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# Monitoring edge-geodetic sets in graphs

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## Abstract

We introduce a new graph-theoretic concept in the area of network monitoring. In this area, one wishes to monitor the vertices and/or the edges of a network (viewed as a graph) in order to detect and prevent failures. Inspired by two notions studied in the literature (edge-geodetic sets and distance-edge-monitoring sets), we define the notion of a monitoring edge-geodetic set (MEG set for short) of a graph  $G$  as an edge-geodetic set  $S \subseteq V(G)$  of  $G$  (that is, every edge of  $G$  lies on some shortest path between two vertices of  $S$ ) with the additional property that for every edge  $e$  of  $G$ , there is a vertex pair  $x, y$  of  $S$  such that  $e$  lies on *all* shortest paths between  $x$  and  $y$ . The motivation is that, if some edge  $e$  is removed from the network (for example if it ceases to function), the monitoring probes  $x$  and  $y$  will detect the failure since the distance between them will increase.

We explore the notion of MEG sets by deriving the minimum size of a MEG set for some basic graph classes (trees, cycles, unicyclic graphs, complete graphs, grids, hypercubes,...) and we prove an upper bound using the feedback edge set of the graph.

## 1 Introduction

We introduce a new graph-theoretic concept, that is motivated by the problem of network monitoring, called *monitoring edge-geodetic sets*. In the area of network monitoring, one wishes to detect or repair faults in a network; in many applications, the monitoring process involves distance probes [1, 2, 3, 9]. Our networks are modeled by finite, undirected simple connected graphs, whose vertices represent systems and whose edges represent the connections between them. We wish to monitor a network such that when a connection (an edge) fails, we can detect the said failure by means of certain probes. To do this, we select a small subset of vertices (representing the probes) of the network such that all connections are covered by the shortest paths between pairs of vertices in the network. Moreover, any two probes are able to detect the current distance that separates them. The goal is that, when an edge of the network fails, some pair of probes detects a change in their distance value, and therefore the failure can be detected. Our inspiration comes from two areas: the concept of *geodetic sets* in graphs and its variants [10], and the concept of *distance edge-monitoring sets* [8, 9].

We now proceed with some necessary definitions. A *geodesic* is a shortest path between two vertices  $u, v$  of a graph  $G$  [15]. The length of a geodesic between two vertices  $u, v$  in  $G$  is the *distance*  $d_G(u, v)$  between them. For an edge  $e$  of  $G$ , we denote by  $G - e$  the graph obtained by deleting  $e$  from  $G$ . An edge  $e$  in a graph  $G$  is a *bridge* if  $G - e$  has more connected components than  $G$ . The *open neighborhood* of a vertex  $v \in V(G)$  is  $N_G(v) = \{u \in V \mid uv \in E(G)\}$  and its *closed neighborhood* is the set  $N_G[v] = N_G(v) \cup \{v\}$ .

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**Monitoring edge-geodetic sets.** We now formally define of our main concept.

**Definition 1.1.** Two vertices  $x, y$  monitor an edge  $e$  in graph  $G$  if  $e$  belongs to all shortest paths between  $x$  and  $y$ . A set  $S$  of vertices of  $G$  is called a monitoring edge-geodetic set of  $G$  (MEG-set for short) if, for every edge  $e$  of  $G$ , there is a pair  $x, y$  of vertices of  $S$  that monitors  $e$ .

We denote by  $\text{meg}(G)$  the size of a smallest MEG-set of  $G$ . We note that  $V(G)$  is always an MEG-set of  $G$ , thus  $\text{meg}(G)$  is always well-defined.

**Related notions.** A set  $S$  of vertices of a graph  $G$  is a *geodetic set* if every vertex of  $G$  lies on some shortest path between two vertices of  $S$  [10]. An *edge-geodetic set* of  $G$  is a set  $S \subseteq V(G)$  such that every edge of  $G$  is contained in a geodesic joining some pair of vertices in  $S$  [14]. A *strong edge-geodetic set* of  $G$  is a set  $S$  of vertices of  $G$  such that for each pair  $u, v$  of vertices of  $S$ , one can select a shortest  $u - v$  path, in a way that the union of all these  $\binom{|S|}{2}$  paths contains  $E(G)$  [13]. It follows from these definitions that any strong edge-geodetic set is an edge-geodetic set, and any edge-geodetic set is a geodetic set (if the graph has no isolated vertices). In fact, every MEG-set is a strong edge-geodetic set. Indeed, given an MEG-set  $S$ , one can choose any shortest path between each pair of vertices of  $S$ , and the set of these paths covers  $E(G)$ . Indeed, every edge of  $G$  is contained in *all* shortest paths between some pair of  $S$ . Hence, MEG-sets can be seen as an especially strong form of strong edge-geodetic sets.

