



**HAL**  
open science

# Consistent discretization of a homogeneous finite-time control for a double integrator

Andrey Polyakov, Denis Efimov, Xubin Ping

► **To cite this version:**

Andrey Polyakov, Denis Efimov, Xubin Ping. Consistent discretization of a homogeneous finite-time control for a double integrator. IEEE Conference on Decision and Control, Dec 2022, Cancun, Mexico. hal-03811973

**HAL Id: hal-03811973**

**<https://hal.science/hal-03811973>**

Submitted on 12 Oct 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Consistent discretization of a homogeneous finite-time control for a double integrator

Andrey Polyakov<sup>1</sup>, Denis Efimov<sup>1</sup>, Xubin Ping<sup>2</sup>

**Abstract**—A discretization of a homogeneous controller for a double integrator is developed. It preserves the finite-time stability property even in the case of the sampled-time implementation of the control law. Theoretical results are supported by numerical simulations.

## I. INTRODUCTION

By definition, the homogeneity is a dilation symmetry known since 18th century, when Leonhard Euler studied a symmetry of functions with respect to the uniform dilation of its argument  $x \mapsto \lambda x$ . The weighted dilation was introduced in 1950s. For instance, in 1958 Vladimir Zubov introduced weighted homogeneous Ordinary Differential Equations (ODEs) [1]. Some extensions of the homogeneity theory of finite-dimensional and infinite-dimensional dynamical models can be found in [2], [3], [4], [5]. Homogeneous differential equations/inclusions form an important class of control system models [6], [7], [8], [9], [10]. They appear as local approximations [11] or set-valued extensions [12] of nonlinear systems and include models of process control [13], mechanical models with frictions [14], etc. Stability and stabilizability problems were studied for both standard [15], [16] and weighted homogeneous [17], [18], [19], [20], [21], [22], [23] systems which are the most popular today [14], [12], [7], [8], [10]. The homogeneous MPC design is studied in [24]. An introduction to homogeneous optimal control can be found in [5, Chapter 12] and an energetically optimal homogeneous stabilization is studied in [25].

An asymptotically stable homogeneous system is finite-time stable in the case of negative homogeneity degree and nearly fixed-time stable in the case of the positive homogeneity degree (see, e.g. [26], [27], [8]). However, the finite/fixed-time stability is a fragile property in the sense that an improper discretization of a finite-time or a fixed-time stable ordinary differential equation (ODE) may result in a chattering [28], [29] or even in a finite-time blow up [30]. Moreover, the explicit discretization (sampled-time implementation) of a finite-time control yields a chattering even if this control law is a continuous function of state [31], [32]. That is why the discretization issues are very important for practical implementation of finite/fixed-time control/estimation algorithms [28], [33], [34], [35], [36], [37], [38], [39].

The concept of consistent discretization introduced in [40] postulates that stability properties of a continuous-time system are preserved in its discrete-time counterpart (approximation). Consistent discretizations for stable generalized homogeneous ODEs were developed in [40], [41] based on Lyapunov function theory. Some schemes with state dependent discretization step were given in [42]. Being efficient for numerical simulations, the mentioned schemes do not allow a consistent discretization (sampled-time implementation) of *finite-time controllers* in the general case. To the best of authors' knowledge, such implementations are developed only for the conventional (first order) sliding mode algorithms [28], [43] as well as for the twisting algorithm [36] and super-twisting algorithm [44] based on the implicit method. This paper presents a consistent discretization for a homogeneous controller studied in [45] for the case of a linear plant modeled by the double integrator. It is shown that the sampled-time implementation of the controller according to the developed scheme preserves the finite-time stability property of the original closed-loop continuous-time system in the disturbance-free case. We also prove the robustness (Input-to-State Stability) of the obtained sampled-time controller with respect to bounded additive perturbations. Numerical simulations show the efficiency of this scheme for rejection of the chattering caused by sampled-time implementation of the continuous-time control algorithm.

*Notation:*  $\mathbb{N}$  is the set of natural numbers including 0;  $\mathbb{R}$  is the field of real numbers;  $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$ ;  $\mathbb{C}$  is the field of complex numbers;  $\mathbf{0}$  is the zero of a vector space (e.g., the zero vector in  $\mathbb{R}^n$  or the zero matrix in  $\mathbb{R}^{n \times m}$ );  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $P \succ 0$  denotes positive definiteness of a matrix  $P = P^\top \in \mathbb{R}^{n \times n}$ ;  $\lambda_{\max}(P)$  is a maximum eigenvalue of a symmetric matrix  $P$ ;  $\|x\| = \sqrt{x^\top P x}$  with  $P \succ 0$  denotes the weighted Euclidean norm in  $\mathbb{R}^n$ ;  $\mathcal{K}$  denotes a class of strictly increasing positive definite functions  $[0, +\infty) \mapsto [0, +\infty)$ ; a function  $\sigma \in \mathcal{K}$  of the class  $\mathcal{K}_\infty$  if  $\sigma(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ ; a function  $\sigma : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$  belongs to the class  $\mathcal{KL}$  if the function  $s \mapsto \sigma(s, \tau)$  belongs to the class  $\mathcal{K}$  for any fixed  $\tau \in [0, +\infty)$  and the function  $\tau \mapsto \sigma(s, \tau)$  is monotonically decreasing to zero for any fixed  $s \in [0, +\infty)$ ; the set of continuous maps  $X \mapsto Y$  is denoted by  $C(X, Y)$ , where  $X, Y$  are subsets of normed vector spaces;  $L^\infty(\mathbb{R}, \mathbb{R}^n)$  - the space of the essential bounded function  $\mathbb{R} \mapsto \mathbb{R}^n$ ;  $\|q\|_{L^\infty((a,b), \mathbb{R}^n)} = \text{ess sup}_{t \in (a,b)} \|q(t)\|$  for  $q \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ .

