$\begin{aligned}\|w_{n-1}\partial_x u_n - f_n^l\|_{H_{00}^{1/2}((x_0,x_1),H_{00_r}^{1/2}(1/4,1)} &\lesssim \|w_{n-1}\|_{H_x^{1/2+\sigma}H_y^{1/2+\sigma}}\|\partial_x u_n\|_{H_x^1H_y^1(x_0,x_1)\times(1/4,1)} + \|h_n^l\|_{H^2}\\ &\lesssim \eta\|w_{n-1}\|_{Q^{1/2}}+\eta^2\|w_{n-2}\|_{Q^{1/2}}.\end{aligned} \tag{6.20}$$

Gathering (6.18), (6.19), (6.20) and (6.22) we obtain

$$\|\bar w_n\|_{Q^{1/2}} \lesssim \eta\|w_{n-1}\|_{Q^{1/2}}+\eta^2\|w_{n-2}\|_{Q^{1/2}}. \tag{6.21}$$

The bound on $\tilde{w}_n$ is easier, and no interpolation result is needed. We observe that the right-hand side belongs to $H^2(\Omega)$, and satisfies the orthogonality conditions. Therefore $\tilde{w}_n \in H_x^{5/3}L_y^2 \cap L_x^2 H_y^4$, and

$$\begin{aligned}\|\tilde{w}_n\|_{H_x^{5/3}L_y^2} + \|\tilde{w}_n\|_{L_x^2 H_y^4} &\lesssim |\nu_{n+1}-\nu_n| + \|h_n\|_{H^2} + \|f_n^l\|_{H^2} \\ &\lesssim |\nu_{n+1}-\nu_n| + \eta\|w_{n-1}\|_{Q^{1/2}} + \eta|\nu_n - \nu_{n-1}|.\end{aligned} \tag{6.22}$$

Hence we also obtain

$$\|\tilde{w}_n\|_{Q^{1/2}} \lesssim \eta\|w_{n-1}\|_{Q^{1/2}} + \eta^2\|w_{n-2}\|_{Q^{1/2}}. \tag{6.23}$$

An induction argument then leads to the estimate announced in the lemma. □

6.2.3 Convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ and definition of the map $\mathcal{U}_\perp$

Classically, for η small enough, we deduce from Lemma 6.5 that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Q^{1/2}$. We recall that it is also uniformly bounded in Q^1. Hence, there exists $u = \mathcal{U}_\perp(\Xi) \in Q^1$ such that

$$\begin{aligned} u_n &\to u \quad \text{strongly in } Q^{1/2}, \\ u_n &\rightharpoonup u \quad \text{weakly in } Q^1. \end{aligned}$$

The strong convergence is sufficient to pass to the limit in (6.6). Furthermore, thanks to the continuity of the linear forms $u \in Q^{1/2} \mapsto \ell_u^i$, we can also pass to the limit in (6.9). We denote by $\nu(f,\delta_0,\delta_1) = \nu(\Xi)$ the limit of the sequence ν_n.

In particular, taking the limit in (6.9) proves that, for $j = 0, 1$,

$$\ell^j_{y+\mathcal{U}_\perp(\Xi)}\left(\Xi + \nu^0(\Xi)\Xi^0 + \nu^1(\Xi)\Xi^1\right) = 0 \quad \forall \Xi \in \mathcal{H}_{\rm sg}^\perp. \tag{6.24}$$

This relation translates the (tautological) fact that the Q^1 solution to the nonlinear problem satisfies the orthogonality conditions of the linearized problem at itself.

Let us prove that this entails the following quadratic estimate on $\nu(\Xi)$, used in Remark 6.2:

$$|\nu(\Xi)| \lesssim \|\Xi\|_{\mathcal{H}}^2. \tag{6.25}$$

Since ν^0, ν^1 and $\mathcal{U}_\perp$ are Lipschitz,

$$\|\mathcal{U}_\perp(\Xi)\|_{Q^1} + |\nu^0(\Xi)| + |\nu^1(\Xi)| \lesssim \|\Xi\|_{\mathcal{H}}. \tag{6.26}$$

Thus, using Proposition 4.1,

$$\|\ell^j_{y+\mathcal{U}_\perp(\Xi)} - \overline{\ell^j}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\mathcal{U}_\perp(\Xi)\|_{Q^1} \lesssim \|\Xi\|_{\mathcal{H}}. \tag{6.27}$$

From (6.24), we have, by linearity, for all $\Xi \in \mathcal{H}_{\rm sg}^\perp$,

$$\begin{aligned} 0 &= \ell^j_{y+\mathcal{U}_\perp(\Xi)}\left(\Xi + \sum_{k=0,1}\nu^k(\Xi)\Xi^k\right) \\ &= \overline{\ell^j}(\Xi) + \sum_{k=0,1}\nu^k(\Xi)\overline{\ell^j}(\Xi^k) + \left(\ell^j_{y+\mathcal{U}_\perp(\Xi)} - \overline{\ell^j}\right)\left(\Xi + \sum_{k=0,1}\nu^k(\Xi)\Xi^k\right) \\ &= 0 + \overline{M}\nu(\Xi) + \mathcal{O}(\|\Xi\|_{\mathcal{H}}^2), \end{aligned} \tag{6.28}$$

where $\overline{M}$ is the invertible matrix defined in Lemma 2.25, so that (6.25) follows.

6.2.4 Lipschitz regularity of the constructed maps.

In this paragraph, we prove that the maps $\Xi \mapsto \mathcal{U}_\perp(\Xi)$ and $\Xi \mapsto \nu(\Xi)$ have Lipschitz regularity. Let $\Xi, \Xi' \in B_\eta$. We use the prime notation to denote all the quantities associated with Ξ' during the nonlinear scheme.

In particular, one has

$$\begin{cases} (y+u_n)\partial_x(u_{n+1}-u'_{n+1}) - \partial_{yy}(u_{n+1}-u'_{n+1}) \\ \qquad = f - f' - (u_n - u'_n)\partial_x u'_{n+1} + (\nu^0_{n+1} - \nu'^0_{n+1})f^0 + (\nu^1_{n+1} - \nu'^1_{n+1})f^1 \\ (u_{n+1}-u'_{n+1})_{|\Sigma_i} = \delta_i - \delta'_i + (\nu^0_{n+1} - \nu'^0_{n+1})\delta^0_i + (\nu^1_{n+1} - \nu'^1_{n+1})\delta^1_i, \\ (u_{n+1}-u'_{n+1})_{|y=\pm 1} = 0. \end{cases} \tag{6.29}$$

Using the same estimates as previously,

$$\|u_{n+1}-u'_{n+1}\|_{Q^{1/2}} \lesssim \eta \|(u_n - u'_n)\|_{Q^{1/2}} + \|\Xi - \Xi'\|_{\mathcal{H}} + |\nu_{n+1} - \nu'_{n+1}| + \eta|\nu_n - \nu'_n|. \tag{6.30}$$

And, using one again Proposition 4.1 together with the definition (6.9),

$$|\nu_{n+1} - \nu'_{n+1}| \lesssim \|\Xi - \Xi'\|_{\mathcal{H}} + \eta \|u_n - u'_n\|_{Q^{1/2}} \tag{6.31}$$

Summing recursively these estimates this leads to the uniform estimates

$$\|u_{n+1}-u'_{n+1}\|_{Q^{1/2}} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}}, \tag{6.32}$$

$$\|\nu_{n+1} - \nu'_{n+1}\|_{\mathbb{R}^2} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}}. \tag{6.33}$$

Passing to the limit as $n \to \infty$, we infer that $\Xi \mapsto \mathcal{U}_\perp(\Xi)$ is Lipschitz from $\mathcal{H}^\perp_{\rm sg}$ to $Q^{1/2}$ and ν is Lipschitz from $\mathcal{H}^\perp_{\rm sg}$ to $\mathbb{R}^2$.

6.2.5 Value of ν^0 and ν^1 at zero.

One checks that, for $\Xi = \underline{0} = (0,0,0)$, the constructed initialization u_0 defined in (6.5) is null. Since $0 \in \mathcal{H}^\perp_{\rm sg}$ and $\ell^j_{y+0} = \overline{\ell^j}$, this leads to $\nu_1 = 0$ (by (6.9)). Hence, in (6.6) for $n = 0$, the system solved by u_1 has vanishing boundary data and vanishing source term. Hence $u_1 = 0$. This property propagates for every $n \geq 0$. Hence $\nu^j(0) = \lim \nu^j_n(0) = 0$.

This concludes the proof of Proposition 6.1.

6.2.6 A variation of the nonlinear scheme

In this paragraph, we discuss the possibility (already mentioned in Section 1.4) to tweak the nonlinear scheme in order to avoid the computation of the linear forms $\ell^j_{\bar{u}}$ at varying flows $\bar{u}$, and only rely on the reference orthogonality conditions $\overline{\ell^j}$. Of course, as described in Section 1.4, relying directly on the linearization scheme (1.16) is not possible, as it involves a loss of derivative.

With the change of variables of Section 3.1, and natural indexed variations of the notations of this section, our nonlinear scheme can be roughly seen as the resolution of:

$$L_n U_{n+1} := (z\partial_x + \gamma_n \partial_z - \alpha_n \partial_{zz})\, U_{n+1} = g_{n+1}, \tag{6.34}$$

for which g_{n+1} is chosen to ensure the varying orthogonality conditions $\ell^j_n(g_{n+1}) = 0$ (say with null boundary data), associated with the operator L_n.

A possible tweak would be to put $\gamma_n \partial_z U_{n+1} - (\alpha_n - 1)\partial_{zz} U_{n+1}$ in the right-hand side, where we replace U_{n+1} by U_n. More precisely, this would suggest the following scheme:

$$(z\partial_x - \partial_{zz})\,\tilde{U}_{n+1} = \tilde{g}_{n+1} + \tilde{\gamma}_n \partial_z \tilde{U}_n + (\tilde{\alpha}_n - 1)\partial_{zz}\tilde{U}_n. \tag{6.35}$$

In this scheme, $\tilde{g}_{n+1}$ would be chosen to satisfy the reference orthogonality conditions

$$\overline{\ell^j}\left(\tilde{g}_{n+1} - \tilde{\gamma}_n \partial_z \tilde{U}_n + (\tilde{\alpha}_n - 1)\partial_{zz}\tilde{U}_n\right) = 0. \tag{6.36}$$

To prove that the sequence $\tilde{U}_n$ converges in some space Q, one would need to make sure that $\tilde{g}_{n+1}$ also converges. In particular, one would need to check that

$$\left|\overline{\ell^j}\left(\tilde{\gamma}_n \partial_z \tilde{U}_n - \tilde{\gamma}_{n-1}\partial_z \tilde{U}_{n-1}\right)\right| \lesssim \eta \|\tilde{U}_n - \tilde{U}_{n-1}\|_Q. \tag{6.37}$$

By Lemma 5.16, $\overline{\ell^j}$ is not continuous below $H_x^{1/6}L_y^2$ tangential regularity. Thus, one would need to prove that $\tilde{\gamma}_n \partial_z \tilde{U}_n$ converges in $H_x^{1/6}L_y^2$. Since $\tilde{\gamma}_n$ involves a term of the form $\partial_x \tilde{U}_n$, this probably requires working at least with $Q = Q^{1/2}$.

In particular, one still needs to establish the $Q^{1/2}$ well-posedness theory, so the identification of the interpolation space of Section 5 cannot be avoided (and this identification relies on the construction of the explicit profiles of Section 2.5). Moreover, the stability estimates performed in Section 4 would mostly still be required, since one would need to estimate $\tilde{\gamma}_n - \tilde{\gamma}_{n-1}$, which is exactly what is done in Section 4.2.

In a nutshell, we expect that, while such a scheme could probably avoid part of the definitions and results of Section 3, the core of the difficulty of the problem would remain the same. Eventually, as mentioned in Section 1.4, our natural approach of "varying linear forms" has the added benefit of yielding a well-posedness theory for the linearized problems, as in Proposition 3.20.

6.3 Local uniqueness of solutions to the nonlinear problem

We prove the local uniqueness statement in Theorem 3. The argument is straightforward: if two strong solutions u and u' exist and are small in Q^1, their difference $w := u - u'$ is the solution to a degenerate elliptic linear equation with null source term and boundary values, so it vanishes identically.

More precisely, let $u, u' \in Q^1$ be two solutions to (1.7) satisfying $\|u\|_{Q^1} \leq \eta$ and $\|u'\|_{Q^1} \leq \eta$ for some small η. Then $w := u - u' \in Q^1$ and solves the system

$$\begin{cases} (y+u)\partial_x w + (\partial_x u')w - \partial_y^2 w = 0, \\ w_{|y=\pm 1} = 0, \\ w_{|\Sigma_i} = 0. \end{cases} \tag{6.38}$$

We follow the arguments of Section 3 and introduce a change of variables Y such that,

$$\forall (x,z) \in \Omega, \quad Y(x,z) + u(x, Y(x,z)) = z.$$

Let $\alpha(x,z) = (1+u_y)^2(x, Y(x,z))$, $\gamma = (zu_x - u_{yy})(x, Y(x,z))$, $\gamma_1' = \bar{u}_x'(x, Y(x,z))$. Then $W(x,z) = w(x, Y(x,z))$ is a solution to

$$\begin{cases} z\partial_x W + \gamma \partial_z W + \gamma_1' W - \alpha \partial_{zz} W = 0, \\ W_{|z=\pm 1} = 0, \\ W_{|\Sigma_i} = 0. \end{cases} \tag{6.39}$$

Furthermore, α, γ, γ_1' satisfy the estimates of Lemma 4.4. Using the results of Appendix A, we infer that $W = 0$. More precisely, we rely on a generalized version of these results which allows a term of order 0 in the equations, of the form βW, where $\|\beta\|_{L^\infty_x L^2_z} \ll 1$. Here, $\beta = \gamma_1'$ and $\|\gamma_1'\|_{L^\infty_x L^2_z} \lesssim \|\bar{u}_x'\|_{H^{2/3}_x L^2_z} \lesssim \|\bar{u}' - y\|_{Q^1}$. This concludes the proof of the uniqueness within the ball of radius η in Q^1.

6.4 Necessity of the orthogonality conditions

At the linear level, Theorem 1 states that $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}^\perp_{\mathrm{sg}}$ is a necessary condition to solve (2.1) with regularity Q^1. Our purpose in this paragraph is to prove that, as stated in Proposition 1.1, at the nonlinear level, if a solution has Q^1 regularity, this necessary condition generalizes to $\Xi \in \mathcal{M}$. In particular, the nonlinear phenomena do not eliminate the need for orthogonality conditions, at least when one tries to obtain solutions with such regularity.

Proof of Proposition 1.1. Let $\eta > 0$ to be chosen small enough later on in the proof. Let $\Xi \in \mathcal{H}$ with $\|\Xi\|_{\mathcal{H}} \leq \eta$ and assume that there exists $u \in Q^1$ with $\|u\|_{Q^1} \leq \eta$ such that u is a solution to (1.7). We introduce

$$\widetilde{\Xi} := \Xi^\perp + \nu^0(\Xi^\perp)\Xi^0 + \nu^1(\Xi^\perp)\Xi^1, \tag{6.40}$$

which can be thought of as a good projection of Ξ on $\mathcal{M}$. Thanks to Proposition 6.1, we introduce $\widetilde{u} := \mathcal{U}_\perp(\Xi^\perp) \in Q^1$, which is a solution to (1.7) with data $\widetilde{\Xi}$ and $\|\widetilde{u}\|_{Q^1} \lesssim \eta$ (by Lipschitz regularity of the solution operator $\mathcal{U}_\perp$).

For $k \in \{0, 1\}$, we also introduce the coefficients $\mu^k := \nu^k(\Xi^\perp) - \langle \Xi^k; \Xi \rangle_{\mathcal{H}}$, which characterize how far Ξ is from $\mathcal{M}$. Then $w := \widetilde{u} - u$ belongs to Q^1 with

$$\|w\|_{Q^1} \lesssim \eta \tag{6.41}$$

and is a solution to

$$\begin{cases} (y + \widetilde{u})\partial_x w - \partial_{yy} w = -w(\partial_x u) + \mu^0 f^0 + \mu^1 f^1, \\ w_{|\Sigma_i} = \mu^0 \delta_i^0 + \mu^1 \delta_i^1, \\ w_{|y=\pm 1} = 0. \end{cases} \tag{6.42}$$

By Proposition 3.20, the fact that $w \in Q^1$ implies that, for $j \in \{0, 1\}$, the following orthogonality conditions are satisfied

$$\begin{aligned} 0 &= \ell^j_{\widetilde{u}}(-w\partial_x u + \mu^0 f^0 + \mu^1 f^1, \mu^0 \delta_0^0 + \mu^1 \delta_0^1, \mu^0 \delta_1^0 + \mu^1 \delta_1^1) \\ &= \ell^j_{\widetilde{u}}(-w\partial_x u, 0, 0) + \mu^0 \ell^j_{\widetilde{u}}(\Xi^0) + \mu^1 \ell^j_{\widetilde{u}}(\Xi^1). \end{aligned} \tag{6.43}$$

We can now apply Proposition 5.1, after a suitable lifting of the traces of $-w\partial_x u$.

