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# Optimal control of bistable travelling waves: looking for the spatial distribution of a killing action to block a pest invasion

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## Abstract

Some pests and vectors of many vector-borne diseases (like mosquitoes for malaria and dengue) are known to invade any homogeneous and favorable territory, following a traveling wave type dynamic. The density of individuals in the field is commonly modeled as the solution of a bistable reaction-diffusion equation on an unbounded domain. In this work, we are interested in finding an optimal strategy to block such a solution by means of a population elimination action in a prescribed subdomain (modeling, for instance the effect of a mechanical action or an insecticide applied in a certain region to reduce the number of individuals in the population). We propose a complete description of the solutions of this problem, based on the precise analysis of the optimality conditions and on arguments for comparison between the possible strategies.

**Keywords:** traveling waves, optimal control, bistable reaction-diffusion equations, population dynamics, pest control

**AMS classification:** 35K57, 35C07, 49K15, 92D25.

## 1 Introduction

Protecting crops against pests or human populations against the vectors of diseases (like malaria or dengue) is one of the main concerns of farmer's and of public health authorities. Many new and more environment-friendly population reduction methods have been tested in the last decades among which the sterile insect technique which has recently attracted a considerable amount of attention among the mathematical community (see, for instance, [2, 3, 5, 9] and the references therein). Nevertheless, for the moment, the most widespread technique is a direct killing action, often using insecticides or phytosanitary products. When such a choice is made, one can at least try to optimize the killing effort (for instance, minimize the quantity of killing product needed for attaining the prescribed objectives) in order to reduce the environmental impact of the intervention - this is the object of the present study.

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The method presented here is applicable to more general pests or disease vectors whose spatial movement can be modeled by diffusion and to more general population reduction actions that can be modelled by an additional death rate of the population at the places where the intervention takes place. Nevertheless, to make the presentation simpler, we will mainly do it in the setting of the important case of using insecticide against mosquitoes - nowadays, this strategy is the main one that is systematically applied against mosquitoes during epidemic outbreaks worldwide. The alternative techniques are mostly used, when possible, in a preventive manner to keep the mosquito population under control hoping that this will prevent the outbreaks from happening.

Many dangerous diseases (such as dengue fever and malaria) are transmitted by certain species of mosquitoes such as *Aedes albopictus* and *Aedes aegypti* among others (see [1]). Since there is no efficient vaccine that can be widely deployed, in order to prevent the emergence of epidemics, health authorities usually act by killing mosquitoes, thus reducing the vector population. This is mainly done through the use of insecticides. This work is devoted to the mathematical study of the use of an insecticide (or any other similar destructive action) in order to block a natural invasion phenomenon.

The main problem with the use of an insecticide is that it does not specifically target one mosquito species but kills (or seriously affects) many other animal species directly or indirectly (including many predators and humans) and has a wide negative impact on the ecosystem. It is therefore ecologically relevant to try to use as little insecticide as possible. Nevertheless, it is necessary to use enough insecticide to prevent invasion phenomena. In this work, we attempt to provide a mathematical answer to this problem in a simple theoretical framework. We model the search for this trade-off using an optimal control problem whose unknown is the action taken to kill the mosquitoes (for example the use of an insecticide).

## 1.1 Model and main results

Let us consider a population of pests whose density  $u$  obeys the simple reaction-diffusion dynamics:

$$\begin{cases} -u''(x) = g(u(x)) - \mu(x)u(x)\mathbb{1}_{\{0 < x < L\}}(x), & x \in \mathbb{R}, \\ u(-\infty) = 1, \quad u(+\infty) = 0, & 0 \leq u \leq 1 \text{ in } \mathbb{R}. \end{cases} \quad (1)$$

In this equation,  $g(u)$  stands for the reaction term in the absence of intervention whereas the killing term in the model is  $-\mu(x)u\mathbb{1}_{\{0 < x < L\}}$  which stands for the use of insecticide in some bounded area  $(0, L)$ . Let us provide some explanations on the choice of each term:

We assume that the natural dynamics of the pests is described by a **reaction term  $g$**  which is bistable, meaning that  $g$  satisfies

$$g(0) = g(\theta) = g(1) = 0, \quad g'(0) < 0, g'(1) < 0, \quad (s - \theta)g(s) > 0 \text{ for } s \in (0, \theta) \cup (\theta, 1). \quad (\mathcal{H}_{bis})$$

This means that they are subject to a certain *Allee* effect: at each point where the population is below the threshold  $\theta \in (0, 1)$ , the contribution of the reaction term makes it decrease towards 0. Furthermore, we assume that when there is no killing agent ( $\mu = 0$ ), the population invades the environment, meaning that

$$\int_0^1 g(u)du > 0. \quad (\mathcal{H}_{inv})$$

Mathematically, when  $\mu = 0$ , there exist stable traveling wave solutions to (1) where the stable state 1 invades the stable state 0 (see [6]).

The term  $-u''$  represents the **diffusion** of the species through the field.

The function  $\mu$  is the **control function** that is determined by the instantaneous amount of insecticide applied at each point - to simplify, we assume here that the death rate due to the insecticide is proportional to the quantity of insecticide and suppose even suppose that  $\mu(x)$  is the quantity of

insecticide used at point  $x$ . We remark that a more general relation between the quantity of insecticide and the death term can also be considered when needed.

Finally, the conditions at  $-\infty$  and  $+\infty$  model that we are looking for a wave with a speed equal to 0 : we look for a control  $\mu$  such that we are able to **block an invasive wave**.

The objective is to determine a solution of the reaction-diffusion system introduced above. Nevertheless, we want to act, through the control term  $\mu$ , in order to use as little insecticide as possible. This translates into the fact that we want to minimize the quantity of insecticide  $\mathcal{N}(\mu)$  given by:

$$\mathcal{N}(\mu) = \int_0^L \mu(x) dx.$$

Moreover, in particular in the case of an insecticide, it is relevant to assume that the quantity used is subject to some restriction (which can be due to ecological and/or logistic reasons), leading to introduce  $C > 0$  such that the control  $\mu$  satisfies the pointwise constraint:  $0 \leq \mu \leq C$ , a.e. in  $[0, L]$ . This corresponds to having a maximum insecticide spread rate at any point in space. Therefore, we define the admissible set of controls as being

$$\mathcal{A}_{L,C} := \{ \mu \in L^\infty(\mathbb{R}), \text{ (1) has a solution and } 0 \leq \mu \leq C \text{ a.e. in } \mathbb{R} \}.$$

Notice that  $\mathcal{A}_{L,C}$  depends also implicitly on the parameter  $L$  through (1). We investigate the optimal control problem

$$\boxed{\inf_{\mu \in \mathcal{A}_{L,C}} \mathcal{N}(\mu)}. \quad (\mathcal{P}_{L,C})$$

We now state the main result of this paper, dedicated to the almost complete and explicit characterization of the solutions of the above problem. This statement takes the form of a discussion because the form of the minimizers is strongly dependent on the parameter regimes considered for  $L$  and  $C$ .

To this aim, it is convenient to introduce the functions  $G$  and  $F$  given by

$$G(u) = \int_0^u g(s) ds \quad \text{and} \quad F(s) = 2G(s) + sg(s). \quad (2)$$

We now define two real numbers denoted  $\gamma_0$  and  $\gamma_1$  using the following result, whose proof will be postponed to Appendix A for the sake of readability.

**Lemma 1.** *Under  $(\mathcal{H}_{bis})$  and  $(\mathcal{H}_{inv})$ , each of the equations  $F(u) = 0$  and  $F(u) = F(1)$  have at least one solution in  $(\theta, 1)$ .*

Moreover, we will assume the following technical assumptions relied on the concavity of  $g$  after  $\theta$ .

$$\begin{aligned} &\text{The equation } F(u) = 0 \text{ (resp. } F(u) = F(1)) \text{ has a unique solution} \\ &\gamma_0 \text{ (resp. } \gamma_1) \text{ in } (\theta, 1). \text{ Moreover } \gamma_0 < \gamma_1. \end{aligned} \quad (\mathcal{H}_\gamma)$$

In what follows, we will be led to consider additional assumptions on the functions  $g$  through  $G$  and  $F$ , which we are now introducing to state them all together in one place.

$$\text{For every } \alpha \in \mathbb{R}, \text{ the equation } F(s) = \alpha \text{ has at most two solutions.} \quad (\mathcal{H}_{eq})$$

$$\text{the mapping } s \mapsto \frac{F'(s)}{s} \text{ is strictly concave.} \quad (\mathcal{H}_{concave})$$

Note that the first assumption above will be specifically used in Proposition 15. The validity of these assumptions for a standard choice of function  $g$  will be discussed at the end of this section.

**Theorem 1.** *Let us assume that*

- $g$  satisfies  $(\mathcal{H}_{bis})$ - $(\mathcal{H}_{inv})$ - $(\mathcal{H}_\gamma)$ - $(\mathcal{H}_{eq})$ - $(\mathcal{H}_{concave})$ ,
- the constant  $C$  is such that

$$C > \max \left\{ \max_{u \in [\theta, 1]} \frac{g(u)u + g'(u)u^2 + 4[G(1) - G(\theta)]}{2\theta^2}, \max_{u \in [0, 1]} g'(u), \max_{u \in [0, 1]} \frac{|g(u)|}{u}, \max_{u \in [0, 1]} \frac{|g(u)|}{\theta} \right\}$$

Then, there exist two constants  $L_*, L^*$  with  $L_* < L^*$  such that:

1. If  $L < L_*$  then  $\mathcal{A}_{L,C} = \emptyset$ .
2. If  $L = L_*$  then  $\mathcal{A}_{L,C} = \{C\mathbb{1}_{[0,L]}\}$ . Furthermore, the function  $u$  solving (1) with  $\mu = C\mathbb{1}_{[0,L]}$  satisfies  $(u(L_*), u'(L_*)) = (\beta, 0)$ , where  $\beta$  denotes the unique real number in  $(0, 1)$  such that  $G(\beta) = 0$ .
3. If  $L \in (L_*, L^*)$  then  $\mathcal{A}_{L,C} \neq \emptyset$ . Furthermore, Problem  $(\mathcal{P}_{L,C})$  has a solution  $\mu$ . There exist two constants  $a, b > 0$  such that  $\mu = C$  in  $[0, a) \cup (b, L]$  and  $\mu$  is a singular arc in  $(a, b)$  given by

$$\mu(x) = \left( \frac{3g(v(x))}{2v(x)} + \frac{g'(v(x))}{2} - h \right) \mathbb{1}_{[a,b]}(x)$$

where  $h$  is a positive constant and  $v$  is the solution of the ODE

$$\begin{cases} v'' = \frac{g(v) + vg'(v)}{2} - hv, \\ v(a) < \gamma_1, \quad v(b) > \gamma_0. \end{cases}$$

4. If  $L \geq L^*$  then  $\mathcal{A}_{L,C} \neq \emptyset$ , there exists a family of solutions to Problem  $(\mathcal{P}_{L,C})$ . Moreover, all the solutions are singular. They are all equal, up to a translation, to the function  $\mu$  defined by

$$\mu(x) = \left( \frac{3g(v(x))}{2v(x)} + \frac{g'(v(x))}{2} \right) \mathbb{1}_{[0,L_0]}(x)$$

where  $v$  is the solution of the following ODE

$$\begin{cases} v'' = \frac{g(v) + vg'(v)}{2}, \\ v(0) = \gamma_1, \quad v'(0) = -\sqrt{2[G(1) - G(\gamma_1)]}, \end{cases}$$

and  $L_0$  is such that  $v(L_0) = \gamma_0, v'(L_0) = -\sqrt{-2G(\gamma_0)}$ .