\*This work was partially supported by the National Natural Science Foundation of China under Grant 62050410352

<sup>1</sup> Andrey Polyakov and Denis Efimov are with Univ. Lille, Inria, CNRS, Centrale Lille, France, e-mail: andrey.polyakov(denis.efimov)@inria.fr

<sup>2</sup> Xubin Ping (the corresponding author) is with Xidian University, Xi'an, Shaanxi, China, e-mail: pingxubin@126.com.

## II. PROBLEM STATEMENT

Let us consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}_+, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is control input and  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  are known matrices.

Recall [46] that a system  $\dot{x} = f(t, x)$ ,  $t \in \mathbb{R}_+$ ,  $x(0) = x_0$  is globally uniformly *finite-time stable* if it is Lyapunov stable and there exists a locally bounded function  $T : \mathbb{R}^n \mapsto [0, +\infty)$  such that any trajectory of the system vanishes to zero in a finite time:  $x(t, x_0) = \mathbf{0}$ ,  $\forall t \geq T(x_0)$ ,  $\forall x_0 \in \mathbb{R}^n$ .

*Definition 1:* Let the system (1) with a feedback  $u \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^m)$  be globally uniformly finite-time stable. A family of functions  $\tilde{u}_h : \mathbb{R}^n \mapsto \mathbb{R}^m$  parameterized by a scalar  $h > 0$  is said to be a *consistent discretization* of  $u$  if

- **Finite-time stability:** the closed-loop system (1) with  $u(t) = \tilde{u}_h(x(t_i))$ ,  $t \in [t_i, t_{i+1})$ ,  $t_i = ih$ ,  $i = 0, 1, \dots$  (2)

is globally uniformly finite-time stable;

- **Approximation:**  $\forall r_2 > \forall r_1 > 0 : \exists \omega_{r_1, r_2} \in \mathcal{K}$  such that

$$\sup_{r_1 \leq \|x\| \leq r_2} \|\tilde{u}_h(x) - u(x)\| \leq \omega_{r_1, r_2}(h). \quad (3)$$

The first condition of Definition 1 asks the sampled-time control system to be finite-time stable for any fixed sampling period  $h > 0$ . The second condition guarantees that the control  $u_h$  is indeed an approximation of  $u$ , i.e.  $u_h \rightarrow u$  as  $h \rightarrow 0^+$  uniformly on compacts from  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ .

In this paper we deal with the controlled double integrator.

*Assumption 1:*  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

The aim of this paper is to develop a consistent (in the sense of the above definition) discretization for a class of homogeneous finite-time controllers (given below).

## III. PRELIMINARIES: HOMOGENEOUS SYSTEMS

### A. Linear dilation and homogeneous norm

The so-called linear (geometric) dilation [5, Chapter 6] in  $\mathbb{R}^n$  is given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{\infty} \frac{(sG_{\mathbf{d}})^i}{i!}, \quad s \in \mathbb{R}, \quad (4)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix<sup>1</sup> known as the *generator of linear dilation*. The latter guarantees that  $\mathbf{d}$  satisfies the limit property,  $\|\mathbf{d}(s)x\| \rightarrow 0$  as  $s \rightarrow -\infty$  and  $\|\mathbf{d}(s)x\| \rightarrow +\infty$  as  $s \rightarrow +\infty$ , required for a group  $\mathbf{d}$  to be a dilation in  $\mathbb{R}^n$  (see, e.g., [3]). The linear dilation introduces an alternative norm topology in  $\mathbb{R}^n$  by means the so-called canonical homogeneous norm.

*Definition 2:* [5] The functional  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  given by  $\|x\|_{\mathbf{d}} = 0$  for  $x = \mathbf{0}$  and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (5)$$

is called the *canonical homogeneous norm* in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a monotone dilation<sup>2</sup>.

<sup>1</sup>A matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is called anti-Hurwitz if  $-G_{\mathbf{d}}$  is Hurwitz.

<sup>2</sup>A dilation in  $\mathbb{R}^n$  is monotone if and only if for any  $x \in \mathbb{R}^n$  the function  $s \mapsto \|\mathbf{d}(s)x\|$ ,  $s \in \mathbb{R}$  is monotone.

Notice that  $\|x\| = 1$  (resp.  $\|x\| \leq 1$ ) is equivalent to  $\|x\|_{\mathbf{d}} = 1$  (resp.  $\|x\|_{\mathbf{d}} \leq 1$ ). For the uniform dilation  $\mathbf{d}(s) = e^{sI_n}$ ,  $s \in \mathbb{R}$  we have  $\|\cdot\| = \|\cdot\|_{\mathbf{d}}$ .

*Theorem 1:* [47] If  $\mathbf{d}$  is a monotone dilation then the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is

- continuous on  $\mathbb{R}^n$ , locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  and there exist  $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}_{\infty}$  such that

$$\underline{\sigma}(\|x\|) \leq \|x\|_{\mathbf{d}} \leq \bar{\sigma}(\|x\|), \quad \forall x \in \mathbb{R}^n;$$

- differentiable on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  provided that  $\|\cdot\|$  is differentiable on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ .

Moreover, for  $\|x\| = \sqrt{x^{\top}Px}$  with the symmetric matrix  $P \in \mathbb{R}^{n \times n}$  satisfying

$$P \succ 0, \quad PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P \succ 0$$

we have

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad \forall x \neq \mathbf{0}. \quad (6)$$

and

$$\left| \|x_1\|_{\mathbf{d}}^{\beta} - \|x_2\|_{\mathbf{d}}^{\beta} \right| \leq \|x_1 - x_2\|, \quad \forall x_i \in \mathbb{R}^n : \|x_i\| \geq 1, i = 1, 2, \quad (7)$$

where  $\beta = 0.5 \lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}^{\top}P^{1/2}) > 0$ .

Below the canonical homogeneous norm is used as a Lyapunov function for analysis and control design.

### B. Homogeneous continuous-time systems

*Definition 3:* [3] A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)f(x)$  (resp.  $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$ ), for all  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}$ .

Homogeneity of a function (operator) is inherited by any other object induced by this function. For example, the Euler homogeneous function theorem implies that the derivative of the homogeneous function is homogeneous as well. If a vector field  $f$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  then solutions of the ODE

$$\dot{x} = f(x) \quad (8)$$

are symmetric [3]:  $x(e^{-\mu s}t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(t, x_0)$ , where  $x(t, z)$  denotes a solution of (8) with  $x(0) = z$ .

*Example 1:* [48] The linear vector field  $x \mapsto Ax$ ,  $A \in \mathbb{R}^{n \times n}$  is  $\mathbf{d}$ -homogeneous of the degree  $\mu \neq 0 \Leftrightarrow A$  is nilpotent  $\Leftrightarrow AG_{\mathbf{d}} = (\mu I_n + G_{\mathbf{d}})A$ .

The homogeneity degree specifies the convergence rate.

*Theorem 2:* [27] Let the vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuous and  $\mathbf{d}$ -homogeneous of a degree  $\mu \in \mathbb{R}$ . If the system (8) is asymptotically stable then it is globally finite-time stable for  $\mu < 0$ .

The homogeneous control systems are robust (input-to-state stable) with respect to a rather large class of perturbations [49], [8].

### C. Homogeneous stabilization of linear plant

The following theorem combines the results of [45], [5].

**Theorem 3:** Let the pair  $\{A, B\}$  be controllable. Then

1) the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0} \quad (9)$$

has a solution with respect to the pair  $Y_0 \in \mathbb{R}^{m \times n}$ ,  $G_0 \in \mathbb{R}^{n \times n}$ , and  $G_0 - I_n$  is invertible,  $G_d = I_n + \mu G_0$  is anti-Hurwitz for any  $\mu \in [-1, 1/k]$ , where  $k$  is minimal natural number such that  $\text{rank}[B, AB, \dots, A^{k-1}B] = n$ , the matrix  $A_0 = A + BY_0(G_0 - I_n)^{-1}$  satisfies

$$A_0G_d = (G_d + \mu I_n)A_0, \quad G_dB = B; \quad (10)$$

2) the system of linear matrix inequalities (LMIs) with the linear matrix equation

$$\begin{aligned} A_0X + XA_0^\top + BY + Y^\top B^\top + \rho(G_dX + XG_d^\top) &= \mathbf{0}, \\ G_dX + XG_d^\top &\succ 0, \quad X = X^\top \succ 0 \end{aligned} \quad (11)$$

has a solution  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{m \times n}$  for any  $\rho \in \mathbb{R}_+$ ;

3) the canonical homogeneous norm  $\|\cdot\|_d$  induced by the weighted Euclidean norm  $\|x\| = \sqrt{x^\top Px}$  with  $P = X^{-1}$  is a Lyapunov function of the system (1) with

$$u(x) = K_0x + \|x\|_d^{1+\mu} K \mathbf{d}(-\ln \|x\|_d)x, \quad (12)$$

$$K_0 = Y_0(G_0 - I_n)^{-1}, \quad K = YX^{-1}, \quad (13)$$

where  $\mathbf{d}$  is a dilation generated by  $G_d$ ; moreover,

$$\frac{d}{dt} \|x\|_d = -\rho \|x\|_d^{1+\mu}, \quad x \neq \mathbf{0}; \quad (14)$$

4) the feedback law  $u$  given by (12) is continuously differentiable on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ ,  $u$  is continuous at zero if  $\mu > -1$  and  $u$  is discontinuous at zero but locally bounded if  $\mu = -1$ ;

5) the closed-loop system (1), (12) is  $\mathbf{d}$ -homogeneous of the degree  $\mu$ .

Obviously, the closed-loop system (1),(12) is finite-time stable if  $\mu < 0$  and it is nearly fixed-time stable if  $\mu > 0$ . For  $\mu = 0$  the control (12) becomes the linear exponentially stabilizing feedback  $u = K_0x + Kx$ .

Such a control law (under some variation and/or simplifications) is well-known in the literature as a solution to a finite-time stabilization problem for linear plants [50], [19], [45]. Using the topological equivalence of any stable  $\mathbf{d}$ -homogeneous system to a standard homogeneous one we prove the following corollary which allows the explicit formula for solutions of the closed-loop system (1), (12) to be presented.

