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Parametric linearization of skew products

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Abstract

We establish a linearization criterion for skew products of contractions in any dimension. We prove their smooth or holomorphic parameter dependence. In the smooth setting, we use the language of tame Fréchet spaces. We apply our result to the linearization of expanding Cantor sets.

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1 Introduction

Linearization is a fundamental problem in dynamics. In the simplest setting of a fixed point p of a differentiable map f, one would like to conjugate the dynamics of f near p to the dynamics of the differential $D_p f$ via a homeomorphism h:

$$h \circ f = D_p f \circ h \ .$$

By the Hartman-Grotman theorem, see [Har82], every contracting fixed point is C^0-linearizable. There are many results that assert more regularity for h, given extra conditions. One of the first results in this direction is [Poi28, pp. XCIX - CV], in which Poincaré established the analytic case under certain non-resonance conditions, see condition $(\mathcal{R})$ in the next section. The Sternberg linearization theorem for contracting fixed points [Ste57] shows the smooth case under the same non-resonance conditions.

These linearization theorems of contracting maps have been generalized in many directions, two of which are the introduction of parameter dependence (see for example [Tak71, Sel85]) and the replacement of a contracting map by a skew products of contracting maps (see for example [MY10, GK98, Ber14, IR16]). The skew products of contracting maps are interesting since they can be used to provide an atlas of charts for expanding compact sets in which the dynamics is linear. In bifurcation theory involving non-trivial expanding compact set, it is useful to have a linearization theorem which deals with both skew products of contracting maps and a nice dependence on the dynamics. This is the subject of our main theorem. While [MY10, Ber14, IR16] deals with one dimensional skew products, our main result will apply to any finite dimensional space. Finite dimensional spaces were studied in [GK98, JV02, KS17] but without the parameter dependence. We will show that the linearization depends smoothly or analytically on the whole space of smooth or analytic skew products. In this aspect our study is finer than [Tak71, Sel85] where the dependence was studied only along finite dimensional families. Working with such infinite dimensional parameter spaces should be useful for defining renormalization operators nearby homoclinic tangencies or investigating the prevalence following Sauer-Hunt-York [HSY92] of some properties such as the Newhouse phenomenon [New79].

2 Main results

2.1 Statement of the main theorem

Let $\mathbb{B}$ be the closed unit ball centered at 0 in $\mathbb{R}^n$. Let $f\colon \mathbb{B} \to \mathbb{B}$ be an analytic or smooth contracting diffeomorphism fixing 0. Assume that $D_0 f = \operatorname{diag} \lambda_i$ is diagonal **without resonances**:

$(\mathcal{R})$ for any $1 \leq i \leq n$ and any multiindex $\underline{k}$ with $|\underline{k}| \neq 1$, we have $|\lambda_i| \neq |\lambda^{\underline{k}}|$.

Then Poincaré's or Sternberg's linearization theorem asserts that f is analytically or respectively smoothly **linearizable**: there exists an analytic or resp. C^∞-diffeomorphism h such that:

$$h_f \circ f(x) = D_0 f \circ h_f(x) \quad \forall x \in \mathbb{B} \ .$$

We will generalize these theorems in Theorem A and Theorem C. In particular, we make the parameter dependence of these results explicit by noting that $f \mapsto h_f$ depends analytically or respectively tamely smooth on f.

In the analytic case, let $\tilde{\mathbb{B}} \subset \mathbb{C}^n$ be the open unit complex ball centered at 0 and consider $f\colon \tilde{\mathbb{B}} \to \tilde{\mathbb{B}}$. Let $\mathcal{B}$ denote the complex Banach space of bounded holomorphic functions from $\tilde{\mathbb{B}}$ into

$\mathbb{C}^n$, vanishing at 0 and with diagonal differential at 0. Then Poincaré proved that $h_f \in \mathcal{B}$. We will show that the operator $f \mapsto h_f$ from an open subset of $\mathcal{B}$ into $\mathcal{B}$ is holomorphic.

In the smooth case, we will consider the Fréchet space $\mathcal{F}$ of C^∞-maps from $\mathbb{B}$ into $\mathbb{R}^n$, which fix 0 and whose derivative at 0 is diagonal. We will show that the operator $f \mapsto h_f$ from an open subset $\mathcal{U} \subset \mathcal{F}$ into $\mathcal{F}$ is **smooth**: there is a family $(L_{i,f})_{i\geq 0}$ of i-multilinear maps $L_{i,f}$ of $\mathcal{F}$, such that $(f, g_1, \dots, g_i) \in \mathcal{U} \times \mathcal{F}^i \mapsto L_{i,f}(g_1, \dots, g_i)$ is continuous and for every $g \in \mathcal{F}$, for the C^r-norm it holds:

$$h_{f+tg} = h_f + \sum_{i=1}^{m} L_{i,f}(g, \dots, g)t^i + o(t^m) \quad \text{when } t \to 0 \ .$$

Moreover we will show $f \mapsto h_f$ is **tamely smooth** in the sense of [Ham82] recalled in Section 4.1.

We generalize these results by considering skew product of contractions. Let $\sigma : \mathsf{A} \to \mathsf{A}$ be a homeomorphism of a compact metric space A. Let $(f_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ be a C^0-family of C^1-self-maps of $\mathbb{B}$. This defines a skew product:

$$f : (\mathsf{a}, x) \in \mathsf{A} \times \mathbb{B} \mapsto (\sigma(\mathsf{a}), f_\mathsf{a}(x)) \in \mathsf{A} \times \mathbb{B}.$$

We now give a sufficient condition to conjugate f to a linear cocycle. Here are the conditions:

(H_1) Each map f_a is a contracting diffeomorphism from $\mathbb{B}$ into $\mathbb{B}$ which fixes 0, for every $\mathsf{a} \in \mathsf{A}$.

(H_2) The differential of f_a at 0 is diagonal; $Df_\mathsf{a}(0) = \operatorname{diag} \lambda_{\mathsf{a},i}$ for every $\mathsf{a} \in \mathsf{A}$.

(H_3) For any $1 \leq i \leq n$ and any multiindex $\underline{k}$ with $|\underline{k}| \neq 1$, we have $|\lambda_{\mathsf{a},i}| \neq |\lambda_\mathsf{a}^{\underline{k}}|$ for all $\mathsf{a} \in \mathsf{A}$, and the sign of the difference does not depend on a.

Theorem A. *If $(f_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ is a C^0-family of C^∞-maps satisfying $(H_1 - H_2 - H_3)$, then there exists a unique C^0-family $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ of C^∞-diffeomorphisms from $\mathbb{B}$ into $\mathbb{R}^n$ such that*

$$D_0 h_\mathsf{a} = \mathrm{id} \ , \quad h_\mathsf{a}(0) = 0 \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_\mathsf{a}(x) = D_0 f_\mathsf{a} \circ h_\mathsf{a}(x), \quad x \in \mathbb{B} \ . \tag{2.1}$$

Moreover, h depends tamely smooth on f.

We will first prove this theorem without parameter dependence in Section 3, and then in Section 5.2 the full theorem (after giving a proper definition of tamely smooth maps).

Remark 2.1. Since we consider contracting diffeomorphisms, we have $0 < |\lambda_{\mathsf{a},i}| < 1$ for every $\mathsf{a} \in \mathsf{A}$. From that it follows that there is an $r \in \mathbb{N}$ such that:

(H_4^r) For any $1 \leq i \leq n$ and any multiindex $\underline{k}$ with $|\underline{k}| \geq r$, we have $|\lambda_{\mathsf{a},i}| > |\lambda_\mathsf{a}^{\underline{k}}|$ for all $\mathsf{a} \in \mathsf{A}$.

Thus (H_3) is an open condition among families satisfying $(H_1 - H_2)$.

As a consequence of the proof, we obtain the following finite regularity result:

Corollary B. *For $r \geq 2$, if $(f_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ is a C^0-family of C^r-maps satisfying $(H_1 - H_2 - H_3 - H_4^r)$, then there exists a unique C^0-family $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ of C^r-diffeomorphisms from $\mathbb{B}$ into $\mathbb{R}^n$ such that*

$$D_0 h_\mathsf{a} = \mathrm{id} \ , \quad h_\mathsf{a}(0) = 0 \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_\mathsf{a}(x) = D_0 f_\mathsf{a} \circ h_\mathsf{a}(x), \quad x \in \mathbb{B} \ . \tag{2.2}$$

Moreover h depends continuously on f.

Let us now consider the case of holomorphic families. In this setting $\tilde{\mathbb{B}}$ is the **open** unit ball around 0 in $\mathbb{C}$ and f_a is a holomorphic map from $\tilde{\mathbb{B}}$ into $\mathbb{C}^n$. Observe that $(H_1 - H_2 - H_3)$ also make sense in this setting. We have the following holomorphic counterpart of Theorem A:

Theorem C. *If $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ is a C^0-family of holomorphic maps satisfying $(H_1 - H_2 - H_3)$ and it is a family of contracting biholomorphisms from $\tilde{\mathbb{B}}$ into $\tilde{\mathbb{B}}$, then there exists a unique C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of biholomorphic maps from $\tilde{\mathbb{B}}$ into $\mathbb{C}^n$ such that*

$$D_0 h_{\mathsf{a}} = \mathrm{id}\ , \quad h_{\mathsf{a}}(0) = 0 \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x), \quad x \in \tilde{\mathbb{B}}\ . \tag{2.3}$$

Moreover, $(h_{\mathsf{a}})_{\mathsf{a}}$ depends holomorphically on f.

We will show this in Section 5.1.

Remark 2.2. If each of the (f_{a}) are real analytic (i.e. $f_{\mathsf{a}}(\tilde{\mathbb{B}} \cap \mathbb{R}^n) \subset \mathbb{R}^n$), then also the maps (h_{a}) are real analytic.

Remark 2.3. The above results are also valid for non-trivial bundles over A: instead of considering a map on the product $\mathsf{A} \times \mathbb{B}$, we could consider a map on a ball bundle over A. For the ease of exposition, the proof is only done in the case of a trivial bundle.

2.2 Applications

The application which motivated this work regards the linearization of expanding compact sets.

Let M be a Riemannian manifold of dimension n and let g be a C^1-self-map of M. We recall a compact subset $K \subset M$ is **invariant** by g if $g(K) \subset K$. It is **expanding** if furthermore:

$$\|D_x g(u)\| > 1\ , \quad \forall x \in K \text{ and unit vector } u \in T_x M\ .$$

Note that by compactness of K, the latter inequality is uniform. Expanding compact subsets are **structurally stable** (see [Shu69] or [Ber10, Thm 0.1]). This means that for every C^1-perturbation $\tilde{g}$ of g, there is a unique embedding $\psi_{\tilde{g}} : K \hookrightarrow M$ that is C^0-close to the canonical inclusion $K \hookrightarrow M$ and such that

$$\psi_{\tilde{g}} \circ g|K = \tilde{g} \circ \psi_{\tilde{g}}\ .$$

The set $K_{\tilde{g}} = \psi_{\tilde{g}}(K)$ is called the **hyperbolic continuation** of K; it is expanding for $\tilde{g}$.