This theorem is proved in Section 4. Let us conclude this section by discussing the assumptions above on the functions  $F$  and  $G$ . We claim that the class of functions  $g$  satisfying all the assumptions above is nonempty.

**Lemma 2.** Let  $\theta \in (0, 1/2)$  and  $g_\theta$  denote the bistable polynomial function given by  $g_\theta(s) = s(1-s)(s-\theta)$ . Then,  $g_\theta$  satisfies the assumptions  $(\mathcal{H}_{bis})$ - $(\mathcal{H}_{inv})$ - $(\mathcal{H}_\gamma)$ - $(\mathcal{H}_{eq})$ - $(\mathcal{H}_{concave})$ .

We have performed some numerical investigation using an optimal interior method through the software `IpOpt` (implementing an interior point method, see [15]) using the interface of `CasADi` ([4]). In Figure 1.1, we recover the three cases 2, 3, 4 stated in Theorem 1 (we do not illustrate case 1 since  $\mathcal{A}_{L,C} \neq \emptyset$  in that case). We also have shown numerically a continuous dependence of the solution  $\mu$  with respect to the size  $L$ . If this numerical observation is true theoretically, then the same holds for  $u$ .

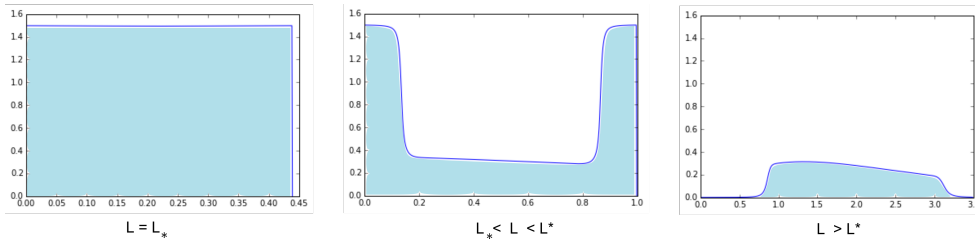


Figure 1: The optimal solution  $\mu$  computed by the software `IpOpt` for three sizes of  $L$  with  $g(u) = u(1 - u)(u - \frac{1}{4})$ ,  $C = \frac{3}{2}$  and three different sizes of  $L$ : (from the left to the right)  $L = 0.44$ ,  $L = 1$ ,  $L = 3.5$ .

## 1.2 Method and comparison with the existing literature

The first step of our approach is to define an equivalent optimal problem in a compact set. After proving the existence of a solution for this auxiliary optimal control problem  $(\mathcal{P}_{L,C})$  by standard arguments, we use the so-called Pontryagin Maximum Principle to obtain a characterization of the minimizers. The second step consists in performing an accurate analysis of this characterization, and we thus recover the three kinds of output that we have showed in Figure 1.1. The last step is to exhibit the relationship between the size of the support of the optimal solution  $\mu$  and the qualitative properties obtained at the second step. This is mostly done through a refined study involving standard arguments in reaction-diffusion such as the comparison principle.

This method is robust enough to be employed in other mathematical frameworks. Indeed, in a forthcoming paper, we will employ this method for a non-affine optimization problem related to pest control by acting on the birth rate: we reduce the birth rate by releasing sterile males or we reduce mechanically the number of egg-laying sites instead of increasing the death rate by spreading insecticide as it is studied here. We also underline that our method allows to take into account  $L^\infty$  bounds on the control and the size of its support.

The study of equations similar to (1) with  $\mu(x) = C$  was initially performed in [7]. In this first article, the authors assume that the effect of the control is strong enough so that the natural dynamics  $g(u)$  can be assumed to be neglectible compared with the death term  $-Cu$ . Therefore, the authors investigate the existence of a blocking wave with speed  $c = 0$ , namely a solution of

$$-u'' = \begin{cases} g(u) & \text{for } x \notin [0, L] \\ -Cu & \text{for } x \in [0, L] \end{cases}, \quad u(-\infty) = 1, \quad u(+\infty) = 0.$$

Using mostly reaction-diffusion arguments, they prove the existence of a solution to the above equation if  $L \geq L_{\text{crit}}$  for some  $L_{\text{crit}} > 0$ . They also characterize the optimal solution for  $L = L_{\text{crit}}$ . Finally, they investigate the stability of the wave. Notice that the assumption that  $g(u) - Cu \sim -Cu$  is relevant in their case but not in the present one since we find out that the optimal solution  $\mu_*$  may be equal to 0 in some regions. Some articles extended the result of [7] to different settings such as systems or other type of controls (see [2, 12, 3]). We only single out here the work [3] where the authors extend the result of [7] by assuming that the interval  $[0, L]$  is moving with a speed  $c \leq 0$ .

Let us also mention the closely related study [8], where the authors investigate the existence of an optimal death term  $\mu$  generating a traveling wave with a prescribed speed  $c$  for the equation

$$\begin{cases} -cu' - u'' = g(u) - \mu(x)u, \\ u(-\infty) = 0, \quad u(+\infty) = 1 \end{cases}$$

while minimizing the  $L^1$  norm  $\int_{\mathbb{R}} \mu$ . One difference with our model is that the authors do not want to only block the invasive wave but either reduce the speed of propagation or reverse the speed of

propagation, this last case corresponding to a local eradication of the pests. Another difference is that we impose  $L^\infty$  pointwise constraints on  $\mu$  and a constraint on the size of the support. The authors of [8] use a very powerful technique based on an analysis of the phase portrait and the use of a Stokes formulae introduced first in [10]. In our opinion, it is not possible to adapt their technique to the case of  $L^\infty$  constraints on the control, since these are difficult to take into account in the phase portrait approach. From the mathematical point of view, the main difficulty of this study comes from the fact that the  $\mu$ -dependence of the admissible set  $\mathcal{A}_{L,C}$  is very implicit, through the existence assumption of solution of the system (1). This poses unusual difficulties and makes the study of the resulting optimal control problem non-standard.

In the only case where  $C > \sup \mu$  and  $L > L^*$  (case 4 of Theorem 1), our result is similar to the main result of [8] (Theorem 3.2) since in that case, none of the constraints are activated. Therefore, in this set of parameters, the question becomes equivalent to the question without imposing any constraint. Eventually, we finish this comparison by stating that the question of the robustness of our method for a moving pesticide intervention with a prescribed speed  $c$  and imposing constraints on the support and the  $L^\infty$  bounds in the spirit of [3] is still open. The main difficulty seems to be at the first step of the method since it is not clear to us how to find an equivalent optimization problem in some compact set.

### 1.3 Outline of the paper

In Section 2, we introduce an optimal control problem, equivalent to Problem  $(\mathcal{P}_{L,C})$  in the sense that both problems share the same solutions. This new optimal control problem has the advantage of involving an ODE on the compact set  $[0, L]$  and not anymore on  $\mathbb{R}$  as in Problem  $(\mathcal{P}_{L,C})$ . This is more adapted to the use of the Pontryagin Maximum Principle. First order optimality conditions are then derived. Next, in Section 3, we analyse such conditions, and lead a discussion on the emergence of singular arcs for minimizers. Finally, in Section 4, we prove Theorem 1 using the analysis developed in Section 3.

Throughout this article, the notation  $|\cdot|$  will denote the one-dimensional Lebesgue measure.

## 2 Existence of a solution and reformulation of Problem $(\mathcal{P}_{L,C})$

First, according to [13], we define  $u$  being a solution of (1) as

$$u(x) = \sup \{ \phi : \phi \text{ is a sub-solution of (1)} \}.$$

Before providing a new equivalent problem, we state that the set  $\mathcal{A}_{L,C}$  is nonempty.

**Proposition 1.** *For  $C > \max_{u \in [0,1]} \frac{g(u)}{u}$ , one has  $\mathcal{A}_{L,C} \neq \emptyset$  whenever  $L > 0$  is large enough.*

*Proof.* We just provide the main steps of the proof and we refer to [7] (or [3]) for more details. The idea is to construct sub and super-solutions that are ordered. One easily checks that a sub-solution is simply given by

$$\underline{u}(x) = \begin{cases} w(x) & \text{for } x < 0, \\ 0 & \text{for } x \geq 0 \end{cases} \quad \text{with} \quad \begin{cases} -w'' = g(w), \\ w(0) = 0, \quad w'(0) = -\sqrt{2G(1)}. \end{cases}$$

Regarding the construction of a super-solution, we underline that since  $g(u) - Cu < -\varepsilon u$  for some well-chosen parameter  $\varepsilon > 0$ , we can apply [7, Theorem 4]: for  $L$  large enough, there exists a solution to the problem

$$-\bar{u}'' = \begin{cases} g(\bar{u}) & \text{for } x \notin (0, L), \\ -\varepsilon \bar{u} & \text{for } x \in (0, L) \end{cases}, \quad \bar{u}(-\infty) = 1, \quad \bar{u}(+\infty) = 0.$$

Next, using [7, Lemma 2], the sub-solution  $\underline{u}$  and the super-solution  $\bar{u}$  are ordered ( $\underline{u} < \bar{u}$ ) and the existence of a solution  $u$  with  $\underline{u} < u < \bar{u}$  follows from the maximum principle. We conclude that  $C\mathbf{1}_{[0,L]}$  belongs to  $\mathcal{A}_{L,C}$ .  $\square$

We remark that,  $C > 0$  being given, if  $\mathcal{A}_{L,C} \neq \emptyset$  for a size  $L > 0$ , then for any size  $L' > L$ , there holds  $\mathcal{A}_{L,C} \subset \mathcal{A}_{L',C}$ .

## 2.1 Towards a compactly supported problem

Recall that  $\beta$  denotes the unique zero of  $G$  in  $(0, 1)$ . First, we prove that Problem  $(\mathcal{P}_{L,C})$  is equivalent to another one involving an ODE on a compact set.