**Corollary 1:** Under conditions of Theorem 3 with  $\mu \neq 0$ , the solution of the closed-loop system (1), (12) is unique and satisfies the identity

$$x(t + \tau) = Q_\tau(\|x(t)\|_d)x(t), \quad (15)$$

where  $\tau, t \geq 0$  and  $Q_\tau(0) = \mathbf{0}$ ,

$$Q_\tau(r) = \begin{cases} e^{G_d \ln r} \hat{Q}_\tau(r) e^{-G_d \ln r} & \text{if } \frac{1}{r^\mu} > -\mu\rho\tau, \\ 0 & \text{if } \frac{1}{r^\mu} \leq -\mu\rho\tau, \end{cases} \quad r > 0, \quad (16)$$

$$\hat{Q}_\tau(r) = e^{\frac{G_d}{\mu} \ln \frac{1}{1+\mu\rho\tau r^\mu}} e^{\frac{(A+B(K_0+K)+\rho G_d) \ln(1+\mu\rho\tau r^\mu)}{\rho\mu}}.$$

Moreover, for  $x \neq \mathbf{0}$  one holds

$$\frac{d}{d\tau} Q_\tau(\|x\|_d)x = A Q_\tau(\|x\|_d)x + B u(Q_\tau(\|x\|_d)x). \quad (17)$$

**Proof.** Denoting  $y = \|x\|_d \mathbf{d}(-\ln \|x\|_d)x$ , we derive  $\|y\| = \|x\|_d \|\mathbf{d}(-\ln \|x\|_d)x\| = \|x\|_d$  and

$$\begin{aligned} \dot{y} &= \frac{d\|x\|_d}{dt} (I_n - G_d) \mathbf{d}(-\ln \|x\|_d)x + \|x\|_d \mathbf{d}(-\ln \|x\|_d) \dot{x} \\ &= \rho \|y\|^\mu (G_d - I_n)y \\ &\quad + \|x\|_d \mathbf{d}(-\ln \|x\|_d) (A_0x + \|x\|_d^{1+\mu} B K \mathbf{d}(-\ln \|x\|_d)x). \end{aligned}$$

Since  $\mathbf{d}(s)A_0 = e^{-\mu s}A_0\mathbf{d}(s)$  and  $\mathbf{d}(s)B = e^s B$  for all  $s \in \mathbb{R}$  then  $\dot{y} = \|y\|^\mu (A_0 + BK + \rho(G_d - I_n))y$ .

In this case, using (11) we conclude

$$\begin{aligned} \frac{d}{dt} \|y(t)\| &= \frac{y^\top P \dot{y}(t)}{\|y(t)\|} = \|y\|^{\mu-1} y^\top P (A_0 + BK + \rho(G_d - I_n))y \\ &= \|y\|^{\mu-1} \left( \frac{y^\top \{P(A_0 + BK + \rho G_d) + (A_0 + BK + \rho G_d)^\top P\} y}{2} - \rho y^\top P y \right) \\ &= -\rho \|y(t)\|^{\mu+1} \end{aligned}$$

and  $\|y(t + \tau)\|^{-\mu} = \|y(t)\|^{-\mu} + \mu\rho\tau$ , for  $\|y(t)\|^{-\mu} + \mu\rho\tau \geq 0$ . Obviously,  $\|y(t + \tau)\| = 0$  if  $\|y(t)\|^{-\mu} + \mu\rho\tau \leq 0$ . The latter corresponds to the negative homogeneity degree  $\mu < 0$  and the finite-time stability of the closed-loop system. Hence, denoting  $K_{lin} = K_0 + K$  we obtain

$$\begin{aligned} y(t + \tau) &= e^{(A+BK_{lin}+\rho(G_d-I_n)) \int_0^\tau \|y(t+\sigma)\|^\mu d\sigma} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_d-I_n)) \int_0^\tau \frac{1}{\|y(t)\|^{-\mu} + \mu\rho\sigma} d\sigma} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_d-I_n)) \frac{1}{\mu\rho} \ln \frac{\|y(t)\|^{-\mu} + \mu\rho\tau}{\|y(t)\|^{-\mu}}} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_d-I_n)) \frac{1}{\mu\rho} \ln(1+\mu\rho\tau \|y(t)\|^\mu)} y(t). \end{aligned}$$

Returning to the original coordinates we derive

$$\begin{aligned} x(t + \tau) &= \mathbf{d}(\ln \|y(t + \tau)\|) \frac{y(t + \tau)}{\|y(t + \tau)\|} \\ &= \mathbf{d}(\ln(\|y(t)\|^{-\mu} + \mu\rho\tau)^{\frac{1}{-\mu}}) \\ &\quad \cdot e^{(A+BK_{lin}+\rho(G_d-I_n)) \frac{1}{\rho\mu} \ln(1+\mu\rho\tau \|y(t)\|^\mu)} \\ &\quad \cdot \frac{y(t)}{(\|y(t)\|^{-\mu} + \mu\rho\tau)^{\frac{1}{-\mu}}} \\ &= \mathbf{d}(\ln(\|x(t)\|_d^\mu + \mu\rho\tau)^{\frac{1}{-\mu}}) \\ &\quad \cdot e^{(A+BK_{lin}+\rho G_d) \frac{1}{\rho\mu} \ln(1+\mu\rho\tau \|x(t)\|_d^\mu)} \\ &\quad \cdot \mathbf{d}(-\ln \|x(t)\|_d)x(t) \\ &= Q_\tau(\|x(t)\|_d)x(t) \end{aligned}$$

for all  $t \geq 0$  and all  $\tau \geq 0$ . The identity (17) follows immediately from the fact that  $\tau \mapsto x(\tau)$  is a solution of (1), (12). ■

The latter corollary presents an important result for a class of nonlinear homogeneous system under consideration, since this is a very rare case that a solution of an essentially nonlinear system in  $\mathbb{R}^n$  can be found explicitly (see the formula (15)). Such a property may be very useful for various analysis.