Mimicking the proof of Lemma 6.5, we infer that

$$\|w\|_{Q^{1/2}} \lesssim |\mu^0| + |\mu^1| + \eta \|w\|_{Q^{1/2}}. \tag{6.44}$$

Thus, for η small enough, we obtain

$$\|w\|_{Q^{1/2}} \lesssim |\mu^0| + |\mu^1|. \tag{6.45}$$

By Proposition 4.1, for η small enough,

$$\|\ell^j_{\widetilde{u}} - \overline{\ell^j}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\widetilde{u}\|_{Q^{1/2}} \leq \|\widetilde{u}\|_{Q^1} \lesssim \eta, \tag{6.46}$$

where $\overline{\ell^j}$ are the orthogonality conditions for the linear shear flow (see Definition 2.9). Hence, recalling Corollary 2.13, we obtain from (6.43),

$$|\mu^0| + |\mu^1| \lesssim \eta(|\mu^0| + |\mu^1|) + \left|\ell^j_{\bar{u}}(w\partial_x u, 0, 0)\right|. \tag{6.47}$$

Using Remark 3.18,

$$\left|\ell^j_{\bar{u}}(w\partial_x u, 0, 0)\right| \lesssim \|w\partial_x u\|_{H^{1/3}_x L^2_y} + \|w\partial_x u\|_{L^2_x H^{2/3}_y} \lesssim \eta\|w\|_{Q^{1/2}}. \tag{6.48}$$

We obtain, using (6.45),

$$|\mu^0| + |\mu^1| \lesssim \eta(|\mu^0| + |\mu^1|). \tag{6.49}$$

We infer that, for $k \in \{0,1\}$, $\mu^k = 0$ so $\nu^k(\Xi^\perp) = \langle \Xi^k; \Xi\rangle_{\mathcal{H}}$, so $\Xi \in \mathcal{M}$ by (6.3). □

A Uniqueness of weak solutions for linear problems

The purpose of this section is to prove the uniqueness of weak solutions in $L^2_x(H^1_0(-1,1))$ to (3.4), which is stated in Proposition 3.3. Such a uniqueness result is also proved in [24, Section 5].

The proof follows the arguments of Baouendi and Grisvard in [9], which concerns the case of the model equation (2.1). For the reader's convenience, we recall the main steps of the proof here, and adapt them to the present (slightly different) context.

We assume that the coefficients α, γ satisfy the assumptions of Proposition 3.3, and we introduce the sets

$$\mathcal{B} := \{U \in L^2_x(H^1_0(-1,1)),\ \ z\partial_x U \in L^2_x(H^{-1}_y)\},$$
$$\mathcal{A} := \mathcal{B} \cap H^1(\Omega).$$

Note that if $U \in L^2_x(H^1_0(-1,1))$ is a weak solution to (3.4), namely

$$\begin{cases} z\partial_x U + \gamma\partial_z U - \alpha\partial_{zz} U = g \in L^2_x H^{-1}_z, \\ U_{|\Sigma_i} = \widetilde{\delta}_i, \\ U_{|z=\pm 1} = 0, \end{cases}$$

then $U \in \mathcal{B}$. Indeed, it follows from the weak formulation (3.12) that for any $V \in H^1_0(\Omega)$,

$$\langle z\partial_x U, V\rangle_{L^2H^{-1}, L^2H^1} = -\int_\Omega (\gamma + \alpha_z) V\partial_z U - \int_\Omega \alpha\partial_z U\partial_z V + \int_\Omega gV.$$

By density, this formula still holds for $V \in L^2_x(H^1_0)$, and therefore $z\partial_x U \in L^2_x(H^{-1}_z)$.

We then recall the following result from [9]:

Lemma A.1. *The set $\mathcal{A}$ is dense in $\mathcal{B}$. Furthermore, there exists a constant C depending only on Ω, such that for $i \in \{0,1\}$,*

$$\forall v \in \mathcal{A}, \quad \int_{-1}^{1} |z|\ |v(x_i, y)|^2\, \mathrm{d}y \le C\|v\|^2_{\mathcal{B}}.$$

As a consequence, the applications

$$v \in \mathcal{A} \mapsto v_{|x=x_i} \in \mathscr{L}^2_z(-1,1)$$

can be uniquely extended into continuous applications on $\mathcal{B}$.

As a consequence, Baouendi and Grisvard [9] obtain the following corollary:

Corollary A.2. *For all $u, v \in \mathcal{B}$,*

$$\langle z\partial_x u, v\rangle_{L^2(H^{-1}),L^2(H_0^1)} + \langle z\partial_x v, u\rangle_{L^2(H^{-1}),L^2(H_0^1)} = \int_0^H (zuv)_{x=x_1} - \int_0^H (zuv)_{x=x_0}. \tag{A.1}$$

Proof. Thanks to Lemma A.1, it suffices to prove the identity when $u, v \in \mathcal{A}$. In that case, the left-hand side is simply

$$\int_\Omega z\partial_x uv + zu\partial_x v = \int_\Omega \partial_x(zuv).$$

The result follows by integration. □

Proof of uniqueness of weak solutions to (3.4). Let $U \in L^2_x(H_0^1)$ be a weak solution to (3.4) with $g = 0$ and $\widetilde{\delta}_i = 0$. As mentioned above, $U \in \mathcal{B}$. According to Corollary A.2, for any $V \in \mathcal{B}$ such that $V = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$-\langle z\partial_x V, U\rangle_{L^2(H^{-1}),L^2(H_0^1)} + \int_\Omega (\gamma + \alpha_z)\partial_z UV + \int_\Omega \alpha\partial_z U\partial_z V = 0.$$

Now, let $h \in C_c^\infty(\Omega)$ be arbitrary, and let $V \in L^2(H_0^1)$ be a weak solution to

$$\begin{cases} -z\partial_x V - \partial_z(\gamma V) - \partial_{zz}(\alpha V) = h, \\ V_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases}$$

(The existence of weak solutions for this adjoint problem is proved in the same way as existence for the direct problem in Proposition 3.3).

Then $V \in \mathcal{B}$, and choosing U as a test function in the variational formulation for V, we obtain

$$\int_\Omega hu = 0.$$

Thus $u = 0$. Uniqueness of weak solutions to (3.1) follows. □

B Proofs of functional analysis results

B.1 An abstract existence principle

As Fichera in [21], we use the following abstract existence principle (see [18, Theorem 1]), which allows skipping a viscous regularization scheme.

Lemma B.1. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}$ be three Hilbert spaces. Let $F_i \in \mathcal{L}(\mathscr{H}_i;\mathscr{H})$ for $i \in \{1,2\}$. Then the following statements are equivalent:*

- $\operatorname{range} F_1 \subset \operatorname{range} F_2$,
- *There exists a constant $C > 0$ such that*

$$\forall h \in \mathscr{H}', \quad \|F_1^* h\|_{\mathscr{H}_1'} \le C\|F_2^* h\|_{\mathscr{H}_2'}. \tag{B.1}$$

- *There exists* $G \in \mathcal{L}(\mathscr{H}_1; \mathscr{H}_2)$ *such that* $F_1 = F_2 G$.

Moreover, when these hold, there exists a unique $G \in \mathcal{L}(\mathscr{H}_1; \mathscr{H}_2)$ *such that* $\ker G = \ker F_1$, $\operatorname{range} G \subset (\operatorname{range} F_2^*)^\perp$ *and* $\|G\| = \inf\{C > 0;\ \text{(B.1) } \textit{holds}\}$.

Indeed, this yields the following weak Lax-Migram result, where the linear right-hand side is assumed to be continuous for the weaker norm.

Lemma B.2. *Let* $\mathscr{U}$ *and* $\mathscr{V}$ *be two Hilbert spaces with* $\mathscr{V}$ *continuously embedded in* $\mathscr{U}$. *Let* a *be a continuous bilinear form on* $\mathscr{U} \times \mathscr{V}$ *and* b *be a continuous linear form on* $\mathscr{U}$. *Assume that there exists a constant* $c > 0$ *such that, for every* $v \in \mathscr{V}$,

$$a(v, v) \geq c\|v\|_{\mathscr{U}}^2. \tag{B.2}$$

Then, there exists $u \in \mathscr{U}$ *such that* $\|u\|_{\mathscr{U}} \leq \frac{1}{c}\|b\|_{\mathcal{L}(\mathscr{U})}$ *and, for every* $v \in \mathscr{V}$, $a(u, v) = b(v)$.

Proof. Set $\mathscr{H} := \mathcal{L}(\mathscr{V})$, $\mathscr{H}_1 := \mathcal{L}(\mathscr{U})$, $F_1 := \mathrm{Id}$ (from $\mathcal{L}(\mathscr{U})$ to $\mathcal{L}(\mathscr{V})$), $\mathscr{H}_2 := \mathscr{U}$ and $F_2 : \mathscr{U} \to \mathcal{L}(\mathscr{V})$ defined by $F_2 u := a(u, \cdot)$. Then $F_1^* = \mathrm{Id}$ (from $\mathscr{V}$ to $\mathscr{U}$) and $F_2^* v = a(\cdot, v)$. Moreover

$$\|F_2^* v\|_{\mathcal{L}(\mathscr{U})} \geq |a(v, v)|/\|v\|_{\mathscr{U}} \geq c\|v\|_{\mathscr{U}} = c\|F_1^* v\|_{\mathscr{U}}. \tag{B.3}$$

So (B.1) holds with $C = 1/c$ and Lemma B.1 yields the existence of $G \in \mathcal{L}(\mathcal{L}(\mathscr{U}); \mathscr{U})$ such that $F_1 = F_2 G$ and $\|G\| \leq \frac{1}{c}$. The conclusions follow by setting $u := Gb$. □

B.2 Product and composition rules in Sobolev spaces

Lemma B.3 (Pointwise multiplication)**.** *Pointwise multiplication is a continuous bilinear map*

- *from* $H^{3/2}(-1,1) \times H^{3/2}(-1,1)$ *to* $H^{3/2}(-1,1)$,
- *from* $H^{1/2}(x_0,x_1) \times H^s(x_0,x_1)$ *to* $H^{1/2}(x_0,x_1)$ *for any* $s > 1/2$,
- *from* $H^{1/2}(x_0,x_1) \times H^s(x_0,x_1)$ *to* $H^{s'}(x_0,x_1)$ *for any* $s' < \min(s, 1/2)$.
- *from* $H^s(x_0,x_1) \times H^{s'}(x_0,x_1)$ *to* $H^{s'}(x_0,x_1)$ *for any* $s > 1/2$, $s \geq s'$.

Proof. These are particular cases of [11, Theorem 7.4]. □

Lemma B.4 (Composition of H^σ functions)**.** *Let* $Y \in H^1(\Omega)$ *such that* $\partial_z Y \in L_z^\infty(H_x^1)$, $\partial_x Y \in L_z^2(H_x^{2/3})$. *Assume that there exists* $m > 0$ *such that* $\partial_z Y(x,z) \in [m, m^{-1}]$ *a.e. and* $Y(x, \pm 1) = \pm 1$. *Then for any* $\sigma, \sigma' \in [0,1]$, *for any* $g \in H_x^\sigma(H_y^{\sigma'}) \cap L_x^2(H_y^{2\sigma+\sigma'})$

$$\|g(x, Y(x,z))\|_{H_x^\sigma(H_z^{\sigma'})} \leq C\left(\|g\|_{H_x^\sigma(H_y^{\sigma'})} + \|g\|_{L_x^2(H_y^{2\sigma+\sigma'})}\right).$$

Proof. Throughout the proof, we set

$$G(x,z) = g(x, Y(x,z)).$$

First, note that for all $g \in L^2(\Omega)$,

$$\|G\|_{L^2(\Omega)}^2 = \int_{x_0}^{x_1}\int_{-1}^{1} g^2(x, Y(x,z))\,\mathrm{d}x\,\mathrm{d}z \leq m^{-1}\int_{x_0}^{x_1}\int_{-1}^{1} g^2(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

so that $\|G\|_{L^2} \leq \|g\|_{L^2}$.

Furthermore,

$$\partial_x G(x,z) = \partial_x g(x, Y(x,z)) + \partial_x Y(x,z) \partial_y g(x, Y(x,z)).$$

Hence

$$\|\partial_x G\|_{L^2(\Omega)} \leq C_m \left(\|\partial_x g\|_{L^2(\Omega)} + \|\partial_x Y\|_{L^2(H^{2/3}_x)} \|(\partial_y g) \circ Y\|_{L^2_x H^1_z} \right),$$

where $\|(\partial_y g) \circ Y\|_{L^2_x H^1_z} \lesssim \|\partial_y g\|_{L^2} + \|\partial_z Y\|_{L^\infty} \|\partial_{yy} g\|_{L^2}$.

Now, note that the application $g \mapsto G$ is linear. By interpolation, we obtain, for any $\sigma \in (0,1)$,

$$\|G\|_{H^\sigma_x(L^2_z)} \leq C_m \left(\|g\|_{H^\sigma_x(L^2_z)} + \|\partial_x Y\|^\sigma_{L^2(H^{2/3}_x)} \|g\|_{L^2(H^{2\sigma}_y)} \right). \tag{B.4}$$

We now prove the same type of estimates for the z derivatives. We have

$$\partial_z G(x,z) = \partial_z Y(x,z) \partial_y g(x, Y(x,z)),$$

and thus $\|G\|_{L^2_x H^1_z} \lesssim \|g\|_{L^2_x H^1_y}$. By interpolation, we infer that for any $\sigma \in (0,1)$,

$$\|G\|_{L^2_x H^\sigma_z} \leq C_m \|g\|_{L^2_x H^\sigma_y}.$$

Additionally,

$$\|\partial_z G\|_{H^\sigma_x L^2_z} \leq \|\partial_y g(x,Y)\|_{H^\sigma_x L^2_z} \|\partial_z Y\|_{L^\infty_z H^1_x}.$$

Using the first step, we infer that

$$\|G\|_{H^\sigma_x H^1_z} \leq C_m (1 + \|\partial_z Y\|_{L^\infty_z H^1_x}) \left(\|g\|_{H^\sigma_x(H^1_y)} + \|g\|_{L^2_x(H^{1+2\sigma}_y)} \right).$$

Interpolating once again, we obtain, for any $\sigma, \sigma' \in (0,1)$,

$$\|G\|_{H^\sigma_x H^{\sigma'}_z} \lesssim \left(\|g\|_{H^\sigma_x(H^{\sigma'}_y)} + \|g\|_{L^2_x(H^{2\sigma+\sigma'}_y)} \right).$$

□

Corollary B.5. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$, and $\bar{u}(x, \pm 1) = \pm 1$. Let $Y = Y(x,z)$ such that $\bar{u}(x, Y(x,z)) = z$ for all $(x,z) \in \Omega$.*

Then $\|Y_z - 1\|_\infty \ll 1$, $Y_z \in L^\infty_z(H^1_x)$, $Y_x \in L^2_z(H^{2/3}_x)$, and $\|Y_x\|_{L^2_z(H^{2/3}_x)} \ll 1$.

Proof. First, observe that $Y_z = 1/\bar{u}_y(x, Y(x,z))$, so that the first estimate follows from Lemma 1.10.

Concerning the estimate on Y_x, we observe that

$$Y_x = -\frac{\bar{u}_x}{\bar{u}_y}(x, Y(x,z)).$$

Let us first assume that $\bar{u}$ is smooth (say C^∞), and then argue by density. If $\bar{u}$ is smooth, then the above formula first shows that $Y_x \in L^\infty$. Differentiating the identity with respect to x once again, we infer that $Y \in W^{2,\infty}$. Furthermore, according to Lemma B.4 (see in particular (B.4))

$$\|Y_x\|_{H^{2/3}_x L^2_z} \lesssim \left\| \frac{\bar{u}_x}{\bar{u}_y} \right\|_{H^{2/3}_x L^2_z} + \|Y_x\|^{2/3}_{H^{2/3}_x L^2_z} \left\| \partial_y^2 \left(\frac{\bar{u}_x}{\bar{u}_y} \right) \right\|_{L^2}.$$

When $\|\bar{u}-y\|_{Q^1} \ll 1$,

$$\left\| \frac{\bar{u}_x}{\bar{u}_y} \right\|_{H^{2/3}_x L^2_z}, \ \left\| \partial_y^2 \left(\frac{\bar{u}_x}{\bar{u}_y} \right) \right\|_{L^2} \ll 1.$$

Using Young's inequality in order to absorb the second term in the right-hand side, we obtain

$$\|Y_x\|_{H^{2/3}_x L^2_z} \lesssim \left\| \frac{\bar{u}_x}{\bar{u}_y} \right\|_{H^{2/3}_x L^2_z} + \left\| \partial_y^2 \left(\frac{\bar{u}_x}{\bar{u}_y} \right) \right\|^3_{L^2} \ll 1.$$

Furthermore,

$$Y_{xz} = -Y_z \frac{\partial}{\partial y} \left(\frac{\bar{u}_x}{\bar{u}_y} \right) (x, Y(x,z)).$$

Recalling that $Y_z \in L^\infty$, $\bar{u}_y, \bar{u}_{yy} \in L^\infty$, and $\bar{u}_{xy} \in L^2_x H^1_y \hookrightarrow L^\infty_y(L^2_x)$, we deduce that

$$\|Y_{xz}\|_{L^\infty_z(L^2_x)} \leq C \|\bar{u}-y\|_{Q^1}.$$

Hence we obtain the result when $\bar{u}$ is smooth and $\|\bar{u}-y\|_{Q^1} \ll 1$. We conclude by density. □

Corollary B.6. *Let $\bar{u} \in Q^1$ such that $\|\bar{u}-y\|_{Q^1} \ll 1$ and $\bar{u}(x,\pm 1) = \pm 1$. For $\varphi \in H^1_x L^2_z \cap L^2_x H^3_z$,*

$$\|\varphi \circ \bar{u}\|_{H^1_x L^2_y} + \|\varphi \circ \bar{u}\|_{L^2_x H^3_y} \lesssim \|\varphi\|_{H^1_x L^2_z} + \|\varphi\|_{L^2_x H^3_z}, \tag{B.5}$$

where $(\varphi \circ \bar{u})(x,y) := \varphi(x, \bar{u}(x,y))$.