A set  $S$  of vertices of a graph  $G$  is a *distance-edge monitoring set* if, for every edge  $e$ , there is a vertex  $x$  of  $S$  and a vertex  $y$  of  $G$  such that  $e$  lies on all shortest paths between  $x$  and  $y$  [8, 9]. Thus, it follows immediately that any MEG-set of a graph  $G$  is also a distance-edge monitoring set of  $G$ .

**Our results.** We start by presenting some basic lemmas about the concept of MEG-sets in Section 2, that are helpful for understanding this concept. We then study in Section 3 the optimal value of  $\text{meg}(G)$  when  $G$  is a tree, cycle, unicyclic graph, complete (multipartite) graph, hypercubes and grids. In Section 4, we show that  $\text{meg}(G)$  is bounded above by a linear function of the *feedback edge set number* of  $G$  (the smallest number of edges of  $G$  needed to cover all cycles of  $G$ , also called *cyclomatic number*) and the number of leaves of  $G$ . This implies that  $\text{meg}(G)$  is bounded above by a function of the *max leaf number* of  $G$  (the maximum number of leaves in a spanning tree of  $G$ ). These two parameters are popular in structural graph theory and in the design of algorithms. We refer to Figure 1 for the relations between parameter  $\text{meg}$  and other graph parameters. Finally, we conclude in Section 5.

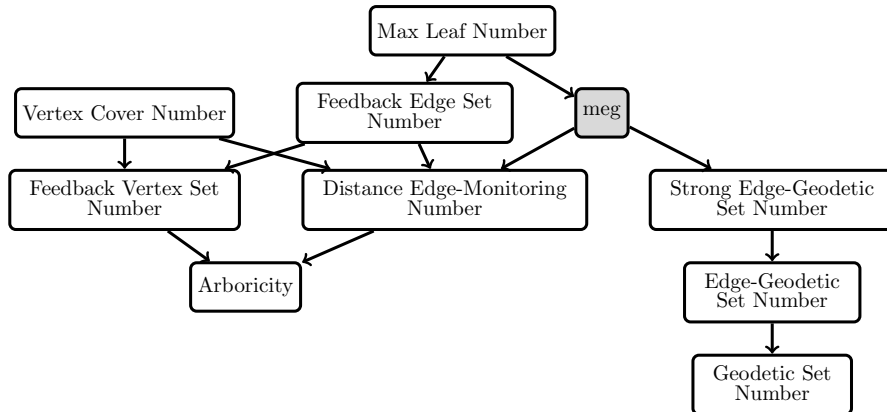


Figure 1: Relations between the parameter  $\text{meg}$  and other structural parameters in graphs (with no isolated vertices). For the relationships of distance edge-monitoring sets, see [8, 9]. Arcs between parameters indicate that the value of the bottom parameter is upper-bounded by a function of the top parameter.

## 2 Preliminary lemmas

We now give some useful lemmas about the basic properties of MEG-sets.

A vertex is *simplicial* if its neighborhood forms a clique.

**Lemma 2.1.** *In a graph  $G$  with at least one edge, any simplicial vertex belongs to any edge-geodetic set and thus, to any MEG-set of  $G$ .*

*Proof.* Let us consider by contradiction an MEG-set of  $G$  that does not contain said simplicial vertex  $v$ . Any shortest path passing through its neighbors will not pass through  $v$ , because all the neighbors are adjacent, hence leaving the edges incident to  $v$  uncovered, a contradiction.  $\square$

Two distinct vertices  $u$  and  $v$  of a graph  $G$  are *open twins* if  $N(u) = N(v)$  and *closed twins* if  $N[u] = N[v]$ . Further,  $u$  and  $v$  are *twins* in  $G$  if they are open twins or closed twins in  $G$ .

**Lemma 2.2.** *If two vertices are twins of degree at least 1 in a graph  $G$ , then they must belong to any MEG-set of  $G$ .*

*Proof.* For any pair  $u, v$  of open twins in  $G$ , for any shortest path passing through  $u$ , there is another one passing through  $v$ . Thus, if  $u, v$  were not part of the MEG-set, then the edges incident to  $u$  and  $v$  would remain unmonitored, a contradiction.

If  $u, v$  are closed twins, if some shortest path contains the edge  $uv$ , then it must be of length 1 and consist of the edge  $uv$  itself (otherwise there would be a shortcut). Thus, to monitor  $uv$ , both  $u, v$  must belong to any MEG-set.  $\square$

The next two lemmas concern cut-vertices and subgraphs, and will be useful in some of our proofs.