A sub-bundle $E \subset TM|K$ is Dg**-invariant** if $Dg(E_x) = E_{g(x)}$ for every $x \in K$. Then Dg induces a map denoted $D[g]$ on the quotient TM/E. We say that the expanding compact set K is **projectively hyperbolic** at E if:

$$\|D_x[g](v)\| > \|D_x g(u)\| \quad \forall x \in K\ , \quad u \in E_x\ , v \in T_x M/E_x, \text{ such that } \|u\| = 1 = \|v\|\ .$$

Note again that by compactness of K, the latter inequality is uniform. We say that the expanding compact set K is **totally projectively hyperbolic** if there exists a flag of invariant sub-bundles:

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = TM|K$$

such that for every $1 \leq i < n$, the set K is **projectively hyperbolic** at E_i. Using cones, one shows that totally projective hyperbolicity is a C^1-open property: for C^1-small perturbations $\tilde{g}$ of g, the hyperbolic continuation of K is also totally projective hyperbolic. Furthermore, this property enables to diagonalize the differential. To this end we consider the **inverse limit** $\overleftarrow{K}$ **of** K:

$$\overleftarrow{K} := \left\{ \mathsf{a} = (\mathsf{a}_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}} \colon g(\mathsf{a}_i) = \mathsf{a}_{i+1} \text{ for all } i \in \mathbb{Z} \right\}\ ,$$

on which g induces a homeomorphism:

$$\overleftarrow{g}\ : \mathsf{a} = (\mathsf{a}_i)_{i\in\mathbb{Z}} \in \overleftarrow{K} \mapsto (\mathsf{a}_{i+1})_{i\in\mathbb{Z}} \in \overleftarrow{K}\ .$$

Using cones, the total projective hyperbolicity implies the existence of a unique splitting $F_{1\mathsf{a}} \oplus F_{2\mathsf{a}} \oplus \cdots \oplus F_{n\mathsf{a}} = T_{\mathsf{a}_0}M$ formed by invariant line bundles that is adapted to the flag:

$$D_{\mathsf{a}_0}g(F_{i\mathsf{a}}) = F_{i\overleftarrow{g}(\mathsf{a})}\ , \quad E_{i\mathsf{a}_0} = F_{1\mathsf{a}} \oplus \cdots \oplus F_{i\mathsf{a}} \quad \forall \mathsf{a} \in \overleftarrow{K}\ .$$

One easily shows that this splitting depends continuously on a. Here is a consequence of the main theorem:

Corollary D. *Let K be an expanding Cantor set for a self-map $g \in C^\infty(M, M)$. Assume that:*

(a) the set K is totally projectively hyperbolic with invariant splitting $F_1 \oplus F_2 \oplus \cdots \oplus F_n \to \overleftarrow{K}$.

(b) for any $1 \le i \le n$ and any multiindex $\underline{k} = (k_1, \ldots, k_n)$ with $|\underline{k}| \neq 1$, it holds $\|D_{\mathsf{a}_0}g|F_i\| \neq \prod_{\ell=1}^n \|D_{\mathsf{a}_0}g|F_\ell\|^{k_\ell}$ and the sign of the difference does not depend on $\mathsf{a} \in \overleftarrow{K}$.

Then for $r > 0$ sufficiently small, there is a unique continuous family $(\varphi_{\mathsf{a}})_{\mathsf{a}\in\overleftarrow{K}}$ of C^∞-charts φ_{a} from the closed r-ball $B_{\mathsf{a}_0}(r)$ of $T_{\mathsf{a}_0}M$ onto a neighborhood of $\mathsf{a}_0 \in M$ such that:

$$\varphi_{\mathsf{a}}(0) = \mathsf{a}_0, \quad D_0\varphi_{\mathsf{a}} = \mathrm{id} \quad \textit{and} \quad \varphi_{\overleftarrow{g}(\mathsf{a})} \circ D_{\mathsf{a}_0}g = g \circ \varphi_{\mathsf{a}} \quad \textit{on } (D_{\mathsf{a}_0}g)^{-1}B_{\mathsf{a}_0}(r)\ . \tag{2.4}$$

Remark 2.4. We will show in Corollary E of Section 4.2 that the resulting linearization depends **smooth tamely** on g. In Corollary F of Section 4.4, we state the holomorphic counterpart of this corollary.

Proof of Corollary D. We are going to apply Theorem A using inverse branches of g to define the fiber dynamics of a skew product over $\mathsf{A} := \overleftarrow{K}$ endowed with the dynamics σ equal to the inverse of $\overleftarrow{g}$:

$$\sigma : \mathsf{a} = (\mathsf{a}_i)_{i\in\mathbb{Z}} \in \mathsf{A} = \overleftarrow{K} \mapsto (\mathsf{a}_{i-1})_{i\in\mathbb{Z}} \in \mathsf{A} = \overleftarrow{K}\ .$$

Since K is a Cantor set, it has a neighborhood U which is diffeomorphic to an open subset of $\mathbb{R}^n$. Likewise $\overleftarrow{K}$ is a Cantor set, so the splitting $F_1 \oplus F_2 \oplus \cdots \oplus F_n \to \overleftarrow{K}$ is trivial. This implies that that there is a continuous family of charts $(\Phi_{\mathsf{a}})_{\mathsf{a}\in\overleftarrow{K}}$ from $\mathbb{B}$ into M and such that:

- $\Phi_{\mathsf{a}}(0) = \mathsf{a}_0$,
- $D_0\Phi_{\mathsf{a}}$ sends the splitting $\mathbb{R} \oplus \cdots \oplus \mathbb{R} = \mathbb{R}^n$ to $F_{1\mathsf{a}} \oplus F_{2\mathsf{a}} \oplus \cdots \oplus F_{n\mathsf{a}}$ and $\|D_0\Phi_{\mathsf{a}}(e_i)\|$ is constant over $\mathsf{a} \in \mathsf{A}$ for every vector e_i in the canonical basis of $\mathbb{R}^n$.

Note that in general, $D_0\Phi_{\mathsf{a}}$ will not be an isometry from $\mathbb{R}^n$ with the standard length to $T_{\mathsf{a}_0}M$. However, with these conditions we have that $D_0(\Phi_{\mathsf{a}}^{-1} \circ g \circ \Phi_{\sigma(\mathsf{a})})$ is a diagonal matrix whose entries have modulus $\|D_{\sigma(\mathsf{a}_0)}g|F_i\|$. In particular, since K is expanding, all these entries have modulus uniformly greater than 1. We shall consider the inverse of these maps to obtain contractions as stated in (H_1). To this end, we consider $\psi_{\mathsf{a}} : x \in \mathbb{B} \mapsto \Phi_{\mathsf{a}}(s \cdot x)$, with $s > 0$ sufficiently small so that $g|\psi_{\sigma(\mathsf{a})}(\mathbb{B})$ is an expanding diffeomorphism onto its image which contains $\psi_{\mathsf{a}}(\mathbb{B})$. Then the following map is well-defined:

$$f_{\mathsf{a}} := (\psi_{\mathsf{a}}^{-1} \circ g \circ \psi_{\sigma(\mathsf{a})})^{-1}|\mathbb{B}\ .$$

By our construction, f_{a} fixes 0 and has diagonal differential whose entries have modulus $\|D_{\sigma(\mathsf{a}_0)}g|F_i\|^{-1}$. For $s > 0$ sufficiently small enough, it follows that f_{a} is contracting, so f_{a} satisfies $(H_1 - H_2)$ of Theorem A. Moreover (H_3) follows from hypothesis (b). Applying Theorem A, let $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ be the continuous family of C^∞-diffeomorphism $h_{\mathsf{a}} : \mathbb{B} \to \mathbb{R}^n$ such that $h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}} = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}$ on $\mathbb{B}$. Then $\varphi_{\mathsf{a}} = \psi_a \circ h_a^{-1} \circ D_0(\psi_a)^{-1}$ satisfies (2.4) on some neighborhood of the zero section of $TM|\overleftarrow{K}$. In particular, for $r > 0$ sufficiently small, we have that φ_{a} is defined on $B_{\mathsf{a}_0}(r)$ for all $\mathsf{a} \in \overleftarrow{K}$. The uniqueness of φ follows from the uniqueness of h. $\square$

Remark 2.5. If f is real analytic we can also arrange the charts ϕ_{a} to be real analytic, this follows from the holomorphic setting below.

3 Proof of main theorem without parameter dependence

The proof of Theorem A without parameter dependence follows from two lemmas:

Lemma 3.1. *For $r \geq 2$ and every continuous family of C^∞-maps $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ satisfying $(H_1 - H_2 - H_3)$, there exists a unique C^0-family of polynomial maps $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of $\mathbb{R}^n$ with degree $\leq r$ such that for every $\mathsf{a} \in \mathsf{A}$:*

$$h_{\mathsf{a}}(0) = 0\ , \quad D_0 h_{\mathsf{a}} = \mathrm{id} \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) + o(x^r)\ . \tag{3.1}$$

Lemma 3.2. *For $r \geq 2$, if f is a continuous family of C^∞-maps satisfies $(H_1 - H_2 - H_3 - H_4^r)$, and furthermore f satisfies*

$$f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}}(x) + o(x^r)\ .$$

Then there exists $\delta > 0$, such that there is a unique C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of C^∞-diffeomorphisms h_{a} from the δ-ball around 0 into $\mathbb{R}^n$ satisfying for following conditions for every $\mathsf{a} \in \mathsf{A}$:

$$h_{\mathsf{a}}(0) = 0\ , h(x) = x + o(x^r) \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x)\ . \tag{3.2}$$

For both of these lemmas, we will first prove a version for finite regularity, and then we explain how to promote it to the C^∞ setting.

Proof of A without parameter dependence. By Remark 2.1 there is an r such that f satisfies (H_4^r). For the existence of h on a small ball around 0, we can do a direct composition of Lemma 3.1 and Lemma 3.2. As we assume that f is a family of contracting diffeomorphisms on $\mathbb{B}$, we can use the relation

$$(D_0 f_{\mathsf{a}})^{-1} \circ h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = h_{\mathsf{a}}(x)$$

to iteratively extend h to $\mathbb{B}$.

The uniqueness of h can be proved similarly: suppose $\tilde{h}$ is another family satisfying (2.1). By the uniqueness part of Lemma 3.1 we get that $h_{\mathsf{a}}(x) = \tilde{h}_{\mathsf{a}}(x) + o(x^r)$, or $\tilde{h}_{\mathsf{a}}(x) \circ h_{\mathsf{a}}^{-1}(x) = x + o(x^r)$. By the uniqueness part of Lemma 3.2 applied to the linear map family, we see that $\tilde{h}_{\mathsf{a}} \circ h_{\mathsf{a}}^{-1}(x) = x$ in a neighbourhood of 0, or $\tilde{h}_{\mathsf{a}}(x) = h_{\mathsf{a}}(x)$. By extension, we also get $\tilde{h}_{\mathsf{a}}(x) = h_{\mathsf{a}}(x)$ on $\mathbb{B}$. □

3.1 Formal linearization

Actually, C^∞-smoothness is not required in Lemma 3.1. So we will show the following version:

Lemma 3.3. *For $r \geq 2$ and every continuous family of C^r-maps $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ satisfying $(H_1 - H_2 - H_3)$, there exists a unique C^0-family of polynomial maps $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of $\mathbb{R}^n$ with degree $\leq r$ such that for every $\mathsf{a} \in \mathsf{A}$:*

$$h_{\mathsf{a}}(0) = 0\ , \quad D_0 h_{\mathsf{a}} = \mathrm{id} \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) + o(x^r)\ . \tag{3.3}$$

Proof of Lemma 3.3. We use the language of r-jets in the proof. Recall that two C^r-maps $g, \tilde{g}\colon \mathbb{B} \to \mathbb{R}^n$ have the same r-jet if $g(x) = \tilde{g}(x) + o(x^r)$. This is an equivalence relation, the resulting quotient space is the space of r-jets J_r. Every r-jet can be represented by a unique polynomial of degree $\leq r$. So we could have stated the lemma also in the language of r-jets: there is a unique C^0-family of r-jets satisfying (3.3). An advantage of jets is that they form a semigroup under composition, and that jets with invertible differential are also invertible in the semigroup.

