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► To cite this version:

Ariane Ravier, Hugo Gilbert, Meltem Öztürk, Olivier Spanjaard. Ordinal dominance with binary interactions for subset choice: axiomatic analysis and complexity issues. M-pref 2022: 13th Multidisciplinary Workshop on Advances in Preference Handling, Jul 2022, Vienna, Austria. hal-03812006

HAL Id: hal-03812006

<https://hal.science/hal-03812006v1>

Submitted on 12 Oct 2022

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Ordinal dominance with binary interactions for subset choice: axiomatic analysis and complexity issues

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Abstract

The article focuses on a subset comparison model introduced by Fishburn and LaValle [1996], called *ordinal dominance* (with binary interactions). After giving an axiomatic analysis pointing out several differences between the studied model and one without interactions, we investigate the problem of determining non-dominated subsets from an algorithmic point of view, with a complexity result (determining if a given subset is non-dominated is co-NP-complete), and present an algorithm to determine a non-dominated subset in polynomial time.

1 Introduction

Subset selection problems are very common in situations where, given a population of individuals or alternatives and preferences between them, one aims at determining the best possible subset of that reference set [Fishburn, 1974; van Rooij *et al.*, 2005]. The applications are diverse, ranging from setting up a working group, to selecting attributes for a machine learning procedure, or picking a set of movies to put on a plane's entertainment system, etc.

An intuitive way to solve such a problem is to use the sum of the utilities of the elements of a subset as its global utility. However, such a representation supposes the independence between the elements of the subset, which is violated in many contexts (two individuals may have positive or negative synergies, two elements may be complementary or redundant, etc.). Different models, handling dependencies between elements, have been proposed in the literature. For instance, Choquet integrals can be used in order to model in a cardinal way dependencies between elements [Choquet, 1954; Grabisch, 1996; Grabisch and Labreuche, 2010], and Sugeno integrals can be used in a similar context if data are ordinal [Dubois *et al.*, 2013]). CP-nets [Boutilier *et al.*, 2004] or CI-nets [Bouveret *et al.*, 2009] can also be used in qualitative contexts, however the dependencies they represent are different from Choquet and Sugeno integrals since instead of synergies, conditional preference dependencies are modeled. But another way, in CP-nets, the dependencies are handled *lexicographically*, which yields *non-compensatory* pref-

erence models, unlike the *compensatory* models such as the Choquet integral.

In many applications, providing numerical data can be very difficult. Hence, we are interested in models where positive and negative synergies between pairs of elements can be handled ordinally. The model we study here was proposed by Fishburn and LaValle [1996], who generalized in an intuitive way the compensatory model of additive utility and designed an ordinal counterpart by reasoning on all numerical utility functions consistent with known preferences on singletons and pairs of elements. This model has the advantage of being simple and close to the additive utility model frequently used by decision makers.

The present work provides an axiomatic as well as an algorithmic analysis of this model. Section 2 starts by introducing the model as presented by Fishburn and LaValle. Section 3 then presents an axiomatic study of the model which stresses differences w.r.t. the ordinal counterpart of the additive utility model. Sections 4 and 5 introduce the notion of non-dominated subsets, analyse the complexity of proving that a subset is non-dominated, propose a polynomial time algorithm to find a non-dominated subset and concludes with some results of numerical tests showing the relations between non-dominated subsets and potentially optimal ones.

2 Preliminaries

We denote by X the reference set of alternatives. Given a set $A \subseteq X$, we denote by $A_{(t)} = \{B \subseteq A : |B| = t\}$ the set of subsets of size t within A . The empty set, singletons and pairs will prove useful to our study, and we therefore note $A^* = \emptyset \cup A \cup A_{(2)}$. The insertion of an external element j in a set A creates new pairs in $A \cup \{j\}$, which we denote by $A_{(2)}^j = \{\{a, j\} \in X_{(2)} : a \in A\}$. To lighten the notations, we may write aj instead of $\{a, j\}$ and a instead of $\{a\}$. Symbol \mathcal{R} refers to the preference relation expressed over X^* . It is assumed that \mathcal{R} is a strict complete order, and \mathcal{R}_i refers to the i -th best element according to this preference relation.

2.1 The binary interaction model

As early as 1972, Fishburn [1972] studied the degree of interdependence of a preference relation defined on a finite subset of a product set. Later on, Fishburn and LaValle [1996] proposed a model allowing for the comparison of subsets, tak-

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ing into account binary interactions between elements. This model is based on a cardinal approach, which supposes the existence of numerical value associated to each subset of X . From this cardinal model, however, the authors then propose an ordinal approach, which allows for a study of less rigid preferences, in more realistic conditions, where one cannot necessarily quantify numerically and precisely one's preference over the elements.

In order to present the ordinal setting on which we focus, we start by presenting the cardinal model on which it is based.

Utility of a subset with binary interactions. The model proposed by Fishburn and LaValle [1996] is a simple generalization of additive utilities where the utility of a subset of elements is the sum of values of singletons plus positive or negative interaction terms between pairs of elements (interactions between more than two elements are supposed to be zero). In order to differentiate between positive and negative values (*i.e.*, “good” and “bad” outcomes), the value of the empty set is fixed to zero: $u(\emptyset) = 0$. The utility of any other subset A of X is given by:

$$u(A) = \sum_{a \in A} u(a) + \sum_{C \in A_{(2)}} \Delta(C), \quad (1)$$

where $\Delta(ij)$ represents the interaction value between i and j . Note that by Equation 1:

$$\Delta(ij) = u(ij) - u(i) - u(j). \quad (2)$$

Using Equation 2, we can express the utility of a subset using only the utility of elements of X^* :

$$u(A) = \sum_{C \in A_{(2)}} u(C) - (|A| - 2) \sum_{a \in A} u(a). \quad (3)$$

Example 1. Let $X = \{a, b, c\}$ and \mathcal{R} be such that:

$$ab \succ bc \succ b \succ c \succ \emptyset \succ a \succ ac.$$

Let u be a utility function respecting the order of \mathcal{R} , with $u(ab) = 4$, $u(bc) = 3$, $u(b) = 2$, $u(c) = 1$, $u(\emptyset) = 0$, $u(a) = -1$ and $u(ac) = -4$. Hence, the interactions between pairs are worth:

$$\begin{aligned} \Delta(ab) &= u(ab) - u(a) - u(b) = 4 + 1 - 2 = 3, \\ \Delta(ac) &= u(ac) - u(a) - u(c) = -4 + 1 - 1 = -4, \\ \Delta(bc) &= u(bc) - u(b) - u(c) = 3 - 2 - 1 = 0, \end{aligned}$$

and:

$$\begin{aligned} u(abc) &= u(a) + u(b) + u(c) + \Delta(ab) + \Delta(ac) + \Delta(bc) \\ &= u(ab) + u(ac) + u(bc) - u(a) - u(b) - u(c) \\ &= 1. \end{aligned}$$

We see that, for this given utility function u , elements a and b have a positive interaction, a and c a negative interaction, while b and c have no interaction. The utility of the complete set $\{a, b, c\}$ is 1 (better than the empty set). \diamond

Sensitivity to small changes in the numerical values. In many applications, the precise knowledge of numerical values may not be available. Let us imagine that there was a mistake or ambiguity on the value of $u(ac)$, while all the other values remain correct. Assume that $u(ac) = -6$ instead of -4 . We still have the same order on X^* (\mathcal{R} remains the same) but the set $\{a, b, c\}$ has now a negative utility ($u(abc) = -1$), thus the empty set would be preferred to abc . It is easy to find many examples showing that small modifications in the values of utilities may drastically change the comparisons on subsets or the sign of interactions. In order to deal with this problem, Fishburn and LaValle have proposed an ordinal counterpart of Equation 1. The subsequent ordinal model leads to an *ordinal dominance relation*.

2.2 Ordinal dominance with binary interactions

The *ordinal dominance relation* [Fishburn and LaValle, 1996], denoted by $\geq_D^{\mathcal{R}}$, expresses a dominance which should hold according to *any utility function* u compatible with \mathcal{R} . More formally, for any $A, B \subseteq X$:

$$\begin{aligned} A \geq_D^{\mathcal{R}} B &\Leftrightarrow \forall u \in \mathcal{U}_{\mathcal{R}}, u(A) \geq u(B), \quad (4) \\ A \geq_D^{\mathcal{R}} B &\Leftrightarrow \forall u \in \mathcal{U}_{\mathcal{R}}, \sum_{C \in A_{(2)}} u(C) + (|B| - 2) \sum_{b \in B} u(b) \\ &\geq \sum_{C \in B_{(2)}} u(C) + (|A| - 2) \sum_{a \in A} u(a). \quad (5) \end{aligned}$$

where $\mathcal{U}_{\mathcal{R}}$ denotes the set of all utility functions compatible with \mathcal{R} .