**Lemma 2.3.** *Let  $G$  be a graph with a cut-vertex  $v$  and  $C_1, C_2, \dots, C_k$  be the  $k$  components obtained when removing  $v$  from  $G$ . If  $S_1, S_2, \dots, S_k$  are MEG-sets of the induced subgraphs  $G[C_1 \cup \{v\}], G[C_2 \cup \{v\}], \dots, G[C_k \cup \{v\}]$ , then  $S = (S_1 \cup S_2, \dots, \cup S_k) \setminus \{v\}$  is an MEG-set of  $G$ .*

*Proof.* Consider any edge  $e$  of  $G$ , say in  $C_1$ . Then, there are two vertices  $x, y$  of  $S_1$  such that  $e$  belongs to all shortest paths between  $x$  and  $y$  in  $G_1 = G[C_1 \cup \{v\}]$ . Assume first that  $v \notin \{x, y\}$ . All shortest paths between  $x$  and  $y$  in  $G$  also exist in  $G_1$ . Thus,  $e$  is monitored by  $\{x, y\} \subseteq S$  in  $G$ . Assume next that  $v \in \{x, y\}$ : without loss of generality,  $v = x$ . At least one edge exists in  $G[C_2 \cup \{v\}]$ , which implies that  $S_2 \setminus \{v\}$  is nonempty, say, it contains  $z$ . Then,  $e$  is monitored by  $y$  and  $z$ , since  $z \in S$ . Thus,  $S$  monitors all edges of  $G$ , as claimed.  $\square$

**Lemma 2.4.** *Let  $G$  be a graph and  $H$  an induced subgraph of  $G$  such that for all vertex pairs  $\{x, y\}$  in  $H$ , no shortest path between them uses edges in  $G - H$ . Then, for any set  $S$  of vertices of  $G$  containing an MEG-set of  $H$ , the edges of  $H$  are monitored by  $S$  in  $G$ .*

*Proof.* Let  $S$  be an MEG-set of  $G$  containing an MEG-set  $S'$  of  $H$ . Let  $e$  be an edge in  $H$  that lies on all shortest paths between some pair  $\{x, y\}$  of vertices of  $S'$ . By our hypothesis, no shortest path between  $x$  and  $y$  in  $G$  uses any edges of  $G - H$ . Thus, the shortest paths between  $x$  and  $y$  in  $H$  are the same as in  $G$ , and therefore in  $G$ ,  $e$  is also monitored by  $\{x, y\} \subseteq S$ .  $\square$

## 3 Basic graph classes and bounds

In this section, we study MEG-sets for some standard graph classes.

### 3.1 Trees

**Theorem 3.1.** *For any tree  $T$  with at least one edge, the only optimal MEG-set of  $T$  consists of the set of leaves of  $T$ .*

*Proof.* The fact that all leaves must be part of any MEG-set follows from Lemma 2.1, as they are simplicial. For the other side, let  $L$  be the set of leaves of  $T$ . Let  $e = xy$  be an edge of  $T$  and consider two leaves of  $T$ ,  $l_x$  and  $l_y$ , such that  $l_x$  is closer to  $x$  than to  $y$  and that  $l_y$  is closer to  $y$  than to  $x$ . We note that  $e$  belongs to the unique (shortest) path between  $l_x$  and  $l_y$ , thus  $e$  is monitored by  $L$ . Hence,  $L$  is an MEG-set of  $T$ .  $\square$

**Corollary 3.2.** *For any path graph  $P_n$ , where  $n \geq 2$ , we have  $\text{meg}(P_n) = 2$*

This provides a lower bound which is tight for path graphs, which have order  $n$  and exactly 2 leaves.

**Corollary 3.3.** *For any tree  $T$  of order  $n \geq 3$ , we have  $2 \leq \text{meg}(T) \leq n - 1$ .*

The upper bound is tight for star graphs, which have order  $n$  and  $n - 1$  leaves.

### 3.2 Cycle graphs

We introduce the following terminology from [7]. In a graph, a vertex is a *core vertex* if it has degree at least 3. A *leg* is a (non-empty) path in a tree graph between a leaf  $v$  and a *core vertex*  $u$  of the graph that is closest to this leaf (note that the leg does not contain the core vertex  $u$  but the leaf  $v$ ).

**Theorem 3.4.** *Given an  $n$ -cycle graph  $C_n$ , for  $n = 3$  and  $n \geq 5$ ,  $\text{meg}(C_n) = 3$ . Moreover,  $\text{meg}(C_4) = 4$ .*

*Proof.* Let us first prove that we need at least three vertices to monitor any cycle. By contradiction, let us assume that two vertices suffice. For any arbitrary vertex pair in the cycle graph, there are two paths joining them, but there is either one single shortest path or two equidistant shortest paths between them. Thus, the edges on at least one of the two paths between the pair will not be monitored by it. Hence, we need at least three vertices in any MEG-set of  $C_n$  ( $n \geq 3$ ).