We show by induction on j from 1 to r that we can find a C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of r-jets h_{a} with

$$h_{\mathsf{a}}(0) = 0\ , \quad D_0 h_{\mathsf{a}} = \mathrm{id} \quad \text{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) + o(x^j)\ . \tag{3.4}$$

For $j = 1$, there is nothing to show. Assume that $j \geq 2$. By induction, if $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ is a family of r-jets satisfying (3.4) for $j-1$, then the composition $(h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}} \circ h_{\mathsf{a}}^{-1})_{\mathsf{a}\in\mathsf{A}}$ is well-defined in the semigroup of r-jets. As we only want to prove the statement on the level of jets, we can replace $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ by $(h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}} \circ h_{\mathsf{a}}^{-1})_{\mathsf{a}\in\mathsf{A}}$ with $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ satisfying (3.4) for $j-1$. So we can assume that (f_{a}) satisfies

$$f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}}(x) + o(x^{j-1})\ . \tag{3.5}$$

Let us recall for every multiindex $\underline{k} = (k_1, \dots, k_n)$ and $x \in \mathbb{R}^n$, $x^{\underline{k}} = \prod_{i=1}^n x_i^{k_i}$ is a real number. Moreover we denote $|\underline{k}| = k_1 + \dots + k_n$, $\underline{k}! = \prod_{i=1}^n k_i!$ and $\partial^{\underline{k}} = \partial_{x_1}^{k_1} \dots \partial_{x_n}^{k_n}$. So we have that $\partial^{\underline{k}} f_{\mathsf{a}}(0)$ is in $\mathbb{R}^n$. Thus, by Eq. (3.5), it holds:

$$f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}}(x) + \sum_{|\underline{k}|=j} \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j)\ .$$

Note that $\partial^{\underline{k}} f_{\mathsf{a}}(0) \ \cdot \frac{x^{\underline{k}}}{\underline{k}!}$ denotes the scalar multiplication of the scalar $\frac{x^{\underline{k}}}{\underline{k}!} \in \mathbb{R}$ and the vector $\partial^{\underline{k}} f_{\mathsf{a}}(0) \in \mathbb{R}^n$. Note that $\mathbb{R}^n$ is a commutative ring endowed with the canonical product structure

$$x \bullet y = (x_i \cdot y_i)_{1 \leq i \leq n}.$$

The ring $\mathbb{R}^n$ endowed with the scalar multiplication, denoted by $\cdot$, it is a $\mathbb{R}$-algebra. In order to stay with 'Taylor-like' development, we will proceed with scalar multiplication on the right, as above. In these notations we have $Df_{\mathsf{a}}(x) = \lambda_{\mathsf{a}} \bullet x$, where $\lambda_{\mathsf{a}} = (\lambda_{\mathsf{a},i})_{1\leq i\leq n}$ and $Df_{\mathsf{a}}(x) = \operatorname{diag} \lambda_{\mathsf{a},i}$. Thus:

$$f_{\mathsf{a}}(x) = \lambda_a \bullet x + \sum_{|\underline{k}|=j} \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j)\ .$$

We look for a continuous family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of polynomials in $\mathbb{R}^n[X_1, \dots, X_n]$ of the form:

$$h_{\mathsf{a}}(x) = x + \sum_{|\underline{k}|=j} q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!},$$

where $q_{\mathsf{a},\underline{k}} \in \mathbb{R}^n$. Here likewise $q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!}$ denote the scalar multiplication of the scalar $\frac{x^{\underline{k}}}{\underline{k}!} \in \mathbb{R}$ and the vector $q_{\mathsf{a},\underline{k}} \in \mathbb{R}^n$. Thus we have:

$$D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) = \lambda_{\mathsf{a}} \bullet x + \sum_{|\underline{k}|=j} \lambda_{\mathsf{a}} \bullet q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j) \tag{3.6}$$

$$
\begin{aligned}
(3.7) \qquad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) &= f_{\mathsf{a}}(x) + \sum_{|\underline{k}|=j} q_{\sigma(\mathsf{a}),\underline{k}} \cdot \frac{(\lambda_{\mathsf{a}} \bullet x)^{\underline{k}}}{\underline{k}!} + o(x^j) \\
&= f_{\mathsf{a}}(x) + \sum_{|\underline{k}|=j} q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j) \\
&= \lambda_{\mathsf{a}} \bullet x + \sum_{|\underline{k}|=j} \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + \sum_{|\underline{k}|=j} q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j) \ .
\end{aligned}
$$

Then subtracting Eq. (3.6) from Eq. (3.7) we obtain:

$$
h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) - D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) = \sum_{|\underline{k}|=j} \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + \sum_{|\underline{k}|=j} q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} - \sum_{|\underline{k}|=j} \lambda_{\mathsf{a}} \bullet q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} + o(x^j) \ .
$$

Hence to prove the lemma, it suffices to show the existence of an C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ such that:

$$
(3.8) \qquad \sum_{|\underline{k}|=j} \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + \sum_{|\underline{k}|=j} q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} = \sum_{|\underline{k}|=j} \lambda_{\mathsf{a}} \bullet q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} \ .
$$

Then with $\lambda_{\mathsf{a}}^{-1} = (\lambda_{\mathsf{a},i}^{-1})_{1\le i\le n}$ it holds:

$$
\sum_{|\underline{k}|=j} \lambda_{\mathsf{a}}^{-1} \bullet \partial^{\underline{k}} f_{\mathsf{a}}(0) \cdot \frac{x^{\underline{k}}}{\underline{k}!} + \sum_{|\underline{k}|=j} \lambda_{\mathsf{a}}^{-1} \bullet q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} = \sum_{|\underline{k}|=j} q_{\mathsf{a},\underline{k}} \cdot \frac{x^{\underline{k}}}{\underline{k}!} \ .
$$

Comparing coefficients in $x^{\underline{k}}$, it is enough to find functions $q_{\mathsf{a},\underline{k}} \in C^0(\mathsf{A}, \mathbb{R}^n)$ for every $|\underline{k}| = j$ with

$$
(3.9) \qquad \lambda_{\mathsf{a}}^{-1} \bullet \partial^{\underline{k}} f_{\mathsf{a}}(0) + \lambda_{\mathsf{a}}^{-1} \bullet q_{\sigma(\mathsf{a}),\underline{k}} \cdot \lambda_{\mathsf{a}}^{\underline{k}} = q_{\mathsf{a},\underline{k}} \ .
$$

Writing $q_{\mathsf{a},\underline{k}}$ in components by $q_{\mathsf{a},\underline{k}} = (q_{\mathsf{a},\underline{k},1}, \dots, q_{\mathsf{a},\underline{k},n})$, and similarly for f, this means we would like to find functions $q_{\mathsf{a},\underline{k},i} \in C^0(\mathsf{A}, \mathbb{R})$ for every $|\underline{k}| = j, 1 \le i \le n$ with

$$
\lambda_{\mathsf{a},i}^{-1} \cdot \partial^{\underline{k}} f_{\mathsf{a},i}(0) + \lambda_{\mathsf{a},i}^{-1} \cdot q_{\sigma(\mathsf{a}),\underline{k},i} \cdot \lambda_{\mathsf{a}}^{\underline{k}} = q_{\mathsf{a},\underline{k},i} \ .
$$

By (H_3) we know that we have $|\lambda_{\mathsf{a},i}^{-1}| \cdot |\lambda_{\mathsf{a}}^{\underline{k}}|$ is either smaller than 1 for all $\mathsf{a} \in A$ or larger than 1 for all $\mathsf{a} \in A$. So we can consider the operator

$$
O_{\underline{k},i} : (q_{\mathsf{a},\underline{k},i})_{\mathsf{a}\in\mathsf{A}} \mapsto \begin{cases} \lambda_{\mathsf{a},i}^{-1} \cdot (q_{\sigma(\mathsf{a}),\underline{k},i} \cdot \lambda_{\mathsf{a}}^{\underline{k}} + \partial^{\underline{k}} f_{\mathsf{a},i}(0)) & \text{if } |\lambda_{\mathsf{a},i}^{-1}| \cdot |\lambda_{\mathsf{a}}^{\underline{k}}| < 1 \\ (\lambda_{\sigma^{-1}(\mathsf{a})}^{\underline{k}})^{-1} \cdot (\lambda_{\sigma^{-1}(\mathsf{a}),i} \cdot q_{\sigma^{-1}(\mathsf{a}),\underline{k},i} - \partial^{\underline{k}} f_{\sigma^{-1}(\mathsf{a}),i}(0)) & \text{otherwise.} \end{cases}
$$

And we let $O_{\underline{k}}$ the operator that is componentwise given by the operators $O_{i,\underline{k}}$. Now $O_{\underline{k}}$ is an affine operator with a contracting linear part by (H_3) and compactness of A. So there is a unique fixed point for $O_{\underline{k}}$, and the fixed point is a solution for (3.9). Doing this for every $|\underline{k}| = r$, we define the (unique) family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ satisfying Eq. (3.8). The uniqueness of $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}} \mod o(x^r)$ follows from the uniqueness of the fixed point of a contracting map. □

Remark 3.4. In fact, $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ depends continuously on f: since composition of r-jets is continuous, it only remains to show that the $q_{\mathsf{a},\underline{k}}$ depend continuously on f. For this we note that the operators $O_{\underline{k},i}$ depend continuously on f, by Lemma A.1, their fixed points depend continuously on f and so does $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$.

3.2 Linearization of flat contractions

We prove below the following finite regularity version of Lemma 3.2:

Lemma 3.5. *For $r \geq 2$, if $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ is a C^0-family of C^r-maps satisfying $(H_1 - H_2 - H_3 - H_4^r)$ and furthermore f satisfies*

$$f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}}(x) + o(x^r) \ .$$

then there exists $\delta > 0$, such that there is a unique C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of C^r-diffeomorphisms h_{a} from the δ-ball around 0 into $\mathbb{R}^n$ satisfying for following conditions for every $\mathsf{a} \in \mathsf{A}$:

$$h_{\mathsf{a}}(0) = 0 \ , \quad h_{\mathsf{a}}(x) = x + o(x^r), \quad \text{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) \ . \tag{3.10}$$

Proof. Let us denote the δ-ball around 0 by $\mathbb{B}_\delta$. Consider the vector space W_δ of C^0-families $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of C^r-maps from $\mathbb{B}_\delta$ into $\mathbb{R}^n$. By (H_1), we have that $f_{\mathsf{a}}(\mathbb{B}_\delta) \subset \mathbb{B}_\delta$ for $\delta \in (0,1)$. Hence the operator

$$\begin{aligned} T_f \colon W_\delta &\to W_\delta \\ (h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}} &\mapsto (D_0 f_{\mathsf{a}})^{-1} \circ h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}} \end{aligned}$$

is well-defined. It is obviously linear. Let V_δ be the linear subspace of C^0-families of C^r-maps on $\mathbb{B}_\delta$ that satisfy $h_{\mathsf{a}}(x) = o(x^r)$ for all $\mathsf{a} \in \mathsf{A}$. Then T_f leaves V_δ invariant as we assumed that f is r-flat.

We consider on V_δ the norm given by

$$\|g\|_V = \|D^r g\|_0 = \sup_{x\in\mathbb{B}_\delta} \|D_x^r g\| \ . \tag{3.11}$$

Here and in the following we use $\|\cdot\|_0$ for the uniform C^0 norm on $\mathbb{B}_\delta$. By Taylor–Lagrange expansion around 0, we see that for $k < r$:

$$\left\| D_x^k g \right\| \leq \delta \cdot \|g\|_V \ , \quad \forall x \in \mathbb{B}_\delta \ . \tag{3.12}$$

This implies that $(V_\delta, \|\cdot\|_V)$ is a Banach space. Let us denote by $\iota \in W_\delta$ the constant family of the canonical inclusion $\mathbb{B}_\delta \subset \mathbb{R}^n$, so $\iota_{\mathsf{a}}(x) = x$ for all $\mathsf{a} \in \mathsf{A}, x \in \mathbb{B}_\delta$. Note that T_f leaves the affine subspace $\iota + V_\delta$ invariant, as $T_f(\iota) \in \iota + V_\delta$ by r-flatness of f. We endow $\iota + V_\delta$ with the distance induced by the norm of V_δ. We will show that, if δ is small enough, then T_f is a contraction on the affine space $\iota + V_\delta$. The resulting fixed point is then a solution for our conjugacy problem.