Remark 1. In a cardinal framework, if there is no interaction between items ($\forall i, j \in X, \Delta(ij) = 0$) then we obtain a special case where $\forall A \in X, u(A) = \sum_{a \in A} u(a)$ which is the basic additive utility model. We call the ordinal version of this model *ordinal dominance without interactions* and denote it by $\geq_{D_0}^{\mathcal{R}}$:

$$A \geq_{D_0}^{\mathcal{R}} B \Leftrightarrow \forall u \in \mathcal{U}_{\mathcal{R}}, \sum_{a \in A} u(a) \geq \sum_{b \in B} u(b). \quad (6)$$

Note that $\geq_{D_0}^{\mathcal{R}}$ amounts to a preference relation between subsets considered by Barteo [1971]

There may exist some \mathcal{R} for which the set $\mathcal{U}_{\mathcal{R}}$ is empty if we want to use the basic additive utility model and not empty with the one with binary interactions. For instance, coming back to Example 1, it is easy to check that if $b \succ c \succ \emptyset \succ a$ it is not possible to have $ab \succ bc$ with a basic additive utility function¹. In the following, when using $\geq_{D_0}^{\mathcal{R}}$, we always assume that $\mathcal{U}_{\mathcal{R}} \neq \emptyset$ for additive utilities. \diamond

For a given linear order relation \mathcal{R} on X^* , Fishburn and LaValle have proposed an efficient method to evaluate whether the ordinal dominance as defined in Equation 5 holds between two subsets of X . To this end, they define a *monotonic dominance relation*, denoted by $\geq_M^{\mathcal{R}}$. Testing whether A monotonically dominates B can be done by a simple polynomial-time three-step method that can be outlined as

¹Naturally, there may also exist some \mathcal{R} with empty $\mathcal{U}_{\mathcal{R}}$ even within the model with binary interactions, for instance if we imperatively need to introduce interactions between more than two items

follows: 1) Take the utilities on the right hand side of Equation 5, and order them (w.r.t. \mathcal{R}) into a vector; 2) Do the same for the left hand side; 3) Check if the right-hand vector componentwise dominates the left-hand vector.

In steps 1 and 2, we start by creating a vector v_A^B (resp. v_B^A), composed of the elements of $A_{(2)}$ (resp. $B_{(2)}$) and of $|B| - 2$ occurrences of each element of B (resp. $|A| - 2$ occurrences of each element of A). If v_A^B and v_B^A are of different sizes, we modify the shortest vector by adding as many occurrences of \emptyset as necessary for vectors v_A^B and v_B^A to be of the same size t . We then proceed to remove elements that are both in v_A^B and v_B^A , and sort the remaining components by order of decreasing preference according to \mathcal{R} . The vectors resulting from these modifications are called respectively V_A^B and V_B^A .

In step 3, we proceed to the monotonic comparison of V_A^B and V_B^A , by confronting the component $(V_A^B)_i$ with the component $(V_B^A)_i$, for $i \in \{0, \dots, t\}$. We say that V_A^B monotonically dominates V_B^A (denoted by $V_A^B \geq_M V_B^A$) if:

$$\forall i \in \{1, \dots, t\}, (V_A^B)_i \succeq (V_B^A)_i \quad (7)$$

where $x \succeq y$ iff $x \succ y$ or $x = y$.

Remark 2. Since we consider \mathcal{R} to be a strict order relation, note that $A \geq_D^{\mathcal{R}} B$ iff $A >_D^{\mathcal{R}} B$ for $A, B \subseteq X$ such that $A \neq B$. The asymmetric part $>_M^{\mathcal{R}}$ of $\geq_M^{\mathcal{R}}$ is written as follows:

$$A >_M^{\mathcal{R}} B \Leftrightarrow \begin{cases} \forall i \in \{1, \dots, t\}, (V_A^B)_i \succeq (V_B^A)_i, \\ \exists j \in \{1, \dots, t\}, (V_A^B)_j \succ (V_B^A)_j. \end{cases} \quad (8)$$

Example 2. Let $X = \{a, b, c, d, e\}$, $A = \{a, b, c, d\}$, $B = \{b, c, e\}$ and \mathcal{R} be such that:

$$\begin{aligned} ac \succ bd \succ de \succ c \succ ae \succ bc \succ ab \succ a \succ \emptyset \succ \\ cd \succ ad \succ be \succ d \succ ce \succ e \succ b. \end{aligned}$$

We have $v_A^B = (ab \ ac \ ad \ bc \ bd \ cd \ b \ c \ e \ \emptyset \ \emptyset)$, $v_B^A = (bc \ be \ ce \ a \ a \ b \ b \ c \ c \ d \ d)$, $V_A^B = (ac \ bd \ ab \ \emptyset \ \emptyset \ cd \ ad \ e)$ and $V_B^A = (c \ a \ a \ be \ d \ d \ ce \ b)$. As $ac \succ c$, $bd \succ a$, $ab \succ a$, $\emptyset \succ be$, $\emptyset \succ d$, $cd \succ d$, $ad \succ ce$ and $e \succ b$, it holds that $V_A^B \geq_M^{\mathcal{R}} V_B^A$. \diamond

The interest of this method stems from the fact that ordinal and monotonic dominance have been proved to be equivalent:

Proposition 1 (Fishburn and LaValle, 1996). *The following equivalence holds for all A and B :*

$$V_A^B \geq_M^{\mathcal{R}} V_B^A \Leftrightarrow A \geq_D^{\mathcal{R}} B.$$

The three-step method can therefore be used to test ordinal dominance in polynomial time.

3 Axiomatic examination

We now investigate some mathematical properties relative to the monotonic dominance relation (equivalent to ordinal dominance $\geq_D^{\mathcal{R}}$, Equation 5), so as to discern its limits and particularities, and to study its differences with the ordinal dominance without interactions ($\geq_{D_0}^{\mathcal{R}}$, Equation 6). In the remainder of the paper, each time we mention only the *ordinal dominance relation*, we mean **with** binary interactions.

Firstly, the reader can easily convince themselves that the ordinal dominance relation defines a partial order:

Proposition 2. *Let X be a set of items and \mathcal{R} a strict order on X^* . The ordinal dominance based on \mathcal{R} (as defined in Equation 4) defines a partial order on 2^X , i.e., a reflexive, transitive and anti-symmetric binary relation that is not necessarily complete.*

Hence, ordinal dominance does not allow us to determine a complete ranking over the subsets of X , and there usually exists a large number of incomparable subsets, which generally increases the number of non-dominated ones.

Axioms 1 to 3 described below seem appealing for any *lifting rule* that extends a binary relation on X^* to a binary relation on 2^X . The first axiom states that, when two subsets A and B are comparable, only the relative positions in \mathcal{R} of the elements of X^* matter in the comparison of A and B , and not the names of the elements of X .

Axiom 1 (Neutrality (N)). *Let X be a set of items, \mathcal{R} and \mathcal{R}' be two preference relations defined on X^* , corresponding to the binary relations \succ and \succ' . Let $>$ and $>'$ be binary relations on 2^X obtained by lifting relations \succ and \succ' from X^* to 2^X . Let $\varphi: X \rightarrow X$ be a bijection defined on X , and $\bar{\varphi}: 2^X \rightarrow 2^X$ be a bijection on 2^X defined from φ as follows:*

$$\begin{aligned} \bar{\varphi}(A) &= \{\varphi(a_1), \dots, \varphi(a_p)\} \text{ if } A = \{a_1, \dots, a_p\} \neq \emptyset, \\ \bar{\varphi}(\emptyset) &= \emptyset. \end{aligned}$$

A lifting rule satisfies the neutrality axiom if

$$A \succ B \Leftrightarrow \bar{\varphi}(A) \succ' \bar{\varphi}(B) \quad \forall A, B \in X^*$$

implies that

$$S \succ T \Leftrightarrow \bar{\varphi}(S) \succ' \bar{\varphi}(T) \quad \forall S, T \in 2^X.$$

The second axiom states that, regarding the pairs, only the ones within the symmetric difference $A \oplus B$ should play a determining role when comparing A and B .

Axiom 2 (Independence to Common pairs (IC)). *Let X be a set of items. Let $A, B \subseteq X$. Any pair $ij \in A_{(2)} \cap B_{(2)}$ may be ignored when comparing A and B using \succ .*

Finally, the third axiom states that the preference over two sets A and B should not depend on elements which are not directly present in either set.

Axiom 3 (Independence to Third-party Alternatives (ITA)). *Let X be a set of items, \mathcal{R} binary relation on X^* . Let $A, B \subseteq X$. Let $S \in X^* \setminus (A^* \cup B^*)$. No modification of the position of S in \mathcal{R} can impact the comparison between A and B .*

The three axioms above obviously hold for the ordinal dominance relation as defined in Equation 5:

Proposition 3. *The ordinal dominance relation satisfies axioms N, IC and ITA.*

In contrast, the following weak independence axiom clearly holds in the case without interactions, but not in the case where the binary interactions are taken into account².

Axiom 4 (Weak independence (WI)). *Let X be a set of items. Let $A, B \subseteq X$ be such that $A > B$, then there exists no $x \in X \setminus (A \cup B)$ such that $B \cup \{x\} > A \cup \{x\}$.*

²The proof of results that are marked with a * are deferred to the appendix.

Proposition 4 (*). *The ordinal dominance relation does not satisfy WI.*

The binary interaction model therefore does not allow us to infer dominance between two sets from the dominance between their subsets. Moreover, the model doesn't ensure that the extension of a set can never be dominated by the same extension applied to a set it dominates. This result is of course a direct consequence of binary interactions, as the addition of an element has consequences linked to the pairs it can form.