We now prove the upper bound. Let  $n \geq 5$  or  $n = 3$ , with the vertices of  $C_n$  from  $v_0$  to  $v_{n-1}$ . Consider the set  $S = \{v_0, v_{\lfloor \frac{n}{3} \rfloor}, v_{\lfloor \frac{2n}{3} \rfloor}\}$ . We show that  $S$  is an MEG-set of  $C_n$ .

Consider every edge of  $C_n$  between a vertex pair  $v_x$  and  $v_y$  in  $S$ , then we note that they lie on every (unique) shortest path between these vertices, which has a length at least one for  $n \leq 5$  and at least 2 otherwise, and at most  $\lfloor \frac{n}{2} - 1 \rfloor$ . Thus,  $\text{meg}(C_n) = 3$  when  $n \geq 5$  or  $n = 3$ .

In the case of  $C_4$ , the above construction does not work. Consider a set of three vertices, say  $v_0, v_1, v_2$  without loss of generality due to the symmetries of  $C_4$ . Notice that the edge  $v_0v_3$  is unmonitored by this set. Thus, we have  $\text{meg}(C_4) = 4$ .  $\square$

### 3.3 Unicyclic graphs

A *unicyclic graph* is a connected graph containing exactly one cycle [11]. We now determine the optimal size of an MEG-set of such graphs.

**Theorem 3.5.** *Let  $G$  be a unicyclic graph where the only cycle  $C$  has length  $k$  and whose set of pendant vertices is  $L(G)$ ,  $|L(G)| = l$ . Let  $V_c^+$  be the set of vertices of  $C$  with degree at least 3. Let  $p(G) = 1$  if  $G[V(C) \setminus V_c^+]$  contains a path whose length is at least  $\lfloor \frac{k}{2} \rfloor$ , and  $p(G) = 0$  otherwise. Then, if  $k \in \{3, 4\}$ ,*

$$\text{meg}(G) = l + k - |V_c^+|.$$

*Otherwise ( $k \geq 5$ ), then*

$$\text{meg}(G) = \begin{cases} 3, & \text{if } |V_c^+| = 0 \\ l + 2, & \text{if } |V_c^+| = 1 \\ l + p(G), & \text{if } |V_c^+| > 1 \end{cases}$$

*Proof.* Let  $G$  be a unicyclic graph where the only cycle  $C$  has length  $k$  and whose set of pendant vertices is  $L(G)$ . By Lemma 2.1, all leaves are part of any MEG-set of  $G$ . This implies that  $\text{meg}(G)$  is at least  $l$ . If  $|V_C^+| = 0$  (i.e.  $l = 0$ ), we are done by Theorem 3.4, so let us assume  $|V_C^+| > 0$  and thus,  $l > 0$ .

Similarly as in the proof of Lemma 2.3, for every vertex  $v$  of  $V_C^+$ , we know that at least one leaf will exist in the tree component  $T_v$  formed if we remove the neighbors of  $v$  in  $C$  from  $G$ . Informally speaking, towards the rest of the graph, this leaf simulates the fact that  $v$  is in the solution set.

If  $k \in \{3, 4\}$ , we consider  $S = L(G)$  and we add to  $S$  all vertices of  $C$  that are of degree 2 in  $G$ . One can easily check that this is an MEG-set. Moreover, one can see that adding these degree 2 vertices is necessary by using similar arguments as in the proof of Theorem 3.4 on cycles.

Next, we assume that  $k \geq 5$ . Let  $v_0, \dots, v_{k-1}$  be the vertices of  $C$ .

When  $|V_C^+| = 1$ , without loss of generality, consider the vertex in  $V_C^+$  to be  $v_0$ . Then, the vertices  $\{v_{\lfloor \frac{k}{3} \rfloor}, v_{\lfloor \frac{2k}{3} \rfloor}\}$  on the cycle are sufficient to monitor the graph, in the same way as in Theorem 3.4. Moreover, by the same arguments as in the proof of Theorem 3.4, one can see that if at most one vertex on  $C$  is chosen in the MEG-set, some edge will not be monitored.

If  $|V_C^+| > 1$  and  $p(G) = 0$ , the  $l$  pendant vertices are sufficient to monitor  $G$ . Indeed, consider an edge  $e$ . If  $e$  is not on  $C$ , let  $v$  be the vertex of  $V_C^+$  closest to  $e$ , and let  $w \neq v$  be the vertex of  $V_C^+$  closest to  $v$  (it exists because  $|V_C^+| > 1$ ). Consider a leaf  $f$  of  $G$  such that  $e$  lies on some path from  $v$  to  $f$ . Since  $p(G) = 0$ , the path from  $w$  to  $f$  is a unique shortest path, and thus,  $e$  is monitored by  $f$  and some leaf whose closest vertex on  $C$  is  $w$ .