**Remark 1:** Under Assumption 1 we have

$$K_0 = \mathbf{0}, \quad A_0 = A, \quad G_d = \begin{bmatrix} 1-\mu & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu \in [-1, 0.5].$$

In this case, the condition (11) with  $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$  and  $Y = [y_1, y_2] \in \mathbb{R}^{1 \times 2}$  becomes

$$\begin{aligned} x_{11} > 0, \quad x_{12} &= -\rho(1-\mu)x_{11}, \quad x_{22} > \frac{(2-\mu)^2 \rho^2 (1-\mu)x_{11}}{4}, \\ y_1 &= \rho^2(2-\mu)(1-\mu)x_{11} - x_{22}, \quad y_2 = -\rho x_{22}. \end{aligned} \quad (18)$$

and the control (12) has the form

$$u(x) = k_1 \|x\|_{\mathbf{d}}^{2\mu} x_1 + k_2 \|x\|_{\mathbf{d}}^{\mu} x_2, \quad K = [k_1, k_2] = YP, \quad (19)$$

where  $x = [x_1, x_2]^{\top} \in \mathbb{R}^2$ ,  $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = X^{-1}$  and the canonical homogeneous  $\|x\|_{\mathbf{d}}$  is induced by the norm  $\|x\| = \sqrt{x^{\top} P x}$ . From (5) we derive

$$x^{\top} \begin{bmatrix} \|x\|_{\mathbf{d}}^{-1+\mu} & 0 \\ 0 & \|x\|_{\mathbf{d}}^{-1} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \|x\|_{\mathbf{d}}^{-1+\mu} & 0 \\ 0 & \|x\|_{\mathbf{d}}^{-1} \end{bmatrix} x = 1.$$

Hence, if  $\mu = \frac{p}{q} \geq 0$ , where  $p, q$  are integers and  $q \geq 1$  then  $\|x\|_{\mathbf{d}} = r^q$ , where  $r \in \mathbb{R}$  is a unique positive root of the following polynomial equation

$$r^{2q} = ar^{2p} + br^p + c \quad (20)$$

where  $a = p_{11}x_1^2$ ,  $b = 2p_{12}x_1x_2$  and  $c = p_{22}x_2^2$ . In particular, for  $q = -p = 1$  we have  $\mu = -1$  and (20) becomes a quartic equation, which can be solved using Ferrari formulas (see. e.g. [51]).

#### IV. CONSISTENT DISCRETIZATION OF HOMOGENEOUS CONTROL FOR DOUBLE INTEGRATOR

Let us consider the discrete-time version of the system (1) with the sample-time control implementation. Obviously, if  $u(t) = u(t_k)$  for  $t \in [t_k, t_{k+1})$  then

$$x_{k+1} = A_h x_k + B_h u(t_k), \quad k = 0, 1, \dots, \quad (21)$$

where  $x_k = x(t_k)$ ,  $t_k = kh$ ,  $A_h = e^{hA}$ ,  $B_h = \left( \int_0^h e^{sA} ds \right) B$ . Using (21) we derive

$$x_{k+2} = B_h u(t_{k+1}) + A_h B_h u(t_k) + A_h^2 x_k. \quad (22)$$

For the double integrator (see Assumption 1) we have  $A_h = I_2 + hA$ ,  $B_h = \left( hI_2 + \frac{h^2}{2}A \right) B$  and, obviously, the matrix

$$W_h = [B_h, A_h B_h] \quad (23)$$

is invertible. Since by Corollary 1 for any solution of the closed-loop continuous-time system (1) with the homogeneous control (12) we have

$$x(t+2h) = Q_{2h}(\|x(t)\|_{\mathbf{d}})x(t)$$

then to guarantee  $x_{k+2} = Q_{2h}(\|x_k\|_{\mathbf{d}})x_k$  we may select

$$\begin{bmatrix} u(t_{k+1}) \\ u(t_k) \end{bmatrix} = W_h^{-1} (Q_{2h}(\|x_k\|_{\mathbf{d}}) - A_h^2) x_k, \quad (24)$$

where  $Q_{\tau}$  is given by (16). By construction, the discrete-time system (21), (24) tracks the states of the closed-loop continuous-time system (1), (12) at discrete time instances  $t_{2k}$ . This means that the discretization (24) preserves the convergence rate of the original system. However, the control (24) is not a conventional feedback law, since  $u(t_{k+1})$

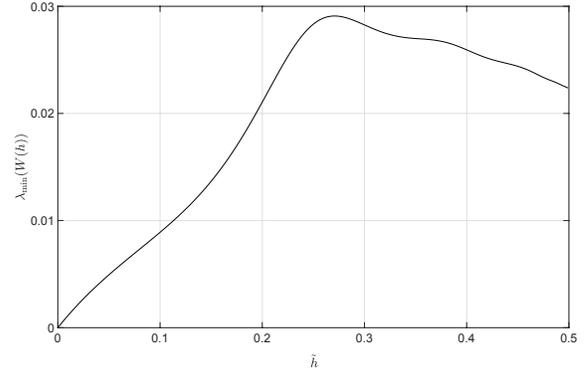


Fig. 1. The minimal eigenvalue of the matrix  $W(\tilde{h})$  for  $\mu = -1/2, \rho = 2$

depends on  $x_k = x(t_k)$  but not on  $x_{k+1}$ . Inspired by (24), we can also consider the control law:

$$\tilde{u}_h(x_k) := \tilde{K}_h(\|x_k\|_{\mathbf{d}})x_k, \quad (25)$$

where  $\tilde{K}_h(r) = [0 \ 1] W_h^{-1} (Q_{2h}(r) - A_h^2)$  for any  $r \geq 0$ .

To show that the static nonlinear feedback (25) is a consistent discretization of the finite-time stabilizing feedback (12), the discrete-time closed-loop system (21), (25) has to be globally finite-time stable for  $\mu < 0$  and the feedback (25) must tend to the control law (12) as  $h \rightarrow 0^+$ .