Following Brams *et al.* [2003], Brams and King [2005] and Bouveret *et al.* [2010], we say that a subset A *pairwise dominates* a subset B of same size if there exists a bijection $\varphi: A \setminus B \rightarrow B \setminus A$ such that $a \succ \varphi(a)$ for all $a \in A \setminus B$. Let us then consider the following axiom expressing compatibility with pairwise dominance:

Axiom 5 (Compatibility with Pairwise dominance (ComP)). *Let X be a set of items. Let $A, B \subseteq X$. If A pairwise dominates B , then $A > B$.*

This axiom is obviously satisfied by $\geq_{D_0}^{\mathcal{R}}$. Nevertheless, it does not hold for $\geq_D^{\mathcal{R}}$, as illustrated by Example 1. Consider indeed $A = bc$ and $B = ab$, and the bijection $\varphi: A \setminus B \rightarrow B \setminus A$ defined by $\varphi(c) = a$. We have $c \succ \varphi(c) = a$ and yet $B \succ A$.

In order to adapt this axiom to the case where binary interactions are taken into account, we say that a subset A *binary pairwise dominates* a subset B of same size if there exists two bijections $\varphi_1: B \setminus A \rightarrow A \setminus B$ and $\varphi_2: A_{(2)} \setminus B_{(2)} \rightarrow B_{(2)} \setminus A_{(2)}$ such that $b \succ \varphi_1(b)$ for all $b \in B \setminus A$ and $ab \succ \varphi_2(ab)$ for all $ab \in A_{(2)} \setminus B_{(2)}$.

Axiom 6 (Compatibility with Binary Pairwise dominance (ComBP)). *Let X be a set of items. Let $A, B \subseteq X$. If A binary pairwise dominates B , then $A > B$.*

Proposition 5 (*). *The ordinal dominance relation satisfies axiom ComBP.*

Thus, when comparing two sets A and B , if it is possible to compare on the one hand all the singletons, and all the pairs on the other hand, we know that the set which dominates over the pairs but is dominated over the singletons will always be the best. This is due to the role that singletons' utilities play in the computation of interactions. We now investigate axioms where the role played by pairs and the role played by singletons are considered separately.

Axiom 7 (Upward Monotonicity w.r.t. Pairs (UMP)). *Let X be a set of items and \mathcal{R} the relation on X^* . Let $A, B \subseteq X$ be such that $A > B$ and let $P \in A_{(2)} \setminus B_{(2)}$. If the preference for P increases (i.e., if P gets closer to the best elements in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 6 (*). *The ordinal dominance relation satisfies UMP.*

Here again, this result seems natural in the context of the study of binary interactions, as the valorisation of a pair *ceteris paribus* implies an improvement of the interaction between its two components. Therefore, the set containing this pair becomes more attractive.

We now focus on the consequence of improving the position of a singleton in \mathcal{R} .

Axiom 8 (Upward Monotonicity w.r.t. Singletons (UMS)). *Let X be a set of items and \mathcal{R} the relation on X^* . Let $A, B \subseteq X$ be such that $A > B$, and let $a \in A \setminus B$. If the preference for a increases (i.e., if a gets closer to the best elements in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 7 (*). *The ordinal dominance relation does not satisfy UMS.*

Actually, improving the position of a singleton in \mathcal{R} has the opposite effect to that of a pair. The reason is that, in the model of utility with binary interactions, a singleton's utility also impacts the appeal of the pairs it may form. Thus, improving the position of a singleton *ceteris paribus* in \mathcal{R} implies the deterioration of its interactions with the other elements in X , in accordance with the definition of interaction provided in Equation 2. Conversely, deteriorating the position of a singleton improves its interactions with the other elements in X , thus the following axiom holds for the ordinal dominance relation:

Axiom 9 (Downward Monotonicity w.r.t. Singletons (DMS)). *Let X be a set of items and \mathcal{R} the relation on X^* . Let $A, B \subseteq X$ be such that $|A| = |B| \geq 2$ and $A > B$, and let $x \in A \setminus B$. If the preference for x decreases (i.e., x gets closer to the worst elements in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 8 (*). *The ordinal dominance relation satisfies DMS.*

Table 1: Axioms satisfied by $\geq_{D_0}^{\mathcal{R}}$ and $\geq_D^{\mathcal{R}}$.

Axioms	$\geq_{D_0}^{\mathcal{R}}$	$\geq_D^{\mathcal{R}}$
(1) [N]	✓	✓
(2) [IC]	N/A	✓
(3) [ITA]	✓	✓
(4) [WI]	✓	✗
(5) [ComP]	✓	✗
(6) [ComBP]	N/A	✓
(7) [UMP]	N/A	✓
(8) [UMS]	✓	✗
(9) [DMS]	✗	✓

Table 1 summarizes the differences between $\geq_{D_0}^{\mathcal{R}}$ and $\geq_D^{\mathcal{R}}$. We observe that, although accounting for interactions within pairs leads to a loss of fundamental properties such as weak independence or upward monotonicity w.r.t. singletons, the ordinal dominance relation $\geq_D^{\mathcal{R}}$ does satisfy new mathematical properties, which are not always meaningful (applicable) for $\geq_{D_0}^{\mathcal{R}}$ (which is denoted by N/A in the table)³.

4 Finding non-dominated subsets

We now use the ordinal dominance relation (with binary interactions) in order to define and find non-dominated subsets.

³Note that, when ordinal dominance without interactions is used, if $\mathcal{U}_{\mathcal{R}} \neq \emptyset$ then IC holds. Axiom UMP is not meaningful for $\geq_{D_0}^{\mathcal{R}}$ because increasing the preference for $P = ij$ independently of i and j may yield $\mathcal{U}_{\mathcal{R}} = \emptyset$, and neither is ComBP because, if $\mathcal{U}_{\mathcal{R}} = \emptyset$ for additive utilities, then no binary pairwise dominance can occur.

Notation ND_k^{\leq} (resp. $ND_k^=$) refers to the set of non ordinally dominated subsets of cardinality at most k (resp. equal to k):

$$\begin{aligned} ND_k^{\bowtie} &= \{A \subseteq X \mid |A| \bowtie k, \nexists B \subseteq X, B \neq A, |B| \bowtie k, \\ &\quad \forall u, u(B) \geq u(A)\} \\ &= \{A \subseteq X \mid |A| \bowtie k, \forall B \subseteq X, B \neq A, |B| \bowtie k, \\ &\quad \exists u, u(A) > u(B)\}. \end{aligned}$$

with $\bowtie \in \{\leq, =\}$.

4.1 Complexity of finding a non ordinally dominated subset

We study two different problems, \bowtie -InOrdND (In the set of Ordinally Non-Dominated solutions) for $\bowtie \in \{\leq, =\}$. These problems are defined as follows:

\bowtie -InOrdND

Input: $X = \{x_1, \dots, x_n\}$, an integer k ,
 $Y \subseteq X$ such that $|Y| \bowtie k$
 \mathcal{R} a preference relation over X^*

Question: Does Y belong to ND^{\bowtie} ?

We now study the complexity of these decision problems.

Theorem 9 (*). \bowtie -InOrdND is co-NP-complete.

Determining whether a given set is non-dominated amongst sets of same or smaller size is thus co-NP-complete.

4.2 Determining a non-dominated subset

Even though we have shown these problems to be NP-complete and, if $P \neq NP$, it is impossible to design an algorithm to determine in polynomial time the entirety of non-dominated subsets, we still try to determine *one* of these subsets, using algorithms running in polynomial time.

We present such an algorithm to find a solution in $ND^=$. There exist different approaches to generate a subset that isn't dominated amongst all subsets of size k . To build the simplest possible algorithm, we choose an iterative approach, based on the monotonic dominance presented in Section 2.2.

4.2.1 k -Selectable pairs

Section 2.2 tells us that, as long as the i -th element of V_S^B is strictly better than that of V_B^S , then B cannot dominate S . This holds regardless of comparisons between the other elements of the vectors.

Since we wish to determine a subset S which could not be dominated by any subset B of same size k , we try to obtain a vector V_S^B composed of at least one element which cannot be dominated in \mathcal{R} , keeping in mind that if the selected element is a singleton, its selection is not advantageous, as its utility is useful to the adversary subset (see Equation 5). For the same reason, if the first pair in \mathcal{R} is such that one of its components precedes it, then its selection may not be advantageous. These observations can be extended to the notion of k -selectable pairs.

Definition 1. A pair P is said to be k -selectable w.r.t. a set S iff it holds that $|P \cup S| \leq k$ and that, for any element $x \in P \setminus S$, $P \succ x$.

Example 3. Let $X = \{1, 2, 3, 4, 5\}$, $k = 4$ and \mathcal{R} such that

$$\begin{aligned} 13 \succ 4 \succ 3 \succ 12 \succ 23 \succ 34 \succ 45 \succ 1 \succ \\ 24 \succ 5 \succ 15 \succ \emptyset \succ 14 \succ 25 \succ 2 \succ 35 \end{aligned}$$

$P = \{1, 2\}$ is 3-selectable w.r.t. $S = \{1, 3\}$ since $12 \succ 2$ ($P \setminus S = \{2\}$), however $P' = \{3, 4\}$ is not 3-selectable w.r.t. $S' = \{1, 2, 3\}$ since $4 \succ 34$ ($P' \setminus S' = \{4\}$).