If  $e$  is an edge of  $C$ ,  $e$  lies on a path between two vertices  $v, w$  of  $V_C^+$ . Since  $p(G) = 0$ , this path is a shortest path, and  $e$  is monitored by two leaves, each of which has  $v$  and  $w$  as its closest vertex of  $C$ , respectively.

Finally, consider the case where  $p(G) = 1$  and  $|V_C^+| > 1$ . Since  $p(G) = 1$ ,  $G[V(C) \setminus V_C^+]$  contains a path  $P$  whose length is at least  $\lfloor \frac{k}{2} \rfloor$  and thus, the edges of  $P$  are not monitored by the set of leaves of  $G$ , which implies that  $\text{meg}(G) \geq l + 1$ . To show that  $\text{meg}(G) \leq l + 1$ , we select as an MEG-set, the set of leaves together with the middle vertex of  $P$  (if  $P$  has even length) or one of the middle vertices of  $P$  (if  $P$  has odd length). One can see that this is an MEG-set by similar arguments as in the previous case.  $\square$

### 3.4 Complete graphs

The following follows immediately from Lemma 2.1, since every vertex of a complete graph is simplicial.

**Theorem 3.6.** *For any  $n \geq 2$ , we have  $\text{meg}(K_n) = n$ .*

### 3.5 Complete multipartite graphs

The complete  $k$ -partite graph  $K_{p_1, p_2, \dots, p_k}$  consists of  $k$  disjoint sets of vertices of sizes  $p_1, p_2, \dots, p_k$ , with an edge between any two vertices from distinct sets.

**Theorem 3.7.** *We have  $\text{meg}(K_{p_1, p_2, \dots, p_k}) = |V(K_{p_1, p_2, \dots, p_k})|$ , with the exceptional case of a bipartite graph  $K_{1, p}$  with an independent set of size 1 (a star graph), for which  $\text{meg}(K_{1, p}) = p$ .*

*Proof.* In a complete  $k$ -partite graph, all vertices in a given partite set are twins. Therefore, by Lemma 2.2, all vertices of any partite set of size at least 2 need to be a part of any MEG-set.

If we have several partite sets of size 1, then the vertices from these sets are closed twins, and again by Lemma 2.2 they all belong to any MEG-set.

Thus, we are done, unless there is a unique partite set of size 1, whose vertex we call  $v$ . If there are at least three partite sets, then note that  $v$  is never part of a unique shortest path, and thus the edges incident with  $v$  cannot be monitored if  $v$  is not part of the MEG-set.

On the other hand, if the graph is bipartite, it is a star  $K_{1, p}$ . Here, we know by Theorem 3.1 that  $\text{meg}(G) = p$ , as claimed.  $\square$

### 3.6 Hypercubes

The *hypercube of dimension  $n$* , denoted by  $Q_n$ , is the undirected graph consisting of  $k = 2^n$  vertices labeled from 0 to  $2^n - 1$  and such that there is an edge between any two vertices if and only if the binary representations of their labels differ by exactly one bit [16]. The *Hamming distance*  $H(A, B)$  between two vertices  $A, B$  of a hypercube is the number of bits where the two binary representations of its vertices differ.

We next show that not only  $C_4$  has the whole vertex set as its only MEG-set (Theorem 3.4), but that this also holds for all hypercubes.

**Theorem 3.8.** *For a hypercube graph  $Q_n$  with  $n \geq 2$ , we have  $\text{meg}(Q_n) = 2^n$ .*

*Proof.* Assume by contradiction that there is an MEG-set  $M$  of size at most  $2^n - 1$ . Let  $v \in V(G)$  be a vertex that is not in  $M$ . It is known that for every vertex pair  $\{v_x, v\}$  with  $H(v_x, v) \leq n$ , there are  $H(v_x, v)$  vertex-disjoint paths of length  $H(v_x, v)$  between them [16]. Thus, there is no vertex pair in  $M$  with a unique shortest path going through the edges incident with  $v$ , and  $M$  is not an MEG-set, a contradiction.  $\square$

### 3.7 Grid graphs

The graph  $G \square H$  is the Cartesian product of graphs  $G$  and  $H$  and with vertex set  $V(G \square H) = V(G) \times V(H)$ , and for which  $\{(x, u), (y, v)\}$  is an edge if  $x = y$  and  $\{u, v\} \in E(H)$  or  $\{x, y\} \in E(G)$  and  $u = v$ . The *grid graph*  $G(m, n)$  is the Cartesian product  $P_m \square P_n$  with vertex set  $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ .

**Theorem 3.9.** *For any  $m, n \geq 2$ , we have  $\text{meg}(G(m, n)) = 2(m + n - 2)$ .*

*Proof.* We claim that the set  $S = \{(i, j) \in V(G(m, n)) \mid i \in \{1, m\} \text{ and } 1 \leq j \leq n \text{ or } j \in \{1, n\} \text{ and } 1 \leq i \leq m\}$  of  $2(m + n - 2)$  vertices of  $G(m, n)$  that form the boundary vertices of the grid, form the only optimal MEG-set.

For the necessity side, let us assume that some vertex  $v = (i, j)$  of  $S$  is not part of the MEG-set. If  $v$  is a corner vertex (without loss of generality say  $v = (1, 1)$ ), the two edges incident with  $v$  are not monitored, as for any shortest path going through them, there is another one going through vertex  $(2, 2)$ . If  $v$  is not a corner vertex (without loss of generality say  $v = (1, j)$  with  $2 \leq j \leq n - 1$ ), then the edge  $e$  between  $v = (1, j)$  and  $(2, j)$  is not monitored, indeed for any shortest path containing  $e$ , there is another one avoiding it, either going through vertex  $(2, j - 1)$  or through  $(2, j + 1)$ .

To see that  $S$  is an MEG-set, first see that each boundary edge is monitored by its endpoints. Next, consider an edge  $e$  that is not a boundary edge, without loss of generality,  $e$  is between  $(i, j)$  and  $(i + 1, j)$ . Then, it is monitored by  $(1, j)$  and  $(m, j)$ , whose unique shortest path goes through  $e$ .  $\square$

## 4 Relation to feedback edge set number

A feedback edge set of a graph  $G$  is a set of edges which when removed from  $G$  leaves a forest. The smallest size of such a feedback edge set of  $G$  is denoted by  $\text{fes}(G)$  and is sometimes called the *cyclomatic number* of  $G$ .

We next introduce the following terminology from [7]. A vertex is a *core vertex* if it has degree at least 3. A path with all internal vertices of degree 2 and whose end-vertices are core vertices is called a *core path*. Do note that we allow the two end-vertices to be equal, but that every other vertex must be distinct. A core path that is a cycle (that is, both end-vertices are equal) is a *core cycle*. For the sake of distinction, a core path that is not a core cycle is called a *proper core path*. We say that a (non-empty) path from a core vertex  $u$  to a leaf  $v$  is a *leg* of  $u$  if all internal vertices of the path have degree 2 ( $u$  is not considered to be a part of the leg). The *base graph* of a graph  $G$  is the graph of minimum degree 2 obtained from  $G$  by iteratively removing vertices of degree 1. A *hanging tree* is a connected subtree of  $G$  which is the union of some legs removed from  $G$  during the process of creating the base graph  $G_b$  of  $G$ .

Thus,  $G$  can be decomposed into its base graph and a set of maximal hanging trees. The root of such a maximal hanging tree  $T$  is the vertex common to  $T$  and  $G$ .

See Figure 2 for a graph whose core vertices are in red. It has two hanging trees, three core cycles, three proper core paths of length 4, and five proper core paths of length 1.

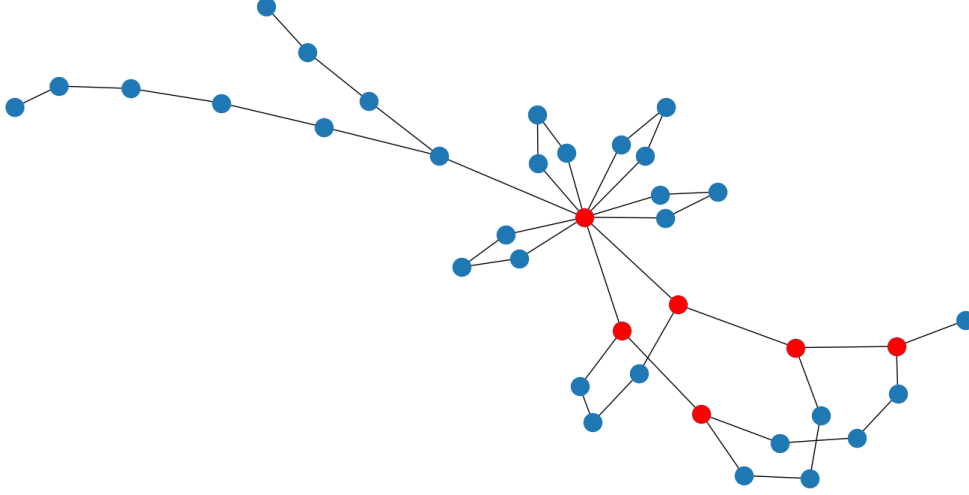


Figure 2: Example of a graph  $G$  with its core vertices in red.