Proof. This stems from Lemma B.4, applied with $Y = \bar{u}$. Indeed, one first has

$$\|\varphi \circ \bar{u}\|_{H^1_x L^2_y} \lesssim \|\varphi\|_{H^1_x L^2_z} + \|\varphi\|_{L^2_x H^2_z}. \tag{B.6}$$

In the vertical direction, write

$$\partial_y^3 (\varphi \circ \bar{u}) = \bar{u}_{yyy} (\varphi_y \circ \bar{u}) + 2 \bar{u}_y \bar{u}_{yy} (\varphi_{yy} \circ \bar{u}) + \bar{u}_y^3 (\varphi_{yyy} \circ \bar{u}). \tag{B.7}$$

Thus, recalling the embeddings of Lemma 1.10,

$$\begin{aligned} \|\partial_y^3 (\varphi \circ \bar{u})\|_{L^2} &\lesssim \|\bar{u}\|_{H^{2/3}_x H^3_y} \|\varphi_y \circ \bar{u}\|_{L^2_x H^1_y} + \|\bar{u}\|^2_{L^\infty_x W^{2,\infty}_y} \|\varphi_{yy} \circ \bar{u}\|_{L^2} + \|\bar{u}\|^3_{L^\infty_x W^{2,\infty}_y} \|\varphi_{yyy} \circ \bar{u}\|_{L^2} \\ &\lesssim \|\varphi\|_{L^2_x H^3_z}, \end{aligned} \tag{B.8}$$

which concludes the proof. □

B.3 Extension operators

We start with Lemma 1.3, which allows extending functions from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.

Proof of Lemma 1.3. Up to translation and rescaling, we can assume that $(x_0, x_1) = (0,1)$.

We start by constructing a continuous horizontal extension operator P_x from $Z^0((0,1)\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,1))$. Let $\chi \in C^\infty(\mathbb{R};[0,1])$ such that $\chi \equiv 1$ on $(0,1)$ and $\operatorname{supp}\chi \subset (-1,2)$. Let $\phi \in Z^0((0,1)\times(-1,1))$. For $x \in (-1,2)$ and $z \in (-1,1)$, let

$$(Q_x\phi)(x,z) := \begin{cases} \phi(-x,z) & \text{if } x \in (-1,0) \\ \phi(x,z) & \text{if } x \in (0,1), \\ \phi(2-x,z) & \text{if } x \in (1,2), \end{cases} \tag{B.9}$$

$$(P_x\phi)(x,z) := \chi(x)(Q_x\phi)(x,z). \tag{B.10}$$

First, $\|P_x\|_{L^2_{x,z}\to L^2_{x,z}} \leq 3$. Moreover, $\partial_z^k(P_x\phi) = P_x\partial_z^k\phi$ for $k \in \{1,2\}$. Hence $\|P_x\|_{L^2_xH^2_z\to L^2_xH^2_z} \leq 3$. Eventually, $z\partial_x(P_x\phi) = P_x(z\partial_x\phi) + \chi' Q_x\phi$. Hence $\|z\partial_x(P_x\phi)\|_{L^2} \leq 3\|z\partial_x\phi\|_{L^2} + 2\|\chi'\|_{L^\infty}\|\phi\|_{L^2}$. Thus P_x defines a continuous extension operator from $Z^0((0,1)\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,1))$.

We now construct a continuous upwards vertical extension operator P_+ from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$. We proceed, as classical (see e.g. [5]), by considering a weighted linear combination of rescaled reflections. For $\phi \in Z^0(\mathbb{R}\times(-1,1))$, $x\in\mathbb{R}$ and $z\in(-1,\infty)$, let

$$(Q_+\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z\in(-1,1),\\ 3\phi(x,2-z) - 2\phi(x,3-2z) & \text{if } z\in(1,2), \end{cases} \tag{B.11}$$

$$(P_+\phi)(x,z) := \chi_+(z)(Q_+\phi)(x,z), \tag{B.12}$$

where $\chi_+ \in C^\infty(\mathbb{R};[0,1])$ is such that $\chi_+ \equiv 1$ on $(-1,1)$ and $\operatorname{supp}\chi_+ \subset (-2,1+\frac14)$. The chosen coefficients ensure that both $Q_+\phi$ and $\partial_z(Q_+\phi)$ are continuous at $z=1$. Hence $P_+\phi \in L^2_xH^2_z$ and

$$\|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,+\infty))} = \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(-1,1))} + \|P_+\phi\|_{L^2_x(\mathbb{R};H^2_z(1,+\infty))} \leq C_+\|\phi\|_{L^2_xH^2_z}, \tag{B.13}$$

for some constant C_+ depending only on $\|\chi_+\|_{W^{2,\infty}}$. Moreover, using that $\chi(z)=0$ for $z>1+\frac14$,

$$\begin{aligned} \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,+\infty))} &= \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,1+\frac14))}\\ &\lesssim \|\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}\\ &\lesssim \|z\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac12,1))}. \end{aligned} \tag{B.14}$$

Hence P_+ is a continuous extension operator from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$.

The extension for $z<-1$ is performed in a similar fashion and left to the reader. □

We will also need the following extension result in this appendix.

Lemma B.7. *There exists a continuous extension operator P from $Z^0(\Omega_+)$ to $Z^0((x_0,x_1)\times\mathbb{R})$ such that, if $\phi_{|x=x_1}=0$ on $(0,1)$, $(P\phi)_{|x=x_1}=0$.*

Proof. We proceed, as in the proof of Lemma 1.3, by extension by reflections and truncation. The reflection at $z=1$ is done in the proof of Lemma 1.3. The truncation is left to the reader. We only check here the reflection at $z=0$ due to the degeneracy of the Z^0 norm at $z=0$.

Let $\phi\in Z^0(\Omega_+)$. We define an extension $Q\phi$ on Ω by

$$(Q\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z\in(0,1),\\ 3\phi(x,-z/2) - 2\phi(x,-z) & \text{if } z\in(-1,0). \end{cases} \tag{B.15}$$

In particular $(Q\phi)(x,0^-) = (Q\phi)(x,0^+)$ and $\partial_z(Q\phi)(x,0^-) = \partial_z(Q\phi)(x,0^+)$, so

$$\|Q\phi\|_{L^2_xH^2_z(-1,1)} \lesssim \|Q\phi\|_{L^2_xH^2_z(-1,0)} + \|Q\phi\|_{L^2_xH^2_z(0,1)} \lesssim \|\phi\|_{L^2_xH^2_z} \leq \|\phi\|_{Z^0}. \tag{B.16}$$

Moreover,

$$\|z\partial_x(Q\phi)\|_{L^2_xL^2_z(-1,0)} \leq 3\|2(z/2)\partial_x\phi(x,z/2)\|_{L^2_xL^2_z(0,1)} + 2\|z\partial_x\phi(x,z)\|_{L^2_xL^2_z(0,1)} \lesssim \|\phi\|_{Z^0}. \tag{B.17}$$

Eventually, if $\phi_{|x=x_1}=0$ on $(0,1)$ (B.15) implies $(Q\phi)_{|x=x_1}=0$ on $(-1,1)$. □

B.4 Critical dualities

Lemma B.8. *For $a, b \in H^1(x_0, x_1)$ such that $a(x_0) = b(x_1) = 0$,*

$$\|ab\|_{H^{1/2}_{00}} \lesssim \|a'\|_{L^2} \|b\|_{H^{1/2}_{00_r}}. \tag{B.18}$$

Proof. On the one hand, by Lemma B.3,

$$\|ab\|_{H^{1/2}} \lesssim \|a\|_{H^1} \|b\|_{H^{1/2}} \lesssim \|a'\|_{L^2} \|b\|_{H^{1/2}_{00_r}}. \tag{B.19}$$

On the other hand, since for every $x \in (x_0, x_1)$, $|a(x)| \leq |x - x_0|^{\frac{1}{2}} \|a'\|_{L^2}$,

$$\int_{x_0}^{x_1} \frac{|a(x)b(x)|^2}{|x - x_0||x - x_1|} \,\mathrm{d}x \leq \|a'\|_{L^2}^2 \int_{x_0}^{x_1} \frac{|b(x)|^2}{|x - x_1|} \lesssim \|a'\|_{L^2}^2 \|b\|_{H^{1/2}_{00_r}}^2. \tag{B.20}$$

Gathering both estimates concludes the proof. □

Lemma B.9. *For $q, v, w \in H^1(x_0, x_1)$ such that $q(x_0) = 0$ and $w(x_1) = 0$,*

$$\left| \int_{x_0}^{x_1} q(\partial_x v) w \right| \lesssim \|q'\|_{L^2} \|v\|_{H^{1/2}} \|w\|_{H^{1/2}_{00_r}}. \tag{B.21}$$

Proof. By [42, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0, x_1)$ to $(H^{1/2}_{00}(x_0, x_1))'$. Hence

$$\left| \int_{x_0}^{x_1} q(\partial_x v) w \right| \lesssim \|v\|_{H^{1/2}} \|qw\|_{H^{1/2}_{00}}. \tag{B.22}$$

The conclusion follows from Lemma B.8. □

Lemma B.10. *For $q, v, w \in H^1(x_0, x_1)$ such that $q(x_0) = 0$ and $w(x_1) = 0$,*

$$\left| \int_{x_0}^{x_1} qv\partial_x w \right| \lesssim \|q'\|_{L^2} \|v\|_{H^{1/2}} \|w\|_{H^{1/2}_{00_r}}. \tag{B.23}$$

Proof. Let $\chi \in C^\infty([x_0, x_1]; [0, 1])$ such that $\chi \equiv 1$ in a neighborhood of x_0 and $\chi \equiv 0$ in a neighborhood of x_i. Let us write

$$\int_{x_0}^{x_1} qv\partial_x w = \int_{x_0}^{x_1} (\chi qv)\partial_x w - \int_{x_0}^{x_1} (1 - \chi) w \partial_x (qv) + \int_{x_0}^{x_1} \chi' wqv. \tag{B.24}$$

By [42, Proposition 12.1], ∂_x is continuous from $H^{1/2}(x_0, x_1)$ to $(H^{1/2}_{00}(x_0, x_1))'$. Thus

$$\left| \int_{x_0}^{x_1} qv\partial_x w \right| \lesssim \|\chi qv\|_{H^{1/2}_{00}} \|w\|_{H^{1/2}} + \|(1 - \chi)w\|_{H^{1/2}_{00}} \|qv\|_{H^{1/2}} + \|\chi' q\|_{L^\infty} \|v\|_{L^2} \|w\|_{L^2}. \tag{B.25}$$

By Lemma B.3, $\|\chi qv\|_{H^{1/2}} \lesssim \|\chi\|_{H^1} \|q\|_{H^1} \|v\|_{H^{1/2}}$, $\|(1 - \chi)w\|_{H^{1/2}} \lesssim \|(1 - \chi)\|_{H^1} \|w\|_{H^{1/2}}$ and $\|qv\|_{H^{1/2}} \lesssim \|q\|_{H^1} \|v\|_{H^{1/2}}$. First, since $\chi \equiv 1$ near x_0, $(1 - \chi(x))^2 \lesssim |x - x_0|$. Thus

$$\int_{x_0}^{x_1} \frac{(1 - \chi(x))^2 |w(x)|^2}{|x - x_0||x - x_1|} \,\mathrm{d}x \lesssim \int_{x_0}^{x_1} \frac{|w(x)|^2}{|x - x_1|} \,\mathrm{d}x \lesssim \|w\|_{H^{1/2}_{00_r}}^2. \tag{B.26}$$

Similarly, since $\chi^2(x) \lesssim |x - x_1|$ and $q^2(x) \lesssim |x - x_0| \|q'\|_{L^2}^2$ by Cauchy–Schwarz, there holds

$$\int_{x_0}^{x_1} \frac{|\chi(x) q(x) v(x)|^2}{|x - x_0||x - x_1|} \,\mathrm{d}x \lesssim \|q'\|_{L^2}^2 \|v\|_{L^2}^2. \tag{B.27}$$

This concludes the proof, since $\|q\|_{H^1} \lesssim \|q'\|_{L^2}$ thanks to the condition $q(x_0) = 0$. □

B.5 Embeddings

We collect in this paragraph various embedding results used throughout the paper.

B.5.1 Embedding of Z^0 in $H^{2/3}_x L^2_z$

Proof of Proposition 1.4. Let $\psi \in C^\infty_c(\mathbb{R})$. By Lemma B.11, one has

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.28}$$

Using standard dimensional analysis arguments (e.g. by introducing the rescaled function $\psi_\lambda : z \mapsto \psi(\lambda z)$ for $\lambda > 0$ and optimizing the choice of λ), one deduces from (B.28) that

$$\|\psi\|_{L^2} \lesssim \|z\psi\|^{\frac{2}{3}}_{L^2} \|\partial_{zz}\psi\|^{\frac{1}{3}}_{L^2}. \tag{B.29}$$

Let $\phi \in C^\infty_c(\mathbb{R}^2)$. Let $\hat\phi(\xi, z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then using (B.29) and Hölder's inequality,

$$\begin{aligned}
\|\phi\|^2_{H^{2/3}_x L^2_z} &= \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{2}{3}} |\hat\phi(\xi,z)|^2 \,\mathrm{d}\xi \,\mathrm{d}z \\
&\lesssim \|\phi\|^2_{L^2} + \int_{\mathbb{R}} |\xi|^{\frac{4}{3}} \|z\hat\phi(\xi,z)\|^{\frac{4}{3}}_{L^2_z} \|\partial_{zz}\hat\phi(\xi,z)\|^{\frac{2}{3}}_{L^2_z} \,\mathrm{d}\xi \\
&\lesssim \|\phi\|^2_{L^2} + \left(\int_{\mathbb{R}^2} |\xi|^2 z^2 |\hat\phi(\xi,z)|^2 \,\mathrm{d}z \,\mathrm{d}\xi \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |\partial_{zz}\hat\phi(\xi,z)|^2 \,\mathrm{d}z \,\mathrm{d}\xi \right)^{\frac{1}{3}} \\
&\lesssim \|\phi\|^2_{L^2} + \|z\partial_x\phi\|^{\frac{4}{3}}_{L^2} \|\partial_{zz}\phi\|^{\frac{2}{3}}_{L^2}.
\end{aligned} \tag{B.30}$$

Hence $\|\phi\|_{H^{2/3}_x L^2_z} \lesssim \|\phi\|_{Z^0}$. This concludes the proof, by density of $C^\infty_c(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$. □

B.5.2 Full domain embeddings

The following inequality is used in the proof of the key result Proposition 1.4.