Proposition 10 (*). Let S be the set obtained by *incrementally* adding the best k -selectable pair w.r.t. itself starting from $S = \emptyset$. Let S' be any subset of size $|S|$ such that $S' \neq S$. Then S' cannot dominate S .

A subset composed only of the best k -selectable pairs can therefore never be dominated by a subset of same size. However, this incremental construction of S can only go on as long as there are k -selectable pairs w.r.t. S , and as such it may stop before S has reached size k . In this case, we complete S by selecting the worst singletons, since we know from Equation 3 that a singleton's utility serves the adversary subset's utility.

Proposition 11 (*). Let S be the set obtained by *incrementally* adding the best k -selectable pair w.r.t. itself starting from $S = \emptyset$. Let W be the set of the $k - |S|$ worst singletons in \mathcal{R} such that $S \cap W = \emptyset$. Then there exist no set S' such that $|S'| = k$ and $S' \geq_{\mathcal{R}_D}^{\mathcal{R}} S \cup W$.

From all these elements, it is now possible to build a procedure to determine a non-dominated subset.

4.2.2 Determining a non-dominated subset in $ND^=$

The procedure unfolds over several steps:

1) *Treatment of "easy" cases* : We start the procedure by considering 4 "easy cases" :

- ◊ if $k = |X|$, then we return X directly,
- ◊ if $|X| < k$ or $k \leq 0$, there exists no solution of size k , therefore we return the empty set,
- ◊ if $k = 1$, we return the best singleton in \mathcal{R} ,
- ◊ if $k = 2$, we return the best pair in \mathcal{R} .

2) *Selection of pairs*: Build a set S , using the greedy strategy over k -selectable pairs explained in Proposition 10.

3) *Completion*: If $|S| < k$, complete S with the worst singletons as explained in Proposition 11.

In Example 3, the selectable pairs chosen to build S (step 2) are 13, then 12 and lastly 23 . Hence, at the end of step 2, $S = \{1, 2, 3\}$. As $|S| < k$, step 3 completes S with the worst singleton outside of S , namely 5. Therefore, at the end of step 3, we obtain the solution $S = \{1, 2, 3, 5\}$. One can easily check that $\{1, 2, 3, 5\}$ is not dominated among same-sized subsets (a formal proof is provided in the Appendix).

Complexity analysis of the procedure. We assume that \mathcal{R} and X are given exhaustively, as they were presented in every example. Moreover, we know that $k \leq n < |\mathcal{R}|$.

Step 1 of the procedure is carried out in linear time, as it only focuses on k and n .

Steps 2 and 3 only needs to browse \mathcal{R} once (checking if a pair is k admissible or if a singleton is in S can be done in constant time), and is therefore performed in $\mathcal{O}(|\mathcal{R}|)$ time.

We conclude that the procedure is indeed carried out in polynomial time, more precisely in $\mathcal{O}(|\mathcal{R}|)$ time.

4.3 Solving by linear programming

The formulation of ordinal dominance using a universal quantifier naturally allows for another formulation using an existential quantifier:

$$A \succeq_D^{\mathcal{R}} B \Leftrightarrow \exists u \in \mathcal{U}_{\mathcal{R}}, u(B) > u(A).$$

Therefore, determining a single function u may suffice to rule on the dominance between A and B . We can therefore draw up a linear program to determine if there exists such a utility function contradicting dominance of A over B . The linear program contains a variable u_x for each element $x \in X^*$, as well as $u(A)$ and $u(B)$. Each of these variables represents a utility which can be associated with the subset under the following constraints:

- *Respect \mathcal{R}* : For each pair of elements of X^* consecutively placed in \mathcal{R} , we add a constraint to verify that the strict order of preferences is satisfied. As such, if for instance $\mathcal{R}_i = a$ and $\mathcal{R}_{i+1} = cd$, we add the constraint

$$u(a) > u(cd).$$

- *Impose the value of \emptyset* : By definition, \emptyset is of null utility, therefore we add a constraint

$$u(\emptyset) = 0.$$

- *Implement utility with binary interactions*: If A (resp. B) is of size greater or equal to 3, we specify that its utility must be computed using the binary interaction model by adding the associated constraint

$$u(A) = \sum_{C \in A_{(2)}} u_C - (|A| - 2) \times \sum_{a \in A} u(a)$$

(resp. $u(B) = \sum_{C \in B_{(2)}} u_C - (|B| - 2) \times \sum_{b \in B} u(b)$).

- *Implement the test*: By adding the constraint

$$u(B) > u(A).$$

It is not necessary to specify a target function: if there exists any feasible solution, there exists an affectation of values to the variables such that all the different constraints are satisfied. This means that there exists a utility function u_1 such that $u_1(B) > u_1(A)$, and therefore $A \not\succeq_D^{\mathcal{R}} B$. If, conversely, the program is infeasible, then the polyhedron is empty, which makes it impossible for such a function u to exist, and ensures the dominance of A over B .

4.4 Study of the polyhedron associated with \mathcal{R}

It is possible to directly exploit the polyhedron of utilities compatible with both utility with binary interactions and \mathcal{R} .

To do this, we use a program containing a variable for each element $x \in X^*$. These variables are subject to the aforementioned constraints, to ensure the respect of the relation \mathcal{R} as well as the imposed value of \emptyset . There is still no need to specify a target function to obtain the polyhedron we want to study. For the sake of polyhedron compacity, however, we can impose that $u(x) \in [-1; 1], \forall x \in X^*$.

Each vertex i of this polyhedron will have a corresponding vector of size $\binom{n}{2} + n + 1$, which indicates the utility u_i associated with each element of \mathcal{R} . We note nb_v the number of vertices of the obtained polyhedron.

Definition 2. The utility vector $s(A)$ of size nb_v is such that each of its components $s(A)_i$ corresponds to the utility of the subset $A \subseteq X$, computed with the utility function u_i associated with the i -th vertex:

$$s(A)_i = u_i(A) = \sum_{C \in A_{(2)}} u_i(C) - (|A| - 2) \sum_{a \in A} u_i(a).$$

Proposition 12. (*) The ordinal dominance (with binary interactions) between two subsets $A, B \subseteq X$ can be determined by simple comparison of $s(A)$ and $s(B)$ according to Pareto dominance:

$$A \succeq_D^{\mathcal{R}} B \iff \forall i \in \{1, \dots, nb_v\}, s(A)_i \geq s(B)_i$$

Using this method, it is possible to consider only a finite number of utility functions to determine the dominance of a subset over another. It also allows for the conversion of the problem of dominance with binary interactions into a multi-criteria maximisation problem.

However, the complexity of this approach depends on the number of vertices on the polyhedron, thus of its shape (determined by the configuration of \mathcal{R}), as well as on the size of X and the value of k .

5 Potential Optimality and its relation with non-dominated subsets

Optimality allows for a wider-scale study of dominance. While monotonic dominance only allows for pairwise comparisons of subsets, an optimal subset is defined in relation to every other subset it can be compared to.

Definition 3. Let X be a set of objects and \mathcal{R} a total order on $X^* = \emptyset \cup X \cup X_{(2)}$, a subset $S \subseteq X$ is said to be *optimal* if, for any utility function $u \in \mathcal{U}_{\mathcal{R}}$ and for any subset $A \subseteq X$, it holds that $u(S) \geq u(A)$.

Just like the ordinal dominance presented in Equation 4, the optimality of a subset is expressed in relation to **any** function u in $\mathcal{U}_{\mathcal{R}}$. To simplify its study, however, we choose to focus on an alternative notion: that of a *potentially optimal* subset.

Definition 4. Let X be a set of objects and \mathcal{R} a total order on $X^* = \emptyset \cup X \cup X_{(2)}$, a subset $S \subseteq X$ is said to be *potentially optimal* (resp. *uniquely potentially optimal*) if there exists a utility function $u \in \mathcal{U}_{\mathcal{R}}$ such that, for any subset $A \subseteq X$, it holds that $u(S) \geq u(A)$ (resp. $u(S) > u(A)$).

The sets UPO_k^{\leq} and PO_k^{\leq} (resp. $UPO_k^=$ and $PO_k^=$) of potentially optimal and uniquely potentially optimal solutions of size equal or inferior to k (resp. equal to k) are defined by:

$$UPO_k^{\leq} = \{A \subseteq X \mid |A| \bowtie k, \exists u, \forall B \subseteq X, B \neq A, |B| \bowtie k, u(A) > u(B)\}$$

$$PO_k^{\leq} = \{A \subseteq X \mid |A| \leq k, \exists u, \forall B \subseteq X, B \neq A, |B| \bowtie k, u(A) \geq u(B)\}$$

with $\bowtie \in \{\leq, =\}$.

Proposition 13. $UPO_k^{\bowtie} \subseteq PO_k^{\bowtie}$, for $\bowtie \in \{\leq, =\}$.⁴

Proof. Trivial. \square

The intersection of PO_k^{\leq} and PO_k^{\leq} or of UPO_k^{\leq} and UPO_k^{\leq} may not be empty, yet we cannot determine any relation of inclusion between PO_k^{\leq} and PO_k^{\leq} , nor between UPO_k^{\leq} and UPO_k^{\leq} . However, we can inspect the relation between these sets and ND_k^{\leq} or ND_k^{\leq} .