Based on the aforementioned, we have the following lemma.

**Lemma 4.1** ([7, 12]). *Let  $G$  be a graph with  $\text{fes}(G) = k \geq 2$ . The base graph of  $G$  has at most  $2k - 2$  core vertices, that are joined by at most  $3k - 3$  edge-disjoint core paths. Equivalently,  $G$  can be obtained from a multigraph  $H$  of order at most  $2k - 2$  and size at most  $3k - 3$  by subdividing its edges an arbitrary number of times and iteratively adding degree 1 vertices.*

**Lemma 4.2.** *Let  $S$  be an MEG-set of the base graph  $G_b$  of  $G$  and  $L(G)$  be the set of leaves in  $G$ . Then,  $S \cup L(G)$  is an MEG-set of  $G$ .*

*Proof.* Let  $G_b$  be a base graph of  $G$ . Consider all vertices that are roots of maximal hanging trees on  $G_b$ . By Theorem 3.1, the optimal MEG-set of each tree consists of all leaves. We repeatedly apply Lemma 2.3 to  $G$ , where for each application of Lemma 2.3, the cut-vertex is the root of a hanging tree in consideration.  $\square$

Lemma 2.1, Theorem 3.1 and Lemma 4.2 together imply that if  $\text{fes}(G) = 0$ , then  $\text{meg}(G) \leq \text{fes}(G) + |L(G)|$ . Moreover, if  $\text{fes}(G) = 1$ , then  $\text{meg}(G) \leq \text{fes}(G) + |L(G)| + 3$ , where  $|L(G)|$  is the number of leaves of  $G$ . We next give a similar bound when  $\text{fes}(G) \geq 2$ .

**Theorem 4.3.** *If  $\text{fes}(G) \geq 2$ , then  $\text{meg}(G) \leq 9\text{fes}(G) + |L(G)| - 8$  where  $|L(G)|$  is the number of leaves of  $G$ .*

*Proof.* Let  $k = \text{fes}(G)$ . We show how to construct a MEG-set  $M$  of  $G_b$  of order at most  $9k - 8$  and, by applying Lemma 4.2 to  $G$ , of order  $9k - 8 + |L(G)|$  for  $G$ . If an edge  $e$  is part of a maximal hanging tree, then by Lemma 2.1 and Lemma 2.3, it is monitored by the leaves of  $G$  on the maximal hanging tree.  $M$  is constructed as follows.

- We let all core vertices of  $G_b$  be part of  $M$ .



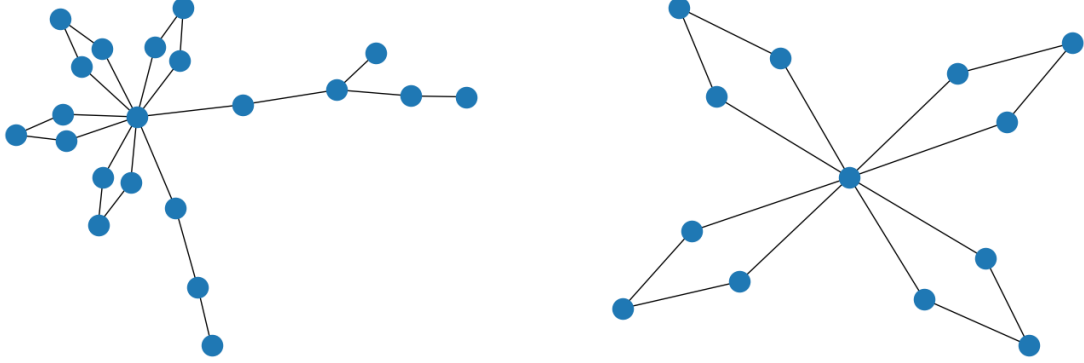


Figure 3: Example of a graph  $G$  and its base graph  $G_b$  with four core cycles.

- One or two internal vertices from each proper core path belongs to  $M$ , only if the length is at least 2, as explained below.
- Two or three internal vertices from each core cycle, as explained below.

Consider a proper core path  $P$ , with core vertex endpoints  $c$  and  $c'$ , and the median vertex  $x_1$  in the case of an odd-length path and  $x_1, x_2$  in the case of an even-length path, with  $d$  edges (on  $P$ ) between the endpoints and the respective medians in  $P$ . Then, we choose the single median vertex  $x_1$  or the two median vertices  $x_1, x_2$  from each of the core paths into  $M$ .