**Proposition 2.** *If  $L > L_*$ , then (1) has a nontrivial solution if, and only if the equation*

$$\begin{cases} -u'' = g(u) - \mu(x)u\mathbf{1}_{\{0 < x < L\}}, \\ u'(0) = -\sqrt{2 \int_{u(0)}^1 g(v)dv}, \quad u'(L) = -\sqrt{-2 \int_0^{u(L)} g(v)dv} \end{cases} \quad (3)$$

has a solution such that  $0 \leq u \leq 1$  in  $(0, L)$ . Moreover, there holds  $u(L) \leq \beta$ .

Lastly, such a solution necessarily satisfies  $u' < 0$  in  $(-\infty, 0) \cup (L, \infty)$ .

*Proof.* Assume first that (3) admits a solution  $u$ . Let us extend  $u$  to  $(L, \infty)$  by taking  $u = v$  where  $v$  denotes the solution of  $-v'' = g(v)$  in  $(L, \infty)$ , with  $v(L) = u(L)$ ,  $v'(L) = u'(L)$ , where we have extended  $g$  linearly outside of  $(0, 1)$ . Multiplying this equation by  $u$  and integrating, we get that  $\frac{(u')^2}{2} + G(u)$  is constant on  $(L, \infty)$ . Moreover, as  $u'(L) = -\sqrt{2 \int_0^{u(L)} g(v)dv}$ , this constant is equal to 0.

Let us now prove that  $u' < 0$  in the complement set of  $[0, L]$ . Let us assume by contradiction that  $u$  admits a local minimum at  $\underline{x} \in (L, \infty)$ . Then  $u'(\underline{x}) = 0$  and thus  $G(u(\underline{x})) = 0$ , which implies that  $u(\underline{x}) \in \{0, \beta\}$ . Assume by contradiction that  $u(\underline{x}) = \beta$ , then  $u$  does not admit any local maximizer  $\bar{x} > \underline{x}$ , otherwise the same arguments would yield  $G(u(\bar{x})) = 0$ , a contradiction since  $u(\bar{x}) > \beta$ . Hence  $u$  would be increasing in  $(\underline{x}, \infty)$ , and as  $-u'' = g(u)$ , one would get  $u(\infty) = 1$ . On the other hand, one has  $G(u(\infty)) = 0$ , a contradiction. Hence  $u(\underline{x}) \neq \beta$  and thus  $u(\underline{x}) = 0$ . But, according to the Cauchy-Lipschitz theorem, one would get  $u(\cdot) = 0$  on  $\mathbb{R}$ , a contradiction with the boundary conditions at  $x = 0$ . It follows that if  $u'(L) < 0$ , then  $u' < 0$  on  $(L, \infty)$ .

Now, if  $u'(L) = 0$ , then one has  $u(L) = \beta$ . Assume by contradiction that  $u$  admits a local maximum at  $\bar{x} \in (L, \infty)$ . Then, one has  $u'(\bar{x}) = 0$  and thus  $G(u(\bar{x})) = 0$ , which implies that  $u(\bar{x}) = \beta$ . Hence, either  $u$  admits a local minimum in  $(L, \bar{x})$ , which has been shown impossible, or  $u(\cdot) = \beta$  in  $(L, \bar{x})$ . In this second case, one would have  $-u'' = 0 = g(\beta)$  on  $(L, \bar{x})$ , a contradiction. Hence,  $u$  is nonincreasing on  $(L, \infty)$  in this case, and applying the strong maximum principle to  $u'$ , we even get that  $u' < 0$  in  $(L, +\infty)$  and  $u'(L) = 0$ .

We conclude that  $u' < 0$  over  $(L, \infty)$  in all cases. Hence,  $u(+\infty)$  is well-defined and since  $\frac{(u')^2}{2} + G(u) = 0$  in  $(L, \infty)$ , one gets that  $u(+\infty) = 0$ .

We could prove that  $u' < 0$  on  $(-\infty, 0)$  and that  $u(-\infty) = 1$  with similar arguments.

Assume next that  $u$  satisfies (1). Then,  $\frac{(u')^2}{2} + G(u)$  is constant in  $(-\infty, 0)$  and  $(L, \infty)$ , and due to the boundary conditions and elliptic regularity, these constants are respectively  $G(1)$  and  $G(0) = 0$ . It immediately follows that  $u(L) \leq \beta$ . Moreover, one can prove as in the previous step that  $u' < 0$  outside  $[0, L]$ . Hence, the boundary conditions in (3) immediately follows by taking the square root of  $\frac{(u')^2}{2} = -G(u)$  in  $(L, \infty)$  and  $\frac{(u')^2}{2} = G(1) - G(u)$  in  $(-\infty, 0)$ .  $\square$

At this step, using Proposition 2 enables us to recast Problem  $(\mathcal{P}_{L,C})$  as

$$\boxed{\inf_{\mu \in \widehat{\mathcal{A}}_{L,C}} \mathcal{N}(u)}. \quad (\mathcal{P}_2)$$

where

$$\widehat{\mathcal{A}}_{L,C} := \{\mu \in L^\infty(\mathbb{R}) : (3) \text{ has a solution, } 0 \leq \mu \leq C \text{ a.e. in } \mathbb{R} \text{ and } \text{supp}(\mu) \subset [0, L]\}.$$

Let us conclude this section by showing the existence of an optimal control  $\mu^*$  by taking advantage of the equivalence between the two problems we have just shown.

**Proposition 3.** *Let  $L > 0$  be given such that  $\mathcal{A}_{L,C} \neq \emptyset$  (or, equivalently, such that  $\widehat{\mathcal{A}}_{L,C} \neq \emptyset$ ). Then, Problem  $(\mathcal{P}_2)$  has a solution  $\mu^*$ .*

For the sake of clarity, the proof of this result is postponed to Appendix B.

## 2.2 Definition of $L_*$ and solution of Problem $(\mathcal{P}_{L,C})$ in the case $L = L_*$

Let  $\mu_L = C\mathbf{1}_{[0,L]}$  denote the family of constant controls indexed by the size of the support  $L$ . We define  $L_*$  as the smallest  $L > 0$  such that (1) admits a solution, that is:

$$L_* := \min\{L > 0 \text{ such that } \mu_L \in \mathcal{A}_{L,C}\} \quad (4)$$

First, the infimum exists according to Proposition 1, and we can easily prove using the compactly supported formulation (Proposition 2) that  $L_*$  is a minimum. Let  $u_*$  be the solution associated with  $\mu_{L_*} = C\mathbf{1}_{(0,L_*)}$ .

**Proposition 4.** *If  $\mathcal{A}_{L,C} \neq \emptyset$ , then  $L \geq L_*$ . Moreover,  $\mathcal{A}_{L_*,C} = \{C\mathbf{1}_{(0,L_*)}\}$ .*

*Proof.* Let  $L$  be such that  $\mathcal{A}_{L,C} \neq \emptyset$ , take  $\mu \in \mathcal{A}_{L,C}$  and  $u$  be the optimal solution associated to (1). Next, consider  $v$  the solution of the ODE

$$\begin{cases} -v'' = g(v) - Cv & \text{for } x > 0, \\ v(0) = u(0), \quad v'(0) = u'(0). \end{cases}$$

The maximum principle implies that  $v(x) \leq u(x)$  for all  $x \in [0, L]$ . We claim that

$$v'(L)^2 + 2G(v(L)) \leq 0. \quad (5)$$

If (5) holds true, we can conclude. Indeed, it would follow that there exists  $\ell \in (0, L)$  such that  $v'(\ell)^2 + 2G(v(\ell)) = 0$ . Thus, if we consider the function,  $\tilde{v}$  defined as the solution of

$$\begin{cases} -\tilde{v}'' = g(\tilde{v}) & \text{for } x > \ell, \\ \tilde{v}(\ell) = v(\ell), \quad \tilde{v}'(\ell) = v'(\ell) \end{cases}$$

we deduce that  $\tilde{v} \rightarrow 0$  as  $x \rightarrow +\infty$ . Next, we define

$$\phi(x) = \begin{cases} u(x) & \text{for } x \leq 0, \\ v(x) & \text{for } x \in [0, \ell], \\ \tilde{v} & \text{for } x \geq \ell. \end{cases}$$

We obviously have that  $\phi$  is solution of (1) for a size  $\ell < L$  and  $\mu = C\mathbf{1}_{(0,\ell)}$ . We conclude that  $L_* \leq \ell < L$ .

It remains to prove (5). A direct computation yields

$$\begin{aligned} v'(L)^2 + 2G(v(L)) &= 2G(v(0)) + C[v(L)^2 - v(0)^2] + v'(0)^2 \\ &= 2G(u(0)) + u'(0)^2 + C[v(L)^2 - u(0)^2] \\ &= 2G(1) + C[v(L)^2 - u(0)^2]. \end{aligned}$$

By multiplying (1) by  $u'$  and integrating over  $(-\infty, L)$ , we deduce that

$$\begin{aligned} u'(L)^2 &= 2G(1) - 2G(u(L)) + 2 \int_0^L \mu(x)u(x)u'(x)dx \\ &\geq 2G(1) - 2G(u(L)) + 2C \int_{u(0)}^{u(L)} udu \\ &= 2G(1) - 2G(u(L)) + C[u(L)^2 - u(0)^2]. \end{aligned}$$

Recalling that  $u(L) \in \mathcal{M}_0$ , we deduce that

$$v'(L)^2 + 2G(v(L)) \leq u'(L)^2 + 2G(u(L)) = 0.$$

Lastly, the reader can notice that this proof also shows that  $v'(L)^2 + 2G(v(L)) < 0$  when  $\mu \neq C1_{(0,L)}$ . Hence, in that case,  $L_* \leq \ell < L$ . This proves that  $\mathcal{A}_{L_*,C} = \{C1_{(0,L_*)}\}$ .  $\square$

**Proposition 5.** *The function  $\mu = C1_{(0,L)}$  minimizes  $\mathcal{N}$  if and only if  $L = L_*$ .*

*Proof.* If  $L = L_*$ , then  $\mathcal{A}_{L_*,C} = \{C1_{(0,L_*)}\}$  and the result immediately follows. Assume now that  $\mu = C1_{(0,L)}$  is optimal, with  $L > L_*$  and let  $u_L$  be the associated solution. Since  $u_*$  is solution of the problem with a size  $L$  associated to  $\mu_{L_*}$ , we have  $\mathcal{N}(\mu_{L_*}) = CL_* < CL = \mathcal{N}(\mu)$ . Therefore, this contradicts the optimality of  $\mu_L$ .  $\square$

**Proposition 6.** *If  $L = L_*$  and  $C > \frac{1}{\beta} \sup_{u \in (0,1)} |g(u)|$  then we have  $u(L) = \beta$ .*

*Proof.* Let  $\mu_L = C1_{[0,L]}$  denote the family of constant controls indexed by the size of the support  $L$ . Let us first determine the minimal size  $L_0$  such that (1) admits a solution for  $\mu = \mu_{L_0}$ . According to the maximum principle (see [7, 3] for details), one has:

*If there exists a solution  $u_{L_1}$  for  $L = L_1$ , then for every  $L_2 > L_1$ , the associated solution  $u_{L_2}$  satisfies  $u_{L_2} < u_{L_1}$ .*

Let us introduce the initial and target manifolds  $\mathcal{M}_0$  and  $\mathcal{M}_1$  defined by

$$\mathcal{M}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^2 - 2 \int_{x_1}^1 g(v) dv = 0\} \quad (6)$$

$$\mathcal{M}_0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^2 + 2 \int_0^{x_1} g(v) dv = 0\}. \quad (7)$$

Let  $\Lambda > 0$  be such that the solution  $u_\Lambda$  satisfies

$$\begin{cases} -u''_\Lambda = g(u_\Lambda) - Cu_\Lambda & \text{for } x \in [0, \Lambda], \\ (u_\Lambda(0), u'_\Lambda(0)) \in \mathcal{M}_1, \\ u_\Lambda(\Lambda) = \beta, \quad u'_\Lambda(\Lambda) = 0. \end{cases}$$

Existence of a such size  $\Lambda$  is shown hereafter. Since  $\beta = \sup \{u : (u, v) \in \mathcal{M}_0\}$ , we deduce that there does not exist any solution  $u_L$  for  $L < \Lambda$  (otherwise, we would have  $\beta = u_\Lambda(\Lambda) < u_L(\Lambda) < u_L(L)$  and  $(u_L(L), u'_L(L)) \in \mathcal{M}_0$ , which is impossible since  $(\beta, 0)$  is an extreme point of  $\mathcal{M}_1$ ). Therefore, we deduce  $L_* = \Lambda$  and the conclusion follows.

It remains to show the existence of  $\Lambda$ . It is enough to prove that the phase portrait of the solution  $w$  of

$$\begin{cases} -w'' = g(w) - Cw, & \text{for } x > 0, \\ w(0) = \beta, \quad w'(0) = 0 \end{cases}$$

crosses the manifold  $\mathcal{M}_1$  whenever  $L$  is large enough. To this aim, notice that for some small  $\varepsilon > 0$ ,  $w'' > 0$  on  $[0, \varepsilon)$  thanks to the assumption on  $C$ . We deduce that  $w' > 0$  on  $(0, \varepsilon)$  and  $w > \beta$  on  $(0, \varepsilon)$ . Using one more time the assumption on  $C$ , we have that  $w'' > 0$  for every  $x \geq 0$  and, as above,  $w > \beta$  for all  $x > 0$ . It follows for any  $x \geq 0$

$$w'(x) \geq \left( C\beta - \sup_{u \in (0,1)} |g(u)| \right) x \quad \text{and} \quad w(x) \geq \beta + \frac{\left( C\beta - \sup_{u \in (0,1)} |g(u)| \right) x^2}{2}.$$

We immediately get that  $(w, w') \rightarrow (+\infty, +\infty)$  as  $x \rightarrow +\infty$ . By remarking that  $\mathcal{M}_1$  is the boundary of a compact set in  $\{u \geq 0\} \times \{u' \geq 0\}$  in the phase portrait plane, we infer that the graph of the phase portrait of  $(w, w')$  crosses  $\mathcal{M}_1$ . The first antecedent of this cross-point defines  $\Lambda$ .  $\square$

### 2.3 Monotonicity of the optimal state

We now show that if  $u$  is the solution to (1) associated to an optimal solution  $\mu_*$  of Problem  $(\mathcal{P}_{L,C})$ , then  $u$  is monotone. This property will be used all along this work.

**Proposition 7.** *For any optimal solution  $\mu_*$  of Problem  $(\mathcal{P}_{L,C})$ , the associated solution  $u_*$  to (1) is nonincreasing.*

*Proof.* Let  $\mu$  be an optimal solution and  $u_\mu$  be the associated solution to (1). According to Proposition 2, it remains only to prove that  $u_\mu$  is nonincreasing on  $[0, L]$ . Assume  $u_\mu$  is not nonincreasing, then we define

$$x_1 \text{ its first local minimizer} \quad \text{and} \quad x_2 = \sup \{x \in \mathbb{R} : u(x) = u(x_1)\}.$$

Existence of  $x_2$  directly comes from the fact that  $u(\infty) = 0$ . Next, we introduce the operator  $\tau_{x_1, x_2}$  defined by

$$\tau_{x_1, x_2} : L^\infty(\mathbb{R}) \ni f \mapsto \begin{cases} f(x) & \text{for } x < x_1, \\ f(x + x_2 - x_1) & \text{for } x \geq x_1. \end{cases}$$

as well as  $\mu_1 = \tau_{x_1, x_2}\mu$  and  $\bar{u} = \tau_{x_1, x_2}u_\mu$ . It is clear that

$$\begin{aligned} -\bar{u}'' &= g(\bar{u}) - \mu_*\bar{u} \quad \text{for } x \in \mathbb{R} \setminus \{x_1\}, \quad \bar{u} \in C^0(\mathbb{R}) \\ \bar{u}'(x_1^-) &= 0 \geq \bar{u}'(x_1^+), \quad \lim_{x \rightarrow -\infty} \bar{u}(x) = 1, \quad \text{and} \quad \lim_{x \rightarrow +\infty} \bar{u}(x) = 0. \end{aligned}$$

According to [7], it follows that  $\bar{u}$  is a super solution of (1) and  $\int \mu_* \leq \int \mu$ . Considering the sub-solution  $\underline{u}$  satisfying  $-\underline{u}'' = g(\underline{u})$  in  $(-\infty, 0)$ ,  $\underline{u}(-\infty) = 1$ ,  $\underline{u}(0) = 0$  as in the proof of Proposition 1, we deduce that (1) has a solution  $u_1$ . Moreover, since  $\mu$  is assumed to be optimal, we deduce that  $\int \mu_* = \int \mu$ . But, we claim that  $\int_{[x_1, x_2]} \mu(x) dx \neq 0$ . Otherwise, by multiplying (1) by  $u'_\mu$  and integrating (1) between  $x_1$  and  $x_2$ , it would follow that

$$(\bar{u}'(x_2))^2/2 = (\bar{u}'(x_2))^2/2 - (\bar{u}'(x_1))^2/2 = G(\bar{u}(x_1)) - G(\bar{u}(x_2)) = 0.$$

As  $x_1$  is the local minimizer, one has  $0 \geq -\bar{u}''(x_1) = g(\bar{u}(x_1))$  and thus  $\bar{u}(x_1) = \bar{u}(x_2) \leq \theta$ . As  $\bar{u}(x) < \bar{u}(x_2)$  for all  $x > x_2$ , one has  $g(\bar{u}(x)) < 0$  for all  $x > x_2$ . Moreover, as  $\mu \geq 0$ , we get from (1) that  $-\bar{u}'' < 0$  on  $(x_2, \infty)$ . Using  $\bar{u}'(x_2) = 0$ , this gives  $\bar{u}' > 0$  on  $(x_2, \infty)$  and thus  $\bar{u}(x) > \bar{u}(x_2)$  on  $(x_2, \infty)$ , a contradiction with  $\bar{u}(+\infty) = 0$ .  $\square$

## 2.4 First properties of any optimal solution

Let us state the first order optimality conditions for Problem  $(\mathcal{P}_2)$ . To this aim, we consider a solution  $\mu$  to Problem  $(\mathcal{P}_2)$  and its associated optimal trajectory  $X = [u, \dot{u}]^\top$ . Let us use the so-called Pontryagin Maximum Principle (see [14, Theorem 1.4.1] and [11]). We introduce the notations  $X = [u \ v]^\top$ ,  $P = [p_u \ p_v]^\top$  as well as the Hamiltonian of Problem  $(\mathcal{P}_2)$  given by

$$\mathcal{H} : \mathbb{R}^2 \times \mathbb{R}^2 \times \{0, -1\} \times \mathbb{R} \ni (X, P, p^0, \mu) \mapsto p_u v + p_v(\mu u - g(u)) + p^0 \mu.$$

According to the Pontryagin Maximum Principle (PMP), there exists an absolutely continuous mapping  $P : [0, T] \rightarrow \mathbb{R}^2$  called adjoint vector such that the so-called extremal  $(X, P, \mu)$  satisfies a.e. in  $[0, L]$ :

*Adjoint equations:*

$$\frac{dP}{dx}(x) = -\frac{\partial \mathcal{H}}{\partial X}(X(x), P(x), p^0, \mu(x)) = \begin{pmatrix} -\mu(x)p_v(x) + g'(u(x))p_v(x) \\ -p_u(x) \end{pmatrix}.$$

*Maximization condition:* for a.e.  $x$  in  $[0, L]$ ,  $\mu(x)$  solves the problem

$$\max_{\mu \in [0, C]} \mu(p^0 + p_v(x)u(x)).$$

*Transversality conditions:* let us recall that the initial and target manifolds  $\mathcal{M}_0$  and  $\mathcal{M}_1$  are defined by (7)-(6). The transversality conditions read  $P(0) \perp T_{X(0)}\mathcal{M}_1$  and  $P(L) \perp T_{X(L)}\mathcal{M}_0$ , where  $T_X\mathcal{M}_i$  denotes the tangent hyperplane to  $\mathcal{M}_i$  ( $i = 0, 1$ ) at  $X$ , and can be written as

$$p_u(0)v(0) - g(u(0))p_v(0) = 0, \quad p_u(L)v(L) - g(u(L))p_v(L) = 0.$$

Notice that, since the optimal control problem is autonomous (neither the cost functional nor the right-hand side of the system solved by the state depend directly on  $x$ ), it follows that

$$\text{for a.e. } x \in [0, L], \quad \max_{\mu \in [0, C]} p_u v + p_v(\mu u - g(u)) + p^0 \mu =: h \text{ is constant.} \quad (8)$$

Before, providing the main results of this section we introduce some notations that will be used all along this work. A function that will play an important role is the so-called *switching function*, given by

$$\psi(x) = u p_v(x). \quad (9)$$

Indeed, the sets  $\{\mu = 0\}$ ,  $\{\mu \in [0, C]\}$  and  $\{\mu = C\}$  are characterized in terms of the the level sets of  $\psi$ , namely up to a set of zero Lebesgue measure, one has

$$\{\psi < -p^0\} \subset \{\mu = 0\} \subset \{\psi \leq -p^0\}, \quad \{\psi > -p^0\} \subset \{\mu = C\} \subset \{\psi \geq -p^0\}$$

and moreover  $\{0 < \mu < C\} \subset \{\psi = -p^0\}$ . Furthermore, it is notable that  $\psi$  satisfies a simple equation that is stated in the next Lemma.