Lemma B.11. *For $\psi \in C^\infty_c(\mathbb{R})$,*

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.31}$$

Proof. On the one hand, for $|z| \geq 1$,

$$\int_{|z|\geq 1} \psi^2 \leq \|z\psi\|^2_{L^2}. \tag{B.32}$$

On the other hand, for every $(z_0, z) \in (-2, 2)$,

$$|\partial_z\psi(z)| \leq |\partial_z\psi(z_0)| + 2\|\partial_{zz}\psi\|_{L^2}. \tag{B.33}$$

Moreover, by classical Sobolev embeddings,

$$\|\partial_z\psi\|_{L^2(1,2)} \lesssim \|\psi\|_{L^2(1,2)} + \|\partial_{zz}\psi\|_{L^2(1,2)} \leq \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.34}$$

Thus, integrating (B.33) for $z_0 \in (1,2)$,

$$\|\partial_z \psi\|_{L^\infty(-2,2)} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}. \tag{B.35}$$

Now, writing $\psi(z) = \psi(z_0) + \int_{z_0}^z \psi'$ and integrating for $z_0 \in (1,2)$ yields

$$\|\psi\|_{L^2(-1,1)} \lesssim \|\psi\|_{L^2(1,2)} + \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}, \tag{B.36}$$

which concludes the proof. □

Lemma B.12. *For $\psi \in C_c^\infty(\mathbb{R})$,*

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.37}$$

Proof. For $|z| \le 2$, (B.35) yields directly

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2(-2,2)} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.38}$$

Let $\chi \in C^\infty(\mathbb{R}_+;[0,1])$ with $\chi \equiv 0$ on $[0,1]$, $\chi \equiv 1$ on $[2;+\infty)$ and $|\partial_z\chi| \le 2$. Then

$$\begin{aligned}\int_2^{+\infty} z|\partial_z\psi(z)|^2\,\mathrm{d}z &\le \int_0^{+\infty} z\chi(z)|\partial_z\psi(z)|^2\,\mathrm{d}z \\ &= -\int_0^{+\infty} z\chi(z)\psi(z)\partial_{zz}\psi(z)\,\mathrm{d}z - \int_0^{+\infty} z\partial_z\chi(z)\psi(z)\partial_z\psi(z)\,\mathrm{d}z \\ &\le \|z\psi\|_{L^2}\|\partial_{zz}\psi\|_{L^2} + 2\|z\psi\|_{L^2}(\|\partial_z\psi\|_{L^2(2;+\infty)} + \|\partial_z\psi\|_{L^2(0,2)})\end{aligned} \tag{B.39}$$

The $\|\partial_z\psi\|_{L^2(0,2)}$ term can be bounded by (B.35) and the $\|\partial_z\psi\|_{L^2(2;+\infty)}$ term can be treated perturbatively via the Peter-Paul inequality. This yields

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2(2,+\infty)} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.40}$$

By symmetry, the same holds on $(-\infty,-2)$, which concludes the proof. □

Lemma B.13. *For $\phi \in Z^0(\Omega)$,*

$$\||z|^{\frac{1}{2}}\partial_z\phi\|_{L^2_z(H^{1/2}_x)} \lesssim \|\phi\|_{Z^0} \tag{B.41}$$

Proof. Thanks to Lemma 1.3, it is sufficient to prove the embedding and (B.41) for $\phi \in Z^0(\mathbb{R}^2)$ with bounded support in the vertical direction, say $\operatorname{supp}\phi \subset \mathbb{R}\times(-5,5)$ (as one can always take a smooth truncation of the extended function). We proceed as in the proof of Proposition 1.4. Let $\psi \in C_c^\infty(\mathbb{R})$. By Lemma B.12, and using standard dimensional analysis arguments, one deduces that

$$\||z|^{\frac{1}{2}}\partial_z\psi\|_{L^2} \lesssim \|z\psi\|_{L^2}^{\frac{1}{2}}\|\partial_{zz}\psi\|_{L^2}^{\frac{1}{2}}. \tag{B.42}$$

Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Let $\hat\phi(\xi,z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then

using (B.42) and Hölder's inequality,

$$\begin{aligned}
\||z|^{\frac12}\partial_z\phi\|^2_{L^2_z(H^{1/2}_x)} &= \int_{\mathbb{R}^2}(1+|\xi|^2)^{\frac12}|z||\partial_z\hat\phi(\xi,z)|^2\,\mathrm{d}\xi\,\mathrm{d}z\\
&\lesssim \||z|^{\frac12}\partial_z\phi\|^2_{L^2} + \int_{\mathbb{R}}|\xi|\|z\hat\phi(\xi,z)\|_{L^2_z}\|\partial_{zz}\hat\phi(\xi,z)\|_{L^2_z}\,\mathrm{d}\xi\\
&\lesssim \||z|^{\frac12}\partial_z\phi\|^2_{L^2} + \left(\int_{\mathbb{R}^2}|\xi|^2z^2|\hat\phi(\xi,z)|^2\,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\partial_{zz}\hat\phi(\xi,z)|^2\,\mathrm{d}z\,\mathrm{d}\xi\right)^{\frac12}\\
&\lesssim \||z|^{\frac12}\partial_z\phi\|^2_{L^2} + \|z\partial_x\phi\|^{\frac12}_{L^2}\|\partial_{zz}\phi\|^{\frac12}_{L^2}.
\end{aligned} \tag{B.43}$$

Moreover, since $\phi(\cdot,z)=0$ for $|z|\geq 5$,

$$\||z|^{\frac12}\partial_z\phi\|_{L^2}\lesssim\|\partial_{zz}\phi\|_{L^2} \tag{B.44}$$

Hence gathering both inequalities proves that $\||z|^{\frac12}\partial_z\phi\|^2_{L^2_z(H^{1/2}_x)}\lesssim\|\phi\|_{Z^0}$. This concludes the proof, by density of $C^\infty_c(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$. □

B.5.3 Embeddings involving the Lions–Magenes space

Lemma B.14. *Let $\phi\in Z^0(\Omega_+)$ such that $\phi_{|x=x_1}=0$. Then $\phi\in C^0_z([0,1];H^{1/2}_{00_r}(x_0,x_1))$ and*

$$\|\phi\|_{L^\infty_z H^{1/2}_{00_r}}\lesssim\|\phi\|_{Z^0}. \tag{B.45}$$

Proof. Thanks to the extension result Lemma B.7, it is sufficient to prove this result with Ω_+ replaced by $\mathcal{O}:=(x_0,x_1)\times\mathbb{R}$. Therefore, let $\phi\in Z^0(\mathcal{O})$ such that $\phi_{|x=x_1}=0$ on $\mathbb{R}$. By Proposition 1.4, $\phi\in L^2_z(\mathbb{R};H^1_{0_r}(x_0,x_1))\cap H^2_z(\mathbb{R};L^2(x_0,x_1))$, where $H^1_{0_r}(x_0,x_1)$ denotes H^1 functions vanishing at $x=x_1$. By the fractional trace theorem [42, Chapter 1, Theorem 4.2 and equation (4.7)], this implies that $\phi\in C^0_z(\mathbb{R};G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0,x_1),L^2(x_0,x_1)]_{\frac14}$ in this reference. By [42, Chapter 1, Theorem 11.7][7], $G=H^{1/2}_{00_r}(x_0,x_1)$. The claimed norm estimate readily follows. □

Lemma B.15. *Let $\phi\in Z^0(\Omega_+)$ such that $\phi_{|x=x_1}=0$. Then $z\partial_z\phi\in L^2_zH^{1/2}_{00_r}$ and*

$$\|z\partial_z\phi\|_{L^2_zH^{1/2}_{00_r}}\lesssim\|\phi\|_{Z^0}. \tag{B.46}$$

Proof. Let $\phi\in Z^0(\Omega_+)$ such that $\phi_{|x=x_1}=0$. We extend ϕ to $Z^0(\mathcal{O})$, where $\mathcal{O}:=(x_0,x_1)\times\mathbb{R}$, thanks to the extension result Lemma B.7. We then truncate the extension for $|z|\geq 2$ thanks to a C^∞_c function, so that the extension is now supported in $(x_0,x_1)\times(-3,3)$, coincides with ϕ on Ω, and belongs to $Z^0(\mathcal{O})$. With a slight abuse of notation, we still denote the extension by ϕ. Note that $\phi_{|x=x_1}=0$ on $\mathbb{R}$.

Let $\psi:=z\phi$. Then $\psi\in L^2_z(\mathbb{R};H^1_{0_r}(x_0,x_1))\cap H^2_z(\mathbb{R};L^2(x_0,x_1))$ with $\|\psi\|\lesssim\|\phi\|_{Z^0}$ for the associated norm. By the intermediate derivative theorem [42, Chapter 1, Theorem 2.3], $\psi\in$

[7] This reference considers the case of vanishing conditions on the full boundary of the domain (so at $x=x_0$ and at $x=x_1$), but the adaptation to functions vanishing only at x_1 is straightforward.

$H^1_z(\mathbb{R};G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0,x_1),L^2(x_0,x_1)]_{\frac12}$ in this reference. By [42, Chapter 1, Theorem 11.7], $G=H^{1/2}_{00_r}(x_0,x_1)$ (see Footnote 7, Page 99). This yields

$$\|\partial_z\psi\|_{L^2_zH^{1/2}_{00_r}}\lesssim\|\phi\|_{Z^0}. \tag{B.47}$$

The conclusion follows since $\partial_z\psi=z\partial_z\phi+\phi$ and $\phi\in L^2_zH^{2/3}_x$ by Proposition 1.4 (with trace $\phi_{|x=x_1}=0$, which makes sense in $H^{2/3}_x$). □

Lemma B.16. *Let $\phi\in L^2((x_0,x_1);H^1_0(-1,1))$ such that $z\partial_x\phi\in L^2((x_0,x_1);H^{-1}(-1,1))$. Assume that $\phi=0$ on $\{x_1\}\times(0,1)\cup\{x_0\}\times(-1,0)$ (in the sense of traces in $\mathscr{L}^2_z(-1,1)$, see Lemma A.1). Then $z\phi\in L^2((0,1);H^{1/2}_{00_r})\cap L^2((-1,0);H^{1/2}_{00_l})$.*

Proof. Let $\psi:=z\phi$. Then $\psi\in H^1_0((-1,1);L^2(x_0,x_1))\cap H^{-1}((-1,1);H^1(x_0,x_1))$ with $\psi=0$ on $\{x_1\}\times(0,1)\cup\{x_0\}\times(-1,0)$ (in the same sense). Moreover, letting $\bar\psi$ denote the restriction to $(x_0,x_1)\times(0,1)$ of ψ, we have $\bar\psi\in H^1((0,1);L^2(x_0,x_1))\cap H^{-1}((0,1);H^1_{0_r}(x_0,x_1))$. We then construct an extension to $(x_0,x_1)\times\mathbb{R}$, still denoted by $\bar\psi$ which satisfies $\bar\psi\in H^1(\mathbb{R};L^2(x_0,x_1))\cap H^{-1}(\mathbb{R};H^1_{0_r}(x_0,x_1))$. By [2, Theorem 4.5.5] (which applies since both $L^2(x_0,x_1)$ and $H^1_{0_r}(x_0,x_1)$ are Hilbert spaces so enjoy the UMD property), $\bar\psi\in L^2(\mathbb{R};G)$ where G is the interpolation space denoted by $[H^1_{0_r}(x_0,x_1),L^2(x_0,x_1)]_{\frac12}$. By [42, Chapter 1, Theorem 11.7], $G=H^{1/2}_{00_r}(x_0,x_1)$ (see Footnote 7, Page 99). This proves the right part of the statement. The left part is proved similarly. □

C Proofs of lemmas of Section 3

C.1 Preliminary remarks on liftings

In Section 3.4.2, we introduced the unknown Θ^j, which is formally linked to Φ^j by $\partial_x\Theta^j=\alpha\partial_x\Phi^j$. We also introduced a lifted version $\Theta^j_\sharp$, which still authorized a jump in $\partial_z\Theta^j_\sharp$ across the line $\{z=0\}$. In this paragraph, we introduce another lift $\Theta^j_\flat$ such that the lifted function will be H^2 in z. This lift will be used in the proofs of this appendix, and is defined in the following way. We write

$$\Theta^j_\sharp=\Theta^j_l+\Theta^j_\flat,$$

where the lifting term Θ^j_l is given by

$$\Theta^j_l=zb_+(x)\chi(z)\mathbf{1}_{\Omega_+}+zb_-(x)\chi(z)\mathbf{1}_{\Omega_-}$$

with some coefficients $b_\pm$ to be determined. The role of Θ^j_l is to ensure that

$$[\Theta^j_\flat]=0,\quad [\partial_z\Theta^j_\flat]=0.$$

Note that the first condition is automatically satisfied with our choice above.

Furthermore, setting $F^j=G[\Theta^j]$,

$$\partial_x[F^j(x,\cdot)]_{|z=0}=\Gamma(x,0)\partial_x[\Theta^j(x,\cdot)]_{|z=0}=0.$$

Hence the jump of F^j is constant across the line $\{z=0\}$, and is equal to

$$\frac{1}{x_1-x_0}\int_{x_0}^{x_1}[F^j(x,\cdot)]_{|z=0}\,\mathrm{d}x.$$

Consequently, we choose $b_\pm$ to be constant in x, and we define

$$b_+^j := -\frac{1}{x_1 - x_0} \int_{x_0}^{x_1} F^j(x, 0^+)\, \mathrm{d}x,$$
$$b_-^j := -\frac{1}{x_1 - x_0} \int_{x_0}^{x_1} F^j(x, 0^-)\, \mathrm{d}x.$$

With this choice,

$$\left[F^j + (b_+^j \mathbf{1}_{z>0} - b_-^j \mathbf{1}_{z<0})\right]_{|z=0} = 0,$$

and therefore $[\partial_z \Theta_\flat^j]_{|z=0} = 0$, and $[F^j + \partial_z \Theta_l^j]_{|z=0} = 0$. Note also that $\partial_x \Theta_l^j = 0$. It follows that

$$-\frac{z}{\alpha}\partial_x \Theta_\flat^j - \partial_{zz}\Theta_\flat^j = S^j \quad \text{in } \Omega, \tag{C.1}$$

where

$$S^j = \mathbf{1}_{z>0}\partial_{zz}\left((\mathbf{1}_{j=0} - z\mathbf{1}_{j=1})\chi(z)\right) + \partial_z\left(\partial_z \Theta_j^l + G[\Theta^j]\right).$$

The boundary conditions for $\Theta_\flat^j$ are

$$\Theta_\flat^j(x, \pm 1) = 0,$$
$$\Theta_\flat^j(x_0, z) = -b_- z\chi(z) \quad \forall z \in (-1, 0),$$
$$\Theta_\flat^j(x_1, z) = -(\delta_{0j} + z(-\mathbf{1}_{j=1} + b_+))\chi(z) \quad \forall z \in (0, 1).$$

C.2 Proof of Lemma 3.11

Throughout the proof, in order to lighten the notation, we set

$$\mu_0 := \|\Gamma_1\|_{L_z^\infty(H_x^{1/2})} + \|\Gamma_1\|_{L_z^2(L_x^\infty)} + \|\Gamma_2\|_{H_x^1 L_z^2} + \|\alpha - 1\|_\infty + \|\alpha_z\|_\infty + \|\alpha_x\|_{L_z^\infty(L_x^2)}.$$

Remark C.1. *Note that $\|\alpha_{xz}\|_{L^2} \ll 1$ implies $\|\alpha_x\|_{L_z^\infty(L_x^2)} \ll 1$. The latter bound will be used several times in the proof.*

Proof of Lemma 3.11. ***Step 1.*** *Uniqueness.* Let $\Theta \in \mathcal{B}$ be a weak solution to (3.51) with $f = 0$, $\eta_i = 0$. We adapt the arguments of Baouendi and Grisvard (see Appendix A). Let $V \in L_x^2(H_0^1(-1, 1))$ be arbitrary. We have, using the weak formulation (3.52)

$$\begin{aligned}
\left\langle z\partial_x\Theta, \frac{V}{\alpha}\right\rangle_{L^2(H^{-1}),L^2(H_0^1)} &= \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H_0^1)} + \langle z\partial_x\Theta, \frac{(1-\alpha)V}{\alpha}\rangle_{L^2(H^{-1},L^2(H_0^1)} \\
&= \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H_0^1)} \qquad \text{(C.2)} \\
&\quad - \int_\Omega G[\Theta]\partial_z((\alpha-1)V) - \int_\Omega \partial_z\Theta\partial_z((\alpha-1)V). \qquad \text{(C.3)}
\end{aligned}$$

Using once again the weak formulation (3.52), we see that the left-hand side is also equal to

$$\int_\Omega G[\Theta]\partial_z V + \int_\Omega \partial_z\Theta\partial_z V.$$

From there, it follows that

$$\begin{aligned}\|z\partial_x\Theta\|_{L^2(H^{-1})} &= \sup_{V\in L^2(H_0^1),\|V\|_{L^2H^1}\leq 1} \langle z\partial_x\Theta, V\rangle_{L^2(H^{-1}),L^2(H^1)} &\text{(C.4)}\\ &\leq (\|G[\Theta]\|_{L^2}+\|\partial_z\Theta\|_{L^2})\,(1+\|\alpha-1\|_\infty+\|\alpha_z\|_{L^2_z(L^\infty_x)}). &\text{(C.5)}\end{aligned}$$

Using Lemma 3.8 together with the smallness assumption on Γ_1, we obtain

$$\begin{aligned}\|z\partial_x\Theta\|_{L^2(H^{-1})} &\lesssim \|\partial_z\Theta\|_{L^2},\\ \|G[\Theta]\|_{L^2} &\leq \mu_0\|\partial_z\Theta\|_{L^2}.\end{aligned} \tag{C.6}$$

Eventually, thanks to Corollary A.2, we observe that

$$\begin{aligned}\langle z\partial_x\Theta,\Theta\rangle_{L^2(H^{-1}),L^2(H^1)} &= \frac12\left(\int_{-1}^1 z\Theta(x_1,z)^2\,\mathrm{d}z-\int_{-1}^1 z\Theta(x_0,z)^2\,\mathrm{d}z\right) &\text{(C.7)}\\ &= \frac12\left(\int_{-1}^0 z\Theta(x_1,z)^2\,\mathrm{d}z-\int_0^1 z\Theta(x_0,z)^2\,\mathrm{d}z\right). &\text{(C.8)}\end{aligned}$$

Now, take $V=\Theta$ in (3.52). Using (C.3), (C.6) and (C.8), we obtain

$$\begin{aligned}&\frac12\left(\int_0^1 z\Theta(x_0,z)^2\,\mathrm{d}z-\int_{-1}^0 z\Theta(x_1,z)^2\,\mathrm{d}z\right)+\int_\Omega(\partial_z\Theta)^2\\ \leq\ &\|G[\Theta]\|_{L^2}\left(\|\partial_z\Theta\|_{L^2}+\|\partial_z((\alpha-1)\Theta)\|_{L^2}\right)+\|\partial_z\Theta\|_{L^2}\|\partial_z((\alpha-1)\Theta\|_{L^2}\\ \lesssim\ &\mu_0\|\partial_z\Theta\|_{L^2}^2.\end{aligned}$$

Hence, for $\mu_0\ll 1$, we infer that $\Theta=0$.