For every a, let $\mu_{\mathsf{a}} := \min_i |\lambda_{\mathsf{a},i}|$ and $\Lambda_{\mathsf{a}} := \max_i |\lambda_{\mathsf{a},i}|$. By (H_4^r), it holds:

$$\mu_{\mathsf{a}} > \Lambda_{\mathsf{a}}^r \ .$$

Note that $\|D_0 f_{\mathsf{a}}\| \leq \Lambda_{\mathsf{a}}$. By compactness of A, there is $C_1 \in (0,1)$ such that:

$$C_1 \cdot \mu_{\mathsf{a}} > \Lambda_{\mathsf{a}}^r \ .$$

Thus for every $C \in (C_1, 1)$, there exists $\delta' > 0$ such that

$$C \cdot \mu_{\mathsf{a}} > \|D_x f_{\mathsf{a}}\|^r \ , \quad \forall x \in \mathbb{B}_{\delta'} \ .$$

Thus we have:

$$C > \mu_{\mathsf{a}}^{-1} \cdot \|D_x f_{\mathsf{a}}\|^r \ , \quad \forall x \in \mathbb{B}_{\delta'} \ . \tag{3.13}$$

Now we consider the Faà di Bruno's formula for D^r:

$$D^r(h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}) = D^r_{f_{\mathsf{a}}} h_{\sigma(\mathsf{a})}(Df_{\mathsf{a}})^{\otimes r} + \sum_{k<r} D^k_{f_{\mathsf{a}}} h_{\sigma(\mathsf{a})}(P_{\mathsf{a},k})$$

where $P_{\mathsf{a},k}$ is a k-tensor whose coefficients are polynomial functions of the $r-1$ one first derivatives of f. Hence we can bound these polynomials by $\mu_{\mathsf{a}} \cdot M$ for a certain $M > 0$. This gives:

$$\|D^r(h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}})\|_0 \ \leq \|D^r_{f_{\mathsf{a}}} h_{\sigma(\mathsf{a})}(Df_{\mathsf{a}})^{\otimes r}\|_0 + \mu_{\mathsf{a}} \cdot M \sum_{k<r} \|D^k h_{\sigma(\mathsf{a})}\|_0 \ .$$

Consequently on $\mathbb{B}_\delta$ with $\delta < \delta'$ we have:

$$\|D^r T_f(h)_{\mathsf{a}}\|_0 \ \leq \mu_{\mathsf{a}}^{-1} \|Df_{\mathsf{a}}|\mathbb{B}_{\delta'} \ \|_0^r \cdot \|D^r(h_{\sigma(\mathsf{a})})\|_0 + M \sum_{k<r} \|D^k h_{\sigma(\mathsf{a})}\|_0 \ .$$

So by (3.12) we have

$$\|D^r T_f(h)_{\mathsf{a}}\|_0 \leq \ (\mu_{\mathsf{a}}^{-1} \|Df_{\mathsf{a}}|\mathbb{B}_{\delta'} \ \|_0^r + r\delta M) \cdot \|D^r(h_{\sigma(\mathsf{a})})\|_0$$

or

$$\|T_f(h)\|_V \leq \ (C + r\delta M) \cdot \|h\|_V \ . \tag{3.14}$$

Hence if δ is small enough such that $C + r\delta M < 1$, we have that T_f is contracting on V_δ.

Hence for δ small enough, we have a unique solution in W_δ for the conclusion of our Lemma. By compactness of A and the local inverse function theorem, we can assure that the solution is a family of C^r-diffeomorphisms by restricting to a smaller δ.

□

Remark 3.6. The operator T_f and the constants C, δ, M depend continuously on f. So by Lemma A.1, the map h depends continuously on f.

We are now ready to give proof of our main statement in the setting of finite regularity.

Proof of Corollary B. This is a direct composition of Lemma 3.3 and Lemma 3.5 and the extension argument we used in the proof of the parameterless version of Theorem A. The continuity follows from Remark 3.4 and Remark 3.6. □

Let us now mention how to obtain a proof of Theorem A without parameters from Corollary B.

Proof of Theorem A without parameters from Corollary B. By Remark 2.1 there is an r such that f satisfies (H_4^r). Then f satisfies (H_4^q) for every $q \geq r$. So for every $q \geq r$, we get by Corollary B a family of C^q-diffeomorphisms $(h_{q,\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ satisfying (2.2). By uniqueness for $(h_{r,\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ the families must all agree. In particular, $(h_{r,\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ is in fact smooth. □

4 Parameter dependence and proof of the application

In this section, we specify the parameter dependence in the tame smooth and analytic setting. We show how to obtain the parametric version of our application from the parametric version of our main theorem.

4.1 Background on tame smooth maps

Let us now specify the norm on the spaces that we consider:

Definition 4.1. *For any $r < \infty$, we endow the space of maps $C^r(\mathbb{B}, \mathbb{R}^n)$ with the norm $\|h\|_r = \max_{0\le i\le r} \sup_{x\in\mathbb{B}} \|D^i_x h\|$, and the space $C^0(\mathsf{A}, C^r(\mathbb{B}, \mathbb{R}^n))$ with the norm $\|(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}\|_r = \max_{\mathsf{a}\in\mathsf{A}} \|h_{\mathsf{a}}\|_r$.*

The spaces $C^r(\mathbb{B}, \mathbb{R}^n)$ and $C^0(\mathsf{A}, C^r(\mathbb{B}, \mathbb{R}^n))$ endowed with these norms are Banach spaces.

Definition 4.2. *We endow the spaces $C^\infty(\mathbb{B}, \mathbb{R}^n)$ and $C^0(\mathsf{A}, C^\infty(\mathbb{B}, \mathbb{R}^n))$ with the family of norms $(\|\cdot\|_r)_{r\ge 0}$.*

The spaces $C^\infty(\mathbb{B}, \mathbb{R}^n)$ and $C^0(\mathsf{A}, C^\infty(\mathbb{B}, \mathbb{R}^n))$ endowed with the given families of norms are graded Fréchet spaces:

Definition 4.3. *A* **graded Fréchet space** *F is a topological vector space endowed with an increasing family of norms $(\|\cdot\|_r)_{r\ge 0}$ generating the topology of F, such that every sequence that is Cauchy with respect to all norms $(\|\cdot\|_r)_{r\ge 0}$ has a limit. If F and G are graded Fréchet spaces, then their direct product $F \times G$ is also a graded Fréchet space endowed with the family of norms $\|(f, g)\|_r = \|f\|_r + \|g\|_r$.*

Example 4.4. Every Banach space $\mathcal{B}$ can be considered as a graded Fréchet space with the constant family of norms $\|\cdot\|_r = \|\cdot\|_{\mathcal{B}}$.

Definition 4.5. *Let F, G be graded Fréchet spaces, $W \subset F$ be open. Let $\psi\colon W \to G$ be a continuous map. We say that ψ is C^1 if there is a continuous function $D\psi\colon W \times F \to G$ such that for every $w \in W, h \in F$, we have that*

$$\lim_{t\to 0} \frac{\psi(w + th) - \psi(w)}{t} = D\psi(w, h) \ .$$

We say that ψ is C^{r+1} if $D\psi$ is C^r. We say that ψ is **smooth** *if ψ is C^r for every finite r.*

We say that ψ is **tame** *if for every $w \in W$, there is a neighborhood $W' \subset W$ and $d, b \in \mathbb{N}$ such that for every $r \ge b$ there exists $C_r > 0$ satisfying $\|\psi(\tilde{w})\|_r \le C_r(1 + \|\tilde{w}\|_{r+d})$ for all $\tilde{w} \in W'$.*

We say that ψ is **smooth tame** *if ψ is smooth and $D^r\psi$ is tame for all $r < \infty$.*

Example 4.6. Let F be a graded Fréchet space, $W \subset F$ be open and $\mathcal{B}$ be a Banach space. Then every continuous map $\psi\colon W \to \mathcal{B}$ from an open subset W is tame (see [Ham82, Example II 2.1.4]). In particular, if $\psi\colon W \to \mathcal{B}$ is smooth, then every derivative $D^r\psi$ is a continuous map into $\mathcal{B}$, so it is tame as well. So smooth maps into Banach spaces are always smooth tame.

The following interpolation inequality is useful in establishing tameness:

Lemma 4.7. *There exists a family of positive constants $(I_j)_{j\ge 1}$ such that for every $h \in C^\infty(\mathbb{B}, \mathbb{R}^n)$ we have*

$$\|h\|_k \le I_j \|h\|_j^{\frac{j-k}{j-1}} |h\|_j^{\frac{k-1}{j-1}} \quad \forall 1 \le k \le j. \tag{4.1}$$

For a proof see [Ham82, Theorem II.2.2.1].

Lemma 4.8. *The following map is smooth tame:*

$$\begin{aligned}\Upsilon\colon C^\infty(\mathbb{B},\mathbb{R}^n)\times C^\infty(\mathbb{B},\mathbb{B}) &\to C^\infty(\mathbb{B},\mathbb{R}^n)\\ (h,f) &\mapsto (h\circ f)\ .\end{aligned}$$

Proof. We sketch the key argument of [Ham82, Lemma II.2.3.4] that the map is tame. In order to show that Υ is tame in $(h_0,f_0)\in C^\infty(\mathbb{B},\mathbb{R}^n)\times C^\infty(\mathbb{B},\mathbb{B})$, we can work in a C^1-neighborhood $\mathcal{W}$ of (h_0,f_0) where $\|h\|_1$ and $\|f\|_1$ are uniformly bounded in $\mathcal{W}$. We can bound $\|h\circ f\|_0\le\|h\|_0\le\|h\|_1$. So in order to show tameness, it suffices to bound $\|D^j(h\circ f)\|_0$ for $j\ge 1$ in terms of $C_j(\|h\|_j+\|f\|_j)$ for some constant $C_j>0$.

We will again use Faà di Bruno's formula, but now in a more explicit fashion: there are constants $c_{j,k,i_1,\dots,i_k}\in\mathbb{Z}$ such that

$$D^j(h\circ f)=\sum_{k\le j}\sum_{i_1+\dots+i_k=j} c_{j,k,i_1,\dots,i_k}D^k_f h\left(D^{i_1}f\otimes\dots\otimes D^{i_k}f\right)\ . \tag{4.2}$$

(with $c_{j,j,1,\dots,1}=1$). By Lemma 4.7 and the fact that $\|h\|_1$ and $\|f\|_1$ are uniformly bounded for $(h,f)\in\mathcal{W}$, there exists constants $I'_j>0$ (depending on $\mathcal{W}$, but not on $(h,f)\in\mathcal{W}$) such that

$$\|h\|_k\le I'_j\|h\|_j^{\frac{k-1}{j-1}}\ ,\quad \|f\|_k\le I'_j\|f\|_j^{\frac{k-1}{j-1}}\quad \forall 1\le k\le j,(h,f)\in\mathcal{W}\ . \tag{4.3}$$

This allows to bound the summands via

$$\|D^k_f h\left(D^{i_1}f\otimes\dots\otimes D^{i_k}f\right)\|\le I_j'^{k+1}\|h\|_j^{\frac{k-1}{j-1}}\|f\|_j^{\frac{j-k}{j-1}}\le I_j'^{k+1}(\|h\|_j+\|f\|_j)\ , \tag{4.4}$$

using for the second step the coarse inequality

$$x^t y^{1-t}\le\max(x,y)\le x+y\quad\forall x,y\ge 0,t\in[0,1]\ . \tag{4.5}$$

So we can bound every term of the right hand side of (4.2) by some constant times $(\|h\|_j+\|f\|_j)$. This shows that Υ is tame.