*Theorem 4:* Let  $\mu \in [-1, 0)$  and the control  $u$  be defined by (12) using Theorem 3. The feedback  $\tilde{u}_h$  given by (25) is a consistent discretization of  $u$  if

$$F(h)X^{-1}F^{\top}(h) < X^{-1}, \quad \forall h \in \left(0, \frac{1}{-2\mu\rho}\right], \quad (26)$$

where

$$F = I_2 + hA + (B + \frac{h}{2}AB) \left[ \frac{1}{h} \ -\frac{1}{2} \right] (Q_{2h}(1) - I_2 - 2hA). \quad (27)$$

To design a consistent discretization (25) for (12), the matrix inequality (11) has to be fulfilled together with (26).

*Remark 2 (On feasibility of matrix inequalities):* In the view of Remark 1 the matrices

$$X = x_{11} \tilde{X}, \quad x_{11} > 0, \quad \tilde{X} = \begin{bmatrix} 1 & -\rho(1-\mu) \\ -\rho(1-\mu) & \frac{7(2-\mu)^2 \rho^2 (1-\mu)}{8} \end{bmatrix}$$

$$Y = \frac{\rho^2(2-\mu)(1-\mu)x_{11}}{8} \begin{bmatrix} 8 - 7(2-\mu) & -7\rho(2-\mu) \end{bmatrix}$$

satisfy (11).

Let us consider the matrix-valued function  $W : (0, 1/2] \rightarrow \mathbb{R}^2$  given by  $W(\tilde{h}) = \tilde{X}^{-1} - F(\tilde{h})^{\top} \tilde{X}^{-1} F(\tilde{h})$ . If the minimal eigenvalue of  $W(\tilde{h})$  is positive for all  $\tilde{h} \in (0, 1/2]$  then the condition (26) is fulfilled provided that  $\mu\rho = -1$ . It has been checked numerically that such a selection of  $X$  the fulfillment of (26) for any  $\mu \in [-1, -1/3]$  and  $\rho = -1/\mu$ . For example, the Fig. 1 depicts the evolution of  $\lambda_{\min}(W(\tilde{h}))$  for the case  $\mu = -1/2, \rho = 2$ .

It is well known [49], [8] that homogeneous systems are Input-to-State Stable (ISS) with respect to sufficiently large class of perturbations. Recall [52] that a system

$$\dot{x} = f(t, x, q), \quad t > t_0 \quad (28)$$

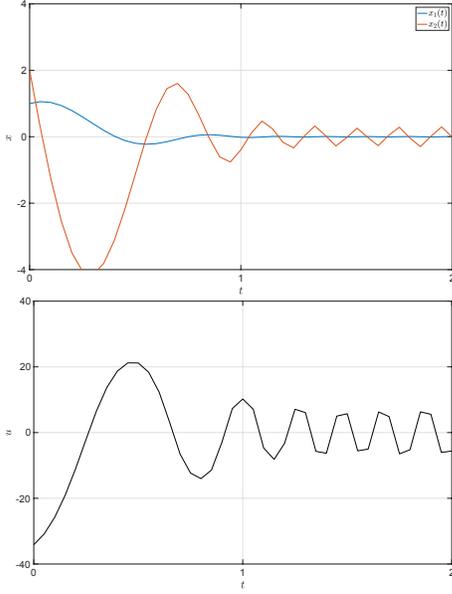


Fig. 2. The evolution of the system (1), (29) for  $h = 0.05$

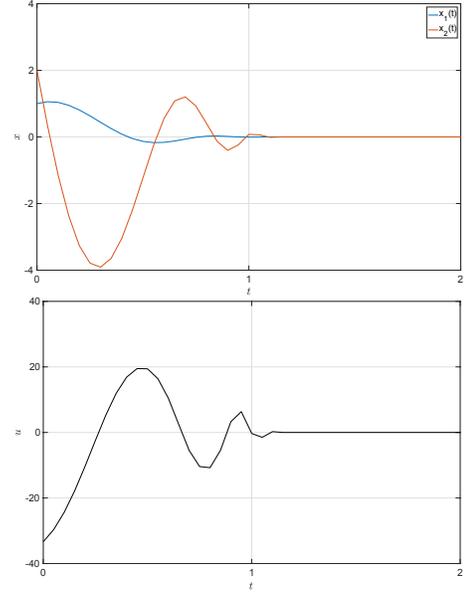


Fig. 3. The evolution of the system (1), (30) for  $h = 0.05$

is ISS with respect to  $q \in L^\infty(\mathbb{R}, \mathbb{R}^m)$  if there exists  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t, x_0)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|q\|_{L^\infty((t_0, t), \mathbb{R}^m)}).$$

Local ISS restricts additionally the set of initial conditions and/or the maximal magnitude of the perturbation  $q$ . The input  $q$  in the above system can be treated as a perturbation.

*Corollary 2:* Under conditions of Theorem 4, the system (1) with the sampled-time control (2), (25) and additive perturbations :

$$\dot{x} = Ax + Bu + q$$

is ISS provided that  $\beta > -\mu$  with  $\beta$  given by Theorem 1.

## V. NUMERICAL SIMULATIONS

The homogeneous control  $u(x)$  is designed using Theorem 3 and Remarks 1, 2 for  $\mu = -0.5, \rho = 2, x_{11} = 1$  under Assumption 1. The evolution of the system (1) with the explicit discretization

$$u(t) = u(x(t_i)), \quad t \in [t_i, t_{i+1}), \quad (29)$$

of the controller  $u$  is shown on Fig. 2. The consistent discretization of  $u$  is given by

$$u(t) = \tilde{u}_h(x(t_i)), \quad t \in [t_i, t_{i+1}), \quad (30)$$

where  $\tilde{u}_h$  is a consistent discretization of  $u$  defined by (25). Fig. 3 shows the evolution of the system (1) with the control (30) and confirms the convergence of the trajectory of the closed-loop system with the control (30) to zero in a finite time.