Proposition 14 (*). *If S is a potentially optimal subset (i.e. $S \in PO_k^{\leq} \cup PO_k^{\leq}$), then S can only be dominated by subsets that are optimal according to the utility S is optimal for.*

Remark 3. Every element of PO_k^{\bowtie} does not necessarily belong to ND_k^{\bowtie} . However, we observe experimentally that PO_k^{\bowtie} is more often contained entirely within ND_k^{\bowtie} .

Proposition 15 (*). $UPO_k^{\bowtie} \subseteq ND_k^{\bowtie}$, for $\bowtie \in \{\leq, =\}$.

Proposition 16 (*). $ND_k^{\bowtie} \not\subseteq UPO_k^{\bowtie}$, for $\bowtie \in \{\leq, =\}$.

Figure 1 synthesises relations of inclusion between the non-dominated, potentially optimal and uniquely potentially optimal solutions.

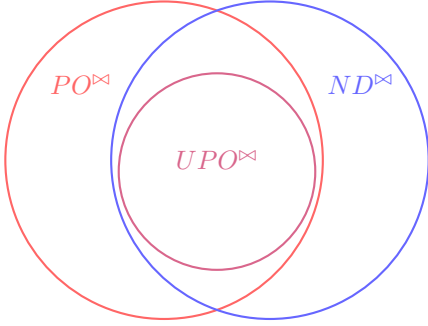


Figure 1: Inclusion relationships between ND_k^{\bowtie} , PO_k^{\bowtie} and UPO_k^{\bowtie} with $\bowtie \in \{\leq, =\}$.

Remark 4. ND_k^{\bowtie} and PO_k^{\bowtie} always contain at least one element, however UPO_k^{\bowtie} may be empty.

5.1 Numerical tests on ND and PO sets

All results presented in Figures 2 and 3 were obtained over 2000 runs, from orders \mathcal{R} generated u.a.r.

The figures only focus on cases where $k > 2$, as there exists only one non-dominated or (uniquely) potentially optimal solution amongst the pairs and singletons. Similarly, the figures studying PO_k^{\leq} , UPO_k^{\leq} and ND_k^{\leq} (Figure 2) do not focus on the case where $k = n$, as we know there exists only one subset of size n : X itself. It is therefore the only non-dominated and (uniquely) potentially optimal subset.

Figures 2 and 3 show that while on average the number of non-dominated subsets and of (uniquely) potentially optimal subsets is relatively low, there exist worst case scenarios in which there is a great number of non-dominated or

⁴We assume that \bowtie takes on the same value every time it is used in a given formulation (i.e. here, it means that either $UPO_k^{\leq} \subseteq PO_k^{\leq}$ or $UPO_k^{\leq} \subseteq PO_k^{\leq}$).

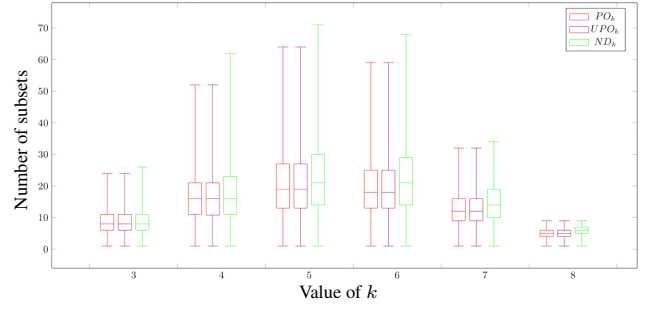


Figure 2: Number of subsets contained in PO_k^{\leq} , UPO_k^{\leq} and ND_k^{\leq} for $n = 9$ and values of k varying in $\{3, 4, 5, 6, 7, 8\}$.

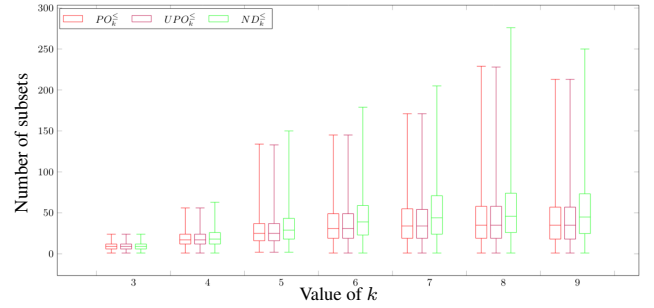


Figure 3: Number of subsets contained in PO_k^{\leq} , UPO_k^{\leq} and ND_k^{\leq} for $n = 9$ and values of k varying in $\{3, 4, 5, 6, 7, 9\}$.

(uniquely) potentially optimal subsets. Sometimes, they can represent nearly half of all possible solutions: in Figure 2, there exist 126 subsets of size exactly 5, and we can see that around 65 are (uniquely) potentially optimal and over 70 are non-dominated. Similarly, in Figure 3, there exist 511 possible subsets of size smaller than or equal to 5, and we can see that around 220 are (uniquely) potentially optimal, and around 275 are non-dominated.

As these large differences in the number of non-dominated and (uniquely) potentially optimal subsets occur even when n and k are fixed, it becomes evident that the source of complexity is \mathcal{R} , the expressed preference order.

Additionally, these two figures reveal a phenomenon we have also observed for other values of n : in most cases, UPO_k^{\bowtie} and PO_k^{\bowtie} are of very similar - if not identical - sizes.

6 Conclusion

We have studied subset selection using an ordinal dominance relation with binary interactions. The originality of the approach is its robustness, by the use of an ordinal model, and its expressiveness, by the consideration of interactions.

For future works, we could go further in the axiomatic analysis and give a characterization of ordinal dominance. The algorithmic study also leaves room for more developments, such as a furthering of the study of optimal subsets determination, or an investigation into more expressive models handling interactions between more than two items, as well as methods for preference elicitation in such a setting.

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Appendix

Omitted Proofs

3 - Axiomatic examination

Axiom 4 (Weak independence (WI)). *Let $A, B \subseteq X$ be such that $A > B$, then there exists no $x \in X \setminus (A \cup B)$ such that $B \cup \{x\} > A \cup \{x\}$.*

Proposition 4. *The ordinal dominance relation does not satisfy WI.*

Proof. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $A = 123$, $B = 456$ and \mathcal{R} be a preference order such that

$$\begin{aligned} 47 > 12 > 1 > 15 > 13 > 24 > \emptyset > 23 > 2 > 3 > 35 > \\ 16 > 4 > 25 > 14 > 45 > 7 > 67 > 6 > 57 > 17 > 46 > \\ 27 > 36 > 37 > 34 > 26 \end{aligned}$$

We start by comparing A and B from their vectors: $V_A^B = (12 \ 13 \ 23 \ 4 \ 5 \ 6)$ and $V_B^A = (1 \ 2 \ 3 \ 45 \ 46 \ 56)$. As $12 > 1$, $13 > 2$, $23 > 3$, $4 > 45$, $5 > 46$ and $6 > 56$, we determine that $V_A^B \geq_M^{\mathcal{R}} V_B^A$, which means that $A \geq_D^{\mathcal{R}} B$.

Let $A' = 1237$ and $B' = 4567$, we now study $V_{A'}^{B'} = (12 \ 13 \ 23 \ 4 \ 4 \ 5 \ 5 \ 6 \ 6 \ 17 \ 27 \ 37)$ and $V_{B'}^{A'} = (47 \ 1 \ 1 \ 2 \ 2 \ 3 \ 3 \ 45 \ 67 \ 57 \ 46 \ 56)$. Since $47 > 12$, $1 > 13$, $1 > 23$, $2 > 4$, $3 > 5$, $45 > 6$, $67 > 6$, $57 > 17$, $46 > 27$ and $56 > 37$, it holds that $V_{A'}^{B'} \leq_M^{\mathcal{R}} V_{B'}^{A'}$, which means that $B' \geq_D^{\mathcal{R}} A'$.

Therefore, by adding the same element to both subsets, we have managed to reverse the dominance relation. \square

Axiom 6 (Compatibility with Binary Pairwise dominance (ComBP)). *Let $A, B \subseteq X$. If A binary pairwise dominates B , then $A > B$.*

Proposition 5. *The ordinal dominance relation satisfies axiom ComBP.*

Proof. Let $A, B \subset X$ be of same size k . From (4), we compute the utility of A when compared to B

$$u(A) = \sum_{C \in A_{(2)}} u(C) + (k-2) \sum_{x \in B} u(x)$$

Similarly, we compute the utility of B when compared to A

$$u(B) = \sum_{C \in B_{(2)}} u(C) + (k-2) \sum_{x \in A} u(x)$$

If A binary pairwise dominates B , it is trivial that $\sum_{C \in A_{(2)}} u(C) > \sum_{C \in B_{(2)}} u(C)$ and $\sum_{x \in B} u(x) > \sum_{x \in A} u(x)$.