To show that Υ is smooth tame, we observe that it is linear in h and its derivative w.r.t. is a a polynomial of derivatives of h composed with f and f, and so a composition of tame smooth operations. □

4.2 Smooth dependence of application

We would like to give a parametric version of Corollary D.

Corollary E. *Let K,g as in Corollary D. Then φ depends tamely smooth on g. More precisely, there exists a neighborhood V of K, a trivialization $TM|V\cong V\times\mathbb{R}^n\subset\mathbb{R}^n\times\mathbb{R}^n$ and a C^∞-neighborhood $\mathcal{U}$ of g such that $\varphi=\varphi_g$ given by Corollary D extends to a map in $C^0(\overleftarrow{K},C^\infty(\mathbb{B},V))\subset C^0(\overleftarrow{K},C^\infty(\mathbb{B},\mathbb{R}^n))$ and depends tamely smooth on $g\in U$.*

Note that the statement is slightly different to Corollary D as we linearize g by a linear map on $\mathbb{B}$. This has the advantage that the statement is simpler for the perturbation.

We begin our discussion by showing smooth dependence of hyperbolic continuation:

Lemma 4.9. *Let K be an expanding compact set for a C^1-map $g\colon M \to M$. Then there exists a C^1-neighborhood $\tilde{\mathcal{U}}$ of g such that the hyperbolic continuation $\psi_{\tilde{g}}\colon K \to M$ depends tame smoothly on $\tilde{g} \in \tilde{\mathcal{U}}$.*

Proof. We can find an ϵ small and a C^1-neighborhood $\tilde{\mathcal{U}}$ of g enough such that the following holds:

- The Riemannian exponential map $\exp_x(v)$ is defined for $x \in K, v \in T_xM, \|v\| < \epsilon$ and the mapping $\exp_x$ from the ϵ-ball in T_xM to the ϵ-ball $B_\epsilon(x)$ is $(1+\epsilon)$ bi-Lipschitz.
- For every $\tilde{g} \in \mathcal{U}$, $k \in K$, $\tilde{g}$ restricts to an $(1+\epsilon)^3$ expanding diffeomorphism on $B_\epsilon(k)$ with image containing $B_\epsilon(g(k))$.

Under this assumptions, we can consider the following space of functions: let $\Gamma^0(K, TM|K)$ be the space of sections of the tangent vector bundle TM over K. This is a Banach space with the uniform C^0-norm $\|s\| = \sup_{k\in K} \|s(k)\|_{T_kM}$. Let $\mathcal{V}$ be the ϵ-ball in this Banach space.

Given $\tilde{g} \in \tilde{\mathcal{U}}$, we can consider the operator

$$\begin{aligned} S_{\tilde{g}}\colon \mathcal{V} &\to \mathcal{V} \\ (\psi(k))_{k\in K} &\mapsto \left(\exp_k^{-1} \circ (\tilde{g}|B_\epsilon(g(k)))^{-1} \circ \exp_{g(k)} \psi(g(k))\right) . \end{aligned}$$

By our assumptions, this operator is well-defined and $(1+\epsilon)^{-1}$ contracting on $\mathcal{V}$. For g we have that the zero section $s_0\colon k \mapsto 0$ is a fixed point. By restricting $\tilde{\mathcal{U}}$ further, we can assume that $\|S_{\tilde{g}}(s_0)\| < \frac{\epsilon}{1+\epsilon}$. Using the Banach fixed point theorem, we obtain that $S_{\tilde{g}}$ has a unique fixed point in $\mathcal{V}$. As $S_{\tilde{g}}$ is smooth, we get by Lemma A.1 that the fixed point depends smoothly on $\tilde{g}$. From this we get that the hyperbolic continuation depends smoothly on $\tilde{g}$. By Example 4.6, the dependence is tame smooth. □

Lemma 4.10. *Let K be an totally projectively hyperbolic expanding Cantor set for a C^1-map $g\colon M \to M$ with splitting $F_1 \oplus \cdots \oplus F_n$ over $\overleftarrow{K}$. Then there exists a C^1-neighborhood $\tilde{\mathcal{U}}$ such that the hyperbolic continuation of the F_i depends tame smoothly on $\tilde{g} \in \tilde{\mathcal{U}}$.*

Proof. We do this in two steps: we show that we can realize E_i and $G_i := F_i \oplus \cdots \oplus F_n$ as fixed points of contracting maps and thus admit a continuation in a small neighborhood of g. Then we continue $F_i = E_i \cap G_i$. Note while E_i is defined over K, G_i is like F_i defined over $\overleftarrow{K}$.

We recall some geometry related to the Grassmannian. We denote by $Gr_{n,i}$ the set of i-dimensional linear subspaces of $\mathbb{R}^n$. This is a smooth manifold, if we are given $E \subset \mathbb{R}^n$ i dimensional and a complement F $n-i$ dimensional, we have charts by graph transforms

$$\begin{aligned} \xi_{E,F}\colon \operatorname{Lin}(E,F) &\to Gr_{n,i} \\ s &\mapsto \{(e, s(e))\colon e \in E\} . \end{aligned}$$

For convenience, we fix a smooth trivialization of $TM|V$ for a neighborhood V of K. In particular, for $\tilde{g}$ close to g, $v \in V$ with $\tilde{g}(v) \in V$, we have $D_v\tilde{g}$ induces a diffeomorphism $(D_v\tilde{g})_*$ on $Gr_{n,i}$.

In particular, for $\tilde{g}$ close to g, we have the following map:

$$\begin{aligned} SE_{\tilde{g}}\colon C^0(K, Gr_{n,i}) &\to C^0(K, Gr_{n,i}) \\ (s(k))_{k\in K} &\mapsto \left((D_{\psi_{\tilde{g}}(k)}\tilde{g})_*^{-1} s(g(k))\right) . \end{aligned}$$

We have that E_i is a fixed point of SE_g. We want to show that $SE_{\tilde{g}}$ has a fixed point close to E_i. For this we will use charts depending on K as follows:

For $k \in K$, we can consider the orthogonal complement of E_i in TM_k (with the Riemannian metric of TM_k), we denote this by $E_i^\perp$. Note that $\mathrm{Lin}(E_i, E_i^\perp)$ is a normed vector bundle over K. We can consider the space of sections $\Gamma^0(K, \mathrm{Lin}(E_i, E_i^\perp))$. We have an open embedding $\Xi\colon \Gamma^0(K, \mathrm{Lin}(E_i, E_i^\perp)) \hookrightarrow C^0(K, Gr_{n,i})$, sending the zero section to E_i.

We claim that there is an open neighborhood $W \subset \Gamma^0(K, \mathrm{Lin}(E_i, E_i^\perp))$ of the zero section such that after restricting $\tilde{\mathcal{U}}$, the composition $(\Xi^{-1} \circ SE_{\tilde{g}} \circ \Xi)|W$ is well-defined and contracting: for g, the existence of such a W is equivalent to cone conditions. This open set also works for a further restriction of $\tilde{\mathcal{U}}$.

Now $\Xi^{-1} \circ SE_{\tilde{g}} \circ \Xi$ depends smoothly on $\tilde{g}$, so we can use Lemma A.1 to get that the resulting fixed point depends smoothly on $\tilde{g}$. So we have constructed a continuation of $E_i(k)$.

Very similarly, we can construct a continuation of G_i. We can consider the operator

$$\begin{aligned} SG_{\tilde{g}}\colon C^0(\overleftarrow{K}, Gr_{n,n-i+1}) &\to C^0(\overleftarrow{K}, Gr_{n,n-i+1}) \\ (s(\mathsf{a}))_{\mathsf{a}\in\overleftarrow{K}} &\mapsto \Big((D_{\psi_{\tilde{g}}(\mathsf{a}_{-1})}\tilde{g})_* s(\sigma(\mathsf{a}))\Big) \end{aligned}$$

such that a fixed point of $SG_{\tilde{g}}$ close to G_i is the continuation of G_i. Now $G_i, G_i^\perp, \mathrm{Lin}(G_i, G_i^\perp)$ are vector bundles over $\overleftarrow{K}$. As for E_i, we can see in the "chart" of sections $\Gamma^0(\overleftarrow{K}, \mathrm{Lin}(G_i, G_i^\perp))$ that G_i is the fixed point of a contracting map that can be continued smoothly.

For g, we know that $G_i(\mathsf{a})$ has transverse intersection with $E_i(\mathsf{a}_0)$ for all $\mathsf{a} \in \overleftarrow{K}$. Restricting $\tilde{\mathcal{U}}$ further, we can assume that the continuation of $G_i(\mathsf{a})$ still has transverse intersection with the continuation of $E_i(\mathsf{a}_0)$, so we obtain the continuation of $F_i(\mathsf{a}) = G_i(\mathsf{a}) \cap E_i(\mathsf{a}_0)$. Since the continuations of G_i and E_i depend smoothly on $\tilde{g}$, so does the continuation of F_i. By Example 4.6, F_i depends tame smooth on $\tilde{g}$. □

Proof of Corollary E. Using the trivialization $TM|V \cong V \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$, let us make the construction in the proof of Corollary D more explicit: the splitting $F_1 \oplus \ldots F_n \to \overleftarrow{K}$ is trivial, so there is are continuous unit vector sections u_i of $F_i \to \overleftarrow{K}$. In particular, for $\mathsf{a} \in \overleftarrow{K}$, there is a unique linear automorphism L_a of $\mathbb{R}^n$ sending the canonical basis vector e_i to u_i. We can then set

$$\Phi_\mathsf{a}(x) = \mathsf{a}_0 + L_\mathsf{a}(x)\ .$$

Continuing with the rest of the proof of Corollary D, we find a $s > 0$ sufficiently small and such that with

$$\psi_\mathsf{a}(x) = \Phi_\mathsf{a}(s \cdot x) = \mathsf{a}_0 + sL_\mathsf{a}(x)$$

we have that

$$f_\mathsf{a} := (\psi_\mathsf{a}^{-1} \circ g \circ \psi_{\sigma(\mathsf{a})})^{-1}|\mathbb{B}$$

is well-defined and satisfies hypotheses $(H_1 - H_2 - H_3)$ of Theorem A.

For a small enough C^1-neighborhood of g, we can continue this construction: By Lemma 4.9 and Lemma 4.10, the line bundle F_i depends tame smoothly on $\tilde{g}$ in some neighborhood of g. So u_i, L_a, Φ_a and finally ψ_a depend tame smoothly on $\tilde{g}$. For a small enough neighborhood of g we have then that the construction of f_a can be continued and depends tamely smooth on $\tilde{g}$. By Theorem A, h depends tamely smooth on $\tilde{g}$.

We can now set $\varphi_\mathsf{a} = \psi_\mathsf{a} \circ h_\mathsf{a}^{-1}$ that depends smoothly on $\tilde{g}$.

□

4.3 Background on holomorphic maps on Banach spaces

Let X, Y be complex Banach spaces, let $U \subset X$ be an open subset. Let us recall that a continuous map $f\colon U \to X$ is **holomorphic** if for every $u \in U$, the map f is complex Fréchet differentiable in u, that is, there is a complex linear map $D_u f\colon X \to Y$ with

$$\lim_{v \to 0} \frac{\|f(u+v) - f(u) - D_u f(v)\|}{|v|} = 0 \ . \tag{4.6}$$

We will state a composition lemma for compositions of holomorphic functions. As above, let $\tilde{\mathbb{B}}$ be the open unit ball in $\mathbb{C}^n$ centered at 0. Let $\mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n)$ be the space of bounded holomorphic functions from $\tilde{\mathbb{B}}$ to $\mathbb{C}^n$. Endowed with the supremum norm, this is a complex Banach space. Let $\mathcal{H}_{cc}(\tilde{\mathbb{B}}, \tilde{\mathbb{B}}) \subset \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n)$ be the open ball of radius 1, so the set of functions from $\tilde{\mathbb{B}}$ into a compact subball of $\tilde{\mathbb{B}}$.