## VI. CONCLUSIONS AND DISCUSSIONS

A consistent discretization of a homogeneous controller for double integrator is developed based on the explicit representation of a solution for the continuous time homogeneous control system (in the unperturbed case). It is shown that such a discretization preserves finite-time stability of the original control system as well as Input-to-State Stability with respect to additive exogenous perturbations.

The closed-loop system with the consistently discretized controller has the following discrete-time representation

$$x_{k+1} = \tilde{F}(\|x_k\|_{\mathbf{d}})x_k,$$

with

$$\tilde{F}(\|x\|_{\mathbf{d}}) = A_h + B_h K(\|x\|_{\mathbf{d}}), \quad x \in \mathbb{R}^n.$$

The key feature of the consistent discretization is the nilpotence of the matrix  $\tilde{F}(\|x\|_{\mathbf{d}})$  for a sufficiently small  $\|x\|_{\mathbf{d}}$ . Indeed, if  $\|x\|_{\mathbf{d}} \leq \frac{1}{-2\mu\rho}$  then

$$\tilde{F}(\|x\|_{\mathbf{d}}) = A_h - [0 \ 1] W_h^{-1} A_h^2.$$

It is easy to check that this matrix is nilpotent, so  $\tilde{F}(\|x_{k+1}\|_{\mathbf{d}})\tilde{F}(\|x_k\|_{\mathbf{d}}) = \mathbf{0}$  if  $\|x_{k+1}\|_{\mathbf{d}} \leq \frac{1}{-2\mu\rho}$  and  $\|x_k\|_{\mathbf{d}} \leq \frac{1}{-2\mu\rho}$ . This feature guarantees the finite-time convergence of the system (1) with the sampled-time controller (2), (25) to zero. The same feature can be utilized for a consistent discretization of some fixed-time stabilizing controllers.

## REFERENCES

- [1] V. Zubov, "On systems of ordinary differential equations with generalized homogeneous right-hand sides," *Izvestia vuzov. Matematika (in Russian)*, vol. 1, pp. 80–88, 1958. [Online]. Available: <http://mi.mathnet.ru/ivm2874>
- [2] V. V. Khomenuk, "On systems of ordinary differential equations with generalized homogenous right-hand sides," *Izvestia vuzov. Matematika (in Russian)*, vol. 3(22), pp. 157–164, 1961.