Step 2. *Energy estimates for equation* (3.51).

First, since we have assumed that the boundary terms η_0, η_1 belong to $H_0^1(-1,1)$, note that we can always lift them with a function $\Theta_\eta(x,z)=\chi(x-x_0)\eta_0(z)+\chi_1(x-x_1)\eta_1(z)$ for some cut-off function $\chi\in C_c^\infty(\mathbb{R})$ as in the proof of Lemma 3.8. This will add a source term in $L^2_x(H^{-1}_z)$ to the equation (coming both from the diffusion term and from the operator G). Hence, in the following, we will consider the case $\eta_i=0$, without loss of generality.

Taking Θ as a test function in the weak formulation (3.52), we obtain

$$\int_\Omega(\partial_z\Theta)^2=\left\langle z\partial_x\Theta,\frac{\Theta}{\alpha}\right\rangle_{L^2(H^{-1}),L^2(H_0^1)}+\langle f,\Theta\rangle_{L^2(H^{-1}),L^2(H_0^1)}-\int_\Omega G[\Theta]\partial_z\Theta. \tag{C.9}$$

As in the previous step, we decompose the first term in the right-hand side as follows

$$\begin{aligned}\left\langle z\partial_x\Theta,\frac{\Theta}{\alpha}\right\rangle_{L^2(H^{-1}),L^2(H_0^1)} &= \langle z\partial_x\Theta,\Theta\rangle_{L^2(H^{-1}),L^2(H_0^1)} &\text{(C.10)}\\ &\quad+\left\langle z\partial_x\Theta,\frac{1-\alpha}{\alpha}\Theta\right\rangle_{L^2(H^{-1}),L^2(H_0^1)}. &\text{(C.11)}\end{aligned}$$

The term (C.10) yields a positive contribution on $\Sigma_0\cup\Sigma_1$, namely

$$-\langle z\partial_x\Theta,\Theta\rangle_{L^2(H^{-1}),L^2(H^1)}=\frac12\int_0^1 z(\Theta(x_0,z))^2\,\mathrm{d}z-\frac12\int_{-1}^0 z(\Theta(x_1,z))^2\,\mathrm{d}z. \tag{C.12}$$

We bound (C.11) as follows

$$\left|\left\langle z\partial_x\Theta, \frac{1-\alpha}{\alpha}\Theta\right\rangle_{L^2(H^{-1}),L^2(H^1_0)}\right| \leq \|z\partial_x\Theta\|_{L^2_x(H^{-1}_z)} \left\|\frac{1-\alpha}{\alpha}\Theta\right\|_{L^2_x(H^1_z)}. \tag{C.13}$$

Using the same arguments as in the first step, we find that

$$\|z\partial_x\Theta\|_{L^2_x(H^{-1}_z)} \lesssim \|\partial_z\Theta\|_{L^2} + \|f\|_{L^2_x(H^{-1}_z)},$$

while

$$\left\|\frac{1-\alpha}{\alpha}\Theta\right\|_{L^2_x(H^1_z)} \lesssim \left(\|\alpha_z\|_{L^2_z(L^\infty_x)} + \|\alpha-1\|_\infty\right)\|\partial_z\Theta\|_{L^2(\Omega_+)} \lesssim \mu_0\|\partial_z\Theta\|_{L^2(\Omega_+)}.$$

Gathering the previous estimates and using once again Lemma 3.8, we obtain

$$\begin{aligned}\|\partial_z\Theta\|^2_{L^2(\Omega)} + \int_0^1 z(\Theta(x_0,z))^2\,\mathrm{d}z - \frac{1}{2}\int_{-1}^0 z(\Theta(x_1,z))^2\,\mathrm{d}z \\ \leq \|f\|_{L^2(H^{-1})}\|\partial_z\Theta\|_{L^2(\Omega)} + C\mu_0\|\partial_z\Theta\|^2_{L^2(\Omega)}\end{aligned} \tag{C.14}$$

Hence, if μ_0 is small enough,

$$\|\Theta\|_{\mathcal{B}} \leq C\|f\|_{L^2(H^{-1})}. \tag{C.15}$$

Step 3. *Existence of weak solutions for smooth coefficients, with a large zero order term.*

We assume in this paragraph that the coefficients are smooth, say $\alpha, \gamma_1, \gamma_2 \in C^2(\overline{\Omega})$.

The purpose is to prove that for $C_0 > 0$ sufficiently large, depending on some norms of the coefficients, the equation

$$\begin{aligned}-\frac{z}{\alpha}\partial_x\Theta - \partial_z(G[\Theta] + \partial_z\Theta) + C_0\Theta = f \quad \text{in } \Omega,\\ \Theta(x,\pm 1) = 0,\\ \Theta(x_1,z) = 0 \quad \forall z > 0,\\ \Theta(x_0,z) = 0 \quad \forall z < 0\end{aligned} \tag{C.16}$$

has a unique solution in $L^2_x(H^1_z)$.

We proceed by viscous regularization and consider, for every $\varepsilon > 0$, the system

$$\begin{aligned}-\frac{z}{\alpha}\partial_x\Theta_\varepsilon - \partial_z G[\Theta_\varepsilon] + \partial_{zz}\Theta_\varepsilon - \varepsilon\partial_{xx}\Theta^j_\varepsilon + C_0\Theta_\varepsilon = f \quad \text{in } \Omega_\pm,\\ \Theta_\varepsilon = 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{C.17}$$

We then perform the same type of estimates as before. Since we allow the coefficients to be smooth and the constant C_0 to be large, the estimates are somewhat simpler. It can be easily proved that if

$$C_0 \gtrsim \|\alpha_x\|_\infty + \|\gamma_x\|_\infty,$$

the equation (C.17) has a unique solution in $H^1_0(\Omega)$, which satisfies uniform in ε bounds in $L^2_x(H^1_z)$. Passing to the limit as $\varepsilon \to 0$, we deduce that for the above choice of C_0, there exists a solution $\Theta \in L^2_x(H^1_0)$ to (C.16). Using the equation, we infer that this solution is such that $z\partial_x\Theta \in L^2_x(H^{-1}_z)$. Using the first step (or a variant including the term C_0), we deduce that this solution is unique.

Step 4. *Existence of weak solutions to* (3.51) *for smooth coefficients and* L^2 *source.*

Now, for any $f \in L^2(\Omega)$, consider the solution to (C.16). As in Appendix C.1, we introduce coefficients $b_\pm$ defined by

$$b_\pm := -\frac{1}{x_1 - x_0} \int_{x_0}^{x_1} G[\Theta](x, 0^\pm)\, \mathrm{d}x.$$

and a function $\Theta_\flat$ such that

$$\Theta = \sum_\pm z b_\pm \chi(z) \mathbf{1}_{\pm z > 0} + \Theta_\flat.$$

Note that

$$|b_\pm| \lesssim \|\partial_x \Gamma_2\|_\infty \|\partial_z \Theta\|_{L^2},$$

so that $\|\partial_z \Theta_\flat\|_{L^2} \lesssim \|\partial_z \Theta\|_{L^2}$. The function $\Theta_\flat$ is a solution to

$$-\frac{z}{\alpha}\partial_x \Theta_\flat - \partial_{zz}\Theta_\flat = S_\flat,$$

where

$$S_\flat = f + \partial_z \left(G[\Theta] - \partial_z \left(\sum_\pm z b_\pm \chi(z) \mathbf{1}_{\pm z > 0} \right) \right) - C_0 \Theta_\flat \in L^2(\Omega).$$

According to the results of Pagani [49], $\Theta_\flat \in Z^0(\Omega)$. In particular, $\Theta_\flat \in H_x^{2/3}(L_z^2)$. Since $\Theta_\flat$ is constant in x, it follows that $\Theta \in H_x^{2/3}(L_z^2)$. As a consequence, the operator $\mathcal{K} : f \in L^2 \mapsto \Theta \in H_x^{2/3}(L_z^2) \cap L_x^2(H_z^1)$, where Θ is the solution to (C.16), is compact.

We now apply the Fredholm alternative to the operator $\mathcal{K}$, which implies the following:

(i) either, for any $f \in L^2(\Omega)$, there exists a unique solution in $H_x^{2/3}(L_z^2) \cap L_x^2(H_z^1)$ to (3.51);

(ii) or there exists a non-trivial solution $\bar{\Theta}$ of

$$\begin{aligned} -\partial_z(\partial_z \bar{\Theta} + G[\bar{\Theta}]) - \frac{z}{\alpha}\partial_x \bar{\Theta} = 0 \quad \text{in } \Omega, \\ \bar{\Theta}(x, \pm 1) = 0, \\ \bar{\Theta}(x_1, z) = 0 \quad \forall z > 0, \\ \bar{\Theta}(x_0, z) = 0 \quad \forall z < 0; \end{aligned}$$

According to the first step (uniqueness for (3.51)), the second case never arises.

We deduce that when the coefficients α, γ are smooth and satisfy the smallness assumptions above, for any source term $f \in L^2$, there exists a unique solution to (3.51).

Step 5. *Existence of weak solutions to* (3.51) *for general coefficients and source terms.*

We argue by density and consider sequences of smooth coefficients $\alpha^n, \gamma_1^n, \gamma_2^n$ converging towards $\alpha, \gamma_1, \gamma_2$ in the relevant norms (i.e. in the norms in which we stated the smallness assumptions.) We also consider a sequence $(f_n)_{n \in \mathbb{N}}$ of L^2 functions such that $f_n \to f$ in $L^2(H^{-1})$.

For any $n \in \mathbb{N}$, there exists a solution Θ^n to (3.51) with the coefficients $\alpha^n, \gamma_1^n, \gamma_2^n$ and source term f_n. Furthermore, the second step (energy estimates) shows that this solution is uniformly bounded in $L^2(H^1)$, and $z\partial_x \Theta^n$ is uniformly bounded in $L^2(H^{-1})$. Hence, we can extract a subsequence such that

$$\Theta^n \rightharpoonup \Theta \quad \text{in } L_x^2(H_z^1).$$

The limit is a weak solution to (3.51). This concludes the proof. □

C.3 Proof of Lemma 3.12

We start with the following technical bound which will be used below.

Lemma C.2. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$. Consider the operator G defined in* (3.40). *For $\Theta \in Z^0(\Omega_\pm)$,*

$$\|\partial_z G[\Theta]\|_{L^2} \lesssim \mu_0 \|\Theta\|_{Z^0(\Omega_\pm)}, \tag{C.18}$$

where

$$\mu_0 := \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} + \|\partial_z \Gamma_1\|_{L^\infty_z (L^2_x)} + \|\Gamma_1\|_{L^2_z (H^{2/3}_x)} + \|\Gamma_2\|_{L^2_z (H^1_x)} + \|\partial_z \Gamma_2\|_{L^2_z (H^{3/5}_x)}. \tag{C.19}$$

Proof. By symmetry, we only treat the case of Ω_+. Differentiating the definition (3.40) of $G[\Theta]$ with respect to z, we have, in Ω_+,

$$\begin{aligned}
\partial_z G[\Theta] &= \Gamma_2 \partial_z \Theta + \int_x^{x_1} \partial_x \Gamma_2 \partial_z \Theta && \text{(C.20)}\\
&\quad + \partial_z \Gamma_2 \Theta + \int_x^{x_1} \partial_{xz} \Gamma_2 \Theta && \text{(C.21)}\\
&\quad + z\Gamma_1 \partial_z \Theta + \int_x^{x_1} \partial_x \Gamma_1 z \partial_z \Theta && \text{(C.22)}\\
&\quad + \Gamma_1 \Theta + \int_x^{x_1} \partial_x \Gamma_1 \Theta && \text{(C.23)}\\
&\quad - \int_x^{x_1} z \partial_z \Gamma_1 \partial_x \Theta. && \text{(C.24)}
\end{aligned}$$

We then evaluate each term of the right-hand side separately.

- The term (C.20) is the easiest. Recalling that $\|\partial_z \Theta\|_{L^\infty((0,1),L^2_x)} \lesssim \|\Theta\|_{Z^0(\Omega_+)}$, we have

$$\|\text{(C.20)}\|_{L^2(\Omega_+)} \lesssim \|\Theta\|_{Z^0(\Omega_+)} \left(\|\Gamma_2\|_{L^2_z (L^\infty_x)} + \|\partial_x \Gamma_2\|_{L^2(\Omega)} \right).$$

- For the term (C.21), we recall that $\|\Theta\|_{L^\infty_z (H^{1/2}_x)} \lesssim \|\Theta\|_{Z^0}$. Using a $H^{-2/5}_x - H^{2/5}_x$ duality, we obtain

$$\|\text{(C.21)}\|_{L^2(\Omega_+)} \|\Theta\|_{L^\infty_z (H^{2/5}_x)} \|\partial_z \Gamma_2\|_{L^2_z (H^{3/5}_x)} \lesssim \|\Theta\|_{Z^0(\Omega_+)} \|\partial_z \Gamma_2\|_{L^2_z (H^{3/5}_x)}.$$

  The term (C.23) is treated in a similar fashion, using a $H^{-1/3} - H^{1/3}$ duality

$$\|\text{(C.23)}\|_{L^2(\Omega_+)} \lesssim \|\Theta\|_{L^\infty_z (H^{1/3}_x)} \|\Gamma_1\|_{L^2_z (H^{2/3}_x)}.$$

- For the term (C.22), the first part is easily bounded, using the embeddings $H^{1/2} \hookrightarrow L^4$ in 1D and Lemma B.13, as

$$\|\Gamma_1 z \partial_z \Theta\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \left\| |z|^{\frac{1}{2}} \partial_z \Theta \right\|_{L^2_z H^{1/2}_x} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|\Theta\|_{Z^0(\Omega_+)}. \tag{C.25}$$

  We write the L^2 norm of the second part in the following way:

$$\begin{aligned}
\left\| \int_x^{x_1} \partial_x \Gamma_1 z \partial_z \Theta \right\|_{L^2(\Omega_+)} &= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \le 1} \int_{\Omega_+} h \int_x^{x_1} \partial_x \Gamma_1 z \partial_z \Theta \\
&= \sup_{h \in L^2(\Omega_+), \|h\|_{L^2} \le 1} \int_0^1 \int_{x_0}^{x_1} \left(\int_{x_0}^x h \right) \partial_x \Gamma_1 z \partial_z \Theta.
\end{aligned} \tag{C.26}$$

By Lemma B.9, for each $z \in (0,1)$,

$$\left| \int_{x_0}^{x_1} \left(\int_{x_0}^{x} h \right) \partial_x \Gamma_1 z \partial_z \Theta \right| \lesssim \|h(\cdot,z)\|_{L^2_x} \|\Gamma_1(\cdot,z)\|_{H^{1/2}_x} \|z\partial_z\Theta(\cdot,z)\|_{H^{1/2}_{00_r}(x_0,x_1)}. \tag{C.27}$$

Integrating over $z \in (0,1)$ yields

$$\int_0^1 \left| \int_{x_0}^{x_1} \left(\int_{x_0}^{x} h \right) \partial_x \Gamma_1 z \partial_z \Theta \right| \lesssim \|h\|_{L^2_{x,z}} \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|z\partial_z\Theta\|_{L^2_z H^{1/2}_{00_r}(x_0,x_1)}. \tag{C.28}$$

Thus, using the embedding of Lemma B.15,

$$\left\| \int_{x}^{x_1} \partial_x \Gamma_1 z \partial_z \Theta \right\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|\Theta\|_{Z^0(\Omega_+)}. \tag{C.29}$$

Combining (C.25) and (C.29) yields

$$\|(\text{C.22})\|_{L^2(\Omega_+)} \lesssim \|\Gamma_1\|_{L^\infty_z H^{1/2}_x} \|\Theta\|_{Z^0(\Omega_+)}. \tag{C.30}$$

- At last, we bound (C.24) by

$$\|(\text{C.24})\|_{L^2(\Omega_+)} \lesssim \|z\partial_x\Theta\|_{L^2(\Omega_+)} \|\partial_z\Gamma_1\|_{L^\infty_z(L^2_x)} \lesssim \|\Theta\|_{Z^0(\Omega_+)} \|\partial_z\Gamma_1\|_{L^\infty_z(L^2_x)}.$$

This concludes the proof. □

Proof of Lemma 3.12. For the proof of the Z^0 estimates, it will be convenient to work with the lift $\Theta^j_\flat$ defined in Appendix C.1. We recall that $\Theta^j_\flat$ satisfies equation (C.1).

Step 1. *Existence of a solution in Z^0 for smooth coefficients.*

We first assume that α and γ are smooth, and we now derive Z^0 estimates for our $L^2(H^1)$ solution. In this case, we notice that S^j belongs to $L^2(\Omega)$.