**Lemma 3.** *The function  $\psi$  solves the equation*

$$-\psi'' = \left( \frac{3g(u)}{u} + g'(u) - 2\mu \right) \psi + 2\mu(-p^0 - \psi) + 2h. \quad (10)$$

*Proof.* A direct computation yields

$$\psi'' = u'' p_v + 2u' p_v' + u p_v'' = (-g(u) + \mu u) p_v - 2v p_u + u(-g'(u) + \mu) p_v.$$

Recalling that  $h = p_u v + p_v(\mu u - g(u)) + p^0 \mu$ , it follows

$$\psi'' = \left( -\frac{g(u)}{u} - g'(u) + 2\mu \right) \psi - 2 \left( h - \mu \psi + \frac{g(u)}{u} \psi - p^0 \mu \right).$$

By ordering the above terms, we obtain (10).  $\square$

Next, it is convenient to introduce the maximal interval  $(l_0, L_0)$  on which  $\psi$  is above  $-p^0$ . Namely

$$l_0 = \min \{x \in [0, L] : \psi(x) \geq -p^0\} \quad \text{and} \quad L_0 = \max \{x \in [0, L] : \psi(x) \geq -p^0\}. \quad (11)$$

Note that this quantity is well-defined since, if  $\psi < -p^0$  everywhere on  $[0, L]$ , then  $\mu = 0$ , which is impossible since  $\int_0^1 g > 0$  ensures that (1) does not admit any solution when  $\mu \equiv 0$ . We underline that the transversality conditions hold true up to the boundary of the support of  $\mu$ .

**Proposition 8.** *There holds:*

$$p_u(l_0)v(l_0) - g(u(l_0))p_v(l_0) = 0, \quad \text{and} \quad p_u(L_0)v(L_0) - g(u(L_0))p_v(L_0) = 0. \quad (12)$$

*Proof.* We recall the transversality conditions that read

$$p_u(0)v(0) - g(u(0))p_v(0) = 0 \quad \text{and} \quad p_u(L)v(L) - g(u(L))p_v(L) = 0.$$

By integrating (respectively) over  $(0, l_0)$  and  $(L_0, L)$  the equations

$$p'_u u' = (g(u))' p_v \quad \text{and} \quad p'_u u' = (g(u))' p_v,$$

it follows that the transversality conditions above also hold true at  $l_0$  and  $L_0$ , whence the result.  $\square$

### 3 Analysis of the optimality conditions provided by the PMP

In this section, we analyze each of the cases provided by the Pontryagin principle introduced in Section 2. Specifically, our approach is divided into the analysis of the following three particular cases:

$$(1) p^0 = 0 \quad (2) p^0 = -1 \text{ and } h = 0 \quad (3) p^0 = -1 \text{ and } h > 0.$$

#### 3.1 The case $p^0 = 0$ : the bang-bang case

First, one proves that the solution is bang-bang. Next, we prove that the set  $\{\mu = C\}$  is non-empty and must be equal to the whole domain  $[0, L]$ . We separate these statements into several propositions and our conclusion is summarized by Corollary 2.

**Proposition 9.** *If  $p^0 = 0$  then any optimal solution is bang-bang and moreover  $|\{\psi = 0\}| = 0$ .*

*Proof.* To prove the *bang-bang* character of the solutions, let us argue by contradiction, assuming the existence of a measurable set  $\mathcal{I}$  of positive measure on which  $0 < \mu < C$ . If  $p^0 = 0$ , then one has  $p_v u = 0$  a.e. on  $\mathcal{I}$ . Hence,  $p_v(\cdot) = 0$  on  $\mathcal{I}$  since  $u > 0$ . It follows that  $p'_v(\cdot) = 0$  a.e. on  $\mathcal{I}$ . According to the Pontryagin Maximum Principle stated in Section 2.4,  $p_v$  solves a linear ODE. We derive from this equation  $p_u(\cdot) = 0$  a.e. on  $\mathcal{I}$ . By using the Cauchy-Lipschitz theorem, we get that  $p_v(\cdot) = 0$  on  $[0, L]$  and  $p_u(\cdot) = 0$  in  $[0, L]$ . However, according to the Pontryagin Maximum Principle, the pair  $(p^0, P)$  cannot be trivial. We have thus reached a contradiction.  $\square$

**Proposition 10.** *If  $p^0 = 0$  then any connected component of the set  $\{\psi < 0\}$  must include 0 or  $L$ .*

*Proof.* We prove it by contradiction. We first notice that, as  $u$  and  $p_v$  belong to  $W^{2,\infty}(0, L)$ , they are continuous and so is  $\psi$ . Hence,  $\{\psi < 0\}$  is an open set.

Assume there exists an interval  $(a, b)$  such that  $0 < a < b < L$  with  $\psi(a) = \psi(b) = 0$  and  $\psi < 0$  in  $(a, b)$ . From this inequality, we deduce two facts (we only provide the justification for  $x = a$ , similar arguments implying the same results for  $b$ ):

1.  $p_v(a) = p_v(b) = 0$  (indeed  $0 = \psi(a) = u(a)p_v(a)$  and  $u(a) \neq 0$  implies the assertion),

2.  $p'_v(a) < 0$  and  $p'_v(b) > 0$  (indeed, this follows by using that the Cauchy-Lipschitz theorem together with  $p_v(a) = 0$  and  $(p_v, p'_v) \neq (0, 0)$ ).

Recalling that  $h = p_u v - p(\mu u - g(u)) + p^0 \mu$  and evaluating this expression at  $a^+$  and  $b^-$ , we deduce

$$\begin{aligned} 0 &> p'_v(a^+)v(a^+) = p'_v(a^+)v(a^+) - p(a^+)g(u(a^+)) = h \\ \text{and } h &= p'_v(b^-)v(b^-) - p(b^-)g(u(b^-)) = p'_v(b^-)v(b^-) > 0. \end{aligned}$$

Indeed, the sign of  $\psi$  changes at  $a$  and  $b$ , and  $\psi < 0$  on  $(a, b)$ . We have thus reached a contradiction.  $\square$

**Corollary 1.** *If  $p^0 = 0$  then there exists a single interval  $(a, b + l_1)$  such that  $\mu = C\mathbb{1}_{(a, b + l_1)}$ .*

Next, we prove that this can happen if and only if  $a = b = 0$  and  $L = l_1$ .

**Proposition 11.** *If  $p^0 = 0$  then the following assertions are equivalent*

$$(1) \psi(0) > 0, \quad (2) \psi(L) > 0, \quad (3) h > 0.$$

*Proof.* Recall that  $-p'_v u' + p_v(\mu u - g(u)) = h$  according to (8). Evaluating this equality at  $x = 0$  and  $x = L$ , and using the transversality conditions provided by Proposition 8, one gets  $h = \psi(0)\mu(0^+) = \psi(L)\mu(L^-)$ . Recall that, as emphasized in the proof of Proposition 10, one has  $\{\psi > 0\} = \{\mu = C\}$  and  $\{\psi < 0\} = \{\mu = 0\}$ . As  $\mu(x)\psi(x) = 0$  if  $\psi(x) < 0$  and  $\psi(x)\mu(x) = C\psi(x) > 0$  if  $\psi(x) > 0$ , the conclusion follows.  $\square$

**Proposition 12.** *Assume that  $C > \max\left(\frac{3g(u)}{2u} + \frac{g'(u)}{2}\right)$ . If  $p^0 = 0$  then we have  $h > 0$ .*

*Proof.* Assume that  $h \leq 0$ . It follows from Proposition 11 that  $\psi(0) = \psi(L) = 0$ . Since  $\mu$  is bang-bang and  $\mu = 0$  is not an admissible solution (since  $\int_0^1 g(s)ds > 0$ ), we deduce the existence of an interval  $(a, b)$  such that  $\psi > 0$  and thus  $\mu = C$  in  $(a, b)$  with  $0 \leq a < b \leq L$  and  $\psi(a) = \psi(b) = 0$ . Hence  $\psi$  takes its maximum in  $(a, b)$  at some interior point  $x_0$ . On one hand, we have  $-\psi''(x_0) \geq 0$ . On the other hand, since  $\psi > 0$  in  $(a, b)$ ,  $h \leq 0$  and  $\frac{3g(u)}{u} + g'(u) - 2C < 0$ , it follows

$$\left(\frac{3g(u)}{u} + g'(u) - 2\mu\right)\psi - 2\mu\psi + 2h < 0 \quad \text{in } (a, b).$$

This contradicts (10).  $\square$

**Corollary 2.** *Assume that  $C > \max\left(\frac{3g(u)}{2u} + \frac{g'(u)}{2}\right)$ . The following assertions are equivalent:*

- (1) *There exists a solution of the adjoint problem such that  $p^0 = 0$ ,*
- (2)  $\mu = C\mathbb{1}_{(0, L)}$ ,
- (3)  $L = L_*$ .

*Proof.* We prove that (1) implies (2), (2) implies (3) and (3) implies (1).

- **(1) implies (2).** From Proposition 12, we deduce that  $h > 0$ . From Proposition 11, it follows that  $\mu = C$  in a neighborhood of 0 and  $L$ . Finally, from Corollary 1, we conclude that  $\mu = C$  in  $[0, L]$ .
- **(2) implies (3).** Proposition 5 provides the result.
- **(3) implies (1).** We have two cases: either  $p^0 = 0$  or  $p^0 = -1$ . The first case corresponds to the desired conclusion. Therefore, we treat only the second case, we prove that we can construct an equivalent adjoint solution that satisfies  $p^0 = 0$ . In this order, we define  $\tilde{X} = [u \ u']^\top$ ,  $\tilde{P} = [p_u \ p_v]^\top$ ,  $\tilde{p}^0 = 0$ ,  $\tilde{\mu} = C\mathbb{1}_{(0, L_*)}$  and one has  $\mathcal{H}(\tilde{X}, \tilde{P}, \tilde{X}, \tilde{p}^0, \tilde{\mu}) = h + C$  a constant. Moreover, it also satisfies the adjoint equations, the maximization condition and the transversality conditions. Since  $\mathcal{N}(\mu)$  is the same for both solutions, the conclusion follows.  $\square$

### 3.2 The case $(p^0, h) = (-1, 0)$ : the singular arc case

We first determine an implicit expression of the singular control  $\mu$  in the set  $\{\mu > 0\}$ . Then, we prove that the values of  $\mu$  at the boundary of this set are prescribed. Finally, we prove that if  $L_0 - l_0$  is less than  $L$  then we automatically have  $(p^0, h) = (-1, 0)$ . We gather all these results in several propositions and we sum-up them into Corollary 3.