Hence, $A \geq_D^{\mathcal{R}} B$ holds. \square

Axiom 7 (Upward Monotonicity w.r.t. Pairs (UMP)). *Let $A, B \subseteq X$ be such that $A > B$ and let $P \in A_{(2)} \setminus B_{(2)}$. If the preference for P increases (i.e., if P gets closer to the best element in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 6. *The ordinal dominance relation satisfies UMP.*

Proof. Since A dominates B , we know from (4) that any utility function verifies that $u(A) \geq u(B)$. However, according to (3), increasing the utility of a pair (i.e. placing it earlier in \mathcal{R}) increases the utility of the set that contains it.

Thus, if P belongs to A and not B , then only the utility of A increases. Let ϵ be this increase of utility, we obtain that $u(A) + \epsilon \geq u(A) \geq u(B)$, which ensures the preservation of the dominance of A over B . \square

Axiom 8 (Upward Monotonicity w.r.t. Singletons (UMS)). *Let $A, B \subseteq X$ be such that $A > B$, and let $a \in A \setminus B$. If the preference for a increases (i.e., if a gets closer to the best elements in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 7. *The ordinal dominance relation does not satisfy UMS.*

Proof. Let $X = \{1, 2, 3, 4\}$ and \mathcal{R} be a preference order such that

$$12 > 34 > 2 > 14 > 13 > 1 > 23 > 3 > \emptyset > 4 > 24$$

Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. We determine the following comparison vectors: $V_A^B = (12 \ 13 \ 4)$ and $V_B^A = (34 \ 1 \ 24)$. Since $12 > 34$, $13 > 1$ and $4 > 24$, it holds that $V_A^B \geq_M^{\mathcal{R}} V_B^A$, hence $A \geq_D^{\mathcal{R}} B$.

However, if the preference for the element 1 increases, all other utilities being equal (i.e. only the Δ are modified, more precisely decreased), and \mathcal{R} is modified such that

$$12 > 34 > 2 > 14 > 1 > 13 > 23 > 3 > \emptyset > 4 > 24$$

Then the monotonic dominance of V_A^B over V_B^A no longer applies, since $1 > 13$, and the dominance of A over B is lost. \square

Axiom 9 (Downward Monotonicity w.r.t. Singletons (DMS)). *Let $A, B \subseteq X$ be such that $|A| = |B| \geq 2$ and $A > B$, and let $x \in A \setminus B$. If the preference for x decreases (i.e., if x gets closer to the worst element in \mathcal{R}) ceteris paribus, then $A > B$ is maintained.*

Proposition 8. *The ordinal dominance relation satisfies DMS.*

Proof. From (5), we know that the value of $u(x)$, $x \in A \setminus B$, impacts the utility of the set B when compared to A . Therefore if $u(x)$ decreases, A remains of identical utility (which is still greater than that of B), while the utility of B decreases, regardless of which utility function u is applied. Consequently, it still holds that $A \geq_D^{\mathcal{R}} B$. \square

4 - Finding non-dominated subsets

Theorem 9. \bowtie -InOrdND is co-NP-complete.

Proof. To prove that \bowtie -InOrdND is co-NP-complete, we show that its complementary problem is NP-complete. Its complementary problem, \bowtie -OrdDominated, has as input:

Input: $X = \{x_1, \dots, x_n\}$, an integer k ,
 $Y \subseteq X$ such that $|Y| \geq k$
 \mathcal{R} a preference relation over X^*

Depending on the value of \bowtie , the question associated to \bowtie -OrdDominated is either:

- \bowtie is = : Is Y dominated in $X_{(k)}$ according to the binary interaction model?
- \bowtie is \leq : Is Y dominated in $\bigcup_{i=0}^k X_{(i)}$ according to the binary interaction model?

We shall use the k -clique problem, which has been proven by [Karp, 2010] to be NP-complete, defined as such:

Input: $G = (S, E)$
 $k \in \mathbb{N}$

Question: Is there a clique of size k in G ?

- \bowtie -OrdDominated \in NP

Let $T = \{t_1, \dots, t_k\}$ be a better solution than Y . We can verify that $T \geq_D^{\mathcal{R}} Y$ in polynomial time using monotonic dominance between the two vectors V_T^Y and V_Y^T . We know it can be checked in polynomial time that $V_T^Y \geq_M^{\mathcal{R}} V_Y^T$.

- k -CLIQUE \leq_P \bowtie -OrdDominated

We transform in polynomial time an instance $(G=(S,E), k)$ of k -CLIQUE into a new instance (X, Y, \mathcal{R}) of \bowtie -OrdDominated such that:

- $X = S \cup Y$, i.e., there is one element per vertex plus k additional elements $Y = \{y_1, \dots, y_k\}$.
- \mathcal{R} is a strict order over X^* such that $E \succ Y_{(2)} \succ \mathcal{C}_1 \succ \mathcal{C}_2 \succ \emptyset \succ Y \succ S$ with

$$\mathcal{C}_1 = \{(i, j) \in X_{(2)} \mid i \in Y \text{ and } j \in S\}$$

$$\mathcal{C}_2 = \{(i, j) \in S_{(2)} \mid (i, j) \notin E\}$$

Note that by construction, adding any element to any set (except \emptyset) will necessarily increase its utility. Hence, Y will be ordinally dominated iff it is ordinally dominated by a set of size k . Thus, in the rest of the reduction we focus on $=$ -OrdDominated, while keeping in mind that the reduction also applies to \leq -OrdDominated.

$\triangleright k$ -CLIQUE $\Rightarrow =$ -OrdDominated.

Let Z be a set of k vertices constituting a clique in G . By definition of \mathcal{R} , Z is of size k and such that every element in $Z_{(2)}$ is preferred to every element in $Y_{(2)}$, and every element in Y is preferred to every element in X . We conclude that $V_Z^Y \geq_M^{\mathcal{R}} V_Y^Z$, i.e. that $Z \geq_D^{\mathcal{R}} Y$.

$\triangleright =$ -OrdDominated $\Rightarrow k$ -CLIQUE

Let $Z \in X_{(k)}$ be a set dominating Y . We know that $V_Z^Y \geq_M^{\mathcal{R}} V_Y^Z$, and it therefore holds that $Z_{(2)} \succ Y_{(2)}$ and $Y \succ Z$ (all relations are strict because \mathcal{R} is a strict order). If Z contains an element of Y as well as an element of S , $Z_{(2)}$

will contain a pair of \mathcal{C}_1 . Yet we know that every element in \mathcal{C}_1 is dominated by every element in $Y_{(2)}$, which prevents us from verifying that $Z_{(2)} \succ Y_{(2)}$. Z must therefore contain either only elements of Y or only elements of S . In the former case, if $Z = Y$, Z cannot dominate Y , which contradicts our initial hypothesis.

We therefore know that Z must contain only elements of S to verify that $Y \succ Z$. However, we know that the only elements of S forming pairs which dominate any pair of $Y_{(2)}$ are the vertices of G linked by an edge. Z must therefore contain only elements of S linked in G in order for $Z_{(2)} \succ Y_{(2)}$ to hold. Such a Z represents a set of k vertices of G connected two by two, i.e., a clique of size k in G . \square

Proposition 10. Let S be the set obtained by **incrementally** adding the best k -selectable pair w.r.t. itself starting from $S = \emptyset$. Let S' be any subset of size $|S|$ such that $S' \neq S$. Then S' cannot dominate S .

Proof. We study the dominance relation between S and S' by comparing the associated vectors $V_S^{S'}$ and $V_{S'}^S$. Four scenarios may arise :

1) Let the first element of $V_S^{S'}$ be a pair P_S and the first element of $V_{S'}^S$ be another pair $P_{S'}$. By construction, $V_S^{S'} \cap V_{S'}^S = \emptyset$, therefore $P_{S'} \notin S$, which means that it is not k -selectable with regards to S . This can mean one of two things: either $P_{S'}$ is not k -selectable w.r.t. S because selecting it made S a set of size greater than k , or because $\exists x \in P_{S'}$ such that $x \succ P_{S'}$ and $x \notin S$.

Let us first assume $P_{S'}$ is not k -selectable w.r.t. S because selecting it makes S a set of size greater than k . Since S and S' are of same size, the presence of $P_{S'}$ in S' , when $P_{S'} \notin S$, implies that at least one element $x \in S$ is not present in S' . Yet x can only be present in S if it is associated to a k -selectable pair w.r.t. S . Let P^* be that pair: if $x \notin S'$, then $P^* \notin S'$, and it holds that $P^* \in V_S^{S'}$. Since we proceed by successively selecting the best k -selectable pairs w.r.t. S , it holds that $P^* \succ P_{S'}$, otherwise $P_{S'}$ would have been selected in place of $P_{S'}$. Since the best element in $V_S^{S'}$ is P_S , it holds that $P_S \succeq P^* \succ P_{S'}$, and $P_{S'}$ cannot be preferred to P_S , which means that $V_S^{S'}$ cannot dominate $V_{S'}^S$, i.e. S' cannot dominate S .

Let us now assume $P_{S'}$ is not k -selectable w.r.t. S because $\exists x \in P_{S'}$ such that $x \succ P_{S'}$ and $x \notin S$, then we know by construction that $x \in V_S^{S'}$, and since the best element in $V_S^{S'}$ is P_S , it holds that $P_S \succ x \succ P_{S'}$. Therefore, $P_{S'}$ cannot be preferred to P_S , which means that $V_S^{S'}$ cannot dominate $V_{S'}^S$, i.e. S' cannot dominate S .