It follows from the work of Pagani that

$$\|z\partial_x\Theta^j_\flat\|_{L^2(\Omega)} + \|\partial_{zz}\Theta^j_\flat\|_{L^2(\Omega)} + \|\Theta^j_\flat\|_{L^\infty_z(H^{1/2}_x)} \tag{C.31}$$

$$\lesssim \|S^j\|_{L^2(\Omega)} + 1 + |b_+| + |b_-| + \|\Theta^j_\flat\|_{L^2(\Omega)}. \tag{C.32}$$

Thus there remains to evaluate each of the terms in the right-hand side. We start with $\|S^j\|_{L^2(\Omega)}$. According to the definition of Θ^j_l, we have

$$\begin{aligned} \|S^j\|_{L^2(\Omega_\pm)} \quad \leq \quad & C\left(1 + |b_+| + |b_-|\right) \\ & + \left(\|\Gamma\|_\infty + \|\partial_z\Gamma\|_{L^2_z(L^\infty_x)}\right) \|\partial_z\Theta^j\|_{L^2(\Omega_\pm)} \\ & + \left(\|\Gamma_x\|_\infty + \|\partial_{xz}\Gamma\|_{L^2_z(L^\infty_x)}\right) \|\partial_z\Theta^j\|_{L^2(\Omega_\pm)}. \end{aligned}$$

Note also that

$$|b_\pm| \leq C\left(\|\Gamma\|_\infty + \|\Gamma_x\|_\infty\right) \|\Theta^j(\cdot,0^\pm)\|_{L^2(x_0,x_1)} \leq \left(\|\Gamma\|_\infty + \|\Gamma_x\|_\infty\right) \|\partial_z\Theta^j\|_{L^2(\Omega_\pm)},$$

and

$$\|\Theta^j_\flat\|_{L^2} \lesssim \|\Theta^j_\sharp\|_{L^2} + |b_+| + |b_-|$$

Since Proposition 3.10 ensures that $\|\partial_z \Theta^j\|_{L^2(\Omega_\pm)} \lesssim 1$, we obtain a bound on $\|S^j\|_{L^2(\Omega_+)}$ and $b_\pm$ (depending on some high order norms of the coefficients). We therefore obtain a first Z^0 estimate on $\Theta^j_\flat$.

Step 2. *Z^0 estimates for smooth coefficients.*

We still assume that the coefficients α, γ are smooth, but our purpose is now to derive a Z^0 bound on the solution that only depends on norms of the coefficients $\alpha, \gamma_1, \gamma_2$ in lower order norms.

We first rewrite the equation for $\Theta^j_\flat$ as

$$-z\partial_x \Theta^j_\flat - \partial_{zz}\Theta^j_\flat = (\alpha - 1)\partial_{zz}\Theta^j_\flat + S^j,$$

so that there exists a universal constant $\bar{C}$ such that

$$\|z\partial_x \Theta^j_\flat\|_{L^2} + \|\partial_{zz}\Theta^j_\flat\|_{L^2} \leq \bar{C}\left(\|\alpha - 1\|_\infty \|\partial_{zz}\Theta^j_\flat\|_{L^2} + \|S^j\|_{L^2} + |b_+| + |b_-| + \|\Theta^j_\flat\|_{L^2}\right).$$

For $\|\alpha - 1\|_\infty \ll 1$, we absorb the first term into the left-hand side. Furthermore,

$$\|S^j\|_{L^2(\Omega)} \lesssim 1 + |b_+| + |b_-| + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_+)} + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_-)}.$$

Note also that $b_\pm$ are the (integrals of the) traces of $G[\Theta^j]$ at $z = 0^\pm$. It follows that if the coefficients α, γ satisfy the smallness assumptions of Proposition 3.10, there exists a universal constant $\bar{C}$ such that

$$\|z\partial_x \Theta^j_\flat\|_{L^2} + \|\partial_{zz}\Theta^j_\flat\|_{L^2} \leq \bar{C}\left(1 + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_+)} + \|\partial_z G[\Theta^j]\|_{L^2(\Omega_-)}\right). \tag{C.33}$$

Hence, using Lemma C.2,

$$|b_\pm| \leq \bar{C}\mu_0 \|\Theta^j\|_{Z^0(\Omega_\pm)},$$

and

$$\|\Theta^j\|_{Z^0(\Omega_\pm)} \leq 1 + \bar{C}\|\Theta^j_\flat\|_{Z^0(\Omega_\pm)}.$$

Substituting these estimates into (C.33), we get

$$\|\Theta^j_\flat\|_{Z^0} \leq \bar{C}(1 + \mu_0 \|\Theta^j_\flat\|_{Z^0}),$$

and thus

$$\|\Theta^j_\flat\|_{Z^0} \leq \bar{C}. \tag{C.34}$$

Step 3. *Z^0 estimates for general coefficients.*

We take a sequence α^n, γ^n of smooth coefficients such that $\gamma^n = z\gamma_1^n + \gamma_2^n$ and γ_k^n (resp. α^n) converges towards γ_k (resp. α) in the relevant norms, namely

$$\begin{aligned}
\|\gamma_1 - \gamma_1^n\|_{L^2_z(H^{2/3}_x)} + \|\gamma_1 - \gamma_1^n\|_{L^\infty_z(H^{1/3}_x)} + \|\partial_z(\gamma_1 - \gamma_1^n)\|_{L^\infty_z(L^2_x)} &\to 0,\\
\|\gamma_2 - \gamma_2^n\|_{L^2_z(L^\infty_x)} + \|\partial_x(\gamma_2 - \gamma_2^n)\|_{L^2} + \|\partial_z(\gamma_2 - \gamma_2^n)\|_{L^2_z(H^{3/5}_x)} &\to 0,\\
\|\alpha - \alpha^n\|_{H^1_z(H^1_x)} + \|\partial_{zz}(\alpha - \alpha^n)\|_{L^2_z(H^{3/5}_x)} &\to 0.
\end{aligned}$$

We consider the profiles Θ^j_n associated with α^n, γ^n. According to the previous step, for all $n \in \mathbb{N}$, $\Theta^j_{n\flat} \in Z^0$, and we have the estimate (C.34). Therefore, we can extract a subsequence and pass to the limit in the equation. It can be easily checked that the limit is a solution to (3.44), and satisfies (C.34). □

C.4 Proof of Lemma 3.14

Proof of Lemma 3.14. We start with uniqueness. Assume that $g = 0$, $\widetilde{\delta}_i = 0$. Let $f \in L^2(H^{-1})$ be arbitrary and let Θ be the unique solution to (3.51) with $\eta_0 = \eta_1 = 0$. It follows from Definition 3.13 that

$$\langle f, V \rangle_{L^2H^{-1}, L^2H_0^1} = 0$$

Since the function f is arbitrary, we infer that $V = 0$.

The same argument allows us to prove *a priori* estimates. Indeed, assume that $V \in L^2(H_0^1)$ is a solution to (3.29) in the sense of Definition 3.13. Consider the unique solution $\Theta \in L^2(H_0^1)$ to (3.51) with $f = -\partial_{zz} V \in L^2_x(H_z^{-1})$ and with $\eta_0 = \eta_1 = 0$. Then, according to Lemma 3.11

$$\left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2(H^{-1})} + \|\Theta\|_{L^2(H_0^1)} \lesssim \|V\|_{L^2(H_0^1)}.$$

Furthermore, combining the weak formulation from Definition 3.13 together with the one of equation (3.51), we deduce that

$$\begin{aligned}
\int_\Omega (\partial_z V)^2 = & \int_\Omega \partial_x \left(\frac{g}{\alpha} \right) \Theta \\
& - \int_{-1}^0 \left[\frac{g}{\alpha}(x_1, z) + \widetilde{\delta}_1''(z) - \widetilde{\delta}_1'(z) \Gamma_2(x_1, z) \right] \Theta(x_1, z) \, \mathrm{d}z \\
& + \int_0^1 \left[\frac{g}{\alpha}(x_0, z) + \widetilde{\delta}_0''(z) - \widetilde{\delta}_0'(z) \Gamma_2(x_0, z) \right] \Theta(x_0, z) \, \mathrm{d}z \\
& + \langle z \partial_x \Theta, \Gamma_1 (\mathbf{1}_{z<0} \delta_1' + \mathbf{1}_{z>0} \delta_0') \rangle_{L^2(H^{-1}), L^2(H_0^1)} \\
& - \int_\Omega (\mathbf{1}_{z<0} \delta_1' + \mathbf{1}_{z>0} \delta_0') \partial_x \Gamma_2 \Theta.
\end{aligned} \tag{C.35}$$

Using the assumptions on the coefficients $\alpha, \gamma_1, \gamma_2$ together with Lemma A.1, we obtain

$$\begin{aligned}
\|\partial_z V\|_{L^2}^2 \lesssim & \left(\|g\|_{H_x^1 L_z^2} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right]^2 \right)^{\frac{1}{2}} + \|\widetilde{\delta}_i'\|_{H^1} \right) \\
& \times \left(\left\| \frac{z}{\alpha} \partial_x \Theta \right\|_{L^2(H^{-1})} + \|\Theta\|_{L^2(H_0^1)} \right).
\end{aligned}$$

Combining the two estimates and using a Cauchy–Schwarz inequality, we deduce that

$$\|V\|_{L^2(H_0^1)} \lesssim \|g\|_{H_x^1 L_z^2} + \sum_i \left(\int_{\Sigma_i} \frac{1}{|z|} \left[\frac{g}{\alpha} + \widetilde{\delta}_i''(z) - \widetilde{\delta}_i'(z) \Gamma_2 \right]^2 \right)^{\frac{1}{2}} + \|\widetilde{\delta}_i'\|_{H^1}.$$

Once *a priori* estimates are available, we can adapt the arguments of Lemma 3.11 to prove the existence of solutions to (3.29) in $L^2(H^1)$. □

D Proofs of lemmas of Section 5

In this section, we prove the decomposition result of Proposition 5.15. The proof will rely in particular on the following estimate.

Lemma D.1. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$, and define the operator G by (3.40). Then, for all $\sigma \in [0,1]$, for all $\Theta \in Z^{\frac{2\sigma}{3}}(\Omega_\pm) = [Z^0(\Omega_\pm), Z^1(\Omega_\pm)]_\sigma$,*

$$\|\partial_z G[\Theta]\|_{H^\sigma_x L^2_z(\Omega_\pm)} \lesssim \|\bar{u} - y\|_{Q^1} \|\Theta\|_{Z^{\frac{2\sigma}{3}}(\Omega_\pm)}.$$

Proof of Lemma D.1. We work separately within each half-domain $\Omega_\pm$. For $\sigma = 0$, the estimate follows directly from Lemma C.2 and the bounds of Lemma 4.4. For $\sigma = 1$, we recall that

$$\partial_x \partial_z G[\Theta] = \partial_z(\Gamma \partial_x \Theta) = \partial_z \Gamma \partial_x \Theta + \Gamma \partial_x \partial_z \Theta.$$

We then decompose Γ as $z\Gamma_1 + \Gamma_2$ and we proceed to evaluate each term. Note that $Z^\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^2$. Furthermore, if $\theta \in Z^0$, then $z\partial_x\theta \in L^2$ and if $\theta \in Z^1$, $z\partial_x\theta \in H^1_x L^2_z \cap L^2_x H^2_z$. By interpolation, if $\theta \in Z^\sigma$, $z\partial_x\theta \in H^\sigma_x L^2_z \cap L^2_x H^{2\sigma}_z$. Now, let $\Theta \in Z^{2/3}$. Then $\partial_z(z\partial_x\Theta) \in H^{1/6}_x L^2_z \cap L^2_x H^{1/3}_z \hookrightarrow L^2_z(L^3_x)$. Since $\Gamma_1 \in L^\infty_z(H^{1/2}_x)$ by Lemma 4.4, it follows that

$$\|\Gamma_1 \partial_z(z\partial_x\Theta)\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1} \|\Theta\|_{Z^{2/3}}.$$

Furthermore, $\partial_z \Gamma_1 \in L^\infty_z(L^2_x)$ and $z\partial_x\Theta \in H^{2/3}_x L^2_z \hookrightarrow L^2_z(L^\infty_x)$. Hence $\|\partial_z\Gamma_1 z\partial_x\Theta\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1}\|\Theta\|_{Z^{2/3}}$. The two terms involving Γ_2 are bounded in a similar way, namely

$$\|\Gamma_2 \partial_x\partial_z\Theta\|_{L^2} \le \|\Gamma_2\|_\infty \|\partial_x\partial_z\Theta\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1}\|\Theta\|_{Z^{2/3}}$$

and

$$\|\partial_z\Gamma_2 \partial_x\Theta\|_{L^2} \le \|\partial_z\Gamma_2\|_{L^2_z(L^\infty_x)} \|\partial_x\Theta\|_{L^\infty_z L^2_x} \lesssim \|\bar{u} - y\|_{Q^1}\|\Theta\|_{Z^{2/3}}.$$

The estimate for $\sigma = 1$ follows. We then deduce the estimate for $\sigma \in (0,1)$ from a straightforward interpolation argument. □

We will also use the following cancellation on $\alpha - 1$.

Lemma D.2. *Let $\bar{u} \in Q^1$ such that $\|\bar{u} - y\|_{Q^1} \ll 1$ and $\bar{u}(x_i, 0) = \bar{u}_{yy}(x_i, 0) = 0$ and $\bar{u}_y(x_i, 0) = 1$. Define α by (3.5). Then, in a neighborhood of $(x_i, 0)$, using the notation r_i of (2.74),*

$$|\alpha(x,z) - 1| \lesssim r_i^{\frac{3}{2}} \|\bar{u} - y\|_{Q^1}. \tag{D.1}$$

Proof. First, $\partial_z\alpha(x_i, \cdot) = 2\bar{u}_{yy}(x_i, Y(x_i, z))$ with the change of variable Y defined by (3.2). Hence $\partial_z\alpha(x_i, \cdot) \in H^{3/2}(-1, 1)$ with $\partial_z\alpha(x_i, 0) = 0$. Therefore, by the Hardy inequality,

$$\int_{-1}^{1} \frac{(\partial_z\alpha(x_i, z))^2}{z^2}\, \mathrm{d}z \lesssim \|\partial_z\alpha(x_i, \cdot)\|_{H^1(-1,1)} \lesssim \|\bar{u} - y\|_{Q^1}. \tag{D.2}$$

It follows that

$$\begin{aligned} |\alpha(x,z) - 1| &= |\alpha(x,z) - \alpha(x_i, 0)| \\ &\le \int_{x_i}^{x} |\alpha_x(x', z)|\, \mathrm{d}x' + \int_0^z |\partial_z\alpha(x_i, z')|\, \mathrm{d}z' \\ &\le |x - x_i|^{\frac{1}{2}} \|\alpha_x\|_{L^\infty_z(L^2_x)} + |z|^{\frac{3}{2}} \|(\partial_z\alpha(x_i, z'))/z'\|_{L^2(-1,1)} \\ &\lesssim r_i^{\frac{3}{2}} \|\bar{u} - y\|_{Q^1}, \end{aligned} \tag{D.3}$$

using the embeddings of Lemma 1.10. □

Lemma D.2 has the following consequence, which will be used in the proof of Proposition 5.15.

Lemma D.3. *Let $\bar{u} \in Q^1$ such that $\|\bar{u}-y\|_{Q^1} \ll 1$ and $\bar{u}(x_i,0) = \bar{u}_{yy}(x_i,0) = 0$ and $\bar{u}_y(x_i,0) = 1$. Define α by (3.5). Then, for $f \in H^1_x L^2_z$,*

$$|\overline{\ell^j}((\alpha-1)f,0,0)| \lesssim \|\bar{u}-y\|_{Q^1}\|f\|_{L^2}. \tag{D.4}$$

Thus, the linear forms $f \mapsto \overline{\ell^j}((\alpha-1)f,0,0)$ can be continuously extended to $L^2(\Omega)$.