Lemma 4.11. *The map*

$$\begin{aligned} C\colon \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n) \times \mathcal{H}_{cc}(\tilde{\mathbb{B}}, \tilde{\mathbb{B}}) &\to \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n) \\ (h, f) &\mapsto h \circ f \end{aligned}$$

is holomorphic.

Proof. It is clear that this map is well-defined and continuous. The map is linear in h. It is enough to check that the map is holomorphic in f. Let $\epsilon > 0$ be such that $f(\tilde{\mathbb{B}}) + \tilde{\mathbb{B}}_{2\epsilon} \subset \tilde{\mathbb{B}}$. We claim that for a pair $(f, h) \in \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n) \times \mathcal{H}_{cc}(\tilde{\mathbb{B}}, \tilde{\mathbb{B}})$ the Fréchet differential in f is given by

$$\begin{aligned} L_{f,h}\colon \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n) &\to \mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n) \\ g(z) &\mapsto D_{f(z)} h(g(z)) \ . \end{aligned}$$

First of all, we can apply the Cauchy integral formula to see that Dh is bounded on $f(\tilde{\mathbb{B}})$ by $\epsilon^{-2}\|h\|$. So $L_{f,h}$ is indeed a bounded operator on $\mathcal{H}_\infty(\tilde{\mathbb{B}}, \mathbb{C}^n)$.

If $\|g\| < \epsilon$, we have by the integral Taylor–Lagrange expansion in h that

$$\|h \circ (f+g)(z) - (h \circ f)(z) - D_{f(z)} h(g(z))\| \leq \frac{1}{2}\|g\|^2 \sup_{f(\tilde{\mathbb{B}}) + \tilde{\mathbb{B}}_\epsilon} \|D^2 h\| \tag{4.7}$$

But by the Cauchy integral formula, we can bound $\sup_{f(\tilde{\mathbb{B}}) + \tilde{\mathbb{B}}_\epsilon} \|D^2 h\|$ by $\epsilon^{-3}\|h\|$. This shows that $L_{f,h}$ is indeed the Fréchet differential. It is complex linear, so the map C is holomorphic. □

4.4 Holomorphic application

Let us also mention the holomorphic setting of our application.

For this, let now M be a complex n dimensional manifold and g a holomorphic self-map of M. We say that an expanding compact set K of g is **complex totally projectively hyperbolic** if there exists a flag of invariant complex sub-bundles:

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = TM|K$$

such that for every $1 \leq i < n$, the set K is **projectively hyperbolic** at E_i.

In this setting, we obtain a splitting $F_{1\mathsf{a}} \oplus F_{2\mathsf{a}} \oplus \cdots \oplus F_{n\mathsf{a}}$ of **complex** line bundles depending continuously on $\mathsf{a} \in \overleftarrow{K}$.

Corollary F. *Let K be an expanding Cantor set for a holomorphic self-map g of M. Assume that:*

(a) the set K is complex totally projectively hyperbolic with splitting $F_1 \oplus F_2 \oplus \cdots \oplus F_n \to \overleftarrow{K}$.

(b) for any $1 \leq i \leq n$ and any multiindex $\underline{k} = (k_1, \ldots, k_n)$ with $|\underline{k}| \neq 1$, we have:

$$\|D_{\mathsf{a}_0} g|F_i\| \neq \prod_{\ell=1}^{n} \|D_{\mathsf{a}_0} g|F_\ell\|^{k_\ell}$$

and the sign of the difference does not depends on $\mathsf{a} \in \overleftarrow{K}$.

Then for $r > 0$ sufficiently small, there is a unique continuous family $(\phi_{\mathsf{a}})_{\mathsf{a} \in \overleftarrow{K}}$ of biholomorphic charts ϕ_{a} from the closed r-ball $B_{a_0}(r)$ of $T_{\mathsf{a}_0} M$ onto a neighborhood of $a_0 \in M$ such that:

$$\varphi_{\mathsf{a}}(0) = \mathsf{a}_0, \quad D_0 \varphi_{\mathsf{a}} = \mathrm{id} \quad \textit{and} \quad \varphi_{\overleftarrow{g}(\mathsf{a})} \circ D_{\mathsf{a}_0} g = g \circ \varphi_{\mathsf{a}} \quad \textit{on } B_{\mathsf{a}_0}(r) \ . \tag{4.8}$$

Moreover the same holds true for the hyperbolic continuation of K after holomorphic perturbation of g, and the linearization depends homomorphically on the perturbation.

The proof of the corollary is the same as the proof of Corollary D, but using Theorem C instead of the main theorem.

Remark 4.12. If M has a real structure given by an antiholomorphic involution $\xi_M \colon M \to M$, and g commutes with ξ_M and ξ_M fixes K, then $\xi_M \circ \varphi_{\mathsf{a}} \circ \xi_{TM}$ also satisfies (4.8):

$$\xi_M \circ \varphi_{\overleftarrow{g}(\mathsf{a})} \circ \xi_{TM} \circ D_{\mathsf{a}_0} g = \xi_M \circ \varphi_{\overleftarrow{g}(\mathsf{a})} \circ D_{\mathsf{a}_0} g \circ \xi_{TM} = \xi_M \circ g \circ \varphi_{\mathsf{a}} \circ \xi_T M = g \circ \xi_M \circ \varphi_{\mathsf{a}} \circ \xi_{TM}. \tag{4.9}$$

So by uniquness of $(\varphi_{\mathsf{a}})_{\mathsf{a} \in \mathsf{A}}$, we obtain $\xi_M \circ \varphi_{\mathsf{a}} \circ \xi_{TM} = \varphi_{\mathsf{a}}$ From this, we recover real analytic dependence for real analytic maps as promised in Remark 2.2.

5 Proof of the parameter dependence of main theorems

We now give the proofs of Theorem A and Theorem C with full parameter dependence. For Theorem C, we work with appropriate Banach spaces of holomorphic maps, while for Theorem A we have to work with Fréchet spaces. Since the proof of Theorem C is closer to the one given in Section 3, we begin with the holomorphic setting.

5.1 Holomorphic version of main theorem

Here is the holomorphic counterpart of Lemma 3.1:

Lemma 5.1. *For $r \geq 1$ and every continuous family of holomorphic maps $(f_{\mathsf{a}})_{\mathsf{a} \in \mathsf{A}}$ satisfying $(H_1 - H_2 - H_3)$, there exists a unique C^0-family of polynomial maps $(h_{\mathsf{a}})_{\mathsf{a} \in \mathsf{A}}$ of $\mathbb{C}^n$ with degree $\leq r$ such that for every $\mathsf{a} \in \mathsf{A}$:*

$$h_{\mathsf{a}}(0) \ , \quad D_0 h_{\mathsf{a}} = \mathrm{id} \ , \quad \textit{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) + o(x^r) \ . \tag{5.1}$$

Moreover (h_{a}) depends holomorphically on f.

Proof. The proof is the holomorphic analogue to the proof of Lemma 3.3, invoking the holomorphic case of Lemma A.1 for holomorphic dependence. □

Let us now show the holomorphic counterpart of Lemma 3.5:

Lemma 5.2. *If $(f_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ is a C^0-family of holomorphic maps maps satisfying $(H_1 - H_2 - H_3 - H_4^r)$ and furthermore f satisfies*

$$f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}}(x) + o(x^r) \ .$$

then there exists $\delta > 0$, such that there is a unique C^0-family $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of biholomorphisms h_{a} on the open δ-ball around 0 in $\mathbb{C}^n$ satisfying for following conditions for every $\mathsf{a} \in \mathsf{A}$:

$$h_{\mathsf{a}}(0) = 0 \ , \quad h(x) = x + o(x^r) \ , \quad \text{and} \quad h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}}(x) = D_0 f_{\mathsf{a}} \circ h_{\mathsf{a}}(x) \ . \tag{5.2}$$

Moreover, h depends holomorphically on f.

Proof. We will adjust the proof of Lemma 3.5 by choosing the right spaces of holomorphic maps. We denote the open δ-ball around 0 in $\mathbb{C}^n$ by $\tilde{\mathbb{B}}_\delta$. Consider the vector space $\tilde{W}_\delta$ of C^0-families $(h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of maps $(g_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ from $\tilde{\mathbb{B}}_\delta$ into $\mathbb{C}^n$ such that $\|D^r g\|$ is uniformly bounded. By $(H_1 - H_2)$, we have that $f_{\mathsf{a}}(\tilde{\mathbb{B}}_\delta) \subset \tilde{\mathbb{B}}_\delta$ for δ small enough. Hence the operator

$$\begin{aligned} T_f \colon \tilde{W}_\delta &\to \tilde{W}_\delta \\ (h_{\mathsf{a}})_{\mathsf{a}\in\mathsf{A}} &\mapsto (D_0 f_{\mathsf{a}})^{-1} \circ h_{\sigma(\mathsf{a})} \circ f_{\mathsf{a}} \end{aligned}$$

is well-defined. Let $\tilde{V}_\delta$ be the subspace of C^0 families of holomorphic maps on $\tilde{\mathbb{B}}_\delta$ that satisfy $h_{\mathsf{a}}(x) = o(x^r)$ for all $\mathsf{a} \in \mathsf{A}$. Then T_f leaves $\tilde{V}_\delta$ invariant as we assumed that f is r-flat.

We consider on $\tilde{V}_\delta$ the norm given by

$$\|g\|_V = \|D^r g\|_0 \ . \tag{5.3}$$

By Taylor–Lagrange expansion around 0, we see that for $k < r$:

$$\left\| D^k g(x) \right\| \leq \delta \cdot \|g\|_V \ , \quad \forall x \in \tilde{\mathbb{B}}_\delta \ . \tag{5.4}$$

This implies that $(\tilde{V}_\delta, \|\cdot\|_V)$ is a Banach space.

Let us denote by $\tilde{\iota} \in \tilde{W}_\delta$ the constant family of the canonical inclusion $\tilde{\mathbb{B}}_\delta \subset \mathbb{C}^n$, so $\tilde{\iota}_{\mathsf{a}}(x) = x$ for all $\mathsf{a} \in \mathsf{A}, x \in \tilde{\mathbb{B}}_\delta$.

A similar computation as in Lemma 3.5 shows that T_f is a contraction on $\tilde{V}_\delta$ for δ small enough. So we find a fixed point in $\iota + \tilde{V}_\delta$ that satisfies (5.2). By restricting δ further, we can obtain that the resulting family is a family of biholomorphisms.

For holomorphic dependence, we get by Lemma 4.11 that T_f depends holomorphically on f, and by the holomorphic case of Lemma A.1 that the fixed point of T_f on $\iota + \tilde{V}_\delta$ depends holomorphically on f. □

Proof of Theorem C. This is now the direct composition of the two previous lemmas, together with a similar extension procedure as in the proof of Theorem A. □

Remark 5.3. The constructions in Lemma 5.1 and Lemma 5.2 preserves real analyticity, so if the family f_{a} is real analytic (i.e. $f_{\mathsf{a}}(\tilde{\mathbb{B}} \cap \mathbb{R}^n) \subset \mathbb{R}^n$), then so are the families (h_{a}) are real. This shows Remark 2.2.

5.2 Smooth dependence for main theorem

The proof with parameter dependence follows the outline given in Section 3, we provide a parametric version of Lemma 3.1 and Lemma 3.2.

Let $\mathcal{F} \subset C^0(\mathsf{A}, C^\infty(\mathbb{B}, \mathbb{R}^n))$ be the subspace of maps $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ such that h_a fixes 0 and has diagonal differential at 0 for all $\mathsf{a} \in \mathsf{A}$. This is a closed vector subspace of $C^0(\mathsf{A}, C^\infty(\mathbb{B}, \mathbb{R}^n))$ and so it is also a graded Fréchet space. Note that the set $\mathcal{U}$ of maps satisfying the hypotheses $(H_1 - H_2 - H_3)$ of Theorem A is an open subset in $\mathcal{F}$.