2) Let the first element of $V_S^{S'}$ be a pair P and the first element of $V_{S'}^S$ be a singleton x . By construction, $P \in S \setminus S'$ and $x \in S' \setminus S$. Yet x can only be present in S if it is associated to a k -selectable pair w.r.t. S . Let P^* be that pair, hence $P^* \succ x$: if $x \notin S'$, then $P^* \notin S'$, and it holds that $P^* \in V_S^{S'}$. Then $P_S \succeq P^* \succ x$ which means that $V_S^{S'}$ cannot dominate $V_{S'}^S$, i.e. S' cannot dominate S .

3) Let the first element of $V_S^{S'}$ be a singleton x and the first element of $V_{S'}^S$ be a pair P . By construction, $P \in S' \setminus S$ and

$x \in S' \setminus S$. As $P \notin S$, we know that it is not k -selectable w.r.t. S . This can mean one of two things: either P is not k -selectable w.r.t. S because selecting it made S a set of size greater than k , or because $\exists y \in P$ such that $y \succ P$ and $y \notin S$.

Let us first assume that P is not k -selectable w.r.t. S because selecting it makes S of size greater than k . Since S and S' are of same size, the presence of P in S' , when $P \notin S$, implies that at least one element $z \in S$ is not present in S' . Yet z can only be present in S if it is in one of the k -selectable pair that was added in the incremental process building S . Let P^* be that pair: if $z \notin S'$, then $P^* \notin S'$, and it holds that $P^* \in V_S^{S'}$. Since we proceed by successively selecting the best k -selectable pairs w.r.t. S , it holds that $P^* \succ P$ (otherwise P would have been added instead⁵). As the best element in $V_S^{S'}$ is x , it holds that $x \succ P^* \succ P$. Therefore, P cannot be preferred to x , which means that V_S^S cannot dominate $V_S^{S'}$, i.e. S' cannot dominate S .

If P is not k -selectable w.r.t. S because $\exists y \in P$ such that $y \succ P$ and $y \notin S$, then we know by construction that $y \in V_S^{S'}$, and since the best element in $V_S^{S'}$ is x , it holds that $x \succeq y \succ P$. Consequently, P cannot be preferred to x , meaning that V_S^S cannot dominate $V_S^{S'}$, i.e. S' cannot dominate S .

4) Let the first element of $V_S^{S'}$ be a singleton x and the first element of V_S^S be another singleton y . By construction, $x \in S' \setminus S$ and $y \in S \setminus S'$, which means that y is in one of the k -selectable pair P that was added in the incremental process building S with $P \succ y$. However, since, by construction, the first element of $V_S^{S'}$ is x , then it must hold that $x \succ P \succ y$. Therefore, V_S^S cannot dominate $V_S^{S'}$, i.e. S' cannot dominate S .

Thus, regardless of the configuration of $V_S^{S'}$ and V_S^S , it is impossible for S' to dominate S . \square

Proposition 11. *Let S be the set obtained by incrementally adding the best k -selectable pair w.r.t. itself starting from $S = \emptyset$. Let W be the set of the $k - |S|$ worst singletons in \mathcal{R} such that $S \cap W = \emptyset$. Then there exist no set S' such that $|S'| = k$ and $S' \succeq_D^{\mathcal{R}} S \cup W$.*

Proof. Once again, we study the dominance relation between $S \cup W$ and S' by comparing the associated vectors $V_{S \cup W}^{S'}$ and $V_{S'}^{S \cup W}$. Four scenarios may arise :

1) Let the first element of $V_{S \cup W}^{S'}$ be a pair P_1 and the first element of $V_{S'}^{S \cup W}$ be a pair P_2 . By construction, we know that $P_1 \in (S \cup W) \setminus S'$ and $P_2 \in S' \setminus (S \cup W)$. While P_1 may or may not be k -selectable w.r.t. S , we know for certain that P_2 is not. This can mean one of two things: either P_2 is not k -selectable w.r.t. S because selecting it made S a set of size greater than k , or because $\exists x \in P_2$ such that $x \succ P_2$ and $x \notin S$.

⁵Note that adding P^* can only make P become selectable if $P^* \succ P$.

If P_2 is not k -selectable w.r.t. S because selecting it made S a set of size greater than k , then necessarily, S is of size $k - 1$ and $S \cap P_2 = \emptyset$. As a consequence, at least one element $x \in S$ is not present in S' . Yet x can only be present in S if it is in one of the k -selectable pair that was added in the incremental process building S . Let P^* be that pair: if $x \notin S'$, then $P^* \notin S'$, and it holds that $P^* \in V_{S \cup W}^{S'}$. Since we proceed by successively selecting the best k -selectable pairs w.r.t. S , it holds that $P^* \succ P$ (otherwise P would have been added instead). In turn, and since the best element in $V_{S \cup W}^{S'}$ is P_1 , it holds that $P_1 \succeq P^* \succ P_2$, and P_2 cannot be preferred to P_1 , which means that $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

If P_2 is not k -selectable w.r.t. S because $\exists x \in P_2$ such that $x \succ P_2$ and $x \notin S$, let x^* be the component of $P_2 \setminus S$ that is most preferred. If x^* were to belong to W , then by construction of W , it would hold that both elements of P_2 belong to W , in which case P_2 would belong to both S' and $S \cup W$, and it could not be the first element of $V_{S'}^{S \cup W}$. It therefore holds that $x^* \notin W$ and $x^* \succ P_2$. By construction, this means that $x^* \in V_{S \cup W}^{S'}$, and since the best element in $V_{S \cup W}^{S'}$ is P_1 , it holds that $P_1 \succ x^* \succ P_2$. Therefore, P_2 cannot be preferred to P_1 , which means that $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

2) Let the first element of $V_{S \cup W}^{S'}$ be a pair P and the first element of $V_{S'}^{S \cup W}$ be a singleton x . By construction, we know that $P \in (S \cup W) \setminus S'$ and $x \in S' \setminus (S \cup W)$. There can be two explanations for x belonging to $S \cup W$: either $x \in S$ and is therefore part of a k -selectable pair $P_x \in S$ that was added in the incremental process building S , or $x \in W$ and has been added to S as one of the worst singletons in \mathcal{R} .

If $x \in S$ and since the best element in $V_{S \cup W}^{S'}$ is P , then it holds that $P \succeq P_x \succ x$, in which case $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

If $x \in W$ and has been added to S as one of the worst singletons in \mathcal{R} , then we know that $\forall y \in X \setminus (S \cup W), y \succ x$. Since $x \notin S'$ but $|S'| = |S \cup W|$, there must exist $z \in S' \setminus (S \cup W)$, which means $z \succ x$. By construction, $z \in V_{S \cup W}^{S'}$, yet the first element of $V_{S'}^{S \cup W}$ is a pair, therefore it must hold that $P \succ z \succ x$. Consequently, $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

3) Let the first element of $V_{S \cup W}^{S'}$ be a singleton x and the first element of $V_{S'}^{S \cup W}$ be a pair P . By construction, we know that $P \in S' \setminus (S \cup W)$ and $x \in S' \setminus (S \cup W)$. Since $P \notin S \cup W$, we know that it is non k -selectable w.r.t. S . This can mean one of two things: either P is not k -selectable w.r.t. S because selecting it made S a set of size greater than k , or because $\exists y \in P$ such that $y \succ P$ and $y \notin S$.

If P is not k -selectable w.r.t. S because selecting it made S of size greater than k , then necessarily, S is of size $k - 1$ and $S \cap P = \emptyset$. As a consequence, at least one element $z \in S$ is not present in S' . Yet z can only be present in S if it is in one of the k -selectable pair that was added in the incremental process building S . Let P^* be that pair: if $z \notin S'$, then $P^* \notin S'$, and it holds that $P^* \in V_{S \cup W}^{S'}$. Since we proceed

by successively selecting the best k -selectable pairs w.r.t. S , it holds that $P^* \succ P$ (otherwise P would have been added instead). In turn, and since the best element in $V_{S \cup W}^{S'}$ is x , it holds that $x \succ P^* \succ P$. Therefore, P cannot be preferred to x , which means that $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

If P is not k -selectable w.r.t. S because $\exists y \in P$ such that $y \succ P$ and $y \notin S$, let y^* be the component of $P \setminus S$ that is most preferred. If y^* were to belong to W , then by construction of W , it would hold that both elements of P belong to W , in which case P would belong to both S' and $S \cup W$, and it could not be the first element of $V_{S'}^{S \cup W}$. It therefore holds that $y^* \notin W$ and $y^* \succ P$. By construction, this means that $y^* \in V_{S \cup W}^{S'}$, and since the best element in $V_{S \cup W}^{S'}$ is x , it holds that $x \succeq y^* \succ P$. Consequently, P cannot be preferred to x , meaning that $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

4) Let the first element of $V_{S \cup W}^{S'}$ be a singleton x and the first element of $V_{S'}^{S \cup W}$ be another singleton y . By construction, we know that $x \in S' \setminus (S \cup W)$ and $y \in (S \cup W) \setminus S'$. If y is in $(S \cup W) \setminus S'$ because it was a part of a k -selectable pair P_y such that $P_y \succ y$, and $P_y \in V_{S \cup W}^{S'}$. Yet since the first element of $V_{S \cup W}^{S'}$ is a singleton, it must hold that $x \succ P_y \succ y$, in which case $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

If y has been added to S as one of the worst singletons in \mathcal{R} , then we know that $\forall z \in X \setminus (S \cup W), y \succ z$. Since $y \notin S'$ but $|S'| = |S \cup W|$, there must exist $s \in S' \setminus (S \cup W)$, which means $s \succ y$. By construction, $s \in V_{S \cup W}^{S'}$, therefore it must hold that $x \succeq s \succ y$. Consequently, $V_{S'}^{S \cup W}$ cannot dominate $V_{S \cup W}^{S'}$, i.e. S' cannot dominate $S \cup W$.