Proof. By Definition 2.9, using the boundary conditions on $\overline{\Phi^j}$ of (2.17),

$$\overline{\ell^j}((\alpha-1)f,0,0) = -\int_\Omega f(\alpha-1)\partial_x\overline{\Phi^j}. \tag{D.5}$$

By Corollary 2.28, there exist $(d_0,d_1) \in \mathbb{R}^2$ and $\Phi_{\mathrm{reg}} \in Q^1$ such that

$$(\alpha-1)\partial_x\overline{\Phi^j} = (\alpha-1)\partial_x\Phi_{\mathrm{reg}} + d_0(\alpha-1)\partial_x\bar{u}^0_{\mathrm{sing}}(x,-z) + d_1(\alpha-1)\partial_x\bar{u}^1_{\mathrm{sing}}(x,-z). \tag{D.6}$$

We then prove that all terms belong to $L^2(\Omega)$. First, $\|(\alpha-1)\partial_x\Phi_{\mathrm{reg}}\|_{L^2} \lesssim \|\alpha-1\|_\infty \lesssim \|\bar{u}-y\|_{Q^1}$. We turn to the second and third terms. Far from the critical point $(x_i,0)$, $(\alpha-1)\partial_x\bar{u}^i_{\mathrm{sing}}$ is easily bounded in L^∞ since $\partial_x\bar{u}_{\mathrm{sing}}$ is smooth away from $(x_i,0)$ by Lemma 2.23. Near $(x_i,0)$, using (2.80), there holds $|\partial_x\bar{u}^i_{\mathrm{sing}}| \lesssim (1+|t_i|)r_i^{-\frac{5}{2}}$. By Lemma D.2, $|\alpha-1| \lesssim r^{\frac{3}{2}}\|\bar{u}-y\|_{Q^1}$. Recalling the Jacobian (2.48), we infer that $\|(\alpha-1)\partial_x\bar{u}^i_{\mathrm{sing}}\|_{L^2} \lesssim \|\bar{u}-y\|_{Q^1}$. □

Proof of Proposition 5.15. Let us first recall that Θ^j can be decomposed as

$$\Theta^j = (\mathbf{1}_{j=1} - z\mathbf{1}_{j=0})\chi(z)\mathbf{1}_{z>0} + \Theta^j_\sharp,$$

where $\Theta^j_\sharp$ is a solution to (3.44). As a consequence, setting $\Theta_c := c_0\Theta^0_\sharp + c_1\Theta^1_\sharp$, we find that $\Theta_c \in \mathcal{B}$ is a weak solution to

$$\begin{cases} -\frac{z}{\alpha}\partial_x\Theta_c - \partial_z G[\Theta_c] - \partial_{zz}\Theta_c = f_c \quad \text{in } \Omega, \\ \Theta_c(\cdot,\pm1) = 0, \\ \Theta_c(x_1,z) = \eta^c_1 \quad \forall z>0, \\ \Theta_c(x_0,z) = \eta^c_0 \quad \forall z<0, \end{cases} \tag{D.7}$$

with

$$f_c = \mathbf{1}_{z>0}\partial_{zz}\left((c_1-c_0z)\chi(z)\right) + \partial_z\left(\mathbf{1}_{z>0}(c_1-c_0z)\chi(z)\Gamma(x_1,z)\right),$$
$$\eta^c_0 \equiv 0, \quad \eta^c_1(z) = -(c_1-c_0z)\chi(z).$$

The argument is formally a perturbation of the one of Corollary 2.26. More precisely, for any $d_0,d_1 \in \mathbb{R}$, we note that $\Theta_{\mathrm{reg}} := \Theta_c - d_0\bar{u}^0_{\mathrm{sing}}(x,-z) - d_1\bar{u}^1_{\mathrm{sing}}(x,-z)$ is a solution to

$$\begin{cases} -z\partial_x\Theta_{\mathrm{reg}} - \partial_{zz}\Theta_{\mathrm{reg}} = \alpha f_c + \alpha\partial_z G[\Theta_c] + (\alpha-1)\partial_{zz}\Theta_c - d_0\overline{f_0}(x,-z) - d_1\overline{f_1}(x,-z) =: g_{\mathrm{reg}}, \\ \Theta_{\mathrm{reg}}(\cdot,\pm1) = 0, \\ \Theta_{\mathrm{reg}}(x_1,z) = \eta^c_1 \quad \forall z>0, \\ \Theta_{\mathrm{reg}}(x_0,z) = \eta^c_0 \quad \forall z<0. \end{cases} \tag{D.8}$$

Let us denote by $\check{\ell}^j$ the linear forms associated with the dual shear flow problem, i.e. with the equation $-z\partial_x - \partial_{zz}$. The idea is to prove that $g_{\mathrm{reg}} \in H^{1/3}_x L^2_z$, and to look for d_0,d_1 so that $\check{\ell}^j(g_{\mathrm{reg}},\eta^c_1,\eta^c_0) = 0$. Using Lemma 5.11, we then deduce that $\Theta_{\mathrm{reg}} \in H^1_x L^2_z$.

The slight difficulty is that we cannot operate immediately on equation (D.8), because we need to bootstrap the regularity of $\Theta_{\rm reg}$ using Lemma D.1. Therefore we will work with a slightly modified version of (D.8), in which the vertical derivatives in the right-hand side have been replaced by finite differences, and we will prove uniform estimates on the solution to our modified equation.

Before proceeding further, we note that we can decompose $g_{\rm reg}$ as follows:

$$\begin{aligned} g_{\rm reg} &= \alpha f_c - d_0\overline{f_0}(x,-z) - d_1\overline{f_1}(x,-z) \\ &\quad + \sum_{i=0,1} d_i\alpha\partial_z G[\bar{u}^i_{\rm sing}(x,-z)] + d_i(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z) \\ &\quad + \alpha\partial_z G[\Theta_{\rm reg}] + (\alpha-1)\partial_{zz}\Theta_{\rm reg}. \end{aligned}$$

Note that the term on the first line in the right-hand side belongs to $H^1_x L^2_z$. Therefore we will focus on the two other terms.

Step 1. *Bound on $\alpha\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]$ and on $(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z)$.* We first obtain bounds in $H^{1/3}_x L^2_z(\Omega_\pm)$ on the two terms involving the singular profiles $\bar{u}^i_{\rm sing}$. We start with the bound on $(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z)$, which we write as $r_i^{-5/4}(\alpha-1)r_i^{5/4}\partial_{zz}\bar{u}^i_{\rm sing}$, with $r_i = (|x-x_i|^{2/3}+z^2)^{1/2}$. With the same arguments as in Lemma 2.24, we find that $r_i^{5/4}\partial_{zz}\bar{u}^i_{\rm sing} \in H^\sigma_x L^2_z \cap L^2_x H^{3\sigma}_z$ for all $\sigma < 7/12$. In particular, $r_i^{5/4}\partial_{zz}\bar{u}^i_{\rm sing} \in L^\infty_z(H^\sigma_x)$ for all $\sigma < 5/12$. We therefore focus on $r_i^{-5/4}(\alpha-1)$, and we prove that this term belongs to $L^2_z W^{1,1}_x \hookrightarrow L^2_z H^{1/2}_x$ by Sobolev embeddings. We have

$$\partial_x \frac{\alpha-1}{r_i^{5/4}} = \frac{\partial_x\alpha}{r_i^{5/4}} - \frac{5\times(-1)^i}{12}\frac{\alpha-1}{r_i^{13/4}}|x-x_i|^{-1/3}.$$

We evaluate each term of the right-hand side in $L^2_z L^1_x$. By the Minkowski inequality, it is sufficient to evaluate them in $L^1_x L^2_z$. For the first term,

$$\begin{aligned} \left\| \frac{\alpha_x}{r_i^{5/4}} \right\|_{L^1_x(L^2_z)} &= \int_{x_0}^{x_1} \left(\int_{-1}^1 \frac{|\alpha_x(x,z)|^2}{(|x-x_i|^{2/3}+z^2)^{5/4}}\,{\rm d}z \right)^{1/2} {\rm d}x \\ &\le \int_{x_0}^{x_1} \|\alpha_x(x,\cdot)\|_{L^2_z}|x-x_i|^{-\frac{5}{12}}\,{\rm d}x \lesssim \|\alpha_x\|_{L^2_x L^2_z} \lesssim \|\bar{u}-y\|_{Q^1}. \end{aligned}$$

As for the other term, using Lemma D.2, we deduce that

$$\begin{aligned} \left\| \frac{\alpha-1}{r_i^{13/4}}|x-x_i|^{-\frac{1}{3}} \right\|_{L^1_x(L^2_z)} &\lesssim \|\bar{u}-y\|_{Q^1} \int_{x_0}^{x_1} |x-x_i|^{-\frac{1}{3}} \left(\int_{-1}^1 \frac{1}{(|x-x_i|^{\frac{2}{3}}+z^2)^{7/4}}\,{\rm d}z \right)^{\frac{1}{2}} {\rm d}x \\ &\lesssim \|\bar{u}-y\|_{Q^1}. \end{aligned} \tag{D.9}$$

Using Lemma B.3, we conclude that

$$\|(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z)\|_{H^{1/3}_x L^2_z} \lesssim \left\| \frac{\alpha-1}{r_i^{5/4}} \right\|_{L^2_z H^{1/2}_x} \|r_i^{5/4}\partial_{zz}\bar{u}^i_{\rm sing}(x,-z)\|_{L^\infty_z H^\sigma_x}$$

for some $\sigma \in (1/3, 5/12)$.

We proceed in a similar way for the term $\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]$. Note that

$$\begin{aligned}\partial_x\partial_z G[\bar{u}^i_{\rm sing}(x,-z)] &= \partial_z((z\Gamma_1+\Gamma_2)\partial_x\bar{u}^i_{\rm sing}(x,-z))\\ &= \partial_z\Gamma_1 z\partial_x\bar{u}^i_{\rm sing}(x,-z)+\Gamma_1\partial_z(z\partial_x\bar{u}^i_{\rm sing}(x,-z))\\ &\quad +\partial_z\Gamma_2\partial_x\bar{u}^i_{\rm sing}(x,-z)+\Gamma_2\partial_x\partial_z\bar{u}^i_{\rm sing}(x,-z).\end{aligned}$$

We evaluate the right-hand side in $L^2_z(L^1_x)$. First, since $z\partial_x\bar{u}^i_{\rm sing}(x,-z)\in L^2(\Omega)$ by Lemma 2.23, we have $\|\partial_z\Gamma_1 z\partial_x\bar{u}^i_{\rm sing}(x,-z)\|_{L^2_zL^1_x}\lesssim\|\partial_z\Gamma_1\|_{L^\infty_z(L^2_x)}\|z\partial_x\bar{u}^i_{\rm sing}\|_{L^2}\lesssim\|\bar{u}-y\|_{Q^1}$. Using the same method as in Lemma 2.24, we see that, for any $\sigma>0$, $r_i^{\sigma+\frac12}\partial_z(z\partial_x\bar{u}^i_{\rm sing}(x,-z))\in L^2(\Omega)$. Hence, taking $0<\sigma<\sigma'<1$ and setting $q=(1+2\sigma')/(1+2\sigma)$,

$$\begin{aligned}\big\|\Gamma_1\partial_z(z\partial_x\bar{u}^i_{\rm sing}(x,-z))\big\|_{L^2_zL^1_x}&\lesssim\left\|\frac{\Gamma_1}{r_i^{\frac12+\sigma}}\right\|_{L^\infty_z(L^2_x)}\\ &\lesssim\|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}\sup_z\left(\int_{x_0}^{x_1}\frac{1}{(|x-x_i|^{2/3}+z^2)^{\frac14+\frac12\sigma'}}\,{\rm d}x\right)^{\frac1q}\\ &\lesssim\|\Gamma_1\|_{L^\infty_z(H^{1/2}_x)}\lesssim\|\bar{u}-y\|_{Q^1}.\end{aligned}$$

The two terms involving Γ_2 are treated in a similar fashion and left to the reader. Eventually, we infer that, within $\Omega_\pm$,

$$\|\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]\|_{H^{1/2}_xL^2_z}\lesssim\|\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]\|_{L^2_z(W^{1,1}_x)}\lesssim\|\bar{u}-y\|_{Q^1},$$

and thus, within $\Omega_\pm$,

$$\big\|(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z)\big\|_{H^{1/3}_xL^2_z}+\|\alpha\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]\|_{H^{1/3}_xL^2_z}\lesssim\|\bar{u}-y\|_{Q^1}. \tag{D.10}$$

Step 2. *Lift of the traces of $G[\bar{u}^i_{\rm sing}(x,-z)]$ and $G[\Theta_{\rm reg}]$.* It follows from the definition of G (see (3.40)) that $G[\bar{u}^i_{\rm sing}(x,-z)]$ and $G[\Theta_{\rm reg}]$ have jumps on the line $z=0$, and therefore $\partial_z G[\bar{u}^i_{\rm sing}(x,-z)]\notin L^2(\Omega)$. More precisely, set

$$\begin{aligned}b^i_\pm:=-\frac{1}{x_1-x_0}\int_{x_0}^{x_1}G[\bar{u}^i_{\rm sing}(x,-z)]_{|z=0^\pm}\,{\rm d}x,\quad b^{\rm reg}_\pm:=-\frac{1}{x_1-x_0}\int_{x_0}^{x_1}G[\Theta_{\rm reg}]_{|z=0^\pm}\,{\rm d}x\\ \Theta^i_l:=z\chi(z)(b^i_+\mathbf{1}_{z>0}+b^i_-\mathbf{1}_{z<0}),\quad \Theta^{\rm reg}_l:=z\chi(z)(b^{\rm reg}_+\mathbf{1}_{z>0}+b^{\rm reg}_-\mathbf{1}_{z<0}),\\ \Theta_\flat:=\Theta_{\rm reg}-d_0\Theta^0_l-d_1\Theta^1_l-\Theta^{\rm reg}_l.\end{aligned}$$

Then $\Theta_\flat$ solves

$$\begin{cases}-z\partial_x\Theta_\flat-\partial_{zz}\Theta_\flat=S_\flat,\\ \Theta_\flat(x,\pm1)=0,\\ \Theta_\flat(x_0,z)=\eta^c_0-z\chi(z)(d_0b^0_-+d_1b^1_-+b^{\rm reg}_-)\quad\forall z\in(-1,0),\\ \Theta_\flat(x_1,z)=\eta^c_1-z\chi(z)(d_0b^0_++d_1b^1_++b^{\rm reg}_+)\quad\forall z\in(0,1),\end{cases} \tag{D.11}$$

where

$$\begin{aligned} S_\flat &= \alpha f_c - d_0\overline{f_0}(x,-z) - d_1\overline{f_1}(x,-z) \\ &\quad + \sum_{i=0,1} d_i(\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z) + d_i\alpha\partial_z\left(G[\bar{u}^i_{\rm sing}(x,-z)] + \partial_z\Theta^i_l\right) \\ &\quad - \sum_{i=0,1} d_i\left(\alpha\partial_z G[\Theta^i_l] + (\alpha-1)\partial_{zz}\Theta^i_l\right) - \alpha\partial_z G[\Theta^{\rm reg}_l] - (\alpha-1)\partial_{zz}\Theta^{\rm reg}_l \\ &\quad + \alpha\partial_z(G[\Theta_\flat] + \partial_z\Theta^{\rm reg}_l) + (\alpha-1)\partial_{zz}\Theta_\flat \\ &=: g_\flat + \alpha\partial_z(G[\Theta_\flat] + \partial_z\Theta^{\rm reg}_l) + (\alpha-1)\partial_{zz}\Theta_\flat. \end{aligned}$$

Note that since Θ^i_l depends only on z,

$$G[\Theta^i_l] = -\Theta^i_l\left(\mathbf{1}_{z>0}\Gamma(x_1,z) + \mathbf{1}_{z<0}\Gamma(x_0,z)\right).$$

The assumptions on Γ ensure that $\partial_z G[\Theta^i_l]$ belongs to L^2 (and therefore to $H^{1/3}_x L^2_z$, since this term does not depend on x). Furthermore, $G[\Theta_{\rm reg}]_{|z=0^\pm} = G[\Theta_\flat]_{|z=0^\pm}$.

At this stage, we have proved that for any choice of the coefficients d_0, d_1, $g_\flat \in H^{1/3}_x L^2_z$.

Step 3. *A priori estimates on $\Theta_{\rm reg}$ in $Z^{1/3}$.* We now derive some *a priori* estimates on $\Theta_{\rm reg}$, which will be justified in the final step. Assume that the coefficients d_0, d_1 are such that

$$\check{\ell^j}(S_\flat, \eta^c_0 - z\chi(z)(d_0 b^0_- + d_1 b^1_-), \eta^c_1 - z\chi(z)(d_0 b^0_+ + d_1 b^1_+)) = 0 \quad \text{for } j = 0,1,$$

and assume that $\Theta_\flat \in Z^{1/3}$. Then according to Lemma 5.11,

$$\|\Theta_\flat\|_{Z^{1/3}} \lesssim \|S_\flat\|_{H^{1/3}_x L^2_z} + |c_0| + |c_1| + |d_0| + |d_1|.$$

Now, according to (D.10) and Lemma D.1,

$$\|S_\flat\|_{H^{1/3}_x L^2_z} \lesssim |c_0| + |c_1| + |d_0| + |d_1| + \|\bar{u} - y\|_{Q^1}\|\Theta_\flat\|_{Z^{1/3}} + \|(\alpha-1)\partial_{zz}\Theta_\flat\|_{H^{1/3}_x L^2_z}.$$

It is easily proved using Lemma B.3 and the same arguments as in Lemma 4.4

$$\|(\alpha-1)\partial_{zz}\Theta_\flat\|_{H^{1/3}_x L^2_z} \lesssim \|\alpha-1\|_{L^\infty_z(H^{3/4}_x)}\|\Theta_\flat\|_{Z^{1/3}} \lesssim \|\bar{u}-y\|_{Q^1}\|\Theta_\flat\|_{Z^{1/3}}.$$

Hence for $\|\bar{u} - y\|_{Q^1} \ll 1$, the two terms involving $\|\Theta_\flat\|_{Z^{1/3}}$ in the right-hand side can be treated perturbatively, and we obtain

$$\|\Theta_\flat\|_{Z^{1/3}} \lesssim |c_0| + |c_1| + |d_0| + |d_1|.$$

There remains to explain how d_0 and d_1 can be chosen, and to prove that $|d_0|+|d_1| \lesssim |c_0|+|c_1|$. This will be done thanks to a finite difference approximation scheme, which will allow us to construct a solution $\Theta_\flat$ in $Z^{1/3}$ satisfying the above *a priori* estimate.

Step 4. *Construction of an approximating sequence.* In this step, we slightly modify the equation. For any $h \in (0,1)$ and $f \in L^2(-2,2)$, we set $\tau_h f := h^{-1}(f(\cdot + h) - f)$, for $z \in (-1,1)$. Now, for $\Theta \in \mathcal{B}$, we extend Θ to $(x_0,x_1)\times(-2,2)$ by setting $\Theta(x,z) = -\Theta(x,2-z)$ for $z \in (1,2)$, $\Theta(x,z) = -\Theta(x,-2-z)$ for $z \in (-2,-1)$. Note that with this extension, if $\Theta \in Z^0(\Omega)$, its extension belongs to $Z^0((x_0,x_1)\times(-2,2))$.