**Proposition 13.** *If  $p^0 = -1$  then the following assertions are equivalent*

$$(1) \psi(0) \leq 1, \quad (2) \psi(L) \leq 1, \quad (3) h = 0.$$

*Proof.* According to (8), one has  $-p'_v u' + p_v(\mu u - g(u)) + p^0 \mu = h$ . It follows that  $\mu(\psi - 1)$  belongs to  $H^1(0, L)$ , which enables us to evaluate these expressions pointwise. Taking this last equality at  $x = 0$  and  $x = L$ , and using the transversality conditions provided by Proposition 8 yields

$$h = (\psi(0) - 1)\mu(0^+) = (\psi(L) - 1)\mu(L^-).$$

As  $\mu(x)(1 - \psi(x)) = 0$  if  $\psi(x) \leq 1$  and  $(1 - \psi(L))\mu(L) = C(1 - \psi(L)) < 0$  if  $\psi(x) > 1$ , the conclusion follows.  $\square$

**Proposition 14.** *If  $(p^0, h) = (-1, 0)$  then we have*

$$\mu = \frac{3g(u)}{2u} + \frac{g'(u)}{2} \quad \text{a.e. in } \{\psi = 1\}. \quad (13)$$

*Proof.* It is a direct application of Lemma 3. Indeed, as  $\psi$  is  $H^2(0, L)$ , one has  $\psi'' = 0$  almost everywhere in the set  $\{\psi = 1\}$ . Since  $h = 0$ , from (10) it follows that

$$0 = \frac{3g(u)}{u} + g'(u) - 2\mu \quad \text{a.e. in this set.}$$

$\square$

**Proposition 15.** *Assume that  $C > \max\left(\frac{3g(u)}{2u} + \frac{g'(u)}{2}\right)$ . If  $(p^0, h) = (-1, 0)$  then  $|\{\psi > 1\}| = 0$ .*

*Proof.* We prove it by contradiction. As  $\psi$  is continuous, we can assume that there exists  $(a, b)$  such that  $\psi > 1$  in  $(a, b)$ . Thanks to Proposition 13, we have  $a > 0$  and  $b < L$ . Moreover,  $a, b$  can be chosen such that  $\psi(a) = \psi(b) = 1$ . We deduce that  $\psi$  takes its maximum in  $(a, b)$  at some interior point  $x_0$ . On one hand, we have  $-\psi''(x_0) \geq 0$ . On the other hand, since  $\psi(x_0) > 1$ ,  $h = 0$  and  $\frac{3g(u(x_0))}{u(x_0)} + g'(u(x_0)) - 2C < 0$ , we have

$$\left( \left( \frac{3g(u)}{u} + g'(u) - 2\mu \right) \psi + 2\mu(1 - \psi) + 2h \right) (x_0) < 0$$

according to Lemma 3, leading to a contradiction. We deduce that  $\psi > 1$  cannot occur.  $\square$

**Proposition 16.** *Assume that  $C > \max\left(\frac{3g(u)}{2u} + \frac{g'(u)}{2}\right)$  and that  $g$  satisfies  $(\mathcal{H}_{eq})$ . Let  $F(s) = sg(s) + 2G(s)$ . If  $(p^0, h) = (-1, 0)$  then the set  $\{\psi = 1\}$  is connected.*

*Proof.* Notice first that  $\{\psi = 1\}$  is nonempty, otherwise by Proposition 15,  $\psi < 1$  everywhere and thus  $\mu(\cdot) = 0$ , a contradiction with  $\int_0^1 g > 0$ . We prove the result by contradiction. Assume that  $\{\psi = 1\}$  is disconnected. Then, there exists an interval  $(a, b)$  such that  $\psi < 1$  in  $(a, b)$  with  $\psi(a) = \psi(b) = 1$  and  $a > 0$ ,  $b < L$  (the case  $\psi > 1$  cannot occur according to Proposition 15).

It follows that  $\mu = 0$  in  $(a, b)$  and by multiplying (1) by  $u'$  and integrating between  $(a, x)$  (for  $x < b$ ) it follows

$$u'(a)^2 + 2G(u(a)) = u'(x)^2 + 2G(u(x)). \quad (14)$$

As  $\psi \leq 1$  by Proposition 15, we know that  $\psi$  reaches local maxima at  $a$  and  $b$  and since these are interior points, one has  $\psi'(a) = \psi'(b) = 0$ . The same holds true for some interior point  $x_0$  where  $\psi$  reaches its minimum. We deduce that for  $z \in \{a, x_0, b\}$ , we have  $p'_v(z) = -u'(z)p_v(z)/u(z)$ . Recalling  $h = -p'_v u' + [\mu u - g(u)]p_v + p^0 \mu$ , it follows

$$p_v(z) [u'(z)^2 - g(u(z))u(z)] = 0.$$

Notice that  $p_v(z) = 0$  is impossible since it would imply that  $p'_v(z) = 0$ . Indeed, in that case, we would also have  $p_u(z) = 0$  and therefore  $P(\cdot) = 0$  so that the pair  $(p^0, P)$  would be trivial. We infer that  $u'(z)^2 = g(u(z))u(z)$  for every  $z \in \{a, x_0, b\}$ .

According to (14), it is clear that  $F(u(a)) = F(u(b)) = F(u(x_0))$ . Since,  $u(b) < u(x_0) < u(a)$ , we end up with a contradiction with  $(\mathcal{H}_{eq})$ .  $\square$

We have seen that, provided that  $C$  be large enough,  $\mu < C$  in  $(l_0, L_0)$ . We will now fully determine  $\mu$  by specifying  $u(l_0)$ . Indeed,  $u$  will be given as a solution of a second order ODE whose initial conditions  $u(l_0), u'(l_0)$  are known.

**Proposition 17.** *If  $(p^0, h) = (-1, 0)$ , then one has  $u(l_0) = \gamma_1$  and  $u(L_0) = \gamma_0$ .*

*Proof.* By multiplying (1) by  $u'$  and integrating between  $(-\infty, l_0)$  and  $(L_0, +\infty)$ , we obtain

$$u'(l_0)^2 = 2[G(1) - G(u(l_0))] \quad \text{and} \quad u'(L_0)^2 = -2G(u(L_0)). \quad (15)$$

Moreover, according to Propositions 14 and 16, we know that the support of  $\mu$  is connected and for  $x \in (l_0, L_0)$ , one has  $\mu(x) = \frac{3g(u(x))}{2u(x)} + \frac{g'(u(x))}{2}$ . It follows that  $u$  satisfies

$$u'' = \frac{1}{2} (g(u) + ug'(u)) \quad \text{in } (l_0, L_0). \quad (16)$$

Multiplying (1) by  $u'$  and integrating between  $(l_0, x)$  for some  $x \in (l_0, L_0)$ , we deduce thanks to (15) that

$$u'(x)^2 = g(u(x))u(x) - F(u(l_0)) + 2G(1). \quad (17)$$

Moreover, since  $p = \frac{1}{u}$  on  $(l_0, L_0)$ , we also have  $p' = -\frac{u'}{u^2}$  and  $p'' = \frac{-u''}{u^2} + \frac{2(u')^2}{u^3}$ . On the one hand, using the equations (16) and (17) on  $u$ , we deduce that for a.e.  $x \in (l_0, L_0)$

$$\begin{aligned} p'' &= p \left( \frac{-g(u)}{2u} - \frac{g'(u)}{2} + 2\frac{g(u)}{u} + \frac{2(2G(1) - F(u(l_0)))}{u^2} \right) \\ &= p \left( \frac{3g(u)}{2u} - \frac{g'(u)}{2} + \frac{2(2G(1) - F(u(l_0)))}{u^2} \right). \end{aligned} \quad (18)$$

On the other hand, using the equation provided by the Pontryagin Maximum Principle and Proposition 14, we have

$$p'' = p (\mu - g'(u)) = p \left( \frac{3g(u)}{2u} - \frac{g'(u)}{2} \right) \quad \text{a.e. in } (l_0, L_0). \quad (19)$$

Combining (18) and (19), we deduce that  $F(u(l_0)) = 2G(1)$ . Next, we set  $x = L_0$  in (17) and we use (15) to obtain  $F(u(L_0)) = 0$ . The conclusion follows.  $\square$

To finish, we prove that if  $L_0 - l_0 < L$  then we can be in the above case thanks to the following Proposition.

**Proposition 18.** *If there exists an optimal  $\mu$  such that  $L_0 - l_0 < L$ , then  $p^0 = -1$  and  $h = 0$ .*

*Proof.* Let  $\mu$  be an optimal solution such that  $L_0 - l_0 < L$  and  $u$  be the associated solution of (1). Let us assume by contradiction that  $p^0 = 0$ . Notice that up to a translation, there is no loss of generality to assume  $0 < l_0 < L_0 < L$ . According to Proposition 8, by evaluating (8) at  $(u(L_0), v(L_0), p_u(L_0), p_v(L_0), \mu(L_0))$ , we have

$$h = p_u(L_0)v(L_0) - g(u(L_0))p_v(L_0) = 0.$$

But we know from Proposition 12 that  $p^0 = 0$  implies  $h > 0$ , leading to a contradiction. Hence  $p^0 = -1$  and  $h = p_u(l_0)v(l_0) - g(u(l_0))p_v(l_0) = 0$ .  $\square$

In the next result, we sum-up all the results obtained in the case  $(p^0, h) = (-1, 0)$ .

**Corollary 3.** *If we know that  $(p^0, h) = (-1, 0)$  then we have a complete procedure to compute explicitly the function  $\mu$ :*

1. We solve the equation 
$$\begin{cases} u'' = \frac{g(u) + ug'(u)}{2}, \\ u(0) = \gamma_1, \quad u'(0) = -\sqrt{2(G(1) - G(\gamma_1))}, \end{cases}$$
2. We can define  $L'_0 = \inf\{x \in (0, L) : u(x) = \gamma_0\}$ ,
3. We keep the solution  $u$  only on the interval  $[0, L'_0]$ ,
4. We define  $\mu$  as

$$\mu(x) = \left( \frac{3g(u(x))}{2u(x)} + \frac{g'(u(x))}{2} \right) \mathbb{1}_{[0, L'_0]}(x).$$

Furthermore, any translation  $\mu(\cdot + a)$ , with  $a \in [0, L - L'_0]$ , is still a maximizer. Lastly, if  $L_0 - l_0 < L$ , we deduce that  $(p^0, h) = (-1, 0)$  and  $\mu$  is defined as above.

### 3.3 The case $p^0 = -1, h > 0$ : an intermediate case

As in the two previous sections, we split the properties of any optimal solutions that satisfy  $p^0 = -1, h > 0$  in several Propositions that we sum up into Corollary 4.

**Proposition 19.** *If  $p^0 = -1$  then the following assertions are equivalent*

$$(1) \psi(0) > 1, \quad (2) \psi(L) > 1, \quad (3) h > 0.$$

The proof of this proposition is exactly similar to the one of Proposition 13 and is therefore left to the reader.

In this case, thanks to Proposition 19 and the optimality conditions provided by the Pontryagin Maximum principle, we know that  $\mu = C$  in a neighborhood of 0 and  $L$ . Numerically, one has observed the emergence of a singular arc in the "middle" of  $(l_0, L_0)$ . The aim of this section is to prove that it is the only type of optimal solutions that can emerge.