For each core cycle, in addition to the core vertex of that cycle, we add three vertices that are as equidistant as possible on the cycle, to be part of  $M$  (as in Theorem 3.4).

Let  $e$  be any edge of  $G$ . We now show that our construction  $M$  monitors any such edge in  $G_b$ . If  $e$  lies on a core cycle, assume an origin core vertex of  $v_0$ . Then, based on Lemma 2.3 and Theorem 3.4, we deduce that in the worst case, four vertices together suffice to monitor the edges.

If the edge  $e$  lies on a proper core path  $P$ , then we have the following cases. Let  $c$  and  $c'$  be the core vertex endpoints of  $P$ , and the median vertex  $x_1$  in the case of an odd-length path and  $x_1, x_2$  in the case of an even-length path and  $d$  edges of  $P$  between the end points and the respective medians in  $P$ . Without loss of generality, let us say that  $e$  lies on the path  $P$  such that its closest core vertex is  $c$  and closest median  $x_1$  in the event of an even-length path. Suppose first that  $d$  is even. Given that the distance between  $c$  and  $x_1$  is  $d$  in  $P$ , the length of any other path between them must be at least  $d + 2$ . Therefore,  $c$  and  $x_1$  monitor  $e$ . We can similarly argue that if the closest core vertex to  $e$  was  $c'$  and the closest median vertex was  $x_2$ , then  $c'$  and  $x_2$  monitor  $e$ . If  $e$  lay in between the median vertices  $x_1$  and  $x_2$ , then we know that those vertices would monitor  $e$  because they are adjacent. If the path was of odd length, then depending on which of the core vertices  $c$  and  $c'$  was closest to  $e$ , the distance between the median and the core vertices would be  $d$  in  $P$  and the length of any other path between them at least  $d + 1$ , ensuring that the median vertex  $x_1$  would monitor the edge apart from the core vertices. This justifies our construction of  $M$  for  $G_b$ .

By Lemma 4.1, the number of core vertices of  $G_b$  is at most  $2k - 2$ , and there are at most  $3k - 3$  core paths.

If we have core cycles in our graph, then we must note that there can be at most  $k$  such cycles in the graph. Indeed, if there were  $k + 1$  core cycles in the graph, since they are all edge-disjoint, we need at least  $k + 1$  edges to be removed from  $G$  to obtain a forest, a contradiction to the fact that  $\text{fes}(G) = k$ .

Let  $n_c$  be the number of core cycles and  $n_p$  be the number of proper core paths. We have  $|M| = 3n_c + 2n_p + 2k - 2$ . Since  $n_c \leq k$  and  $n_c + n_p \leq 3k - 3$  by Lemma 4.1, we get  $|M| \leq 3k + 2(2k - 3) + 2k - 2 = 9k - 8$ .  $\square$

Recall that the *max leaf number* of  $G$ , denoted  $\text{mln}(G)$ , is the maximum number of leaves in a spanning

tree of  $G$ . It can be seen as a refinement of the feedback edge set number of  $G$  [5]. We get the following corollary.

**Corollary 4.4.** *For any graph  $G$ , we have  $\text{meg}(G) \leq 10 \text{mln}(G)$ , where  $\text{mln}(G)$  is the max leaf number of  $G$ .*

*Proof.* It is known that  $\text{fes}(G) \leq \text{mln}(G)$  [5], and clearly,  $|L(G)| \leq \text{mln}(G)$ , thus the bound follows from Theorem 4.3.  $\square$

**Proposition 4.5.** *For any integer  $k \geq 2$ , there exists a graph  $G$  with  $\text{fes}(G) = k$  and  $\text{meg}(G) = 3k + |L(G)|$ .*

*Proof.* Consider  $G$  and its base graph  $G_b$  in Figure 3. We know that the leaves must be part of any MEG-set by Lemma 2.1. The MEG-set for  $G_b$  consists of all the vertices in each of the core cycles (each a  $C_4$ ) in  $G_b$ , except the common core vertex. It is easy to check that no smaller set can work. The size of the optimal MEG-set in this example is  $3k + |L(G)|$  and therefore, this is an instance where this proposition holds.  $\square$

## 5 Conclusion

Inspired by a network monitoring application, we have defined the new concept of MEG-sets of a graph, which is a common refinement of the popular concept of a geodetic set and its variants, and of the previously studied distance-edge-monitoring sets.

We have studied the concept on basic graph classes. It is interesting to note that there are many graph classes which require the entire vertex set in any MEG-set: complete graphs, complete multipartite graphs, and hypercubes. It could thus be a difficult, but interesting, question, to characterize all such graphs.

Our upper bound using the feedback edge set number is probably not tight. What is a tight bound on this regard?

Finally, it remains to investigate computational aspects of the problem.

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