For all $c_0, c_1, d_0, d_1 \in \mathbb{R}$, and for $h \in (0,1)$, we consider the system

$$\begin{cases} -z\partial_x \Theta_h - \partial_{zz}\Theta_h = g_\flat + \alpha\tau_h \left(G[\Theta_h] + \partial_z \Theta_l^{\mathrm{reg}}\right) + (\alpha-1)\tau_h \partial_z \Theta_h, \\ \Theta_h(x, \pm 1) = 0, \\ \Theta_h(x_0, z) = \eta_0^c - z\chi(z)(d_0 b_-^0 + d_1 b_-^1 + b_-^{\mathrm{reg}}) \quad \forall z \in (-1, 0), \\ \Theta_h(x_1, z) = \eta_1^c - z\chi(z)(d_0 b_+^0 + d_1 b_+^1 + b_+^{\mathrm{reg}}) \quad \forall z \in (0, 1). \end{cases} \tag{D.12}$$

In the above formula, the coefficients $b_\pm^{\mathrm{reg}}$ within the definition of Θ_l^{reg} have been replaced by $-(x_1 - x_0)^{-1} \int_{x_0}^{x_1} G[\Theta_h]_{|z=0\pm}$. Note that $|b_\pm^{\mathrm{reg}}| \lesssim \|\partial_z G[\Theta_h]\|_{L^2(\Omega_\pm)} \lesssim \|\Theta_h\|_{Z^0}$.

Using the same method and estimates as in the proof of Proposition 3.10 and Lemma 3.12, it can be proved that the above system has a unique solution $\Theta_h \in Z^0$, and that

$$\|\Theta_h\|_{Z^0} \lesssim |c_0| + |c_1| + |d_0| + |d_1|.$$

As a consequence

$$\|g_\flat\|_{H_x^{1/3} L_z^2} \lesssim |c_0| + |c_1| + |d_0| + |d_1|.$$

Furthermore, for any $h > 0 >$, $\alpha\tau_h G[\Theta_h], (\alpha-1)\tau_h \partial_z \Theta_h \in H_x^{1/3} L_z^2$. Indeed, since $\Theta_h \in Z^0 \hookrightarrow Q^0$, $\partial_z \Theta_h \in H_x^{1/3} L_z^2 \cap L_x^2 H_z^1$, and therefore $(\alpha-1)\tau_h \partial_z \Theta_h \in H_x^{1/3} L_z^2$. Concerning the other term, we write, for $z > 0$,

$$G[\Theta_h](x,z) = (\Gamma_1 z \Theta_h)(x_1, z) - \int_x^{x_1} \Gamma_1 z \partial_x \Theta_h + \Gamma_2 \Theta_h + \int_x^{x_1} \partial_x \Gamma_2 \Theta_h.$$

The second term is bounded in $L_z^2(W_x^{1,1})$ by $\|\Gamma_1\|_{L_z^\infty(L_x^2)} \|z\partial_x \Theta_h\|_{L^2} \lesssim \|\bar{u} - y\|_{Q^1} \|\Theta_h\|_{Z^0}$. Hence we obtain a bound in $H_x^{1/2} L_z^2$. The three other terms are easily bounded. We obtain

$$\|\alpha\tau_h G[\Theta_h]\|_{H_x^{1/3} L_z^2(\Omega_\pm)} \lesssim h^{-1} \|\bar{u} - y\|_{Q^1} \|\Theta_h\|_{Z^0}.$$

It then follows immediately from Lemma 5.11 that for all $\sigma < 1/6$, there exists a constant C_σ such that

$$\|V_h\|_{Z^\sigma} \leq C_\sigma \left\| g_\flat + \alpha\tau_h \left(G[\Theta_h] + \tau_h \Theta_l^{\mathrm{reg}}\right) + (\alpha-1)\tau_h \partial_z \Theta_h \right\|_{H_x^\sigma L_z^2}.$$

Using Lemma D.1 together with the bounds on $b_\pm^{\mathrm{reg}}$,

$$\left\| \alpha\tau_h \left(G[\Theta_h] + \partial_z \Theta_l^{\mathrm{reg}}\right) \right\|_{H_x^\sigma L_z^2} \lesssim \left\| \alpha \partial_z \left(G[\Theta_h] + \partial_z \Theta_l^{\mathrm{reg}}\right) \right\|_{H_x^\sigma L_z^2} \lesssim \|\bar{u} - y\|_{Q^1} \|\Theta_h\|_{Z^{\frac{2\sigma}{3}}}.$$

Furthermore, following the same argument as in the third step,

$$\|(\alpha-1)\partial_z \tau_h \Theta_h\|_{H_x^\sigma L_z^2} \lesssim \|\bar{u} - y\|_{Q^1} \|\Theta_h\|_{Z^\sigma}.$$

Since $\|\bar{u} - y\|_{Q^1} \ll 1$, we obtain that for all $\sigma < 1/6$, there exists a constant C_σ such that for all c_0, c_1, d_0, d_1, for all $h > 0$,

$$\|V_h\|_{Z^\sigma} \leq C_\sigma \left(|c_0| + |c_1| + |d_0| + |d_1|\right).$$

Let us now prove that for any $h > 0$, we can choose $d_0, d_1 = O(|c_0| + |c_1|)$ such that for $j = 0, 1$

$$\check{\ell}^j(g_\flat + \alpha\tau_h \left(G[\Theta_h] + \partial_z \Theta_l^{\mathrm{reg}}\right) + (\alpha-1)\tau_h \partial_z \Theta_h, \\ \eta_0^c - z\chi(z)(d_0 b_-^0 + d_1 b_-^1 + b_-^{\mathrm{reg}}), \eta_1^c - z\chi(z)(d_0 b_+^0 + d_1 b_+^1 + b_+^{\mathrm{reg}})) = 0. \tag{D.13}$$

By linearity, it can be easily checked that the left-hand side is equal to

$$\check{\ell^j}(\alpha f_c, \eta_0^c, \eta_1^c) - d_0\check{\ell^j}(\overline{f_0}(x,-z),0,0) - d_1\check{\ell^j}(\overline{f_1}(x,-z),0,0) + \sum_{i=0,1} N_{ij}d_i + \sum_{i=0,1} P_{ij}c_i,$$

where N_{ij} and P_{ij} are real numbers. We recall that the numbers c_0, c_1 are given and fixed, and that we look for d_0, d_1. According to Lemma 2.25, the matrix $(\check{\ell^j}(\overline{f_i}(x,-z),0,0))_{0\le i,j\le 1}$ is invertible. Hence it suffices to prove that the numbers N_{ij} are small, uniformly in h, and that P_{ij} are uniformly bounded in h. Thanks to the first step and to Remark 2.29,

$$\left|\check{\ell^j}\left((\alpha-1)\partial_{zz}\bar{u}^i_{\rm sing}(x,-z) + \alpha\partial_z(G[\bar{u}^i_{\rm sing}(x,-z)] + \partial_z\Theta^i_l), 0, 0\right)\right| \lesssim \|\bar{u}-y\|_{Q^1},$$
$$\left|\check{\ell^j}\left(\alpha\partial_z G[\Theta^i_l] + (\alpha-1)\partial_{zz}\Theta^i_l, 0, 0\right)\right| \lesssim \|\bar{u}-y\|_{Q^1}.$$

Furthermore, using Lemma D.1, we find that $\partial_z G[\Theta_h]$ is bounded in $H^\sigma_x L^2_z(\Omega_\pm)$, uniformly in h, for all $\sigma < 1/4$. Taking for instance $\sigma = \frac{5}{24} \in (\frac{1}{6}, \frac{1}{4})$, we obtain

$$\left|\check{\ell^j}\left(\alpha\tau_h\left(G[\Theta_h] + \partial_z\Theta^{\rm reg}_l\right), 0, 0\right)\right| \lesssim \|\bar{u}-y\|_{Q^1}\|\Theta_h\|_{Z^{5/36}} \lesssim \|\bar{u}-y\|_{Q^1}\left(|c_0|+|c_1|+|d_0|+|d_1|\right).$$

Eventually, using Lemma D.3,

$$\left|\check{\ell^j}\left((\alpha-1)\partial_z\tau_h\Theta_h, 0, 0\right)\right| \lesssim \|\bar{u}-y\|_{Q^1}\|\Theta_h\|_{Z^0} \lesssim \|\bar{u}-y\|_{Q^1}\left(|c_0|+|c_1|+|d_0|+|d_1|\right).$$

Hence it follows that $|N_{ij}|, |P_{ij}| \lesssim \|\bar{u}-y\|_{Q^1}$, and the estimate is uniform in h. As a consequence, for all $(c_0, c_1) \in \mathbb{R}^2$, for all $h > 0$, there exists d_0^h, d_1^h (which depend on h) such that the orthogonality conditions (D.13) are satisfied. Furthermore,

$$|d_0^h| + |d_1^h| \lesssim |c_0| + |c_1|,$$

uniformly in h. Using Lemma 5.11, we deduce that

$$\|V_h\|_{Z^{1/3}} \lesssim |c_0| + |c_1|,$$

and the estimate is uniform in h.

Step 5. *Conclusion.* By compactness, we extract a subsequence (still denoted by V_h) that converges weakly in $Z^{1/3}$, and such that the sequence (d_0^h, d_1^h) also converges in $\mathbb{R}^2$. Passing to the limit in the equation, we obtain the desired decomposition. Furthermore, if $d_0 = d_1 = 0$, then

$$\check{\ell^j}(\mathbf{1}_{z>0}\partial_{zz}((c_1 - c_0 z)\chi(z)), 0, -(c_1 - c_0 z)\chi(z)) = O(\|\bar{u}-y\|_{Q^1}(|c_0|+|c_1|)).$$

Using the same argument as in Corollary 2.28, we find that

$$c_0 = O(\|\bar{u}-y\|_{Q^1}(|c_0|+|c_1|)), \quad c_1 = O(\|\bar{u}-y\|_{Q^1}(|c_0|+|c_1|)),$$

and therefore, since $\|\bar{u}-y\|_{Q^1} \ll 1$, $c_0 = c_1 = 0$. □

List of notations

Functional spaces

$\mathcal{B}$ Baouendi-Grisvard solution space of (1.28), used in Appendix A, *p. 15*

$\mathscr{H}^1_z$ Weighted H^1 space for boundary data with norm (1.20), *p. 13*

$\mathcal{H}$ Hilbert space with norm (1.3) of triples (f, δ_0, δ_1) considered as data for the PDE, *p. 4*

$\mathcal{H}^\perp_{\mathrm{sg}}$ Subspace of codimension 2 in $\mathcal{H}$ satisfying the shear flow orthogonality conditions, *p. 23*

$H^{1/2}_{00}$ Interpolation space of $H^{1/2}$ functions with "vanishing trace" at both endpoints, *p. 13*

$H^{1/2}_{00_l}$ Interpolation space of $H^{1/2}$ functions with "vanishing trace" at the left endpoint, *p. 13*

$H^{1/2}_{00_r}$ Interpolation space of $H^{1/2}$ functions with "vanishing trace" at the right endpoint, *p. 13*

$\mathscr{L}^2_z$ Weighted L^2 space for boundary data with norm (1.19), *p. 13*

Q^0 Solution space $L^2_x(H^2_y) \cap H^{2/3}_x(L^2_y)$, *p. 15*

Q^1 Solution space $L^2_x(H^5_y) \cap H^{5/3}_x(L^2_y)$, *p. 15*

$Q^{1/2}$ Intermediate solution space $L^2_x(H^{7/2}_y) \cap H^{7/6}_x(L^2_y)$, *p. 15*

$\mathcal{Y}_0$ Notation for $L^2(\Omega)$ during discussions on interpolation, *p. 64*

$\mathcal{Y}_1$ Notation for $H^1_x L^2_y \cap L^2_x H^3_y$ with $f_{|\Sigma_0 \cup \Sigma_1} = 0$ during discussions on interpolation, *p. 64*

$\mathcal{Y}_1^{\bar{u}}$ Subspace of $f \in \mathcal{Y}_1$ satisfying the orthogonality conditions at $\bar{u}$, *p. 64*

Z^0 Pagani solution space such that u, $z\partial_x u$ and $\partial_{zz} u$ are L^2, with norm (1.23), *p. 14*

Z^1 Solution space such that $u, \partial_x u \in Z^0$, with norm (1.24), *p. 14*

Other

α Diffusion coefficient of the PDE in the Y variable, $\alpha = (\bar{u}_y)^2 \circ Y \approx 1$, *p. 36*

γ Coefficient of the ∂_z term for the PDE in the Y variable, $\gamma = (z\bar{u}_x - \bar{u}_{yy}) \circ Y \ll 1$, *p. 36*

γ_1, γ_2 Decomposition of $\gamma(x,z)$ as $z\gamma_1(x,z) + \gamma_2(x,z)$, $\gamma_1 = \bar{u}_x \circ Y$, $\gamma_2 = -\bar{u}_{yy} \circ Y$, *p. 36*

Γ Quotient $\Gamma = \gamma/\alpha$ involved in the definition of the operator G, *p. 42*

Γ_1, Γ_2 Quotients $\Gamma_k = \gamma_k/\alpha$ involved in the equations of the dual profiles, *p. 42*

δ_i Boundary data at the inflow boundary Σ_i, *p. 4*

$\widetilde{\delta}_i$ Boundary data for the PDE in the Y variable, $\widetilde{\delta}_i = \delta_i \circ Y$, *p. 36*

Δ_i Boundary data for $\partial_x u$, given by $\Delta_i = (f + \delta_i'')/z$, see (2.9), *p. 18*

Θ^j Related with the dual profile Φ^j by $\partial_x \Theta^j = \alpha \partial_x \Phi^j$, *p. 42*

$\Theta^j_\sharp$ Lifted version of the conjugate dual profile Θ^j, *p. 43*

$\Theta^j_\flat$ Lifted version of the conjugate dual profile Θ^j, *p. 100*

Θ^j_l Lifted version of the conjugate dual profile Θ^j, *p. 100*

Λ_k Angular profile of the k-th explicit singular solution in the half-plane, *p. 27*

Ξ Shorthand for a data triplet $\Xi = (f, \delta_0, \delta_1) \in \mathcal{H}$, *p. 80*

Σ_0 Left inflow boundary $\{x_0\} \times (0,1)$, see Fig. 1, *p. 3*

Σ_1 Right inflow boundary $\{x_1\} \times (-1,0)$, see Fig. 1, *p. 3*

$\overline{\Phi^j}$ Dual profiles of Lemma 2.6 involved in orthogonality conditions for the shear flow, *p. 19*

Φ^j Dual profiles for the linearized problem, *p. 41*

χ_i Cut-off function localized near $(x_i, 0)$, *p. 30*

Ω Physical rectangular domain $(x_0, x_1) \times (-1, 1)$, see Fig. 1, *p. 3*

$\Omega_\pm$ Upper and lower halves of the domain Ω, *p. 13*

$\overline{f_i}$ Smooth source term associated with the singular solution $\bar{u}^i_{\text{sing}}$, *p. 31*

G Operator mapping the unknown Θ to $(\partial_x)^{-1}\left(\frac{\gamma}{\alpha}\partial_x\Theta\right)$, *p. 42*

g Source term for the PDE in the Y variable, $g = f \circ Y$, *p. 36*

$\overline{\ell^j}$ Linear forms on $\mathcal{H}$ giving the orthogonality conditions for the shear flow, *p. 21*

$\ell^j_{\bar{u}}$ Linear forms on $\mathcal{H}$ of the orthogonality conditions for the linearized problem at $\bar{u}$, *p. 48*

$\overline{M}$ Invertible matrix relating the singular solutions $\bar{u}^i_{\text{sing}}$ with the dual profiles $\overline{\Phi^j}$, *p. 33*

r Radial-like variable given by $r = (z^2 + x^{\frac{2}{3}})^{\frac{1}{2}}$, *p. 26*

r_i Radial-like variable near $(x_i, 0)$ given by $r_i = (z^2 + |x - x_i|^{\frac{2}{3}})^{\frac{1}{2}}$, *p. 30*

t Angular-like variable given by $t = zx^{-\frac{1}{3}}$, *p. 26*

t_i Angular-like variable near $(x_i, 0)$, given by $t_i = (-1)^i z|x - x_i|^{-\frac{1}{3}}$, *p. 30*

$\bar{u}^i_{\text{sing}}$ Reference singular solution localized near $(x_i, 0)$, *p. 30*

U Solution to the PDE in the Y coordinate, $U = u \circ Y$, *p. 36*

v_k k-th explicit singular solution in the half plane, $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$, *p. 27*

Y Change of vertical coordinate, depending on x, such that $\bar{u}(x, Y(x,z)) = z$, *p. 36*

Acknowledgements

The authors thank Jérémie Szeftel and Frédéric Hérau for nice discussions about this problem.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program Grant agreement No 637653, project BLOC "Mathematical Study of Boundary Layers in Oceanic Motion". This work was supported by the SingFlows project, grants ANR-18-CE40-0027 of the French National Research Agency (ANR). A.-L. D. acknowledges the support of the Institut Universitaire de France. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while A.-L. D. participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2021 semester.

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