**Proposition 20.** *If  $p^0 = -1, h > 0$  and  $C > \max_{[0,1]} g'$  then  $u(L) \geq \theta$ .*

*Proof.* We know that  $\psi(L) > 1$  according to Proposition 19. We consider two cases:  $\psi > 1$  on  $(0, L)$  or  $\psi - 1$  changes sign.

**Case 1:**  $\psi > 1$  on  $(0, L)$ . In this case, we have  $\mu(\cdot) = C$  on  $(0, L)$  and  $u(L) = \beta$  by Proposition 6. The conclusion follows.

**Case 2:**  $\psi - 1$  changes its sign. We define  $l = \sup\{x < L : \psi(x) \leq 1\}$ . One has  $\mu = C$  in  $[l, L]$ ,  $\psi(l) = 1$  and  $\psi'(l) \geq 0$ . We deduce that  $p_v(l) = \frac{1}{u(l)} > 0$  and

$$p_u(l) \leq \frac{v(l)p_v(l)}{u(l)} < 0. \quad (20)$$

One has  $\psi \geq 1$  and thus  $p_v \geq \frac{1}{u} > 0$  in  $[l, L]$ . By noticing that  $p'_u = [g'(u) - C]p_v < 0$  (by assumption on  $C$ ), we deduce that  $p_u < 0$  in the whole interval  $[l, L]$ .

Next, let us assume by contradiction that the conclusion does not hold:  $u(L) < \theta$ , then the following inequalities hold:

$$p_u(L) < 0, \quad v(L) < 0, \quad g(u(L)) < 0 \quad \text{and} \quad p_v(L) > 0.$$

The transversality conditions lead to  $(p_u v - g(u)p_v)(L) = 0$ , whence a contradiction.  $\square$

Before getting into the qualitative properties of the optimal solution  $\mu$ , we also prove that  $\psi > 0$  which will be useful all along this part.

**Lemma 4.** *If  $p^0 = -1$  and  $h > 0$  then there holds  $\psi > 0$ .*

*Proof.* Assume by contradiction that  $|\{\psi \leq 0\}| > 0$ . Hence, there exists an interval  $[a, b]$  such that  $\psi < 1$  in  $[a, b]$  and  $\psi(a) = \psi(b) = 1$  since  $\psi(0) > 1$  and  $\psi(L) > 1$  by Proposition 19. First, we remark that, in the set  $[a, b]$ ,  $h$  rewrites

$$h = \frac{-u'\psi'}{u} + \frac{u'^2}{u^2}\psi - \frac{g(u)}{u}\psi.$$

Indeed, in  $[a, b]$ ,  $h = -p'_v u' - \psi \frac{g(u)}{u}$  and  $\psi' = p'_u u + u' p_v$ . Next, let  $x_1 = \inf\{x > a : \psi(x) = 0\}$ . We deduce that  $a < x_1 < b$  and  $\psi > 0$  in  $[a, x_1[$ . On the one hand we have  $\psi'(x_1) \leq 0$ . On the other hand, we have

$$0 < h = \frac{-u'(x_1)\psi'(x_1)}{u(x_1)}.$$

Since  $\frac{-u'(x_1)}{u(x_1)} > 0$ , we deduce that  $\psi'(x_1) > 0$  and a contradiction follows.  $\square$

Next, we state that  $|\{\psi < 1\}| = 0$ .

**Proposition 21.** *If  $p^0 = -1$ ,  $h > 0$  and  $(u \mapsto \frac{F'(u)}{u})$  is concave then we have  $|\{\psi < 1\}| = 0$ .*

*Proof.* Let us recall that  $h = -p'_v u' - g(u)p_v + \mu[\psi + p^0]$ . According to Proposition 19, one has  $\psi(0) > 1$  and  $\psi(L) > 1$ . Let us hence argue by contradiction, assuming that  $\psi < 1$  in some interval  $(a, b)$ . It follows  $\psi(a) = \psi(b) = 1$  and we deduce the existence of  $x_0 \in (a, b)$  such that  $\psi(x_0) = \min_{x \in (a, b)} \psi(x)$ . Hence, we deduce  $\psi'(x_0) = 0$ . A direct computation yields

$$-p'_v(x_0) = \left( \frac{u' p_v}{u} \right) (x_0).$$

Recalling the formula for  $h$ , we deduce

$$0 < hu(x_0) = p_v(x_0)[u'(x_0)^2 - g(u)u(x_0)].$$

Next, by multiplying (1) by  $u'$  and integrating over  $(0, x_0)$ , we obtain

$$\begin{aligned} u'(x_0)^2 &\leq u'(0)^2 + 2[G(u(0)) - G(u(x_0))] + 2 \int_0^{x_0} \mu u' u \\ &\leq 2[G(1) - G(u(x_0))] + C[u(a)^2 - u(0)^2]. \end{aligned}$$

Next, we argue that  $0 < u(a) < u(0)$  and we obtain

$$0 < hu(x_0) + Cp_v(x_0)[u(0)^2 - u(a)^2] \leq p_v(x_0)[2G(1) - F(u(x_0))].$$

Since  $p_v(x_0) > 0$ , it follows  $F(u(x_0)) < 2G(1)$ . We conclude that  $\theta < u(x_0) < \gamma_1$ . We claim that it implies  $F'(u(x_0)) > 0$ . From this claim, noticing that  $-\psi''(x_0) \leq 0$ , we deduce thanks to (10) that  $-\psi''(x_0) = \frac{F'(u(x_0))}{u(x_0)}\psi(x_0) + 2h > 0$  which is a contradiction.

It remains to prove the claim  $F'(u(x_0)) > 0$ . We simply remark that  $F'(\theta)/\theta = 0$  and according to  $(\mathcal{H}_\gamma)$ , we have  $F'(\gamma_1)/\gamma_1 \geq 0$ . Then, the hypothesis  $(\mathcal{H}_{\text{concave}})$  implies that  $F'(u)/u > 0$  for any  $u \in (\theta, \gamma_1)$  and  $F'(u(x_0)) > 0$  follows. □

**Proposition 22.** *Let us assume that  $g$  satisfies  $(\mathcal{H}_{\text{concave}})$  and*

$$C > \max_{u \in [\theta, 1]} \frac{g(u)u + g'(u)u^2 + 4[G(1) - G(u)]}{2u^2}.$$

If  $p^0 = -1$ ,  $h > 0$ , then  $\psi$  does not have a maximum point inside the set  $(0, L)$ .

*Proof.* Assume by contradiction that the conclusion does not hold. Thanks to Proposition 19, we know that  $\min(\psi(0), \psi(L)) > 1$ . Therefore, if  $\psi$  takes a maximum point at some  $x_0 \in (0, L)$ , we deduce that  $\psi(x_0) > 1$ . In that case  $\mu = C$  on some interval  $(a, b)$  such that  $x_0 \in (a, b)$ . Notice first that we have  $-\psi''(x_0) \geq 0$ . We will now evaluate the right-hand side in (10) to reach a contradiction. On the other hand, we have  $\psi'(x_0) = 0$ . It follows  $(u'p_v + up'_v)(x_0) = 0$ . Recalling that

$$\begin{aligned} h &= (-u'p'_v + C(\psi - 1) - p_v g(u))(x_0) \\ &= \left( \left[ \frac{u'(x_0)}{u(x_0)} \right]^2 - \frac{g(u(x_0))}{u(x_0)} \right) \psi(x_0) + C(\psi(x_0) - 1) \end{aligned}$$

leads to

$$\left( \frac{F'(u(x_0))}{u(x_0)} - 2C \right) \psi(x_0) + 2h + 2C(1 - \psi(x_0)) = \left( \frac{g(u(x_0))}{u(x_0)} + g'(u(x_0)) + 2 \left[ \frac{u'(x_0)}{u(x_0)} \right]^2 - 2C \right) \psi(x_0).$$

Next, by multiplying (1) by  $u'$  and integrating over  $(0, x_0)$ , we deduce that

$$u'(x_0)^2 \leq u'(0)^2 + 2[G(u(0)) - G(u(x_0))] + C[u(x_0)^2 - u(0)^2].$$

Recalling Proposition 7, it follows  $u(x_0)^2 - u(0)^2 < 0$  and

$$u'(x_0)^2 \leq u'(0)^2 + 2[G(u(0)) - G(u(x_0))] = 2[G(1) - G(u(x_0))].$$

From Proposition 20 and the above inequality, we deduce

$$\begin{aligned} &\left( \frac{F'(u(x_0))}{u(x_0)} - 2C \right) \psi(x_0) + 2h + 2C(1 - \psi(x_0)) \\ &= \left( \frac{g(u(x_0))}{u(x_0)} + g'(u(x_0)) + 2 \left[ \frac{u'(x_0)}{u(x_0)} \right]^2 - 2C \right) \psi(x_0) \\ &\leq \left( \frac{g(u(x_0))u(x_0) + g'(u(x_0))u(x_0)^2 + 4[G(1) - G(u(x_0))]}{u(x_0)^2} - 2C \right) \psi(x_0) < 0 \end{aligned}$$

which contradicts (10). □

From Proposition 22, we immediately deduce

**Proposition 23.** *Let us assume that  $g$  satisfies  $(\mathcal{H}_{\text{concave}})$  and*

$$C > \max_{u \in [\theta, 1]} \frac{g(u)u + g'(u)u^2 + 4[G(1) - G(u)]}{2u^2}.$$

If  $p^0 = -1$ ,  $h > 0$ , then the set  $\{\psi = 1\}$  is connected.

We close this section with

**Corollary 4.** *Under the above assumptions, the following properties hold true for  $\mu$ : there exist  $a, b > 0$  such that  $\mu = C$  on  $]0, a[ \cup ]b, L[$  and  $\mu$  is a singular arc on  $(a, b)$ . Moreover,  $\mu$  verifies*

$$\mu = \frac{3g(u)}{2u} + \frac{g'(u)}{2} + h$$

with  $u$  the optimal solution associated to  $\mu$  which satisfies  $u(a), u(b) \in (\gamma_0, \gamma_1)$ .

*Proof.* The shape of the optimal solution is a direct consequence of Propositions 19, 23 and 21. The equation of the singular arc follows directly from Lemma 3. It remains to prove  $\gamma_0 < u(b) < u(a) < \gamma_1$ . First, we prove that  $u(a) < \gamma_1$ . Next, we prove  $u(b) > \gamma_0$ . Since  $u(a) > u(b)$  the conclusion will follow.