We can now state the parametric version of Lemma 3.1:

Lemma 5.4. *The C^0-family of polynomial maps $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ of $\mathbb{R}^n$ with degree $\leq r$ constructed in Lemma 3.1 depends smoothly tame on $f \in \mathcal{U}$.*

Proof. Let us first show that $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ depends smoothly on $f \in \mathcal{U}$: as composition of r-jets is a smooth operation, the only thing left to show is that the fixed points of $O_{\underline{k},i}$ depend smoothly on $f \in \mathcal{U}$. As the operator $O_{\underline{k},i}$ depends smoothly on f, so does their fixed points by Lemma A.1. Finally, we use Example 4.6 to promote smooth dependence to smooth tame dependence. □

Let $\mathcal{F}_r \subset \mathcal{F}$ be the subspace of maps $(h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}}$ such that $h_\mathsf{a}(x) = D_0 h_\mathsf{a}(x) + o(x^r)$ for every $\mathsf{a} \in \mathsf{A}$. Note that these are all linear conditions, so $\mathcal{F}_r$ is really a closed vector subspace of $\mathcal{F}$. We denote by $\mathcal{U}_r$ the set of maps in $\mathcal{F}_r$ satisfying the hypotheses $(H_1 - H_2 - H_3 - H_4^r)$ of Lemma 3.2. This is again an open subset of $\mathcal{F}_r$.

Let us now state the parametric version of Lemma 3.2:

Lemma 5.5. *If $\mathring{f} \in \mathcal{U}_r$. Then there exists a $\delta > 0$, and a C^r-neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}_r$ of $\mathring{f}$ such that for every $f \in \mathcal{U}'$, there exists a unique C^0-family $(h_{f,\mathsf{a}})_{\mathsf{a}\in\mathsf{A}}$ of C^∞-diffeomorphisms $h_{f,\mathsf{a}}$ from the δ-ball around 0 into $\mathbb{R}^n$ satisfying for following conditions for every $\mathsf{a} \in \mathsf{A}$:*

$$h_{f,\mathsf{a}}(0) = 0\ , \quad h_{f,\mathsf{a}}(x) = x + o(x^r)\ , \quad \text{and} \quad h_{f,\sigma(\mathsf{a})} \circ f_\mathsf{a}(x) = D_0 f_\mathsf{a} \circ h_{f,\mathsf{a}}(x)\ . \tag{5.5}$$

Moreover, the map $g \mapsto h_{g,\mathsf{a}}$ is tamely smooth.

We will proof this lemma in the rest of this section. Let us give a brief overview: the main idea is to follow the lines of the proof of Lemma 3.5: we construct an operator T_f and show it has a unique fixed point on some affine space, providing a solution to Eq. (5.5). To this end, we show that $\mathrm{id} - T_f$ is invertible with inverse R_f. Estimates similar to the proof of Lemma 4.8 intertwined with the contraction of T_f on the $\|\cdot\|_r$-norm are used to show that R_f is tame. From this we conclude that the fixed point of T_f depends smooth tame on f. We start with the construction of T_f:

Lemma 5.6. *Let $r \geq 2$, let $\mathring{f} \in \mathcal{U}_r$. Then there exist $\delta' > 0$, $0 < C < C' < 1$, a C^r-neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}_r$ of $\mathring{f}$, and a family $(M_j)_{j\geq r}$ of positive constants such that with*

$$\tilde{V}_{\delta'} = \left\{ h \in C^0(\mathsf{A}, C^\infty(\mathbb{B}_{\delta'}, \mathbb{R}^n)) : h(x) = o(x^r) \right\}\ ,$$

the following operator:

$$T_f \colon (h_\mathsf{a})_{\mathsf{a}\in\mathsf{A}} \in \tilde{V}_{\delta'} \mapsto ((D_0 f_\mathsf{a})^{-1} \circ h_{\sigma(\mathsf{a})} \circ f_\mathsf{a})_{\mathsf{a}\in\mathsf{A}} \in \tilde{V}_{\delta'}$$

is well-defined for every $f \in \tilde{\mathcal{U}}$ and $T \colon (f, h) \in \tilde{\mathcal{U}} \times \tilde{V}_{\delta'} \mapsto T_f(h) \in \tilde{V}_{\delta'}$ is smooth tame. Moreover T_f satisfies the following bounds:

$$\|D^r T_f(h)\|_0 \ \leq C' \cdot \|D^r h\|_0\ , \tag{5.6}$$

$$\|D^j T_f(h)\|_0 \ \leq C \cdot \|D^j h\|_0 \ + M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} \cdot \|D^k h\|_0 \ \textit{ for all } j \geq r \ . \tag{5.7}$$

Proof. The construction of δ', C and M in Lemma 3.5 depended only on C^r-bounds of f. So we can find a C^r-neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}_r$ of f_0, $\delta' > 0$, $0 < C < C' < 1$ such that the following holds:

- for every $f \in \tilde{\mathcal{U}}$, $\mathsf{a} \in \mathsf{A}$, the map f_a is contracting on $\mathbb{B}_{\delta'}$,
- with $\tilde{\mu}_\mathsf{a} = \inf_{f\in\tilde{\mathcal{U}}} \min_i \lambda_{f,\mathsf{a},i}$, we have
$$C \cdot \tilde{\mu}_\mathsf{a} > \|D_x f_\mathsf{a}\|^r \ , \quad \forall x \in \mathbb{B}_{\delta'} \ .$$
- The operator $T_f \colon V_{\delta'} \to V_{\delta'}$ is C'-contracting for all $f \in \tilde{\mathcal{U}}$, where $V_{\delta'}$ is the Banach space $V_{\delta'} = \left\{h \in C^0(\mathsf{A}, C^r(\mathbb{B}_{\delta'}, \mathbb{R}^n)) : h(x) = o(x^r)\right\}$,

In particular $\|D_x f_\mathsf{a}\| < 1$ for every $x \in \mathbb{B}_{\delta'}$. Thus for every $j \geq r$, we have:

$$C > \tilde{\mu}_\mathsf{a}^{-1} \cdot \|D_x f_\mathsf{a}\|^j \ , \quad \forall x \in \mathbb{B}_{\delta'}, f \in \tilde{\mathcal{U}} \ . \tag{5.8}$$

Since every f_a is contracting on $\mathbb{B}_{\delta'}$ and fixes 0, we see that T is well-defined. Similar to Lemma 4.8 one shows that T is smooth tame. Now Eq. (5.6) follows from the C'-contraction on $V_{\delta'}$.

We will again use explicit Faà di Bruno's formula Eq. (4.2): there are constants $c_{j,k,i_1,\ldots,i_k} \in \mathbb{Z}$ such that

$$D^j(h_{\sigma(\mathsf{a})} \circ f_\mathsf{a}) = D^j_{f_\mathsf{a}} h_{\sigma(\mathsf{a})}(Df_\mathsf{a})^{\otimes j} + \sum_{k<j} \sum_{i_1+\cdots+i_k=j} c_{j,k,i_1,\ldots,i_k} D^k_{f_\mathsf{a}} h_{\sigma(\mathsf{a})} \left(D^{i_1} f_\mathsf{a} \otimes \cdots \otimes D^{i_k} f_\mathsf{a}\right) \ .$$

By Lemma 4.7 and that $\|D^1 f\|_0 < 1$, there is a positive constant I_j such that for all $f \in \mathcal{U}$ we have $\|D^i f_\mathsf{a}\|_0 \leq I_j \|f_\mathsf{a}\|_j^{\frac{i-1}{j-1}}$. So for $i_1 + \cdots + i_k = j$ we can bound

$$\|D^{i_1} f_\mathsf{a}\|_0 \cdots \|D^{i_k} f_\mathsf{a}\|_0 \leq I_j^k \|f_\mathsf{a}\|_j^{\frac{j-k}{j-1}} \ .$$

So we can find constants $M_j > 0$ such that

$$\sum_{i_1+\cdots+i_k=j} \|c_{j,k,i_1,\ldots,i_k} D^k_{f_\mathsf{a}} h_{\sigma(\mathsf{a})} \left(D^{i_1} f_\mathsf{a} \otimes \cdots \otimes D^{i_k} f_\mathsf{a}\right)\|_0 \leq \tilde{\mu}_\mathsf{a} M_j \|f\|_j^{\frac{j-k}{j-1}} \|D^k h_{\sigma(\mathsf{a})}\|_0 \ .$$

This gives:

$$\|D^j(h_{\sigma(\mathsf{a})} \circ f_\mathsf{a})\|_0 \ \leq \|D^j_{f_\mathsf{a}} h_{\sigma(\mathsf{a})}(Df_\mathsf{a})^{\otimes j}\| + \tilde{\mu}_\mathsf{a} \cdot M_j \sum_{k<j} \|f\|_j^{\frac{j-k}{j-1}} \|D^k h_{\sigma(\mathsf{a})}\|_0 \ .$$

Consequently:

$$\|D^j T_f(h)_\mathsf{a}\|_0 \ \leq \tilde{\mu}_\mathsf{a}^{-1} \|Df_\mathsf{a}|\mathbb{B}_{\delta'}\|_0^j \cdot \|D^j h_{\sigma(\mathsf{a})}\|_0 + M_j \sum_{k<j} \|f\|_j^{\frac{j-k}{j-1}} \|D^k h_{\sigma(\mathsf{a})}\|_0 \ .$$

From Eq. (5.8) we obtain the sought result. □

Lemma 5.7. *Let $\tilde{U}, \delta', C, C', M_j$ as resulting from Lemma 5.6. The following map is well-defined and tame:*

$$R : (f,h) \in \tilde{\mathcal{U}} \times \tilde{V}_{\delta'} \mapsto R_f(h) :=: \sum_{m\geq 0} T_f^m(h) \in \tilde{V}_{\delta'} \ .$$

In particular, $\mathrm{id} - T_f$ *is invertible on* $\tilde{V}_{\delta'}$ *with inverse* $R_f(h)$.

Proof. We already know that every summand $T_f^k(h)$ is well-defined. We will show the following: there is a family $(\tilde{M}_j)_{j\geq r}$ of positive constants (depending on $\tilde{\mathcal{U}}, \delta', C, M_j$, but not on $f \in \tilde{\mathcal{U}}$) such that

$$\sum_{m\geq 0} \|T_f^m(h)\|_j \leq \tilde{M}_j(\|h\|_r \|f\|_j + \|h\|_j) \text{ for all } f \in \tilde{U}, h \in \tilde{V}_{\delta'}, j \geq r \ . \tag{5.9}$$

With this it is clear that $R_f(h)$ is well-defined, and equal to the inverse of $\mathrm{id} - T_f$ on $V_{\delta'}$. As we can bound $\|h\|_r$ in a C^r-neighborhood of (f,h), this then also shows that $R_f(h)$ is tame.

Let us inductively show the existence of $(\tilde{M}_j)_{j\geq r}$ satisfying (5.9). Let $s_j = \sum_{m\geq 0} \|D^j T_f^m(h)\|_0$. For $j \leq r$ we can bound

$$\|D^j T_f^m(h)\|_0 \leq \|D^r T_f^m(h)\|_0 \leq C'^m \|h\|_r$$

by (3.12) and (5.6) and so

$$s_j \leq s_r = \sum_{m\geq 0} \|T_f^m(h)\|_r \leq \frac{1}{1-C'} \|h\|_r \quad \forall j \leq r \ . \tag{5.10}$$

In particular we can start with $\tilde{M}_r = \frac{1}{1-C'}$.

For $j > r$, we note that

$$\sum_{m\geq 0} \|T_f^m(h)\|_j \leq s_j + \sum_{m\geq 0} \|T_f^m(h)\|_{j-1} \leq s_j + \tilde{M}_{j-1}(\|h\|_r \|f\|_{j-1} + \|h\|_{j-1}) \leq s_j + \tilde{M}_{j-1}(\|h\|_r \|f\|_j + \|h\|_j)$$

since the norms $\|\cdot\|_j$ are an increasing family. So it is enough to bound s_j by a constant times $(\|h\|_r \|f\|_j + \|h\|_j)$.