Thus, regardless of the configuration of $V_{S \cup W}^{S'}$ and $V_{S'}^{S \cup W}$, it is impossible for S' to dominate $S \cup W$. \square

Proposition 12. *The relation of dominance with binary interactions between two subsets $A, B \subseteq X$ can be determined by simple comparison of $s(A)$ and $s(B)$ according to Pareto dominance:*

$$A \geq_D^{\mathcal{R}} B \iff \forall i \in \{1, \dots, nb_v\}, s(A)_i \geq s(B)_i$$

Proof. $\diamond s(A) \succeq_P s(B) \Rightarrow A \geq_D^{\mathcal{R}} B$

If $s(A) \succeq_P s(B)$, we know that A is of higher utility than B over every vertex of the polyhedron. By definition, any point of the polyhedron (here, any utility verifying \mathcal{R}) can be written as a linear combination of its extreme points. Therefore, let u_i be the utility defined on vertex i , we know that, for any utility function u verifying \mathcal{R} , there exists a set of weights p_i such that $\sum_{i=1}^{nb_s} p_i = 1$, from which we can express $u(A)$ by

$$u(A) = \sum_{i=1}^{nb_s} p_i \times u_i(A).$$

Since, by construction, $s(A) \succeq_P s(B)$ means that $\forall i \in \{0, \dots, nb_v\}, u_i(A) \geq u_i(B)$, it holds that

$$\sum_{i=1}^{nb_s} u_i(A) \geq \sum_{i=1}^{nb_s} u_i(B)$$

which implies that, for any function u verifying \mathcal{R} , $u(A) \geq u(B)$.

Using (4), we then obtain that $s(A) \succeq_P s(B) \Rightarrow A \geq_D^{\mathcal{R}} B$.

$\diamond A \geq_D^{\mathcal{R}} B \Rightarrow s(A) \succeq_P s(B)$

Trivial: from (4), we know that if $A \geq_D B$, then let u_i be the utility function fixed on vertex i , $\forall i \in \{1, \dots, nb_s\}$, it holds that $u_i(A) \geq u_i(B)$. As A is of higher utility than B on every vertex, it holds that $s(A)_i \geq s(B)_i, \forall i \in \{1, \dots, nb_s\}$, i.e. $s(A) \succeq_P s(B)$.

Pareto dominance over the vertices of the polyhedron determined from \mathcal{R} is therefore equivalent to the dominance relation $\geq_D^{\mathcal{R}}$. \square

Example 3 (Complete version). Let $X = \{1, 2, 3, 4, 5\}$, $k = 4$ and \mathcal{R} such that

$$\begin{aligned} 13 \succ 4 \succ 3 \succ 12 \succ 23 \succ 34 \succ 45 \succ 1 \succ \\ 24 \succ 5 \succ 15 \succ \emptyset \succ 14 \succ 25 \succ 2 \succ 35 \end{aligned}$$

The selectable pairs chosen to build S (step 2.) are 13, then 12 and lastly 23. Hence, at the end of step 2, $S = \{1, 2, 3\}$. As $|S| < k$, step 3 completes S with the worst singleton outside of S , namely 5. Therefore, at the end of step 3, we obtain the solution $S = \{1, 2, 3, 5\}$. We check that $\{1, 2, 3, 5\}$ is not dominated among same-sized subsets.

$V_{1235}^{1234} = (4 \ 4 \ 15 \ 25 \ 35)$ and $V_{1234}^{1235} = (34 \ 24 \ 5 \ 5 \ 14)$: since $4 \succ 34$, 1234 does not dominate 1235,

$V_{1235}^{1245} = (13 \ 4 \ 4 \ 23 \ 35)$ and $V_{1245}^{1235} = (3 \ 3 \ 45 \ 24 \ 14)$: since $13 \succ 3$, 1245 does not dominate 1235,

$V_{1235}^{1345} = (4 \ 4 \ 12 \ 23 \ 25)$ and $V_{1345}^{1235} = (34 \ 45 \ 14 \ 2 \ 2)$: since $4 \succ 34$, 1345 does not dominate 1235,

$V_{1235}^{2345} = (13 \ 4 \ 4 \ 12 \ 15)$ and $V_{2345}^{1235} = (34 \ 45 \ 1 \ 1 \ 24)$: since $13 \succ 34$, 2345 does not dominate 1235.

Consequently, 1235 is not dominated amongst same-sized sets.

5 - Potential Optimality and its relation with non-dominated subsets

Proposition 14. *If S is a potentially optimal subset (i.e. $S \in PO_k^= \cup PO_k^<$), then S can only be dominated by subsets that are optimal according to the utility S is optimal for.*

Proof. Let $S \in PO_k$ (without loss of generality). Then, by definition, there exists a utility function u_1 such that $u_1(S) \geq u_1(A), \forall A \in X_{(k)}$. We also know that, in order for A to be dominated by a subset B , the latter needs to be such that, for any utility function u , it holds that $u(B) \geq u(S)$. However, for u_1 , no subset is of higher utility than S . To ensure the dominance relation, B must therefore be of utility equal to that of S according to u_1 , i.e. $u_1(B) = u_1(S) \geq$

$u_1(A), \forall A \in X_{(k)}$. By definition, this means B also belongs to PO_k . More precisely, it means that B , like S , is optimal for u_1 . \square

Proposition 15. $UPO_k^\bowtie \subseteq ND_k^\bowtie$, for $\bowtie \in \{\leq, =\}$.

Proof. Let $A \in UPO_k^\bowtie$. By definition, there exists u_1 such that $\forall B \subseteq X$ different from A and of size $\bowtie k$, $u_1(A) > u_1(B)$. This means that for any subset $B \subseteq X$ different from A and of size $\bowtie k$, we have at least one utility function such that $u(A) > u(B)$, which allows us to deduce that $A \in ND_k^\bowtie$. \square

Proposition 16. $ND_k^\bowtie \not\subseteq UPO_k^\bowtie$, for $\bowtie \in \{\leq, =\}$.

Proof. Figures 2 and 3 provide computational proofs for the proposition. Additionally, we provide a counter-example for the case $\bowtie = \leq$. Let $X = \{1, 2, 3, 4, 5\}$, $k = 4$ and \mathcal{R} be a preference order such that

$$\begin{aligned} \emptyset \succ 2 \succ 12 \succ 3 \succ 34 \succ 23 \succ 14 \succ 5 \succ \\ 4 \succ 24 \succ 13 \succ 35 \succ 1 \succ 15 \succ 45 \succ 25, \end{aligned}$$

it is determined that 145 is non-dominated (i.e. $145 \in ND_4^{\leq}$) but not potentially optimal (i.e. $145 \notin PO_4^{\leq}$).

Indeed, if on the one hand we compare 145 with the set 124, we obtain the following comparison vectors: $V_{145}^{124} = (2 \ 15 \ 45)$ and $V_{124}^{145} = (12 \ 5 \ 24)$. Since $2 \succ 12$ but $15 \prec 5$ and $45 \prec 24$, for a utility function u to verify $u(145) \geq u(124)$ would require for

$$u(2) - u(12) \geq (u(5) - u(15)) + (u(24) - u(45))$$

to hold.

On the other hand, if we compare 145 with the set 1345, we obtain the following comparison vectors: $V_{145}^{1345} = (3 \ 3 \ 5 \ 4 \ 1)$ and $V_{1345}^{145} = (\emptyset \ \emptyset \ 34 \ 13 \ 35)$. As $4 \succ 13$, but $3 \prec \emptyset$, $5 \prec 34$ and $1 \prec 35$, for a utility function u to verify $u(145) \geq u(1345)$ would require for

$$u(4) - u(13) \geq 2 \times (u(\emptyset) - u(3)) + (u(34) - u(5)) + (u(35) - u(1))$$

to hold.

Yet we observe that the gap between $u(4)$ and $u(13)$ is included in the gap between $u(5)$ and $u(15)$; similarly, the gap between $u(2)$ and $u(12)$ is included in the gap between $u(\emptyset)$ and $u(3)$. Consequently, the increase of the gap between one of the two equations u must satisfy to make 145 the best coalition, has the opposite effect on the other equation, and it is therefore impossible to determine a utility function u that would verify both that $u(145) \geq u(124)$ and that $u(145) \geq u(1345)$.

Thus, there exists no utility function u for which 145 is potentially optimal, even though we know 145 is not dominated. \square