We first focus on  $u(a) < \gamma_1$ . From the above equation, we deduce that  $u$  satisfies in  $(a, b)$ :

$$u'' = \frac{g(u)}{2} + \frac{ug'(u)}{2} + uh.$$

It follows  $u'(x)^2 = (g(u)u)(x) - (g(u)u)(a) + h[u(x)^2 - u(a)^2] + u'(a)^2$ . Moreover by integrating (1) over  $(-\infty, a)$ , we obtain  $u'(a)^2 = 2[G(1) - G(u(a))] + C[u(a)^2 - u(0)^2]$ . It follows

$$u'(x)^2 = (g(u)u)(x) - F(u(a)) + h[u(x)^2 - u(a)^2] + 2G(1) + C[u(a)^2 - u(0)^2].$$

Moreover, on one hand, noticing that  $p = \frac{1}{u}$  on  $(a, b)$ , we have

$$\begin{aligned} p'' &= -\frac{u''}{u^2} + \frac{2u'}{u^3} \\ &= p \left( \frac{3g(u)}{2u} - \frac{g'(u)}{2} + h + \frac{2(2G(1) + C[u(a)^2 - u(0)^2] - hu(a)^2 - F(u(a)))}{u^2} \right). \end{aligned}$$

On the other hand, we have

$$p'' = p(\mu - g'(u)) = p \left( \frac{3g(u)}{2u} - \frac{g'(u)}{2} + h \right).$$

We deduce that  $2G(1) + C[u(a)^2 - u(0)^2] - hu(a)^2 - F(u(a)) = 0$ . Since,  $C[u(a)^2 - u(0)^2] - hu(a)^2 \leq 0$ , we deduce that  $u(a) \leq \gamma_1$ .

Next, we focus on  $u(b) > \gamma_0$ . From the previous computations, we deduce that  $u'(x)^2 = g(u)u(x) - hu(x)^2$  for  $x \in [a, b]$ . A direct integration provides  $u'(b)^2 = -2G(u(b)) + C[u(b)^2 - u(L)^2]$ . We deduce that  $G(u(b)) = C[u(b)^2 - u(L)^2] + hu(b)^2 \geq 0$  and we conclude that  $u(b) \geq \gamma_0$ .  $\square$

## 4 Proof of Theorem 1

*Proof of Theorem 1.* According to the above analysis (Corollaries 2, 3 and 4), the optimal problem  $(\mathcal{P}_{L,C})$  may have three types of solution

- Type A :  $\mu = C\mathbf{1}_{[0,L]}$  which corresponds to the case  $h > 0$ ,  $p^0 = 0$ ,
- Type B :  $\mu = C\mathbf{1}_{[0,a[} + \tilde{\mu}\mathbf{1}_{(a,b)} + C\mathbf{1}_{]b,L]}$  (for  $0 < a < b < L$  and  $\tilde{\mu}$  a singular arc) which corresponds to the case  $h > 0$ ,  $p^0 = -1$ ,
- Type C :  $\mu = \tilde{\mu}\mathbf{1}_{]l_0,L_0]}$  (for some  $0 < l_0 < L_0 < L$  and  $\tilde{\mu}$  a singular arc) which corresponds to the case  $h = 0$ ,  $p^0 = -1$ .

Notice that the optimal solution is of Type A if and only if  $L = L_*$  by Proposition 5. Such solutions are investigated in details at section 2.2 which, by the same token contains the proof of the first point of Theorem 1. In what follows, we hence only consider the cases  $L > L_*$ . Notice also that the optimal solution is of Type C for all  $L \geq L^*$  where  $L^* = \inf\{L > 0 : h = 0\}$ . Note also that the case where  $L > L_*$  is completely covered by Proposition 5.

**Step 1:** *proof that  $L^* < +\infty$ .*

Assume that the claim is false. It follows that for all  $L > L_*$ , we have that the optimal solution  $\mu$  is of Type B and  $\int_0^L \mu(x)dx$  is bounded by  $\int_{l_0}^{L_0} \mu_C(x)dx$  (where  $\mu_C$  designed the unique optimal solution of type C described in Corollary 3). We recall that thanks to Corollary 4, we have  $\gamma_0 \leq u(b) < u(a) \leq \gamma_1$ . It follows

$$\int_0^L \mu(x)dx \geq L \left( \min_{u \in (\gamma_0, \gamma_1)} \frac{3g(u)}{2u} + \frac{g'(u)}{2} + h \right) > L \left( \min_{u \in (\gamma_0, \gamma_1)} \frac{3g(u)}{2u} + \frac{g'(u)}{2} + h \right).$$

Since  $\frac{3g(u)}{u} + g'(u) = \frac{F'(u)}{u}$ , one has  $\min_{u \in (\gamma_0, \gamma_1)} \frac{3g(u)}{2u} + \frac{g'(u)}{2} > 0$ . This is in contradiction with the upper bound stated above. The conclusion follows.

**Step 2:** *investigation of the special case  $L = L^*$ .*

First, we remark that  $\min_{\mu \in \mathcal{A}_{L,C}} \mathcal{N}(\mu) = \mathcal{N}(\mu_{L+\varepsilon})$  for any  $\varepsilon > 0$  where  $\mu_{L+\varepsilon}$  is any solution of type C in a domain of size  $L + \varepsilon$  described in Corollary 3. Let  $u^\varepsilon$  the associated solution, letting  $\varepsilon \rightarrow 0$ , using similar argument than in Proposition 1, we deduce that there exists an unique optimal solution for  $L = L_*$  which is fully described in Corollary 3 (we omit here the technical details).  $\square$

## A Proofs of the Lemmas 1 and 2

**Proof of Lemma 1** Let us assume that  $g$  satisfies  $(\mathcal{H}_{bis})$  and  $(\mathcal{H}_{inv})$ . Since  $F'(s) = 3g(s) + sg'(s)$ , one has  $F'(1) < 0$ . Since  $F(\theta) = 2G(\theta) < 0$  and  $F(1) = 2G(1) > 0$ , we conclude by Bolzano's theorem.

**Proof of Lemma 2.** Assumption  $(\mathcal{H}_{bis})$  is obviously satisfied. Let us assume that  $\theta \in (0, 1/2)$ . An easy computation yields  $\int_0^1 g\theta = \frac{1}{12} - \frac{\theta}{6} > 0$ , whence the trueness of  $(\mathcal{H}_{inv})$ . Furthermore, one computes  $F(s) = -\frac{3}{2}s^4 + \frac{5}{3}(1+\theta)s^3 - 2\theta s^2$  and thus,  $F''(s) = -18s^2 + 10(1+\theta)s - 4\theta$ . It follows that  $F''$  is increasing on  $(0, 5(1+\theta)/18)$  and decreasing on  $(5(1+\theta)/18, 1)$ . The maximal value of  $F''$  on  $[0, 1]$  is

$$F'' \left( \frac{5(1+\theta)}{18} \right) = \frac{25(1+\theta)^2}{36} - 2\theta.$$

A straightforward study yields that  $F'' \left( \frac{5(1+\theta)}{18} \right) > 0$  for all  $\theta \in (0, 1/2)$  Using that  $F''(0) < 0$  and  $F''(1) < 0$ , it follows that  $F'$  decreases on  $(0, z_-)$ , increases in  $(z_-, z_+)$  and decreases on  $(z_+, 1)$ , with

$$z_{\pm} = \frac{5(1+\theta) \pm \sqrt{25(1+\theta)^2 - 72\theta}}{18}.$$

A lengthly but easy computation enables us to show that for every  $\theta \in (0, 1/2)$ ,  $F'(z_-) < 0$  and  $F'(z_+) > 0$ . Since  $F'(0) = 0$  and  $F'(1) = -1 + \theta < 0$ , we infer that  $F$  is successively decreasing, and then increasing on  $(0, 1)$ . Since  $F'(\theta) = \theta^2(1 - \theta) > 0$ , we get that  $F$  is increasing and then decreasing on  $(\theta, 1)$ . By observing that  $F(\theta) < 0$ , we get at the same time from this whole reasoning that  $(\mathcal{H}_\gamma)$  and  $(\mathcal{H}_{eq})$  are true.

Finally,  $(\mathcal{H}_{concave})$  is clearly true since  $d^2/ds^2[F'(s)/s] = -6$ .

## B Proof of Proposition 3

According to Proposition 1,  $\widehat{\mathcal{A}}_{L,C}$  is nonempty which allows us to consider a minimizing subsequence  $(\mu_n)_{n \in \mathbb{N}}$  for Problem  $(\mathcal{P}_2)$ . Then, according to the Banach-Alaoglu-Bourbaki theorem, it converges weakly star in  $L^\infty(0, L)$ , up to a subsequence, towards some element  $\mu^*$  in  $\widehat{\mathcal{A}}_{L,C}$ . For the sake of simplicity, in what follows, we will denote similarly any sequence and its subsequence.

According to Proposition 2, one has  $0 \leq u_n \leq 1$  a.e. in  $[0, L]$ . Using this estimate and noting furthermore that one has

$$u'_n(x) = -\sqrt{2 \int_{u_n(0)}^1 g} + \int_0^x (-g(u_n) + \mu_n u_n)$$

for every  $x \in [0, L]$ , we infer that  $(u'_n)_{n \in \mathbb{N}}$  is bounded in  $L^2(0, L)$  and that  $(u_n)_{n \in \mathbb{N}}$  is bounded in  $W^{1,\infty}(0, L)$ . By using the Arzela-Ascoli theorem, we infer that  $(u_n)_{n \in \mathbb{N}}$  converges towards some element  $u^* \in W^{1,\infty}(0, L)$ , weakly in  $H^1(0, L)$  and strongly in  $C^0([0, L])$ . Now, let us observe that equation (3) can be equivalently rewritten under the variational form: find  $u_n \in H^1(0, L)$  such that

$$\int_0^L u'_n \varphi' - \int_0^L g(u_n) \varphi = \int_0^L \mu_n u_n \varphi + \sqrt{2 \int_{u_n(L)}^0 g} \varphi(L) - \sqrt{2 \int_{u_n(0)}^1 g} \varphi(0)$$

for every  $\varphi$  in  $H^1(0, L)$ . Let us fix  $\varphi$  in  $H^1(0, L)$ . According to the convergence results above, by letting  $n$  tend to  $+\infty$ , we obtain that  $u^*$  solves the problem

$$\int_0^L (u^*)' \varphi' - \int_0^L g(u^*) \varphi = \int_0^L \mu^* u^* \varphi + \sqrt{2 \int_{u^*(L)}^0 g} \varphi(L) - \sqrt{2 \int_{u^*(0)}^1 g} \varphi(0),$$

It follows that  $u^*$  is the solution of (3) associated to the control function  $\mu = \mu^*$ . Finally, we conclude by noting that

$$\lim_{n \rightarrow +\infty} \mathcal{N}(\mu_n) = \lim_{n \rightarrow +\infty} \langle \mu_n, 1 \rangle_{L^\infty, L^1} = \langle \mu^*, 1 \rangle_{L^\infty, L^1} = \mathcal{N}(\mu^*).$$

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