We will first show the following bound:

$$s_j \leq \frac{\|h\|_j + M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} s_k}{1-C} \ . \tag{5.11}$$

For this, we first note that we can obtain

$$\|D^j T_f^m(h)\|_0 \leq C^m \|D^j h\|_0 + \sum_{0\leq l<m} C^{m-l-1} M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} \|D^k T_f^l(h)\|_0 \quad \forall m \geq 0 \tag{5.12}$$

by induction over m from (5.7).

Summing up (5.12) we obtain

(5.13)
$$s_j = \sum_{m\geq 0} \|D^j T_f^m(h)\|_0 \leq \sum_{m\geq 0} \left(C^m \|D^j(h)\|_0 + \sum_{0\leq l<m} C^{m-l-1} M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} \|D^k T_f^l(h)\|_0 \right)$$

$$= \frac{1}{1-C}\|D^j(h)\|_0 + \sum_{m\geq 0}\sum_{0\leq l<m} C^{m-l-1} M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} \|D^k T_f^l(h)\|_0 \tag{5.14}$$

$$= \frac{1}{1-C}\|D^j(h)\|_0 + \sum_{m'\geq 0} C^{m'} M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} \sum_{l\geq 0} \|D^k T_f^l(h)\|_0 \tag{5.15}$$

$$= \frac{1}{1-C}(\|D^j(h)\|_0 + M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} s_j) \tag{5.16}$$

$$\leq \frac{1}{1-C}(\|h\|_j + M_j \sum_{1\leq k<j} \|f\|_j^{\frac{j-k}{j-1}} s_j) \tag{5.17}$$

where we replaced the summation index m by $m' := m-l-1$ from (5.15) to (5.16). Since all terms in our summations are nonnegative, we can freely change the summation order. This establishes (5.11).

Let us now insert (5.10) for $1 \leq k \leq r$ and inductively (5.9) combined with $s_k \leq \sum_{m\geq 0} \|T_f^m(h)\|_j$ for $r < k < j$ into (5.11) to obtain

$$s_j \leq \frac{\|h\|_j + \sum_{1\leq k<j} M_j \|f\|_j^{\frac{j-k}{j-1}} s_k}{1-C} \tag{5.18}$$

$$\leq \frac{1}{1-C}(\|h\|_j + \sum_{1\leq k\leq r} M_j \|f\|_j^{\frac{j-k}{j-1}} \tilde{M}_r \|h\|_r + \sum_{1\leq k<j} M_j \|f\|_j^{\frac{j-k}{j-1}} \tilde{M}_k (\|h\|_r \|f\|_k + \|h\|_k)) \ . \tag{5.19}$$

We can bound every monomial appearing in the right hand side of (5.19) by a constant times $\|h\|_r\|f\|_j + \|h\|_j$: Trivially, $\|h\|_j \leq \|h\|_r\|f\|_j + \|h\|_j$. We also have

$$\|f\|_j^{\frac{j-k}{j-1}} \|h\|_r \leq (\|f\|_j + 1)\|h\|_r \leq \|f\|_j \|h\|_r + \|h\|_j \ .$$

By Lemma 4.7 there is a $I_j > 0$ such that

$$\|f\|_k \leq I_j \|f\|_j^{\frac{k-1}{j-1}} \|f\|_1^{\frac{j-k}{j-1}} \ , \quad \|h\|_k \leq I_j \|h\|_j^{\frac{k-1}{j-1}} \|h\|_1^{\frac{j-k}{j-1}} \quad \forall f \in \tilde{\mathcal{U}}, h \in \tilde{V}_{\delta'}.$$

So we have the following:

$$\|f\|_j^{\frac{j-k}{j-1}} \|h\|_r \|f\|_k \leq I_j \|f\|_j^{\frac{j-k}{j-1}} \|h\|_r \|f\|_j^{\frac{k-1}{j-1}} \|f\|_1^{\frac{j-k}{j-1}} = I_j \|f\|_j \|h\|_r \|f\|_1^{\frac{j-k}{j-1}} \leq I_j \|f\|_j \|h\|_r$$

where we also used $\|f\|_1 < 1$. Finally

$$\|f\|_j^{\frac{j-k}{j-1}} \|h\|_k \leq I_j (\|f\|_j^{\frac{j-k}{j-1}} \|h\|_1^{\frac{j-k}{j-1}}) \|h\|_j^{\frac{k-1}{j-1}} \leq I_j (\|f\|_j \|h\|_1 + \|h\|_j) \leq I_j (\|f\|_j \|h\|_r + \|h\|_j)$$

using also (4.5). So we can bound every monomial appearing in the right hand side of (5.19) by a constant times $\|h\|_r\|f\|_j + \|h\|_j$. From this, the existence of $\tilde{M}_j$ follows. □

Lemma 5.8. *Under the assumptions of Lemma 5.7, R is smooth tame.*

Proof. Let us first show that R is C^1. We follow the proof strategy of [Ham82, Theorem I.5.3.1]. As R is linear in h, we only have to show that R is C^1 in f. Let us compute

$$\frac{R_{f+tg}h - R_f h}{t} = \frac{R_{f+tg}\left((\mathrm{id} - T_f)R_f h - (\mathrm{id} - T_{f+tg})R_f h\right)}{t} \quad = \frac{R_{f+tg}\left(T_{f+tg}R_f h - T_f R_f h\right)}{t} .$$

For $t \to 0$ we have $\frac{\left(T_{f+tg}R_f h - T_f R_f h\right)}{t} \to \partial^1 T_f(g) R_f h$ as T is C^1. Since we know that R is continuous, we obtain

$$\frac{R_{f+tg}\left(T_{f+tg}R_f h - T_f R_f h\right)}{t} \to R_f \partial^1 T_f(g) R_f h .$$

As T is smooth tame, and R is tame, it follows from this formula and the chain rule that all higher derivatives of R exist and are tame, so R is also smooth tame. □

Proof of Lemma 5.5. We use $\tilde{\mathcal{U}}$, δ', T and R as in the previous lemmas. As in the proof of Lemma 3.5, we let $\iota \in C^0(\mathsf{A}, C^\infty(\mathbb{B}_{\delta'}, \mathbb{R}^n))$ be the constant family of the canonical inclusion $\mathbb{B} \subset \mathbb{R}^n$, so $\iota_{\mathsf{a}}(x) = x$ for all $\mathsf{a} \in \mathsf{A}, x \in \mathbb{B}_{\delta'}$. The operator T_f also acts on the affine space $\iota + \tilde{V}_{\delta'}$, as $T_f(\iota) \in \iota + \tilde{V}_{\delta'}$. Since the operator $\mathrm{id} - T_f$ has inverse R_f on $\tilde{V}_{\delta'}$, we know that there is a unique fixed point in $\iota + \tilde{V}_{\delta'}$. This fixed point satisfies (5.5).

In fact, we have the follwing explicit formula for the fixed point:

$$h_f = \iota - R_f(\iota - T_f(\iota)) .$$

To see this, we show that $(\mathrm{id} - T_f)h_f$ vanishes:

$$(\mathrm{id} - T_f)h_f = (\mathrm{id} - T_f)(\iota) - (\mathrm{id} - T_f)R_f(\iota - T_f(\iota)) = \iota - T_f - (\iota - T_f(\iota)) = 0$$

As R and T are tamely smooth by Lemma 5.6 and Lemma 5.5, h depends tamely smooth on f as a family of C^∞-maps on the δ'-ball around 0. By the local inverse function theorem and restricting $\tilde{\mathcal{U}}$ further if necessary, we thereby constructed a family of C^∞-diffeomorphisms. Uniqueness already follows from the uniqueness of Lemma 3.5. □

Proof of Theorem A. We follow the proof of Corollary B using our smooth version of the lemmas. We obtain smooth tame dependence of h on a small ball $\mathbb{B}_{\delta'}$. Note that the extension process used in the proof of Corollary B is also smooth tame.

□

A Smooth dependence of contractions

Here we state smooth dependence of fixed points of contractions. We follow the lines of [Yoc95, Appendix A] that we extend to the case of a complex Banach and a Fréchet parameter space.

Lemma A.1. *Let U be an open subspace of a Fréchet space $\mathcal{F}$, let V be a open subset of an Banach space $\mathcal{B}$. Let $\rho\colon U \times \overline{V} \to V$ be a continuous map such that for every $u \in U$, the map $\rho_u\colon \overline{V} \to V$ is a k-contraction for a fixed $k \in (0,1)$ and fixed point $\phi(u) \in V$.*

Then the map $\phi\colon U \to V$ is continuous. Moreover, if ρ is C^r or smooth, then so ϕ. If $\mathcal{F}$ and $\mathcal{B}$ are complex Banach spaces and ρ is holomorphic, then ϕ is also holomorphic.

Proof. We first observe that ϕ is continuous. By the proof of the Banach fixed point theorem we have

$$\|v-\phi(u)\| \leq \sum_{i=0}^{\infty} \|\rho_u^i(v)-\rho_u^{i+1}(v)\| \leq \frac{1}{1-k}\|v-\rho(u,v)\| \quad \forall u \in U, v \in V \ .$$

In particular we have

$$\|\phi(u')-\phi(u)\| \leq \frac{1}{1-k}\|\phi(u')-\rho(u,\phi(u'))\| = \frac{1}{1-k}\|\rho(u',\phi(u'))-\rho(u,\phi(u'))\| \ .$$

So the continuity of ϕ at u' follows from the continuity of ρ at $(u',\phi(u'))$.

Let us now further suppose that ρ is C^1. We will compute the differential of ϕ as follows: let $u \in U$, $h \in \mathcal{F}$. By continuity of ϕ we know that for $|t|$ small enough, we have that the segment $[u, u+t\cdot h]$ is in U and $[\phi(u), \phi(u+t\cdot h)]$ is in V. Let $\gamma\colon [0,1] \to U \times V$ be given by $\gamma(s) = (u+s\cdot t\cdot h, s\cdot\phi(u+t\cdot h)+(1-s)\cdot\phi(u))$. We can then compute

$$\begin{aligned}\phi(u+h\cdot t)-\phi(u) &= \rho(u+h\cdot t,\phi(u+h\cdot t))-\rho(u,\phi(u))\\ =&\rho(\gamma(1))-\rho(\gamma(0)) = \int_0^1 D_{\gamma(s)}\rho(\gamma'(s))ds = \int_0^1 D^{(1)}_{\gamma(s)}\rho(h\cdot t)+D^{(2)}_{\gamma(s)}\rho(\phi(u+t\cdot h)-\phi(u))ds \ .\end{aligned}$$

Here $D^{(1)}_{\gamma(s)}\rho\colon \mathcal{F}\to\mathcal{B}, D^{(2)}_{\gamma(s)}\rho\colon \mathcal{B}\to\mathcal{B}$, are the partial derivatives of ρ at $\gamma(s)$. We note that $D^{(2)}_{\gamma(s)}\rho$ has norm bounded by k. In particular, $1-\int_0^1 D^{(2)}_{\gamma(s)}\rho ds$ is an invertible operator on $\mathcal{B}$. Hence we obtain

$$\frac{\phi(u+h\cdot t)-\phi(u)}{t} = (1-\int_0^1 D^{(2)}_{\gamma(s)}\rho ds)^{-1}\int_0^1 D^{(1)}_{\gamma(s)}\rho(h)ds \ .$$

As ρ is C^1, the right hand side converges to $(1-D^{(2)}_{(u,\phi(u))}\rho)^{-1}D^{(1)}_{(u,\phi(u))}\rho(h)$ for $t\to 0$. This is a continuous function in u, so ϕ is also C^1.

From the formula the statements for higher regularity or holomorphic maps follow directly. □

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