- [3] M. Kowski, "Families of dilations and asymptotic stability," *Analysis of Controlled Dynamical Systems*, pp. 285–294, 1991.
- [4] G. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups," *Arkiv for Matematik*, vol. 13, no. 1-2, pp. 161–207, 1975.
- [5] A. Polyakov, *Generalized Homogeneity in Systems and Control*. Springer, 2020.
- [6] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.
- [7] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: application to secure communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 356–360, 2008.
- [8] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous Approximation, Recursive Observer Design, and Output Feedback," *SIAM Journal of Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.
- [9] D. Efimov and W. Perruquetti, "Oscillations conditions in homogenous systems," in *Proc. IFAC NOLCOS Symp.*, 2010, pp. 1379–1384.
- [10] A. Polyakov, D. Efimov, and W. Perruquetti, "Robust stabilization of MIMO systems in finite/fixed time," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 1, pp. 69–90, 2016.
- [11] H. Hermes, "Nilpotent approximations of control systems and distributions," *SIAM Journal of Control and Optimization*, vol. 24, no. 4, p. 731, 1986.
- [12] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, no. 5, pp. 823–830, 2005.
- [13] K. Zimenko, D. Efimov, A. Polyakov, and W. Perruquetti, "A note on delay robustness for homogeneous systems with negative degree," *Automatica*, vol. 79, no. 5, pp. 178–184, 2017.
- [14] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2005.
- [15] V. I. Zubov, *Methods of A.M. Lyapunov and Their Applications*. Noordhoff, Leiden (Translated from Russian: V.I. Zubov, *Metody Lyapunova i ih primeneniya*, Leningrad: LGU, 1957), 1964.
- [16] A. Andreini, A. Bacciotti, and G. Stefani, "Global stabilizability of homogenous vector fields of odd degree," *Systems & Control Letters*, vol. 10, pp. 251–256, 1988.
- [17] J.-M. Coron and L. Praly, "Adding an integrator for the stabilization problem," *Systems & Control Letters*, vol. 17, no. 2, pp. 89–104, 1991.
- [18] H. Hermes, "Homogeneous feedback controls for homogeneous systems," *Systems & Control Letters*, vol. 24, pp. 7–11, 1995.
- [19] L. Praly, "Generalized weighted homogeneity and state dependent time scale for linear controllable systems," in *Proc. IEEE Conference on Decision and Control*, San Diego, USA, 1997, pp. 4342–4347.
- [20] R. Sepulchre and D. Aeyels, "Homogeneous Lyapunov Functions and Necessary Conditions for Stabilization," *Mathematics of Control, Signals and Systems*, vol. 9, pp. 34–58, 1996.
- [21] —, "Stabilizability does not imply homogeneous stabilizability for controllable homogeneous systems," *SIAM Journal of Control and Optimization*, vol. 34, no. 5, pp. 1798–1813, 1996.
- [22] L. Grüne, "Homogeneous state feedback stabilization of homogeneous systems," *SIAM Journal of Control and Optimization*, vol. 38, no. 4, pp. 1288–1308, 2000.
- [23] N. Nakamura, H. Nakamura, and Y. Yamashita, "Homogeneous stabilization for input-affine homogeneous systems," in *Conference on Decision and Control*, 2007, pp. 80–85.
- [24] J.-M. Coron, L. Grüne, and K. Worthmann, "Model predictive control, cost controllability, and homogeneity," *SIAM Journal on Control and Optimization*, vol. 58, no. 5, pp. 2979–2996, 2020.
- [25] A. Polyakov, D. Efimov, and X. Ping, "On energetically optimal finite-time stabilization," in *Conference on Decision and Control*, 2021 (submitted).
- [26] H. Nakamura, Y. Yamashita, and H. Nishitani, "Smooth Lyapunov functions for homogeneous differential inclusions," in *Proceedings of the 41st SICE Annual Conference*, 2002, pp. 1974–1979.
- [27] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.
- [28] V. Acary and B. Brogliato, "Implicit euler numerical scheme and chattering-free implementation of sliding mode systems," *Systems & Control Letters*, vol. 59, no. 5, pp. 284–293, 2010.
- [29] A. Levant, "Chattering analysis," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1380–1389, 2010.
- [30] —, "On fixed and finite time stability in sliding mode control," in *IEEE Conference on Decision and Control*, 2013, pp. 4260–4265.
- [31] D. Efimov, A. Polyakov, A. Levant, and W. Perruquetti, "Realization and discretization of asymptotically stable homogeneous systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5962–5969, 2017.
- [32] F. Josse, E. Bernuau, E. Moulay, and P. Coirault, "Robustness of sampled-data homogeneous systems," *Automatica*, vol. 123, p. 109345, 2021.
- [33] R. Kikuuwe, S. Yasukouchi, H. Fujimoto, and M. Yamamoto, "Proxy-based sliding mode control: A safer extension of pid position control," *IEEE Transactions on Robotics*, vol. 26, no. 4, pp. 670–683, 2010.
- [34] M. Livne and A. Levant, "Proper discretization of homogeneous differentiators," *Automatica*, vol. 50, no. 8, pp. 2007–2014, 2014.
- [35] S. Koch, M. Reichhartinger, M. Horn, and L. Fridman, "Discrete-time implementation of homogeneous differentiators," *IEEE Transactions on Automatic Control*, vol. 65, no. 2, pp. 757–762, 2019.
- [36] O. Huber, V. Acary, and B. Brogliato, "Lyapunov stability analysis of the implicit discrete-time twisting control algorithm," *IEEE Transactions on Automatic Control*, vol. 65, no. 6, pp. 2619–2626, 2020.
- [37] B. Brogliato and A. Polyakov, "Digital implementation of sliding-mode control via the implicit method: A tutorial," *International Journal of Robust and Nonlinear Control*, vol. 31, pp. 3528–3586, 2021.
- [38] L. Michel, M. Chanes, F. Plestan, J. Aoustin, and J.-P. Barbot, "A noise less-sensing semi-implicit discretization of a homogeneous differentiator : principle and application," in *European Control Conference (ECC)*, 2021.
- [39] A. Hanan, A. Jbara, and A. Levant, "Low-chattering discretization of sliding mode control," in *the 60th IEEE Conference on Decision and Control*, 2021.
- [40] A. Polyakov, D. Efimov, and B. Brogliato, "Consistent discretization of finite-time and fixed-time stable systems," *SIAM Journal of Control and Optimization*, vol. 57, no. 1, pp. 78–103, 2019.
- [41] T. Sanchez, A. Polyakov, and D. Efimov, "Lyapunov-based consistent discretisation of stable homogeneous systems," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 9, pp. 3587–3605, 2021.
- [42] D. Efimov, A. Polyakov, and A. Aleksandrov, "Discretization of homogeneous systems using euler method with a state-dependent step," *Automatica*, vol. 109, no. 11, p. 108546, 2019.
- [43] O. Huber, V. Acary, and B. Brogliato, "Lyapunov stability and performance analysis of the implicit discrete sliding mode control." *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3016–3030, 2016.
- [44] B. Brogliato, A. Polyakov, and D. Efimov, "The implicit discretization of the super-twisting sliding-mode control algorithm," *IEEE Transactions on Automatic Control*, vol. 65, no. 8, pp. 3707–3713, 2020.
- [45] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: Implicit Lyapunov function approach," *Automatica*, vol. 51, no. 1, pp. 332–340, 2015.
- [46] S. Bhat and D. Bernstein, "Finite time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.
- [47] A. Polyakov, "Sliding mode control design using canonical homogeneous norm," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 3, pp. 682–701, 2018.
- [48] K. Zimenko, A. Polyakov, D. Efimov, and W. Perruquetti, "Robust feedback stabilization of linear mimo systems using generalized homogenization," *IEEE Transactions on Automatic Control*, 2020.
- [49] Y. Hong, " $H_\infty$  control, stabilization, and input-output stability of nonlinear systems with homogeneous properties," *Automatica*, vol. 37, no. 7, pp. 819–829, 2001.
- [50] V. Korobov, "A solution of the synthesis problem using controllability function," *Doklady Akademii Nauk SSSR*, vol. 248, pp. 1051–1063, 1979.
- [51] A. Polyakov and I. Chairez, "A new homogeneous quasi-continuous second order sliding mode control," in *XVI Congreso Latinoamericano de Control Automático*, 2014.
- [52] E. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, pp. 435–443, 1989.
- [53] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 856–869, 2001.